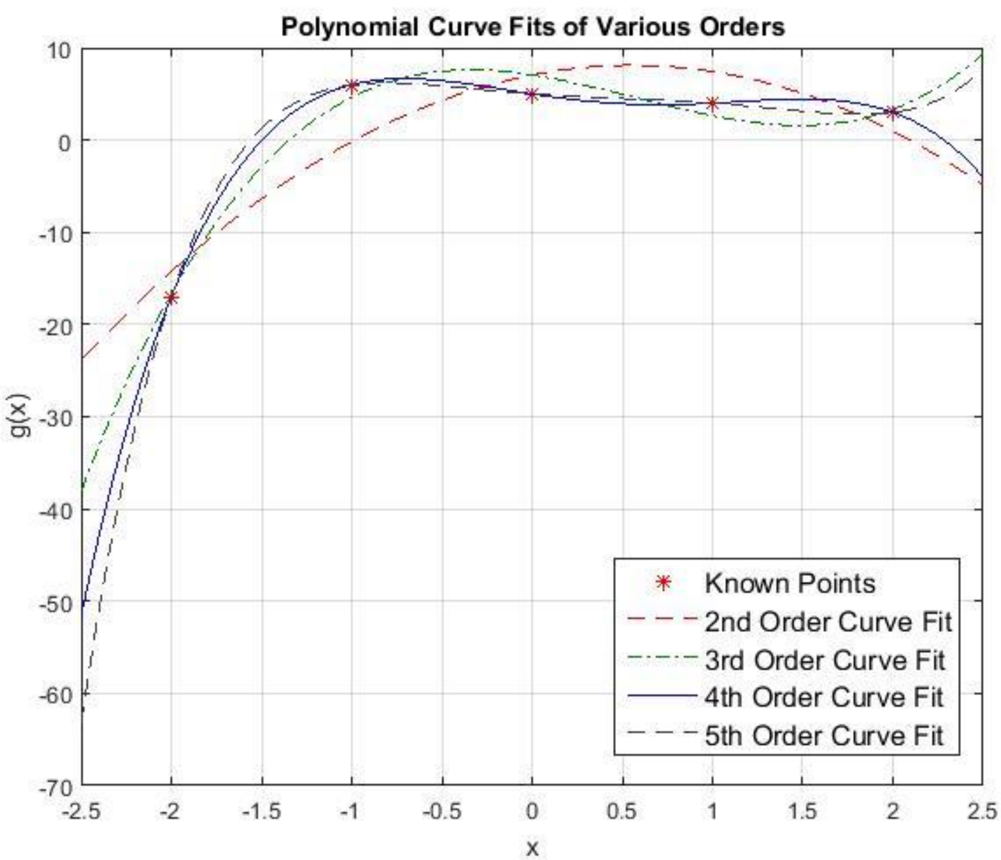
UNIVERSITY OF TORONTO
FACULTY OF APPLIED SCIENCE & ENGINEERING
First Year Program – Core 8 and TrackOne

FIRST YEAR PROGRAM
ENGINEERING PROBLEM SOLVING LABS

MAT188: Laboratory #5
Mathematical Modelling and Curve Fitting



CURVE FITTING AND MODELLING

In this lab, you will learn how to use polynomial curve fitting tools in MATLAB to mathematically model a system. You will learn how to use known data points to fit curves and generate functions for interpolation or extrapolation. This lab will also end with a 10 minute online quiz.

Learning Outcomes

By the end of this lab students will…

1) Understand simple polynomial curve-fitting methods, and
2) Develop a mathematical model from a curve fit based on known or measured data.

Preparation *(Required doing* ***before*** *you come to the laboratory session)*

1. Watch the video demonstration, *Analyzing and Visualizing Data with MATLAB (3:26)* http://www.mathworks.com/videos/analyzing-and-visualizing-data-with-matlab-70942.html
2. Read through this lab document.

Review – Lab #3 and #4

At this point you should be comfortable with creating multiple plots in a figure and writing simple scripts to save and modify your code.

For example, you could use the following commands to produce plots of $\sin(2x-1)$ *and its derivative,* $2\cos(2x-1)$*, over the interval* $[0, 2\pi]$ *on the same figure.*

```
>> x=linspace(0,2*pi,101) % x has 101 elements equally spaced from 0 to 2pi
>> y=sin(2*x-1);  % calculates the value of the function
>> ydot=2*cos(2*x-1);  % calculates the values of the first derivative
>> plot(x,y,'r-',x,ydot,'b--')  % Plots both curves on the same figure with different styles and colours. Alternatively plot y first then use hold and plot ydot.
```

MATLAB Skills and Knowledge:
Fitting polynomial curves to available data

The `polyval` function is useful for defining a vector of polynomial function values more easily. Instead of typing the whole function, you can provide a vector of the coefficients. `polyval(p,x)` generates a vector of polynomial values with coefficients given in vector `p` and input range given in vector `x`. For example, to define the function $f(x) = 2x^4 - 4x^3 + 7x - 5$ over the interval $[-5, 5]$:

```
>> x=linspace(-5,5,101);
>> coefs=[2 -4 0 7 -5];
>> y=polyval(coefs,x);
```

`y` now contains the values of the polynomial for each value of `x`. Notice how the x^2 coefficient must be included in the vector as a 0.

