Robust Optimization

Annkathrin Krämmer, Sabina Przioda, Christian Kreipl,
Johannes Milz

Technische Universität München

May 27, 2015

Contents

Motivation

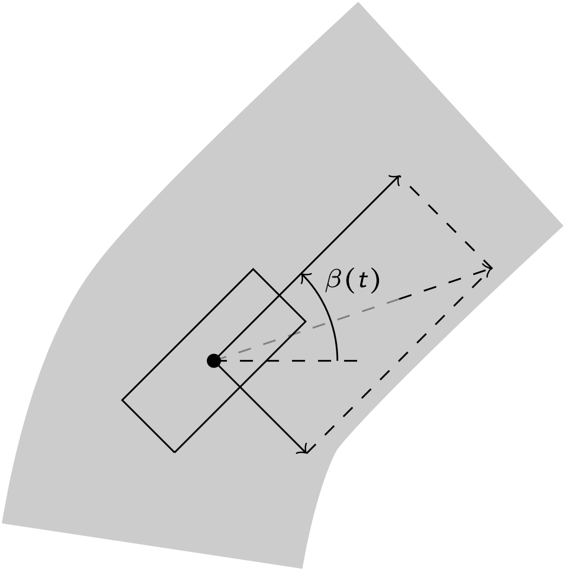
- steering a car
- minimize fuel consumption
- constraints
  - avoid crashes
  - dynamics

Problem

Dynamics change considerably, e.g. for different weather.

➔ Robust Optimization

Model of Car



Dynamical System

$$\dot{x}(t) = G(x(t), u(t), p) \qquad\qquad x(0) = x_0,$$

where

$$x \in C^1([0, t_f], \mathbb{R}^n) \text{ is the state variable,}$$
$$u \in C^1([0, t_f], \mathbb{R}^m) \text{ is the control variable,}$$
$$p \in \mathbb{R}^{n_p} \text{ is the uncertain parameter vector.}$$

Optimal Control Problem

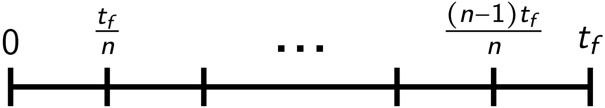
$$
\begin{aligned}
& \min_{x(\cdot)\in\mathbb{R}^{n_x},\, u(\cdot)\in\mathbb{R}^{n_u}} && f_0(x, u) && \\
\text{s.t.} \quad & && \dot{x}(t) = G(x(t), u(t), p) && \forall t \in [0, t_f], \\
& && x(0) = x_0, && \\
& && f_i(x(t), u(t)) \le 0 && \text{for } i = 1, \dots, n_f, \\
& && && \forall t \in [0, t_f].
\end{aligned}
$$

Discretization

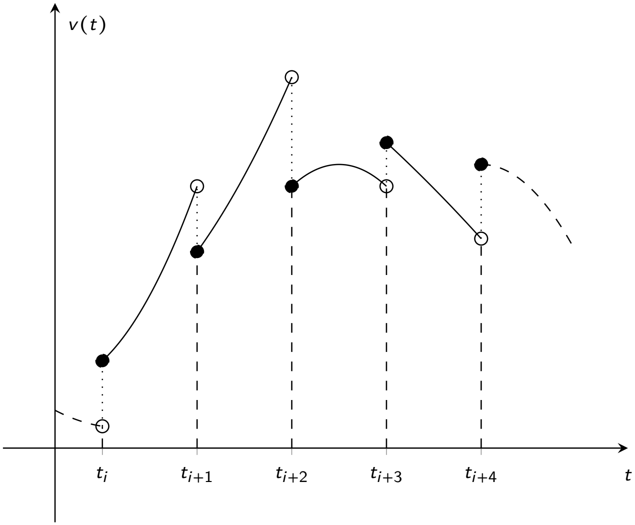
- infinte dimensional optimization problem

