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Spectral collocation methods in solid mechanics

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Content

- Spectral collocation methods
- Two-dimensional boundary value problem
  - Deployment of boundary conditions
    - Resampling technique
    - Moving technique
    - Overdetermined system - the least squares method
- Deflection of a semi-infinite elastic solid by a surface load
- Displacement formulations
- Stress formulations
- Conclusion

Spectral collocation methods

- Lagrange interpolation at Chebyshev points
- Differentiation matrix

$$\frac{\mathrm{d}f}{\mathrm{d}x}(x)\bigg|_{x=x_j} \approx \frac{\mathrm{d}p}{\mathrm{d}x}(x)\bigg|_{x=x_j} \qquad \Longrightarrow \qquad \boldsymbol{f}'_N = \boldsymbol{D}_{N\times N}\boldsymbol{f}_N$$

- Resampling matrix $\boldsymbol{P}^{N,-m}$, transformation from $\{x_k\}_{k=1}^{N} \Longrightarrow \{y_j\}_{j=1}^{N-m}$

$$\boldsymbol{f}_{N-m} = \boldsymbol{P}^{N,-m}\boldsymbol{f}_N$$

$\boldsymbol{f}_{N-m}$ values of f at the 1st kind points
$\boldsymbol{f}_N$ values of f at the 2st kind points

Two-dimensional boundary value problem

- Chebyshev grid, $f(x_i, y_j) = f_{ij}$

$$\boldsymbol{f}_{N_x N_y} = \begin{bmatrix} \boldsymbol{f}_{i1} & \boldsymbol{f}_{i2} & \dots & \boldsymbol{f}_{iN_y} \end{bmatrix}^\top$$

- Partial derivatives

$$\frac{\partial}{\partial x} \quad \Longrightarrow \quad \boldsymbol{I}_{N_y \times N_y} \otimes \boldsymbol{D}_{N_x \times N_x}$$

$$\frac{\partial}{\partial y} \quad \Longrightarrow \quad \boldsymbol{D}_{N_y \times N_y} \otimes \boldsymbol{I}_{N_x \times N_x}$$

- Deployment of boundary conditions, for example

$$\begin{aligned} Lu(x,y) &= F(x,y) && \text{in } \Omega = (-1,1) \times (-1,1) \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

Deployment of boundary conditions

- Overdetermined system - the least squares method

$$\begin{bmatrix} \boldsymbol{B}_{2N_x+2N_y-4\times N_xN_y} \\ \boldsymbol{L}_{N_xN_y\times N_xN_y} \end{bmatrix} \boldsymbol{u}_{N_xN_y} = \begin{bmatrix} \mathbf{0} \\ \boldsymbol{F}_{N_xN_y} \end{bmatrix}$$

- Resampling technique

$$\begin{bmatrix} \boldsymbol{B}_{2N_x+2N_y-4\times N_xN_y} \\ \boldsymbol{P}_{(N_x-2)(N_y-2)\times N_xN_y}\boldsymbol{L}_{N_xN_y\times N_xN_y} \end{bmatrix} \boldsymbol{u}_{N_xN_y} = \begin{bmatrix} \mathbf{0} \\ \boldsymbol{F}_{(N_x-2)(N_y-2)} \end{bmatrix}$$

- Moving technique – Dirichlet boundary conditions

$$\boldsymbol{M}\boldsymbol{u}_{\text{withoutB}} = \boldsymbol{F} - \boldsymbol{C}\boldsymbol{u}_{\text{B}}$$

$\boldsymbol{C}$ – matrix $\boldsymbol{L}_{N_xN_y}$ with removed rows
$\boldsymbol{M}$ – matrix $\boldsymbol{C}$ with removed columns

Deflection of a semi-infinite elastic solid

- Plane strain $\boldsymbol{u} = (u, v, 0)$
- Analytical solution – Flamant solution

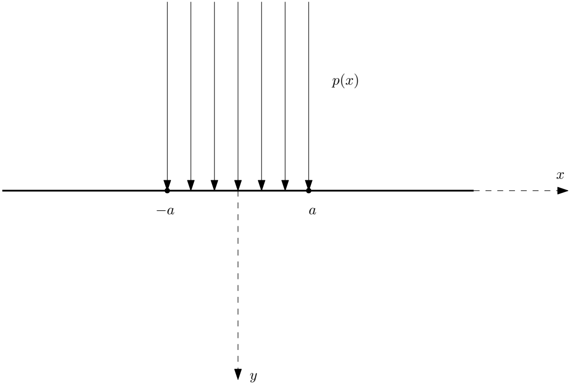

Figure: Pressure applied to the surface of two-dimensional elastic solid.

