Mathematical Specialists

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Contents

The pourpose

This document was written to be used as a summary to help revise the content covered mathematical specialists. For any inquiries, feedback, and further explanations, contact lachlanprivate@duck.com or through the discord server: https://discord.gg/6P8rddkXFr . I encourage you to let me know of any topic I missed, how I could explain it better, or how it could be reworded or formatted to be more helpful in it's purpose. The goal of this document is to be a comprehensive summary of everything you need to know.

0.1 Important symbols

Symbol	Mathematical definition	Simple definition
$\cup$	Union	*A and B*, or think as in 'add'.
$\cap$	Intersection	*A or B*, or think as in 'multiply'.
$n!$	$n \times (n-1) \times (n-2) \times (n-3)... \times 3 \times 2 \times 1$	The product of integers between the given value and 1.
^nP_r	$\frac{n!}{(n-r)!}$	The number of combinations there are of length r from a group of length n where the order matters.
^nC_r	$\frac{\frac{n!}{(n-r)!}}{r!}$ or $\frac{n!}{r!(n-r)!}$	The number of combinations there are of length r from a group of length n where the order does not matter.
$\binom{n}{r}$	^nC_r	The same as the combinations but in a different notation.

0.2   Addition principle

If there are n ways of performing operation A, and m ways of performing operation B, there are $n + m$ ways of performing operation $A \cap B$.

As in, let there be 2 ways to perform task A, and 3 ways to perform task B. If there is an option to perform A *or* B, there is a total of $2 + 3 = 5$ ways to perform an operation.

0.3   Multiplication principle

If there are n ways of performing operation A, and m ways of performing operation B, there are $n \times m$ ways of performing operation $A \cup B$.

As in, let there be 4 ways to perform task A, and 5 ways to perform task B. With considering 1 way to perform A, there are 5 ways to perform B, repeat the process through the number of ways to perform A.

	$B, 1$	$B, 2$	$B, 3$	$B, 4$	$B, 5$
$A, 1$	$(1, 1)$	$(1, 2)$	$(1, 3)$	$(1, 4)$	$(1, 5)$
$A, 2$	$(2, 1)$	$(2, 2)$	$(2, 3)$	$(2, 4)$	$(2, 5)$
$A, 3$	$(3, 1)$	$(3, 2)$	$(3, 3)$	$(3, 4)$	$(3, 5)$
$A, 4$	$(4, 1)$	$(4, 2)$	$(4, 3)$	$(4, 4)$	$(4, 5)$

Where there are $4 \times 5 = 20$ ways to perform the operations.

0.4   Factorials

Factorials is the result of multiplying all of the integers between the given integer and 1. This is given the notation and formula:

$$n! = n \times (n - 1) \times (n - 2) \times (n - 3)... \times 3 \times 2 \times 1$$

Example:

$$5! = 5 \times 4 \times 3 \times 2 \times 1$$
$$= 120$$

0.4.1   Dividing factorials

When given some factorial over another factorial in such cases as:

$$\frac{4!}{2!}$$

$$\frac{4 \times 3 \times 2 \times 1}{2 \times 1}$$

$$\frac{4 \times 3 \times \cancel{2 \times 1}}{\cancel{2 \times 1}}$$

$$4 \times 3$$

$$12$$

0.4.2   Special cases

$$0! = 1$$

0.5 Permutations

Permutations is the number of ways of choosing r things from n distinct things where *order matters*. This is given the notation and formula:

$$^nP_r = \frac{n!}{(n-r)!}$$

Example:

$$
\begin{aligned}
^6P_4 &= \frac{6!}{(6-4)!} \\
&= \frac{6!}{2!} \\
&= \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{2 \times 1} \\
&= \frac{6 \times 5 \times 4 \times 3 \times \cancel{2 \times 1}}{\cancel{2 \times 1}} \\
&= 6 \times 5 \times 4 \times 3 \\
&= 360
\end{aligned}
$$