Discretization

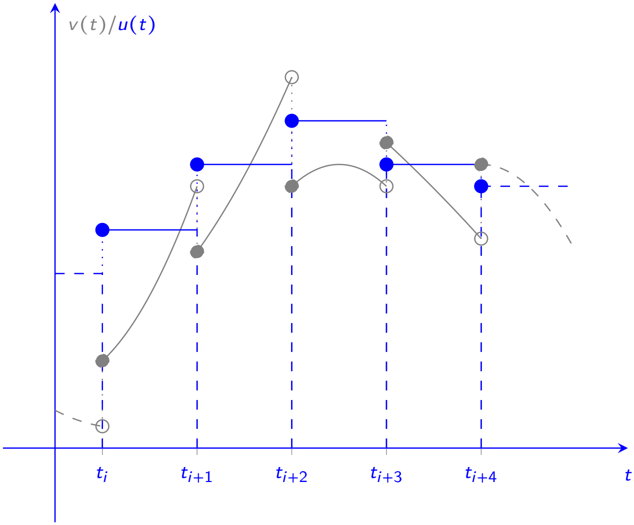
- infinte dimensional optimization problem



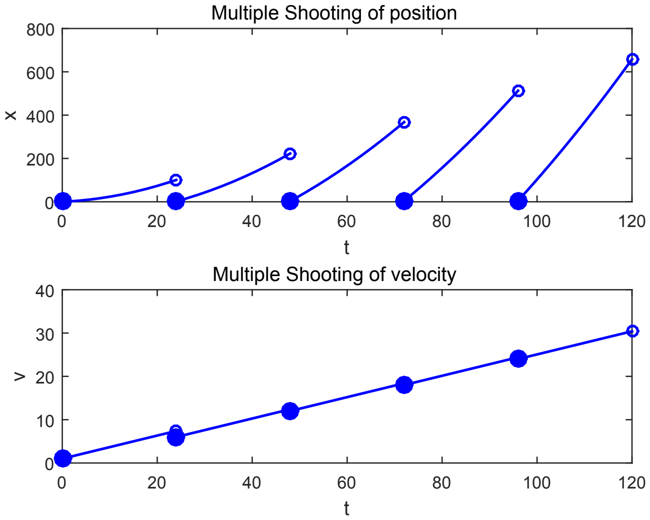
Multiple Shooting



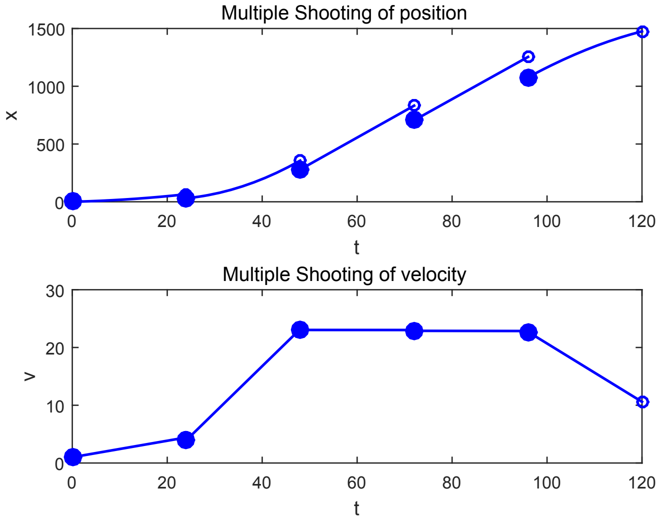
Multiple Shooting



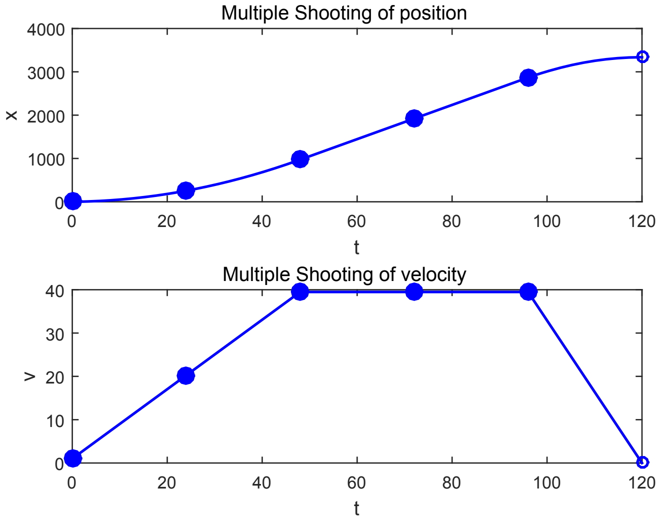
Multiple Shooting



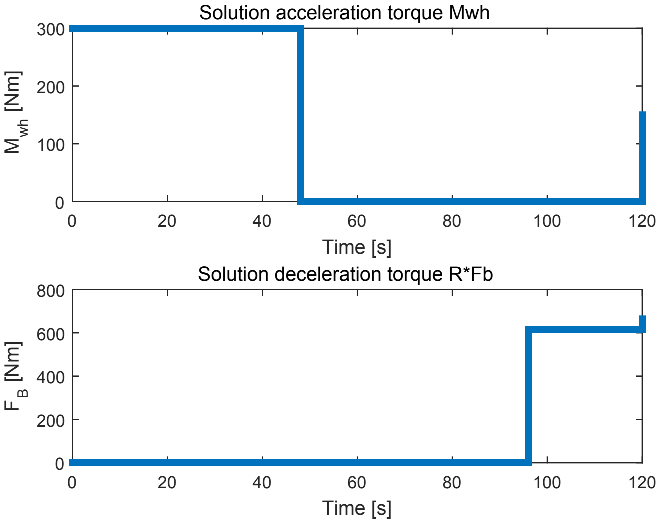
Multiple Shooting



Multiple Shooting



Multiple Shooting



Problem Formulation

Discrete Optimization Problem

$$
\begin{aligned}
&\min_{x \in \mathbb{R}^{n_x}, u \in \mathbb{R}^{n_u}} \quad f_0(x, u) & \\
\text{s.t.} \quad & f_i(x, u) \le 0 & \text{for } i = 1, \ldots, n_f \\
& g_j(x, u, p) = 0 & \text{for } j = 1, \ldots, n_x,
\end{aligned}
$$

where $p \in \mathbb{R}^{n_p}$ is an uncertain parameter vector

Problem Formulation

Discrete Optimization Problem

$$
\begin{aligned}
&\min_{x \in \mathbb{R}^{n_x}, u \in \mathbb{R}^{n_u}} && f_0(x, u) && \\
&\text{s.t.} && f_i(x, u) \leq 0 && \text{for } i = 1, \dots, n_f \\
& && g_j(x, u, p) = 0 && \text{for } j = 1, \dots, n_x,
\end{aligned}
$$

where $p \in \mathbb{P}_{box} = \{ p \in \mathbb{R}^{n_p} \,|\, p_{lower} \leq p \leq p_{upper} \}$

Problem Formulation

Discrete Optimization Problem

$$
\begin{aligned}
& \min_{x\in\mathbb{R}^{n_x},\, u\in\mathbb{R}^{n_u}} && f_0(x,u) && \\
\text{s.t.} \quad & && f_i(x,u) \le 0 && \text{for } i = 1,\dots,n_f \\
