# Morse Theory on Path Spaces

## A brief introduction

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# Table of Contents

# Definitions

Let $M$ be a smooth manifold, $f\colon M \to \mathbb{R}$ be a smooth function, we say $p \in M$ is a critical point if $Df\colon T_pM \to T_{f(p)}\mathbb{R}$ is not surjective.

A critical point $p$ is called non-degenerate if the Hessian matrix $\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(p)\right)$ is non-singular. We can also define the Hessian of $F$ at $p$ as following:

$$\mathrm{Hess}\, f_p(v, w) = D_V(D_W f)$$

where $v, w \in T_pM$, $V$ and $W$ are extensions of $v$ and $w$. This is a well defined symmetric bilinear form on $T_pM$. The index of a non-degenerate critical point is the number of negative eigenvalues of the Hessian.

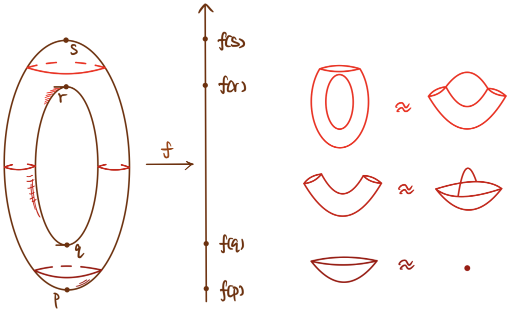
**Morse Lemma**

$p$ is a non-degenerate critical point of index $\lambda$ if and only if:

$$f = f(p) - x_1^2 - \cdots - x_\lambda^2 + x_{\lambda+1}^2 + \cdots + x_n^2$$

after proper coordinate changing.

## Example




$M^a = f^{-1}((\infty, a])$ is attached by a $\lambda$-cell when $a$ is passing a critical value, where $\lambda$ is the index of the corresponding critical point.

# Main theorem

**Theorem**

Let $f\colon M \to \mathbb{R}$ be a Morse function, $p$ be a critical point of $f$ with index $\lambda$. Suppose $f(p) = c$, $f^{-1}(c-\epsilon, c+\epsilon)$ is compact and contains no critical points other than $p$, for some $\epsilon > 0$. Then if $\epsilon$ is sufficiently small, $M^{c+\epsilon}$ has homotopy type of $M^{c-\epsilon}$ with a $\lambda$-cell attached.

More generally suppose there are $k$ critical points contained in $f^{-1}(c)$ with index $\lambda_1, \ldots, \lambda_k$, under same conditions $M^{c+\epsilon}$ has homotopy type of $M^{c-\epsilon}$ attaching $\lambda_i$-cell for each critical point.

If each $M^a$ is compact, the homotopy type of $M$ can be recovered, in a sense of homotopy equivalence with a $CW$-complex.

## Table of Contents

## Path spaces

Suppose $(M, g)$ is a connected Riemannian manifold. Let $p$ and $q$ be two points of $M$, we define the **path space** of $M$ to be the set of all piecewise smooth paths from $p$ to $q$, which is denoted by $\Omega_{p,q}$, or simply $\Omega$ for simplicity.

We formally define the **tangent space** $T_\omega\Omega$ of $\omega \in \Omega$ as the vector space of all piecewise smooth vector fields along $\omega$ vanishing at the endpoints. For $W \in T_\omega\Omega$, denote $W(t) := W|_{\omega(t)}$.

The **energy functional** defined on $\Omega$ is given by:

$$E(\omega) = \int_0^1 g\left(\frac{d\omega}{dt}, \frac{d\omega}{dt}\right) dt = \int_0^1 \left\|\frac{d\omega}{dt}\right\|^2 dt, \quad \omega \in \Omega$$

which will be our analogy of Morse function on $\Omega$.

# Variation

**Definition**

A **"1-parameter" variation** of $\omega \in \Omega_{p,q}$ is a family of paths $\bar{\omega}: (-\epsilon, \epsilon) \times I \to M$ for some $\epsilon > 0$, such that:

- $\bar{\omega}(0, t) = \omega(t)$ for all $t \in I$.
- $\bar{\omega}(s, t) \in \Omega_{p,q}$ for fixed $s \in (-\epsilon, \epsilon)$.
- $\bar{\omega}$ is continuous and there is subdivision "$0 = t_0 < \cdots < t_k = 1$" of $I$ such that each map $\bar{\omega}|_{(-\epsilon,\epsilon)\times[t_i, t_{i+1}]}$ is differentiable of class $C^\infty$.

