I. INTRODUCTION

This report examines the relationship between the mass of an object attached to a spring and its angular frequency when the mass exhibits simple harmonic motion caused by an initial external force (Figure 1). A sonar sensor installed at the bottom of the mass will record position data as the mass oscillates up and down as a function of time to determine its angular frequency. The objective of this experiment is to experimentally determine the spring constant k of the unknown spring. Two methods will be used to calculate k: by using Hooke's Law (Figure 2) and by fitting the graph of w^2 as a function of $\frac{1}{m}$ to a linear fit from a derived equation (Figure 3).

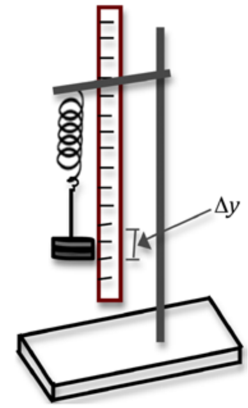

Figure 1: *A spring-mass system in oscillation whereby Δy is the change in position of the mass between the equilibrium point of the spring and the lowest point. (Advanced & North 2013)*

$$\boldsymbol{F = -kx}$$

Figure 2: *Hooke's Law equation, where F is the applied force, k is the spring constant, and x is the extension of the material.*

$$\boldsymbol{w^2 = \frac{k}{m}}$$

Figure 3: *Derived equation, where w is the angular frequency, k is the spring constant, and m is the mass of the object attached to the spring.*

The following derives the equation in Figure 3: equating Newton's Second Law ($F = ma$)[1] to Hooke's Law ($F = -kx$)[2], one attains $a + \frac{kx}{m} = 0$. The equation of the acceleration of a spring is $a = -Aw^2\cos(wt)$ which is attained by finding the second derivative of its displacement equation ($x = A\cos(wt)$)[3]. Thus, $0 = -Aw^2\cos(wt) + \frac{kA\cos(wt)}{m} = -w^2 + \frac{k}{m} \rightarrow w^2 = \frac{k}{m}$.

II. PROCEDURE

The experiment apparatus consists of the Pasco Scientific Hooke's Law Set along with a computer which runs the Pasco Capstone Software. Initially, the mass of the mass hanger and the three masses were measured using a measuring scale which corresponds to 0.0247 kg, 0.0501 kg, 0.0993 kg,

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[1] Where F is the force, m is the mass, and a is the acceleration.
[2] Where F is the force, k is the spring constant, and x is the extension of the material.
[3] Where a is the acceleration, k is the spring constant, x is the extension of the material, and m is the mass.

0.1017 kg, respectively, which a measuring uncertainty of ±0.0001 kg. To set up the apparatus, the sonar sensor, secured to the bottom of the base support, is connected to the Universal Interface which is connected to the computer. The spring is then attached to the rod stand and was measured to be 0.318±0.001 m in length using a measuring tape. Five different loads were attached to the spring for experimentation using different combinations of the masses: 0.0247±0.0001 kg, 0.0748±0.0001 kg, 0.1240±0.0001 kg, 0.1741±0.0002 kg, and 0.2010±0.0001 kg, with the corresponding length of the deformed spring to be 0.332±0.001 m, 0.370±0.001 m, 0.400±0.001 m, 0.435±0.001 m, and 0.470±0.001 m. The uncertainty for the loads is calculated by adding the absolute uncertainties of the masses in the combination. Upon recording using the sonar sensor, the attached load on the spring experiences an initial external upwards force allowing it to exhibit simple harmonic motion. With the collection frequency set to 50 Hz, one trial, lasting around twenty oscillations, was conducted for each load (Figure 4).

