T1 summary

Thursday, February 10, 2022 3:49 PM

3 Precise Definition of an Infinite Limit The notation $\lim_{n\to\infty} a_n = \infty$ means that for every positive number M there is an integer N such that

$$\text{if} \quad n > N \quad \text{then} \quad a_n > M$$

4 Theorem If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim_{n\to\infty} a_n = L$.

Limit Laws for Sequences Suppose that $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant. Then

1. $\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$
2. $\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$
3. $\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n$
4. $\lim_{n\to\infty} (a_nb_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$
5. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ if $\lim_{n\to\infty} b_n \neq 0$

6) if f is cont. $\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n)$

Power Law $\lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty} a_n\right]^p$ if $p > 0$ and $a_n > 0$

The Squeeze Theorem can also be adapted for sequences as follows (see Figure 9).

Squeeze Theorem for Sequences If $a_n \le b_n \le c_n$ for $n \ge n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

6 Theorem If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

7 Theorem If $\lim_{n\to\infty} a_n = L$ and the function f is continuous at L, then

$$\lim_{n\to\infty} f(a_n) = f(L) \quad \text{ie } f(\lim_{n\to\infty} a_n) = \lim_{n\to\infty} f(a_n)$$

9 The sequence $\{r^n\}$ is convergent if $-1 < r \le 1$ and divergent for all other values of r.

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

10 Definition A sequence $\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all $n \ge 1$, that is, $a_1 < a_2 < a_3 < \cdots$. It is called **decreasing** if $a_n > a_{n+1}$ for all $n \ge 1$. A sequence is called **monotonic** if it is either increasing or decreasing.

11 Definition A sequence $\{a_n\}$ is **bounded above** if there is a number M such that

$$a_n \le M \qquad \text{for all } n \ge 1$$

A sequence is **bounded below** if there is a number m such that

$$m \le a_n \qquad \text{for all } n \ge 1$$

If a sequence is bounded above and below, then it is called a **bounded sequence**.

12 Monotonic Sequence Theorem Every bounded, monotonic sequence is convergent.

In particular, a sequence that is increasing and bounded above converges, and a sequence that is decreasing and bounded below converges.

4 The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \qquad |r| < 1 \quad \text{ie } -1 < r < 1$$

If $|r| \ge 1$, the geometric series is divergent.

Compare with the improper integral

$$\int_1^\infty f(x)\,dx = \lim_{t\to\infty} \int_1^t f(x)\,dx$$

To find this integral we integrate from 1 to t and then let $t \to \infty$. For a series, we sum from 1 to n and then let $n \to \infty$.

6 Theorem If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

7 Test for Divergence If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

8 Theorem If Σa_n and Σb_n are convergent series, then so are the series Σca_n (where c is a constant), $\Sigma(a_n + b_n)$, and $\Sigma(a_n - b_n)$, and

(i) $\sum_{n=1}^{\infty} ca_n = c \sum_{n=1}^{\infty} a_n$

(ii) $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

(iii) $\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

The Integral Test Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^\infty f(x)\,dx$ is convergent. In other words:

(i) If $\int_1^\infty f(x)\,dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

(ii) If $\int_1^\infty f(x)\,dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

1 The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \le 1$.

2 Remainder Estimate for the Integral Test Suppose $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \ge n$ and Σa_n is convergent. If $R_n = s - s_n$, then

$$\int_{n+1}^{\infty} f(x)\,dx \le R_n \le \int_n^\infty f(x)\,dx$$

$$s_n + \int_{n+1}^{\infty} f(x)\,dx \le s \le s_n + \int_n^\infty f(x)\,dx$$

$$\sum_{i=1}^{\infty} a_i - \sum_{i=1}^{n} a_i = R_n = a_{n+1} + a_{n+2} + \cdots \le \int_n^\infty f(x)\,dx \qquad R_n = \sum_{i=n+1}^{\infty} a_i$$

$$\sum_{i=1}^{n} a_i + \int_{n+1}^{\infty} f(n) \le \underbrace{\sum_{i=1}^{\infty} a_i}_{s_n + R_n = s} \le \sum_{i=1}^{n} a_i + \int_n^\infty f(n)$$

The Direct Comparison Test Suppose that Σa_n and Σb_n are series with positive terms.

