MA3B8 Complex Analysis

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Abstract

Personal lecture notes for the MA3B8 course on Complex Analysis at the University of Warwick. These notes are based on the lectures given by Dr. Peter Topping in the academic year 2023-24. The notes are written in a concise manner and may contain errors; please PM me on discord `@.lads.` if you find any.

1 Review of basic complex analysis 1

1.1 Complex differentiability

Definition 1.1 Suppose $\Omega \subseteq \mathbb{C}$ is open. Then a function $f : \Omega \to \mathbb{C}$ is complex differentiable at $z \in \Omega$ if the limit

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \tag{1.1.1}$$

exists. We denote the the limit by $f'(z)$ and call it the derivative of f at z.

Notation We will sometimes view Ω as a subset of $\mathbb{R}^2$ instead of $\mathbb{C}$, i.e. we consider $(x, y) \in \mathbb{R}^2 : x + \mathrm{i}y \in \Omega$. We may also decompose f into real and imaginary parts:

$$f(x + \mathrm{i}y) = f(x, y) = u(x, y) + \mathrm{i}v(x, y)$$

For $u, v : \Omega \to \mathbb{R}$.

If the corresponding function $F : \Omega \to \mathbb{R}^2$ defined by

$$F(x, y) := \big(u(x, y), v(x, y)\big)$$

is differentiable in the sense of multivariable calculus, we call F real differentiable. Complex differentiability is a stronger condition than real differentiability. In fact, it is equivalent to real differentiability with the extra condition that

$$f'(z) = \frac{\partial f}{\partial x} = -\mathrm{i}\frac{\partial f}{\partial y} \tag{1.1.2}$$

which are the Cauchy-Riemann equations.

Definition We define the ***Wirtinger derivatives*** of f

$$f_{\bar{z}} := \frac{\partial f}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial f}{\partial x} + \mathrm{i}\frac{\partial f}{\partial y}\right) \tag{1.1.3}$$

and

$$f_z := \frac{\partial f}{\partial z} := \frac{1}{2}\left(\frac{\partial f}{\partial x} - \mathrm{i}\frac{\partial f}{\partial y}\right) \tag{1.1.4}$$

which make sense at any point where f is real differentiable.

Thus the C.R. equations can be written as

$$f_{\bar{z}} = 0. \tag{1.1.5}$$

If f is complex differentiable we find that $f_z = \frac{1}{2}\left(\frac{\partial f}{\partial x} - \mathrm{i}\frac{\partial f}{\partial y}\right) = \frac{\partial f}{\partial x} = -\mathrm{i}\frac{\partial f}{\partial y} = f'(z)$, such that

$$f_{\bar{z}} = 0 \quad \text{and} \quad f'(z) = f_z. \tag{1.1.6}$$

Remark 1.1 Let's rewrite the C.R. equations as

$$\mathrm{i}\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y}. \tag{1.1.7}$$

By definition $\frac{\partial f}{\partial x}$ is the velocity vector of the path $x \mapsto f(x + \mathrm{i}y)$ and $\frac{\partial f}{\partial y}$ is the velocity vector of the path $y \mapsto f(x + \mathrm{i}y)$. They are related by a factor of i which geometrically corresponds to an anti-clockwise rotation by 90 degrees.

Remark 1.2 At a point z where f is complex differentiable, the derivative of F is a linear map from $\mathbb{R}^2 \to \mathbb{R}^2$ that is a rotation and dilation. In complex notation, this linear map is given by $w \to f'(z)w$. It preserves orthogonality and is invertible, provided $f'(z) \neq 0$.

Definition 1.2 Suppose $\Omega \subseteq \mathbb{C}$ is open. Then a function $f : \Omega \to \mathbb{C}$ is ***holomorphic*** if it is complex differentiable at every point $z \in \Omega$. In the case that $\Omega = \mathbb{C}$, we say that f is ***entire***.

Being holomorphic is a much stronger condition than being merely continuously differentiable from $\mathbb{R}^2$ to $\mathbb{R}^2$. As an example, there are many continuously differentiable functions from $\mathbb{R}^2$ to $\mathbb{R}^2$ that are 'real valued' in that they map into $\mathbb{R} \times \{0\}$, but the only ones of these that come from holomorphic functions are the constant functions.

Generally, a function or map preserving angles is called conformal. We will settle on a precise definition of 'conformal' that is adapted to this course in Section 2.9.

1.2 Product and chain rules

By the product rule we have

$$(f \cdot g)_{\bar{z}} = f \cdot g_{\bar{z}} + f_{\bar{z}} \cdot g \quad \text{and} \quad (f \cdot g)_z = f \cdot g_z + f_z \cdot g. \tag{1.2.1}$$

If f and g are both complex differentiable so is $(f \cdot g)$, and

$$(f \cdot g)' = f \cdot g' + f' \cdot g. \tag{1.2.2}$$

Lemma 1.1 Suppose $f : \Omega \to \mathbb{C}$, for $\Omega \subseteq \mathbb{C}$ open, and $\gamma : I \to \Omega$, for $I \subseteq \mathbb{R}$ some open interval. If γ is differentiable at $t \in I$ and f is real differentiable at $\gamma(t)$, then $f \circ \gamma : I \to \mathbb{C}$ is differentiable at t and

$$(f \circ \gamma)'(t) = f_z\big(\gamma(t)\big)\gamma'(t) + f_{\bar{z}}\big(\gamma(t)\big)\overline{\gamma'(t)}. \tag{1.2.3}$$

If f is *complex* differentiable at $\gamma(t)$ then

$$(f \circ \gamma)'(t) = f'\big(\gamma(t)\big)\overline{\gamma'(t)}. \tag{1.2.4}$$

Proof If we write $\gamma(t) = u(t) + iv(t)$, then

$$(f \circ \gamma)'(t) = f_x\big(\gamma(t)\big)u'(t) + f'\big(\gamma(t)\big)v'(t)$$

It can easily be shown that this is equal to (1.2.3). ■

Lemma 1.2 Suppose that $\Omega_1, \Omega_2 \subseteq \mathbb{C}$ are open sets. If $g : \Omega_1 \to \Omega_2$ is real differentiable at $z \in \Omega_1$, and $f : \Omega_2 \to \mathbb{C}$ is complex differentiable at $g(z)$, then $f \circ g$ is real differentiable at z, and we have the two chain rules

$$(f \circ g)_z(z) = f'\big(g(z)\big)g_z(z), \tag{1.2.5}$$

and

$$(f \circ g)_{\bar{z}}(z) = f'\big(g(z)\big)g_{\bar{z}}(z). \tag{1.2.6}$$

If g is also *complex* differentiable at z, then $f \circ g$ is *complex* differentiable at z, and we have the chain rule

$$(f \circ g)'(z) = f'\big(g(z)\big)g'(z). \tag{1.2.7}$$

2 Möbius transformations

2.1 Riemann sphere

We extend the complex plane by adding a point at infinity $\mathbb{C}_\infty \coloneqq \mathbb{C} \cup \{\infty\}$. We can equip this extended complex plane with a topology: a sequence $z_i \in \mathbb{C} \subset \mathbb{C}_\infty$ converges to $\infty \in \mathbb{C}_\infty$ if and only if $z_i \to \infty$ in the usual sense.

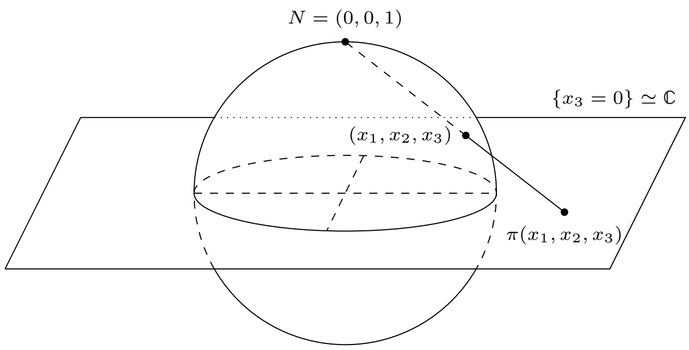
We would like to give $\mathbb{C}_\infty$ enough geometric structure so that it makes sense to talk about a function being complex differentiable at ∞ and at points that map to ∞. To do this we consider *stereographic projection*, which will turn $\mathbb{C}_\infty$ into the *Riemann sphere*.

2.2 Stereographic projection

We want to find a correspondence between $\mathbb{C}$ and $S^2 \setminus N$, where

$$S^2 \coloneqq \big\{(x_1, x_2, x_3) : {x_1}^2 + {x_2}^2 + {x_3}^2 = 1\big\} \subset \mathbb{R}^3$$

and $N := (0,0,1)$ is the 'north pole'. We do this by mapping each point (x_1, x_2, x_3) on the unit sphere, other than N, to the unique point on the plane that is on the line through N and (x_1, x_2, x_3), and mapping infinity to the north pole.



Definition 2.1 We define ***stereographic projection*** $\pi : S^2 \setminus N \to \mathbb{C}$ by

$$\pi(x_1, x_2, x_3) = \frac{x_1 + \mathrm{i}x_2}{1 - x_3}. \tag{2.2.1}$$

It extends to a bijection $\pi : S^2 \mapsto \mathbb{C}_\infty$ by sending $N \in S^2$ to $\infty \in \mathbb{C}_\infty$. The inverse of π can be computed to be the map $\pi^{-1} : \mathbb{C} \to S^2 \setminus N$ given by

$$\pi^{-1}(x + \mathrm{i}y) = \left(\frac{2x}{1 + |z|^2}, \frac{2y}{1 + |z|^2}, \frac{|z|^2 - 1}{1 + |z|^2} \right) \tag{2.2.2}$$

Warning Remember, $z = x + \mathrm{i}y$.

The bijection $\pi : S^2 \mapsto \mathbb{C}_\infty$ can be used to transfer the standard topology on S^2 to a topology on $\mathbb{C}_\infty$; this coincides with the topology we alluded to in Section 2.1. With this topology in hand we can say, for example, that the function $z \mapsto 1/z$ is a homeomorphism $\mathbb{C}_\infty \to \mathbb{C}_\infty$, without worrying about any singularity at 0 (now mapped to the point ∞ in the target $\mathbb{C}_\infty$) and without worrying about the function omitting 0 in the range (now ∞ in the domain is mapped to 0). In fact, the function $z \mapsto 1/z$ corresponds to a rotation of the sphere by 180°.

Remark 2.1 π maps circles in S^2 to circles/lines in $\mathbb{C}$. More precisely circles that pass through N are mapped to circles, otherwise they are mapped to lines. The inverse correspondence holds for π^{-1}.

Definition 2.2 A circle in $\mathbb{C}$ is any subset of $\mathbb{C}$ that arises as the image under π of the intersection of S^2 with any plane that intersects the open unit ball in R^3. In other words, it is either a circle in $\mathbb{C}$ or a line in $\mathbb{C}$ together with the point $\infty \in \mathbb{C}_\infty$.

2.3 Möbius transformations

Definition 2.3 ***Möbius transformations*** are homeomorphisms $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ of the form

$$f(z) = \frac{az + b}{cz + d},$$

for $a, b, c, d \in C$ such that $ad - bc \neq 0$. This restriction is to ensure that f does not map to a single point and that they are invertible.

Lemma 2.1 The Möbius transformation in definition 2.3 is invertible, and the inverse is also a mobius transformation given by

$$f^{-1}(z) = \frac{dz - b}{-cz + a}. \tag{2.3.1}$$

Lemma 2.2 Let f_1, f_2 be Möbius transformations given by

$$f_i = \frac{a_i z + b_i}{c_i z + d_i} \qquad i = 1, 2.$$

Then $f_1 \circ f_2$ is again a Möbius transformation and is given by

$$f_1 \circ f_2(z) = \frac{(a_1 a_2 + b_1 c_2)z + (a_1 b_2 + b_1 d_2)}{(c_1 a_2 + d_1 c_2)z + (c_1 b_2 + d_1 d_2)}. \tag{2.3.2}$$

2.4 $PSL(2, \mathbb{C})$

Consider the map from $GL(2, \mathbb{C})$, i.e. the group of invertible 2×2 matrices with complex entries, to the set of Möbius transformations, given by

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \longmapsto \qquad f_M(z) := \frac{az + b}{cz + d}. \tag{2.4.1}$$

Note The assumption $ad - bc \neq 0$ in the definition of Möbius transformation is precisely the condition that the matrix on the left-hand side of (2.4.1) is invertible.

Suppose $M_1, M_2 \in GL(2, \mathbb{C})$ are defined as

$$M_i := \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \qquad \text{where} \qquad i = 1, 2$$

then because the product has determinant $\det(M_1, M_2) = \det(M_1)\det(M_2) \neq 0$, it also lies in $GL(2, \mathbb{C})$ and thus gives rise to a Möbius transformation $f_{M_1 M_2}$. We can calculate

$$M_1 M_2 = \begin{pmatrix} a_1 a_2 + b_1 c_2 & a_1 b_2 + b_1 d_2 \\ c_1 a_2 + d_1 c_2 & c_1 b_2 + d_1 d_2 \end{pmatrix},$$

and so the Möbius transformation $f_{M_1 M_2}$ is exactly $f_1 \circ f_2$, which proves Lemma 2.2.

Notation We may also write the composition of Möbius transformations as

$$f_{M_1} \circ f_{M_2} = f_{M_1 M_2}. \tag{2.4.2}$$

It is now clear from (2.4.2) where the formula for $f^{-1}(z)$ in Lemma 2.2 comes from

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ c & a \end{pmatrix}.$$

The map in (2.4.1) is a group homomorphism but not a group isomorphism because it isn't injective: for any $M \in GL(2, \mathbb{C})$ and $\lambda \in \mathbb{C} \setminus \{0\}$, $\lambda M \in GL(2, \mathbb{C})$ will give the same Möbius transformation under (2.4.1). If we restrict ourselves to $SL(2, \mathbb{C})$, where $\det(M) = 1$, then we still have the issue that $-M \in SL(2, \mathbb{C})$ gives the same Möbius transformation.

Note $\det(\lambda M) = \lambda^2 \det(M)$.

Thus, we can turn this homomorphism into an isomorphism by factoring out the kernel $\{\pm I\}$ and considering the quotient group

$$PSL(2, \mathbb{C}) := SL(2, \mathbb{C})/\{\pm I\},$$

where the P stands for *projective*.

Lemma 2.3 The map from $PSL(2, \mathbb{C})$ to the group of Möbius transformations induced by (2.4.1) is a group isomorphism by the first isomorphism theorem.

2.5 Decomposition of Möbius transformations

Definition 2.4 (Elementary transformations)

(i) *Translations*: $f(z) = z + b$ where $b \in \mathbb{C}$. These maps shift a point z by the complex number b.

(ii) *Rotations*: $f(z) = e^{i\theta} z$ where $\theta \in \mathbb{R}$. These maps correspond to an anti-clockwise rotation about the origin by an angle θ.

(iii) *Dilations*: $f(z) = \lambda z$ where $\lambda > 0$. These maps act as an expansion when $\lambda > 1$ and contraction when $\lambda < 1$.

(iv) *Complex inversion*: $f(z) = {}^1/_z$. The effect of this map can be understood as a map from S^2 to S^2 using stereographic projection. In that viewpoint, it is a rotation by $180°$ about the x_1-axis in $\mathbb{R}^3$. In other words, the map $(x_1, x_2, x_3) \mapsto (x_1, -x_2, -x_3)$.

To show this, we first note that $z \mapsto {}^1/_z$ in coordinates is $x + iy \mapsto \frac{x - iy}{x^2+y^2}$ and that for $(x_1, x_2, x_2) \in S^2$, $x_1^2 + x_2^2 = 1 - x_3^2 = (1 - x_3)(1 + x_3)$. Then

$$(x_1, x_2, x_3) \quad \overset{\pi}{\mapsto} \quad \frac{x_1 + ix_2}{1 - x_3} \quad \xmapsto{z \mapsto {}^1/_z} \quad (1 - x_3)\frac{x_1 - ix_2}{x_1^2 + x_2^2} \quad \xmapsto{\pi^{-1}} \quad (x_1, -x_2, -x_3).$$

Lemma 2.4 Every Möbius transformation can be expressed as the composition of elementary Möbius transformations.

Proof For $c = 0$ the Möbius transformation would be $z \mapsto {}^{az+b}/_d$, which can be decomposed into elementary transformations

$$z \ \mapsto\ az \ \mapsto\ az + b \ \mapsto\ \frac{az + b}{d}.$$

For $c \neq 0$ we can rewrite

$$\frac{az + b}{cz + d} \quad \text{as} \quad \frac{a}{c} + \frac{b - \frac{ad}{c}}{cz + d}$$

which can be decomposed as follows

$$z \ \mapsto\ cz \ \mapsto\ cz + d \ \mapsto\ \frac{1}{cz + d} \ \mapsto\ \frac{b - \frac{ad}{c}}{cz + d} \ \mapsto\ \frac{a}{c} + \frac{b - \frac{ad}{c}}{cz + d}. \tag{2.5.1}$$

■

Theorem 2.1 The image of every circle in $\mathbb{C}_\infty$ under any Möbius transformation is also a circle in $\mathbb{C}_\infty$.

Proof Using Lemma 2.4 we can consider the effect of different elementary transformations on circles in $\mathbb{C}_\infty$. Circles are clearly preserved under translations, rotations and dilations. The property for complex inversion follows from its interpretation as a $180°$ rotation, together with the preservation of circles/lines by stereographic projection given in Remark 2.1. ■

2.6 Three points to determine a Möbius transformation

Theorem 2.2 Given three distinct points $z_1, z_2, z_3 \in \mathbb{C}_\infty$ and another three distinct points $w_1, w_2, w_3 \in \mathbb{C}_\infty$, there exists a unique Möbius transformation f such that $f(z_i) = w_i$ for $i = 1, 2, 3$.

The proof of Theorem 2.2 is given after the following sub-results.

