Mathematical Specialists

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April 8, 2023

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Abstract

This document was written to be used as a summary to help revise the content covered mathematical specialists. For any inquiries, contact lachlanprivate@duck.com or through the discord server: https://discord.gg/6P8rddkXFr

0.1 Important symbols

Symbol	Mathematical definition	Simple definition
$\cup$	Union	*A and B*, or think as in 'add'.
$\cap$	Intersection	*A or B*, or think as in 'multiply'.
$n!$	$n \times (n-1) \times (n-2) \times (n-3)... \times 3 \times 2 \times 1$	The product of integers between the given value and 1.
^nP_r	$\frac{n!}{(n-1)!}$	The number of combinations there are of length r from a group of length n where the order matters.
^nC_r	$\frac{\frac{n!}{(n-1)!}}{r!}$	The number of combinations there are of length r from a group of length n where the order does not matter.

0.2 Addition principle

If there are n ways of performing operation A, and m ways of performing operation B, there are $n + m$ ways of performing operation $A \cap B$.

As in, let there be 2 ways to perform task A, and 3 ways to perform task B. If there is an option to perform A *or* B, there is a total of $2 + 3 = 5$ ways to perform an operation.

0.3 Multiplication principle

If there are n ways of performing operation A, and m ways of performing operation B, there are $n \times m$ ways of performing operation $A \cup B$.

As in, let there be 4 ways to perform task A, and 5 ways to perform task B. With considering 1 way to perform A, there are 5 ways to perform B, repeat the process through the number of ways to perform A.

	B, 1	B, 2	B, 3	B, 4	B, 5
A, 1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)
A, 2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)
A, 3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)
A, 4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)

Where there are $4 \times 5 = 20$ ways to perform the operations.

0.4 Factorials

Factorials is the result of multiplying all of the integers between the given integer and 1. This is given the notation and formula:

$$n! = n \times (n-1) \times (n-2) \times (n-3)... \times 3 \times 2 \times 1$$

Example:

$$\begin{aligned} 5! &= 5 \times 4 \times 3 \times 2 \times 1 \\ &= 120 \end{aligned}$$

0.4.1 Dividing factorials

When given some factorial over another factorial in such cases as:

$$\frac{4!}{2!}$$

$$\frac{4 \times 3 \times 2 \times 1}{2 \times 1}$$

$$\frac{4 \times 3 \times \cancel{2 \times 1}}{\cancel{2 \times 1}}$$

$$4 \times 3$$

$$12$$

0.4.2 Special cases

$$0! = 1$$

0.5 Permutations

Permutations is the number of ways of choosing r things from n distinct things where *order matters.* This is given the notation and formula:

$$^{n}P_{r} = \frac{n!}{(n-r)!}$$

Example:

$$\begin{aligned}
^{6}P_{4} &= \frac{6!}{(6-4)!} \\
&= \frac{6!}{2!} \\
&= \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{2 \times 1} \\
&= \frac{6 \times 5 \times 4 \times 3 \times \cancel{2 \times 1}}{\cancel{2 \times 1}} \\
&= 6 \times 5 \times 4 \times 3 \\
&= 360
\end{aligned}$$

0.5.1 Permutations in a circle

In cases where the positions being concerned is a circle such as the permutations of a circular seating arrangement, if the standard permutations formula is applied, there are several over-counting of the like arrangement in a different perspective. In these cases apply the formula:

$$\frac{^{n}P_{r}}{r} = \frac{\frac{n!}{(n-r)!}}{r}$$

0.5.2 Like objects repetitions

The number of ways of arranging n objects made up of indistinguishable objects, n_1 in the first group, n_2 in the second group and so on, is:

$$\frac{n!}{n_1!n_2!n_3!...n_r!}$$

Example:

Find the number of permutations of the letters in the world WOOLLOOMOOLOO.
There are 8 'O's and 3 'L's
Permutations:

$$\frac{13!}{8!3!}$$

$$\frac{13 \times 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{(8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1) \times (3 \times 2 \times 1)}$$

$$\frac{13 \times 12 \times 11 \times 10 \times 9 \times \cancel{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}}{\cancel{(8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1)} \times (3 \times 2 \times 1)}$$

$$\frac{13 \times 12 \times 11 \times 10 \times 9}{3 \times 2}$$

$$\frac{13 \times \cancelto{6 \times \cancel{2}}{12} \times 11 \times 10 \times \cancelto{3 \times \cancel{3}}{9}}{\cancel{3 \times 2}}$$

$$13 \times 6 \times 11 \times 10 \times 3$$

$$25,740$$

0.5.3 Restrictions

When considering restrictions, deal with the restrictions first.
Example:

