# 星系动力学作业 4

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## 1

### (a)

$M(r) = \int_0^r \rho_0 \frac{r_0{}^{\gamma}}{R^{\gamma}} 4\pi R^2 \, dR = 4\pi\rho_0 \; r_0{}^{\gamma} \frac{1}{3-\gamma} (R^{3-\gamma}) |_0{}^r$

为了使质量积分有意义，$\gamma$需要小于3

### (b)

$$0 \le \gamma < 3; \; M[r\_] := 4\pi\rho 0 \, r0^{\gamma} \frac{r^{3-\gamma}}{3-\gamma}; \; \rho[r\_] := \rho 0 \frac{r0^{\gamma}}{r^{\gamma}};$$

$$\text{DSolve}\left[D[\rho[r] \times \sigma 2[r], r] == -\rho[r] \, G \frac{M[r]}{r^2}, \sigma 2[r], r\right]$$

求解… 偏导

Out[ ]= $$\left\{\left\{\sigma 2[r] \to \frac{4 G \pi r^{2-\gamma} r0^{\gamma} \rho 0}{(2-2\gamma)(-3+\gamma)} + r^{\gamma} c_1\right\}\right\}$$

In[ ]:= $$\sigma 2[r\_] := \frac{4 G \pi r^{2-\gamma} r0^{\gamma} \rho 0}{(2-2\gamma)(-3+\gamma)} + r^{\gamma} c_1;$$

$$\text{Solve}\left[\sigma 2[r0] == \sigma 0^2, c_1\right]$$

解方程

Out[ ]= $$\left\{\left\{c_1 \to \frac{r0^{-\gamma} \left(2 G \pi r0^2 \rho 0 + 3 \sigma 0^2 - 4 \gamma \sigma 0^2 + \gamma^2 \sigma 0^2\right)}{(-3+\gamma)(-1+\gamma)}\right\}\right\}$$

$$\sigma 2[r\_] := \frac{4 G \pi r^{2-\gamma} r0^{\gamma} \rho 0}{(2-2\gamma)(-3+\gamma)} + r^{\gamma} \frac{r0^{-\gamma} \left(2 G \pi r0^2 \rho 0 + 3 \sigma 0^2 - 4 \gamma \sigma 0^2 + \gamma^2 \sigma 0^2\right)}{(-3+\gamma)(-1+\gamma)};$$

### (c)

从 $\sigma^2(r)$ 表达式可以看出，当 $r \to 0$ 时

$\gamma$=2 使 $\sigma^2(r)$ 与r 无关； $0 \le \gamma < 2$ 使 $\sigma^2(r) \to 0$ ; $2 < \gamma < 3$ 时 $\sigma^2(r)$发散

对于NFW轮廓，$r \to 0$ 时接近于 $\gamma$=1 所以 $\sigma^2(r) = 0$

## 2

### (a)

由 (4.81)

$N(\epsilon, L) = 16\pi^2 \int dr \frac{L f(\epsilon, L)}{\sqrt{2(\epsilon - \Phi) - L^2/r^2}} = 8\pi^2 L f(\epsilon, L) \, T_r(\epsilon, L)$

## (b)

将分布函数的自变量由 $\epsilon$,L 换元为$\epsilon$,e

$$T_r = 2\pi\sqrt{\frac{a^3}{GM}} = \pi GM\sqrt{-\frac{1}{2\epsilon^3}} \qquad L = GM\sqrt{-\frac{1-e^2}{2\epsilon}}$$

$$N(\epsilon, e)\, d\epsilon de = N(\epsilon, L)\, d\epsilon dL = 8\pi^3 GM\sqrt{-\frac{1-e^2}{2\epsilon}}\, f(\epsilon)\, \frac{GM}{2}\sqrt{-\frac{1}{2\epsilon^3}}\, \frac{GM}{2\sqrt{-\frac{1-e^2}{2\epsilon}}}\, \frac{e}{\epsilon}\, d\epsilon de$$

$$= 2\pi^3 G^3 M^3 f(\epsilon)\sqrt{-\frac{1}{2\epsilon^3}}\, \frac{e}{\epsilon}\, d\epsilon de$$

$$= \pi^3 G^3 M^3 f(\epsilon)\sqrt{-\frac{1}{2\epsilon^3}}\, \frac{e}{\epsilon}\, d\epsilon\, 2e\, de$$