Lemma 2.5 Every Möbius transformation $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ other than the identity $f(z) = z$ has at least one, but at most two, fixed points. In particular, if f is a Möbius transformation and $z_1, z_2, z_3 \in \mathbb{C}_\infty$ are distinct points such that $f(zi) = zi$, then f is the identity.

Note For a Möbius transformation such as $f(z) = z + 1$, ∞ is a fixed point.

Proof As usual, we write $f(z) = \frac{az+c}{cz+d}$. We assume f is not the identity, where the identity corresponds to the case $a = d \neq 0$ and $b = c = 0$.

First we consider the case $c = 0$, then $f = \frac{a}{d}z + \frac{b}{d}$, which has a fixed point at ∞ and a second fixed point at $z = \frac{b}{d-a}$ if $a \neq d$.

If $c \neq 0$

$$\begin{aligned} f(z) = z \quad &\Rightarrow \quad (az+b) = (cz+d)z \\ &\Rightarrow \quad cz^2 + (d-a)z - b = 0 \end{aligned}$$

which provides one or two distinct solutions. ■

Proposition 2.1 Given three distinct points $z_1, z_2, z_3 \in \mathbb{C}_\infty$, there exists a Möbius transformation f that maps z_1, z_2, z_3 to $1, 0, \infty$ respectively. In the case that $z_i \neq \infty$ for $i = 1, 2, 3$, then it is

$$f(z) \coloneqq \frac{(z-z_2)(z_1-z_3)}{(z-z_3)(z_1-z_2)}. \tag{2.6.1}$$

In the case that $z_1 = \infty$, we set

$$f(z) = \frac{z-z_2}{z-z_3}, \tag{2.6.2}$$

if $z_2 = \infty$, we set

$$f(z) = \frac{z_1-z_3}{z-z_3}, \tag{2.6.3}$$

if $z_3 = \infty$, we set

$$f(z) = \frac{z-z_2}{z_1-z_2}. \tag{2.6.4}$$

Proof By inspection, we find that these Möbius transformations map z_i to the required image points. ■

Proof of Theorem 2.2

Existence: Let f_1 be the function from Proposition 2.1 that sends z_1, z_2, z_3 to $1, 0, \infty$ respectively. Let f_2 be the function from Proposition 2.1 that sends w_1, w_2, w_3 to $1, 0, \infty$ respectively. The Möbius transformation f we seek is simply $f_2^{-1} \circ f_1$.

Uniqueness: Suppose that we have two Möbius transformations f and g, both of which send the points z_i to w_i respectively. Then $g^{-1} \circ f$ is a Möbius transformation that has all three distinct points z_i as fixed points. Lemma 2.5 then tells us that $g^{-1} \circ f$ is the identity, i.e. $f \equiv g$. ■

2.7 Examples and special classes of Möbius transformations

Definition (Open unit disc) $D \coloneqq \{z \in \mathbb{C} : |z| < 1\}$.

Definition (Upper half-space) $H_+ \coloneqq \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$.

Example 2.1 (The Cayley transform) A Möbius transformation that gives a bijection from H_+ to D given by

$$f(z) = \frac{z - \mathrm{i}}{z + \mathrm{i}}.$$

Proof $f(z) \in D \;\Leftrightarrow\; f(z) < 1 \;\Leftrightarrow\; |z - \mathrm{i}| < |z + \mathrm{i}| \;\Leftrightarrow\; z \in H_+$ ■

Example 2.2 (Möbius transformation that gives a bijection from D to H_+) Since Möbius transformations are homeomorphisms from $\mathbb{C}_\infty$ to itself, f will map the boundary of D, i.e. the unit circle, to the boundary of H_+, i.e. the real axis plus ∞. By Theorem 2.2 we can pick any three points on ∂D, e.g. $1, -\mathrm{i}, \mathrm{i}$ and map them to $1, 0, \infty$ respectively. In this case the map would be given by (2.6.1) as

$$f(z) = \frac{z + \mathrm{i}}{\mathrm{i}z + 1}.$$

Proof By Theorem 2.1, f will map ∂D to $\mathbb{R} \cup \{\infty\}$. f is a homeomorphism, therefore it must send the disc D either to the upper half-plane H_+, or the lower half-plane H_-. Since it sends 0 to i, it must be the former case, as required. ■

Example 2.3 (Möbius transformations that give bijections from D to D) Consider

$$f(z) = \frac{z-w}{\overline{w}z-1} \qquad \text{for } w \in \mathbb{C}, \text{ with } |w| < 1. \tag{2.7.1}$$

Proof Observe the identity

$$|z-w|^2 = |\overline{w}z-1|^2 - (1-|z|^2)(1-|w|^2)$$

and use it to compute

$$|f(z)|^2 = \frac{|z-w|^2}{|\overline{w}z-1|^2} = 1 - \frac{(1-|z|^2)(1-|w|^2)}{|\overline{w}z-1|^2}.$$

Because we are assuming that $1-|w|^2 > 0$, we see that $|f(z)| < 1$ if and only if $|z| < 1$, and $|f(z)| = 1$ if and only if $|z| = 1$. Surjectivity follows from $f^{-1} = f$. ■

Remark 2.2 We can generalise the class of Möbius transformations from Example 2.3, by composing with a rotation about the origin

$$f(z) = \mathrm{e}^{\mathrm{i}\theta}\frac{z-w}{\overline{w}z-1} \qquad \text{with } w \in D \text{ and } \theta \in (-\pi, \pi]. \tag{2.7.2}$$

In fact, every holomorphic map $D \mapsto D$ is of the form (2.7.2).

Example 2.4 (Möbius transformations that give bijections from H_+ to H_+) A Möbius transformation $g : D \to D$ can be converted into a Möbius transformation $h := H_+ \to H_+$, defined as $h := f^{-1} \circ g \circ f$, where $f : D \to H_+$ is the Cayley transform from Example 2.1.

Consider the subgroup $PSL(2, \mathbb{R}) := SL(2, \mathbb{R})/\{\pm I\}$, which is the restriction of $PSL(2, \mathbb{C})$, defined in Section 2.4, to real matrices. We claim that this subgroup of Möbius transformations map $H_+ \to H_+$ bijectively.

Proof Bearing in mind that $a, b, c, d \in \mathbb{R}$, we rewrite

$$f(z) = \frac{az+b}{cz+d} = \frac{(az+b)(c\overline{z}+d)}{(cz+d)(c\overline{z}+d)} = \frac{ac|z|^2 + adz + bc\overline{z} + bd}{|cz+d|^2}.$$

Therefore

$$\operatorname{Im}\big(f(z)\big) = \frac{\operatorname{Im}(adz + bc\overline{z})}{|cz+d|^2} = \frac{(ad-bc)\operatorname{Im}(z)}{|cz+d|^2} = \frac{\operatorname{Im}(z)}{|cz+d|^2},$$

which implies that $\operatorname{Im}(z) > 0 \Leftrightarrow \operatorname{Im}\big(f(z)\big) > 0$ or equivalently $z \in H_+ \Leftrightarrow f(z) \in H_+$. To show that f is bijective, one can verify that f^{-1} is also an element of $PSL(2, \mathbb{R})$, and thus mapping $H_+ \to H_+$.

In fact, every Möbius transformation that maps $H_+ \to H_+$ bijectively is of this form. ■

2.8 Conformal maps

What is the right notion of equivalence for domains in $\mathbb{C}$?

Note By domain here we mean a *nonempty, open and connected subset.*

Definition 2.5 (Conformal map) Given an open set $\Omega \subseteq \mathbb{C}$, a function $f : \Omega \to \mathbb{C}$ is said to be a ***conformal map***, if f is holomorphic and $f'(z) \mapsto 0$ for all $z \in \Omega$.

Note f is not necessarily injective. Consider the function $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ defined as $f(z) = z^2$.

Definition 2.6 (Biholomorphic) A function $f : \Omega_1 \to \Omega_2$ where $\Omega_1, \Omega_2 \in \mathbb{C}$, is said to be ***biholomorphic*** if it is a bijection such that both f and f^{-1} are conformal maps.

In fact, any bijective holomorphic function $f : \Omega_1 \to \Omega_2$ is automatically biholomorphic, as we will see later.

Definition 2.7 (Conformally equivalent) Two domains $\Omega_1, \Omega_2 \in \mathbb{C}$ are said to be ***conformally equivalent*** if there exists a biholomorphic function $\varphi : \Omega_1 \to \Omega_2$.

If Ω_1 and Ω_2 are conformally equivalent via $\varphi : \Omega_1 \to \Omega_2$, and Ω_2 and Ω_3 are conformally equivalent via $\psi : \Omega_2 \to \Omega_3$, then Ω_1 and Ω_3 are conformally equivalent via $\psi \circ \varphi : \Omega_1 \to \Omega_3$.

Which domains are conformally equivalent to the unit disc D.

Example 2.5 We claim that the upper right quarter of the complex plane

$$Q := \{z \in \mathbb{C} : \mathrm{Re}(z) > 0 \text{ and } \mathrm{Im}(z) > 0\}$$

is conformally equivalent to D.

Proof Q is conformally equivalent to the upper half plane H_+ via the map $z \mapsto z^2$, which is biholomorphic from Q to the upper half plane. The upper half plane is then conformally equivalent to D via the Cayley transform of Example 2.1. ■

Example 2.6 We claim that the upper half disc

$$D_+ := \{z \in \mathbb{C} : |z| < 1 \text{ and } \mathrm{Im}(z) > 0\}$$

is conformally equivalent to the whole disc D.

Proof

Warning We cannot use the map $z \mapsto z^2$ as this show that D_+ is conformally equivalent to $D \setminus [0, 1)$

Instead we show that D_+ is conformally equivalent to Q. See the lecture notes... ■

Example 2.7 (Domains not conformally equivalent to D)

1. $\Omega = \mathbb{C}$ If we could find a biholomorphic function from $\mathbb{C}$ to D, then this would be a bounded holomorphic function on $\mathbb{C}$, and therefore constant by Liouville's theorem, i.e. not surjective.
2. $\Omega = \{z \in \mathbb{C} : a < |z| < b\}$, where $0 < a < b < \infty$. This domain is not even homeomorphic to D

Definition 2.8 (Homotopic) Two paths $\gamma_1, \gamma_2 : [a, b] \to \Omega$ are said to be ***homotopic*** if there exists a continuous map $h : [0, 1] \times [a, b] \to \Omega$ such that $h(0, t) = \gamma_1(t)$ and $h(1, t) = \gamma_2(t)$ for all $t \in [a, b]$, i.e. the paths interpolate between γ_1 and γ_2. They must also have the same endpoints.

Definition 2.9 (Closed path) A path $\gamma : [a, b] \to \Omega$ is ***closed*** if $\gamma(a) = \gamma(b)$.

Definition 2.10 (Simply connected) An open set $\Omega \subseteq \mathbb{C}$ is said to be ***simply connected*** if it is connected and every closed continuous path $\gamma : [a, b] \to \Omega$ is homotopic to the constant path $\tilde{\gamma} : [a, b] \to \Omega$ defined by $\tilde{\gamma}(t) = \gamma(a) = \gamma(b)$.

3 Review of basic complex analysis 2

3.1 Power series

We consider power series of the form $\sum_{n=0}^{\infty} a_n z^n$.

Theorem 3.1 Given a complex-valued sequence (a_n), define the so-called ***radius of convergence*** by

$$R := \frac{1}{\limsup |a_n|^{1/n}} \in [0, \infty].$$

Then the power series $\sum_{n=0}^{\infty} a_n z^n$ converges pointwise for all $|z| < R$ and diverges for all $|z| > R$.

Theorem 3.2 If the radius of convergence of a power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is $R \in (0, \infty]$, then within the open ball $B_R := \{z \in \mathbb{C} : |z| < R\}$, f is holomorphic and

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$$

with radius of convergence R.

Corollary 3.1 If the radius of convergence of a power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is $R \in (0, \infty]$, then within the open ball $B_R := \{z \in \mathbb{C} : |z| < R\}$, f is infinitely differentiable and

$$f^{(n)}(0) = a_n n!. \tag{3.1.1}$$

Theorem 3.3 If the radius of convergence of a power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is $R \in (0, \infty]$, then for all $r \in (0, R)$, the convergence

$$\sum_{n=0}^{k} a_n z^n \to \sum_{n=0}^{\infty} a_n z^n$$

is uniform within B_r as $k \to \infty$.

3.2 Definitions of $\exp(z), \sin(z), \cos(z), \sinh(z)$ and $\cosh(z)$

Definition 3.1 We define $\exp : \mathbb{C} \to \mathbb{C}$ or $z \mapsto \mathrm{e}^z$ by

$$\mathrm{e}^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

The radius of convergence is $R = \infty$.

If we differentiate term by term we obtain the property $(\mathrm{e}^z)' = \mathrm{e}^z$.

Lemma 3.1 For all $a, b \in \mathbb{C}$, we have

$$\mathrm{e}^{a+b} = \mathrm{e}^a \mathrm{e}^b. \tag{3.2.1}$$

Proof Consider the function $f(z) := \mathrm{e}^{a+b-z}\mathrm{e}^z$. The derivative is given by

$$\begin{aligned} f'(z) &= \mathrm{e}^{a+b-z}(\mathrm{e}^z)' + (\mathrm{e}^{a+b-z})'\mathrm{e}^z \\ &= \mathrm{e}^{a+b-z}\mathrm{e}^z - \mathrm{e}^{a+b-z}\mathrm{e}^z = 0. \end{aligned}$$

Therefore, f is constant by Exercise 1.3 and $\mathrm{e}^{a+b} = f(0) = f(b) = \mathrm{e}^a \mathrm{e}^b$. ∎

Definition 3.2 We define the entire functions

$$\sinh(z) := \frac{\mathrm{e}^z - \mathrm{e}^{-z}}{2} \qquad \text{and} \qquad \cosh(z) := \frac{\mathrm{e}^z + \mathrm{e}^{-z}}{2},$$

as well as

$$\sin(z) := \frac{\mathrm{e}^{\mathrm{i}z} - \mathrm{e}^{-\mathrm{i}z}}{2\mathrm{i}} \qquad \text{and} \qquad \cos(z) := \frac{\mathrm{e}^{\mathrm{i}z} + \mathrm{e}^{-\mathrm{i}z}}{2}.$$

3.3 Argument and logarithm

If we write $z = r\mathrm{e}^{\mathrm{i}\theta}$, then $\arg(z) = \theta$ which takes values in $\mathbb{R}/2\pi\mathbb{Z}$. Generally we take a *branch cut* by removing some ray $\{r\mathrm{e}^{\mathrm{i}\theta} : r > 0\}$. It can be shown to have the property $\arg(zw) = \arg(z) + \arg(w)$.

Definition (Complex logarithm)

$$\log(z) := \log|z| + \mathrm{i}\arg(z),$$

where $z \neq 0$. Branch cut $\{x \leqslant 0\} \subset \mathbb{C}$.

It can be shown to have the properties $\mathrm{e}^{\log z} = z = \log(\mathrm{e}^z)$ and $\log(zw) = \log(z) + \log(w)$.

3.4 Complex integration

Given $f[a,b] \in \mathbb{C}$, where $[a,b] \in \mathbb{R}$, we define the integral

$$\int_a^b f(t)dt := \int_a^b \mathrm{Re}[f(t)]dt + \mathrm{i}\int_a^b \mathrm{Im}[f(t)]dt.$$

Property

$$\left|\int_a^b f(t)dt\right| \leqslant \int_a^b |f(t)|\, dt$$

Definition 3.3 We say that $\gamma : [a,b] \to \mathbb{C}$ is a $\mathcal{C}^1$ curve if it is continuous over $[a,b]$, the derivative γ' exists on (a,b) and extends to a continuous function $\gamma' : [a,b] \to \mathbb{C}$.

Definition 3.4 (Contour integral) Given a continuous function $f : \Omega \to \mathbb{C}$, and a $\mathcal{C}^1$ curve $\gamma : [a,b] \to \mathbb{C}$, we define

$$\int_\gamma f(z)dz = \int_a^b f\big(\gamma(t)\big)\gamma'(t)dt.$$

Property (Invariance under reparametrisation) For any $\mathcal{C}^1$ curve $\tilde{\gamma} : [\tilde{a}, \tilde{b}] \to \Omega$ where $\tilde{\gamma} = \gamma(\phi(t))$ for some $\mathcal{C}^1$ bijection $\phi : [\tilde{a}, \tilde{b}] \to [a,b]$ with $\phi' : [\tilde{a}, \tilde{b}] \to \mathbb{R}$ positive, then

$$\int_\gamma f(z)dz = \int_{\tilde{\gamma}} f(z)dz.$$

If instead $\phi' < 0$ we have

$$\int_\gamma f(z)dz = -\int_{\tilde{\gamma}} f(z)dz.$$

An important fact is if $|f(z)| \leqslant M$ then

$$\left|\int_\gamma f(z)dz\right| \leqslant M\int_a^b |\gamma'(t)|dz = ML(\gamma), \tag{3.4.1}$$

where $L(\gamma) := \int_a^b |\gamma'(t)|\, dt$ is the length of the image of γ.

Definition 3.5 We say that $\gamma : [a,b] \to \mathbb{C}$ is a piecewise $\mathcal{C}^1$ curve if it is continuous on $[a,b]$ and there exists finitely many intermediate points $a = c_0 < c_1 < \cdots < c_n = b$ such that the restriction of γ to each interval $[c_i, c_{i+1}]$ is a $\mathcal{C}^1$ curve.

Example 3.1 For a triangle T with boundary ∂T.

Note Convention is to parametrise in the anti-clockwise direction.

Notation We write such an integral as

$$\int_{\partial T} f(z)dz.$$

For the integral around an open ball around a point a, that is $B_r(a) := \{z \in \mathbb{C} : |z-a| < r\}$, we write

$$\int_{\partial B_r(a)} f(z)dz := \int_\gamma f(z)dz$$

where $\gamma : [0, 2\pi] \to \mathbb{C}$ is defined as $\gamma(\theta) := a + r\mathrm{e}^{\mathrm{i}\theta}$.