Given $W \in T_\omega\Omega$, there exists a variation $\bar{\omega}$ such that:

$$\left.\frac{\partial \bar{\omega}}{\partial s}\right|_{(0,t)} = W(t)$$

Indeed, one can set $\bar{\omega}(s, t) = \exp_{\omega(t)}(sW_t)$. We then give an analogous definition for Jacobian.

## Jacobian of the energy

For $W \in T_\omega\Omega$, let $\bar{\omega}$ be a variation of $\omega \in \Omega$ such that $\frac{\partial \bar{\omega}}{\partial s}\Big|_{(0,t)} = W(t)$, the induced map $DE_\omega \colon T_\omega\Omega \to T_{E(\omega)}\mathbb{R}$ follows:

$$DE_\omega(W) := \left( \frac{dE(\omega_s)}{ds}\Big|_{s=0} \right) \cdot \frac{d}{dt}\Big|_{t=E(\omega)}$$

where $\omega_s(t) := \bar{\omega}(s,t)$ is a path in $\Omega$. This is a well defined linear map by the first variation formula:

$$\frac{1}{2}\,\frac{dE(\omega_s)}{ds}\Big|_{s=0} = -\int_0^1 g\left(W(t), \frac{d^2\omega}{dt^2}\right)\, dt - \sum_{i=1}^{k-1} g\left(W(t_i), \Delta\frac{d\omega}{dt}(t_i)\right)$$

where $\Delta\frac{d\omega}{dt}(t) := \frac{d\omega}{dt}(t^+) - \frac{d\omega}{dt}(t^-)$.

**Claim.** $\omega \in \Omega$ is a critical point of $E$ if and only if it is a geodesic.

## Hessian of the energy at critical points

**Let $\gamma \in \Omega$ be a geodesic.** For $W_1, W_2 \in T_\gamma\Omega$, let $\tilde{\gamma}\colon U \times I \to M$ be a "2-parameter" variation satisfies the similar condition where $U \subset \mathbb{R}^2$ now is an open neighborhood of $(0,0)$, such that $\partial\tilde{\gamma}/\partial s_i|_{(0,0,t)} = W_i$ for $i = 1, 2$. Define:

$$\text{Hess}\, E_\gamma(W_1, W_2) := \left.\frac{\partial^2 E(\gamma_{s_1,s_2})}{\partial s_1 \partial s_2}\right|_{s_1=s_2=0}$$

where $\gamma_{s_1,s_2} := \tilde{\gamma}(s_1, s_2, t)$ is a path in $\Omega$. This is a well defined bi-linear functional by the second variation formula (also symmetric):

$$\frac{1}{2}\left.\frac{\partial^2 E(\gamma_{s_1,s_2})}{\partial s_1 \partial s_2}\right|_{(0,0)} = -\int_0^1 g\left(W_2(t), \frac{d^2 W_1}{dt^2} + \mathcal{R}(V, W_1)V\right) dt - \sum_{i=1}^{k-1} g\left(W_2(t_i), \Delta\frac{dW_1}{dt}(t_i)\right)$$

**Claim.** $W \in T_\gamma\Omega$ belongs to $\text{Null}\,(\text{Hess}\, E_c)$ if and only if it is a Jacobi field.

# The index theorem

## Definition

- Along geodesic $\gamma$, two point $\gamma(a)$ and $\gamma(b)$ with $a \neq b$ is **conjugate** if there exists a non-zero Jacobi field along $c$ vanishing at $\gamma(a)$ and $\gamma(b)$.
- If $\gamma(a)$ and $\gamma(b)$ are conjugate, we say the corresponding **multiplicity** is the dimension of the vector space consisting of all such Jacobi fields.

## Theorem (Morse)

The index $\lambda$ of Hess $E_c$ is **finite** and equal to the number of points $\gamma(t)$ with $0 < t < 1$, such that $\gamma(t)$ is conjugate to $\gamma(0)$, **each such conjugate point is counted with its multiplicity**.

## A finite dimensional approximation

We topologize $\Omega$ by the following metric:

$$d(\omega, \omega') = \max_{t \in [0,1]} \rho\left(\omega(t), \omega'(t)\right) + \left( \int_0^1 \left( \left\| \frac{d\omega}{dt} \right\| - \left\| \frac{d\omega'}{dt} \right\| \right)^2 dt \right)^{\frac{1}{2}}$$

$\rho$ is the topological metric on $M$ induced by the Riemannian metric $g$:

$$\rho(p', q') = \inf\{L(\omega) \colon \omega \in \Omega_{p',q'}\}$$

where $L$ is the **arclength functional**.