Deflection of a semi-infinite elastic solid

- Linearised elasticity

$$
\begin{aligned}
\boldsymbol{\epsilon} &= \frac{1}{2}\left(\operatorname{grad}\boldsymbol{u} + (\operatorname{grad}\boldsymbol{u})^\top\right) && \text{linearised strain tensor} \\
\boldsymbol{t} &= \lambda\,\mathrm{Tr}(\boldsymbol{\epsilon})\boldsymbol{I} + 2\mu\boldsymbol{\epsilon}, && \text{Hooke law} \\
\operatorname{div}\boldsymbol{t} &= 0 && \text{balance of the linear momentum}
\end{aligned}
$$

- Analytical solution – Fourier transform
- Displacements can diverge at infinity
- Numerical solution – `Matlab` – `Chebfun`

Deflection of a semi-infinite elastic solid

Formulations:

- Displacement formulation with mixed conditions
  - the resampling technique
- Displacement formulation with Dirichlet conditions
  - the resampling technique
  - the moving technique
- Stress formulation with Dirichlet conditions
  - two alternatives of the resampling technique
  - two alternatives of the moving technique
  - overdetermined system solved by the least-squares method

Displacement formulations

- Equations for $\boldsymbol{u} = (u, v, 0)$

$$\mu \Delta \boldsymbol{u} + (\lambda + \mu)\, \mathrm{grad}\, \mathrm{div}\, \boldsymbol{u} = 0$$

- Mixed conditions on the rectangle $[0, c] \times [0, b]$

$$\begin{array}{lll} \sigma_y = -p(x) & \tau_{xy} = 0 & \text{on } y = 0 \\ u = 0 & \frac{\partial v}{\partial x} = 0, & \text{on } x = 0 \\ u = u_{\text{analytic}} & v = v_{\text{analytic}} & \text{on } x = c \\ u = u_{\text{analytic}} & v = v_{\text{analytic}} & \text{on } y = b \end{array}$$

- Non-homogeneous Dirichlet conditions on the rectangle $[0, c] \times [0, b]$

$$\begin{array}{lll} u = u_{\text{analytic}} & v = v_{\text{analytic}} & \text{on } x = c \text{ and } y = 0 \\ u = u_{\text{analytic}} & v = v_{\text{analytic}} & \text{on } y = b \text{ and } x = 0 \end{array}$$

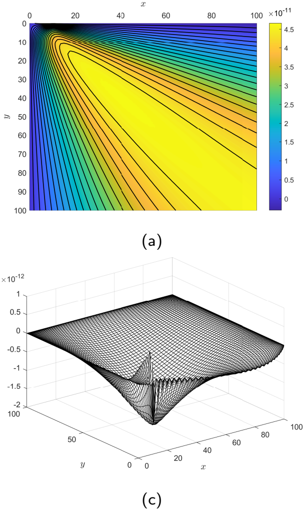

Figure: Displacement field, resampling, mixed conditions: $p = 1\,\mathrm{Pa}$, $\nu = 0.49$, $E = 10\,\mathrm{GPa}$, 80×80 points, $a = 10$, $b = c = 100$: (a) u, (b) v, (c) distribution of error u, (d) distribution of error v

Stress formulations

- Equations for stress field

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0$$

$$\frac{\partial^2}{\partial x^2}\left[\sigma_y - \nu(\sigma_x + \sigma_y)\right] + \frac{\partial^2}{\partial y^2}\left[\sigma_x - \nu(\sigma_x + \sigma_y)\right] = 2\frac{\partial^2 \tau_{xy}}{\partial x \partial y}$$

- non-homogeneous Dirichlet conditions on the rectangle $[-c, c] \times [0, b]$

$$\sigma_y = \sigma_{y,\text{analytic}} \qquad \tau_{xy} = \tau_{xy,\text{analytic}} \qquad \text{on } y = 0 \text{ and } y = b$$

$$\sigma_x = \sigma_{x,\text{analytic}} \qquad \tau_{xy} = \tau_{xy,\text{analytic}} \qquad \text{on } x = \pm c$$