(i) If Σb_n is convergent and $a_n \le b_n$ for all n, then Σa_n is also convergent.

(ii) If Σb_n is divergent and $a_n \ge b_n$ for all n, then Σa_n is also divergent.

In using the Direct Comparison Test we must, of course, have some known series Σb_n for the purpose of comparison. Most of the time we use one of these series:

- A p-series [$\Sigma 1/n^p$ converges if $p > 1$ and diverges if $p \le 1$; see (11.3.1)]
- A geometric series [Σar^{n-1} converges if $|r| < 1$ and diverges if $|r| \ge 1$; see (11.2.4)]

The Limit Comparison Test Suppose that Σa_n and Σb_n are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

Notice that in testing many series we find a suitable comparison series Σb_n by keeping only the highest powers in the numerator and denominator.

Alternating Series Test If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots \qquad (b_n > 0)$$

doesn't matter

satisfies the conditions

postve, dec ↓ (i) $b_{n+1} \le b_n$ for all n

(ii) $\lim_{n\to\infty} b_n = 0$

then the series is convergent.

Alternating Series Estimation Theorem If $s = \Sigma(-1)^{n-1}b_n$, where $b_n > 0$, is the sum of an alternating series that satisfies

postve, dec↓ (i) $b_{n+1} \le b_n$ and (ii) $\lim_{n\to\infty} b_n = 0$

then $|(-1)^n b_n| = b_n$ $\quad |R_n| = |s - s_n| \le b_{n+1}$ th term

1 Definition A series Σa_n is called **absolutely convergent** if the series of absolute values $\Sigma |a_n|$ is convergent.

2 Definition A series Σa_n is called **conditionally convergent** if it is convergent but not absolutely convergent; that is, if Σa_n converges but $\Sigma |a_n|$ diverges.

3 Theorem If a series Σa_n is absolutely convergent, then it is convergent.

The Ratio Test

(dec ↓)

(i) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

The Ratio Test is usually conclusive if the nth term of the series contains an exponential or a factorial, as in Examples 1 and 2. The test will

Therefore, in the case where f is positive and has a continuous derivative, we define the **surface area** of the surface obtained by rotating the curve $y = f(x)$, $a \le x \le b$, about the x-axis as

4 $$S = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\,dx$$

With the Leibniz notation for derivatives, this formula becomes

5 $$S = \int_a^b 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

If the curve is described as $x = g(y)$, $c \le y \le d$, then the formula for surface area becomes

6 $$S = \int_c^d 2\pi y\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy$$

Both formulas 5 and 6 can be summarized symbolically, using the notation for arc length given in Section 8.1, as

7 $$S = \int 2\pi y\,ds$$

For rotation about the y-axis we can use a similar procedure to obtain the following symbolic formula for surface area:

$x = g(y)$ $\qquad S = \int_c^d 2\pi x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy$

8 $$S = \int 2\pi x\,ds$$

where, as before (see Equations 8.1.7 and 8.1.9), we can use either

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx \qquad \text{or} \qquad ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy$$

$$F = mg = \rho gAd$$

where g is the acceleration due to gravity. The **pressure** P on the plate is defined to be the force per unit area:

$$P = \frac{F}{A} = \rho gd$$

The SI unit for measuring pressure is a newton per square meter, which is called a pascal (abbreviation: $1\ \text{N/m}^2 = 1$ Pa). Since this is a small unit, the kilopascal (kPa) is often used. For instance, because the density of water is $\rho = 1000\ \text{kg/m}^3$, the pressure at the bottom of a swimming pool 2 m deep is

$$P = \rho gd = 1000\ \text{kg/m}^3 \times 9.8\ \text{m/s}^2 \times 2\ \text{m}$$
$$= 19{,}600\ \text{Pa} = 19.6\ \text{kPa}$$

62.5 instead of 9800

When using US Customary units, we write $P = \rho gd = \delta d$, where $\boxed{\delta = \rho g}$ is the *weight density* (as opposed to ρ, which is the *mass density*). For instance, the weight density of water is $\delta = 62.5\ \text{lb/ft}^3$, so the pressure at the bottom of a swimming pool 8 ft deep is $P = \delta d = 62.5\ \text{lb/ft}^3 \times 8\ \text{ft} = 500\ \text{lb/ft}^2$.

