Homework Set 2
"Galactic Dynamics" course at SJTU

Due Date: 2pm on Nov. 1 (Wednesday), 2023

Warning: please start working on these problems as early as possible. 本次作业需要编程计算并画图。编程语言可自选，建议使用 python 编程语言。

1. (5 points) [Problem 2.3 of BT08] Show that the potential of an infinite razor-thin sheet of surface density Σ in the plane $z = 0$ is $\Phi = 2\pi G\Sigma|z| +$ constant, (a) using Gauss's theorem, and (b) from Poisson's equation.

2. (10 points) [BT08, Problem 2.16] Prove that the potential $\Phi(r)$ is a non-decreasing function of r in any spherical system. Does the same conclusion hold in an axisymmetric razor-thin disk? If so, prove it; if not, find a counter-example.

3. (10 points) [BT08, Problem 2.10] Consider an axisymmetric body whose density distribution is $\rho(R, z)$ and total mass is $M = \int d^3\boldsymbol{r}\rho(R, z)$. Assume that the body has finite extent, $\rho(R, z) = 0$ for $r^2 = R^2 + z^2 > r^2_{\max}$, and is symmetric about its equator, that is, $\rho(R, -z) = \rho(R, z)$.

   (a) Show that at distances large compared to $r_{\max}$, the potential arising from this body can be written in the form

$$\Phi(R, z) \simeq -\frac{GM}{r} - \frac{G}{4}\frac{R^2 - 2z^2}{r^5}\int d^3\boldsymbol{r'}\rho(R', z')(R'^2 - 2z'^2), \qquad (1)$$

   where the error is of order $(r_{\max}/r)^2$ smaller than the second term.
   (b) Show that at large distances from an exponential disk with surface density $\Sigma(R) = \Sigma_0 \exp(-R/R_d)$, the potential has the form

$$\Phi(R, z) \simeq -\frac{GM}{r}[1 + \frac{3R_d^2(R^2 - 2z^2)}{2r^4} + \mathrm{O}(R_d^4/r^4)], \qquad (2)$$

   where M is the mass of the disk.

4. (10 points) [Problem 2.6 of BT08] Defining prolate spheroidal coordinates (u, v) by $R = a \sinh u \sin v$, $z = a \cosh u \cos v$, where $a > 0$ is a constant, show that $R^2 + (a + |z|)^2 = a^2(\cosh u + |\cos v|)^2$. Hence show that the potential (2.68a) of the Kuzmin disk can be written

$$\Phi_{\mathrm{K}}(u, v) = -\frac{GM}{a}\frac{\cosh u - |\cos v|}{\sinh^2 u + \sin^2 v}.$$

   In §3.5.3 we show that this potential is an example of a Stäckel potential, in which orbits admit an extra isolating integral.

5. (15 points) The Gaussian disk. A razor-thin disk has surface density $\Sigma(R) = \Sigma_0 \exp\left(-R^2/2a^2\right)$. Compute its potential $\Phi(R)$ in the disk plane.
You may need the following formulae of from the book by Gradshteyn & Ryzhik (2000):

$$\int_0^\infty e^{-\alpha x^2} J_\nu(\beta x) \mathrm{d}x = \frac{\sqrt{\pi}}{2\sqrt{\alpha}} \exp\left(-\frac{\beta^2}{8\alpha}\right) I_{\frac{1}{2}\nu}\left(\frac{\beta^2}{8\alpha}\right) \quad [\mathrm{Re}\,\alpha > 0, \quad \beta > 0, \quad \mathrm{Re}\,\nu > -1] \tag{3}$$

$$\int_0^\infty x^{\nu+1} e^{-\alpha x^2} J_\nu(\beta x) \mathrm{d}x = \frac{\beta^\nu}{(2\alpha)^{\nu+1}} \exp\left(-\frac{\beta^2}{4\alpha}\right) \quad [\mathrm{Re}\,\alpha > 0, \quad \mathrm{Re}\,\nu > -1] \tag{4}$$

6. (20 points) For a spherical system with the NFW mass density profile

$$\rho = \frac{\rho_s r_s^3}{r\,(r + r_s)^2}$$

.

(1). Derive its circular rotation curve profile $v_{\mathrm{circ}}(r)$.
(2). With the help of numerical integrals, plot the projected surface density profile $\Sigma(R)$. Scale Σ and R with $\rho_s r_s$ and r_s, respectively.