Robust Optimization

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Contents

Motivation

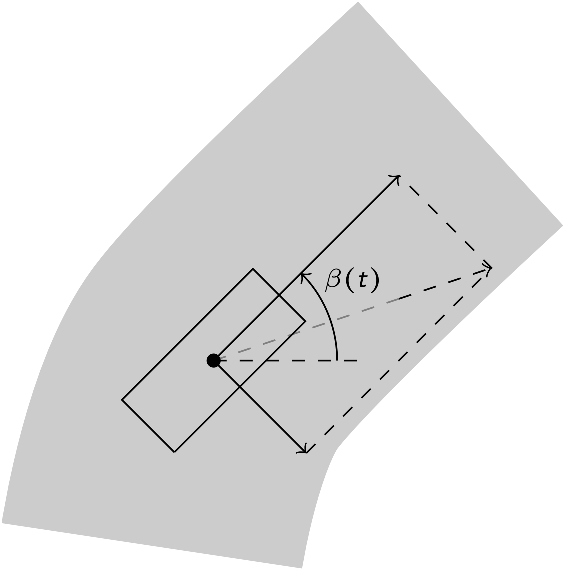
- steering a car
- minimize fuel consumption
- constraints
  - avoid crashes
  - dynamics

Problem

Dynamics change considerably, e.g., for different weather.

➔ Robust Optimization

Model of Car



Dynamical System

$$\dot{x}(t) = G(x(t), u(t), p) \qquad\qquad x(0) = x_0,$$

where

$x \in C^1([0, t_f], \mathbb{R}^n)$ is the state variable,

$u \in C^1([0, t_f], \mathbb{R}^m)$ is the control variable,

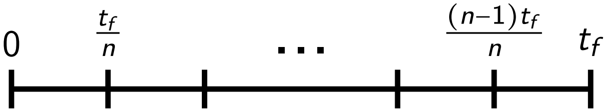
$p \in \mathbb{R}^{n_p}$ is the uncertain parameter vector.

Optimal Control Problem

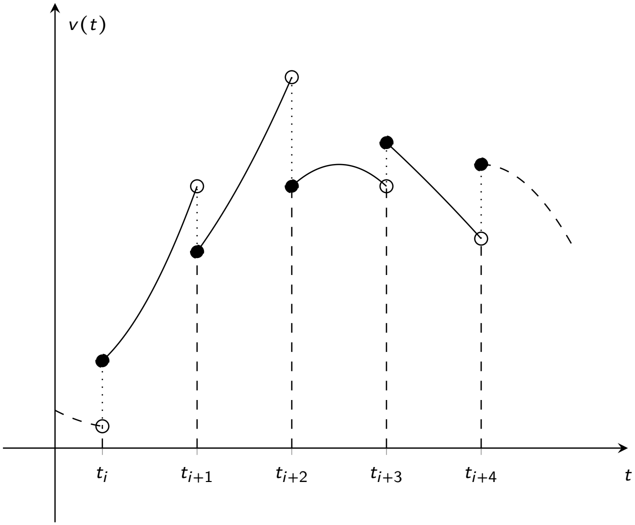
$$
\begin{aligned}
&\min_{x(\cdot)\in\mathbb{R}^{n_x},\,u(\cdot)\in\mathbb{R}^{n_u}} \quad f_0(x,u) && \\
\text{s.t.} \quad & \dot{x}(t) = G(x(t),u(t),p) && \forall t \in [0,t_f], \\
& x(0) = x_0, && \\
& f_i(x(t),u(t)) \le 0 && \text{for } i = 1,\dots,n_f, \\
& && \forall t \in [0,t_f].
\end{aligned}
$$

