

Mathematics in Deep Learning

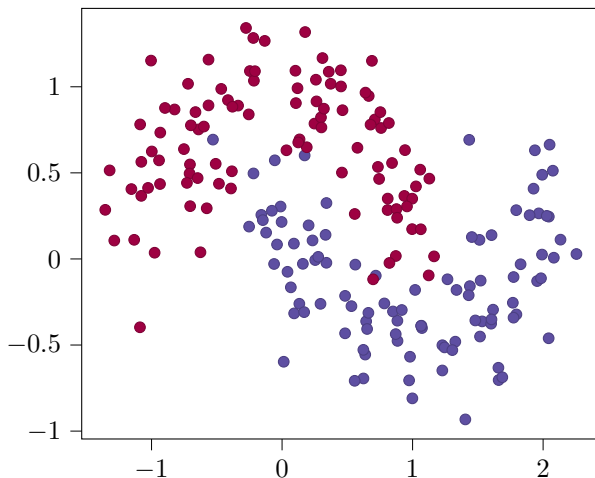
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MATH199

2024 Mid-Year Workshop

A classification problem

Let's build some networks to separate this dataset of red and blue dots



Simple perceptron networks

Perceptrons are the building block of Neural Networks

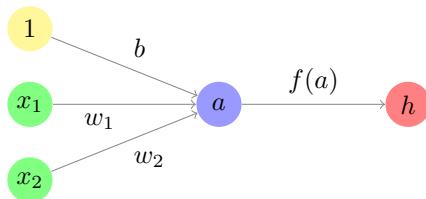


Figure: Simple perceptron network with two inputs

Simple perceptron networks

Perceptrons are the building block of Neural Networks

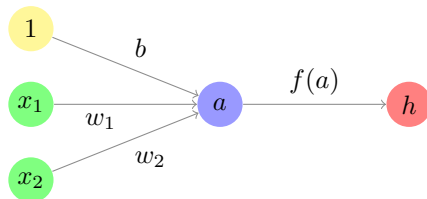


Figure: Simple perceptron network with two inputs

- $a = w_1 \cdot x_1 + w_2 \cdot x_2 + b$

Simple perceptron networks

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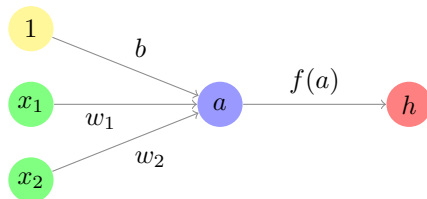


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- $a = w_1 \cdot x_1 + w_2 \cdot x_2 + b$
- f is a **non-linear activation function**

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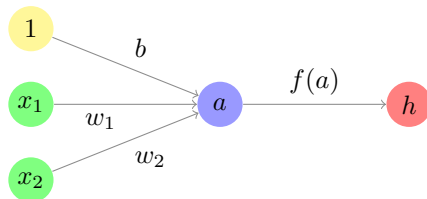


Figure: Simple perceptron network with two inputs

- $a = w_1 \cdot x_1 + w_2 \cdot x_2 + b$
- f is a **non-linear activation function**
- the weights w_1 and w_2 and bias b are **parameters**

Simple perceptron networks

Limited to linear decision boundaries:

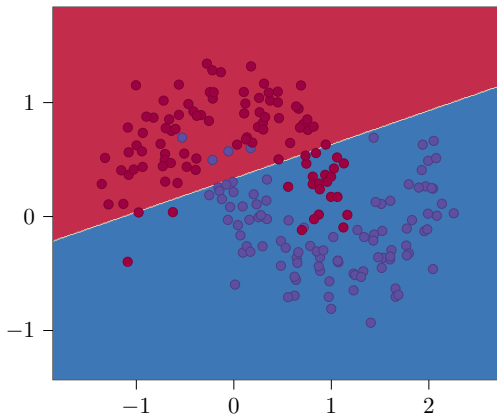
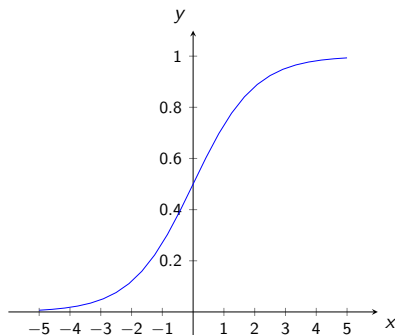