Find the number of arrangements of the letters of the word DARWIN beginning and ending with a vowel.
Number of letters = 6
Number of positions = 6
Beginning and end must be a vowel, so the available positions are decreased by 2: number of positions - 2 = 4
Since 2 vowels must be used: number of letters - 2 = 4
Permutations:

$$^4P_4$$

$$4!$$

$$4 \times 3 \times 2 \times 1$$

$$24$$

0.5.4 Grouped items

When items are grouped together, treat each group as a single object. Find the number of arrangements of the groups, then multiply by the number of arrangement within each group.
Example:

Find the number of arrangement of the letters of the word EQUALS if the vowels are kept together.
Number of vowels = 3
Number of letters = 6
Number of positions = 6
As the 3 vowels are grouped together, the number of positions decreases:
number of positions - $3_{\text{for the members of the group}} + 1_{\text{for the group}} = 4$
Permutations within vowel group:

$$^3P_3 = 3! = 3 \times 2 = 6$$

Permutations of all positions:

$$^4P_4 = 4! = 4 \times 3 \times 2 = 24$$

Multiply together:

$$6 \times 24 = 144$$

0.5.5 Special cases

$$^nP_n = n!$$

$$^nP_0 = 1$$

0.6 Combinations

Combinations is the number of ways of choosing or selecting r objects from n distinct objects where *order does not matter*. This is given the notation and formula:

$$^nC_r = \frac{^nP_r}{r!}$$

$$^nC_r = \frac{\frac{n!}{(n-r)!}}{r!}$$

Example:

$$
\begin{aligned}
{}^4C_2 &= \frac{{}^4P_2}{2!} \\
&= \frac{\frac{4!}{(4-2)!}}{2!} \\
&= \frac{\frac{4!}{2!}}{2!} \\
&= \frac{\frac{4\times3\times2}{2}}{2} \\
&= \frac{\frac{4\times3\times\cancel{2}}{\cancel{2}}}{2} \\
&= \frac{4\times3}{2} \\
&= \frac{12}{2} \\
&= 6
\end{aligned}
$$

0.7 Pascal's triangle

The Pascal's triangle is a pattern formed by adding the top 2 adjacent numbers and a 1 is placed on either side of the bottom row to resemble a triangle:

$$
\begin{array}{r}
n=0: \\ n=1: \\ n=2: \\ n=3: \\ n=4: \\ n=5: \\ n=6: \\ n=7: \\ n=8: \\ n=9: \\ n=10:
\end{array}
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\
1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1 \\
1 \quad 9 \quad 36 \quad 84 \quad 126 \quad 126 \quad 84 \quad 36 \quad 9 \quad 1 \\
1 \quad 10 \quad 45 \quad 120 \quad 210 \quad 252 \quad 210 \quad 120 \quad 45 \quad 10 \quad 1
\end{array}
$$

Each element in the Pascal's triangle can be used to calculate combinations, hence, the triangle can be written using Combinations notation (${}^nC_r$):

$$
\begin{array}{r}
n=0: \\ n=1: \\ n=2: \\ n=3: \\ n=4: \\ n=5:
\end{array}
\begin{array}{c}
{}^0C_0 \\
{}^1C_0 \quad {}^1C_1 \\
{}^2C_0 \quad {}^2C_1 \quad {}^2C_2 \\
{}^3C_0 \quad {}^3C_1 \quad {}^3C_2 \quad {}^3C_3 \\
{}^4C_0 \quad {}^4C_1 \quad {}^4C_2 \quad {}^4C_3 \quad {}^4C_4 \\
{}^5C_0 \quad {}^5C_1 \quad {}^5C_2 \quad {}^5C_3 \quad {}^5C_4 \quad {}^5C_5
\end{array}
$$

Pascal's triangle shows that the r^{th} element of the n^{th} row of Pascal's triangle is given by ${}^nC_r$. It is assumed that the 1 at the beginning of each row is the 0^{th} element. This gives the *Pascal's identity*:

$$ {}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r \text{ for } 0 < r < n $$

The Pascal's triangle can be extended to the binomial theorem, where the rule for expanding an expression such as $(a+b)^n$. Where:

$$
\begin{aligned}
(a+b)^0 &= 1a^0b^0 \\
(a+b)^1 &= 1a^1b^0 + 1a^0b^1 \\
(a+b)^2 &= 1a^2b^0 + 2a^1b^1 + 1a^0b^2 \\
(a+b)^3 &= 1a^3b^0 + 3a^2b^1 + 3a^1b^2 + 1a^0b^3 \\
(a+b)^4 &= 1a^4b^0 + 4a^3b^1 + 6a^2b^2 + 4a^1b^3 + 1a^0b^4 \\
(a+b)^5 &= 1a^5b^0 + 5a^4b^1 + 10a^3b^2 + 10a^2b^3 + 5a^1b^4 + 1a^0b^5
\end{aligned}
$$