Discretization

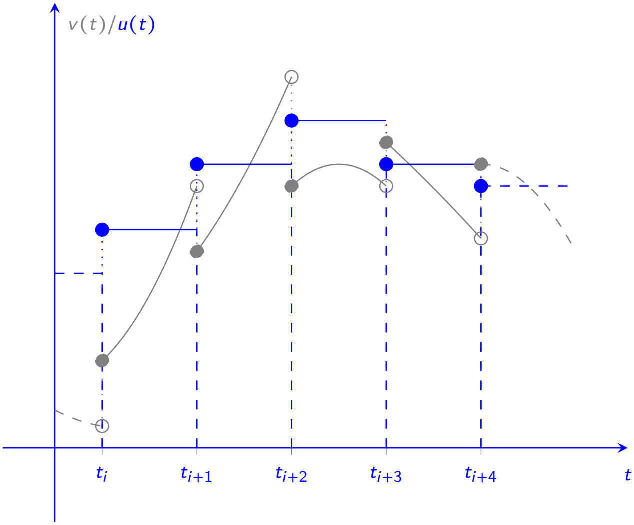
- infinte dimensional optimization problem
  ➔ approximate the optimal control problem



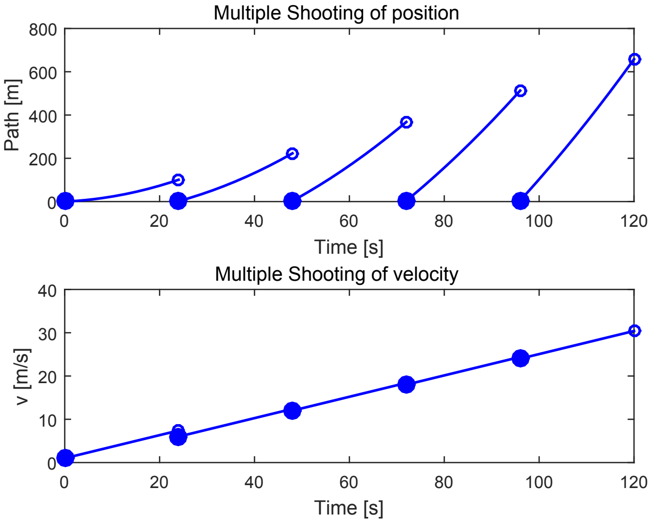
Multiple Shooting



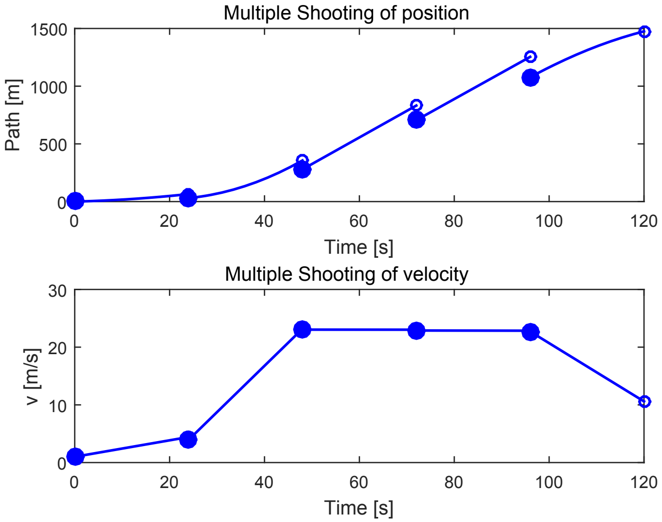
Multiple Shooting



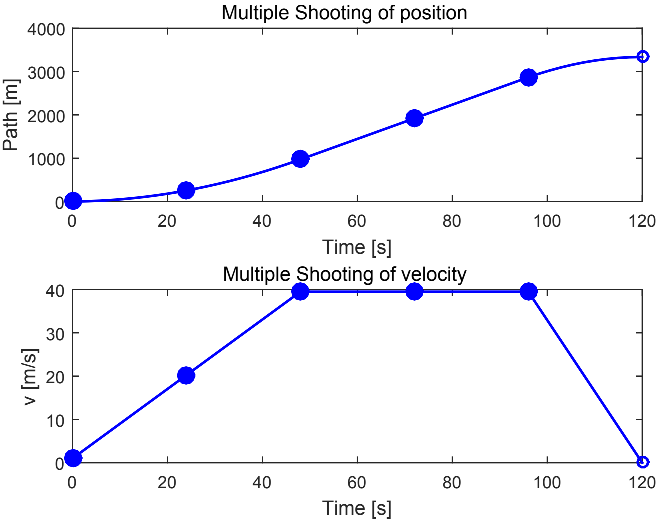
Multiple Shooting



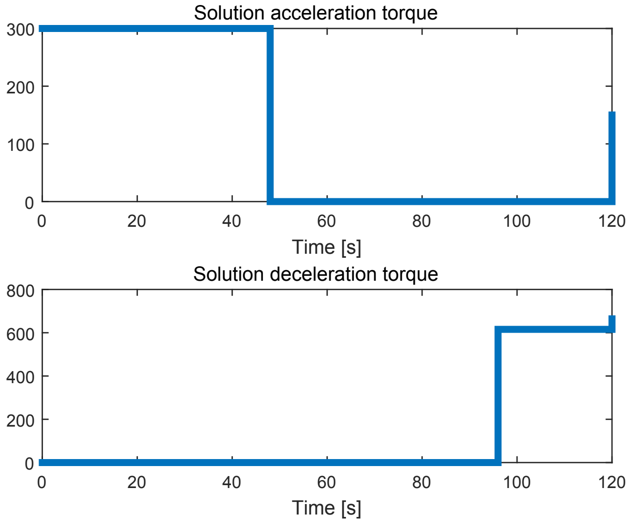
Multiple Shooting



Multiple Shooting



Multiple Shooting



Problem Formulation

Discrete Optimization Problem

$$
\begin{aligned}
&\min_{x \in \mathbb{R}^{n_x}, u \in \mathbb{R}^{n_u}} && f_0(x, u) & \\
\text{s.t.} \quad & && f_i(x, u) \leq 0 && \text{for } i = 1, \ldots, n_f \\
& && g_j(x, u, p) = 0 && \text{for } j = 1, \ldots, n_x,
\end{aligned}
$$

where $p \in \mathbb{R}^{n_p}$ is an uncertain parameter vector

Problem Formulation

Discrete Optimization Problem

$$
\begin{aligned}
& \min_{x\in\mathbb{R}^{n_x},\, u\in\mathbb{R}^{n_u}} f_0(x,u) & \\
\text{s.t.} \quad & f_i(x,u) \leq 0 & \text{for } i = 1,\dots,n_f \\
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\end{aligned}
$$

where $p \in \mathbb{P}_{box} = \left\{ p \in \mathbb{R}^{n_p} \middle| \, p_{lower} \leq p \leq p_{upper} \right\}$