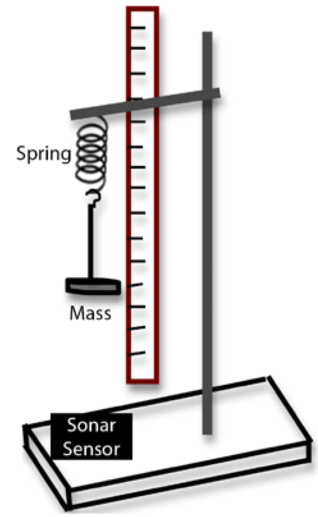

Figure 4: *The experiment apparatus where the sonar sensor, which measures position data of the mass, is placed at bottom of the base support. Loads of different mass are attached to the spring, which is attached to the rod stand.*

Using the heaviest load (0.2010±0.0001 kg) and the associated change in the deformation of the spring (0.470±0.001 m), the spring constant is calculated to be 12.97±0.09 Nm^{-1} using:

$$-mg = -k\Delta x \rightarrow (0.2010)(9.80678) = k(0.470 - 0.318) \rightarrow k = 12.97 \text{ Nm}^{-1}$$

where m is the mass, g is the gravitational field strength, k is the spring constant, and x is the change in the extension of the spring. The value of the gravitational field strength in Toronto, Canada is 9.806 78±0.000 01 m/s^2 (Wolfram Alpha 2017). The uncertainty is calculated using the relative uncertainties of m, g, and x. The heaviest load is used since it carries the lowest relative uncertainty.

Using the Capstone Software, the position of the mass as it oscillates is graphed as a function of time (Figure 5) and is fitted to a sine function. The same is performed to its velocity-time graph (Figure 6). From Figure 5, the amplitude A, the principal axis x_o, phase constant φ_o, and angular frequency w

are estimated to be 0.039 m, $(0.326 + 0.404) \div 2 = 0.365$ m, 5.2243 s, and 10.4720 rad^{2}s^{-2}. The phase constant φ_o is determined by setting $t = 0$:

$$x = Asin(wt + \varphi_o) + x_o \rightarrow x_{int} = Asin(\varphi_o) + x_o \rightarrow \varphi_o = \sin^{-1}\left(\frac{0.329 - 0.365}{0.039}\right) \rightarrow \varphi_o = 5.1072\ s$$

where x_{int} is the x-intercept which is 0.329 (Figure 5), x_o is the principal axis, and A is the amplitude. The angular frequency w is determined using the equation:

$$w = \frac{2\pi}{T}$$

where T is the period, which is $1.240 - 0.640 = 0.60$ s. These estimated values are plotted in the form of $x = Asin(wt + \varphi_o) + x_o$ as the user-defined equation in Figure 5.

Using the Capstone Software, the position data from the sonar sensor is also inputted into the fast Fourier transform (FFT) algorithm whereby it decomposes the position as a function of time into the corresponding frequencies (Figure 7). The frequency at the spike is $\frac{1.6116+1.6601}{2} = 1.64 \pm \frac{1.6601-1.6116}{2} = 0.02$ s^{-1}.

