星系动力学 作业 3

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1.

已知 $\frac{d\Omega R^2}{dR}<0$ 因为 Ω $R^2>0$ 所以 $\frac{d\Omega^2 R^4}{dR}<0$

$\frac{d\Omega^2 R^4}{dR}<0 \Rightarrow R^4\frac{d\Omega^2}{dR}+4R^3\frac{d\Omega^2}{dR}<0 \Rightarrow R^3(R\frac{d\Omega^2}{dR}+4\ \Omega^2)<0 \Rightarrow R\frac{d\Omega^2}{dR}+4\ \Omega^2<0$

即 $\kappa^2=\left(\frac{\partial^2\Phi_{\mathrm{eff}}}{\partial R^2}\right)|_{(R_g,0)}=R\frac{d\Omega^2}{dR}+4\ \Omega^2<0$ 所以 κ 对应虚数解，即对于扰动不稳定

2.

$\frac{\partial^2\Phi}{\partial z^2}=4\pi G\rho_0-\frac{1}{R}\frac{\partial}{\partial R}(R\frac{\partial\Phi}{\partial R})$

$\frac{1}{R}\frac{\partial}{\partial R}(R\frac{\partial\Phi}{\partial R})=\frac{1}{R}\frac{\partial}{\partial R}(\Omega^2R^2)=2R\Omega\frac{\partial\Omega}{\partial R}+2\Omega^2=2\Omega(R\frac{\partial\Omega}{\partial R}+\Omega)=2(A-B)(A+B)$

故 $\frac{\partial^2\Phi}{\partial z^2}=4\pi G\rho_0-2(A^2-B^2)$

3.

(a)

$$\Delta\psi=\frac{2\pi}{\kappa}\Omega=\frac{2\pi\Omega}{\sqrt{4\Omega^2+R\frac{d\Omega^2}{dR}}}=\frac{2\pi}{\sqrt{4+\frac{R}{\Omega^2}\frac{d\Omega^2}{dR}}}=2\pi\left(4+\frac{d\ln\Omega^2}{d\ln R}\right)^{-1/2}$$

(b)

$$\pi<\frac{2\pi}{\sqrt{4+\frac{R}{\Omega^2}\frac{d\Omega^2}{dR}}}<2\pi \Rightarrow 1<4+\frac{R}{\Omega^2}\frac{d\Omega^2}{dR}<4\Rightarrow -3<\frac{R}{\Omega^2}\frac{d\Omega^2}{dR}<0$$

$\Omega^2=\frac{GM}{R^3}$ $\frac{d\Omega^2}{dR}=\frac{4\pi G\rho}{R}-\frac{3GM}{R^4}$ $\frac{R}{\Omega^2}\frac{d\Omega^2}{dR}=\frac{4\pi\rho R^3}{M}-3<\frac{4\pi\rho R^3}{4/3\,\pi R^3\rho}-3=0$

同时 $\rho>0$ 故 $\frac{R}{\Omega^2}\frac{d\Omega^2}{dR}>-3$

4.

(a)

由 (2. 48) $v_c{}^2=\frac{GMr^2}{(b+a)^2a}$

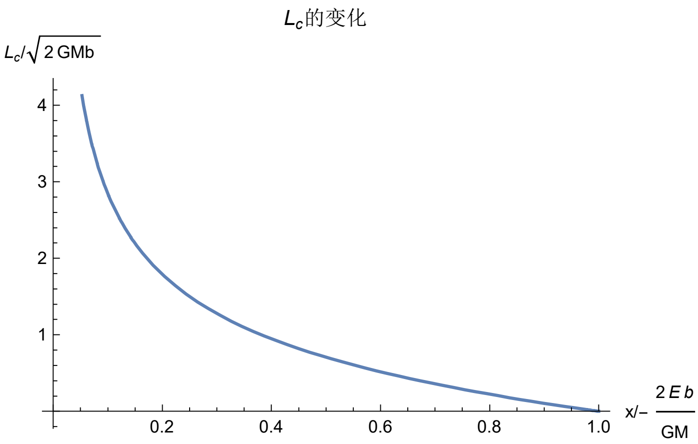
$E=\frac{GMr^2}{2(b+a)^2a}-\frac{GM}{a+b}=\frac{GMr^2-2a(a+b)GM}{2(b+a)^2a}=\frac{GM(a^2-b^2)-2a(a+b)GM}{2(b+a)^2a}=-\frac{GM}{2a}$ $x=-\frac{2Eb}{GM}=\frac{b}{a}$