对 $\epsilon$ 的积分归一化，所以 f(e) =2ede

ref : https://joe-antognini.github.io/astronomy/thermal-eccentricities

https://articles.adsabs.harvard.edu/pdf/1919MNRAS..79..408J

## 3.

$$\frac{\partial^2\Phi(z)}{\partial z^2} = 4\pi G\,\rho(z)$$

$$\rho(z) = \int \rho_0 \frac{1}{\sqrt{2\pi\sigma_z^2}} \exp\left(-\frac{1/2\, v_z^2 + \Phi(z)}{\sigma_z^2}\right) dv_z = \int \rho_0 \frac{1}{\sqrt{2\pi\sigma_z^2}} \exp\left(-\frac{1/2\, v_z^2}{\sigma_z^2}\right) dv_z\, e^{-\phi} = \rho_0 e^{-\phi}$$

In[ ]:=
```
Integrate[ρ0/Sqrt[2 π σz^2] Exp[-vz^2/(2 σz^2)], {vz, -∞, +∞}]
```
积分 指数形式

Out[ ]=
```
ConditionalExpression[ρ0 Sqrt[1/σz^2] Sqrt[σz^2], Re[σz^2] > 0]
```

$$\frac{\partial^2\Phi(z)}{\partial z^2} = 4\pi G\,\rho_0 e^{-\phi} \Rightarrow 2\frac{\partial^2\Phi(z)/\sigma_z^2}{\partial(z/z_0)^2} = e^{-\phi} \Rightarrow 2\frac{d^2\phi}{d\zeta^2} = e^{-\phi}$$

In[ ]:=
```
DSolve[{2 ϕ''[ξ] == Exp[- ϕ[ξ]], ϕ'[0] == 0, ϕ[0] == 0}, ϕ[ξ], ξ]
```
求解微分方程 指数形式

Out[ ]=
```
{{ϕ[ξ] → Log[-1/4 (-1 - Cosh[ξ] - Sinh[ξ])^2 (-Cosh[ξ] + Sinh[ξ])]}}
```

In[ ]:=
```
Simplify[Exp[-Log[-1/4 (-1 - Cosh[ξ] - Sinh[ξ])^2 (-Cosh[ξ] + Sinh[ξ])]]]
```
化简 指⋯ 对数 双曲余弦 双曲正弦

Out[ ]=
```
Sech[ξ/2]^2
```

$$\rho(z) = \rho_0\, \mathrm{sech}^2\left(\frac{z}{2z_0}\right)$$

In[ ]:= Integrate$\left[2\,\rho 0\ \text{Sech}\left[\frac{z}{2\,z0}\right]^2, \{z, 0, +\infty\}\right]$
积分

Out[ ]= ConditionalExpression$\left[4\,z0\,\rho 0, \left(\text{Re}\left[(-1)^{z0}\right] \geq 1 \,||\, \text{Re}\left[(-1)^{z0}\right] \leq 0 \,||\, (-1)^{z0} \notin \mathbb{R}\right) \&\&\ \text{Re}[z0] > 0\right]$

故有 $\Sigma = \frac{\sigma_z^2}{2\pi G z_0} = 4\,\rho_0\, z_0$

## 4.

### (a)

In[ ]:= $\Phi[r\_] := \frac{4}{3}\pi r^2 \rho 0\, G;\ \rho 0 > 0;\ G > 0;\ A > 0;$

$\rho(r) = \int \frac{\rho_1}{(2\pi\sigma^2)^{3/2}} \exp\left(-\frac{1/2\,(v_x^2+v_y^2+v_z^2)+\Phi(r)}{\sigma^2}\right) dv^3 = \int 4\pi v^2 \frac{\rho_1}{(2\pi\sigma^2)^{3/2}} \exp\left(-\frac{v^2}{2\sigma^2}\right) dv$

