Robust Optimization

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- Our work is divided into 4 parts
  - Model of a Car
  - Mathematical problem formulation
  - Implementation
  - Visualization

Motivation

- steering a car
- minimize fuel consumption
- constraints
  - avoid crashes
  - dynamics

Problem

Dynamics change considerably, e.g. for different weather.

→ Robust Optimization

Motivation

- steering a car
- minimize fuel consumption
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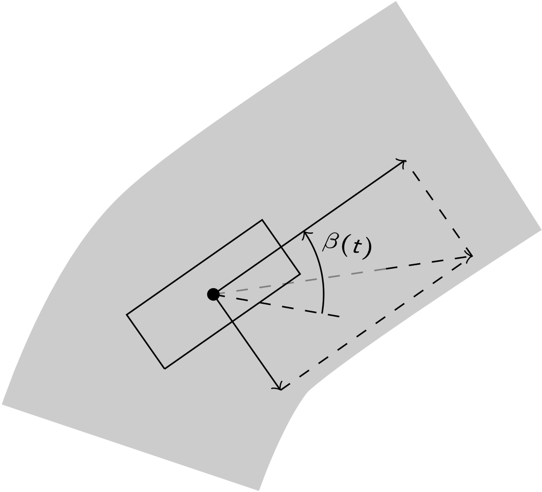
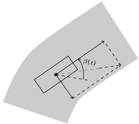
Problem
Dynamics change considerably, e.g. for different weather.
→ Robust Optimization

- Our goal is to steer a car such that the fuel consumption is minimal
- Contraints: Avoid Crashes
  - our car is not allowed to leave the road
  - hit other cars or construction sites
- Dynamics of the car
  - motion of the car
    - slipping behavior
    - acceleration behavior
- Problem: Changing weather has a major impact on the slipping behavior
  - we cannot drive fast on icy roads
  - Rain causes aquaplaning
- Way out: Robust Optimization
  - This Optimization method takes changing parameters such as weather into account.
  - Car is steered save for changing conditions such as weather, different roads...

Model of Car



Model of Car



- Our car is described as a pointmass, for simplicity
- Road is visualized in gray
- Car as a rectangle
- steering angle is denoted by $\beta(t)$
  - We applied Newton's mechanic
  - The motion/ direction of motion is described with
    - The steering angle $\beta(t)$
    - The acceleration force pushing the car forward
    - The central forces
    - If the speed of the car is to high, it slides and may leave the road.

Dynamical System

$$\dot{x}(t) = G(x(t), u(t), p) \qquad x(0) = x_0,$$

where

$x \in C^1([0, t_f], \mathbb{R}^{n_x})$ is the state variable,
$u \in C^1([0, t_f], \mathbb{R}^{n_u})$ is the control variable,
$p \in \mathbb{R}^{n_p}$ is the uncertain parameter vector.

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- Motion of the car is described as a dynamical system, i.e., with a ODE and an initial value
- t_f is the final time
- our states: path, velocity, steering angle
- control: brake, motor torque (gas petal), steering angle velocity
- parameter: weather, rolling friction coefficients, uncertain sea level due to wrong NAVI data

Optimal Control Problem

$$
\begin{aligned}
& \min_{x(\cdot)\in\mathbb{R}^{n_x},\,u(\cdot)\in\mathbb{R}^{n_u}} && f_0(x,u) && \\
\text{s.t.}\quad & && \dot{x}(t) = G(x(t),u(t),p) && \forall\, t \in [0,t_f], \\
& && x(0) = x_0, && \\
& && f_i(x(t),u(t)) \le 0 && \text{for } i = 1,\dots,n_f, \\
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Discretization

- infinte dimensional optimization problem

Discretization

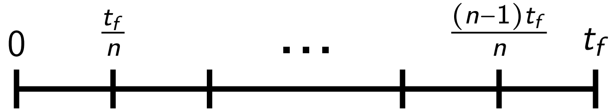
- infinte dimensional optimization problem

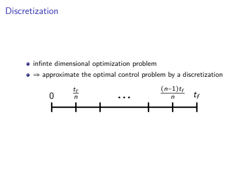
For consistency, I would not use several slides to present this content. I am not a fan of that. What is your opinion?
What do we approximate?
I suggest not to ask questions in the slide, just answer them and mention the questions in your talk (consistency)

We should try to have the same layout for our graphics. I thought that LaTeX does this one its own....