3.5 Anti-derivatives, and a baby version of Cauchy's theorem

Lemma 3.2 Suppose $\Omega \subset \mathbb{C}$ is open and that $f : \Omega \to \mathbb{C}$ is continuous and $F : \Omega \to \mathbb{C}$ is holomorphic where $F'(z) = f(z)$. If γ is a piecewise $\mathcal{C}^1$ curve in Ω, then

$$\int_\gamma f(z)dz = 0.$$

Lemma 3.3 Suppose $F : \Omega \to \mathbb{C}$ is holomorphic, with F' continuous. If $\gamma : [a, b] \to \Omega$ is a piecewise $\mathcal{C}^1$ curve, then

$$\int_\gamma F'(z)dz = F\big(\gamma(b)\big) - F\big(\gamma(a)\big).$$

In particular, for a closed curved (i.e. $\gamma(a) = \gamma(b)$) we have $\int_\gamma F'(z)dz = 0$.

Proof

$$\int_\gamma F'(z)dz = \int_a^b F'\big(\gamma(t)\big)\gamma'(t)dt = \int_a^b \frac{d}{dt}F\big(\gamma(t)\big)dt = F\big(\gamma(b)\big) - F\big(\gamma(a)\big).$$

■

Note In the last step have used the assumption that F' is continuous.

Corollary 3.2 Suppose $n \in \mathbb{Z}$ does not equal -1. Then for $\gamma : [a, b] \to \mathbb{C} \setminus \{0\}$ any piecewise $\mathcal{C}^1$ closed curve, we have

$$\int_\gamma z^n dz = 0.$$

Proof Define $F(z) := {}^{z^{n+1}}\!/_{n+1}$, then $F'(z) = z^n$ and the result follows from Lemma 3.2. ■

Example 3.2 For $r > 0$ and $k \in \mathbb{Z}$ let $\gamma : [0, 2\pi] \to \mathbb{C}$ be the closed $\mathcal{C}^1$ curve $\gamma(\theta) = r\mathrm{e}^{\mathrm{i}k\theta}$ that travels anti-clockwise k times around the circle of radius r. Then

$$\int_\gamma \frac{dz}{z} = 2\pi\mathrm{i}k.$$

4 Winding numbers

4.1 Winding numbers of continuous closed paths

Lemma 4.1 (Lifting lemma) Suppose $\gamma : [a, b] \to \mathbb{C} \setminus \{0\}$ is continuous, and fix $\theta_0 \in \mathbb{R}$ such that $\gamma(a) = |\gamma(a)|\, \mathrm{e}^{\mathrm{i}\theta_0}$. Then there exists a continuous function $\theta : [a, b] \to \mathbb{R}$ such that $\theta(a) = \theta_0$ and $\gamma(t) = |\gamma(t)|\, \mathrm{e}^{\mathrm{i}\theta(t)}$ for all $t \in [a, b]$.

Definition 4.1 Suppose $\gamma : [a, b] \to \mathbb{C} \setminus \{0\}$ is continuous, and let $\theta : [a, b] \to \mathbb{R}$ be a function arising from Lemma 4.1. We define

$$\measuredangle(\gamma) := \theta(b) - \theta(a).$$

Note The function θ was only defined up to a constant multiple of 2π that was determined by θ_0. However, when we subtract θ_a from θ_b this unknown multiple of 2π will disappear, making $\measuredangle(\gamma)$ well-defined.

Definition 4.2 (Winding number) Suppose $\gamma : [a, b] \to \mathbb{C} \setminus \{0\}$ is a *closed* continuous path. Then we define the ***winding number*** (or index) of γ around 0 to be

$$I(\gamma, 0) := \frac{1}{2\pi}\measuredangle(\gamma) \in \mathbb{Z}.$$

More generally, if $w \in \mathbb{C}$ and $\gamma : [a, b] \to \mathbb{C} \setminus \{w\}$ then

$$I(\gamma, w) \coloneqq I(\gamma_w, 0),$$

where $\gamma_w : [a, b] \to \mathbb{C} \setminus \{0\}$ is the path γ translated to send w to the origin, i.e. $\gamma_w(t) \coloneqq \gamma(t) - w$.

Example 4.1 For $n \in \mathbb{Z}$, consider the curve $\gamma : [0, 2\pi] \to \mathbb{C}$ defined by $\gamma(\theta) = r\mathrm{e}^{in\theta}$, for some $r > 0$. Then

$$I(\gamma, 0) = n.$$

Remark 4.1 Suppose that $\gamma : [a, b] \to \mathbb{C}\setminus\{0\}$ is a closed continuous path taking values within a region on which we can make a global continuous choice of $\arg(z)$. For example, for some $\alpha \in \mathbb{R}$, γ might map into the slit plane $\mathbb{C} \setminus \{-r\mathrm{e}^{\mathrm{i}\alpha} : r \geqslant 0\}$, in which case we could decide to insist that $\arg(z) \in (\alpha - \pi, \alpha + \pi)$. Then one possibility for the function $\theta(t)$ of Lemma 4.1 would be $\arg(\gamma(t))$, and hence $\theta(a) = \arg(\gamma(a)) = \arg(\gamma(b)) = \theta(b)$ and we deduce that $I(\gamma, 0) = 0$. The branch cut $\{-r\mathrm{e}^{\mathrm{i}\alpha} : r \geqslant 0\}$ prevents γ from winding around the origin. By translation of this picture we see that if $\gamma : [a, b] \to \mathbb{C}$ is a closed continuous path that avoids a radial line from some point $w \in \mathbb{C}$ out to infinity then $I(\gamma, w) = 0$.

4.2 Nearby closed paths have the same winding number

In this section we prove that if we have a closed path $\gamma : [a, b] \to \mathbb{C} \setminus \{0\}$, then a small-enough perturbation of γ will wind round 0 the same number of times as γ itself.

Lemma 4.2 (Dog walking lemma) Suppose $\gamma : [a, b] \to \mathbb{C} \setminus \{0\}$ and $\tilde{\gamma} : [a, b] \to \mathbb{C} \setminus \{0\}$ are continuous closed paths, with $|\gamma(t) - \tilde{\gamma}(t)| < |\gamma(t)|$ for every $t \in [a, b]$. Then

$$I(\gamma, 0) = I(\tilde{\gamma}, 0).$$

Proof Let $\theta(t)$ and $\tilde{\theta}(t)$ be lifts of the arguments of $\gamma(t)$ and $\tilde{\gamma}(t)$, respectively, as given by Lemma 4.1. Define a continuous function $\alpha : [a, b] \to \mathbb{R}$ by $\alpha(t) \coloneqq \tilde{\theta}(t) - \theta(t)$. If we consider a new continuous closed path $\sigma : [a, b] \to \mathbb{C} \setminus \{0\}$ defined by

$$\sigma(t) \coloneqq \frac{\tilde{\gamma}(t)}{\gamma(t)}$$

then $\sigma(t) = |\sigma(t)|\mathrm{e}^{\mathrm{i}\alpha(t)}$, so $\alpha(t)$ is a lift of the argument of $\sigma(t)$. By definition of winding number, we have

$$\begin{aligned} I(\tilde{\gamma}, 0) - I(\gamma, 0) &= \frac{1}{2\pi}\left[\tilde{\gamma}(b) - \tilde{\gamma}(a)\right] - \frac{1}{2\pi}\left[\theta(b) - \theta(a)\right] \\ &= \frac{1}{2\pi}\left(\alpha(b) - \alpha(a)\right) \\ &= I(\sigma, 0), \end{aligned}$$

so we are reduced to proving $I(\sigma, 0) = 0$. But

$$|1 - \sigma(t)| = \left|\frac{\gamma(t) - \tilde{\gamma}(t)}{\gamma(t)}\right| < 1,$$

so $\sigma(t) \in B_1(1)$ and $I(\sigma, 0) = 0$ by Remark 4.1. ■

Lemma 4.3 Suppose $\gamma : [a, b] \to \mathbb{C}$ is a continuous closed path. Then on each connected component of $\mathbb{C} \setminus ([a, b])$, the function $w \mapsto I(\gamma, w)$ is constant.

Note As the continuous image of a compact set, we know that $\gamma([a, b])$ is compact, and therefore (being a subset of $\mathbb{C}$) it is closed. We deduce that $\mathbb{C} \setminus \gamma([a, b])$ is open.

Proof of Lemma 4.3 Exercise 1.10 reduces the proof to showing that $I(\gamma, w)$ is constant in the neighbourhood of every point $w \in \mathbb{C} \setminus \gamma([a, b])$. Define $\varepsilon \coloneqq \min_{t\in[a,b]} |\gamma(t)|$, then WLOG. we may assume that $w = 0$, such that $I(\gamma, w)$ is constant for $w \in B_\varepsilon(0)$, or equivalently $I(\gamma, w) = I(\gamma, 0), \forall w \in B_\varepsilon(0)$. Define $\tilde{\gamma} : [a, b] \to \mathbb{C} \setminus \{0\}$ by $\tilde{\gamma}(t) = \gamma(t) - w$, by definition $I(\gamma, w) = I(\tilde{\gamma}, 0)$. We have $|\gamma(t) - \tilde{\gamma}(t)| = |w| < \varepsilon \leqslant |\gamma(t)|, \forall t \in [a, b]$, so

Lemma 4.2 implies

$$I(\gamma, 0) = I(\tilde{\gamma}, 0) = I(\gamma, w),$$

as required. ■

4.3 Winding number under homotopies

Theorem 4.1 Let $w \in \mathbb{C}$. If $\gamma_1, \gamma_2 : [a,b] \to \mathbb{C} \setminus \{w\}$ are homotopic continuous closed paths, then $I(\gamma_1, w) = I(\gamma_2, w)$. In particular, if $\gamma : [a,b] \to \mathbb{C} \setminus \{w\}$ is a continuous closed path that is homotopic to a constant path, then $I(\gamma, w) = 0$.

Corollary 4.1 If an open set $\Omega \subset \mathbb{C}$ is simply connected then for every $w \in \mathbb{C} \setminus \Omega$ and every continuous closed path $\gamma : [a,b] \to \Omega$, we have $I(\gamma, w) = 0$.

Proof of Theorem 4.1 WLOG. we may assume that $w = 0$. Since that γ_0 and γ_1 are homotopic continuous closed paths there exists a continuous map $h : [0,1] \times [a,b] \to \mathbb{C} \setminus \{0\}$ such that

$$h(0,t) = \gamma_1(t) \qquad \text{and} \qquad h(1,t) = \gamma_2(t) \qquad \text{for all } t \in [a,b], \tag{4.3.1}$$

i.e. the homotopy starts at γ_1 and ends at γ_2, and such that

$$h(s,a) = z_0 \qquad \text{and} \qquad h(s,b) = z_1 \qquad \text{for all } s \in [0,1], \tag{4.3.2}$$

where $z_0 \coloneqq \gamma_1(a) = \gamma_2(a) = \gamma_1(b) = \gamma_2(b)$ is the fixed end point. In particular for each $s \in [0,1]$, we have a continuous closed curve $\gamma_s : [a,b] \to \mathbb{C} \setminus \{0\}$ defined by $\gamma_s(t) \coloneqq h(s,t)$. It suffices to prove that the winding number $I(\gamma_s, 0)$ is constant in s.

By compactness of $[0,1] \times [a,b]$, there exists an $\varepsilon > 0$ such that $|h| \geqslant \varepsilon$. Because h is continuous on its compact domain, it is also uniformly continuous. Therefore, we can pick a $\delta > 0$ such that for all $t \in [a,b]$ and $s_1, s_2 \in [0,1]$ with $|s_1 - s_2| < \delta$, we have

$$|h(s_1,t) - h(s_2,t)| < \varepsilon,$$

and therefore

$$|\gamma_{s_1}(t) - \gamma_{s_2}(t)| = |h(s_1,t) - h(s_2,t)| < \varepsilon \leqslant |\gamma_{s_1}(t)|\,.$$

By Lemma 4.2, $I(\gamma_{s_1}, 0) = I(\gamma_{s_2}, 0)$ where $|s_1 - s_2| < \delta$, which implies that the map $s \mapsto I(\gamma_s, 0)$ is locally constant and thus constant for all $s \in [0,1]$. ■

4.4 The winding number as an integral

The winding number is often defined as an integral.

Lemma 4.4 If $w \in \mathbb{C}$ and $\gamma : [a,b] \to \mathbb{C} \setminus \{w\}$ is a closed piecewise $\mathcal{C}^1$ curve, then

$$I(\gamma, w) = \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{dz}{z - w}.$$

Proof WLOG. we may assume that $w = 0$. Let's assume that γ is $\mathcal{C}^1$. By definition

$$\int_\gamma \frac{dz}{z} = \int_a^b \frac{\gamma'(t)}{\gamma(t)} dt.$$

By the Lifting lemma, there exists a continuous function $\theta(t)$ such that $\gamma(t) = |\gamma(t)| \mathrm{e}^{\mathrm{i}\theta(t)}$. We compute

$$\gamma'(t) = \mathrm{e}^{\mathrm{i}\theta(t)} \frac{d}{dt} |\gamma(t)| + |\gamma(t)| \mathrm{i}\theta'(t) \mathrm{e}^{\mathrm{i}\theta(t)},$$

and so

$$\frac{\gamma'(t)}{\gamma(t)} = \frac{d}{dt} \log |\gamma(t)| + \mathrm{i}\theta'(t).$$

Integrating yields

$$\int_\gamma \frac{dz}{z} = \int_a^b \left[\frac{d}{dt}\log|\gamma(t)| + \mathrm{i}\theta'(t)\right] = 0 + \mathrm{i}\left[\theta(b) - \theta(a)\right] = \mathrm{i}\measuredangle(\gamma) = 2\pi\mathrm{i}I(\gamma, 0)$$

since γ is closed. ∎

Example 4.2 Consider the same curve as in Example 4.1. We have $I(\gamma, 0) = \frac{1}{2\pi\mathrm{i}}\int_\gamma \frac{dz}{z} = n$, as expected.

5 Cauchy's Theorem

5.1 Preamble

Theorem 5.1 (Cauchy's theorem on simply connected domains) Suppose $\Omega \subset \mathbb{C}$ is open and ***simply connected***. Suppose further that $f : \Omega \to \mathbb{C}$ is holomorphic and γ is a piecewise $\mathcal{C}^1$ closed curve in Ω. Then

$$\int_\gamma f(z)dz = 0.$$

If we drop the requirement that Ω be simply connected, then Cauchy's theorem fails.

Example 5.1 Consider the holomorphic function $f(z) = \frac{1}{z}$ on $\Omega := \mathbb{C} \setminus \{0\}$, and for $r > 0$ let $\gamma : [0, 2\pi] \to \mathbb{C}$ be the closed $\mathcal{C}^1$ curve $\gamma(\theta) = r\mathrm{e}^{\mathrm{i}\theta}$ that travels anti-clockwise around a circle of radius r. Then

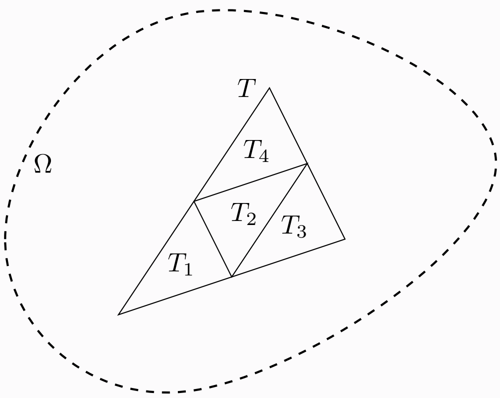
$$\int_\gamma f(z)dz = \int_0^{2\pi} r^{-1}\mathrm{e}^{-\mathrm{i}\theta}\mathrm{i}r\mathrm{e}^{\mathrm{i}\theta}d\theta = \mathrm{i}\int_0^{2\pi} d\theta = 2\pi\mathrm{i} \neq 0.$$

5.2 Goursat's theorem - Cauchy's theorem on triangles

Theorem 5.2 (Goursat's theorem) Suppose $\Omega \subset \mathbb{C}$ is open and contains a closed triangle T. Suppose further that $f : \Omega \to \mathbb{C}$ is holomorphic. Then

$$\int_{\partial T} f(z)dz = 0.$$

Proof We can decompose T into four congruent triangles $T_1, \ldots, T_4$. Due to cancellation of inner edges we have



$$\int_{\partial T} f(z)dz = \sum_{i=1}^{4}\int_{\partial T_i} f(z)dz$$

and the triangle inequality tells us that

$$\left|\int_{\partial T} f(z)dz\right| \leqslant \sum_{i=1}^{4}\left|\int_{\partial T_i} f(z)dz\right|.$$

Let T^1 be the sub-triangle such that

$$\left|\int_{\partial T} f(z)dz\right| \leqslant 4\left|\int_{\partial T^1} f(z)dz\right|.$$

We can repeat this process for T^1 to obtain a smaller triangle T^2 with the property that

$$\left|\int_{\partial T^1} f(z)dz\right| \leqslant 4\left|\int_{\partial T^2} f(z)dz\right|.$$

Iterating yields a sequence of triangles T^n whose diameters and boundary lengths decay geometrically, that is

$\operatorname{diam}(T^n) = 2^{-n} \operatorname{diam}(T)$ and $L(\partial T^n) = 2^{-n} L(\partial T)$, such that

$$\left| \int_{\partial T} f(z) dz \right| \leqslant 4^n \left| \int_{\partial T^n} f(z) dz \right|. \tag{5.2.1}$$

Now pick, for each $n \in \mathbb{N}$, a point $z_n \in T^n$. Because the triangles are nested, with diameter converging to zero, z_n is a Cauchy sequence and thus has a limit $z_\infty \in T \subset \Omega$. By definition of the complex differentiability of f at z_∞, for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $z \in B_\delta(z_\infty)$ we have

$$f(z) = f(z_\infty) + f'(z_\infty)(z - z_\infty) + R(z),$$

where the remainder is controlled by $|R(z)| \leqslant \varepsilon |z - z_\infty|$. For sufficiently large n, we have $T^n \subset B_\delta(z_\infty)$, and therefore

$$\begin{aligned} \int_{\partial T^n} f(z) dz &= \int_{\partial T^n} \left[f(z_\infty) + f'(z_\infty)(z - z_\infty) + R(z) \right] dz \\ &= \left(f(z_\infty) - f'(z_\infty) z_\infty \right) \int_{\partial T^n} dz + f'(z_\infty) \int_{\partial T^n} z\, dz + \int_{\partial T^n} R(z) dz \\ &= \int_{\partial T^n} R(z) dz \end{aligned}$$

where we have used that

$$\int_{\partial T^n} dz = 0 \qquad \text{and} \qquad \int_{\partial T^n} z\, dz = 0$$

by Corollary 3.2. Therefore, by (3.4.1), we have

$$\left| \int_{\partial T^n} f(z) dz \right| \leqslant L(\partial T^n) \varepsilon \sup_{\partial T^n} |z - z_\infty| \leqslant 2^{-n} L(\partial T) \varepsilon \operatorname{diam}(T^n) \leqslant 4^{-n} \varepsilon L(\partial T) \operatorname{diam}(T).$$

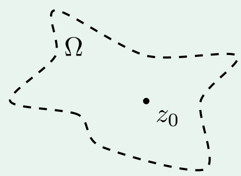
Substituting into (5.2.1) gives

$$\left| \int_{\partial T} f(z) dz \right| \leqslant \varepsilon L(\partial T) \operatorname{diam}(T),$$

and because ε is arbitrary, this completes the proof. ■

5.3 Goursat's conclusion gives us an anti-derivative

Definition 5.1 (Star-shaped domain)
An open set $\Omega \subset \mathbb{C}$ is called a ***star-shaped domain*** if there exists $z_0 \in \Omega$ such that for all $z \in \Omega$, the line segment $[z_0, z]$ connecting z_0 to z also lies in Ω. We call such a point z_0 a central point.



