1 Simplicial sets

1.1 Definition

Let $\mathbf{\Delta}$ be the **simplicial category** where objects are finite ordered sets $[n] = \{0 < 1 < 2 < \cdots < n\}$ for $n \geq 0$, and morphisms are (weakly) order-preserving maps. A **simplicial object** in category $\mathcal{C}$ is a functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$, and a simplicial object in **Set** is called a **simplicial set**.

The category of simplicial sets is denoted by $\mathcal{S}$, where morphisms are natural transformations between functors. For $C \in \mathcal{S}$, we write C_n instead of $C([n])$, and each element of C_n is called an n**-simplex** of C. A **simplicial subset** $D \subset C$ admits a level-wise inclusion $D_n \subset C_n$, such that $D(f) = C(f)|_{D_n}$ for each $f \in \mathbf{\Delta}([m], [n])$.

There is an alternative description for simplicial sets (objects). Let $d^i \colon [n-1] \to [n]$ be the injection skipping i, called the i-th **coface map**, and $s^i \colon [n] \to [n-1]$ be the surjection identifying i and $i+1$, called the i-th **codegeneracy map**.

Lemma 1.1 Each $\phi \in \mathbf{\Delta}([m], [n])$ can be uniquely written as a "standard" composite $d^{i_k} d^{i_{k-1}} \cdots d^{i_1} s^{j_1} s^{j_2} \cdots s^{j_l}$, in such a way that $i_1 < i_2 \cdots < i_k$ and $j_1 < j_2 < \cdots < j_l$.

Proof. Consider $\{i_1 < i_2 < \cdots i_k\} = [n] - \operatorname{im}\phi$ and $\{j_1 < j_2 < \cdots < j_l\} = \{j \in [m] \colon \phi(j) = j+1\}$. (^^)

Obvious relations $d^j d^i = d^i d^{j-1}$ for $i < j$, $s^j s^i = s^{i-1} s^j$ for $i > j$, and

$$s^j d^i = \begin{cases} d^i s^{j-1} & \text{for } i < j \\ id & \text{for } i = j, j+1 \\ d^{i-1} s^j & \text{for } i > j+1 \end{cases}$$

are called the **cosimplicial relations**.

Proposition 1.2 The simplicial category $\mathbf{\Delta}$ is generated by the coface and codegeneracy maps subject to the cosimplicial relations.

Proof. Let $\widetilde{\mathbf{\Delta}}$ be the category freely generated by objects $\{[n]\}_{n \geq 0}$ and morphisms $\{\widetilde{d^i} \colon [n-1] \to [n]\}$, $\{\widetilde{s^i} \colon [n] \to [n-1]\}$. Let $\overline{\mathbf{\Delta}}$ be the quotient of $\widetilde{\mathbf{\Delta}}$ subject to the cosimplicial relations. The obvious functor $F \colon \overline{\mathbf{\Delta}} \to \mathbf{\Delta}$ sending each $[n]$ to itself, such that $F(\widetilde{d^i}) = d^i$ and $F(\widetilde{s^i}) = s^i$, is then an isomorphism of categories, provided by Lemma 1.1 and the fact that every morphism in $\overline{\mathbf{\Delta}}$ can be lifted to a composite in the standard form. (^^)

For a simplicial set C, we call $d_i = C(d^i)$ the **face maps** and $s_i = C(s^i)$ the **degeneracy maps**. By Proposition 1.2, C is just a sequence of sets $\{C_n\}_{n \geq 0}$ together with maps $\{d_i \colon C_n \to C_{n-1}\}$ and $\{s_i \colon C_{n-1} \to C_n\}$ satisfying the **simplicial relations**, which are duals of the cosimplicial relations.

