星系动力学 作业二

023426910029 胡旭凡 2023.10.9

1.

(a) Gauss’s theorem

在距离平面为 ± z 处平行设置无限大高斯面

$2E\cdot S = -4\pi G\, S\, \Sigma$

$E = -2\pi G\, \Sigma$

$\Phi = -\int E\, \mathrm{d}z = 2\pi\, G\, \Sigma\, |z| + C$

(b) Possion’s equation

$\nabla^2\cdot\Phi = 4\pi G\rho\, \delta^3(z)$

$\frac{\partial^2}{\partial z^2}\Phi = 4\pi G\, \rho\, \delta^3(z)$

$\frac{\partial}{\partial z}\Phi = 2\,\pi G\, \Sigma(I(z>0)-I(z<0))$

$\Phi = 2\pi G\, \Sigma\, |z| + C_2$

2.

外部为均匀球壳，引力为0

设在 r 处， 引力势为 $\Phi(r)$, 在 r+dr 处, 引力势的改变量为 $d\,\Phi(r) = \int \frac{G\,\rho\,(r,\theta,\phi)}{r^2}\, \mathrm{d}V = 4\pi G\rho\, dr$

对环面这并不成立，设某点位于 r 处， 圆盘半径 r_0， 则其引力强度为（已假定向外为正方向）

$E(r) = -\int_0^r dR \int_0^{2\pi} d\theta\ \ G\, R\,\Sigma(R)\, \frac{r-\cos\theta\, R}{(R^2+r^2-2Rr\cos\theta)^{5/2}} + \int_r^{r_0} dR \int_0^{2\pi} d\theta\ \ G\, R\Sigma(R)\, \frac{\cos\theta\, R-r}{(R^2+r^2-2Rr\cos\theta)^{5/2}} =$

$\int_0^{r_0} dR \int_0^{2\pi} d\theta\ \ G\, R\, \Sigma(R)\, \frac{\cos\theta\, R-r}{(R^2+r^2-2Rr\cos\theta)^{5/2}}$

不妨设 $\Sigma(R) = \Sigma_0$, r =5 r_0=10

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In[ ]:= r = 5; r0 = 10;
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In[ ]:= NIntegrate[NIntegrate[R (Cos[θ] R - r)/(R^2 + r^2 - 2 R r Cos[θ])^(5/2), {R, 0, r0}], {θ, 0, 2 π}]
        数值积分       数值积分
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Out[ ]= 7.49166×10^{7}

数值积分大于0，即引力势向外衰减

3.

(a)

单极矩 $\Phi_0 = -\frac{GM}{r}$

四极矩 $\Phi_2 = -4\pi\frac{G}{5r^3}\sum_{m=-2}^{2} Q_{2m}\, Y_2^m(\theta,\phi)$

$Q_{22}\, Y_2^2(\theta,\phi)+Q_{2-2}\, Y_2^{-2}(\theta,\phi)= \frac{15}{32\pi} \sin^2\theta\, e^{2 i\phi} \int d^3 r\, r^2 \rho(r) \sin^2\theta\, e^{2 i\phi}+\frac{15\pi}{32} \sin^2\theta\, e^{-2 i\phi} \int d^3 r\, r^2 \rho(r) \sin^2\theta\, e^{-2 i\phi}$

$= \frac{15}{16\pi} \sin^2\theta(\cos 2\phi \int d^3 r\, r^2 \rho(r) \sin^2\theta \cos 2\phi\text{-}\sin 2\phi \int d^3 r\, r^2 \rho(r) \sin^2\theta \sin 2\phi)$

$Q_{21}\, Y_2^1(\theta,\phi)+Q_{2-1}\, Y_2^{-1}(\theta,\phi)= \frac{15}{8\pi} \sin^2\theta\, e^{i\phi} \int d^3 r\, r^2 \rho(r) \sin\theta \cos\theta\, e^{i\phi}+ \frac{15\pi}{8} \sin^2\theta\, e^{i\phi} \int d^3 r\, r^2 \rho(r) \sin\theta \cos\theta\, e^{-i\phi}$

$= \frac{15}{4\pi} \sin\theta \cos\theta(\cos\phi \int d^3 r\, r^2 \rho(r) \sin\theta \cos\theta \cos\phi\text{-}\sin\phi \int d^3 r\, r^2 \rho(r) \sin\theta \cos\theta \sin\phi)$

$Q_{20}\, Y_2^0(\theta,\phi)= \frac{5}{16\pi} (3\cos^2\theta\text{-}1) \int d^3 r' r'^2 \rho(r') (3\cos^2\theta\text{-}1)$