Note This is more general than convexity.

Theorem 5.3 (The output of Goursat's theorem implies the existence of an anti-derivative) Suppose that Ω is a star-shaped domain, and $f : \Omega \to \mathbb{C}$ is a continuous function. Suppose that for every closed triangle $T \subset \Omega$, we have

$$\int_{\partial T} f(z) dz = 0.$$

Then there exists a holomorphic function $F : \Omega \to \mathbb{C}$ such that $F'(z) = f(z)$. Indeed, if z_0 is a central point of the star-shaped domain then we can take F defined by

$$F(z) = \int_{[z_0, z]} f(w) dw, \tag{5.3.1}$$

where we defined this to be the integral of f over the $\mathcal{C}^1$ curve $\gamma : [0,1] \to \Omega$ given by $\gamma(t) = z_0 + t(z - z_0)$.

Proof Let's fix a point $z \in \Omega$, such that we can choose $r > 0$ with $B_r(z) \subset \Omega$.

For all $h \in B_r(0)$, the segment $[z, z+h]$ must lie in Ω. Since Ω is star-shaped wrt. to z_0, the closed triangle T with vertices z_0, z and $z+h$ must lie in Ω. By assumption,

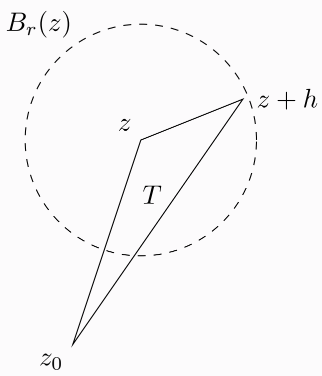
$$\int_{\partial T} f(w)dw = 0,$$

and hence

$$F(z+h) - \int_{[z,z+h]} f(w)dw - F(z) = 0.$$

Keeping in mind that

$$\int_{[z,z+h]} dw = \int_0^1 \gamma'(t)dt = \gamma(1) - \gamma(0) = (z+h) - z = h.$$



We can use (5.2.1) to compute

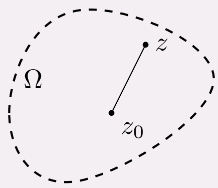
$$\left|\frac{F(z+h)-F(z)}{h} - f(z)\right| = \left|\frac{1}{h}\int_{[z,z+h]} (f(w)-f(z))\,dw\right| \leqslant \max_{w\in[z,z+h]} |f(w)-f(z)| \to 0$$

as $h \to 0$ since f is continuous at z. Thus, F is complex differentiable at z and $F'(z) = f(z)$. ■

5.4 Cauchy's theorem on star-shaped domains

Corollary 5.1
Suppose that Ω is a star-shaped domain, and $f : \Omega \to \mathbb{C}$ is a holomorphic function. Then there exists a holomorphic function $F : \Omega \to \mathbb{C}$ such that $F'(z) = f(z)$. If z_0 is a central point of the star-shaped domain then we can take F defined by

$$F(z) = \int_{[z_0,z]} f(w)dw.$$



Combining this corollary with Lemma 3.2, immediately yields an accurate proof of Cauchy's theorem 5.1 in the special case that Ω is star-shaped.

Theorem 5.4 (Cauchy's theorem on a star-shaped domain) Suppose that Ω is a star-shaped domain, $f : \Omega \to \mathbb{C}$ is holomorphic and γ is a piecewise $\mathcal{C}^1$ closed curve in Ω. Then

$$\int_\gamma f(z)dz = 0.$$

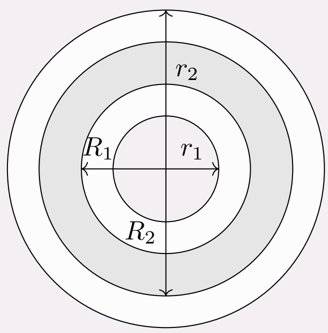
5.5 Cauchy's theorem on annuli

Corollary 5.2 (Cauchy's theorem on annuli)
Suppose that $0 \leqslant r_1 < R_1 < R_2 < r_2$, and that f is a holomorphic function on the annulus $A_{r_1,r_2} \coloneqq \{z \in \mathbb{C} : r_1 < |z| < r_2\}$.
Then writing $A \coloneqq \{z \in \mathbb{C} : R_1|z| < R_2\}$, we have

$$\int_{\partial A} f(z)dz = 0, \tag{5.5.1}$$

or equivalently that

$$\int_{\partial B_{R_2}(0)} f(z)dz = \int_{\partial B_{R_1}(0)} f(z)dz. \tag{5.5.2}$$

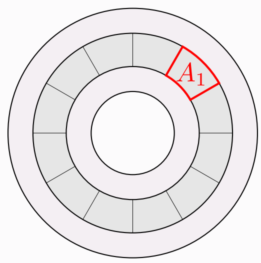


Proof

We can divide A_{R_1,R_2} into n sections $A_1, \dots, A_n$ such that

$$\int_{\partial A} f(z)dz = \sum_{i=1}^{n} \int_{\partial A_i} f(z)dz,$$

due to cancellation of inner edges. For n sufficiently large, each A_i will fit within a star-shaped domain within A_{R_1,R_2} such that $\int_{\partial A_i} f(z)dz = 0 \Rightarrow \int_{\partial A} f(z)dz = 0$.



■

5.6 Cauchy's integral formula

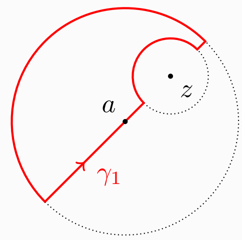
Theorem 5.5 (Cauchy's integral formula on a disc) Suppose $\Omega \subset \mathbb{C}$ is open and $f : \Omega \to \mathbb{C}$ is holomorphic. Suppose that the closed disc/ball $B_r(a)$ of radius $r > 0$, centred at $a \in \Omega$, lies within Ω. Then for every $z \in B_r(a)$ we have

$$f(z) = \frac{1}{2\pi\mathrm{i}} \int_{\partial B_r(a)} \frac{f(w)}{w-z} dw.$$

Proof For the given $z \in B_r(a)$, choose $\delta > 0$ small enough such that $B_\delta(z) \subset B_r(a)$.

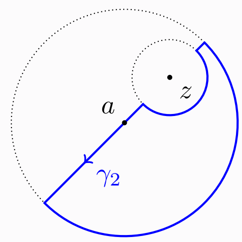
The function $\frac{f(w)-f(z)}{w-z}$ is defined and holomorphic on $\Omega \backslash \{z\}$, and by Cauchy's theorem for star-shaped domains, Theorem 5.4, we have

$$\int_{\gamma_1} \frac{f(w)-f(z)}{w-z} dw = 0.$$



We can repeat for γ_2, and add to give

$$\int_{\partial B_r(a)} \frac{f(w)-f(z)}{w-z} dw = \int_{\partial B_\delta(z)} \frac{f(w)-f(z)}{w-z} dw. \tag{5.6.1}$$



Note The integrals cancel along the diameter.

Because $\lim_{w\to z_0} \frac{f(w)-f(z)}{w-z} = f'(z)$, and the length of $\partial B_\delta(z)$ is $2\pi\delta$, we see that the RHS. of (5.6.1) converges to 0 as $\delta \to 0^+$, by (3.4.1). Therefore

$$\int_{\partial B_r(a)} \frac{f(w)}{w-z} dw = \int_{\partial B_r(a)} \frac{f(z)}{w-z} dw = f(z) 2\pi\mathrm{i} I(\partial B_{r(a)}, z) = f(z)2\pi\mathrm{i},$$

by the integral characterisation of winding number given in Lemma 4.4, and the formula for the winding number derived in Q. 4.2. ■

6 Taylor series and applications

6.1 Taylor series - main result

Theorem 6.1 (Taylor's theorem) Let $\Omega \subset \mathbb{C}$ be open, and let $f : \Omega \to \mathbb{C}$ be holomorphic. Suppose $z_0 \in \Omega$ and $r > 0$ such that $B_r(z_0) \subset \Omega$. Then for all $z \in B_r(z_0)$, we have

$$f(z) = \sum_{k=0}^{\infty} a_k (z-z_0)^k, \tag{6.1.1}$$

where,

$$a_k = \frac{1}{2\pi\mathrm{i}} \int_{\partial B_r(z_0)} \frac{f(w)}{(w-z_0)^{k+1}} dw. \tag{6.1.2}$$

Proof is given in the next subsection.

Remark 6.1 By Cauchy's theorem on annuli, Corollary 5.2, we could reduce the radius of the circle over which we are integrating in (6.1.2) to give

$$a_k = \frac{1}{2\pi \mathrm{i}} \int_{\partial B_s(z_0)} \frac{f(w)}{(w - z_0)^{k+1}} dw$$

for any $s \in (0, r]$. In particular, the Taylor coefficients a_k do not depend on r.

Remark 6.2 Taylor's theorem tells us that a holomorphic function is analytic (can be written as a power series in some ball about each point $z_0 \in \Omega$) and Theorem 3.2 gives the converse. Hence, the terms holomorphic and analytic are often used interchangeably.

From this we can deduce the following result.

Corollary 6.1 If $f : \Omega \to \mathbb{C}$ is a holomorphic function on an open set $\Omega \subset \mathbb{C}$ then it is infinitely differentiable.

Corollary 6.2 (cf. Cauchy's integral formula) If $f : \Omega \to \mathbb{C}$ is a holomorphic on an open set $\Omega \subset \mathbb{C}$, and $B_r(z_0) \subset \Omega$ for some $z_0 \in \Omega$ and $r > 0$, then for each $n \in \mathbb{N}$ we have

$$f^{(n)}(z_0) = \frac{n!}{2\pi \mathrm{i}} \int_{\partial B_r(z_0)} \frac{f(w)}{(w - z_0)^{n+1}} dw.$$

This is immediate from Corollary 3.1.

6.2 Taylor's theorem - proof

Proof of Taylor's theorem By translation we may assume that $z_0 = 0$. Cauchy's integral formula tells us that for all $z \in B_r(0)$ we have

$$f(z) = \frac{1}{2\pi \mathrm{i}} \int_{\partial B_r(0)} \frac{f(w)}{w - z} dw. \tag{6.2.1}$$

We can rewrite

$$\frac{1}{w - z} = \frac{1}{w} \left[\frac{1}{1 - z/w} \right] \tag{6.2.2}$$

and since $z \in B_r(0)$ and $w \in \partial B_r(0)$, $\frac{|z|}{|w|} < 1$ and we may write the part in square brackets in (6.2.1) as a geometric series

$$\frac{1}{1 - z/w} = \sum_{k=0}^{\infty} \left(\frac{z}{w} \right)^k. \tag{6.2.3}$$

Therefore

$$\begin{aligned} f(z) &= \frac{1}{2\pi \mathrm{i}} \int_{\partial B_r(0)} f(w) \left(\sum_{k=0}^{\infty} \frac{z^k}{w^{k+1}} \right) dw \\ &= \frac{1}{2\pi \mathrm{i}} \sum_{k=0}^{\infty} \left(\int_{\partial B_r(0)} \frac{f(w)}{w^{k+1}} dw \right) z^k \\ &= \sum_{k=0}^{\infty} a_k z^k. \end{aligned}$$

The interchange of the summation and the integration is justified because the sum converges uniformly in the integration variable w. ■

6.3 Basic consequences of Taylor's theorem. Liouville's theorem.

Corollary 6.3 Suppose that $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is holomorphic on $B_R(0)$ for some $R > 0$, and that for all

$z \in B_R(0)$ we have $|f(z)| \leqslant M \leqslant \infty$. Then for all k

$$|a_k| \leqslant \frac{M}{R^k}. \tag{6.3.1}$$

Proof For each $r \in (0, R)$, Taylor's theorem applied on $\overline{B_r(0)}$ gives $f(z)$ as a power series. By uniqueness of the Taylor coefficients, a_k is given by (6.1.2). Hence,

$$|a_k| \leqslant \frac{1}{2\pi} \left| \int_{\partial B_r(0)} \frac{f(w)}{w^{k+1}} dw \right| \leqslant \frac{1}{2\pi} 2\pi r \frac{M}{r^{k+1}} = \frac{M}{r^k}, \tag{6.3.2}$$

by (3.4.1). Then let $r \uparrow R$. ■

Corollary 6.4 (Liouville's theorem) Any bounded entire function is constant.

Proof By applying Taylor's theorem with $z_0 = 0$ and arbitrarily large R, we can write our entire function $f : \mathbb{C} \to \mathbb{C}$ as a Taylor series $f(z) = \sum_{k=0}^{\infty} a_k z^k$. We are assuming that f is bounded, i.e. $|f(z)| \leqslant M$ for all $z \in \mathbb{C}$, so by Corollary 6.3 we have $|a_k| \leqslant \frac{M}{R^k}$, for every $k \in \mathbb{N}$ and $R > 0$. By taking $R \to \infty$ we deduce that $a_k = 0$ for each $k \geqslant 1$ and $f = a_0$. ■

Corollary 6.5 (Fundamental Theorem of Algebra) Every non-constant polynomial has at least one zero in $\mathbb{C}$.

Proof This proof is non-examinable, so I will only provide a sketch proof. Essentially the idea is that if a polynomial $p(z)$ does not have a zero then one can show that $1/p$ is a bounded entire function, and must therefore be constant by Liouville's theorem. ■

6.4 Morera's Theorem

The following is an inverse to Goursat's theorem.

Theorem 6.2 (Morera's theorem) Suppose $\Omega \subset \mathbb{C}$ is open and $f : \Omega \to \mathbb{C}$ is a continuous function. Suppose that for all closed triangles $T \subset \Omega$ we have

$$\int_{\partial T} f(z) dz = 0,$$

then f is holomorphic on Ω.

Proof We need to show that f is complex differentiable at an arbitrary point $a \in \Omega$. Pick $r > 0$ sufficiently small so that $B_r(a) \subset \Omega$. By Theorem 5.3, we can construct a holomorphic function $F : B_r(a) \to \mathbb{C}$ with $F'(z) = f(z)$ for all $z \in B_r(a)$. Because F is holomorphic, Corollary 6.1 tells us that it is infinitely complex differentiable. In particular, $f = F'$ is complex differentiable at a. ■

6.5 Local invertibility of holomorphic functions

Lemma 6.1 Suppose $\Omega \subset \mathbb{C}$ is open and $f : \Omega \to \mathbb{C}$ is holomorphic with $f'(z_0) \neq 0$ at some $z_0 \in \Omega$. Then there exists a neighbourhood $V_0 \subset \Omega$ of z_0 and a neighbourhood $V_1 \subset \Omega$ of $f(z_0)$ such that the restriction of f to V_0 is a biholomorphic map from V_0 to V_1.

The requirement that $f'(z_0) \neq 0$ is immediate if we consider $f \equiv 0$ or $f(z) = z^2$ with $z_0 = 0$.