Discretization

- infinte dimensional optimization problem
- $\Rightarrow$ approximate the optimal control problem by a discretization





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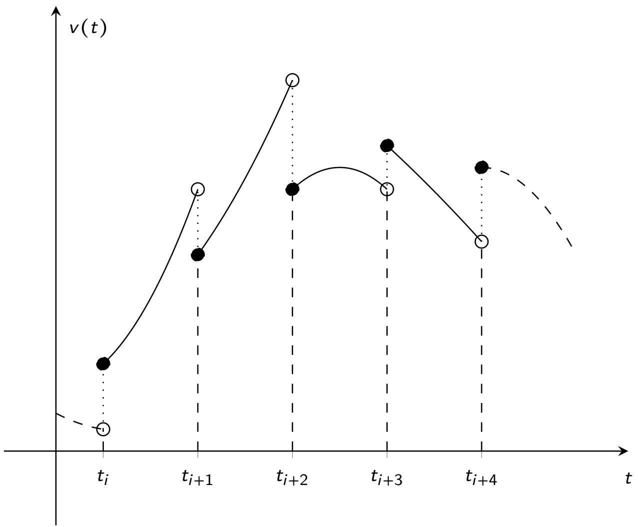

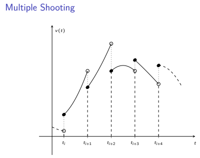

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- The t is above the axis
- It might be good to add another blue y -axis!
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- I would not present MATLAB graphics (consistency) - just another tikz graphic with continuous state.
- Will the solution output a continuous control? (might be a question of the audience.
- I would omit the plots for Mwh and R*Fb. Why are the vertical lines not dashed? (consistency)

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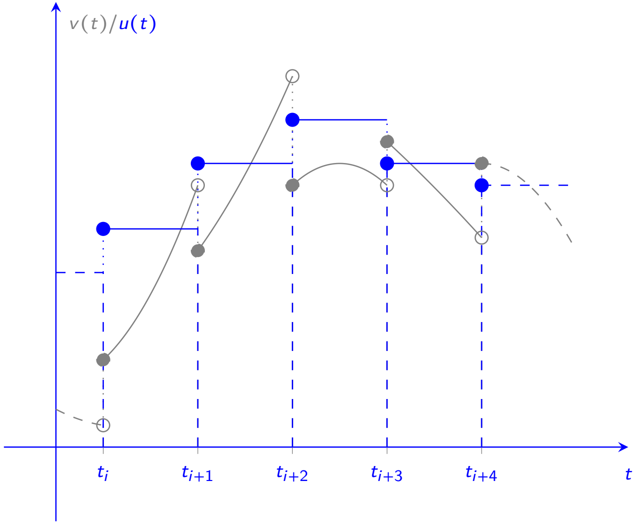

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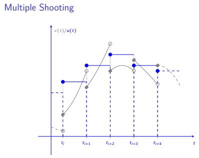

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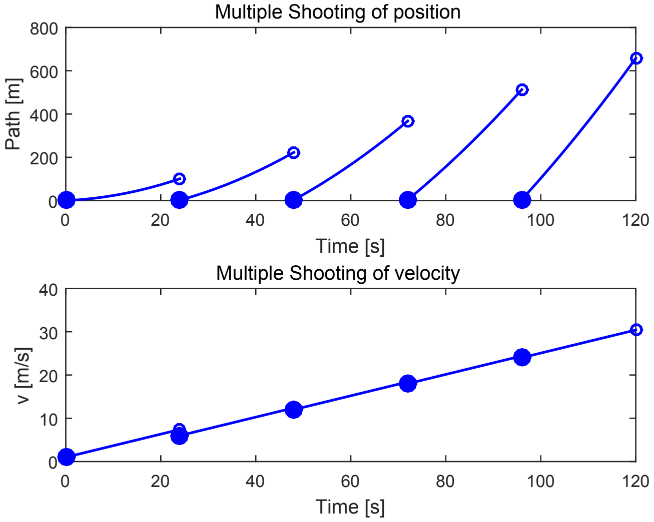

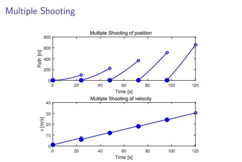

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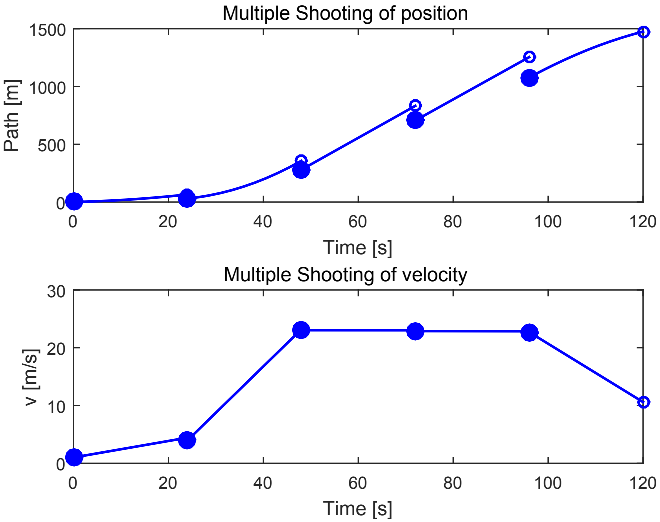