Problem Formulation

Discrete Optimization Problem

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\begin{aligned}
&\min_{x \in \mathbb{R}^{n_x}, u \in \mathbb{R}^{n_u}} \quad f_0(x, u) \\
\text{s.t.} \quad & f_i(x, u) \leq 0 \qquad \text{for } i = 1, \dots, n_f \\
& g_j(x, u, p) = 0 \qquad \text{for } j = 1, \dots, n_x,
\end{aligned}
$$

where $p \in \mathbb{P}_{box} = \{ p \in \mathbb{R}^{n_p} | \, p_{lower} \leq p \leq p_{upper} \}$

$$
\begin{aligned}
\rightsquigarrow \Phi_i(u) := \quad & \max_{x \in \mathbb{R}^{n_x}, p \in \mathbb{R}^{n_p}} \quad f_i(x, u) \qquad \text{for } i = 0, \dots, n_f \\
\text{s.t.} \quad & g(x, u, p) = 0 \\
& p \in \mathbb{P}_{box}
\end{aligned}
$$

Worst Case Formulation

$$\begin{aligned} \Phi_i(u) := \quad & \max_{x \in \mathbb{R}^{n_x}, p \in \mathbb{R}^{n_p}} f_i(x, u) && \text{for } i = 0, \dots, n_f \\ \text{s.t.} \quad & g(x, u, p) = 0 \\ & p \in \mathbb{P}_{box} \end{aligned}$$

Robust Counterpart

$$\begin{aligned} & \min_{u \in \mathbb{R}^{n_u}} \Phi_0(u) \\ \text{s.t.} \quad & \Phi_i(u) \le 0 && \text{for } i = 1, \dots, n_f. \end{aligned}$$

→ bilevel structure!