0.5.1 Permutations in a circle

In cases where the positions being concerned is a circle such as the permutations of a circular seating arrangement, if the standard permutations formula is applied, there are several over-counting of the like arrangement in a different perspective. In these cases apply the formula:

$$\frac{^nP_r}{r} = \frac{\frac{n!}{(n-r)!}}{r}$$

0.5.2 Like objects repetitions

The number of ways of arranging n objects made up of indistinguishable objects, n_1 in the first group, n_2 in the second group and so on, is:

$$\frac{n!}{n_1!n_2!n_3!...n_r!}$$

Example:

Find the number of permutations of the letters in the world WOOLLOOMOOLOO.
There are 8 'O's and 3 'L's
Permutations:

$$\frac{13!}{8!3!}$$

$$\frac{13 \times 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{(8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1) \times (3 \times 2 \times 1)}$$

$$\frac{13 \times 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{(8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1) \times (3 \times 2 \times 1)}$$

$$\frac{13 \times 12 \times 11 \times 10 \times 9}{3 \times 2}$$

$$\frac{13 \times 12 \times 11 \times 10 \times 9}{3 \times 2}$$

$$13 \times 6 \times 11 \times 10 \times 3$$

$$25,740$$

0.5.3 Restrictions

When considering restrictions, deal with the restrictions first.
Example:

Find the number of arrangements of the letters of the word DARWIN beginning and ending with a vowel.
Number of letters = 6
Number of positions = 6
Beginning and end must be a vowel, so the available positions are decreased by 2: number of positions - 2 = 4
Since 2 vowels must be used: number of letters - 2 = 4
Permutations:

$$^4P_4$$

$$4!$$

$$4 \times 3 \times 2 \times 1$$

$$24$$

0.5.4 Grouped items

When items are grouped together, treat each group as a single object. Find the number of arrangements of the groups, then multiply by the number of arrangement within each group.
Example:

Find the number of arrangement of the letters of the word EQUALS if the vowels are kept together.
Number of vowels = 3
Number of letters = 6
Number of positions = 6
As the 3 vowels are grouped together, the number of positions decreases:
number of positions - $3_{\text{for the members of the group}}$ + $1_{\text{for the group}}$ = 4
Permutations within vowel group:

$$^3P_3 = 3! = 3 \times 2 = 6$$

Permutations of all positions:

$$^4P_4 = 4! = 4 \times 3 \times 2 = 24$$

Multiply together:

$$6 \times 24 = 144$$

0.5.5 Special cases

$$^nP_n = n!$$
$$^nP_0 = 1$$

0.6 Combinations

Combinations is the number of ways of choosing or selecting r objects from n distinct objects where *order does not matter*. This is given the notation and formula:

$$^nC_r = \frac{^nP_r}{r!}$$

$$^nC_r = \frac{\frac{n!}{(n-r)!}}{r!}$$

Example:

$$^4C_2 = \frac{^4P_2}{2!}$$
$$= \frac{\frac{4!}{(4-2)!}}{2!}$$
$$= \frac{\frac{4!}{2!}}{2!}$$
$$= \frac{\frac{4\times3\times2}{2}}{2}$$
$$= \frac{\frac{4\times3\times\cancel{2}}{\cancel{2}}}{2}$$
$$= \frac{4\times3}{2}$$
$$= \frac{12}{2}$$
$$= 6$$

0.6.1 Combinations with restrictions

Combinations with specific terms

When given a restriction that something must be something, consider the restriction first:
As an example:

Grace belongs to a group of 8 workers. How many ways can a team of four workers be selected if Grace must be on the team?

If there is no restriction, it would be equal to $^8C_4$. But as Grace must be on the team, n must be equal to $8-1$ and r must be also $4-1$

Therefore, the answer is:

$$^7C_3 = 35$$

35 total combinations.