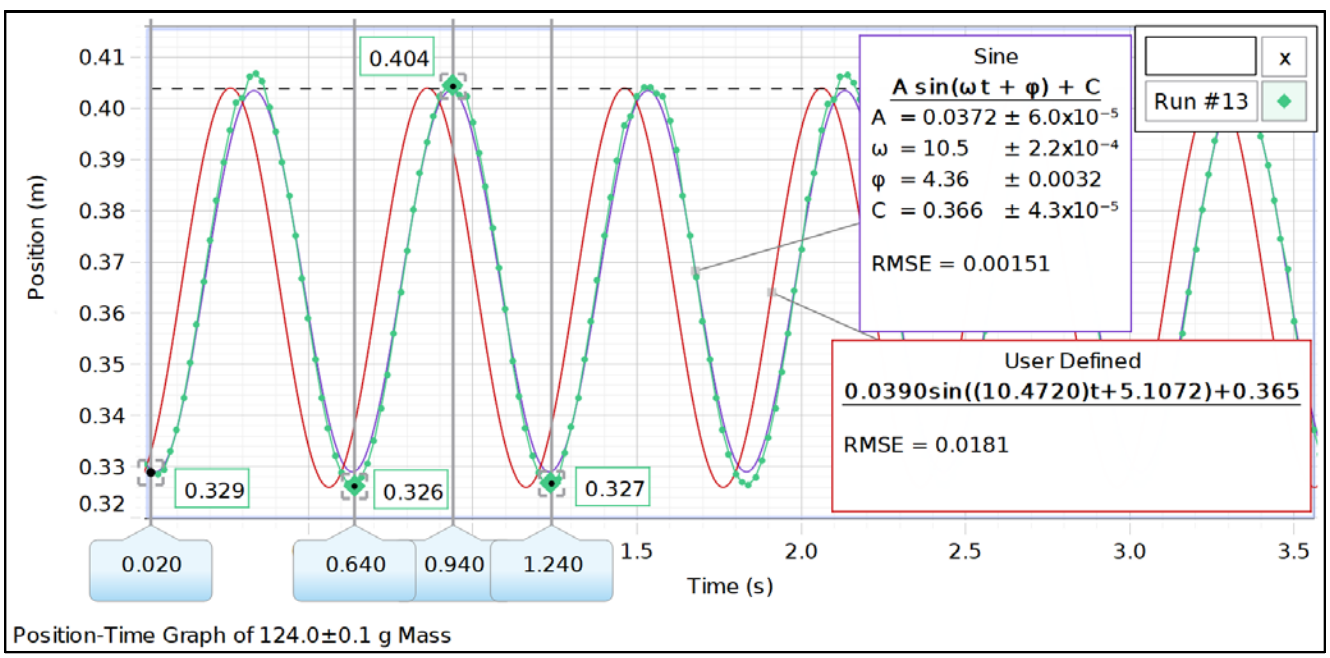

Figure 5: *Position versus time graph created using the Capstone Software from the trial with 0.1240±0.0001 kg as the load. The green curve is the position data collected by the sonar sensor. The purple curve is the fitted sine equation by the Capstone Software. The red curve is the user-defined equation derived from the estimated values.*