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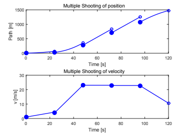

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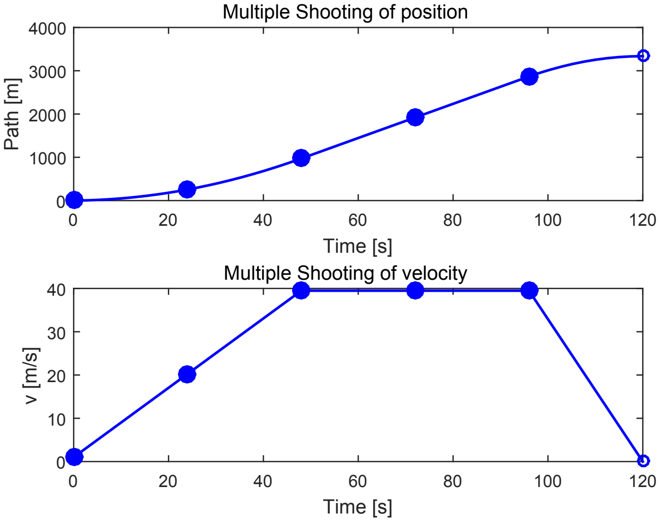

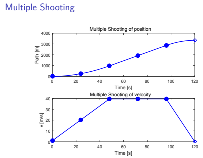

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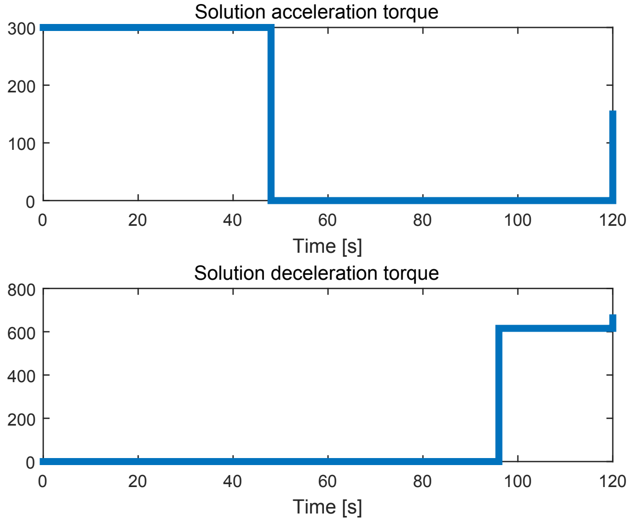

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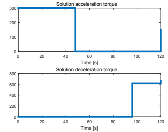

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Problem Formulation

Discrete Optimization Problem

$$
\begin{aligned}
& \min_{x \in \mathbb{R}^{n_x}, u \in \mathbb{R}^{n_u}} f_0(x, u) \\
\text{s.t.} \quad & f_i(x, u) \leq 0 && \text{for } i = 1, \ldots, n_f \\
& g_j(x, u, p) = 0 && \text{for } j = 1, \ldots, n_x
\end{aligned}
$$

with $p \in \mathbb{R}^{n_p}$ uncertain parameter vector

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$$\text{with } p \in \mathbb{P}_{box} = \{ p \in \mathbb{R}^{n_p} | \, p_{lower} \le p \le p_{upper} \}$$

Problem Formulation
Discrete Optimization Problem
$$\min_{x \in \mathbb{R}^{n_x}, u \in \mathbb{R}^{n_u}} f_0(x,u)$$
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$$
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$$

$$
\begin{aligned}
\rightsquigarrow \Phi_i(u) := \quad & \max_{x\in\mathbb{R}^{n_x},\,p\in\mathbb{R}^{n_p}} && f_i(x,u) \qquad \text{for every } i = 0,\dots,n_f \\
\text{s.t.} \quad & && g(x,u,p) = 0 \\
& && p \in \mathbb{P}_{box}
\end{aligned}
$$

Problem Formulation
Discrete Optimization Problem
$\min_{x\in\mathbb{R}^{n_x},u\in\mathbb{R}^{n_u}} f_0(x,u)$
s.t. $f_i(x,u)\le 0$ for $i=1,\dots,n_f$
$g_j(x,u,p)=0$ for $j=1,\dots,n_x$
with $p\in\mathbb{P}_{box}=\{p\in\mathbb{R}^{n_p}| p_{lower}\le p\le p_{upper}\}$
$\rightsquigarrow \Phi_i(u) := \max_{x\in\mathbb{R}^{n_x},p\in\mathbb{R}^{n_p}} f_i(x,u)$ for every $i=0,\dots,n_f$
s.t. $g(x,u,p)=0$
$p\in\mathbb{P}_{box}$

Worst Case Formulation

$$\begin{aligned} \Phi_i(u) := \quad & \max_{x\in\mathbb{R}^{n_x}, p\in\mathbb{R}^{n_p}} f_i(x,u) \qquad \text{for every } i = 0,\dots,n_f \\ \text{s.t.} \quad & g(x,u,p) = 0 \\ & p \in \mathbb{P}_{box} \end{aligned}$$

Robust Counterpart

$$\begin{aligned} & \min_{u\in\mathbb{R}^{n_u}} \Phi_0(u) \\ \text{s.t.} \quad & \Phi_i(u) \le 0 \qquad \text{for } i = 1,\dots,n_f. \end{aligned}$$

$\Rightarrow$ bilevel structure!