Combinations from multiple groups

Permutations and Combinations combined

0.7 Pascal's triangle

The Pascal's triangle is a pattern formed by adding the top 2 adjacent numbers and a 1 is placed on either side of the bottom row to resemble a triangle:

$n=0$:										1										
$n=1$:									1		1									
$n=2$:								1		2		1								
$n=3$:							1		3		3		1							
$n=4$:						1		4		6		4		1						
$n=5$:					1		5		10		10		5		1					
$n=6$:				1		6		15		20		15		6		1				
$n=7$:			1		7		21		35		35		21		7		1			
$n=8$:		1		8		28		56		70		56		28		8		1		
$n=9$:	1		9		36		84		126		126		84		36		9		1	
$n=10$:	1	10		45		120		210		252		210		120		45		10		1

Each element in the Pascal's triangle can be used to calculate combinations, hence, the triangle can be written using Combinations notation (^nC_r):

$n = 0$: $\qquad\qquad\qquad\qquad\qquad\qquad ^0C_0$

$n = 1$: $\qquad\qquad\qquad\qquad\qquad ^1C_0 \qquad\quad ^1C_1$

$n = 2$: $\qquad\qquad\qquad\qquad ^2C_0 \qquad\quad ^2C_1 \qquad\quad ^2C_2$

$n = 3$: $\qquad\qquad\qquad ^3C_0 \qquad\quad ^3C_1 \qquad\quad ^3C_2 \qquad\quad ^3C_3$

$n = 4$: $\qquad\qquad ^4C_0 \qquad\quad ^4C_1 \qquad\quad ^4C_2 \qquad\quad ^4C_3 \qquad\quad ^4C_4$

$n = 5$: $\qquad ^5C_0 \qquad\quad ^5C_1 \qquad\quad ^5C_2 \qquad\quad ^5C_3 \qquad\quad ^5C_4 \qquad\quad ^5C_5$

Pascal's triangle shows that the rth element of the nth row of Pascal's triangle is given by ^nC_r. It is assumed that the 1 at the beginning of each row is the 0th element. This gives the *Pascal's identity*:

$$^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r \text{ for } 0 < r < n$$

The Pascal's triangle can be extended to the binomial theorem, where the rule for expanding an expression such as $(a + b)^n$. Where:

$$(a + b)^0 = 1a^0b^0$$
$$(a + b)^1 = 1a^1b^0 + 1a^0b^1$$
$$(a + b)^2 = 1a^2b^0 + 2a^1b^1 + 1a^0b^2$$
$$(a + b)^3 = 1a^3b^0 + 3a^2b^1 + 3a^1b^2 + 1a^0b^3$$
$$(a + b)^4 = 1a^4b^0 + 4a^3b^1 + 6a^2b^2 + 4a^1b^3 + 1a^0b^4$$
$$(a + b)^5 = 1a^5b^0 + 5a^4b^1 + 10a^3b^2 + 10a^2b^3 + 5a^1b^4 + 1a^0b^5$$

0.8 Non-right-ange-triangles

0.8.1 Sine rule

The sine rule is:
$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$
Where a, b, and c are lengths of sides of a triangle, and A, B, and C are angles opposite the sides.

0.8.2 Cosine rule

The cosine rule is:
$$c^2 = a^2 + b^2 - 2ab\cos\theta$$
Where c is the length of side opposite θ, and a and b are the other two sides.

0.9 Vectors

Vectors have a magnitude and a direction. There are three ways for writing vectors, either the Cartesian form/component from or the polar form. Vectors are represented by $\underset{\sim}{a}$. the Cartesian form is written as:

$$\underset{\sim}{v} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

The component form notation is written as:
$$\underset{\sim}{v} = 2\hat{i} + 3\hat{j}$$

And polar from is written as:
$$\underset{\sim}{v} = (r, \theta)$$

Where r is the radius and θ is the direction in angles.

$\hat{i}$ is the unit vector in the x-axis, and $\hat{j}$ is the unit vector in the y-axis. Unit vectors are the basis vectors and are multiplied by scalars to move the tip of the vector and represent different vectors.