Proof From Corollary 6.1, we know that f is a $\mathcal{C}^1$ function when viewed as a real-differentiable function. Our hypothesis $f'(z_0) \neq 0$ tells us that the real derivative of f is a rotation and/or dilation. In particular, the real derivative is invertible and the Inverse Function Theorem applies. That is, there exist neighbourhoods V_0 and V_1 of z_0 and $f(z_0)$ respectively, and $f^{-1} : V_1 \to V_0$ is also $\mathcal{C}^1$. We may assume that $f'(z) \neq 0$ for all $z \in V_0$ for sufficiently small V_0, by continuity of f. We need to show that f^{-1} is holomorphic, i.e. that it satisfies the C.R. equations, and that its derivative is non-zero. Using (1.2.6), we have that for all any $\mathcal{C}^1$ function $g : V_1 \to V_0$

$$(f \circ g)_{\bar{z}}(z) = f'(g(z))\, g_{\bar{z}}(z)$$

for all $z \in V_1$. If we let $g = f^{-1}$, we deduce that

$$0 = z_{\bar{z}} = f'\left(f^{-1}(z)\right)^{-1}_{\bar{z}}(z)$$

which implies $(f^{-1})_{\bar{z}} \equiv 0$ since $f' \neq 0$. Since f^{-1} is holomorphic we can use (1.2.7) to deduce that

$$1 = z'(z) = f'\left(f^{-1}(z)\right)(f^{-1})'(z)$$
$$\Rightarrow (f^{-1})'(z) = \frac{1}{f'(f^{-1}(z))} \neq 0$$

since $f' \neq 0$. ■

7 Zeros of holomorphic functions

7.1 Basic structure

Consider the holomorphic function $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = z^n$, for some $n \in \mathbb{N}$. This function is zero precisely at $z = 0$. The *order* of the zero, defined below, will be n.

Definition 7.1 (Order) Let $\Omega \subset \mathbb{C}$ be open and let $f : \Omega \to \mathbb{C}$ be a holomorphic with $f(z_0) = 0$ for some $z_0 \in \Omega$. We define the ***order*** of f at z_0 to be

$$\operatorname{ord}(f, z_0) := \begin{cases} \infty & \text{if } f^{(k)}(z_0) = 0 \text{ for all } k \in \mathbb{N}, \\ \min\{k \in \mathbb{N} : f^{(k)}(z_0) \neq 0\} & \text{otherwise.} \end{cases} \tag{7.1.1}$$

Example 7.1 If $g : \Omega \to \mathbb{C}$ is a holomorphic function for which $g(z_0) \neq 0$, and we define the holomorphic function $f(z) := (z - z_0)^n g(z)$, then the order of the zero of f at z_0 is n. This is because as we differentiate $k < n$ times, using the product rule, each of the resulting terms will have at least a factor $(z - z_0)^{n-k}$ within it, so will vanish at z_0. But if we differentiate n times, and evaluate at z_0, then there will be one non-zero term $n!g(z_0)$.

Theorem 7.1 Suppose that $\Omega \subset \mathbb{C}$ is open and $f : \Omega \to \mathbb{C}$ is a holomorphic function that has zero of *finite* order $n \in \mathbb{N}$ at $z_0 \in \Omega$. Then there exists a holomorphic function $g : \Omega \to \mathbb{C}$ such that

$$f(z) = (z - z_0)^n g(z),$$

and g is non-zero in a neighbourhood of z_0. In particular, each zero of finite order is an *isolated* point of the set of zeros.

Proof Pick $r > 0$ such that $\overline{B_r(z_0)} \subset \Omega$, we can use Taylor's theorem to write

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k.$$

Corollary 3.1 tells us that $a_k = \frac{f^{(k)}(z_0)}{k!}$, and because f has a zero of order n at z_0, we must have $a_k = 0$ for $k < n$, and $a_n \neq 0$. So, we can write, for all $z \in B_r(z_0)$

$$f(z) = \sum_{k=n}^{\infty} a_k (z - z_0)^k = (z - z_0)^n \sum_{k=0}^{\infty} a_{k+n} (z - z_0)^k = (z - z_0)^n g(z), \tag{7.1.2}$$

where $g(z) := \sum_{k=0}^{\infty} a_{k+n}(z - z_0)^k$ is defined on $B_r(z_0)$, and $g(z_0) \neq 0$. We can extend g to the rest of Ω by setting $g(z) = f(z)(z - z_0)^{-n}$, which only disagrees at z_0. ■

Theorem 7.2 Suppose that $\Omega \subset \mathbb{C}$ is open *and connected*, and $f : \Omega \to \mathbb{C}$ is a holomorphic function that has a zero of *infinite* order at some point $z_0 \in \Omega$. Then $f \equiv 0$.

Proof Consider the set $\Omega_0 := \{z \in \Omega : f \text{ has a zero of order infinity at } z\}$. Our aim is to prove that $\Omega_0 = \Omega$. We know that Ω_0 is non-empty, since $z_0 \in \Omega_0$. By connectedness of Ω, it suffices to show that Ω_0 is open and closed in Ω.

Pick an arbitrary point $w_0 \in \Omega_0$. The Taylor series of f at $z = w_0$, given by Taylor's theorem, has coefficients $a_k = \frac{f^{(k)}(z_0)}{k!} = 0$ using Corollary 3.1. Hence, $f \equiv 0$ in any ball $B_r(w_0) \subset \Omega_0 \subseteq \Omega$ and Ω_0 must be open.

If we take a sequence $z_i \in \Omega_0$ that converges to some $z_\infty \in \Omega$, then $f(z_\infty) = 0$ by continuity of f. But z_∞ cannot be a zero of finite order since we have seen that such zeros are isolated within the set of all zeros. Therefore, $z_\infty \in \Omega_0$, and we can deduce that Ω_0 is (relatively) closed. ■

7.2 The identity theorem

Definition We call $z_\infty \in \Omega$ an ***accumulation point*** of Σ if there exists a sequence $z_i \in \Sigma \setminus \{z_\infty\}$ such that $z_i \to z_\infty$. This is the opposite of an isolated point.

Theorem 7.3 (Identity theorem) Let $\Omega \subset \mathbb{C}$ be open *and connected* and let f_1 and f_2 be holomorphic functions $\Omega \mapsto \mathbb{C}$. If the set $\Sigma := \{z \in \mathbb{C} : f_1(z) = f_2(z)\}$ has at least one accumulation point in Ω, then $f_1 \equiv f_2$ throughout Ω.

Equivalently, two holomorphic functions on an open *and connected* set are either identical or agree only at isolated points.

Proof By hypothesis, the function $g := f_1 - f_2$ is holomorphic and has a non-isolated zero. By Theorem 7.1, this zero must be of infinite order (not isolated), and then by Theorem 7.2, we must have $g \equiv 0$ throughout Ω, i.e. $f_1 \equiv f_2$. ■

Example 7.2 Suppose f is a holomorphic function on the ball $B_2(0) \subset \mathbb{C}$, and suppose we know that $f(\frac{1}{n}) = 0$ for all $n \in \mathbb{N}$. Then we can deduce that f is identically zero on $B_2(0)$.

Example 7.3 Consider the function $f(z) := z^2 \sin\left(\frac{\pi}{z}\right)$ on $\mathbb{C}$. This is clearly holomorphic on $\{\mathrm{Re}(z) > 0\}$ and $f(\frac{1}{n}) = 0, \forall n \in \mathbb{N}$. We can construct a sequence of zeros which accumulates at $\{\mathrm{Re}(z) = 0\}$, the boundary of $\{\mathrm{Re}(z) > 0\}$. Thus, we can't deduce that $f \equiv 0$.

7.3 Refined structure

Theorem 7.4 Let $\Omega \subset \mathbb{C}$ be open and $f : \Omega \to \mathbb{C}$ holomorphic. If f has a zero of finite order $k \geqslant 1$ at $z_0 \in \Omega$, then there exists a neighbourhood $V_0 \subset \Omega$ of z_0, a radius $r > 0$ and a biholomorphic function $h : V_0 \to B_r(0)$ such that for every $z \in V_0$, we have

$$f(z) = \big(h(z)\big)^k. \tag{7.3.1}$$

In particular, f is locally k-to-one near z_0. This means that f takes every value in $B_{r^k}(0) \setminus \{0\}$ exactly k times in V_0.

Proof is given below.

Intuitively, we would like to define $h(z) := (z - z_0)\mathrm{e}^{\frac{1}{k}\log g(z)}$ but require an unambiguous definition of the logarithm.

Lemma 7.1 Suppose $\Omega \subset \mathbb{C}$ is open and connected, and $g : \Omega \to \mathbb{C} \setminus \{0\}$ is a holomorphic function such that the 'holomorphic derivative' $\frac{g'(z)}{g(z)}$ admits an anti-derivative. That is, we assume that there exists a holomorphic $F : \Omega \to \mathbb{C}$ such that $F'(z) = \frac{g'(z)}{g(z)}$. Then there exists $w_0 \in \mathbb{C}$ so that when we define a holomorphic function $\ell : \Omega \to \mathbb{C}$ by

$$\ell(z) := F(z) + w_0, \tag{7.3.2}$$

we have

$$g(z) = \mathrm{e}^{\ell(z)} \qquad \text{for all } z \in \Omega. \tag{7.3.3}$$

The function ℓ is unique up to an additive constant $2\pi i n$ for $n \in \mathbb{Z}$.

Corollary 7.1 Suppose $\Omega \subset \mathbb{C}$ is *star-shaped* and $g : \Omega \to \mathbb{C} \setminus \{0\}$ is holomorphic. Then there exists a holomorphic function $\ell : \Omega \to \mathbb{C}$, unique up to an integer multiple of $2\pi i$, such that

$$g(z) = \mathrm{e}^{\ell(z)}.$$

In particular, for $k \in \mathbb{N}$, the function $z \mapsto \mathrm{e}^{\frac{1}{k}\ell(z)}$ gives a holomorphic function on Ω whose k^{th} power is $g(z)$.

Proof of Lemma 7.1 Fix an arbitrary $z_0 \in \Omega$. As $g(z_0) \neq 0$ by assumption, we can pick $w_0 \in \mathbb{C}$ such that $\mathrm{e}^{w_0} = g(z_0)\mathrm{e}^{-F(z_0)}$. We compute

$$\left(g(z)\mathrm{e}^{-\ell(z)}\right)' = g'(z)\mathrm{e}^{-\ell(z)} - g(z)\mathrm{e}^{-\ell(z)}\ell'(z) = \mathrm{e}^{-\ell(z)}\left(g'(z) - g(z)F'(z)\right) = 0, \tag{7.3.4}$$

which tells us that $g(z)\mathrm{e}^{-\ell(z)}$ is constant and equals 1 at z_0, such that $g(z) = \mathrm{e}^{\ell(z)}$ as required. ■

Proof of Theorem 7.4 By Theorem 7.1, we can write

$$f(z) = (z - z_0)^k g(z), \tag{7.3.5}$$

where $g \neq 0$ on a neighbourhood $B_s(z_0) \subset \Omega$ of z_0, for some $s > 0$. We apply Corollary 7.1 (with $\Omega = B_s(z_0)$ there), to obtain a holomorphic function $\ell : B_s(z_0) \to \mathbb{C}$ such that $g(z) = \mathrm{e}^{\ell(z)}$, and then defining

$$h(z) = (z - z_0)\mathrm{e}^{\frac{1}{k}\ell(z)}.$$

So, $h(z)^k = f(z)$ as required. Observe that $h'(z_0) = \mathrm{e}^{\frac{1}{k}\ell(z_0)} \neq 0$. Applying the local invertibility lemma, we can find neighbourhoods $V_0 \subset B_s(z_0)$ of z_0 and V_1 of $h(z_0) = 0$, so that the restriction $h : V_0 \to V_1$ is biholomorphic. By shrinking these neighbourhoods, we may assume that $V_1 = B_r(0)$ for some $r > 0$. More precisely, we take $r > 0$ small enough so that $B_r(0) \subset V_1$, and then redefine $V_1 = B_r(0)$ and $V_0 = h^{-1}(B_r(0))$.

To see the k-to-one property, pick an arbitrary point $w \in B_{r^k}(0) \setminus \{0\}$. Then there are precisely k points $\xi_1, \ldots, \xi_k$, all lying in $B_r(0)$, such that $\xi_j^k = w$ for each $j \in \{1, \ldots, k\}$. So within V_0, precisely the k points $h^{-1}(\xi_j)$ are mapped to w by f. ■

7.4 Open mapping theorem; Maximum modulus principle; Mean value property

Theorem 7.5 (Open mapping theorem) Suppose $\Omega \subset \mathbb{C}$ is open and connected, and $f : \Omega \to \mathbb{C}$ is holomorphic but not constant. Then the image $f(\Omega)$ of Ω under f is also open and connected.

Proof Topology tells us that the image of every connected set under a continuous function is connected. RTP that the image is open. That is, for any point $w_0 = f(z_0) \in f(\Omega)$ there exists a neighbourhood of w_0 contained in $f(\Omega)$.

The function $g(z) \coloneqq f(z) - w_0$ has a zero at z_0, which must be of finite order. Otherwise, g would be identically zero by Theorem 7.2, and we would have $f = w_0$ which contradicts our assumption. By Theorem 7.4, locally we have that $f(z) = w_0 + (h(z))^k$, where h is a biholomorphic map onto $B_r(0)$. Therefore, the image of f contains the ball $B_{r^k}(0)$. ■

Corollary 7.2 (Maximum modulus principle) Suppose $\Omega \subset \mathbb{C}$ is open and connected, and $f : \Omega \to \mathbb{C}$ is holomorphic but not constant. Then $|f|$ does not have any local maxima.

Proof Suppose that $|f|$ attains a local maximum at $z_0 \in \Omega$. By the Open mapping theorem, the image of any neighbourhood of z_0 is a neighbourhood of $f(z_0)$, and therefore must contain points with larger absolute value, contradicting our assumption. ■

Lemma 7.2 (Mean value property) Suppose $\Omega \subset \mathbb{C}$ is open with $\overline{B_r(z_0)} \subset \Omega$, for some $r > 0$ and $z_0 \in \Omega$.

Suppose that $f : \Omega \to \mathbb{C}$ is holomorphic. Then

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + r\mathrm{e}^{\mathrm{i}\theta})d\theta, \tag{7.4.1}$$

that is, $f(z_0)$ equals the average of f over the circle of radius r centred at z_0.

Proof By Cauchy's integral formula, we have

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi\mathrm{i}}\int_{\partial B_r(z_0)} \frac{f(z)}{z - z_0}dz = \frac{1}{2\pi\mathrm{i}}\int_0^{2\pi} \frac{f(z_0 + r\mathrm{e}^{\mathrm{i}\theta})\mathrm{i}r\mathrm{e}^{\mathrm{i}\theta}}{r\mathrm{e}^{\mathrm{i}\theta}}d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + r\mathrm{e}^{\mathrm{i}\theta})d\theta. \end{aligned}$$

■

7.5 Injective holomorphic functions are biholomorphic onto their image

Recall our definition of a biholomorphic function.

Theorem 7.6 Suppose $\Omega \subset \mathbb{C}$ is a domain (open and connected) and $f : \Omega \to \mathbb{C}$ is both injective and holomorphic. Then $f(\Omega)$ is a domain and $f : \Omega \to f(\Omega)$ is a biholomorphic map.

Proof Since Ω is connected and f is continuous, the image $f(\Omega)$ is connected. By the Open mapping theorem, $f(\Omega)$ is an open subset of $\mathbb{C}$. (f cannot be constant as it is injective). Therefore, $f(\Omega)$ is a domain.

Suppose that we could find some $z_0 \in \Omega$ such that $f'(z_0) = 0$. Then the function $\phi(z) = f(z) - f(z_0)$ would satisfy $\phi(z_0) = \phi'(z_0) = 0$. That is, ϕ would have a zero of order $k \geqslant 2$ at z_0, and Theorem 7.4 would then imply that ϕ could not be injective and therefore that f could not be injective, which is a contraction.

Using that $f'(z) \neq 0$ in Ω and Lemma 6.1, we find that f is locally biholomorphic. Since f is bijective, it must be (globally) biholomorphic. ■

7.6 Schwarz lemma

Theorem 7.7 (Schwarz lemma) Let $f : D \to D$ be holomorphic on D with $f(0) = 0$. Then

(i) $|f'(0)| \leqslant 1$, and

(ii) $|f(z)| \leqslant |z|$ for all $z \in D$.

If we have equality in (i), or (ii) for some $z \in D \setminus \{0\}$, then $f(z) = \mathrm{e}^{\mathrm{i}\theta}z$ for some $\theta \in \mathbb{R}$.

Proof If the zero of f at $z = 0$ is of infinite order, then $f \equiv 0$ and the theorem is trivial. So by Theorem 7.1, there exists a holomorphic function $g : D \to \mathbb{C}$ such that for all $z \in D$ we have $f(z) = zg(z)$.

Suppose $r \in (0, 1)$. For all z with $|z| = r$ we have

$$1 \geqslant |f(z)| = |z||g(z)| = r|g(z)|, \tag{7.6.1}$$

and hence $|g(z)| < \frac{1}{r}$. By the Maximum modulus principle, $|g|$ must attain its maximum over the ball $B_r(0)$ on the boundary $\{|z| = r\}$, and thus we have $|g(z)| < \frac{1}{r}$ for all $|z| \leqslant r$. By taking the limit $r \to 1^+$, we obtain $|g(z)| \leqslant 1$ throughout D. This implies (i) because $f'(0) = g(0)$, and implies (ii) because $|f(z)| = |z||g(z)|$.

If instead we have equality in (i) or (ii), then we have $|g(z)| = 1$ for some $z \in D$. Then the Maximum modulus principle implies that g must be constant, i.e. we can write $g(z) = \mathrm{e}^{\mathrm{i}\theta}$ for some $\theta \in \mathbb{R}$. ■

Corollary 7.3 (Classification of biholomorphic maps of the disc) Every biholomorphic function $f : D \to D$ is a Möbius transformation of the form (2.7.2).

Proof Suppose that $f(0) = 0$ such that we can apply the Schwarz lemma to both f and f^{-1}. Then $|f(z)| \leqslant |z|$ and $|z| = |f^{-1}(f(z))| \leqslant |f(z)|$, i.e. $|f(z)| = |z|$. Since we have equality then f is a rotation.