$e^{-\Phi(r)/\sigma^2} = \rho_1\ e^{-\Phi(r)/\sigma^2}$

假设质光比为A， j(r)= $A\ e^{-\Phi(r)/\sigma^2}$

### (b)

In[ ]:= 2 Integrate$\left[\frac{A\,r}{\sqrt{r^2 - R^2}} \text{Exp}\left[\frac{-\Phi[r]}{\sigma^2}\right], \{r, R, +\infty\}\right]$
积分　　　　　指数形式

Out[ ]= ConditionalExpression$\left[\frac{\sqrt{3}\,A\,e^{-\frac{4 G \pi R^2 \rho 0}{3\sigma^2}}}{2\sqrt{\frac{G\rho 0}{\sigma^2}}}, \text{Re}\left[\frac{G\rho 0}{\sigma^2}\right] \geq 0\ \&\&\ \text{Re}[R] > 0\ \&\&\ \text{Im}[R] == 0\right]$

$I(R) = \frac{\sqrt{3}}{2}\frac{A}{\sqrt{G\rho_0/\sigma^2}}\exp(-\frac{4\pi G\rho_0 R^2}{3\sigma^2})$

### (c)

In[ ]:= $\text{Id}[R\_] := \frac{\sqrt{3}}{2}\frac{A}{\sqrt{G\rho 0/\sigma^2}}\text{Exp}\left[-\frac{4\pi G\rho 0 R^2}{3\sigma^2}\right];$
指数形式

In[ ]:= Solve[Id[Rc] == $\frac{1}{2}$ Id[0], Rc]
解方程

Out[ ]= $\left\{\left\{Rc \to \text{ConditionalExpression}\left[-\frac{\sqrt{\frac{3}{\pi}}\,\sigma\sqrt{2\,i\,\pi\,c_1+\text{Log}[2]}}{2\sqrt{G}\,\sqrt{\rho 0}},\ c_1\in\mathbb{Z}\right]\right\},\left\{Rc \to \text{ConditionalExpression}\left[\frac{\sqrt{\frac{3}{\pi}}\,\sigma\sqrt{2\,i\,\pi\,c_1+\text{Log}[2]}}{2\sqrt{G}\,\sqrt{\rho 0}},\ c_1\in\mathbb{Z}\right]\right\}\right\}$

In[ ]:= Solve[Rc == $\sqrt{\frac{3\,\text{Log}[2]}{4\,\pi\,G\,\rho 0}}$ σ, ρ0]
解方程

Out[ ]= $\left\{\left\{\rho 0 \to \frac{3\,\sigma^2\,\text{Log}[2]}{4\,G\,\pi\,Rc^2}\right\}\right\}$

## (d)

$R_c = \sqrt{\frac{9\,\sigma^2}{4\,\pi\,G\,\rho_{0,K}}}$ $\rho_{0,K} = \frac{9\,\sigma^2}{4\,\pi\,G\,R_c{}^2}$

$\frac{\rho_0}{\rho_{0,K}} = \frac{\ln 2}{3}$

# 5

## (a)

突然吹走物质，速度分布不变，但是势能和动能发生突变

In[ ]:= W = Integrate[$-4\,\pi\,r^2\,\rho\,\frac{4}{3}\,\pi\,r^2\,\rho\,G$, {r, 0, R}]
积分

Out[ ]= $-\frac{16}{15}\,G\,\pi^2\,R^5\,\rho^2$

In[ ]:= $-\frac{16}{15}\,G\,\pi^2\,R^5\,\rho^2$ /. $\left\{\rho \to \frac{M}{4/3\,\pi\,R^3}\right\}$

Out[ ]= $-\frac{3\,G\,M^2}{5\,R}$

In[ ]:= Solve[$M\,\sigma^2 - \frac{3\,G\,M^2}{5\,R}$ == 0, σ]
解方程

Out[ ]= $\left\{\left\{\sigma \to -\frac{\sqrt{\frac{3}{5}}\,\sqrt{G}\,\sqrt{M}}{\sqrt{R}}\right\},\left\{\sigma \to \frac{\sqrt{\frac{3}{5}}\,\sqrt{G}\,\sqrt{M}}{\sqrt{R}}\right\}\right\}$