0.9.1   Addition of vectors

Let $\underset{\sim}{a} = 3\hat{i} + -2\hat{j}$, and $\underset{\sim}{b} = 7\hat{i} + 1\hat{j}$
And the addition of these vectors $\underset{\sim}{a} + \underset{\sim}{b} = \underset{\sim}{c}$

$$\underset{\sim}{c} = a + b$$
$$\underset{\sim}{c} = 3\hat{i} + -2\hat{j} + 7\hat{i} + 1\hat{j}$$
$$\underset{\sim}{c} = 10\hat{i} + -1\hat{j}$$

0.9.2   Multiplication of vectors by a scalar

Let $\underset{\sim}{x} = 4\hat{i} + -5\hat{j}$
And $\underset{\sim}{s} = 2\underset{\sim}{x}$

$$\underset{\sim}{s} = 2x$$
$$\underset{\sim}{s} = 2(4\hat{i} + -5\hat{j})$$
$$\underset{\sim}{s} = 8\hat{i} + -10\hat{j}$$

0.9.3   Calculating the magnitude of a vector

The magnitude of a vector is written as $|\underset{\sim}{v}|$, and to calculate the magnitude of a vector use the pythagoras's theorem:
Let $\underset{\sim}{w} = 3\hat{i} + 4\hat{j}$

$$\underset{\sim}{w} = 3\hat{i} + 4\hat{j}$$
$$|\underset{\sim}{w}| = \sqrt{3^2 + 4^2}$$
$$|\underset{\sim}{w}| = 5$$

0.9.4   Calculating the angle of a vector

To calculate the angle of a vector, use the sine, cosine, and tangent ratios.

The trigonometric ratio are as follows:

Name	Ratio
Sine	$\frac{\text{Opposite}}{\text{Hypotenous}}$
Cosine	$\frac{\text{Adjecent}}{\text{Hypotenous}}$
Tangent	$\frac{\text{Opposite}}{\text{Adjecent}}$

For example:
Let $\underset{\sim}{a} = 3\hat{i} + 4\hat{j}$
To calculate the angle of $\underset{\sim}{a}$, I will use the tangent rule:

$$\underset{\sim}{a} = 3\hat{i} + 4\hat{j}$$
$$\tan\theta = \frac{4}{3}$$
$$\tan^{-1}(\frac{4}{3}) = \theta$$
$$= 53.1301024°$$

0.9.5   The negative of a vector

The negative of a vector is the the vector of same length, but pointing in the opposite direction.
For example:
Let $\underset{\sim}{a} = -1\hat{i} + 3\hat{j}$
The inverse or the negative vector would be $-\underset{\sim}{a} = 1\hat{i} + -3\hat{j}$
As you can see, $\underset{\sim}{a}$ points off to the left and up, and $-\underset{\sim}{a}$ points to the right and down.

0.9.6 Converting between component form and polar form

To convert between the component form and polar from, use the trigonometric ratios.
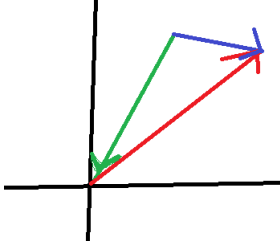If given a polar vector, the component form is $\underset{\sim}{v} = r\cos(\theta)\hat{i} + r\sin(\theta)\hat{j}$

0.9.7 How to find the vector between points that is at the tip of other vectors

If given vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ and points A and B that lie at the tips of the vectors, and told to find the vector $\vec{AB}$, it will be equal to $-\underset{\sim}{a} + \underset{\sim}{b}$.
For example:
Let vectors $\underset{\sim}{a} = 1\hat{i} + 4\hat{j}$, $\underset{\sim}{b} = -2\hat{i} + 0\hat{j}$, and points $A = (1,4)$, $B = (-2,0)$
To find vector $\vec{AB}$:

Where the blue vector is $\vec{AB}$, red is $\underset{\sim}{b}$, and green is $-\underset{\sim}{a}$

0.9.8 Unit vectors

To use a vector as the unit vector for others, use the formula:

$$\hat{\underset{\sim}{u}} = \frac{\underset{\sim}{u}}{|\underset{\sim}{u}|}$$

0.9.9 Dot products

The dot product represents one vector being projected onto the other vector, and multiplying the length of the projection by the length of the other vector. This is calculated by:

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta$$

or

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = \underset{\sim}{a}_{\hat{i}}\underset{\sim}{b}_{\hat{i}} + \underset{\sim}{a}_{\hat{j}}\underset{\sim}{b}_{\hat{j}}$$

NOTE: If the result is positive, the angle between the two vectors is between 0 to 90 °, if it is 0, it is at 90 °, and if it is a negative, it is between 90 to 180 °.
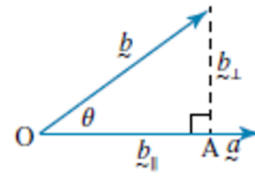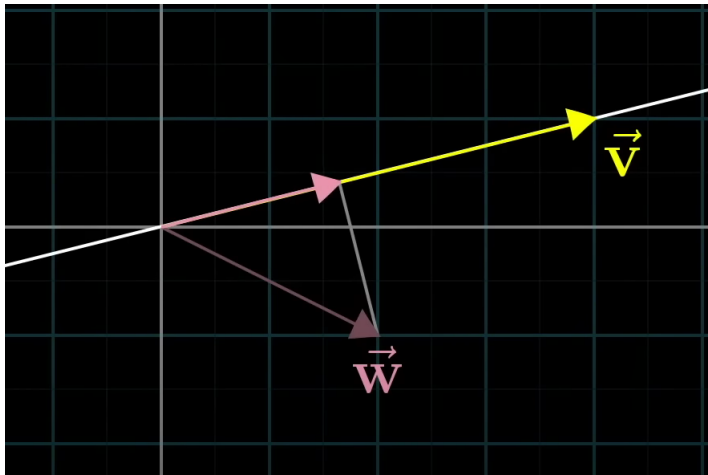
0.9.10 Angle between two vectors

Using the two formulas for calculating the dot product, the equation for cosine can be derived:

$$\cos\theta = \frac{\underset{\sim}{a}_{\hat{i}}\underset{\sim}{b}_{\hat{i}} + \underset{\sim}{a}_{\hat{j}}\underset{\sim}{b}_{\hat{j}}}{|\underset{\sim}{a}||\underset{\sim}{b}|}$$

0.9.11 Scalar resolutes

The scalar resolute is the length of the resultant vector if it were to be projected onto the other vector where a line from the tip of the vector projected to the tip of the projection is perpendicular to the vector being projected onto:

This can be calculated using the formula:

$$\vec{OA} = |\underset{\sim}{b}| \cos\theta$$

Where $\vec{OA}$ is the line between the origin and the tip of the projected vector, $\underset{\sim}{b}$ is the vector being projected, and θ is the angle between the two vectors. Or:

$$\vec{OA} = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}|}$$

And also:

$$\vec{OA} = \hat{\underset{\sim}{a}} \cdot \underset{\sim}{b}$$

0.9.12 Vector resolutes

The vector resolute is the same concept as the scalar resolute, but only that that we are concerned with the vector instead of the magnitude of the vector. As we are projecting a vector onto another vector, we can simply multiply the result of the scalar resolute with the unit vector of the vector we are projecting onto:

$$\underset{\sim}{b}_{\parallel} = (\hat{\underset{\sim}{a}} \cdot \underset{\sim}{b})\hat{\underset{\sim}{a}}$$

Additionally, the vector perpendicular to the vector being projected to is given by:

$$\underset{\sim}{b}_{\perp} = \underset{\sim}{b} - (\hat{\underset{\sim}{a}} \cdot \underset{\sim}{b})\hat{\underset{\sim}{a}}$$

0.9.13 Applications of vectors

Displacement and velocity

Triangle of forces

0.9.14 Relative vectors

0.10 Proofs

0.10.1 Propositions

0.10.2 Direct proofs

0.10.3 Proof by contraposition

Proof by contraposition is done by proving the contrapositive

0.10.4 Proof by contradiction

Proof by contradiction is done by assuming that it is false initially.