For subdivision "$0 = t_0 < \cdots < t_k = 1$", let $\Omega(t_0, \ldots, t_k)$ be the subspace of $\Omega$ consisting of corresponding piecewise geodesic paths. Then given $c > 0$, if the subdivision is fine enough, $\operatorname{Int}\Omega(t_0, \ldots, t_k)^c = (\operatorname{Int}\Omega^c) \cap \Omega(t_0, \ldots, t_k)$ can be given a smooth manifold structure in a natural way.

## A finite dimensional approximation

To study $\operatorname{Int}\Omega^c$, it suffices to understand $B = \operatorname{Int}\Omega(t_0, \ldots, t_k)^c$. Let $E_0$ be the energy functional restricted to $B$, we have the following theorem.

**Theorem**

- $E_0 \colon B \to \mathbb{R}$ is smooth.
- $E_0^{-1}[0, a]$ is compact and is a deformation retract of $\Omega^a$, for all $a < c$.
- The critical points of $E_0$ and $E$ are precisely the same.
- The index (or the nullity) of $\operatorname{Hess} E_0$ and $\operatorname{Hess} E$ are equal at each critical point correspondingly.

**Theorem**

Let M be a **complete Riemannian manifold** and $p, q \in M$ be two points which are not conjugate along any geodesic of length $\leq \sqrt{a}$. Then $\Omega^a$ has the homotopy type of a finite $CW$-complex, with one cell of dimension $\lambda$ for each geodesic in $\Omega^a$ that is of index $\lambda$.

## The full path space

What we really care about is the space of all paths in $M$ from $p$ to $q$, denoted by $\Omega^*$, equipped with the compact open topology induced by the metric:

$$d^*(\omega, \omega') = \max_{t\in[0,1]} \rho\left(\omega(t), \omega'(t)\right)$$

In fact, the natural inclusion $\Omega \hookrightarrow \Omega^*$ is a homotopy equivalence.

### Fundamental theorem of Morse Theory

Let M be a **complete Riemannian manifold** and $p, q \in M$ be two points which are not conjugate along any geodesic. Then $\Omega$ (or $\Omega^*$) has the homotopy type of a countable $CW$-complex, with one cell of dimension $\lambda$ for each geodesic from $p$ to $q$ that is of index $\lambda$.

# In case of symmetric space

**Definition**

A **symmetric space** is a connected Riemannian manifold $M$ such that, for each $p \in M$ there is an isometry $I_p \colon M \to M$ fixing $p$, which reserves geodesics through $p$, i.e., $I_p(\gamma(t)) = \gamma(-t)$ if $\gamma$ is a geodesic where $\gamma(0) = p$.

- $M$ is automatically complete since $I_{\gamma(c)} I_p(\gamma(t)) = \gamma(t + 2c)$, where $c$ is small enough that $\gamma(c)$ makes sense. $I_p$ is unique due to completeness.
- $R(U, V)W$ is a parallel vector field along $\gamma$ if so are $U$, $V$ and $W$.

It turns out that on symmetric spaces the Jacobi equation has a rather simple form, in which case we can solve it explicitly.

## In case of symmetric spaces

We have:

1. The map $R(v, \cdot)v \colon T_pM \to T_pM$ is self-adjoint, where $v = V(0)$.
2. There is an orthonormal basis $\{e_i\}_{i=1}^m$ of $T_pM$ such that $R(v, e_i)v = k_i e_i$.
3. $R(V, E_i)V = k_i E_i$ by extending to parallel vector fields.
4. For any vector field $W$ along $\gamma$, we have $W(t) = \sum_{i=1}^m w_i(t)E_i(t)$.

Hence the Jacobi equation becomes:

$$\sum_{i=1}^m \frac{d^2 w_i}{dt^2} E_i + \sum_{i=1}^m k_i E_i = 0 \iff \frac{d^2 w_i}{dt^2} + k_i = 0 \quad (\forall i)$$

The point $q = \gamma(c)$ is conjugate to $p$ if and only if $c$ is a multiple of $\frac{\pi}{\sqrt{k_i}}$ for some $k_i > 0$, and the corresponding multiplicity is the number of such $k_i$.