Worst Case Formulation

$$
\begin{aligned}
\Phi_i(u) := \quad & \max_{x \in \mathbb{R}^{n_x}, p \in \mathbb{R}^{n_p}} f_i(x, u) \qquad \text{for every } i = 0, \ldots, n_f \\
\text{s.t.} \quad & g(x, u, p) = 0 \\
& p \in \mathbb{P}_{box}
\end{aligned}
$$

Robust Counterpart

$$
\begin{aligned}
& \min_{u \in \mathbb{R}^{n_u}} \Phi_0(u) \\
\text{s.t.} \quad & \Phi_i(u) \leq 0 \qquad \text{for } i = 1, \ldots, n_f.
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$$

➔ bilevel structure!

Worst Case Formulation

$$\begin{aligned}
\Phi_i(u) := \quad & \max_{x \in \mathbb{R}^{n_x}, p \in \mathbb{R}^{n_p}} f_i(x, u) && \text{for every } i = 0, \ldots, n_f \\
\text{s.t.} \quad & g(x, u, p) = 0 \\
& p \in \mathbb{P}_{box}
\end{aligned}$$

Robust Counterpart

$$\begin{aligned}
& \min_{u \in \mathbb{R}^{n_u}} \tilde{\Phi}_0(u) \\
\text{s.t.} \quad & \tilde{\Phi}_i(u) \leq 0 && \text{for } i = 1, \ldots, n_f.
\end{aligned}$$

~~→ bilevel structure!~~

Approximation Technique

Linearization

at a point $(\bar{x}, u, \bar{p})$ with $\bar{p} = \frac{p_{lower} + p_{upper}}{2}$ and $g(\bar{x}, u, \bar{p}) = 0$:

$$\begin{aligned}
\tilde{\Phi}_i(u) := \quad & \max_{(x-\bar{x}) \in \mathbb{R}^{n_x}, (p-\bar{p}) \in \mathbb{R}^{n_p}} f_i(\bar{x}, u) + \frac{\partial f_i}{\partial x}(\bar{x}, u)(x - \bar{x}) \\
\text{s.t.} \quad & \frac{\partial g}{\partial x}(\bar{x}, u, \bar{p})(x - \bar{x}) + \frac{\partial g}{\partial p}(\bar{x}, u, \bar{p})(p - \bar{p}) = 0 \\
& p \in \mathbb{P}_{box}
\end{aligned}$$

Approximation Technique

Linearization

at a point $(\bar{x}, u, \bar{p})$ with $\bar{p} = \frac{p_{lower}+p_{upper}}{2}$ and $g(\bar{x}, u, \bar{p}) = 0$:

$$
\begin{aligned}
\tilde{\Phi}_i(u) := \quad & \max_{(x-\bar{x})\in\mathbb{R}^{n_x},(p-\bar{p})\in\mathbb{R}^{n_p}} f_i(\bar{x}, u) + \frac{\partial f_i}{\partial x}(\bar{x}, u)(x-\bar{x}) \\
\text{s.t.} \quad & \frac{\partial g}{\partial x}(\bar{x}, u, \bar{p})(x-\bar{x}) + \frac{\partial g}{\partial p}(\bar{x}, u, \bar{p})(p-\bar{p}) = 0 \\
& p \in \mathbb{P}_{box}
\end{aligned}
$$

With a unique optimal solution $\tilde{\Phi}_i(u)$:

Standard Optimization Problem

$$
\begin{aligned}
& \min_{u\in\mathbb{R}^{n_u},\bar{x}\in\mathbb{R}^{n_x}} \tilde{\Phi}_0(u) \\
\text{s.t.} \quad & \tilde{\Phi}_i(u) \leq 0 \qquad \text{for } i = 1, \ldots, n_f \\
& g(\bar{x}, u, \bar{p}) = 0
\end{aligned}
$$

Visualization

What do we want?

Visualization

- sell our results to you
- nice graphs
- rendered video

Visualization

What can we improve?

Visualization

- textures
- nonlinear street design
- fit acceleration to optimization results
- fit resolution to output device
- follow the car with the camera

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