To solve the linear differential equation $y' + P(x)y = Q(x)$, multiply both sides by the **integrating factor** $I(x) = e^{\int P(x)\,dx}$ and then integrate both sides.

2 Theorem The only solutions of the differential equation $dy/dt = ky$ are the exponential functions

$$y(t) = y(0)e^{kt}$$

Newton's Law of Cooling as a differential equation:

$$\frac{dT}{dt} = k(T - T_s)$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \qquad \text{if} \quad \frac{dx}{dt} \neq 0$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

5 Theorem If a curve C is described by the parametric equations $x = f(t)$, $y = g(t)$, $\alpha \le t \le \beta$, where f' and g' are continuous on $[\alpha, \beta]$ and C is traversed exactly once as t increases from α to β, then the length of C is

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$v(t) = s'(t) = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \qquad s(t) = \int_\alpha^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2}\,du$$

If C is rotated about the x-axis, then the area of the resulting sur-

$$S = \int_\alpha^\beta 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

from the figure, we have $\cos\theta = x/r$ and $\sin\theta = y/r$. So to find the Cartesian coordinates (x, y) when the polar coordinates (r, θ) are known, we use the equations

1 $$x = r\cos\theta \qquad y = r\sin\theta$$

To find polar coordinates (r, θ) when the Cartesian coordinates (x, y) are known, we use the equations

2 $$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x}$$

1 Definition Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b). Then we say that the **limit of $f(x, y)$ as (x, y) approaches (a, b)** is L and we write

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

if $(x, y) \in D$ and $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ then $|f(x, y) - L| < \varepsilon$

4 Definition If f is a function of two variables, its **partial derivatives** are the functions f_x and f_y defined by

$$f_x(x, y) = \lim_{h\to 0} \frac{f(x + h, y) - f(x, y)}{h} \qquad y \text{ fixed}$$

$$f_y(x, y) = \lim_{h\to 0} \frac{f(x, y + h) - f(x, y)}{h} \qquad x \text{ fixed}$$

Notations for Partial Derivatives If $z = f(x, y)$, we write

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_1f = D_xf$$

$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}f(x, y) = \frac{\partial z}{\partial y} = f_2 = D_2f = D_yf$$

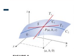

FIGURE 1
The partial derivatives of f at (a, b) are the slopes of the tangents to C_1 and C_2.

Interpretations of Partial Derivatives

To give a geometric interpretation of partial derivatives, we recall that the equation $z = f(x, y)$ represents a surface S (the graph of f). If $f(a, b) = c$, then the point $P(a, b, c)$ lies on S. By fixing $y = b$ we are restricting our attention to the curve C_1 in which the vertical plane $y = b$ intersects S. (In other words, C_1 is the trace of S in the plane $y = b$.) Likewise, the vertical plane $x = a$ intersects S in a curve C_2. Both of the curves C_1 and C_2 pass through the point P. (See Figure 1.)

Note that the curve C_1 is the graph of the function $g(x) = f(x, b)$, so the slope of its tangent T_1 at P is $g'(a) = f_x(a, b)$. The curve C_2 is the graph of the function $G(y) = f(a, y)$, so the slope of its tangent T_2 at P is $G'(b) = f_y(a, b)$.

Thus the partial derivatives $f_x(a, b)$ and $f_y(a, b)$ can be interpreted geometrically as the slopes of the tangent lines at $P(a, b, c)$ to the traces C_1 and C_2 of S in the planes $y = b$ and $x = a$.