& && g_j(x,u,p) = 0 && \text{for } j = 1,\dots,n_x,
\end{aligned}
$$

where $p \in \mathbb{P}_{box} = \{\, p \in \mathbb{R}^{n_p} \,|\, p_{lower} \le p \le p_{upper} \,\}$

$$
\begin{aligned}
\rightsquigarrow \Phi_i(u) := \quad & \max_{x\in\mathbb{R}^{n_x},\, p\in\mathbb{R}^{n_p}} && f_i(x,u) && \text{for } i = 0,\dots,n_f \\
\text{s.t.} \quad & && g(x,u,p) = 0 && \\
& && p \in \mathbb{P}_{box} &&
\end{aligned}
$$

Worst Case Formulation

$$
\begin{aligned}
\Phi_i(u) := \quad & \max_{x \in \mathbb{R}^{n_x}, p \in \mathbb{R}^{n_p}} f_i(x, u) \qquad && \text{for } i = 0, \ldots, n_f \\
\text{s.t.} \quad & g(x, u, p) = 0 \\
& p \in \mathbb{P}_{box}
\end{aligned}
$$

Robust Counterpart

$$
\begin{aligned}
& \min_{u \in \mathbb{R}^{n_u}} \Phi_0(u) \\
\text{s.t.} \quad & \Phi_i(u) \leq 0 \qquad && \text{for } i = 1, \ldots, n_f .
\end{aligned}
$$

➔ bilevel structure!

Worst Case Formulation

$$\begin{aligned}
\Phi_i(u) := \quad & \max_{x\in\mathbb{R}^{n_x}, p\in\mathbb{R}^{n_p}} f_i(x,u) \qquad \text{for every } i = 0,\ldots,n_f \\
\text{s.t.} \quad & g(x,u,p) = 0 \\
& p \in \mathbb{P}_{box}
\end{aligned}$$

Robust Counterpart

$$\begin{aligned}
& \min_{u\in\mathbb{R}^{n_u}} \Phi_0(u) \\
\text{s.t.} \quad & \Phi_i(u) \le 0 \qquad \text{for } i = 1,\ldots,n_f.
\end{aligned}$$

➔ bilevel structure!

