Extended Kalman Filter (EKF) — Full Mathematical Description

1. True State (Ground Truth)

$$\mathbf{x}_t = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix} \in \mathbb{R}^3$$

This represents the actual (unobservable) system state.

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2. State Estimate (Mean and Covariance)

We assume a Gaussian belief over the state:

$$\mathbf{x}_t \sim \mathcal{N}(\boldsymbol{\mu}_t, \Sigma_t)$$

Where:

- $\boldsymbol{\mu}_t \in \mathbb{R}^3$: estimated state mean
- $\Sigma_t \in \mathbb{R}^{3 \times 3}$: covariance of the estimate

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3. Control Input

$$\mathbf{u}_t = \begin{bmatrix} v \\ \omega \end{bmatrix} \in \mathbb{R}^2$$

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4. Motion Model (Nonlinear)

The robot follows a nonlinear motion model:

$$\bar{\boldsymbol{\mu}}_t = g(\boldsymbol{\mu}_{t-1}, \mathbf{u}_t) = \begin{bmatrix} x + v \cdot \Delta t \cdot \cos(\theta) \\ y + v \cdot \Delta t \cdot \sin(\theta) \\ \theta + \omega \cdot \Delta t \end{bmatrix}$$

In reality, the motion is subject to Gaussian noise:

$$\mathbf{x}_t = g(\mathbf{x}_{t-1}, \mathbf{u}_t) + \mathbf{w}_t, \quad \mathbf{w}_t \sim \mathcal{N}(\mathbf{0}, R_t)$$

With Jacobian (w.r.t. state):

$$G_t = \frac{\partial g}{\partial \boldsymbol{\mu}}\bigg|_{\boldsymbol{\mu}_{t-1}, \mathbf{u}_t} = \begin{bmatrix} 1 & 0 & -v \cdot \Delta t \cdot \sin(\theta) \\ 0 & 1 & v \cdot \Delta t \cdot \cos(\theta) \\ 0 & 0 & 1 \end{bmatrix}$$

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5. Observation Model

Assuming we can observe position and orientation (x, y, θ):

$$\hat{\mathbf{z}}_t = h(\bar{\boldsymbol{\mu}}_t) = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix}$$

And the actual observation is:

$$\mathbf{z}_t = h(\mathbf{x}_t) + \mathbf{v}_t, \quad \mathbf{v}_t \sim \mathcal{N}(\mathbf{0}, Q_t)$$

With observation Jacobian:

$$H_t = \frac{\partial h}{\partial \mathbf{x}}\bigg|_{\bar{\boldsymbol{\mu}}_t} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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6. Covariances

- R_t: process noise covariance matrix (uncertainty in motion model)
- Q_t: measurement noise covariance matrix (sensor uncertainty)

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7. EKF Algorithm (with Formula Blocks)

$$\begin{aligned}
&\textbf{Prediction step:} \\
&\qquad \bar{\boldsymbol{\mu}}_t = g(\boldsymbol{\mu}_{t-1}, \mathbf{u}_t) \\
&\qquad \bar{\Sigma}_t = G_t\, \Sigma_{t-1}\, G_t^\top + R_t \\
&\textbf{Correction step:} \\
&\qquad K_t = \bar{\Sigma}_t\, H_t^\top \left(H_t\, \bar{\Sigma}_t\, H_t^\top + Q_t \right)^{-1} \\
&\qquad \boldsymbol{\mu}_t = \bar{\boldsymbol{\mu}}_t + K_t \left(\mathbf{z}_t - h(\bar{\boldsymbol{\mu}}_t) \right) \\
&\qquad \Sigma_t = (I - K_t H_t)\, \bar{\Sigma}_t
\end{aligned}$$