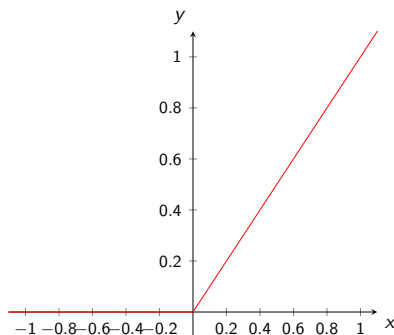
Figure: Simple perceptron decision boundary

Non-linear activation functions

Two examples:



(a) Sigmoid function



(b) Rectified linear unit

Remark

Deep learning doesn't work without non-linear activation functions.

- Multi-layered perceptron
- Combine multiple perceptrons together into a large network

MLP networks

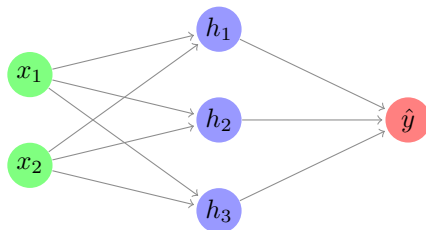


Figure: MLP network with two inputs and one hidden layer

Remark

We've tidied up our diagram to not show biases and activation functions, but they're still there and being used in calculations.

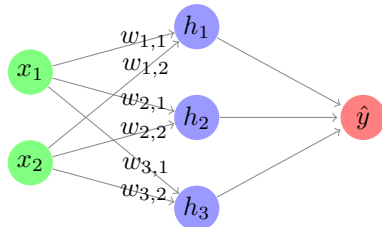
MLP networks

Calculations:

$$h_1 = w_{1,1} \cdot x_1 + w_{1,2} \cdot x_2 + b_1$$

$$h_2 = w_{2,1} \cdot x_1 + w_{2,2} \cdot x_2 + b_2$$

$$h_3 = w_{3,1} \cdot x_1 + w_{3,2} \cdot x_2 + b_3$$



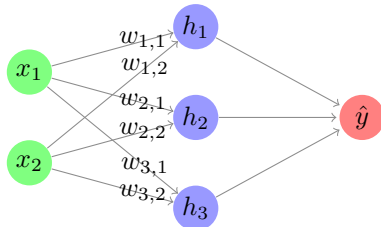
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This is a typical system of equations, so we can tidy things up with matrix algebra:

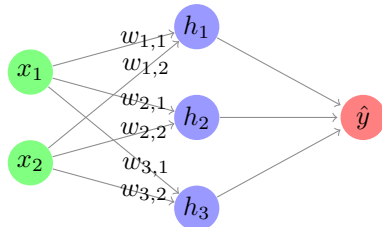
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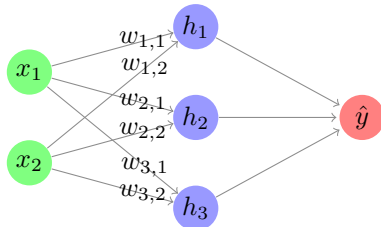
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With an activation function, the output of the hidden layer is:

$$\underline{h} = f(W \cdot \underline{x} + \underline{b})$$

MLP networks

Typically, architecture is much more complex, with many hidden layers. This is what **deep** learning is referring to.

