Homework Set 3
“Galactic Dynamics” course at SJTU

Due Date: 5pm, Dec 13 (Wednesday), 2023

Warning: please start working on these problems as early as possible. Warning: please start working on these problems as early as possible.

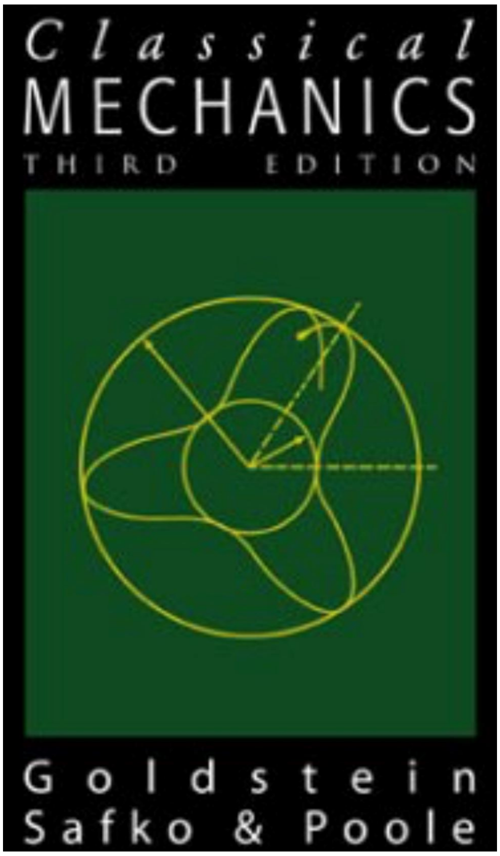
1. (10 points) [BT08 Problem 3.8] Prove that circular orbits in a given potential are unstable if the angular momentum per unit mass on a circular orbit decreases outward. Hint: evaluate the epicycle frequency.

2. (10 points) [BT08 Problem 3.18] Let $\Phi(R, z)$ be the Galactic potential. At the solar location, $(R, z) = (R_0, 0)$, prove that
$$\frac{\partial^2 \Phi}{\partial z^2} = 4\pi G \rho_0 + 2\left(A^2 - B^2\right)$$
where ρ_0 is the density in the solar neighborhood and A and B are the Oort constants. Hint: use equation (2.73).

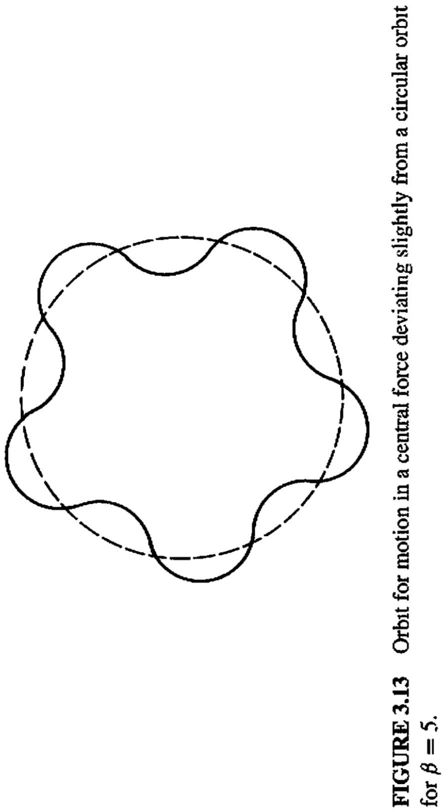
3. (10 points) [BT08, Problem 3.10 (b) and 3.16] $\Delta\psi$ denotes the increment in azimuthal angle during one complete radial cycle of an orbit.
(a) Prove in the epicycle approximation that along orbits in a potential with circular frequency $\Omega(R)$,
$$\Delta\psi = 2\pi(4 + \frac{\mathrm{d}\ln\Omega^2}{\mathrm{d}\ln R})^{-1/2}.$$
(b) Using the epicycle approximation, prove that the azimuthal angle $\Delta\psi$ between successive pericenters lies in the range $\pi \le \Delta\psi \le 2\pi$ in the gravitational field arising from any spherical mass distribution in which the density decreases outwards.

4. (15 points) [BT08, Problem 3.3] Show that the energy of a circular orbit in the isochrone potential (2.47) is $E = -GM/(2a)$, where $a = \sqrt{b^2 + r^2}$. Let the angular momentum of this orbit be $L_\mathrm{c}(E)$. (1). Show that
$$L_\mathrm{c} = \sqrt{GMb}\left(x^{-1/2} - x^{1/2}\right), \quad \text{where} \quad x \equiv -\frac{2Eb}{GM}.$$
(2) Plot out $L_c/(\sqrt{GMb})$ as a function of x.

5. (10 points) Prove that the orbits in Figure 1(a) and Figure 1(b) are impossible in a spherical potential (they are the cover page of the internationally renowned textbook *Classical Mechanics* (Goldstein et al. 2001), and Figure 3.13 in their book). You may use the curvature of a plane curve $R(\phi)$ in polar coordinates:
$$\kappa = \frac{R^2 + 2(R')^2 - RR''}{[R^2 + (R')^2]^{3/2}},$$
where$R' = \mathrm{d}R/\mathrm{d}\phi$. The local radius of curvature is κ^{-1}. Prove that the curvature of an orbit with energy E and angular momentum L in the spherical potential $\Phi(r)$ is
$$\kappa = \frac{L\mathrm{d}\Phi/\mathrm{d}r}{r[2(E - \Phi(r))]^{3/2}}.$$



(a) The cover of *Classical Mechanics*



(b) Figure 3.13 in *Classical Mechanics*

Figure 1: The impossible orbits

Hint: use Equation 3.10 in BT08.

6. (15 points) A Freeman bar is a razor-thin elliptical disk that are stationary in a frame that rotates at angular speed Ω_b (Freeman 1966). In this frame, the outer boundary of the disk is elliptical, $x^2/a^2 + y^2/b^2 = 1$, and the potential is given by

$$\Phi(x, y) = \frac{1}{2}\left(\Omega_x^2 x^2 + \Omega_y^2 y^2\right)$$

   (a) Is there a bar corotation radius?

   (b) Derive the analytical form of the orbits in the rotating Freeman bar potential, and relate them to the terminology of x_1, x_2, x_3 x_4 orbits.

7. (15 points) [BT08 Problem 3.41] From equations (3.39b) and (3.190), show that the radial action J_r of an orbit in the isochrone potential (2.47) is related to the energy E and angular momentum L of this orbit by

$$J_r = \sqrt{GMb}\left[x^{-\frac{1}{2}} - f(L)\right],$$

where $x \equiv -2Eb/(GM)$ and f is some function. Use equation (3.327) to show that $f(L) = \left(\sqrt{l^2+1} - l\right)^{-1} = \sqrt{l^2+1} + l$, where $l \equiv |L|/(2\sqrt{GMb})$, and hence show that the isochrone Hamiltonian can be written in the form (3.226a).