当E>0时，系统不再受束缚

In[ ]:= Reduce$\left[\frac{1}{2} M (1-\epsilon)\frac{3 G M}{5 R}-\frac{3 G M^2 (1-\epsilon)^2}{5 R}>0 \&\& G>0 \&\& M>0 \&\& R>0, \epsilon\right]$
约化

Out[ ]= $R>0 \&\& M>0 \&\& G>0 \&\& \frac{1}{2}<\epsilon<1$

In[ ]:= Solve$\left[\frac{1}{2} M (1-\epsilon)\frac{3 G M}{5 R}-\frac{3 G M^2 (1-\epsilon)^2}{5 R}==\frac{1}{2} W, W\right]$
解方程

Out[ ]= $\left\{\left\{W\to -\frac{3 G M^2 (-1+\epsilon)(-1+2\epsilon)}{5 R}\right\}\right\}$

In[ ]:= Solve$\left[-\frac{3 G M^2 (-1+\epsilon)(-1+2\epsilon)}{5 R}==-\frac{3 G M^2 (1-\epsilon)^2}{5 r}, r\right]$
解方程

Out[ ]= $\left\{\left\{r\to \frac{-R+R\epsilon}{-1+2\epsilon}\right\}\right\}$

系统会膨胀 r'= r $\frac{M'}{2M'-M}$

## (b)

如果每次只移走少许质量，就等效于(a) 中 $\epsilon$<<1 的情形，所以系统将永远稳定 。
假设每次移动了 $\eta$，一共移动 n 次,且 $\eta=\frac{x}{n}$

M'= $\lim_{n\to\infty}(1-\eta)^n$ M =$\lim_{n\to\infty}(1-\frac{x}{n})^n$ M = $e^{-x}$ M = $(1-\epsilon)$ M

$e^{-x}=1-\epsilon$

$r_n=r_{n-1}\frac{1-\eta}{1-2\eta}$= r $\left(\frac{1-\eta}{1-2\eta}\right)^n$

r'=$\lim_{n\to\infty}$ r $\left(\frac{1-\eta}{1-2\eta}\right)^n$ = $\lim_{n\to\infty}$ r $\left(1+\frac{1}{\frac{1-2\eta}{\eta}}\right)^n$ = $\lim_{n\to\infty}$ r $\left(1+\frac{x}{n-2x}\right)^n$= $e^x$ r = $\frac{r}{1-\epsilon}$

In[2]:= Solve$\left[\frac{1-\eta}{1-2\eta}==1+\frac{1}{x}, x\right]$
解方程

Out[2]= $\left\{\left\{x\to\frac{1-2\eta}{\eta}\right\}\right\}$

In[3]:= Solve$\left[\frac{1}{m}==\frac{1-2\frac{x}{n}}{\frac{x}{n}}, n\right]$
解方程

Out[3]= $\left\{\left\{n\to\frac{(1+2m)x}{m}\right\}\right\}$

## 6.

不妨设 $r_s$=1  $4\pi G\rho_s r_s^2$=1 选取积分区域为[0.0001,10000]之间，边界条件为 $\sigma^2(10000)=0$

In[ ]:=
```
Integrate[r0/(r0 + 1)^2, {r0, 0, r}]
积分
```

Out[ ]=
```
ConditionalExpression[-1 + 1/(1 + r) + Log[1 + r], Re[r] > -1 || r ∉ ℝ]
```

In[ ]:=
```
GM[r_] := -1 + 1/(1 + r) + Log[1 + r]; ρ[r_] := 1/(r (r + 1)^2);
                           对数
```

In[ ]:=
```
β = 0;
σr = NDSolveValue[
     数值解的值
  {D[ρ[r] × σ[r], r] + 2 ρ[r]/r σ[r] β == -ρ[r] GM[r]/r^2, σ[10000] == 0}, σ, {r, 0.0001, 10000}]
   偏导
```

Out[ ]=
```
InterpolatingFunction[ Domain: {{0.0001, 1.00×10^4}} Output: scalar ]
```

In[ ]:=
```
β2 = 1;
σr2 = NDSolveValue[{D[ρ[r] × σ[r], r] + 2 ρ[r]/r σ[r] β2 == -ρ[r] GM[r]/r^2, σ[10000] == 0},
      数值解的值      偏导
  σ, {r, 0.0001, 10000}]
```