In the general case $f(0) = w$, we set $w = f^{-1}(0)$ and define $\varphi(z) = \frac{z-w}{\overline{w}z-1}$. Considering Example 2.3, we see that $f \circ \varphi$ is a biholomorphic map from D to itself that maps 0 to 0, so it is of the form $z \mapsto e^{i\theta} z$. Furthermore, φ is its own inverse. Therefore,

$$f(z) = f \circ \varphi \circ \varphi(z) = e^{i\theta} \frac{z - w}{\overline{w}z - 1}$$

as required. ∎

8 Isolated singularities

8.1 Riemann's removable singularity theorem

Definition 8.1 (Isolated singularity) A function f that is holomorphic on $B_r(a) \setminus \{a\} \subset \mathbb{C}$, for some $r > 0$ and $a \in \mathbb{C}$, is said to have an ***isolated singularity*** at a.

Theorem 8.1 (Riemann's removable singularity theorem) Let $f : B_r(a) \setminus \{a\} \to \mathbb{C}$ be a holomorphic function from a ball of radius $r > 0$ centred at $a \in \mathbb{C}$. Suppose that

$$|f(z)| \leqslant M \text{ for some } M < \infty \text{ and every } z \in B_r(a) \setminus \{a\},$$

or more generally that

$$\lim_{z \to a} (z - a) f(z) = 0.$$

Then we can extend f to a holomorphic function $f : B_r(a) \to \mathbb{C}$.

Proof Define $g : B_r(a) \to \mathbb{C}$ by

$$g(z) = \begin{cases} (z-a)^2 f(z) & \text{for } z \in B_r(a) \setminus \{a\} \\ 0 & \text{for } z = a. \end{cases} \tag{8.1.1}$$

By the product rule, g is holomorphic when restricted to $B_r(a)$. We can compute

$$\frac{g(z) - g(a)}{z - a} = (z - a) f(z) \to 0$$

as $z \to a$, by our assumption (8.1.1). Thus, g is complex differentiable at $z = a$ with $g'(a) = 0$ such that this zero is of order at least 2. If the zero at a is of infinite order then $g \equiv f \equiv 0$ by Theorem 7.2. Otherwise, we apply Theorem 7.1 to obtain

$$g(z) = (z - a)^n h(z),$$

for holomorphic $h : B_r(a) \to \mathbb{C}$ with $h(a) \neq 0$. But then $(z - a)^{n-2} h(z)$ is a holomorphic function on $B_r(a)$ that equals f on $B_r(a) \setminus \{a\}$. ∎

8.2 Classification of isolated singularities; description of poles

Definition 8.2 A holomorphic function $f : B_r(z_0) \setminus \{z_0\} \to \mathbb{C}$ is said to have a

(1) *removable singularity* at z_0 if $f(z)$ has a limit in $\mathbb{C}$ as $z \to z_0$,

(2) *pole* at z_0 if $f(z) \to \infty$ as $z \to z_0$,

(3) *essential singularity* at z_0 if neither of the previous two cases hold.

Example 8.1 The function $f(z) = {}^1\!/_{z^n}$ on $D \setminus \{0\}$ has a pole (of order n) at 0.

Example 8.2 For any entire function g, the function $h(z) := g({}^1\!/_z)$ will have an essential singularity at 0.

If $f : B_r(z_0) \setminus \{z_0\} \to \mathbb{C}$ is holomorphic and has a pole at z_0, then for sufficiently small $r > 0$ we may assume that $|f(z)| \geqslant 1$ for all $z \in B_r(z_0)$. Therefore, ${}^1\!/_{f(z)}$ is bounded and holomorphic on $B_r(z_0) \setminus \{z_0\}$. By Riemann's removable singularity theorem, it is the restriction of some holomorphic function $\phi : B_r(z_0) \to \mathbb{C}$ with $\phi(z_0) = 0$. The zero of ϕ at z_0 must be of finite order (otherwise $\phi \equiv 0$). Thus, we can apply Theorem 7.1 to obtain $\phi(z) = (z - z_0)^n \psi(x)$, where ψ is holomorphic on $B_r(z_0)$ with $\psi(z_0) \neq 0$. Defining $g(z) = {}^1/_{\psi(z)}$ gives another holomorphic and non-zero function on $B_r(z_0)$, and we see that we have proved an analogue of Theorem 7.1 for poles:

Theorem 8.2 Suppose that a holomorphic $f : B_r(z_0) \setminus \{z_0\} \to \mathbb{C}$ has a pole at z_0. Then there exists $n \in \mathbb{N}$ and a holomorphic function $g : B_r(z_0) \to \mathbb{C}$ such that

$$f(z) = \frac{g(z)}{(z - z_0)^n}.$$

The integer n is called the ***order*** of the pole. If $n = 1$ then f is said to have a ***simple pole***.

A ***meromorphic*** function is, loosely speaking, a function that does not have essential singularities.

Definition 8.3 (Meromorphic function) Suppose $\Omega \subset \mathbb{C}$ is open. A holomorphic function $f : \Omega \setminus \mathcal{P} \to \mathbb{C}$, where $\mathcal{P} \subset \Omega$ is a discrete subset, is said to be ***meromorphic*** if it has a pole at each point in $\mathcal{P}$.

Example 8.3 The function $f(z) = \frac{1}{e^z - 1}$ is meromorphic on $\mathbb{C}$ with poles at $z = 2\pi i n$ for $n \in \mathbb{Z}$.

8.3 Essential singularities

Theorem 8.3 (Casorati-Weierstrass theorem) Suppose that $f : B_r(z_0) \setminus \{z_0\} \to \mathbb{C}$ is holomorphic and has an essential singularity at z_0. Then however small we take $\delta \in (0, r)$, the image $f\big(B_\delta(z_0) \setminus \{z_0\}\big)$ is dense in $\mathbb{C}$.

Proof Suppose that $f\big(B_\delta(z_0) \setminus \{z_0\}\big)$ is not dense in $\mathbb{C}$. Then there exists $w \in \mathbb{C}$ and $\epsilon > 0$ such that $|f(z) - w| \geqslant \epsilon$ for all $z \in B_\delta(z_0) \setminus \{z_0\}$. The function

$$h(z) = \frac{1}{f(z) - w}$$

is holomorphic on $B_\delta(z_0) \setminus \{z_0\}$ and bounded by ${}^1/_\epsilon$. By Riemann's removable singularity theorem, h extends to a holomorphic function on $B_\delta(z_0)$. If $h(z_0) \neq 0$, then we can rewrite $f(z) = w + {}^1/_{h(z)}$ on $B_\delta(z_0)$, and f has a removable singularity at z_0. If $h(z_0) = 0$, then h must have a zero of finite order $n \in \mathbb{N}$ at z_0 and by Theorem 7.1

$$h(z) = (z - z_0)^n g(z)$$

for some non-zero holomorphic $g : B_\delta(z_0) \to \mathbb{C}$. But then

$$f(z) = w + \frac{1}{(z - z_0)^n g(z)}$$

on $B_\delta(z_0)$, and we see that f has a pole at z_0. In either case, f does not have an essential singularity at z_0. ∎

Remark 8.1 The *Great Picard's theorem* states that however small a neighbourhood of an essential singularity we take, the image of our holomorphic function will be **all** of $\mathbb{C}$ except possibly one point.

8.4 Laurent series 1

Given a holomorphic function $f : B_r(z_0) \setminus \{z_0\} \to \mathbb{C}$, we can't write down a power series expansion for f about z_0 as in Taylor's theorem. Consider for example the function $f(z) = {}^1/_z$ on $D \setminus \{0\}$. However, we can write down a double-ended power series expansion

$$\sum_{k \in \mathbb{Z}} a_k z^k,$$

if we allow k to be negative.

Definition 8.4 A double-ended power series $\sum_{k \in \mathbb{Z}} a_k z^k$ is said to converge to $\ell \in \mathbb{C}$ if $\sum_{k=0}^\infty a_k$ converges to ℓ_+, $\sum_{k=1}^\infty a_{-k}$ converges to ℓ_-, and $\ell = \ell_+ + \ell_-$.

This allows us to makes sense of the double-ended power series $\sum_{k \in \mathbb{Z}} a_k z^k$ and highlights that we are really considering two normal power series. The first is $f_+(z) = \sum_{k=0}^\infty a_k z^k$ and the second is $f_-(z) = \sum_{k=1}^\infty a_{-k} z^{-k}$. We can see that they both converge on the annulus ${}^1/_{R_-} < |z| < R_+$.

8.5 Cauchy's integral formula for annuli

Theorem 8.4 (Cauchy's integral formula for annuli) Suppose $\Omega \subset \mathbb{C}$ is open and $f : \Omega \to \mathbb{C}$ is holomorphic. If Ω contains the closure of an annulus

$$A = \{z \in \mathbb{C} : R_1 < |z| < R_2\}, \tag{8.5.1}$$

then for any $w \in A$ we have

$$f(w) = \frac{1}{2\pi \mathrm{i}} \int_{\partial A} \frac{f(z)}{z - w} dz. \tag{8.5.2}$$

Corollary 8.1 (Corollary of Riemann's removable singularity theorem) Suppose that $\Omega \subset \mathbb{C}$ is open, $w \in \Omega$ and $f : \Omega \to \mathbb{C}$ is holomorphic. Then the function

$$z \mapsto \frac{f(z) - f(w)}{z - w},$$

is holomorphic on $\Omega \setminus \{w\}$ and has a removable singularity at w, i.e. it extends to a holomorphic function on Ω.

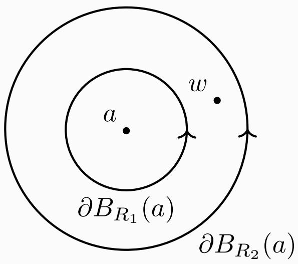
Proof The function is clearly holomorphic on $\Omega \setminus \{w\}$, and by applying Riemann's removable singularity theorem on a ball around w, we are done. ■

Proof of Cauchy's integral formula on annuli Fix $w \in A$. Plugging our function $z \mapsto \frac{f(z)-f(w)}{z-w}$ into Cauchy's theorem for annuli gives

$$\int_{\partial A} \frac{f(z) - f(w)}{z - w} dz = 0,$$

and so

$$\begin{aligned}\int_{\partial A} \frac{f(z)}{z - w} dz &= \int_{\partial A} \frac{f(w)}{z - w} \\ &= f(w) \left(\int_{\partial B_{R_2}(a)} \frac{dz}{z - w} - \int_{\partial B_{R_1}(a)} \frac{dz}{z - w} \right) \\ &= 2\pi \mathrm{i} f(w) \left(I\left(\partial B_{R_2}(a), w\right) - I\left(\partial B_{R_1}(a), w\right) \right) \\ &= 2\pi \mathrm{i} f(w),\end{aligned}$$



where we use the notation $\partial B_R(a)$ to refer to the curve $t \mapsto a + Re^{\mathrm{i}t}$ for $t \in [0, 2\pi]$. The winding numbers are evident from the sketch. ■

8.6 Laurent series 2

Theorem 8.5 (Laurent's theorem) Suppose $0 \leqslant r_1 < r_2$, $a \in \mathbb{C}$ and f is holomorphic on the annulus $A = \{z \in \mathbb{C} : r_1 < |z - a| < r_2\}$. Then there exist unique coefficients $a_k \in \mathbb{C}$ for $k \in \mathbb{Z}$ such that the double-ended power series $\sum_{k \in \mathbb{Z}} a_k (z - a)^k$ converges to $f(z)$ for all $z \in A$. Moreover, the coefficients are given by

$$a_k = \frac{1}{2\pi \mathrm{i}} \int_{\partial B_r(a)} \frac{f(w)}{(w - a)^{k+1}} dw, \tag{8.6.1}$$

for any $r \in (r_1, r_2)$.

Proof WLOG. $a = 0$. Fix $z \in A$, then choose R_1, R_2 such that $r_1 < R_1 < |z| < R_2 < r_2$. By Cauchy's theorem for annuli

$$a_k = \frac{1}{2\pi \mathrm{i}} \int_{\partial B_{R_1}(0)} \frac{f(w)}{w^{k+1}} dw = \frac{1}{2\pi \mathrm{i}} \int_{\partial B_{R_2}(0)} \frac{f(w)}{w^{k+1}} dw.$$

Cauchy's integral formula for annuli implies

$$f(z) = \frac{1}{2\pi \mathrm{i}} \int_{\partial B_{R_2}(0)} \frac{f(w)}{w - z} dw - \frac{1}{2\pi \mathrm{i}} \int_{\partial B_{R_1}(0)} \frac{f(w)}{w - z} dw.$$

The first integral can be handled as in the proof of Taylor's theorem, giving

$$\frac{1}{2\pi \mathrm{i}} \int_{\partial B_{R_2}(0)} \frac{f(w)}{w-z} dw = \sum_{k=0}^{\infty} \frac{1}{2\pi \mathrm{i}} \int_{\partial B_{R_2}(0)} \frac{f(w)}{w^{k+1}} dw z^k = \sum_{k=0}^{\infty} a_k z^k.$$

For the second integral, we can write

$$\frac{1}{w-z} = \frac{1}{z}\left[\frac{1}{1-{}^{w}\!/\!_{z}}\right] = \frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{w}{z}\right)^k,$$

for any $w \in \partial B_{R_1}(0)$ which converges since $|w| < |z|$. Hence,

$$\begin{aligned}\frac{1}{2\pi \mathrm{i}} \int_{\partial B_{R_1}(0)} \frac{f(w)}{w-z} dw &= \frac{1}{2\pi \mathrm{i}} \int_{\partial B_{R_1}(0)} \sum_{k=0}^{\infty} \frac{f(w)}{w^{k+1}} dw z^k \\ &= \sum_{k=-\infty}^{-1} \left(\frac{1}{2\pi \mathrm{i}} \int_{\partial B_{R_1}(0)} \frac{f(w)}{w^{k+1}} dw\right) z^k \\ &= \sum_{k=-\infty}^{-1} a_k z^k.\end{aligned}$$

■

Example 8.4 The function $f(z) = \frac{1}{1-z}$ is holomorphic on the disc D and on the annulus $\mathbb{C} \setminus \overline{D}$. On D we have the Taylor series $f(z) = \sum_{k=0}^{\infty} z^k$, and on $\mathbb{C} \setminus \overline{D}$ we instead have the Laurent series

$$\frac{1}{1-z} = -\frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{1}{z}\right)^k = \sum_{k=-\infty}^{-1}(-1)z^k.$$

8.7 Classification of isolated singularities using Laurent series

Suppose $f : D \setminus \{0\} \to \mathbb{C}$ is holomorphic with Laurent series $f(z) = \sum_{k \in \mathbb{Z}} a_k z^k$ on $D \setminus \{0\}$. We define the order of f at some point a to be

$$\operatorname{ord}(f, a) := \inf\{n \in \mathbb{Z} : a_n \neq 0\}.$$

Let $a = 0$, if

(1) $\operatorname{ord}(f, 0) \geqslant 0$ then the Laurent series is a Taylor series, giving a holomorphic function on the whole of D. Hence, f has a ***removable singularity*** at 0.

(2) $\operatorname{ord}(f, 0) < 0$ and $\operatorname{ord}(f, 0) \neq -\infty$ iff. f has a ***pole*** at 0. In this case, $\operatorname{ord}(f, 0) = -n$.

(3) $\operatorname{ord}(f, 0) = -\infty$ iff. f has an ***essential singularity*** at 0.

Note If f is identically zero then we define $\operatorname{ord}(f, 0) = \infty$.

8.8 Classification of injective entire functions

Theorem 8.6 (Injective entire functions are linear) Suppose that $f : \mathbb{C} \to \mathbb{C}$ is holomorphic and injective. Then f is linear, i.e. $f(z) = \alpha z + \beta$ for some $\alpha, \beta \in \mathbb{C}$ with $a \neq 0$.

Proof Define $g : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ defined by $g(z) = f({}^{1}\!/\!_{z})$ is clearly holomorphic and injective by composition.

We claim that g has a pole at 0. If the isolated singularity at 0 were removable then g would be bounded on some neighbourhood of 0, say $\overline{D} \setminus \{0\}$, and hence f would be bounded on some neighbourhood of ∞, say $\mathbb{C} \setminus D$. But f is continuous and thus bounded on $\overline{D}$, so f would be bounded on $\mathbb{C}$. By Liouville's theorem, f would be constant, contradicting the injectivity of f.

If g had an essential singularity at 0, then by the Casorati-Weierstrass theorem, $g(D \setminus \{0\})$ would be dense in $\mathbb{C}$. This implies that $f(\mathbb{C} \setminus \overline{D})$ is dense in $\mathbb{C}$. But $f(D)$ is an open set by the open mapping theorem, and so there must be some intersection of $f(\mathbb{C} \setminus \overline{D})$ and $f(D)$, which contradicts the injectivity of f.

If we Taylor expand f about 0 we get

$$f(z) = \sum_{k=0}^{\infty} a_k z^k,$$

then the Laurent series of g about 0 is

$$g(z) = \sum_{k=-\infty}^{0} a_k z^k.$$

Since g has a pole at 0, we must have $a_k = 0$ for all $k > n$. Thus, f is a polynomial.

