NATIONAL RESEARCH UNIVERSITY
HIGHER SCHOOL OF ECONOMICS

Faculty of Mathematics

Project proposal

Connections in holomorphic bundles
Связности в голоморфных расслоениях

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Abstract

And more, and more, and more

Write something Anfel Let me tell you about the Riemann-Hilbert problem on elliptic curve. Let us take a meromorphic matrix function $A(z)$ called *the matrix of coeffitients* over a complex manifold and solve the equation $\dot{y} = A(z)y$. One will get a linear space of solutions, but the solutions have a monodromy map associated with singular points of the coefficients, which can be represented as a set of linear maps on the space of solutions. Each monodromy map correspodns to a specific singular point of A. Taking these maps together, a representation of the fundamental group of punctured surface is constructed. One assigns every loop class to the corresponding monodromy of the equation. But, given the monodromy representation, how can one find A which corresponds to this representation? This question is a naive statement of a Riemann-Hilbert problem, or Inverse Monodromy problem.

It turns out that one can generalize the above construction and treat the solutions as the sections of a semistable zero-degree bundle and A as the matrix of a flat connection on the bundle. In this more general setting lies a huge opportuity, because the language of bundles and connections is inherently algebraic, contrary to the analytic one which was presented in the beginning. I confine the generalization to the case of semistable bundles since their Chern classes behave better than for a non-stable one, and if the bundle has suficiently large degree (the degree of the divisor of the arbitrary section), then the bundle has a set of global sections which form a basis at every point. The profit of this confinement will not be seen in the current inquisitions, but this property gives much in terms of algebraic structure of the bundle. Let us confine the considerations to the case of logarithmic (Fuchsian) connections. These are connections which have simple poles only. One can try to construct the connections and solutions explicitly.

I want to construct such connections in a two-dimensional exceptional (non-splitting) (see Thm5 in Atyiah's paper [?]) bundle over an elliptic curve. I expect to get the explicit formulas for the connections, and find which monodromies they represent.

Preliminaries

Определения и теоремы, которые входят в Introduction Далее, про постановку задачи есть небольшая проблема. Теоретически она из двух частей. Первая это выяснить как вообще устроены расслоения на эллиптической кривой,: сначала одномерные а потом двумерные, как их удобно описывать, как устроены логарифмические связности в них, как они пишутся в явном виде через тэта-функции, причём здесь вообще тэта-функции.

Let me first provide a couple of examples. Let us condider a fundamental matrix, and construct the equation corresponding to this fundamental matrix.

$$: \tag{1}$$

I will assume that the definition of the vector bundle is known.

Вторая часть - это после освоения первой части порешать какую-нибудь практическую задачу. Например, найти общий вид связности в исключительном двумерном расслоении или описать калибровочные симметрии связности в полустабильном, или найти аналог формулы с экспонентой вычета для монодромий по а- и б- циклам.

Methods of computations

Introduction

Let us have an elliptic curve which is a holomorphic one-dimensional manifold of genus 1. In my bachelor thesis I will treat flat logarighmic connections in holomorphic bundles of rank 2 and degree 0 over elliptic curve. The curve has a fixed modular parameter τ. The bundle will be called E and the base will be called B. Horizontal sections of the connection are such sections that $\nabla s = 0$ or $ds = Adz \otimes s$ holds where both sides belong to $E \otimes T^*M$. The matrix A is called the *connection matrix* in this setting. Any meromorphic matrix that is changed along the bundle sections such that the horizontal section stays horizontal with respect to this modified matrix is a connection matrix for a particular bundle. Matveeva and Poberezhny in their article [?] consider the case of a connection over the stable direct sum of line bundles. They find a Fuchsian connection corresponding to a rigid representation (here: the representation induced from a sphere) with 3 branch points. They find explicitly a conection which is associated to the stable bundle $\mathcal{O}_\lambda(0) \oplus \mathcal{O}_{-\lambda}(0)$ for given λ. The logarithm of a local monodromy matrix of the found connection is conjugated to the residue of the matrix of the connection at the given monodromy point. Then, using these facts they find the solution for local monodromies and give an integral equation that should be satisfied by a conjugated (deformed) connection matrix so the global monodromy was trivial. See Preliminaries for additional

There is a couple of thoughts I have not yet elaborated. First, it would be marvelous to infer some general properties of those connections using Serre Duality, Riemann-Roch cohomology theorem, Chern characters and *Ext* groups, which are standard algebraic instruments when one is studying complex manifolds. Second, I like to think about connections as sections itself, since they are actually special elements of $F = Hom(E, E \otimes T^*M)$ which is a bundle itself. The Leibniz rule and the flattness are two conditions which I should restate in this settings so I could distinguish "good" sections from "bad" ones. The Fuchsianness of the connection means that there are no divisor points with multiplicity more than 1.

The classic analytic way to operate with the sections are Jacoby θ-functions, as far as I know they were discovered in the beginning of XIX century. Any line-bundle has sections, represented by a product of theta-functions. I will heavily use it in our constructions. Another classic object is Hypergeometric Differential Equation. The canonical form is

$$z(1-z)\frac{d^2y}{dz^2} + [c - (a+b+1)z]\frac{dy}{dz} - aby$$

This equation has regular singular points at $0, 1, \infty$, and different monodromies depending on a, b, c. This second-order linear differential equation can be represented as

$$\begin{pmatrix} 0 & 1 \\ \frac{ab}{z(z-1)} & \frac{c-(a+b+1)z}{z(z-1)} \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} = \frac{d}{dz} \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} \tag{2}$$

this equation has a two-dimensional space of solution. The Riemann-Hilbert problem has a rich history, that will not be covered here. Riemann in 1857 has published a paper where he considered the hypergeometric equations and its monodromies. Hilbert in 1905 solved the regular Riemann-Hilbert problem for rank 2 bundles. The Riemann-Hilbert problem on the Riemann sphere was solved negatively by Andrey Bolybruch in 1988, using the techniques leading to many other results. [?] He has provided an explicit counterexample of a punctured Riemann sphere fundamental group representation that can not be a monodromy representation of a Fuchsian system. Bolybruch has obtained other results, one of them is that for any *irreducible* representation can be obtained as a monodromy. The Riemann-Hilbert problem has connections with integrable systems [?] and algebraic geometry.

Main Results