Curve Fitting (Guided Exercise)

In MAT188, you will soon solve problems on systems of linear equations, which can be applied to polynomial curve fitting[1].

Consider the following problem that we will solve:

> Find a polynomial of the form $g(x) = ax^4 + bx^3 + cx^2 + dx + e$, such that it goes through the points $g(-2) = -17$, $g(-1) = 6$, $g(0) = 5$, $g(1) = 4$, $g(2) = 3$.

If you were to solve this problem on paper, you would obtain a system of five linear equations and solve it to find the five unknown coefficients[2]:

$a = -1, b = 2, c = 1, d = -3, e = 5$

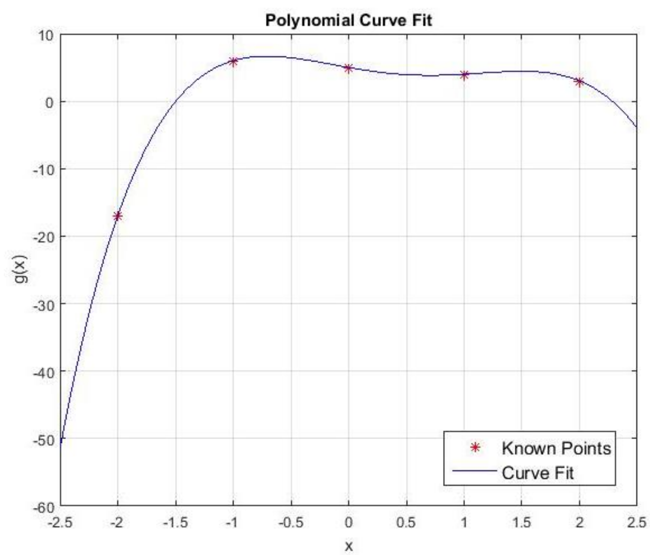


Figure 1. Fourth-order polynomial curve fit

Using these coefficients and `polyval`, generate a plot of this 4th-order polynomial, similar to Figure 1. Include the five known points as a separate function on your plot with a different line style (refer to the MATLAB Summary document for Lab 3).

The `polyfit` function calculates the coefficients of the *optimal curve fit*, in which the differences between the polynomial curve and the known data points are minimized.

This enables us to predict the behaviour of a system based on only a few known or measured pieces of data. This process is known as *interpolation* or *extrapolation*, and the curve fit is a *mathematical model* of that system.

Consider the same generic polynomial as above, $g(x)$, with the same set of known data points. Use the `polyfit` function in MATLAB to determine the coefficients for a fourth-order polynomial fit:

```
>> xpts=[-2,-1,0,1,2];
>> ypts=[-17,6,5,4,3];
>> coefs=polyfit(xpts,ypts,4)
```

How do the elements of the `coefs` *vector compare to the solution given above?*

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[1] If you are interested a simple example using MATLAB can be found at: https://www.youtube.com/watch?v=y2EF-d-15Oo
We are extending these ideas here to deal with polynomial curve fits as opposed to linear (or straight line) fit in the video.
[2] Following the same process as shown in the video above, can you come up with these five linear equations?

A good engineer is able to identify the simplest possible mathematical model of a system that still predicts its behaviour accurately. We will now explore different polynomial curve fits for this system.

Use `polyfit` to determine the coefficients of a 2nd-order, a 3rd-order, and a 5th-order polynomial curve fit for this set of known data points. *Did you get any errors in these calculations?*

Create a script that obtains the coefficients of various polynomial curve fits (2nd, 3rd, 4th, and 5th order), then plots the known data points and the four curve fits on the same figure, similar to Figure 2. After getting the coefficients using `polyfit`, use `polyval` to create the `y`-vectors. Note that for the modelled functions based on the curve fits, you should create a proper input vector `x` using `linspace` and not use `xpts` as your input vector.