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Robust Optimization
- Robust Optimization Problem
  - Worst Case Formulation

Worst Case Formulation

$\Phi_i(u) := \max_{x\in\mathbb{R}^{n_x}, p\in\mathbb{R}^{n_p}} f_i(x,u)$ for every $i = 0,\dots,n_f$ s.t. $g(x,u,p) = 0$, $p \in \mathbb{P}_{box}$

Robust Counterpart

$\min_{u\in\mathbb{R}^{n_u}} \Phi_0(u)$ s.t. $\Phi_i(u) \le 0$ for $i = 1,\dots,n_f$.

$\Rightarrow$ bilevel structure!

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\text{s.t.} \quad & g(x,u,p) = 0 \\
& p \in \mathbb{P}_{box}
\end{aligned}
$$

Robust Counterpart

$$
\begin{aligned}
& \min_{u\in\mathbb{R}^{n_u}} \tilde{\Phi}_0(u) \\
\text{s.t.} \quad & \tilde{\Phi}_i(u) \le 0 && \text{for } i = 1,\dots,n_f.
\end{aligned}
$$

$\Longrightarrow$ ~~bilevel structure!~~

Worst Case Formulation
$\Phi_i(u) := \max_{x\in\mathbb{R}^{n_x},\, p\in\mathbb{R}^{n_p}} f_i(x,u)$ for every $i = 0,\dots,n_f$
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Approximation Technique

Linearization

at a point $(\bar{x}, u, \bar{p})$ with $g(\bar{x}, u, \bar{p}) = 0$ and $\bar{p} = \frac{p_{lower} + p_{upper}}{2}$:

$$
\begin{aligned}
\tilde{\Phi}_i(u) := \quad & \max_{(x-\bar{x}) \in \mathbb{R}^{n_x}, (p-\bar{p}) \in \mathbb{R}^{n_p}} f_i(\bar{x}, u) + \frac{\partial f_i}{\partial x}(\bar{x}, u)(x - \bar{x}) \\
\text{s.t.} \quad & \frac{\partial g}{\partial x}(\bar{x}, u, \bar{p})(x - \bar{x}) + \frac{\partial g}{\partial p}(\bar{x}, u, \bar{p})(p - \bar{p}) = 0 \\
& p \in \mathbb{P}_{box}
\end{aligned}
$$

Approximation Technique
Linearization
at a point $(\bar{x}, u, \bar{p})$ with $g(\bar{x}, u, \bar{p}) = 0$ and $\bar{p} = \frac{p_{lower} + p_{upper}}{2}$:
$\tilde{\Phi}_i(u) := \max_{(x-\bar{x}) \in \mathbb{R}^{n_x}, (p-\bar{p}) \in \mathbb{R}^{n_p}} f_i(\bar{x}, u) + \frac{\partial f_i}{\partial x}(\bar{x}, u)(x - \bar{x})$
s.t. $\frac{\partial g}{\partial x}(\bar{x}, u, \bar{p})(x - \bar{x}) + \frac{\partial g}{\partial p}(\bar{x}, u, \bar{p})(p - \bar{p}) = 0$
$p \in \mathbb{P}_{box}$

Approximation Technique

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& p \in \mathbb{P}_{box}
\end{aligned}
$$

With a unique optimal solution $\tilde{\Phi}_i(u)$:

Standard Optimization Problem

$$
\begin{aligned}
& \min_{u \in \mathbb{R}^{n_u}, \bar{x} \in \mathbb{R}^{n_x}} \tilde{\Phi}_0(u) \\
\text{s.t.} \quad & \tilde{\Phi}_i(u) \le 0 \qquad \text{for } i = 1, \ldots, n_f \\
& g(\bar{x}, u, \bar{p}) = 0
\end{aligned}
$$

Approximation Technique
Linearization
at a point $(\bar{x}, u, \bar{p})$ with $g(\bar{x}, u, \bar{p}) = 0$ and $\bar{p} = \frac{p_{lower} + p_{upper}}{2}$:
With a unique optimal solution $\tilde{\Phi}_i(u)$:
Standard Optimization Problem

Visualization

References

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