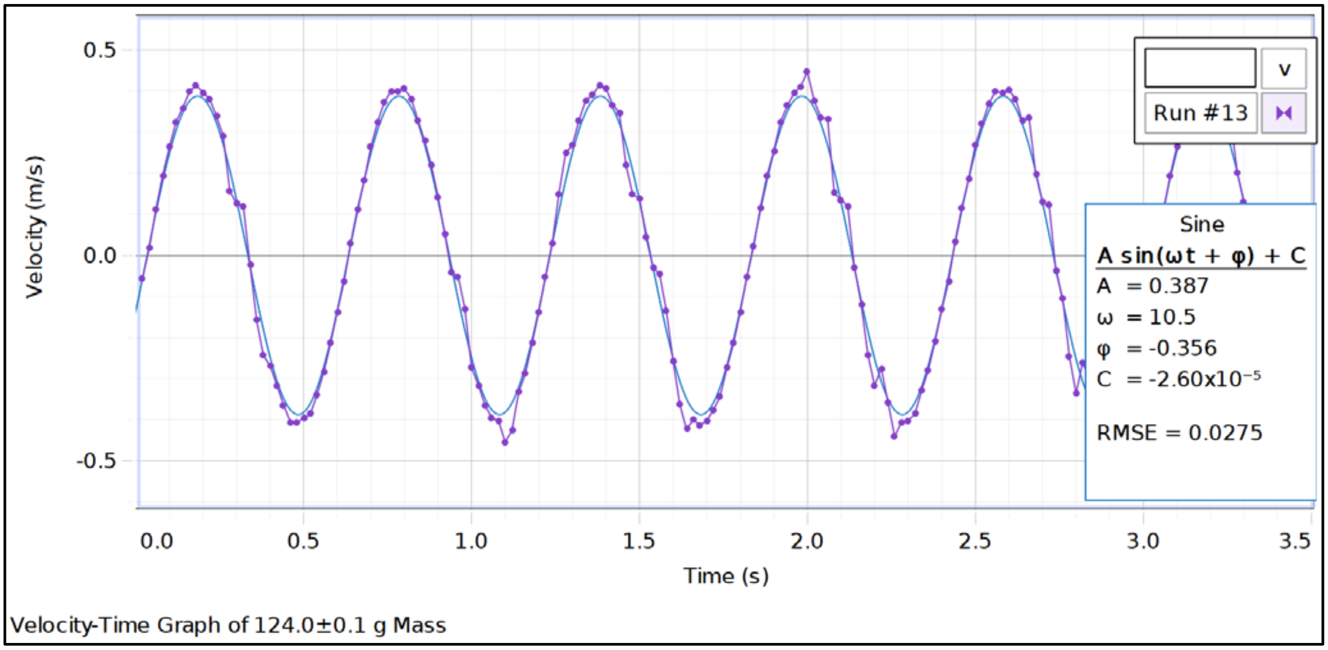

Figure 6: *Velocity versus time graph created using the Capstone Software from the trial with 0.1240±0.0001 kg as the load. The purple curve is the velocity data collected by the sonar sensor. The blue curve is the fitted sine equation by the Capstone Software.*

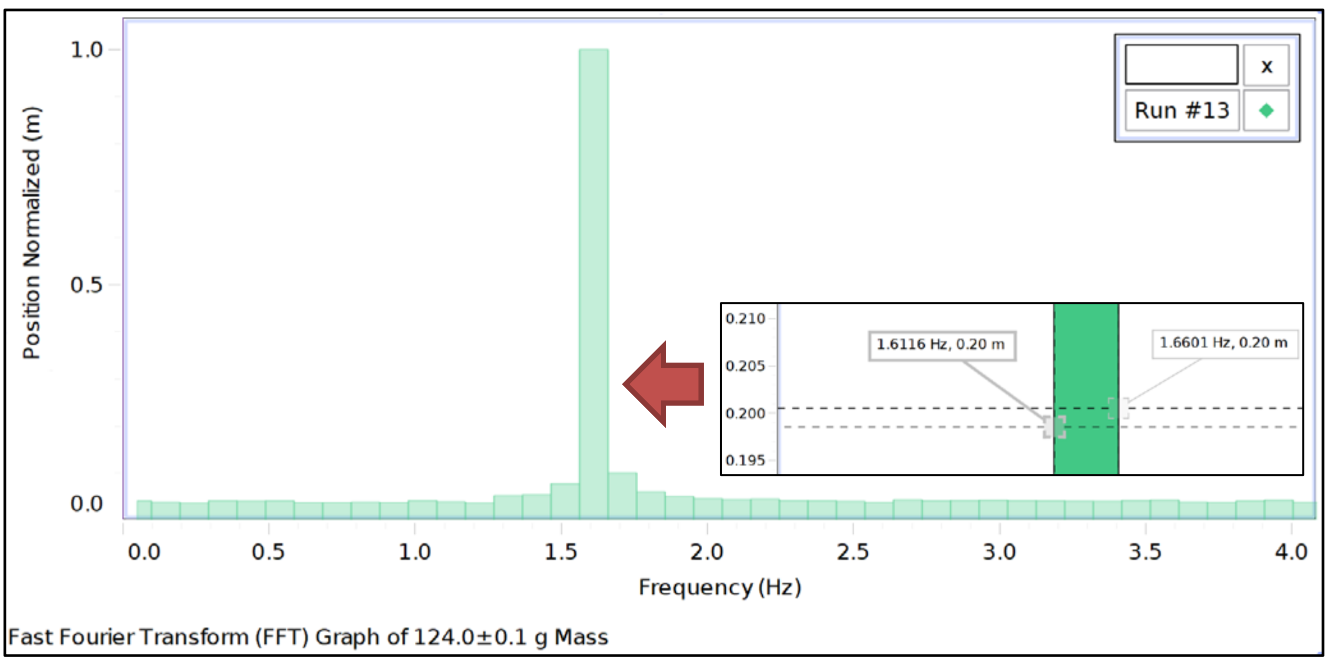

Figure 7: *FFT graph created using the Capstone Software from the trial with 0.1240±0.0001 kg as the load.*

Additionally, to measure the rate of energy loss from the spring-mass system to its environment, the 0.1240±0.0001 kg mass was allowed to oscillate for around eleven minutes while position data is collected. Upon fitting the position-time graph onto a damped sine wave, the decay constant B is found to be 0.003 775±0.000 005 s^{-1} (Figure 8).