The Fundamental theorem of algebra implies that f is a polynomial of degree at most 1, i.e. $f(z) = \alpha z + \beta$. ■

9 The general form of Cauchy's theorem

9.1 Chains and cycles

Suppose $\Omega \subset \mathbb{C}$ is open and $f : \Omega \to \mathbb{C}$ is a continuous function. Given two piecewise $\mathcal{C}^1$ curves $\gamma_1 : [a, b] \to \Omega$ and $\gamma_2 : [a', b'] \to \Omega$, we can make formal definition

$$\int_{\gamma_1+\gamma_2} f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz.$$

More generally, given a finite collection of piecewise $\mathcal{C}^1$ curves $\gamma_1, \ldots, \gamma_n$ and weights $\alpha_1, \ldots, \alpha_n \in \mathbb{Z}$, we can consider the formal linear combination

$$\gamma \coloneqq \alpha_1\gamma_1 + \cdots + \alpha_n\gamma_n : [a, b] \to \Omega$$

and define the integral

$$\int_{\gamma} f(z)dz \coloneqq \sum_{k=1}^{n} \alpha_k \int_{\gamma_k} f(z)dz.$$

Loosely speaking, we can think of γ as a ***chain***. A ***cycle*** is, strictly speaking, one of these chains that can be represented in terms of curves $\gamma_1, \ldots, \gamma_n$ that are each *closed* curves. This allows us to define the winding number as

$$I(\gamma, w) \coloneqq \sum_{k=1}^{n} \alpha_k I(\gamma_k, w).$$

Definition 9.1 Let $\Omega \subset \mathbb{C}$ be open. A cycle γ in Ω is *homologous to zero* in Ω if for any $a \in \mathbb{C} \setminus \Omega$ we have

$$I(\gamma, a) = 0.$$

Example 9.1 Let $\gamma_1, \gamma_2 : [0, 2\pi] \to \Omega \coloneqq \mathbb{C}\{0\}$ be the curves given by $\gamma_1(\theta) = \mathrm{e}^{\mathrm{i}\theta}$ and $\gamma_2(\theta) = 2\mathrm{e}^{\mathrm{i}\theta}$. Then the cycle $\gamma = \gamma_1 - \gamma_2$ is homologous to zero in Ω but the individual curves γ_1 and γ_2 are not.

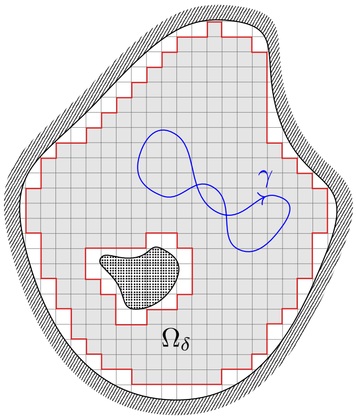
9.2 The homological version of Cauchy's theorem

Theorem 9.1 (Cauchy's theorem - homological version) Let $\Omega \subset \mathbb{C}$ be open and $f : \Omega \to \mathbb{C}$ holomorphic. For any cycle γ that is homologous to zero in Ω we have

$$\int_{\gamma} f(z)dz = 0.$$

If γ isn't homologous to zero, then there exists some $a \in \mathbb{C} \setminus \Omega$ such that $I(\gamma, a) \neq 0$, and we can define a holomorphic function $f : \Omega \to \mathbb{C}$ by $f(z) = \frac{1}{z-a}$, giving

$$\int_{\gamma} f(z)dz = \int_{\gamma} \frac{dz}{z-a} = I(\gamma, a) \neq 0.$$



Sketch of the proof for the homological version of Cauchy's theorem.

Proof WLOG. we may assume that Ω is bounded. By compactness, we may define

$$2\delta := \inf\{|x-y| : x \in \gamma, y \in \partial\Omega\} > 0.$$

We would like to shrink Ω to a smaller domain Ω_δ such that $\partial\Omega_\delta \subset \Omega$ is a piecewise $\mathbb{C}^1$ curve on which f is defined. This motivates the following.

Consider a grid of squares with width δ on $\mathbb{C}$

$$\mathcal{G} := \{[x, x+\delta] \times [\mathrm{i}y, \mathrm{i}(y+\delta)] : x, y \in \mathbb{Z}\}.$$

Denote by $\{Q_j\}_{j=1}^J$ the finite collection of squares fully contained in $\mathcal{G} \cap \Omega$. Define $\Omega_\delta := \mathrm{interior}(\cup_{j=1}^J Q_j)$, such that $\partial\Omega_\delta \subset \Omega$ is a piecewise $\mathbb{C}^1$ curve (see sketch).

Pick w in the interior of some Q_{j_0}. By Cauchy's integral formula (on squares) we have

$$f(w) = \frac{1}{2\pi\mathrm{i}} \int_{\partial Q_{j_0}} \frac{f(z)}{z-w} dz.$$

For any other square Q_j we have

$$\frac{1}{2\pi\mathrm{i}} \int_{\partial Q_j} \frac{f(z)}{z-w} dz = 0$$

by Cauchy's theorem on star-shaped domains. By cancellation, we have that for any $w \in \Omega_\delta$

$$f(w) = \frac{1}{2\pi\mathrm{i}} \int_{\partial\Omega_\delta} \frac{f(z)}{z-w} dz. \tag{9.2.1}$$

By definition of δ, the image of γ is fully contained in Ω_δ. Also, for every $z \in \mathbb{C} \setminus \Omega_\delta$, and in particular for every $z \in \partial\Omega_\delta$, we have

$$I(\gamma, z) = 0, \tag{9.2.2}$$

as follows from Lemma 4.3 and our assumption that γ is homologous to zero. Integrating (9.2.1) over γ gives

$$\begin{aligned} \int_\gamma f(w) dw &= \int_\gamma \frac{1}{2\pi\mathrm{i}} \left(\int_{\partial\Omega_\delta} \frac{f(z)}{z-w} dz \right) dw = \int_{\partial\Omega_\delta} f(z) \left(\frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{dw}{z-w} \right) dz \\ &= -\int_{\partial\Omega_\delta} f(z) I(\gamma, z) dz = 0, \end{aligned}$$

by (9.2.2). ■

9.3 The general version of Cauchy's integral formula

Corollary 9.1 (Cauchy's integral formula - general version) Let $\Omega \subset \mathbb{C}$ be open and $f : \Omega \to \mathbb{C}$ holomorphic.

For any cycle γ that is homologous to zero in Ω and any $w \in \Omega$ with $w \notin \gamma$ we have

$$f(w)I(\gamma, w) = \frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{f(z)}{z-w} dz.$$

Proof By Corollary 8.1, the function $g(z) = \frac{f(z)-f(w)}{z-w}$ can be extended to a holomorphic function on $\Omega \setminus \{w\}$. The homological version of Cauchy's theorem we have $\int_\gamma g(z)dz = 0$ and hence

$$\frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{f(z)}{z-w} dz = \frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{f(w)}{z-w} dz = f(w)I(\gamma, w).$$

■

9.4 The deformation theorem; Cauchy's theorem on simply connected domains

If Ω is simply connected, then by Corollary 4.1 then any closed curve $\gamma \in \Omega$ has zero winding number around any $w \in \mathbb{C} \setminus \Omega$. I.e. any cycle γ is homologous to zero in Ω. This implies Cauchy's theorem on simply connected domains.

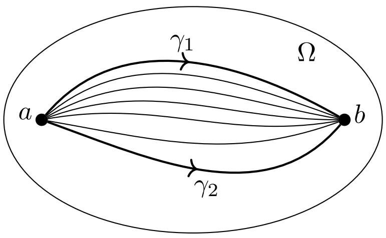
One way of constructing a closed curve γ in Ω is to start with a curve γ_1 in Ω and deform it continuously to a curve γ_2 in Ω such that γ_1 and γ_2 have the same endpoints. This is the content of the following theorem.

Theorem 9.2 (Deformation theorem on simply connected domains) Let $\Omega \subset \mathbb{C}$ be simply connected and $f : \Omega \to \mathbb{C}$ holomorphic. If $\gamma_1, \gamma_2 : [a, b] \to \Omega$ are two piecewise $\mathcal{C}^1$ curves with $\gamma_1(a) = \gamma_2(a)$ and $\gamma_1(b) = \gamma_2(b)$, then

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz.$$

Lemma 9.1 Suppose $\Omega \subset \mathbb{C}$ is an open set and $\gamma_1, \gamma_2 : [a, b] \to \Omega$ are two piecewise $\mathcal{C}^1$ curves with $\gamma_1(a) = \gamma_2(a)$ and $\gamma_1(b) = \gamma_2(b)$. Then $\gamma_1 - \gamma_2$, viewed as a closed curve, is homotopic to a constant curve iff. γ_1 is homotopic to γ_2.

Proof is non-examinable.



Homotopy of curves

Theorem 9.3 (Deformation theorem) Let $\Omega \subset \mathbb{C}$ be open and $f : \Omega \to \mathbb{C}$ holomorphic. If $\gamma_1, \gamma_2 : [a, b] \to \Omega$ are two piecewise $\mathcal{C}^1$ curves that are homotopic, then

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz.$$

Note The intermediate curves do not retain the piecewise $\mathcal{C}^1$ nature of γ_1 and γ_2, so integration along these curves is not well-defined (with the technology we have developed).

9.5 Residue theorem

Theorem 9.4 (Residue) Suppose that $f : B_\delta(z_0) \setminus \{z_0\} \to \mathbb{C}$ is holomorphic, for some $\delta > 0$, $z_0 \in \mathbb{C}$. The ***residue*** of f at z_0 is defined by

$$\mathrm{Res}(f, z_0) := \frac{1}{2\pi\mathrm{i}} \int_{\partial B_\varepsilon(z_0)} f(z)dz, \tag{9.5.1}$$

for any $\varepsilon \in (0, \delta)$.

Theorem 9.5 (Residue theorem) Let $\Omega \subset \mathbb{C}$ be open. Suppose that f is holomorphic on $\Omega \setminus \mathcal{S}$ where $\mathcal{S}$ is a discrete set (finite number of isolated singularities) that is closed in $\mathbb{C}$. Let γ be a cycle in $\Omega \setminus \mathcal{S}$ that is homologous to zero in Ω. Then there are only finitely many $a \in \mathcal{S}$ such that $I(\gamma, a) \neq 0$, and

$$\int_\gamma f(z)dz = 2\pi \mathrm{i} \sum_{a \in \mathcal{S}} I(\gamma, a) \operatorname{Res}(f, a). \tag{9.5.2}$$

Proof Assume for contraction that

$$\mathcal{A} := \{a \in \mathcal{S} : I(\gamma, a) \neq 0\}$$

is infinite. Then we can pick a sequence $\{a_n\}_{n=1}^\infty \subset \mathcal{A}$ such that $a_n \neq a_m$ for $n \neq m$. By boundedness of $\mathcal{A}$, we can find a subsequence $\{a_{n_k}\}_{k=1}^\infty$ such that $a_{n_k} \to a_\infty \in \partial\Omega$. More precisely $a_\infty \in \partial\Omega \notin \Omega$ because otherwise it would be an accumulation point in $\mathcal{S}$ which we require to be discrete. So by Lemma 4.3, $I(\gamma, a_\infty) = 0$ for all a in some neighbourhood of a_∞. But this contradicts the assumption that $a_{n_k} \in \mathcal{A}$ for all $k \in \mathbb{N}$. Hence, $\mathcal{A}$ is finite.

We may write $\mathcal{A} = \{a_1, \dots, a_n\}$ and choose $\varepsilon > 0$ such that $B_{2\varepsilon}(a_k) \setminus \{a_k\} \subset \Omega \setminus \mathcal{S}$ for all $k \in \{1, \dots, n\}$. Let $\gamma_k : [0, 1] \to \Omega \setminus \mathcal{S}$ be the circle of radius ε centred at a_k, that is

$$\gamma_k(\theta) = a_k + \varepsilon \mathrm{e}^{2\pi \mathrm{i}\theta}.$$

Notice that $I(\gamma_k, a_k) = 1$, and $I(\gamma_k, a) = 0$ for all $a \in \mathcal{S} \setminus \{a_k\}$.

Define $n_k := I(\gamma, a_k)$ and consider the cycle

$$\Gamma = \gamma - \sum_{k=1}^n n_k \gamma_k.$$

By construction, $I(\Gamma, a) = 0$ for all $a \in \mathcal{S}$. Moreover, $I(\Gamma, a) = 0$ for all $a \in \mathbb{C} \setminus \Omega$. Hence, by the general Cauchy theorem applied on $\Omega \setminus \mathcal{S}$ we have $\int_\Gamma f(z)dz = 0$, i.e.

$$\int_\gamma f(z)dz = \sum_{k=1}^n n_k \int_{\gamma_k} f(z)dz = \sum_{k=1}^n I(\gamma, a_k) \operatorname{Res}(f, a_k).$$

■

9.6 Evaluation of residues

In practice, it may be hard to compute residues directly from the integral definition in (9.5.1).

Removable singularities

If f has a removable singularity at z_0, then f is bounded in a neighbourhood of z_0, and hence the integral in (9.5.1) is zero.

Example 9.2 Consider $f(z) = \frac{\sin z}{z}$. Then f has a removable singularity at $z = 0$, and $\operatorname{Res}(f, 0) = 0$.

Simple poles

If f has a simple pole at z_0, then by Theorem 8.2 we can write $f(z) = \frac{g(z)}{z - z_0}$ for some holomorphic function $g : B_\delta(z_0) \to \mathbb{C}$ with $g(z_0) \neq 0$. Substituting into (9.5.1) gives

$$\operatorname{Res}(f, z_0) = \frac{1}{2\pi \mathrm{i}} \int_{\partial B_\varepsilon(z_0)} \frac{g(z)}{z - z_0} dz = g(z_0),$$

by Cauchy's integral formula.

Example 9.3 Consider $f(z) = \frac{1}{z^2+1} = \frac{1}{(z-\mathrm{i})(z+\mathrm{i})}$, with simple poles at $z_0 = \pm\mathrm{i}$. So $\operatorname{Res}(f, \pm\mathrm{i}) = g(\pm\mathrm{i}) = \pm\frac{1}{2\mathrm{i}}$.

Rationals, with at worst a simple pole

If $f(z) = \frac{p(z)}{q(z)}$ where $p, q : B_\delta(z_0) \to \mathbb{C}$ are holomorphic functions with $q(z_0) = 0$ but $q'(z_0) \neq 0$, i.e. f has a simple pole at z_0, then we can write

$$f(z) = \frac{g(z)}{z - z_0} \quad \text{for} \quad g(z) = \frac{p(z)}{\left(\frac{q(z)}{z-z_0}\right)}.$$

Then g has a removable singularity at z_0, and can be extended to $B_\delta(z_0)$ by setting

$$g(z_0) = \lim_{z \to z_0} g(z) = \frac{p(z_0)}{q'(z_0)}.$$

Hence, by the previous case, we have

$$\operatorname{Res}(f, z_0) = \frac{p(z_0)}{q'(z_0)}.$$

Example 9.4 The residue of $\frac{1}{\sin z}$ at $z = 0$ is $\frac{1}{\cos 0} = 1$.

Higher order poles

If f has a pole of order n at z_0, then by Theorem 8.2 we can write $f(z) = \frac{g(z)}{(z-z_0)^n}$ for some holomorphic function $g : B_\delta(z_0) \to \mathbb{C}$ with $g(z_0) \neq 0$. Substituting into (9.5.1) gives

$$\operatorname{Res}(f, z_0) = \frac{1}{2\pi \mathrm{i}} \int_{\partial B_\varepsilon(z_0)} \frac{g(z)}{(z - z_0)^n} dz = \frac{g^{(n-1)}(z_0)}{(n-1)!}, \tag{9.6.1}$$

by Cauchy's integral formula. We can rewrite this in terms of f, giving

$$\operatorname{Res}(f, z_0) = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} \Big[(z - z_0)^n f(z)\Big].$$

Example 9.5 Consider $f(z) = \frac{1}{z^2(z-1)}$. Then f has a pole of order 2 at $z = 0$ and a simple pole at $z = 1$. So $\operatorname{Res}(f, 0) = \frac{1}{1!} \lim_{z \to 0} \frac{d}{dz} \left[z^2 f(z)\right] = \lim_{z \to 0} \frac{d}{dz} \left[\frac{1}{z-1}\right] = -1$.

Example 9.6 $\operatorname{Res}\left(\frac{\cos z}{z^3}, 0\right) = \frac{1}{2!} \lim_{z \to 0} \frac{d^2}{dz^2} \left[z^3 \frac{\cos z}{z^3}\right] = \frac{1}{2!} \lim_{z \to 0} \frac{d^2}{dz^2} [\cos z] = -\frac{1}{2}$.

Essential singularities

If f has an essential singularity at z_0, then we cannot compute the residue of f at z_0 directly from the integral definition in (9.5.1). Instead, we appeal to (8.6.1) to find that

$$\operatorname{Res}(f, z_0) = a_{-1}.$$

Example 9.7 Consider $f(z) = \mathrm{e}^{\frac{1}{z}}$. Then f has an essential singularity at $z = 0$, and $\operatorname{Res}(f, 0) = a_{-1} = \frac{1}{0!} = 1$.

9.7 Computing real integrals using residues

We claim that

$$I := \int_0^{2\pi} \frac{4 \sin^2 \theta}{5 + 4 \cos \theta} d\theta = \pi.$$

We can rewrite this as a complex integral over the $\mathcal{C}^1$ curve $\gamma : [0, 2\pi] \to \mathbb{C}$ given by $\gamma(\theta) = \mathrm{e}^{\mathrm{i}\theta}$, i.e.

$$I = \int_\gamma \frac{f(z)}{\mathrm{i}z} dz = \int_0^{2\pi} \frac{f(\gamma(\theta))}{\mathrm{i}\gamma(\theta)} \gamma'(\theta) d\theta = \int_0^{2\pi} f(\mathrm{e}^{\mathrm{i}\theta}) d\theta.$$

Recalling Definition 3.2, we define $z = \mathrm{e}^{\mathrm{i}\theta}$ such that the integrand becomes

$$f(z) = \frac{4\left(\frac{z-\frac{1}{z}}{2\mathrm{i}}\right)^2}{5+4\left(\frac{z+\frac{1}{z}}{2}\right)} = \frac{-z\left(z-\frac{1}{z}\right)^2}{(2z+1)(z+2)},$$

and thus

$$I = \int_\gamma F(z)dz, \qquad \text{where } F(z) = \frac{\mathrm{i}\left(z-\frac{1}{z}\right)^2}{(2z+1)(z+2)}.$$

γ encloses a double pole at 0 and a simple pole at $-\frac{1}{2}$. We can compute

$$\operatorname{Res}\left(F(z), -\frac{1}{2}\right) = \frac{3}{4}\mathrm{i},$$

by rewriting

$$F(z) = \left[\frac{\mathrm{i}\left(z-\frac{1}{z}\right)^2}{2(z+2)}\right]\frac{1}{z-\left(\frac{1}{2}\right)}.$$

If we expand brackets of the numerator as z^2+2+z^{-2} we notice that the first two terms are holomorphic near $z=0$. Therefore,

$$\operatorname{Res}(F,0) = \operatorname{Res}\left(\frac{\mathrm{i}}{z^2(2z+1)(z+2)}, 0\right).$$

We can write $G(z) = \frac{g(z)}{z^2}$ for $g(z) = \frac{\mathrm{i}}{(2z+1)(z+2)}$, and use (9.6.1) to find

$$\operatorname{Res}(G,0) = \frac{1}{2-1!}g'(0) = -\mathrm{i}\frac{4(0)+5}{((2(0)+1)(0+2))^2} = -\frac{5\mathrm{i}}{4}.$$

By the Residue theorem we have

$$I = 2\pi\mathrm{i}\left(-\frac{5\mathrm{i}}{4}+\frac{3\mathrm{i}}{4}\right) = \pi.$$

9.8 The argument principle

Definition 9.2 A closed continuous path $\gamma : [a,b] \to \mathbb{C}$ is ***simple*** if its restriction to $[a,b)$ is injective, i.e. $\gamma(t_1) \neq \gamma(t_2)$ for all $t_1, t_2 \in [a,b)$ with $t_1 \neq t_2$.