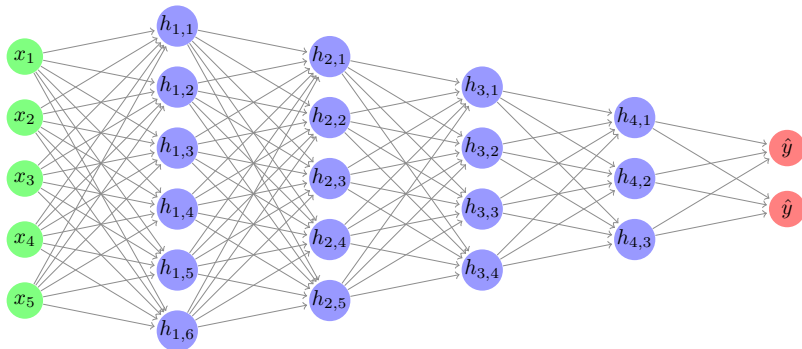


Figure: Example of what a deep NN could look like

Generally:

$$\underline{h}_i = f(W_i \cdot \underline{h}_{i-1} + \underline{b}_i)$$

Remark

We can think of deep neural networks as a recurrence relation, passing the output of one hidden layer as the input to the next layer.

MLP networks

The combination of **hidden layers** and **non-linear activation functions** gives us a **non-linear decision boundary**:

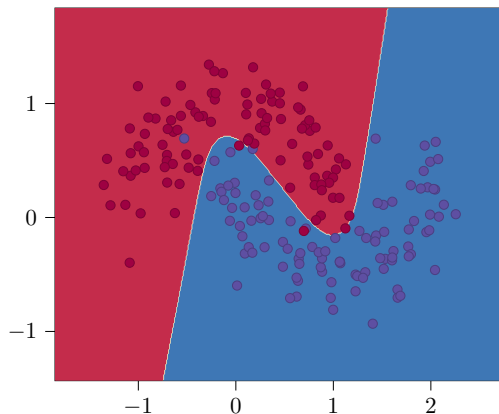


Figure: MLP decision boundary

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Training

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- **Training** is achieved by updating our **parameters**, the weights and biases.

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- **Training** is achieved by updating our **parameters**, the weights and biases.
- We achieve this using **calculus**.

We define a **loss function** that depends on the parameters:

$$J(\theta_1, \dots, \theta_n)$$

A loss function compares the error of a network's current prediction against human labelling of data.

We want to find a local minima of this function.

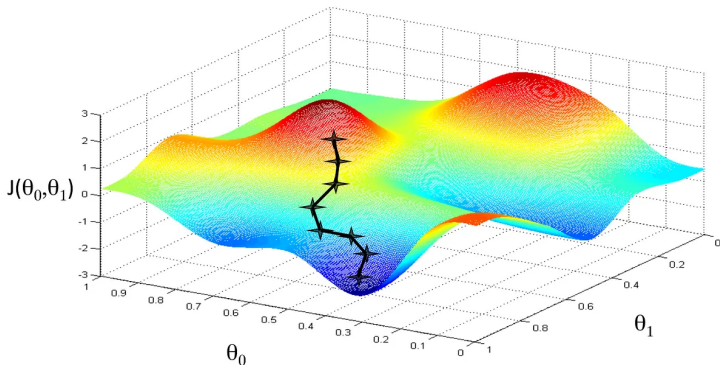
We can compute the **gradient** of the loss function, which tells us how to make small updates to our parameters:

$$\nabla J(\theta_1, \dots, \theta_n) = \begin{bmatrix} \frac{\partial J}{\partial \theta_1} \\ \frac{\partial J}{\partial \theta_2} \\ \vdots \\ \frac{\partial J}{\partial \theta_n} \end{bmatrix}$$

These are partial derivatives with respect to each parameter. You'll learn about partial derivatives later in semester 2. We need the chain rule to compute most of these derivatives.

Training

A great way to visualise this is walking down a mountain, the gradient tells us the steepest way down.



Each small step is the gradient being computed for a batch of examples from our training data.

Summary

The maths we've seen being used:

- Matrix algebra
- Recurrence relations
- Functions
- Partial differentiation
- Chain rule

These are all things you learn about in MATH199!