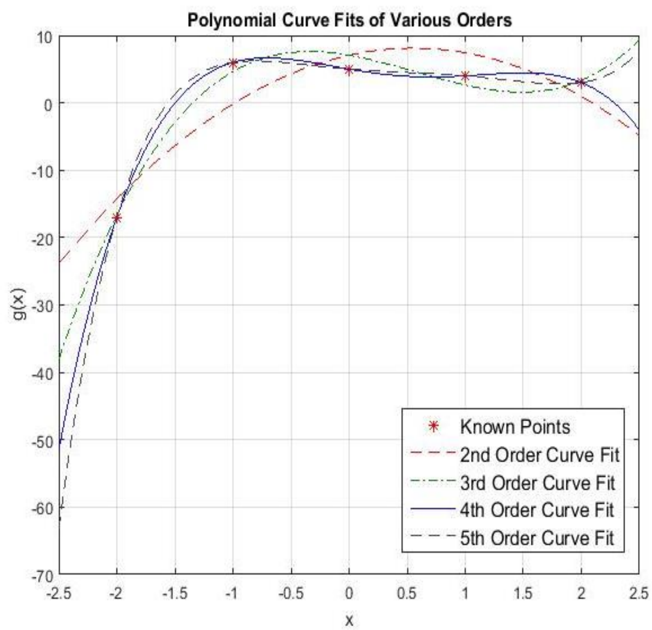

Figure 2. Comparison of various polynomial curve fits

What do you notice about the 2nd and 3rd order curve fits that is different from the 4th and 5th order curve fits?

If you were to use these models to predict or extrapolate the behavior of the system for values of x>2.5, which curve fit appears to be the best?

Engineering Applications

The Engineer's Toolbox - Mathematical Modelling

One of the most fundamental components of an "*engineer's toolbox*" is the ability to apply the principles of math and science to a problem to identify an optimal solution. Using *mathematical modelling*, an engineer creates an approximate description of a system using equations, graphs, or other tools, often based on fundamental principles combined with measured data. With such a model, the engineer can predict the behaviour of the system beyond what is known from direct measurements, and can virtually experiment with a wide range of design approaches or ideas without the need for physical prototyping or experimentation.

In this part of the lab, you will use curve fitting for a specific application. For a situation, you will plot the original data, determine and compare polynomial and exponential models for the data, choose which one better models the situation, and use the better model to predict future behaviour.

Choose one of the two missions below, and submit your response via the MAT188 PRA website for credit.

Engineering Mission #1: Chillin' with the Penicillin.

As an agent of SKULE, your mission is to investigate the concentration of penicillin in the bloodstream over time, based on five experimental data points available on the internet. We will use this information to engineer a model and predict the optimal time to wait between effective penicillin injections. We will also critically examine some limitations of this approach.

Write a script to do the following:

a. Plot the five experimental data points in the table at http://mathinsight.org/penicillin_clearance_model.
b. On the same figure, develop and plot a polynomial fit for this data. Experiment with different order curve fits and choose the best one.
c. On the same figure, develop and plot an exponential fit for this data ($y = ae^{bt}$, where t is the time in minutes). You can determine a and b from two of the data points; do this by manual calculation on paper or in the MATLAB command window.
d. Which type of fit seems to be more realistic as the better model of the real-life situation? Explain in point form; be clear, concise, and thoughtful.
e. Using the better model:
    i. Predict when the penicillin concentration would drop below 40 µg/ml.
    ii. Suppose it is known that for penicillin to remain effective at killing bacteria, a level of >20 µg/ml must be maintained within a person's bloodstream. Identify the longest time one should wait between penicillin injections.

Submit the following:
As one PDF document, each student should submit figures, responses to questions, and in clearly and concisely written language, articulate any limitations to your work, and why you think it is important to consider them. Submit to MAT188 PRA website. Total submission must be a single page.

Engineering Mission #2: Population Investigation

As an agent of SKULE, your mission will be to investigate the growth of the US population over time, as recorded by census data. We will use this information to help predict relationships between time and population, and the potential impact this will have on engineering.

Write a script to do the following:

a. Load the US Census data of US population in millions from 1790 to 1990 by typing `load census` in the MATLAB command window. This will create the vectors `cdate` and `pop` in the workspace. Plot this data.
b. On the same figure, develop and plot a polynomial fit for this data. Experiment with different order curve fits and choose the best one.

c. On the same figure, develop and plot an exponential fit for this data ($y = ae^{bt}$, where t is the time in years). You can determine a and b from two of the data points; do this by manual calculation on paper or in the MATLAB command window.
d. Which type of fit seems to be more realistic as the better model of the real-life situation?
e. Using the better model estimate the US population in 2017. Does this match what is expected (see http://www.worldometers.info/world-population/us-population/)?
f. If it is expected that the current US agricultural system and economy will not be able to sustain a population over 750 million, estimate how many years the country's engineers, scientists, and government officials have to identify a long-term solution to these critical issues, and why. Keep your responses very brief.

Submit the following:
As a PDF document, each student should submit figures, responses to questions, and, in roughly 2-3 bullet points, clearly and concisely articulate any limitations to your work, and why you think it is important to consider them. Submit to MAT188 PRA website. Total submission must be a single page.

You will submit these two items through the MAT188 PRA Portal page:
Winter-2018-MAT188H1-S-PRA0101: APP. LINEAR ALGEBRA

Ideally, you will submit this assignment by the end of your Lab #5 session, but at the very latest you can submit this by Wednesday, February 14th at 11:59PM.