Stress formulations

- Resampling technique

$$
\begin{aligned}
\boldsymbol{R}_1 &= \boldsymbol{I}_{N_y \times N_y} \otimes \boldsymbol{P}^{N_x,-2} \\
\boldsymbol{R}_2 &= \boldsymbol{P}^{N_y,-2} \otimes \boldsymbol{I}_{N_x \times N_x} \\
\boldsymbol{R}_3 &= (\boldsymbol{P}^{N_y,-2} \otimes \boldsymbol{I}_{N_x-2 \times N_x-2})(\boldsymbol{I}_{N_y \times N_y} \otimes \boldsymbol{P}^{N_x,-2})
\end{aligned}
$$

$$
\boldsymbol{P}_A = \begin{bmatrix} \boldsymbol{R}_2 & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{R}_3 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{R}_1 \end{bmatrix} \qquad \boldsymbol{P}_B = \begin{bmatrix} \boldsymbol{R}_3 & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{R}_1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{R}_2 \end{bmatrix}
$$

- Moving technique

$$
\boldsymbol{E}_A = \begin{bmatrix} \boldsymbol{E}_2 \\ \boldsymbol{E}_3 \\ \boldsymbol{E}_1 \end{bmatrix} \qquad \boldsymbol{E}_B = \begin{bmatrix} \boldsymbol{E}_3 \\ \boldsymbol{E}_1 \\ \boldsymbol{E}_2 \end{bmatrix}
$$

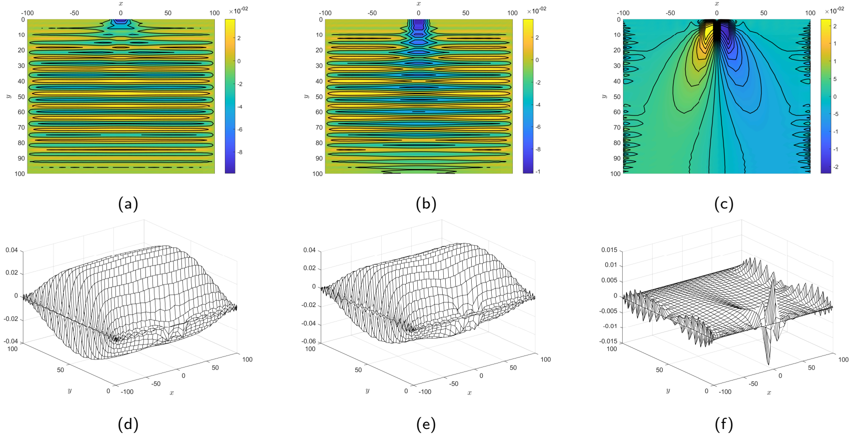

Figure: Stress field, resampling $\boldsymbol{P}_A$, Dirichlet conditions: $p = 0.1\,\mathrm{Pa}$, $\nu = 0.49$, $E = 10\,\mathrm{GPa}$, 40×40 points, $a = 10$, $b = c = 100$: (a) σ_x, (b) σ_y, (c) τ_{xy}, (d) distribution of error σ_x, (e) distribution of error σ_y, (f) distribution of error τ_{xy}

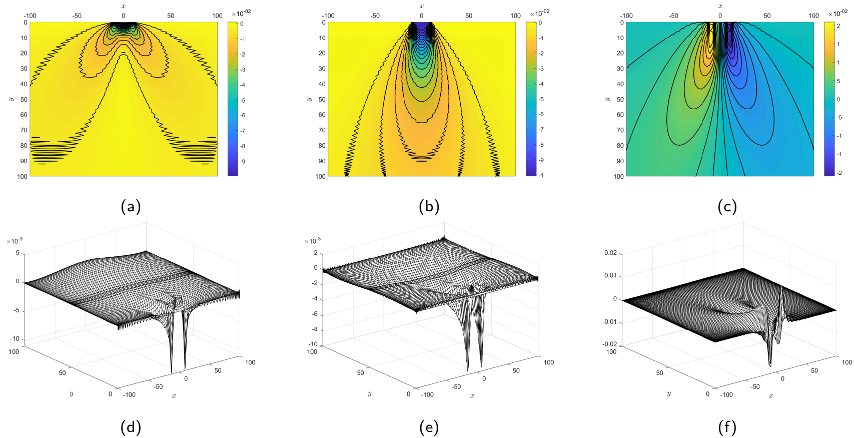

Figure: Stress field, resampling $\boldsymbol{P}_A$, `smoothdata`, Dirichlet conditions: $p = 0.1\,\mathrm{Pa}$, $\nu = 0.49$, $E = 10\,\mathrm{GPa}$, 100×100 points, $a = 10$, $b = c = 100$: (a) σ_x, (b) σ_y, (c) τ_{xy}, (d) distribution of error σ_x, (e) distribution of error σ_y, (f) distribution of error τ_{xy}

