# (a, b)-tree experiment

## Test program

The test program evaluates the implementation of (2, 3)-tree and (2, 4)-tree.
It performs three types of tests:

1. Insert test
2. Min test
3. Random test

## Performance

Performance is measured in terms of the average number of structural changes.
Structural change is either a node split (in insert) or merging of two nodes (in delete).

- #INSERT the number of performed insert operations
- #DELETE the number of performed delete operations

- #SPLIT the number of performed split operations
- #MERGE the number of performed merge operations

- #NODES the number of nodes

## Plots

Each plot shows the dependence of the average number of structural changes ASC on the set size #INSERT.

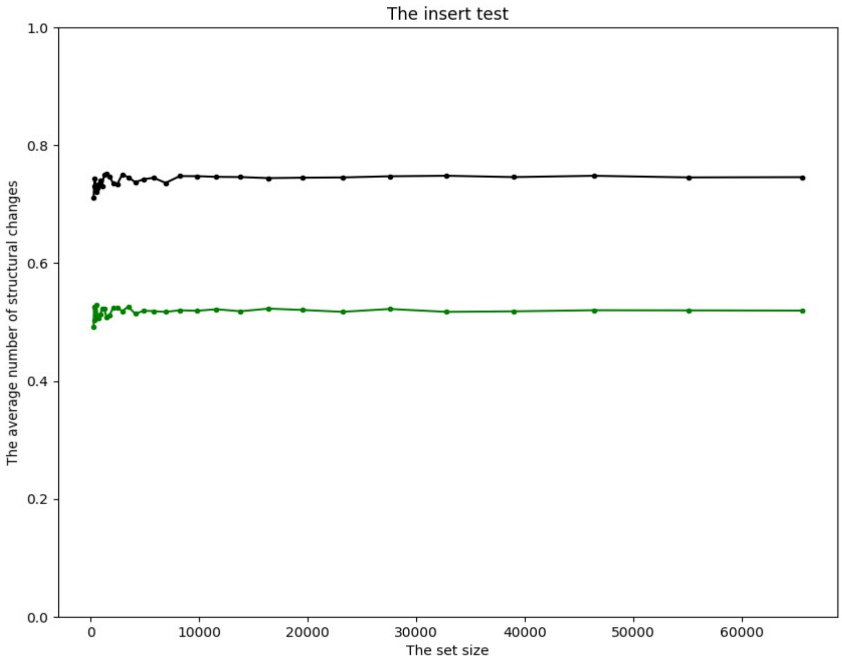
- (2, 3)-tree the black curve
- (2, 4)-tree the green curve

## Theoretical facts

- The height of (a, b)-tree is at least $\log_b$ (#INSERT + 1).

## 1. Insert test

Insert n elements in random order.




**Estimation of bounds for the average number of structural changes:**

- $1 / (b – 1) - \log_b (\#INSERT + 1) / \#INSERT \quad <= \#SPLIT / \#INSERT \quad < 1$.
  - $1 / (b – 1) - \log_b (\#INSERT + 1) / \#INSERT \quad <= \#SPLIT / \#INSERT$:
    - #INSERT <= (b - 1) * #NODES.
      - Each node has at most (b – 1) keys.
    - #NODES - #HEIGHT <= #SPLIT.
      - Each split creates at least one node.
      - When we split the root, another new node is created and added as a new root.
    - #SPLIT / #INSERT >= (#NODES - #HEIGHT) / #INSERT
      = #NODES / #INSERT - #HEIGHT / #INSERT
      >= #NODES / ((b - 1) * #NODES) - #HEIGHT / #INSERT
      = 1 / (b – 1) - #HEIGHT / #INSERT
      >= 1 / (b – 1) - $\log_b$ (#INSERT + 1) / #INSERT.

- #SPLIT / #INSERT < 1:

  - Consider the worst case:
    - (2, 3)-tree: insert sequence in ascending or descending order
    - (2, 4)-tree: insert sequence in ascending order

  - There is only one branch, to that the keys are inserted.
    Let's say it has internal nodes $n_1$, $n_{2, \ldots,} n_h$ , where $n_l$ is the lowest internal node and $n_k$ is root.
  - For each such node holds:
    - except $n_l$ (initial root), each node was created with (b - 2) key.
    - it has (b - 2) keys after each split.
    - after each insertion to this node without a split, the next insertion is performed with a split, because the node can't contain more than (b - 1) keys.

  - First insertion creates $n_l$ with one key.
  - Next (b -2) insertions adds keys to $n_{1.}$
  - Next insertion splits $n_1$ and creates $n_2$ (as a new root) with one key.