由绕z轴对称性，积分里含 ϕ 的都为零

$\Sigma_{m=-2}^2\, Q_{2m}\, Y_2^m(\theta,\phi)= \frac{5}{16\pi} (3\cos^2\theta\text{-}1) \int d^3 r'\, r'^2 \rho(r') (3\cos^2\theta\text{-}1) = \frac{5}{16\pi r^2} (2 z^2 - R^2) \int d^3 r'\, \rho(r') (2 z'^2 - R'^2)$

故考虑到二阶 $\Phi= - \frac{GM}{r} - \frac{G}{4 r^5} (2 z^2 - R^2) \int d^3 r'\, \rho(r') (2 z'^2 - R'^2)$

(b)

单极矩 $\Phi_0= - \frac{GM}{r}$

四极矩 $\Phi_2= - \frac{G}{4 r^5} (2 z^2 - R^2) \int d^3 r'\, \rho(r') (2 R^2 - z^2)$

In[ ]:=
```
Integrate[Integrate[ Integrate[Σ0 Exp[-R / Rd] R (2 z^2 - R^2) DiracDelta[z], {R, 0, +∞}],
  {z, -∞, +∞}], {ϕ, 0, 2 π}]
```
积分 积分 积分 指数形式 狄拉克δ函数

Out[ ]= ConditionalExpression[-12 π Rd^4 Σ0, Re[Rd] > 0]

In[ ]:=
```
M = Integrate[Integrate[
  Integrate[Σ0 Exp[-R / Rd] R DiracDelta[z], {R, 0, +∞}], {z, -∞, +∞}], {ϕ, 0, 2 π}]
```
积分 积分 指数形式 狄拉克δ函数

Out[ ]= ConditionalExpression[2 π Rd^2 Σ0, Re[Rd] > 0]

$\Phi_2= - \frac{G}{4 r^5} (2 z^2 - R^2)\ (-6\ R_d^2\ M) = - \frac{3 GM}{r^5}\ R_d^2\ (R^2 - 2 z^2)$