III. RESULTS AND DISCUSSION

Damped Sine

$Ae\hat{}(-Bt)\,(\sin(\omega t+\varphi)) + C$

A	= 0.03517	± 3x10⁻⁵
B	= 0.003775	± 5x10⁻⁶
ω	= -10.54	± 5x10⁻⁶
φ	= 5.299	± 9x10⁻⁴
C	= 0.2936	± 7x10⁻⁶

RMSE = 0.00125

Figure 8: *Damped sine wave fit equation where the decay constant B is 0.003 775±0.000 005 s*$^{-1}$.

The relationship between the position-time (Figure 5) and velocity-time (Figure 6) graphs involve a step of differentiation. Since the derivative of sine is cosine and that sine and cosine functions can be expressed by the other with $n(\frac{\pi}{2})$, for n to be any integer, as the phase shift, by analyzing the phase shifts of the two graphs, n is roughly 3.

The user-defined fit function for the position-time graph (Figure 5) resembles the data curve in terms of the amplitude, period, and the principal axis, with the phase shift having a noticeable deviation. When compared to the software fit, the user-defined fit is much less accurate. Furthermore, since the estimated value of the period is 0.60 s, the frequency results to $\frac{1}{0.60} = 1.67$ s^{-1}. This value is very similar to the frequency value attained from the FTT graph (Figure 7), 1.64±0.02 s^{-1}, despite being slightly outside of uncertainty. The inaccuracies of both the user-defined fit and the estimated period value are justified due to the uncertainty associated with estimation of the position-time graph.

The energy lost from the spring-mass system during simple harmonic motion is the result of the transfer of mechanical work to internal work in forms such as thermal energy.

The spring constant k can also be attained by using the equation in Figure 3 whereby a linear fit is performed on the graph of w^2 as a function of $\frac{1}{m}$ (Figure 9). The resulting slope, 11.5±0.3 Nm^{-1}, is the spring constant. This value somewhat differs with the original measurement of k, 12.97±0.09 Nm^{-1}, calculated using Hooke's Law. The original measurement can be justified as less accurate since the calculation was performed on only one trial and the result of random error associated with measuring the deformation of the spring. Nevertheless, the linear fit should attain a y-intercept value b of zero since when $\frac{1}{m}$ equals zero when the limit of $\frac{1}{m}$ as m approaches infinity, w^2 should be zero. This evidences that the spring constant derived from the linear fit is also not quite as accurate.