1 Definition of an Improper Integral of Type 1

(a) If $\int_a^t f(x)\,dx$ exists for every number $t \ge a$, then

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

provided this limit exists (as a finite number).

(b) If $\int_t^b f(x)\,dx$ exists for every number $t \le b$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

provided this limit exists (as a finite number).

The improper integrals $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^b f(x)\,dx$ are called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If both $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ are convergent, then we define

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$

In part (c) any real number a can be used (see Exercise 88).

2 $\int_1^\infty \frac{1}{x^p}\,dx$ is convergent if $p > 1$ and divergent if $p \le 1$.

3 Definition of an Improper Integral of Type 2 (V.A)

(a) If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$$

if this limit exists (as a finite number).

(b) If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$$

if this limit exists (as a finite number).

The improper integral $\int_a^b f(x)\,dx$ is called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then we define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Comparison Theorem Suppose that f and g are continuous functions with $f(x) \ge g(x) \ge 0$ for $x \ge a$.

(a) If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is convergent.

(b) If $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is divergent.

centered at

5 Theorem If f has a power series representation (expansion) at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n(x - a)^n \qquad |x - a| < R \qquad \text{G.S}$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}$$

Substituting this formula for c_n back into the series, we see that *if f has a power series expansion at a, then it must be of the following form.*

6 $$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x - a)^n$$
$$= \underbrace{\overset{C_0}{f(a)}}_{0^{\text{th}} \text{ derivative}} + \underbrace{\frac{f'(a)}{1!}}_{C_1}(x - a) + \underbrace{\frac{f''(a)}{2!}}_{C_2}(x - a)^2 + \underbrace{\frac{f'''(a)}{3!}}_{C_3}(x - a)^3 + \cdots$$

The series in Equation 6 is called the **Taylor series of the function f at a** (or **about a** or **centered at a**). For the special case $a = 0$ the Taylor series becomes

3 Theorem If a series Σa_n is absolutely convergent, then it is convergent.

11.6

The Ratio Test

(dec ↓)

(i) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(inc ↑)

(ii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of Σa_n.

The Ratio Test is usually conclusive if the nth term of the series contains an exponential or a factorial, as in Examples 1 and 2. The test will always fail for p-series, as in Example 3.

→ Preferred when terms have common power.

The Root Test

(i) If $\lim_{n\to\infty}\sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n\to\infty}\sqrt[n]{|a_n|} = L > 1$ or $\lim_{n\to\infty}\sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty}\sqrt[n]{|a_n|} = 1$, the Root Test is inconclusive.

If $\lim_{n\to\infty}\sqrt[n]{|a_n|} = 1$, then part (iii) of the Root Test says that the test gives no information. The series Σa_n could converge or diverge. (If $L = 1$ in the Ratio Test, don't try the Root Test because L will again be 1. And if $L = 1$ in the Root Test, don't try the Ratio Test because it will fail too.)

11.8

Power Series

A **power series** is a series of the form

$$\boxed{1} \qquad \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

$$\boxed{2} \qquad \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots$$

More generally, a series of the form

$$\boxed{3} \qquad \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

4 Theorem For a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, there are only three possibilities:

(i) The series converges only when $x = a$.
(ii) The series converges for all x.
(iii) There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

	Series	Radius of convergence	Interval of convergence
Geometric series	$\sum_{n=0}^{\infty} x^n$	$R = 1$	$(-1, 1)$
Example 1	$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$	$R = 1$	$[2, 4)$
Example 2	$\sum_{n=0}^{\infty} n!\, x^n$	$R = 0$	$\{0\}$
Example 3	$\sum_{n=0}^{\infty} \frac{x^n}{(2n)!}$	$R = \infty$	$(-\infty, \infty)$

11.9

Representations of Functions using Geometric Series

We will obtain power series representations for several functions by manipulating geometric series. We start with an equation that we have seen before.

$$\boxed{1} \qquad \frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n \qquad |x| < 1$$

centered at a

2 Theorem If the power series $\Sigma c_n(x-a)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a - R, a + R)$ and

(i) $f'(x) = \overset{0\,+}{c_1} + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \overset{0\,+}{\sum_{n=1}^{\infty}} nc_n(x-a)^{n-1} = \sum_{0}^{\infty} nc_n(x-a)^{n-1}$

(ii) $\int f(x)\,dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots$

$$= C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

The radii of convergence of the power series in Equations (i) and (ii) are both R.