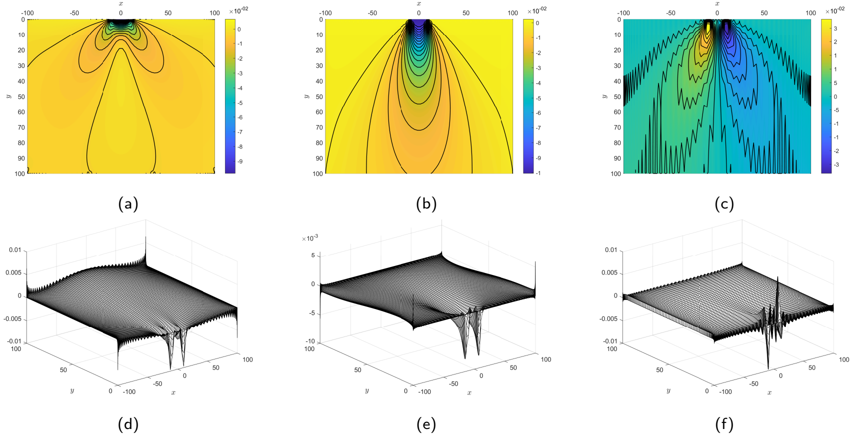

Figure: Stress field, moving $\boldsymbol{E}_A$, Dirichlet conditions: $p = 0.1\,\mathrm{Pa}$, $\nu = 0.49$, $E = 10\,\mathrm{GPa}$, 100×100 points, $a = 10$, $b = c = 100$: (a) σ_x, (b) σ_y, (c) τ_{xy}, (d) distribution of error σ_x, (e) distribution of error σ_y, (f) distribution of error τ_{xy}

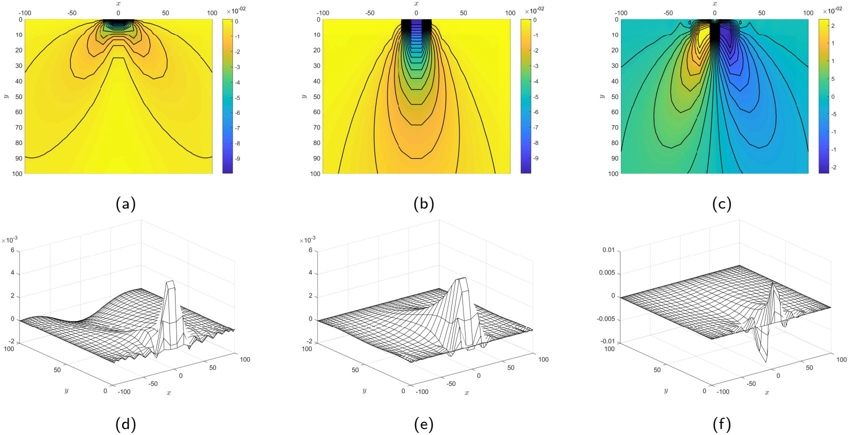

Figure: Stress field, least-squares method, Dirichlet conditions: $p = 0.1\,\mathrm{Pa}$, $\nu = 0.49$, $E = 10\,\mathrm{GPa}$, 30×30 points, $a = 10$, $b = c = 100$: (a) σ_x, (b) σ_y, (c) τ_{xy}, (d) distribution of error σ_x, (e) distribution of error σ_y, (f) distribution of error τ_{xy}

Conclusion

- Analytical solution – the displacements can diverge at infinity
- More accurate results for the displacement formulations than for the stress formulations (unphysical oscillations)
- The oscillations can be caused by the presence of discontinuities on the boundary (a variant of Gibbs phenomenon)
- Numerical experiments of the resampling technique for the Poisson equation with discontinuities on a boundary do not contain oscillations
- Different formulations lead to different numerical solutions with different features such as absolute, relative errors and oscillations.

Conclusion

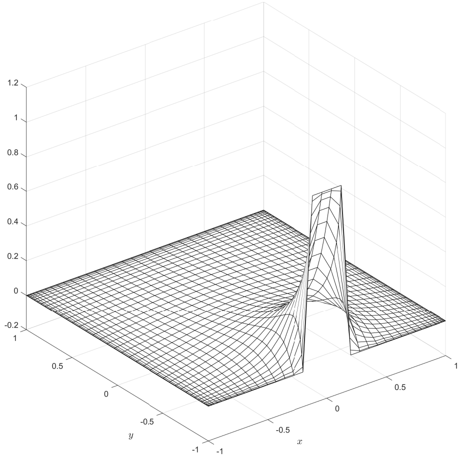

Figure: Solution of the Poisson equation on $[-1, 1] \times [-1, 1]$, resampling, 40×40 points