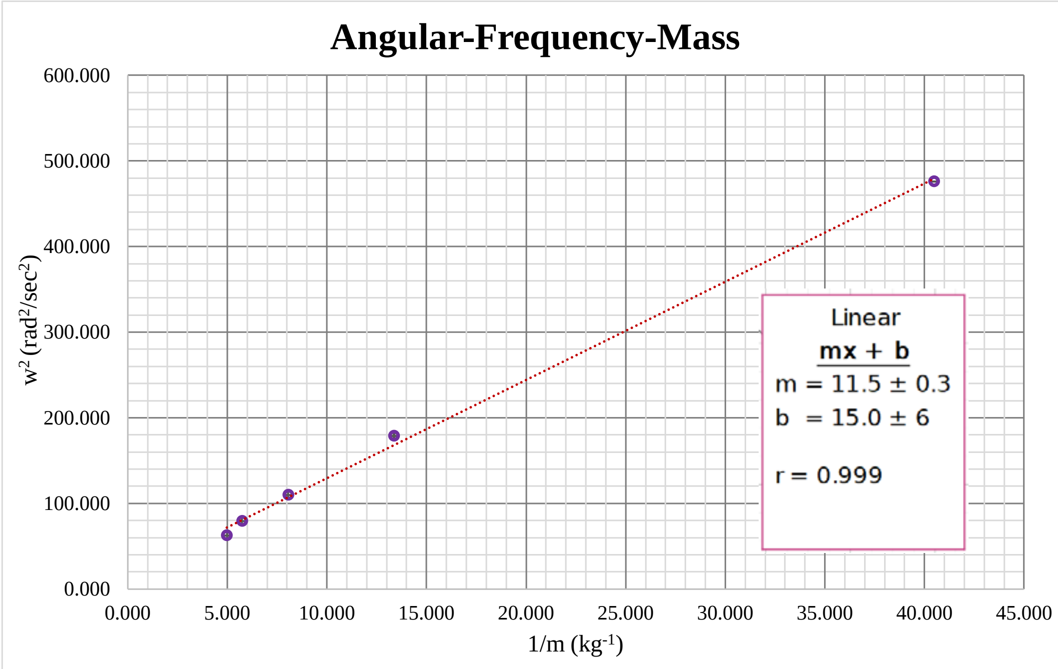

Figure 9: *Angular-frequency-mass graph from* w^2 *as a function of* $\frac{1}{m}$*. The slope of the linear fit is the spring constant. The error bars for both axis is too insignificant to be identified.*

The uncertainties involved in this experimentation consists of the measurement uncertainty associated with the sonar sensor (±0.001 m), the weight of each mass (±0.0001 kg), and the deformation of the spring (±0.001 m). Measurement uncertainty also includes the estimation by inspection of the position-time graph for the user-defined fit and the FFT graph (Figure 7) for the frequency. The three statistical uncertainties consist of the trigonometric fitting for the position-time (Figure 5) and velocity-time (Figure 6) graphs for each load on the spring along with the linear fit of the angular-frequency-mass graph (Figure 9). The linear fit likely provides the dominant statistical uncertainty since the y-intercept theoretically should be zero. Nevertheless, the dominant uncertainty among all is likely associated with the random error in the measurement of the deformation of the spring due to the holding position and by how only one measurement was taken for each load.

IV. CONCLUSION

The experiment has succeeded in identifying a linear relationship between the inverse of the mass of an object attached to a spring and the square of its angular frequency. In other words, the mass and the angular frequency is related by an inverse power relationship. The objective of this experiment has been met by experimentally determining the spring constant using two methods involving Hooke's Law and by fitting the graph of w^2 as a function of $\frac{1}{m}$ to a linear fit despite the value inconsistency. The first method determined the spring constant to be 11.5±0.3 Nm^{-1} while the latter is 12.97±0.09 Nm^{-1}.

The first method to determine the spring constant utilized Hooke's Law and by measuring the mass of the loads and the corresponding deformation of the spring. For the second method, the position data as a function of time of the mass attached to the spring during harmonic motion were initially fitted to a sine function. The angular frequency w is obtained for each of the five loads and its square was graphed as a function of the inverse of its corresponding mass. The slope of the graph's linear fit is the spring constant.

The experimental value of the spring constant is limited by both the random error of the measurement of the deformation of the spring and the statistical error associated with the linear fitting of the angular-frequency-mass graph. While more trials and more loads can be explored to improve the accuracy in addition to using a more sensitive sonar sensor, other methods that exclude using Hooke's Law and linear fitting can be used to determine the spring constant, such as using the period for $T = 2\pi\sqrt{\frac{m}{k}}$.

The experiment can be extended by allowing the mass to oscillate until it stops due to the loss of mechanical energy. Different masses can be explored to find a relationship between the mass and the time it takes for the spring-mass system to stop oscillating given an initial displacement from the equilibrium position.

References

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