NOTE 1 Equations (i) and (ii) in Theorem 2 can be rewritten in the form

(iii) $\frac{d}{dx}\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] = \sum_{n=0}^{\infty} \frac{d}{dx}[c_n(x-a)^n]$

(iv) $\int\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right]dx = \sum_{n=0}^{\infty} \int c_n(x-a)^n\,dx$

11.11

In Section 11.10 we introduced the notation $T_n(x)$ for the nth partial sum of this series and called it the nth-degree Taylor polynomial of f at a. Thus

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$|R_n(x)| = |f(x) - T_n(x)|$$

There are three possible methods for estimating the size of the error:

1. We can use a calculator or computer to graph $|R_n(x)| = |f(x) - T_n(x)|$ and thereby estimate the error.
2. If the series happens to be an alternating series, we can use the Alternating Series Estimation Theorem.
3. In all cases we can use Taylor's Inequality (Theorem 11.10.9), which says that if $|f^{(n+1)}(x)| \leqslant M$, then

$$|R_n(x)| \leqslant \frac{M}{(n+1)!}|x-a|^{n+1}$$

$$= \underbrace{\overset{c_0}{f(a)}}_{\text{0th derivative}} + \underbrace{\frac{f'(a)}{1!}}_{c_1}(x-a) + \underbrace{\frac{f''(a)}{2!}}_{c_2}(x-a)^2 + \underbrace{\frac{f'''(a)}{3!}}_{c_3}(x-a)^3 + \cdots$$

The series in Equation 6 is called the **Taylor series of the function f at a** (or **about a** or **centered at a**). For the special case $a = 0$ the Taylor series becomes

$$\boxed{7} \qquad f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\underset{(x-0)^n}{x^n} = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}\underset{(x-0)^2}{x^2} + \cdots$$

This case arises frequently enough that it is given the special name **Maclaurin series**.

centered at a = 0.

In Section 11.10 we introduced the notation $T_n(x)$ for the nth partial sum of this series and called it the nth-degree Taylor polynomial of f at a. Thus

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

8 Theorem If $f(x) = T_n(x) + R_n(x)$, where T_n is the nth-degree Taylor polynomial of f at a, and if

$$\lim_{n\to\infty} R_n(x) = 0$$

for $|x-a| < R$, then f is equal to the sum of its Taylor series on the interval $|x-a| < R$. ROC

In trying to show that $\lim_{n\to\infty} R_n(x) = 0$ for a specific function f, we usually use the following theorem.

$|R_n(x)| \leqslant \frac{M}{(n+1)!}|x-a|^{n+1}$

* NOT $f^{(n+1)}(a)$

9 Taylor's Inequality If $|f^{(n+1)}(x)| \leqslant M$ for $|x-a| \leqslant d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x)| \leqslant \frac{M}{(n+1)!}|x-a|^{n+1} \qquad \text{for } |x-a| \leqslant d$$

When we apply Theorems 8 and 9 it is often helpful to make use of the following fact.

$$\boxed{10} \qquad \lim_{n\to\infty}\frac{x^n}{n!} = 0 \qquad \text{for every real number } x$$

17 The Binomial Series If k is any real number and $|x| < 1$, then

$$(1+x)^k = \sum_{n=0}^{\infty}\binom{k}{n}x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$$

$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$	$R = 1$
$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$	$R = \infty$
$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$	$R = \infty$
$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$	$R = \infty$
$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$	$R = 1$
$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$	$R = 1$
$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = \underset{0!}{1} + \underset{1!}{kx} + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$	$R = 1$