$L_c=v_cr=\frac{\sqrt{GM}\ r^2}{(b+a)\sqrt{a}}=\sqrt{\frac{GM}{a}\frac{a^2-b^2}{a+b}}=\sqrt{GMb}\sqrt{\frac{1}{ab}}(a-b)=\sqrt{GMb}\left(\sqrt{\frac{a}{b}}-\sqrt{\frac{b}{a}}\right)=\sqrt{GMb}\,(x^{-1/2}-x^{1/2})$

(b)

```
In[]:= Plot[x^(-1/2) - x^(1/2), {x, 0, 1},
        AxesLabel → {"x/- 2Eb/GM", "Lc/√(2GMb)"}, PlotLabel → "Lc的变化"]
```

Out[]=



5.

由 (3.10) $\left(\frac{L}{r}\right)^2 \frac{d}{d\phi}\left(\frac{1}{r^2}\frac{dr}{d\phi}\right) - \frac{L^2}{r^3} = -\frac{d\phi}{dr} \Rightarrow \frac{L^2}{r^2}\left(\frac{1}{r^2}r'' - \frac{2}{r^3}(r')^2\right) - \frac{L^2}{r^3} = -\frac{d\phi}{dr} \Rightarrow r^2 + 2(r')^2 - r\,r'' = \frac{r^5}{L^2}\frac{d\phi}{dr}$

$$\kappa = \frac{r^2+2(r')^2-rr''}{\left[r^2+(r')^2\right]^{3/2}} = \frac{r^5\,d\phi/dr}{\left[r^2+(r')^2\right]^{3/2}L^2} = \frac{Ld\phi/dr}{r\left[L^2/(2r^2)+L^2(r')^2/(2r^4)\right]^{3/2}}$$

对于球对称势 $d\phi/dr>0$ 即曲率不为0，故 $\kappa>0$，然而图中出现了曲率半径向外的轨迹（$\kappa<0$），所以矛盾

6.

(a)

不失一般性的，假设 $\Omega_x > \Omega_y$

$\Phi = \frac{1}{2}(\Omega_x^2 x^2 + \Omega_y^2 y^2)$

$= \frac{1}{2}\Omega_y^2 (x^2+y^2) + \frac{1}{2}(\Omega_x^2 - \Omega_y^2)x^2$

$= \frac{1}{2}\Omega_y^2 R^2 + \frac{1}{2}(\Omega_x^2-\Omega_y^2)R^2\frac{1+\cos 2\phi}{2}$

$= \frac{1}{4}(\Omega_x^2+\Omega_y^2)R^2 + \frac{1}{4}(\Omega_x^2-\Omega_y^2)R^2\cos 2\phi$

$= \Phi_0(R) + \Phi_1(R,\phi)$

不妨假设共转半径为 R_b，如果 $\Phi_0(R) >> \Phi_1(R,\phi)$，有

$\Omega_b^2 R_b = \frac{d\Phi_0(R)}{dR}\Big|_{R_b} \Rightarrow \frac{1}{2}(\Omega_x^2+\Omega_y^2)R_b = \Omega_b^2 R_b$

当 $\Omega_b = \sqrt{\frac{1}{2}(\Omega_x^2+\Omega_y^2)}$ 时 R_b 存在

(b)

$\Phi_{eff}\,(x,y) = \frac{1}{2}(\,\Omega_x{}^2\, x^2 + \Omega_y{}^2\, y^2) - \frac{1}{2}\,\Omega_b{}^2(\,x^2 + y^2)$