  - After creating $n_i$, all nodes in the tree have exactly one key and $n_i$ is the new root.
    We need to perform next $2^i$ insertions { $\text{insert}_1$, $\text{insert}_2$,..., $\text{insert}_{2^i}$ } until $n_{i+1}$ is created and added as a new root:

    - For j: 1 <= j <= $2^i$ each $\text{insert}_j$ starts in $n_{1.}$
    - For j: 1 <= j <= $2^{i-1}$ each $\text{insert}_{2j}$ is performed with splitting of $n_{1.}$
      After this, some key is inserted to $n_2$.
    - For j: 1 <= j <= $2^{i-2}$ each $\text{insert}_{2^2 j}$ is performed with splitting of $n_1$ and $n_2$.
      After this, some key is inserted to $n_3$.
    - Last insertion, $\text{insert}_{2^i}$, is performed with splitting of $n_1$,...,$n_i$.
      After this, $n_{i+1}$ is created and added as a new root.

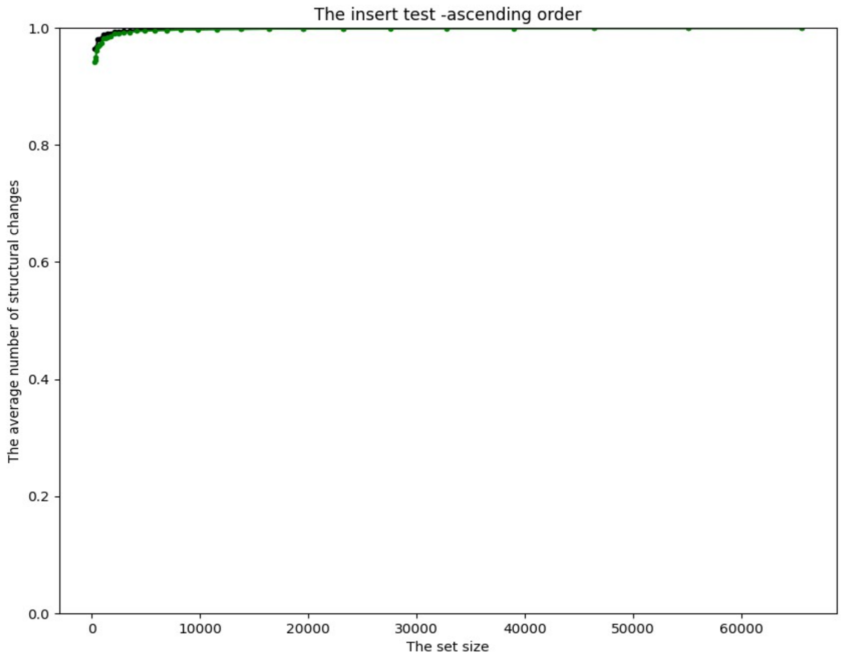
    We perform exactly $2^{i-1}$ splits on node $n_i$ during these $2^i$ insertions.

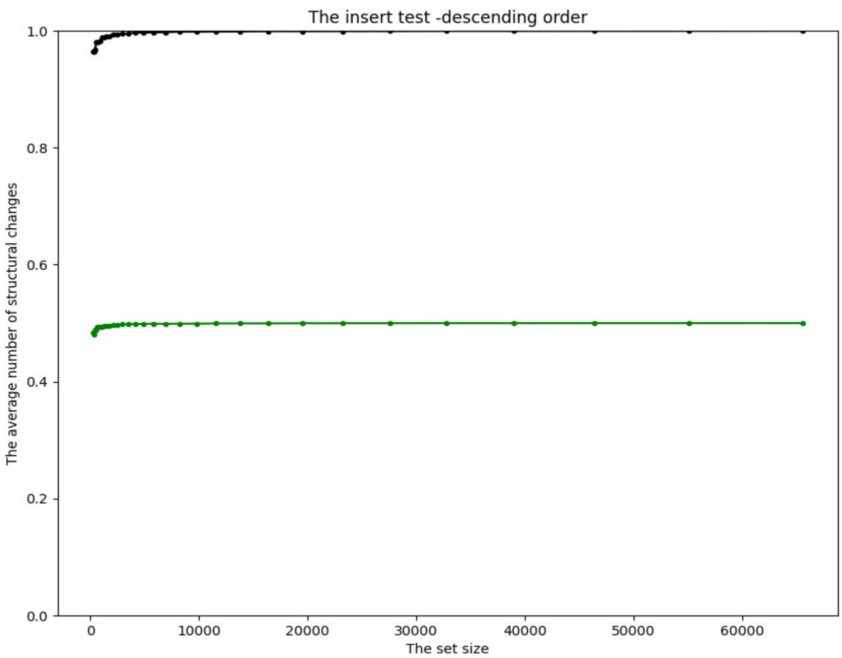
    - Sum of the number of splitting over $n_1$,...,$n_i$ is:
      $2^{i-1} + 2^{i-2} + 2^{i-3} + \ldots + 2^0 = 2^i - 1$.

  - #SPLIT / #INSERT < 1.
    - We perform exactly $2^i$ insertions until new root $n_{i+1}$ is created, the average number of structural changes for each level i is $(2^i - 1) / 2^i < 1$.

