Mathematical Specialists

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Abstract

This document was written to be used as a summary to help revise the content covered mathematical specialists. For any inquiries, contact lachlanprivate@duck.com or through the discord server: https://discord.gg/6P8rddkXFr

0.1 Important symbols

Symbol	Mathematical definition	Simple definition
$\cup$	Union	*A and B*, or think as in 'add'.
$\cap$	Intersection	*A or B*, or think as in 'multiply'.
$n!$	$n \times (n-1) \times (n-2) \times (n-3) ... \times 3 \times 2 \times 1$	The product of integers between the given value and 1.
${}^nP_r$	$\frac{n!}{(n-r)!}$	The number of combinations there are of length r from a group of length n where the order matters.
${}^nC_r$	$\frac{\frac{n!}{(n-r)!}}{r!}$ or $\frac{n!}{r!(n-r)!}$	The number of combinations there are of length r from a group of length n where the order does not matter.
$\binom{n}{r}$	${}^nC_r$	The same as the combinations but in a different notation.

0.2 Addition principle

If there are n ways of performing operation A, and m ways of performing operation B, there are $n+m$ ways of performing operation $A \cap B$.

As in, let there be 2 ways to perform task A, and 3 ways to perform task B. If there is an option to perform A *or* B, there is a total of $2+3=5$ ways to perform an operation.

0.3 Multiplication principle

If there are n ways of performing operation A, and m ways of performing operation B, there are $n \times m$ ways of performing operation $A \cup B$.

As in, let there be 4 ways to perform task A, and 5 ways to perform task B. With considering 1 way to perform A, there are 5 ways to perform B, repeat the process through the number of ways to perform A.

	B, 1	B, 2	B, 3	B, 4	B, 5
A, 1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)
A, 2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)
A, 3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)
A, 4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)

Where there are $4 \times 5 = 20$ ways to perform the operations.

0.4 Factorials

Factorials is the result of multiplying all of the integers between the given integer and 1. This is given the notation and formula:

$$n! = n \times (n-1) \times (n-2) \times (n-3) ... \times 3 \times 2 \times 1$$

Example:

$$\begin{aligned} 5! &= 5 \times 4 \times 3 \times 2 \times 1 \\ &= 120 \end{aligned}$$

0.4.1 Dividing factorials

When given some factorial over another factorial in such cases as:

$$\frac{4!}{2!}$$

$$\frac{4 \times 3 \times 2 \times 1}{2 \times 1}$$

$$\frac{4 \times 3 \times \cancel{2 \times 1}}{\cancel{2 \times 1}}$$

$$4 \times 3$$

$$12$$

0.4.2 Special cases

$$0! = 1$$

0.5 Permutations

Permutations is the number of ways of choosing r things from n distinct things where *order matters.* This is given the notation and formula:

$$^nP_r = \frac{n!}{(n-r)!}$$

Example:

$$\begin{aligned}
^6P_4 &= \frac{6!}{(6-4)!} \\
&= \frac{6!}{2!} \\
&= \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{2 \times 1} \\
&= \frac{6 \times 5 \times 4 \times 3 \times \cancel{2 \times 1}}{\cancel{2 \times 1}} \\
&= 6 \times 5 \times 4 \times 3 \\
&= 360
\end{aligned}$$

0.5.1 Permutations in a circle

In cases where the positions being concerned is a circle such as the permutations of a circular seating arrangement, if the standard permutations formula is applied, there are several over-counting of the like arrangement in a different perspective. In these cases apply the formula:

$$\frac{^nP_r}{r} = \frac{\frac{n!}{(n-r)!}}{r}$$

0.5.2 Like objects repetitions

The number of ways of arranging n objects made up of indistinguishable objects, n_1 in the first group, n_2 in the second group and so on, is:

$$\frac{n!}{n_1!n_2!n_3!...n_r!}$$

Example:

Find the number of permutations of the letters in the world WOOLLOOMOOLOO.
There are 8 'O's and 3 'L's
Permutations:

$$\frac{13!}{8!3!}$$

$$\frac{13 \times 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{(8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1) \times (3 \times 2 \times 1)}$$

$$\frac{13 \times 12 \times 11 \times 10 \times 9 \times \cancel{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}}{(\cancel{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}) \times (3 \times 2 \times 1)}$$

$$\frac{13 \times 12 \times 11 \times 10 \times 9}{3 \times 2}$$

$$\frac{13 \times \cancelto{6 \times \cancel{2}}{12} \times 11 \times 10 \times \cancelto{3 \times \cancel{3}}{9}}{\cancel{3 \times 2}}$$

$$13 \times 6 \times 11 \times 10 \times 3$$

$$25,740$$

0.5.3 Restrictions

When considering restrictions, deal with the restrictions first.
Example:

Find the number of arrangements of the letters of the word DARWIN beginning and ending with a vowel.
Number of letters = 6
Number of positions = 6
Beginning and end must be a vowel, so the available positions are decreased by 2: number of positions - 2 = 4
Since 2 vowels must be used: number of letters - 2 = 4
Permutations:

$$^4P_4$$

$$4!$$

$$4 \times 3 \times 2 \times 1$$

$$24$$

0.5.4 Grouped items

When items are grouped together, treat each group as a single object. Find the number of arrangements of the groups, then multiply by the number of arrangement within each group.
Example:

Find the number of arrangement of the letters of the word EQUALS if the vowels are kept together.
Number of vowels = 3
Number of letters = 6
Number of positions = 6
As the 3 vowels are grouped together, the number of positions decreases:
number of positions - $3_{\text{for the members of the group}} + 1_{\text{for the group}} = 4$
Permutations within vowel group:

$$^3P_3 = 3! = 3 \times 2 = 6$$

Permutations of all positions:

$$^4P_4 = 4! = 4 \times 3 \times 2 = 24$$

Multiply together:

$$6 \times 24 = 144$$

0.5.5 Special cases

$$^nP_n = n!$$

$$^nP_0 = 1$$

0.6 Combinations

Combinations is the number of ways of choosing or selecting r objects from n distinct objects where *order does not matter*. This is given the notation and formula:

$$^nC_r = \frac{^nP_r}{r!}$$

$$^nC_r = \frac{\frac{n!}{(n-r)!}}{r!}$$

Example:

$$
\begin{aligned}
{}^{4}C_2 &= \frac{{}^{4}P_2}{2!} \\
&= \frac{\frac{4!}{(4-2)!}}{2!} \\
&= \frac{\frac{4!}{2!}}{2!} \\
&= \frac{\frac{4\times3\times2}{2}}{2} \\
&= \frac{\frac{4\times3\cancel{\times2}}{\cancel{2}}}{2} \\
&= \frac{4\times3}{2} \\
&= \frac{12}{2} \\
&= 6
\end{aligned}
$$

0.6.1 Combinations with restrictions

Combinations with specific terms

When given a restriction that something must be something, consider the restriction first:
As an example:

Grace belongs to a group of 8 workers. How many ways can a team of four workers be selected if Grace must be on the team?
If there is no restriction, it would be equal to ${}^{8}C_4$. But as Grace must be on the team, n must be equal to $8-1$ and r must be also $4-1$
Therefore, the answer is:

$$ {}^{7}C_3 = 35 $$

35 total combinations.

Combinations from multiple groups

Permutations and Combinations combined

0.7 Pascal's triangle

The Pascal's triangle is a pattern formed by adding the top 2 adjacent numbers and a 1 is placed on either side of the bottom row to resemble a triangle:

$n=0$: 1

$n=1$: 1 1

$n=2$: 1 2 1

$n=3$: 1 3 3 1

$n=4$: 1 4 6 4 1

$n=5$: 1 5 10 10 5 1

$n=6$: 1 6 15 20 15 6 1

$n=7$: 1 7 21 35 35 21 7 1

$n=8$: 1 8 28 56 70 56 28 8 1

$n=9$: 1 9 36 84 126 126 84 36 9 1

$n=10$: 1 10 45 120 210 252 210 120 45 10 1

Each element in the Pascal's triangle can be used to calculate combinations, hence, the triangle can be written using Combinations notation (${}^nC_r$):

$n=0$:						${}^0C_0$					
$n=1$:					${}^1C_0$		${}^1C_1$				
$n=2$:				${}^2C_0$		${}^2C_1$		${}^2C_2$			
$n=3$:			${}^3C_0$		${}^3C_1$		${}^3C_2$		${}^3C_3$		
$n=4$:		${}^4C_0$		${}^4C_1$		${}^4C_2$		${}^4C_3$		${}^4C_4$	
$n=5$:	${}^5C_0$		${}^5C_1$		${}^5C_2$		${}^5C_3$		${}^5C_4$		${}^5C_5$

Pascal's triangle shows that the r^{th} element of the n^{th} row of Pascal's triangle is given by ${}^nC_r$. It is assumed that the 1 at the beginning of each row is the 0^{th} element. This gives the *Pascal's identity*:

$$^nC_r =^{n-1} C_{r-1} +^{n-1} C_r \text{ for } 0 < r < n$$

The Pascal's triangle can be extended to the binomial theorem, where the rule for expanding an expression such as $(a+b)^n$. Where:

$$\begin{aligned}
(a+b)^0 &= 1a^0b^0 \\
(a+b)^1 &= 1a^1b^0 + 1a^0b^1 \\
(a+b)^2 &= 1a^2b^0 + 2a^1b^1 + 1a^0b^2 \\
(a+b)^3 &= 1a^3b^0 + 3a^2b^1 + 3a^1b^2 + 1a^0b^3 \\
(a+b)^4 &= 1a^4b^0 + 4a^3b^1 + 6a^2b^2 + 4a^1b^3 + 1a^0b^4 \\
(a+b)^5 &= 1a^5b^0 + 5a^4b^1 + 10a^3b^2 + 10a^2b^3 + 5a^1b^4 + 1a^0b^5
\end{aligned}$$