The simplicial set $\Delta^n = \mathbf{\Delta}(-, [n]) \in \mathbf{sSet}$ is called the **standard** n**-simplex**. By Yoneda's lemma, there is a natural bijection $\mathcal{S}(\Delta^n, C) \simeq C_n$ for each $C \in \mathcal{S}$ and $n \geq 0$, given by $\varphi \mapsto \varphi(id_{[n]})$. An important simplicial subset of Δ^n is the one with simplices generated by $\{d_i(id_{[n]})\}_{0 \leq i \leq n}$, called the **Bundary** of Δ^n and denoted by $\partial\Delta^n$. It is clear that $(\partial\Delta^n)_j$ consists of non-serjective maps in $\mathbf{\Delta}([j], [n])$. The simplicial subset with simplices generated by $\{d_i(id_{[n]})\}_{0 \leq i \leq n, i \neq k}$ for $0 \leq k \leq n$ is called the k**-th horn** of Δ^n, denoted by Λ^n_k.

1.2 Realization

A **cosimplicial object** in category $\mathcal{C}$ is a functor $\mathbf{\Delta} \to \mathcal{C}$, and alternatively, a sequence of objects in $\mathcal{C}$ together with maps $\{d^i\}$ and $\{s^i\}$ satisfying the cosimplicial relations. There is a standard **cosimplicial topological space** given by

$$[n] \mapsto |\Delta^n| = \left\{ (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \colon \sum_{i=0}^{n} x_i = 1, x_i \geq 0 \right\}$$

and with the i-th coface map $|\Delta^{n-1}| \to |\Delta^n|$ being $(x_0, \ldots, x_{n-1}) \mapsto (x_0, \ldots, 0, \ldots, x_{n-1})$ inserting a 0 in the i-th position, the i-th codegeneracy map $|\Delta^n| \to |\Delta^{n-1}|$ being $(x_0, \ldots, x_n) \mapsto (x_0, \ldots, x_i + x_{i+1}, \ldots, x_n)$.

Let $\mathbf{\Delta} \downarrow C$ be the **simplex category** of a simplicial set C, with objects being simplices encoded by tuples (x, n) referring to $x \in C_n$. A map $(n, x) \to (m, x')$ is a map $f \colon [n] \to [m]$ such that $C(f)(x') = x$.

Lemma 1.3 Let C be a simplicial set. There is an isomorphism $C \simeq \varinjlim_{\mathbf{\Delta} \downarrow C} \Delta^n$, where the colimit is taken over the functor $\mathbf{\Delta} \downarrow C \to \mathcal{S}$ given by $(x, n) \mapsto \Delta^n$.

Proof. Let $f_{(x,n)}$ be the simplicial map $\Delta^n \to C$ sending $id_{[n]}$ to x, it is then easy to verify that C together with $f_{(x,n)}$ satisfy the universal property of colimit. (^^)

The **realization** of C is then defined as $\varinjlim_{\mathbf{\Delta} \downarrow C} |\Delta^n|$, which tells us how these $|\Delta^n|$ are glued together, keeping track of the simplicial structure of C. We may write the realization of a simplicial set C as $|C|$ in general, as suggested by the following example.

Example. The realization of Δ^n is $|\Delta^n|$, since $(id_{[n]}, n)$ is terminal in $\mathbf{\Delta} \downarrow \Delta^n$.

Example. Let $S^n = \Delta^n / \partial\Delta^n$ be the simplicial n-sphere, then $|S^n|$ is homeomorphic to the topological n-sphere.

It is clear that $|-|\colon \mathcal{S} \to \mathbf{Top}$ defines a functor, since any simplicial map $C \to Q$ induces a functor $\mathbf{\Delta} \downarrow C \to \mathbf{\Delta} \downarrow Q$. An explicit construction of the realization is given as

$$|C| = \left(\coprod_n C_n \times |\Delta^n| \right) \Big/ \sim$$

where we identify two points $(x, u) \in X_n \times |\Delta^n|$ and $(x', u') \in X_m \times |\Delta^m|$ if and only if there is $f\colon [n] \to [m]$, such that $C(f)(x') = x$ and $|\Delta^n(f)|(u) = u'$.

The realization has a right adjoint $\mathrm{Sing}\colon \mathbf{Top} \to \mathcal{S}$, induced by the assignment $\mathrm{Sing}\colon X \mapsto ([n] \mapsto \mathbf{Top}(|\Delta^n|, X))$ of a simplicial set to a topological space, called the **singular complex**.