$\vec{r}\,'' = -\nabla\,\Phi_{eff} - 2\,\vec{\Omega}_b \times \vec{r}\,'$

$\ddot{x} = (\Omega_b{}^2 - \Omega_x{}^2)\,x - 2\,\Omega_b\,\dot{y}$

$\ddot{y} = (\Omega_b{}^2 - \Omega_y{}^2)\,y + 2\,\Omega_b\,\dot{x}$

设 $x = X\,e^{\lambda t}$　$y = Y\,e^{\lambda t}$　$\Phi_{xx} = \Omega_b{}^2 - \Omega_x{}^2$　$\Phi_{yy} = \Omega_b{}^2 - \Omega_y{}^2$ 为了让结果有意义，需要系数行列式不为0

$$\begin{vmatrix} \lambda^2 + \Phi_{xx} & -2\,\lambda\,\Omega_b \\ 2\,\lambda\,\Omega_b & \lambda^2 + \Phi_{yy} \end{vmatrix} = 0$$

$\lambda^4 + \lambda^2(\Phi_{xx} + \Phi_{yy} + 4\,\Omega_b{}^2) + \Phi_{xx}\Phi_{yy} = 0$

In[ ]:=
```
Solve[λ2^2 + λ2 (Φxx + Φyy + 4 Ωb^2) + Φxx Φyy == 0, λ2]
```
解方程

Out[ ]=

$$\left\{\left\{\lambda 2 \to \frac{1}{2}\left(-\Phi xx - \Phi yy - 4\,\Omega b^2 - \sqrt{-4\,\Phi xx\,\Phi yy + \left(\Phi xx + \Phi yy + 4\,\Omega b^2\right)^2}\right)\right\},\right.$$
$$\left.\left\{\lambda 2 \to \frac{1}{2}\left(-\Phi xx - \Phi yy - 4\,\Omega b^2 + \sqrt{-4\,\Phi xx\,\Phi yy + \left(\Phi xx + \Phi yy + 4\,\Omega b^2\right)^2}\right)\right\}\right\}$$

为了形成轨道，λ^2需为负实数：

（1） $\lambda_1{}^2\,\lambda_2{}^2 = \Phi_{xx}\Phi_{yy} > 0$ （2） $\lambda_1{}^2 + \lambda_2{}^2 = -(\Phi_{xx} + \Phi_{yy} + 4\,\Omega_b{}^2\,) < 0$ （3） $\left(\Phi xx + \Phi yy + 4\,\Omega b^2\right)^2 > -4\,\Phi xx\,\Phi yy$

设 四个根分别为 $\pm\,i\alpha$,$\pm\,i\beta$ $(\alpha<\beta)$ 解直接写为

$x(t) = X_1\cos(\alpha\,t + \phi_1) + X_2\cos(\beta\,t + \phi_2)$

$y(t) = Y_1\cos(\alpha\,t + \phi_1) + Y_2\cos(\beta\,t + \phi_2)$

把解代入最初方程组，可得：

$Y_1 = \frac{\Phi_{xx}-\alpha^2}{2\,\Omega_b\,\alpha}\,X_1 = \frac{2\,\Omega_b\,\alpha}{\Phi_{yy}-\alpha^2}\,X_1$

$Y_2 = \frac{\Phi_{xx}-\beta^2}{2\,\Omega_b\,\beta}\,X_2 = \frac{2\,\Omega_b\,\beta}{\Phi_{yy}-\beta^2}\,X_2$

因为 $\Omega_x > \Omega_y$ 所以 $\Phi_{xx} < \Phi_{yy}$ ，将$\lambda^4 + \lambda^2(\Phi_{xx} + \Phi_{yy} + 4\,\Omega_b{}^2) + \Phi_{xx}\Phi_{yy} = 0$ 关于纵轴对称，得到以 α^2，β^2 为零点的函数 $f(x) = x^2 + x(\Phi_{xx} + \Phi_{yy} + 4\,\Omega_b{}^2) + \Phi_{xx}\Phi_{yy}$ 容易验证 $f(\Phi_{xx}) < 0$　$f(\Phi_{yy}) < 0$，

所以有 $\alpha^2 < \Phi_{xx} < \Phi_{yy} < \beta^2$

$X_2 = Y_2 = 0$, $Y_1 / X_1 > 0$　prograde　$|\,X_1\,| > |\,Y_1|$ //bar　为 x_1轨道

$X_1 = Y_1 = 0$, $Y_2 / X_2 < 0$　retrograde　$|\,X_2\,| < |\,Y_2|$ ⊥bar　为 x_4轨道

7.