0.8 Vectors

Vectors have a magnitude and a direction. There are three ways for writing vectors, either the Cartesian form/component from or the polar form. Vectors are represented by $\underset{\sim}{a}$. the Cartesian form is written as:

$$\underset{\sim}{v} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

The component form notation is written as:

$$\underset{\sim}{v} = 2\hat{\imath} + 3\hat{\jmath}$$

And polar from is written as:

$$\underset{\sim}{v} = (r, \theta)$$

Where r is the radius and θ is the direction in angles.

$\hat{\imath}$ is the unit vector in the x-axis, and $\hat{\jmath}$ is the unit vector in the y-axis. Unit vectors are the basis vectors and are multiplied by scalars to move the tip of the vector and represent different vectors.

0.8.1 Addition of vectors

Let $\underset{\sim}{a} = 3\hat{\imath} + -2\hat{\jmath}$, and $\underset{\sim}{b} = 7\hat{\imath} + 1\hat{\jmath}$
And the addition of these vectors $\underset{\sim}{a} + \underset{\sim}{b} = \underset{\sim}{c}$

$$\begin{aligned}
\underset{\sim}{c} &= \underset{\sim}{a} + \underset{\sim}{b} \\
\underset{\sim}{c} &= 3\hat{\imath} + -2\hat{\jmath} + 7\hat{\imath} + 1\hat{\jmath} \\
\underset{\sim}{c} &= 10\hat{\imath} + -1\hat{\jmath}
\end{aligned}$$

0.8.2 Multiplication of vectors by a scalar

Let $\underset{\sim}{x} = 4\hat{\imath} + -5\hat{\jmath}$
And $\underset{\sim}{s} = 2\underset{\sim}{x}$

$$\begin{aligned}\underset{\sim}{s} &= 2\underset{\sim}{x}\\ \underset{\sim}{s} &= 2(4\hat{\imath} + -5\hat{\jmath})\\ \underset{\sim}{s} &= 8\hat{\imath} + -10\hat{\jmath}\end{aligned}$$

0.8.3 Calculating the magnitude of a vector

The magnitude of a vector is written as $|\underset{\sim}{v}|$, and to calculate the magnitude of a vector use the pythagoras's theorem:
Let $\underset{\sim}{w} = 3\hat{\imath} + 4\hat{\jmath}$

$$\begin{aligned}\underset{\sim}{w} &= 3\hat{\imath} + 4\hat{\jmath}\\ |\underset{\sim}{w}| &= \sqrt{3^2 + 4^2}\\ |\underset{\sim}{w}| &= 5\end{aligned}$$

0.8.4 Calculating the angle of a vector

0.8.5 The negative of a vector

0.8.6 Converting between component form and polar form

To convert between the component form and polar from, use the trigonometric ratios.
If given a polar vector, the component form is $\underset{\sim}{v} = r\cos(\theta)\hat{\imath} + r\sin(\theta)\hat{\jmath}$

0.8.7 How to find the vector between points that

0.8.8 Unit vectors

0.8.9 Dot products

0.8.10 Angle between two vectors

0.8.11 Scalar resolutes

0.8.12 Vector resolutes

0.8.13 Sine rule

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0.8.15 Applications of vectors

Displacement and velocity

Triangle of forces

0.8.16 Relative vectors

0.9 Proofs

0.9.1 Propositions

0.9.2 Direcct proofs

0.9.3 Proof by contraposition

Proof by contraposition is done by proving the contrapositive

0.9.4 Proof by contradiction

Proof by contradiction is done by assuming that it is false initially.

1. Assume (incorrectly) that what we want to prove is true, is actually false
2. Show that the assumption eventually leads to mathematical nonsense
3. Conclude that we are wrong to assume it is false
4. realize that if it is not false, then it must be true

Example:
x satisfies $5^x = 2$, show that x is irrational.++++

1. Assume that x is rational.
2. $x = \frac{p}{q}, \in \int$(with no common factors) as all rational numbers can be represented by a fraction of integers
3. $5^{\frac{p}{q}} = 2$
4. $(5^{\frac{p}{q}})^q = (2)^q$
5. $5^p = 2^p$
6. odd = even
7. $\therefore$ Contradiction, assumption was wrong.
8. $\therefore x$ is irrational.

0.10 Circles

0.10.1 Terms

0.10.2 Theorems

Theorem	Diagram	Symbol
The angle at the centre of the circle is twice the angle at the circumference subtended on the same arc.