Homework Set 4
"Galactic Dynamics" course at SJTU

Due Date: 2pm, Jan. 8 (Monday), 2024

Warning: please start working on these problems as early as possible. 本次作业需要编程计算并画图。编程语言可自选，建议使用 *python* 编程语言。

1. (10 points) The goal of this problem is to explore the behavior of the velocity dispersion near the center of a spherical galaxy. At radii $r < r_0$ assume that the density has the power-law form

$$\rho(r) = \rho_0 \frac{r_0^\gamma}{r^\gamma}, \quad 0 \leq \gamma < 3. \tag{1}$$

Assume that the velocity dispersion is isotropic at all radii and equal to σ_0^2 at r_0.

(a) Why is the constraint $\gamma < 3$ necessary?

(b) What is the dispersion $\sigma^2(r)$ for $r < r_0$?

(c) For what value(s) of γ is $\sigma^2(r)$ independent of r as $r \to 0$? For what range of γ does $\sigma^2(r) \to 0$ as $r \to 0$? For what range of γ does $\sigma^2(r)$ diverge as $r \to 0$? What is the case for the NFW profile?

2. (20 points) [BT08, Problem 4.8] Consider a spherical system with DF $f(\mathcal{E}, L)$. Let $N(\mathcal{E}, L)\mathrm{d}\mathcal{E}\mathrm{d}L$ be the fraction of stars with $\mathcal{E}$ and L in the ranges $(\mathcal{E}, \mathcal{E} + d\mathcal{E})$ and $(L, L + \mathrm{d}L)$.
(a) Show that

$$N(\mathcal{E}, L) = 8\pi^2 L f(\mathcal{E}, L) T_r(\mathcal{E}, L). \tag{2}$$

where T_r is the radial period defined by equation (3.17).
(b) A spherical system of test particles with ergodic DF surrounds a point mass. Show that the fraction of particles with eccentricities in the range $(e, e + \mathrm{d}e)$ is $2e\,\mathrm{d}e$.

3. (20 points) [BT08, Problem 4.21] We may study the vertical structure of a thin axisymmetric disk by neglecting all radial derivatives and assuming that all quantities vary only in the coordinate z normal to the disk. Thus we adopt the form $f = f(E_z)$ for the DF, where $E_z \equiv \frac{1}{2}v_z^2 + \Phi(z)$. Show that for an isothermal disk in which $f = \rho_0 \left(2\pi\sigma_z^2\right)^{-1/2} \exp\left(-E_z/\sigma_z^2\right)$, the approximate form (2.74) of Poisson's equation may be written

$$2\frac{\mathrm{d}^2\phi}{\mathrm{d}\zeta^2} = \mathrm{e}^{-\phi}, \quad \text{where} \quad \phi \equiv \frac{\Phi}{\sigma_z^2}, \quad \zeta \equiv \frac{z}{z_0}, \quad \text{and} \quad z_0 \equiv \frac{\sigma_z}{\sqrt{8\pi G\rho_0}}$$

By solving this equation subject to the boundary conditions $\phi(0) = \phi'(0) = 0$, show that the density in the disk is given by (Spitzer 1942)

$$\rho(z) = \rho_0 \,\mathrm{sech}^2\left(\frac{1}{2}z/z_0\right). \tag{3}$$

Show further that the surface density of the disk is

$$\Sigma = \frac{\sigma_z^2}{2\pi G z_0} = 4\rho_0 z_0. \tag{4}$$

4. (20 points) Dark matter usually dominate over baryons in most dwarf spheroidal galaxies, thus stars in such a galaxy may be considered as massless test particles in a background dark matter potential. Consider a collection of massless but luminous particles (stars) inside a large uniform-density dark matter halo (thus we have effectively assumed the core radius of dark matter halo is much larger than that of the stars). Suppose that the particles have an isothermal distribution function:

$$f(\varepsilon) = \frac{\rho_1}{(2\pi\sigma^2)^{3/2}} \exp\left(\frac{\varepsilon}{\sigma^2}\right)$$

Where ε is the relative energy per unit mass and σ is the one-dimensional velocity dispersion.

(a) Derive the spherical luminosity density distribution $j(r)$.

(b) Derive the projected surface brightness distribution $I(R)$.

(c) Derive the central dark matter density ρ_0 as a function of the core radius R_{c} and σ. Note that the core radius R_{c} is defined as $I(R_{\mathrm{c}}) = 1/2\, I(0)$.

(d) In the previous problem, we used King's core fitting method to derive the central mass density of a galaxy (here we call it $\rho_{0,K}$), where we effectively assumed the dark matter halo has the same R_{c} as the stars. What is the ratio of $\rho_0/\rho_{0,K}$?

5. (20 points) Consider an isolated spherical system that is initially in dynamical equilibrium. Its total mass is M and its radius is r. Assume that the density of the system is nearly uniform.

(a) Now *instantaneously* remove a fraction ϵ of its mass, $0 \leq \epsilon \leq 1$, so that the final mass is $M' = (1-\epsilon)M$. This is a common practice in some galaxy formation simulations with strong supernova feedback: the first supernovae in a young dwarf galaxy can instantaneously blow away all gas that has not already turned into stars. Will the system collapse or expand? Calculate the final equilibrium radius r' as a function of r and ϵ. For which values of ϵ is the system bound and for which values is it unbound? *Hint: apply the Virial Theorem.*

(b) Consider the same situation as in (a), but suppose instead that the mass is removed slowly, so the system always has time to re-adjust to the mass loss and (if it can) stay in equilibrium. Now what happens to the system? Calculate the final equilibrium radius r' as a function of r and the final mass M'. For which values of $M' = (1-\epsilon)M$ is the system bound and for which values is it unbound?

6. (30 points) For a spherical system with the NFW mass density profile

$$\rho = \frac{\rho_s r_s^3}{r\,(r+r_s)^2}$$

With the help of numerical integrals, plot the 3D velocity dispersion profile $\sigma^2(r)/(4\pi G\rho_s r_s^2)$ and projected velocity dispersion profile $\sigma_p^2(R)/(4\pi G\rho_s r_s^2)$ for 3 orbital anisotropy parameters ($\beta = 0$, $\beta = 1$, and $\beta = -10$). Please scale the radius by r_s.