The worst case:

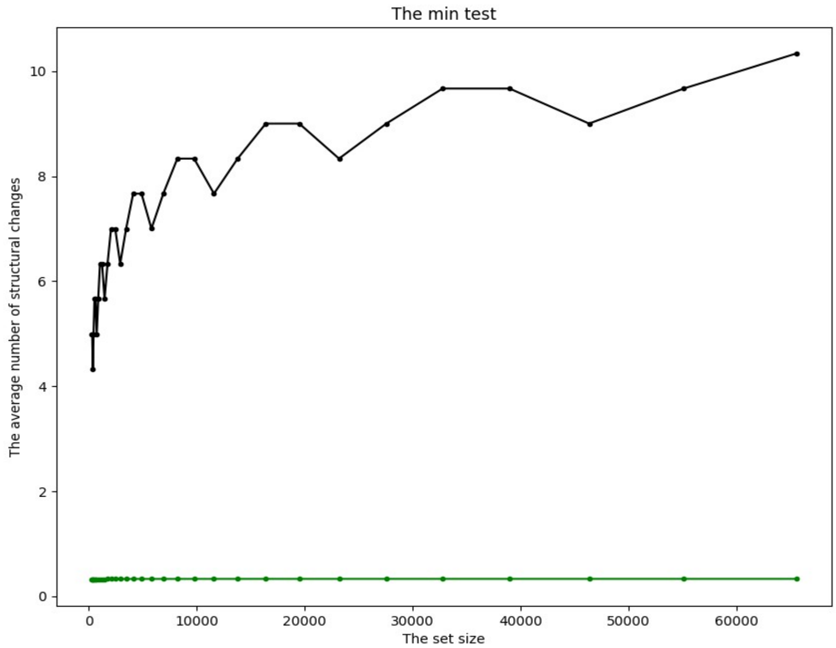
- (2, 3)-tree: insert sequence in ascending or descending order
- (2, 4)-tree: insert sequence in ascending order



**2. Min test**

Insert n elements sequentially and then n times repeat: remove the minimal element in the tree and then insert it back.




**(2, 3)-tree:**

- insert n elements sequentially (the worst case ):
  - there is only one branch (the right one), to that the keys are inserted
  - only nodes on this branch can have more than one key
  - #SPLIT / #INSERT ~ 1.

- n times remove the minimal element in the tree and then insert it back:
  - the minimal element is removed from the left branch with (log(n) - 1) merges and insert back to the left branch with (log(n) – 1) splits.
  - #MERGE / #DELETE ~ #SPLIT / #INSERT ~ (log(n) – 1).

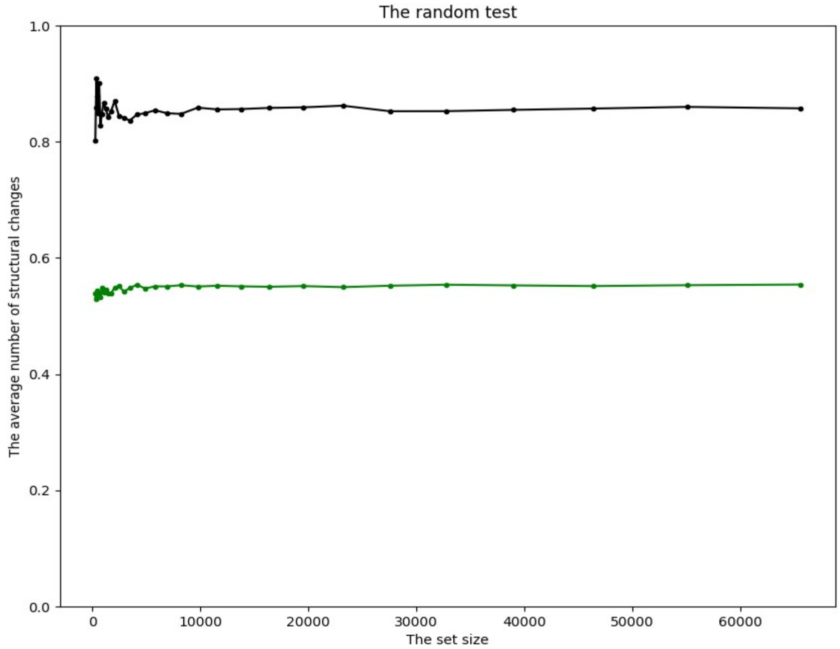
**(2, 4)-tree:**

- insert n elements sequentially (the worst case ):
  - #SPLIT / #INSERT ~ 1.

- n times remove the minimal element in the tree and then insert it back:
  - the minimal element is removed from the lowest internal node (with two keys) on the left branch and insert back to the same node without any merge or split.

## 3. Random test

Insert n elements sequentially and then n times repeat: remove random element from the tree and then insert random element into the tree.
Removed element is always present in the tree and inserted element is always not present in the tree.




**(2, 3)-tree, (2, 4)-tree:**

- insert n elements sequentially (the worst case ):
  - #SPLIT / #INSERT ~ 1.
- n times remove random element and then insert random element into the tree:
  - #SPLIT / #INSERT is similar as in Insert test
  - #MERGE / #DELETE ~ #SPLIT / #INSERT