1. Assume (incorrectly) that what we want to prove is true, is actually false

2. Show that the assumption eventually leads to mathematical nonsense

3. Conclude that we are wrong to assume it is false

4. realize that if it is not false, then it must be true

Example:
x satisfies $5^x = 2$, show that x is irrational.++++

1. Assume that x is rational.

2. $x = \frac{p}{q}, \in \int$ (with no common factors) as all rational numbers can be represented by a fraction of integers

3. $5^{\frac{p}{q}} = 2$

4. $(5^{\frac{p}{q}})^q = (2)^q$

5. $5^p = 2^p$

6. odd = even

7. $\therefore$ Contradiction, assumption was wrong.

8. $\therefore x$ is irrational.

0.11   Circles
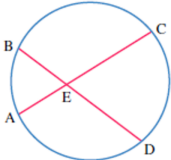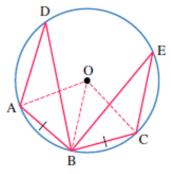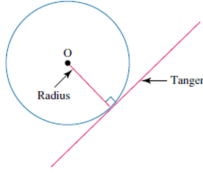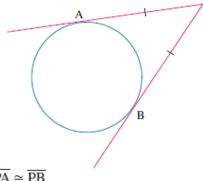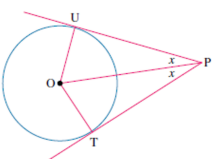
0.11.1   Terms

Some important terms used with circles are:

Term	Meaning
Circumference	The distance around the circle, or it's perimeter
Centre	The point denoted O which is the middle point of the circle
Chord	A line between two points on the circumference
Diameter	The length of a chord that intersects the centre
Radius	The length from the centre to the circumference
Arc	The part of a circle created by a chord $\overset{\frown}{abd}$ is
Angle at the centre	The angle made by two radii
Secant	A line the cuts the circumference of the circle at two points
Tangent	A line that touches the circle at a point
Segment	A region of a circle cut by a chord
Cyclic quadrilateral	A quadrilateral (shape with four sides) whose vertices all lie on a circle

0.11.2 Theorems

Theorem	Diagram	Symbol
The angle at the centre of the circle is twice the angle at the circumference subtended on the same arc.		
Angles at the circumference subtended by the same arc are congruent.		
The angle in a semicircle is a right angle.		
If a line from the center bisects a chord, it is perpendicular to the chord. The converse implication is also true.		
If the chords are congruent, then they are equidistant from the centre of the center.		

Theorem	Diagram	Symbol
If two chords intersect inside the circle, then the point of intersection divides each chord into two segments so that the product of the lengths of the segments for both chords is the same.	$BE \cdot DE = AE \cdot CE$	
If chords are congruent, they subtend equal angles at the centre.	Congruent chords subtend equal angles at the centre. If $\overline{AB} \cong \overline{BC}$, $\angle AOB \cong \angle BOC$.	
A tangent and a radius intersects at right angles.	Radius — Tangent	
Two tangents drawn from an external point to a circle are congruent.	$\overline{PA} \cong \overline{PB}$	
The angle formed by two tangents meeting at an external point is bisected by a straight line joining the centre of the circle to that external point.		

Theorem	Diagram	Symbol
The angle formed by a tangent and a chord is congruent to the angle in the alternate segment.	Angle in the alternate segment, Chord, Tangent	
If a secant (meeting the circle at A and B) and a tangent (meeting at circle at T) are drawn from an external point P, the square of the length of the tangent equals the product of the length to the circle on the secant.	$PT^2 = PA \cdot PB$	
If two secants intersect outside the circle, then the product of one external secant segment length and it's external length is equal to the product of the other secant lengths.	$PA \cdot PB = PC \cdot PD$	
The opposite angles of a cyclic quadrilateral are supplementary.	$\angle ABC + \angle CDA = 180°$ and $\angle BCD + \angle DAB = 180°$	
The exterior angle of a cyclic quadrilateral is congruent to the inner opposite angle.	$\angle CDE = \angle ABC$	