Definition 9.3 A simple closed continuous path $\gamma : [a,b] \to \mathbb{C}$ bounds an open set $A \subset \mathbb{C}$ in a ***positive*** direction if $\mathbb{C} \setminus \gamma([a,b])$ has two connected components, one of which is A, and $I(\gamma, z) = 1$ for every $z \in A$.

Suppose that $f : \Omega \to \mathbb{C}_\infty$ is meromorphic on $\Omega \subset \mathbb{C}$ and $f \neq 0$. Let $\mathcal{P} \subset \Omega$ be the set of poles of f, and $\mathcal{Z} \subset \Omega$ be the set of zeros of f. Both are discrete and closed sets.

Definition Given some $A \subset \Omega$, we define

$$\mathcal{Z}_A(f) = \sum_{z\in\mathcal{Z}\cap A} \operatorname{ord}(f,z), \qquad \mathcal{P}_A(f) = \sum_{z\in\mathcal{P}\cap A}[-\operatorname{ord}(f,z)],$$

i.e. the number of zeros and poles counting multiplicity.

Theorem 9.6 (Argument principle) Suppose that $f : \Omega \to \mathbb{C}_\infty$ is meromorphic on $\Omega \subset \mathbb{C}$ and $f \neq 0$. Let $\gamma : [a,b] \to \Omega \setminus (\mathcal{P} \cup \mathcal{Z})$ be a piecewise $\mathcal{C}^1$ simple closed path that bounds an open set $A \subset \Omega$ in a positive direction. Then

$$\mathcal{Z}_A(f) - \mathcal{P}_A(f) = \frac{1}{2\pi\mathrm{i}}\int_\gamma \frac{f'(z)}{f(z)}dz. \tag{9.8.1}$$

In fact, we can rewrite the integral in (9.8.1) as

$$\int_a^b \frac{f'(\gamma(t))}{f(\gamma(t))}\gamma'(t)dt = \int_a^b \frac{(f\circ\gamma)'(t)}{f\circ\gamma(t)}dt = \int_{f\circ\gamma}\frac{dz}{z}.$$

Corollary 9.2 (Argument principle) $\mathcal{Z}_A(f) - \mathcal{P}_A(f) = I(f \circ \gamma, 0)$. (9.8.2)

Proof Near a zero or a pole of f (at z_0), we have $f(z) = (z - z_0)^n g(z)$ for some $n \in \mathbb{Z} \setminus \{0\}$ and some holomorphic function $g : B_\delta(z_0) \to \mathbb{C}$ with $g(z_0) \neq 0$. Then

$$\begin{aligned} \frac{f'(z)}{f(z)} &= \frac{n(z-z_0)^{n-1}g(z) + (z-z_0)^n g'(z)}{(z-z_0)g(z)} \\ &= \frac{n}{z-z_0} + \frac{g'(z)}{g(z)}. \end{aligned}$$

Hence, the integrand in (9.8.1) has a simple pole at z_0 with residue n. By the Residue theorem we have

$$\operatorname{Res}\left(\frac{f'(z)}{f(z)}, z_0\right) = n = \operatorname{ord}(f, z_0).$$

■

9.9 Rouché's theorem

Theorem 9.7 (Rouché's theorem) Suppose that $g, G : \Omega \to \mathbb{C}$ are holomorphic on $\Omega \subset \mathbb{C}$ and $\gamma : [a,b] \to \Omega$ is a piecewise $\mathcal{C}^1$ simple closed curve that bounds an open set $A \subset \Omega$ in a positive direction. If $\forall z \in \gamma$, $|G(z)| > |g(z)|$, then G and $G + g$ have the same number of zeros in A.

Note Neither G nor $G + g$ have zeros on the image of γ.

Proof We can apply the dog walking lemma to the curves $G \circ \gamma$ and $(G + g) \circ \gamma$ to find that

$$I(G \circ \gamma, 0) = I((G+g) \circ \gamma, 0).$$

By the argument principle we have $\mathcal{Z}_A(G) = \mathcal{Z}_A(G + g)$. ■

Example 9.8 Show that $z^5 + 15z + 1$ has exactly four zeros in the annulus $\{z \in \mathbb{C} : 1 < |z| < 2\}$.

Proof First, we apply Rouché with $\Omega = \mathbb{C}, A = B_2(0), \gamma : [0, 2\pi] \to \mathbb{C}$ given by $\gamma(\theta) = 2\mathrm{e}^{\mathrm{i}\theta}$, $G(z) = z^5$, and $g(z) = 15z + 1$. We have

$$\left|z^5\right| = 32 > 31 = 15|z| + 1 \geqslant |15z + 1| \quad \text{for all } z \in \gamma.$$

Hence, $z^5 + 15z + 1$ has exactly five zeros in $B_2(0)$.

Next, we apply Rouché with $\Omega = \mathbb{C}, A = B_1(0), \gamma : [0, 2\pi] \to \mathbb{C}$ given by $\gamma(\theta) = \mathrm{e}^{\mathrm{i}\theta}$, $G(z) = 15z$, and $g(z) = z^5 + 1$. We have

$$|15z| = 15 > 2 = |z|^5 + 1 \geqslant \left|z^5 + 1\right| \quad \text{for all } z \in \gamma.$$

Therefore, $z^5 + 15z + 1$ has exactly one zero in $B_1(0)$.

We conclude that $z^5 + 15z + 1$ has exactly four zeros in the annulus $\{z \in \mathbb{C} : 1 < |z| < 2\}$. ■

10 Sequences of holomorphic functions

10.1 Weierstrass convergence theorem

Theorem 10.1 (Weierstrass convergence theorem) Let $\Omega \subset \mathbb{C}$ be open and let $f_n : \Omega \to \mathbb{C}$ be a sequence of holomorphic functions converging locally uniformly to a function $f : \Omega \to \mathbb{C}$. Then

(i) f is holomorphic,

(ii) $f_n^{(k)} \to f^{(k)}$ locally uniformly $\forall k \in \mathbb{N}$.

Recall weierstrass A sequence f_n converges locally uniformly to f if for every compact subset $K \subset \Omega$, the convergence is uniform on K, i.e. $f_n|_K \rightrightarrows f|_K$.

Proof of (i) Since f is the local uniform limit of holomorphic functions, it is continuous. By Morera's theorem, it suffices to show that the integral of f around any triangle is zero. Let T be a triangle in Ω. By Goursat's theorem, we have

$$\int_{\partial T} f(z)dz = \lim_{n\to\infty} \int_{\partial T} f_n(z)dz = 0.$$

■

Proof of (ii) Suppose $K \subset \Omega$ is the compact set on which we would like to show that $f_n' \rightrightarrows f'$. Choose $\delta > 0$ such that $B_{2\delta}(z) \subset \Omega$ for all $z \in K$. Define the compact set

$$K_\delta := \bigcup_{z\in K} \overline{B_\delta(z)}.$$

By the Cauchy integral formula, for all $z \in K$ and $k \in \mathbb{N}$, we have

$$f_n^{(k)}(z) - f^{(k)}(z) = \frac{k!}{2\pi i} \int_{\partial B_\delta(z)} \frac{f_n(w) - f(w)}{(w-z)^{k+1}} dw,$$

and so by (3.4.1), we have

$$|f_n^{(k)}(z) - f^{(k)}(z)| \leqslant \frac{k!}{2\pi}(2\pi\delta) \sup_{w\in\partial B_\delta(z)} \frac{|f_n(w) - f(w)|}{\delta^{k+1}} \leqslant \frac{k!}{\delta^k} \sup_{w\in\partial K_\delta} |f_n(w) - f(w)|.$$

Since $f_n \to f$ locally uniformly on K_δ, we have that $f_n^{(k)} \to f^{(k)}$ locally uniformly on K. ■

10.2 Hurwitz's theorem

Theorem 10.2 (Hurwitz's theorem) Let $\Omega \subset \mathbb{C}$ be open and connected, and let $f_n : \Omega \to \mathbb{C}$ be a sequence of holomorphic functions converging locally uniformly to a holomorphic function $f : \Omega \to \mathbb{C}$. If for some $k \in \mathbb{N}_0$, none of the functions f_n has more than k zeros (counting multiplicities), then either f has at most k zeros or $f \equiv 0$.

Proof Assume for contradiction that f has more than k zeros, without being identically zero. Theorem 7.2 tells us that all zeros of f must be if finite order. Let $z_1, \dots, z_K$ be the (isolated) zeros of f with multiplicities $m_1, \dots, m_K$, such that $\sum_{j=1}^K m_j > k$.

We can choose $\delta > 0$ such that the K closed balls $\overline{B_\delta(z_j)}$ are pairwise disjoint and contained in Ω, with no zeros in any $\overline{B_\delta(z_j)} \setminus \{z_j\}$. Let

$$\Sigma := \bigcup_{j=1}^K \partial B_\delta(z_j).$$

By compactness of Σ and continuity of $|f|$, we can define

$$\varepsilon := \max_{z\in\Sigma} |f(z)| > 0.$$

Since $f_n \to f$ locally uniformly, we can choose $N \in \mathbb{N}$ such that $|f_n(z) - f(z)| < \varepsilon$ for all $z \in \Sigma$ and $n \geqslant N$. Rouché's theorem implies that f_n has exactly m_j zeros in $B_\delta(z_j)$ for all $n \geqslant N$, such that $\sum_{j=1}^K m_j > k$, which is a contradiction. ■

Corollary 10.1 Any function $f : \Omega \to \mathbb{C}$ that is the local uniform limit of ***injective*** holomorphic functions $f_n : \Omega \to \mathbb{C}$, is either injective or constant.

Proof Suppose that f is neither injective nor constant. Then there exist $z_1 \neq z_2$ such that $f(z_1) = f(z_2) := w$.

For each $n \in \mathbb{N}$, the function $f_n(z) - w$ has at most one zero, by injectivity of f_n. Since f is not constant, $f(z) - w \not\equiv 0$. Hurwitz's theorem implies that $f(z) - w$ has at most one zero, which is a contradiction. ■

10.3 Compactness: Montel's theorem

Definition 10.1 Let $\Omega \subset \mathbb{C}$ be open. A sequence of functions $f_n : \Omega \to \mathbb{C}$ is ***locally uniformly bounded*** if for all compact $K \subset \Omega$, there exists $M < \infty$ such that $|f_n(z)| \leqslant M$ for all $z \in K$ and $n \in \mathbb{N}$.

Definition 10.2 Let $K \subset \mathbb{C}$ be compact. A sequence of functions $f_n : K \to \mathbb{C}$ is ***equicontinuous*** if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $|z - w| < \delta$ implies $|f_n(z) - f_n(w)| < \varepsilon$ for all $n \in \mathbb{N}$.

Theorem 10.3 (Ascoli-Arzelà's theorem) Let $K \subset \mathbb{C}$ be compact. A sequence of functions $f_n : K \to \mathbb{C}$ is equicontinuous and uniformly bounded, then it has a uniformly convergent subsequence.

Theorem 10.4 (Montel's theorem) Let $\Omega \subset \mathbb{C}$ be open. Every locally uniformly bounded sequence of holomorphic functions $f_n : \Omega \to \mathbb{C}$ has a locally uniformly convergent subsequence.

Proof For each $k \in \mathbb{N}$, let $K_k \subset \Omega$ be the compact set $\overline{B_k(0)} \cap \Omega$, such that $B_{2^{-k}}(z) \subset \Omega$ for all $z \in K_k$. This is a sequence of compact sets such that $K_k \subset K_{k+1}$ for all $k \in \mathbb{N}$, and $\bigcup_{k\in\mathbb{N}} K_k = \Omega$.

Fix $k \in \mathbb{N}$. We want to show that the sequence $f_n|_{K_k}$ is uniformly equicontinuous. Let $\varepsilon > 0$. By local uniform boundedness, there exists $M < \infty$ such that $|f_n(z)| \leqslant M$ for all $z \in K_k$ and $n \in \mathbb{N}$. By equicontinuity, there exists $\delta > 0$ such that $|z - w| < \delta$ implies $|f_n(z) - f_n(w)| < \varepsilon$ for all $n \in \mathbb{N}$.

We appeal to the uniform boundedness on K_{k+1} to find $M < \infty$ such that $|f_n(z)| \leqslant M$ for all $z \in K_{k+1}$ and $n \in \mathbb{N}$. Define $\eta := 2^{-k-1}$ and let $z_1, z_2 \in K_k$ such that $|z_1 - z_2| < \frac{\eta}{2}$. Hence, $B_{2\eta}(z_1) \subset \Omega$ and $B_\eta(z_1) \subset K_{k+1}$. In particular, $|f_n(w)| < M$ for all $w \in B_\eta(z_1)$ and $n \in \mathbb{N}$.

Cauchy's integral formula tells us that for any n, we have

$$\begin{aligned} f_n(z_1) - f_n(z_2) &= \frac{1}{2\pi i} \int_{\partial B_\eta(z_1)} \frac{f_n(w)}{w - z_1} dw - \frac{1}{2\pi i} \int_{\partial B_\eta(z_1)} \frac{f_n(w)}{w - z_2} dw \\ &= \frac{z_1 - z_2}{2\pi i} \int_{\partial B_\eta(z_1)} \frac{f_n(w)}{(w - z_1)(w - z_2)} dw. \end{aligned}$$

For $w \in \partial B_\eta(z_1)$, we have $|w - z_1| = \eta$ and $|w - z_2| \geqslant |w - z_1| - |z_1 - z_2| \geqslant \frac{\eta}{2}$, and so

$$\left| \frac{1}{(w - z_1)(w - z_2)} \right| \leqslant \frac{2}{\eta^2}.$$

Therefore, by (3.4.1), we have

$$|f_n(z_1) - f_n(z_2)| \leqslant \frac{|z_1 - z_2|}{2\pi} (2\pi\eta) \frac{2M}{\eta^2} = |z_1 - z_2| M 2^{k+2}.$$

Hence, the sequence $f_n|_{K_k}$ is uniformly equicontinuous. By Ascoli-Arzelà's theorem, we can extract a subsequence $f_{n_j}|_{K_k}$ that converges uniformly on K_k. We can repeat this process for all $k \in \mathbb{N}$ to obtain a subsequence that converges locally uniformly on any compact set $K \subset K_k \subset \Omega$, for large enough k. ■

11 The Riemann mapping theorem

11.1 Statement and final ingredients

Theorem 11.1 (Riemann mapping theorem) Let $\Omega \subset \mathbb{C}$ be a simply connected open set that is not all of $\mathbb{C}$. Then Ω is conformally equivalent to the unit disk D, i.e. there exists a biholomorphic function $f : D \to \Omega$.

We generalise Corollary 7.1 in the following lemma.

Lemma 11.1 Let $\Omega \subset \mathbb{C}$ be a simply connected open set and let $g : \Omega \to \mathbb{C} \setminus \{0\}$ be holomorphic. Then there exists a holomorphic function $\ell : \Omega \to \mathbb{C}$ such that $g(z) = \mathrm{e}^{\ell(z)}$ for all $z \in \Omega$.

Furthermore, for all $k \in \mathbb{N}$, the function $\psi(z) := \mathrm{e}^{\frac{1}{k}\ell(z)}$ is a holomorphic k^{th} root of $g(z)$, i.e. $\psi^k(z) = g(z)$ for all $z \in \Omega$. If g is injective, then ψ is injective.

Remark 11.1 If $\Omega \subset \mathbb{C} \setminus \{0\}$ is simply connected, then we can take $g(z) = z$ and $\ell(z) = \log z$, giving a well-defined holomorphic function $\log : \Omega \to \mathbb{C}$.

Proof of Lemma 11.1 By Lemma 7.1 it suffices to find an anti-derivative $F(z)$ of the function $f(z) := \frac{g'(z)}{g(z)}$. Fix $z_0 \in \Omega$ and let γ be a piecewise $\mathcal{C}^1$ curve in Ω from z_0 to any other point z. Define

$$F(z) := \int_\gamma f(w)dw.$$

By the Deformation theorem, this definition is independent of the choice of γ, and so F is well-defined. To see that F is holomorphic and satisfies $F'(z) = f(z)$, we write

$$F(z+h) = F(z) + \int_{[z,z+h]} f(w)dw.$$

Differentiating with respect to h and evaluating at $h = 0$ gives $F'(z) = f(z)$, by Corollary 5.1.

If $\psi(z_1) = \psi(z_2)$ for some $z_1, z_2 \in \Omega$, then taking the k^{th} power of both sides gives $g(z_1) = g(z_2)$. Hence, injectivity of g implies injectivity of ψ. ∎

11.2 Proof of the Riemann mapping theorem

To be done.