Worst Case Formulation

$$
\begin{aligned}
\Phi_i(u) := \quad & \max_{x\in\mathbb{R}^{n_x}, p\in\mathbb{R}^{n_p}} f_i(x,u) && \text{for every } i = 0,\dots,n_f \\
\text{s.t.} \quad & g(x,u,p) = 0 \\
& p \in \mathbb{P}_{box}
\end{aligned}
$$

Robust Counterpart

$$
\begin{aligned}
& \min_{u\in\mathbb{R}^{n_u}} \tilde{\Phi}_0(u) \\
\text{s.t.} \quad & \tilde{\Phi}_i(u) \le 0 && \text{for } i = 1,\dots,n_f\,.
\end{aligned}
$$

→ ~~bilevel structure!~~

Approximation Technique

Linearization

at a point $(\bar{x}, u, \bar{p})$ with $\bar{p} = \frac{p_{lower} + p_{upper}}{2}$ and $g(\bar{x}, u, \bar{p}) = 0$:

$$
\begin{aligned}
\tilde{\Phi}_i(u) := \quad & \max_{(x-\bar{x}) \in \mathbb{R}^{n_x}, (p-\bar{p}) \in \mathbb{R}^{n_p}} f_i(\bar{x}, u) + \frac{\partial f_i}{\partial x}(\bar{x}, u)(x - \bar{x}) \\
\text{s.t.} \quad & \frac{\partial g}{\partial x}(\bar{x}, u, \bar{p})(x - \bar{x}) + \frac{\partial g}{\partial p}(\bar{x}, u, \bar{p})(p - \bar{p}) = 0 \\
& p \in \mathbb{P}_{box}
\end{aligned}
$$

Approximation Technique

Linearization

at a point $(\bar{x}, u, \bar{p})$ with $\bar{p} = \frac{p_{lower} + p_{upper}}{2}$ and $g(\bar{x}, u, \bar{p}) = 0$:

$$\begin{aligned}
\tilde{\Phi}_i(u) := \quad & \max_{(x-\bar{x}) \in \mathbb{R}^{n_x}, (p-\bar{p}) \in \mathbb{R}^{n_p}} f_i(\bar{x}, u) + \frac{\partial f_i}{\partial x}(\bar{x}, u)(x - \bar{x}) \\
\text{s.t.} \quad & \frac{\partial g}{\partial x}(\bar{x}, u, \bar{p})(x - \bar{x}) + \frac{\partial g}{\partial p}(\bar{x}, u, \bar{p})(p - \bar{p}) = 0 \\
& p \in \mathbb{P}_{box}
\end{aligned}$$

With a unique optimal solution $\tilde{\Phi}_i(u)$:

Standard Optimization Problem

$$\begin{aligned}
& \min_{u \in \mathbb{R}^{n_u}, \bar{x} \in \mathbb{R}^{n_x}} \tilde{\Phi}_0(u) \\
\text{s.t.} \quad & \tilde{\Phi}_i(u) \leq 0 \qquad \text{for } i = 1, \ldots, n_f \\
& g(\bar{x}, u, \bar{p}) = 0
\end{aligned}$$

Visualization

What do we want?

Visualization

- sell our results to you
- nice graphs
- rendered video

Visualization

What can we improve?

Visualization

- textures
- nonlinear street design
- fit acceleration to optimization results
- fit resolution to output device
- follow the car with the camera

References I

Ben-Tal, A., El Ghaoui, L., and Nemirovski, A. (2009).
Robust Optimization.
Princeton University Press.

Diehl, M., Bock, H. G., Diedam, H., and Wieber, P.-B. (2005).

Fast direct multiple shooting algorithms for optimal robot control.
Fast Motions in Biomechanics and Robotics.

Diehl, M., Bock, H. G., and Kostina, E. (2006).
An approximation technique for robust nonlinear optimization.
Math. Program., Ser. B 107, pages 213–230.

Gerdts, M. (WiSe 2009/2010).
Optimale Steuerung.
Universität Würzburg.

References II

Mitschke, M. and Wallentowitz, H. (2014).
Dynamik der Kraftfahrzeuge.
Springer.

Tipler, P. A. and Mosca, G. (2015).
Physik für Wissenschaftler und Ingenieure.
SpringerSpektrum.