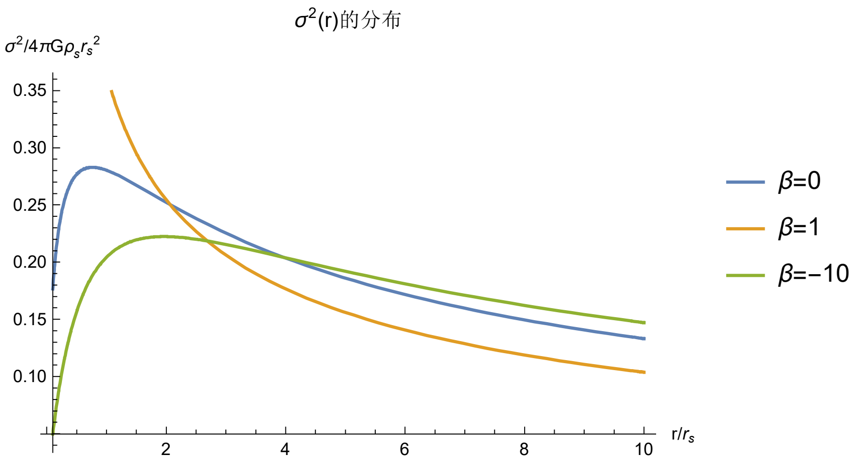
Out[ ]=
```
InterpolatingFunction[ Domain: {{0.0001, 1.00×10^4}} Output: scalar ]
```

In[ ]:=
```
β3 = -10;
σr3 = NDSolveValue[{D[ρ[r] × σ[r], r] + 2 ρ[r]/r σ[r] β3 == -ρ[r] GM[r]/r^2, σ[10000] == 0},
      数值解的值      偏导
  σ, {r, 0.0001, 10000}]
```

Out[ ]=
```
InterpolatingFunction[ Domain: {{0.0001, 1.00×10^4}} Output: scalar ]
```

In[ ]:= 
```
Plot[{(3 - 2 β) σr[r], (3 - 2 β2) σr2[r], (3 - 2 β3) σr3[r]},
  {r, 0.1, 10}, PlotLabel → "σ^2 (r)的分布",
  PlotLegends → {"β=0", "β=1", "β=-10"}, AxesLabel → {"r/r_s", "σ^2/4πGρ_s r_s^2"}]
```
绘图 绘图标签 绘图的图例 坐标轴标签

Out[ ]=




In[ ]:=

$$Ib[R\_] := 2\,NIntegrate\left[\frac{\rho[r]\,r}{\sqrt{r^2 - R^2}}, \{r, R, 10000\}\right];$$

数值积分

$$\sigma p[R] := 2\,NIntegrate\left[\left(1 - \beta\frac{R^2}{r^2}\right)\frac{\rho[r]\times\sigma r[r]\,r}{\sqrt{r^2 - R^2}}, \{r, R, 10000\}\right] \Big/ Ib[R];$$

数值积分

$$\sigma p2[R] := 2\,NIntegrate\left[\left(1 - \beta 2\frac{R^2}{r^2}\right)\frac{\rho[r]\times\sigma r2[r]\,r}{\sqrt{r^2 - R^2}}, \{r, R, 10000\}\right] \Big/ Ib[R];$$

数值积分

$$\sigma p3[R] := 2\,NIntegrate\left[\left(1 - \beta 3\frac{R^2}{r^2}\right)\frac{\rho[r]\times\sigma r3[r]\,r}{\sqrt{r^2 - R^2}}, \{r, R, 10000\}\right] \Big/ Ib[R];$$

数值积分

In[ ]:= 
```
Plot[{σp[R], σp2[R], σp3[R]}, {R, 0.1, 10}, PlotLabel → "σp^2(r)的分布",
 PlotLegends → {"β=0", "β=1", "β=-10"}, AxesLabel → {"R/rs", "σp^2/4πGρs rs^2"}]
```
绘图 绘图标签 坐标轴标签

Out[ ]=