$E = H = \Phi + \frac{L^2}{2\,r^2} + \frac{1}{2}\,p_r{}^2$　$\Phi = -\frac{G\,M}{b + \sqrt{b^2 + r^2}}$

$J_r = \frac{1}{\pi}\int_{r_1}^{r_2} p_r\,dr = \frac{1}{\pi}\int_{r_1}^{r_2}\sqrt{2\,(E - \Phi) - L^2 / r^2}\;dr$

令 $s = -\frac{GM}{b\Phi} = 1 + \sqrt{1 + \frac{r^2}{b^2}}$

In[ ]:= Simplify[$\sqrt{2\left(e+\frac{GM}{b+\sqrt{b^2+r^2}}\right)-\frac{L^2}{r^2}}$ /. {r → b $\sqrt{(s-1)^2-1}$}]
化简

Out[ ]= $\sqrt{2e+\frac{2GM}{b+\sqrt{b^2(-1+s)^2}}-\frac{L^2}{b^2(-2+s)s}}$

In[ ]:= D[b $\sqrt{(s-1)^2-1}$, s]
偏导

Out[ ]= $\frac{b(-1+s)}{\sqrt{-1+(-1+s)^2}}$

In[ ]:= Simplify[$\sqrt{2e+\frac{2GM}{b+b(s-1)}-\frac{L^2}{b^2(-2+s)s}}\ \frac{b(-1+s)}{\sqrt{-1+(-1+s)^2}}$]
化简

Out[ ]= $\frac{b(-1+s)\sqrt{\frac{-L^2+2b(-2+s)(GM+bes)}{b^2(-2+s)s}}}{\sqrt{(-2+s)s}}$

In[ ]:= Collect[$\left(-L^2+2b(-2+s)(GM+bes)\right)/b^2$, s]
合并同类项

Out[ ]= $\frac{-L^2-4bGM}{b^2}+\frac{\left(-4b^2e+2bGM\right)s}{b^2}+2es^2$

整理为 $J_r=\frac{1}{\pi}b\sqrt{-2E}\int_{s_1}^{s_2}\frac{s-1}{s(s-2)}\sqrt{(s_2-s)(s-s_1)}\,ds$

In[ ]:= Integrate[$\frac{s-1}{s(s-2)}\sqrt{(s2-s)(s-s1)}$, {s, s1, s2}, Assumptions → s1 > 0 && s2 > 0]
积分 假设

Out[ ]= ConditionalExpression[$\frac{1}{2}\pi\left(-2+s1-\sqrt{-2+s1}\sqrt{-2+s2}+s2-\sqrt{s1\,s2}\right)$,
s1 < s2 && ((s2 > 2 && s1 > 2) || s2 < 2)]

$s_1+s_2=2-\frac{GM}{Eb}$ $s_1 s_2=-\frac{4GM/b+L^2/b^2}{2E}$

$J_r=\frac{1}{2}b\sqrt{-2E}\left(-2+s_1+s_2-\sqrt{s_1s_2-2(s_1+s_2)+4}-\sqrt{s_1s_2}\right)$

In[ ]:= **Simplify**[
化简

$$\frac{1}{2} b\sqrt{-2e}\left(-2+2-\frac{GM}{eb}-\sqrt{-\frac{4GM/b+L^2/b^2}{2e}-2\left(2-\frac{GM}{eb}\right)+4}-\sqrt{-\frac{4GM/b+L^2/b^2}{2e}}\right)]$$

Out[ ]=

$$\frac{2GM+\sqrt{2}\,b\,e\left(\sqrt{-\frac{L^2}{b^2 e}}+\sqrt{-\frac{L^2+4bGM}{b^2 e}}\right)}{2\sqrt{2}\sqrt{-e}}$$

即 $J_r=\frac{GM}{\sqrt{-2E}}-\frac{1}{2}\left(L+\sqrt{L^2+4GMb}\right)$

$\Rightarrow J_r=\frac{GM}{\sqrt{-2H}}-\frac{1}{2}\left(L+\sqrt{L^2+4GMb}\right)$

$\Rightarrow \frac{GM}{\sqrt{-2H}}=J_r+\frac{1}{2}\left(L+\sqrt{L^2+4GMb}\right)$

$\Rightarrow H=-\frac{(GM)^2}{2\left[J_r+\frac{1}{2}\left(L+\sqrt{L^2+4GMb}\right)\right]^2}$