In[ ]:=
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For[i = -3, i < 4, i++, Print[SphericalHarmonicY[3, i, θ, ϕ]]]
```
For循环 打印 球面谐函数

$\frac{1}{8} e^{-3 i \phi} \sqrt{\frac{35}{\pi}} \text{Sin}[\theta]^3$

$\frac{1}{4} e^{-2 i \phi} \sqrt{\frac{105}{2\pi}} \text{Cos}[\theta] \text{Sin}[\theta]^2$

$\frac{1}{8} e^{-i \phi} \sqrt{\frac{21}{\pi}} \left(-1 + 5 \text{Cos}[\theta]^2\right) \text{Sin}[\theta]$

$\frac{1}{4} \sqrt{\frac{7}{\pi}} \left(-3 \text{Cos}[\theta] + 5 \text{Cos}[\theta]^3\right)$

$-\frac{1}{8} e^{i \phi} \sqrt{\frac{21}{\pi}} \left(-1 + 5 \text{Cos}[\theta]^2\right) \text{Sin}[\theta]$

$\frac{1}{4} e^{2 i \phi} \sqrt{\frac{105}{2\pi}} \text{Cos}[\theta] \text{Sin}[\theta]^2$

$-\frac{1}{8} e^{3 i \phi} \sqrt{\frac{35}{\pi}} \text{Sin}[\theta]^3$

考虑 三阶项（八级矩），同样由绕z轴对称性， 只有 Y_3^0 不为0；再考虑 密度以z=0为对称面，$\cos\theta$ 的奇次方的积分必然为0，故不存在八级矩。

Φ=- $\frac{GM}{r}[1+\frac{3R_d^2}{2r^4}(R^2\text{-}2z^2)]$

4.

$$R^2+(a+|z|)^2 = a^2 \sinh^2 u \sin^2 v + (a + a\cosh u\ |\cos v|)^2$$
$$= a^2 \sinh^2 u\,(1\text{-}\cos^2 v) + a^2 + a^2\cosh^2 u\,|\cos v|^2 + 2a^2\cosh u\,|\cos v|$$
$$= a^2\sinh^2 u + a^2 + a^2\,|\cos v|^2 + 2a^2\cosh u\,|\cos v|$$
$$= a^2\cosh^2 u + a^2\,|\cos v|^2 + 2a^2\cosh u\,|\cos v|$$
$$= a^2(\cosh u + |\cos v|)^2$$

$$\Phi_K = \text{-GM}\ \frac{1}{\sqrt{R^2+(a+|z|)^2}} = \frac{-GM}{a}\frac{1}{\cosh u + |\cos v|} = \frac{-GM}{a}\frac{\cosh u - |\cos v|}{\cosh^2 u - \cos^2 v} = \frac{-GM}{a}\frac{\cosh u - |\cos v|}{\sinh^2 u + \sin^2 v}$$

5.

$S(k) = -2\pi G \int_0^\infty \Sigma_0 \exp\left(-R^2/2a^2\right) J_0(kR)\, R\, dR = -2\pi G\,\Sigma_0\,\frac{1}{1/a^2}\exp(-k^2a^2/2) = -2\pi G\,\Sigma_0\,a^2 \exp(-k^2a^2/2)$

$\Phi(R) = -\frac{1}{2\pi G}\int_0^\infty -2\pi G\,\Sigma_0\, a^2 \exp\left(-k^2 a^2/2\right) J_0(kR)\, dk = \Sigma_0\, a^2 \sqrt{\frac{2\pi}{a^2}}/2 \exp\left(-\frac{R^2}{4a^2}\right) I_0\left(\frac{R^2}{4a^2}\right) =$

$\Sigma_0\, a \sqrt{\frac{\pi}{2}} \exp\left(-\frac{R^2}{4a^2}\right) I_0\left(\frac{R^2}{4a^2}\right)$

6.

(a)

In[ ]:= $M = \text{Integrate}\left[4\pi R^2 \frac{\rho s\, rs^3}{R\,(R+rs)^2}, \{R, 0, r\}\right]$
积分

Out[ ]= $\text{ConditionalExpression}\left[4\pi\, rs^3 \rho s \left(-1 + \frac{rs}{r+rs} - \text{Log}[rs] + \text{Log}[r+rs]\right), \left(\left(\text{Re}\left[\frac{rs}{r}\right] \geq 0 \,\&\&\, \frac{rs}{r} \neq 0\right) \,||\, \frac{rs}{r} \notin \mathbb{R} \,||\, \text{Re}\left[\frac{rs}{r}\right] < -1\right) \,\&\&\, \left((\text{Im}[r] \geq 0 \,\&\&\, \text{Im}[rs] \geq 0) \,||\, (\text{Im}[r] \leq 0 \,\&\&\, \text{Im}[rs] \leq 0) \,||\, \frac{\text{Im}[rs]}{\text{Im}[r]} \leq -1 \,||\, \frac{\text{Im}[rs]\,\text{Re}[r]}{\text{Im}[r]} \leq \text{Re}[rs]\right)\right]$

In[ ]:= $v = \sqrt{G\frac{M}{r}}$ // Simplify
化简

Out[ ]= $\text{ConditionalExpression}\left[2\sqrt{\pi}\sqrt{\frac{G\, rs^3 \rho s \left(-1 + \frac{rs}{r+rs} - \text{Log}[rs] + \text{Log}[r+rs]\right)}{r}}, \left(\left(\text{Re}\left[\frac{rs}{r}\right] \geq 0 \,\&\&\, \frac{rs}{r} \neq 0\right) \,||\, \frac{rs}{r} \notin \mathbb{R} \,||\, \text{Re}\left[\frac{rs}{r}\right] < -1\right) \,\&\&\, \left((\text{Im}[r] \geq 0 \,\&\&\, \text{Im}[rs] \geq 0) \,||\, (\text{Im}[r] \leq 0 \,\&\&\, \text{Im}[rs] \leq 0) \,||\, \frac{\text{Im}[rs]}{\text{Im}[r]} \leq -1 \,||\, \frac{\text{Im}[rs]\,\text{Re}[r]}{\text{Im}[r]} \leq \text{Re}[rs]\right)\right]$

$$v(r) = 2\sqrt{\frac{\pi r_s^3 \rho_s(-1+r_s/(r+r_s))+\ln((r+r_s)/r_s)}{r}}$$

(c)

In[ ]:= $\rho[R\_, z\_] := \frac{1}{\sqrt{R^2+z^2}\left(\sqrt{R^2+z^2}+1\right)^2};$

In[ ]:= $\Sigma[R\_] := \text{NIntegrate}[\rho[R, z], \{z, -\infty, +\infty\}]$
数值积分

In[ ]:= $\Sigma[0]$

Out[ ]= 297.479

In[ ]:= $\Sigma[E]$
自然常数

Out[ ]= 0.165157

```
In[ ]:= Plot[Σ[R], {R, 0, 2}, AxesLabel → {"r/rs", "ρsrs"}, PlotLabel → "Σ(R)的可视化"]
         绘图                 坐标轴标签                    绘图标签
```

Out[ ]=