Robust Optimization

Annkathrin Krämmer, Sabina Przioda, Christian Kreipl, Johannes Milz

Technische Universität München

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Contents

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- Our work is divided into 4 parts
  - Model of a Car
  - Mathematical problem formulation
  - Implementation
  - Visualization

Motivation

- steering a car
- minimize fuel consumption
- constraints
  - avoid crashes
  - dynamics

Problem

Dynamics change considerably, e.g. for different weather.

→ Robust Optimization

Motivation
steering a car
minimize fuel consumption
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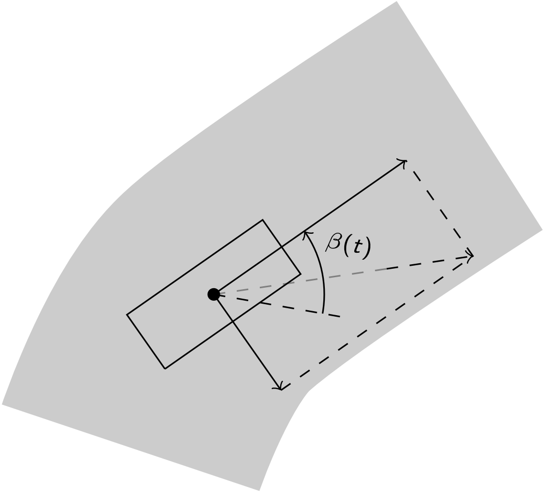
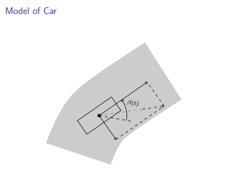
dynamics
Problem
Dynamics change considerably, e.g. for different weather.
→ Robust Optimization

- Our goal is to steer a car such that the fuel consumption is minimal
- Contraints: Avoid Crashes
  - our car is not allowed to leave the road
  - hit other cars or construction sites
- Dynamics of the car
  - motion of the car
    - slipping behavior
    - acceleration behavior
- Problem: Changing weather has a major impact on the slipping behavior
  - we cannot drive fast on icy roads
  - Rain causes aquaplaning
- Way out: Robust Optimization
  - This Optimization method takes changing parameters such as weather into account.
  - Car is steered save for changing conditions such as weather, different roads...

Model of Car





- Our car is described as a pointmass, for simplicity
- Road is visualized in gray
- Car as a rectangle
- steering angle is denoted by $\beta(t)$
  - We applied Newton's mechanic
  - The motion/ direction of motion is described with
    - The steering angle $\beta(t)$
    - The acceleration force pushing the car forward
    - The central forces
    - If the speed of the car is to high, it slides and may leave the road.

Dynamical System

$$\dot{x}(t) = G(x(t), u(t), p) \qquad\qquad x(0) = x_0,$$

where

$$x \in C^1([0, t_f], \mathbb{R}^{n_x}) \text{ is the state variable,}$$
$$u \in C^1([0, t_f], \mathbb{R}^{n_u}) \text{ is the control variable,}$$
$$p \in \mathbb{R}^{n_p} \text{ is the uncertain parameter vector.}$$

Dynamical System

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- Motion of the car is described as a dynamical system, i.e., with a ODE and an initial value
- t_f is the final time
- our states: path, velocity, steering angle
- control: brake, motor torque (gas petal), steering angle velocity
- parameter: weather, rolling friction coefficients, uncertain sea level due to wrong NAVI data

Optimal Control Problem

$$
\begin{aligned}
&\min_{x(\cdot)\in\mathbb{R}^{n_x},u(\cdot)\in\mathbb{R}^{n_u}} f_0(x,u) \\
\text{s.t.}\quad & \dot{x}(t) = G(x(t),u(t),p) && \forall t\in[0,t_f],\\
& x(0) = x_0,\\
& f_i(x(t),u(t)) \le 0 && \text{for } i=1,\dots,n_f,\\
& && \forall t\in[0,t_f].
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Discretization

- Why?

Discretization

- Why?

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What do we approximate?
I suggest not to ask questions in the slide, just answer them and mention the questions in your talk (consistency)

We should try to have the same layout for our graphics. I thought that LaTeX does this one its own....

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- Why?

  infinte dimensional optimization problem

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Discretization

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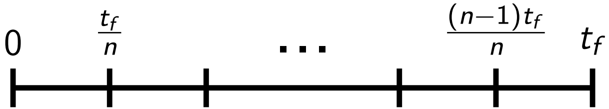
  infinte dimensional optimization problem

- How?



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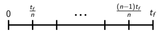

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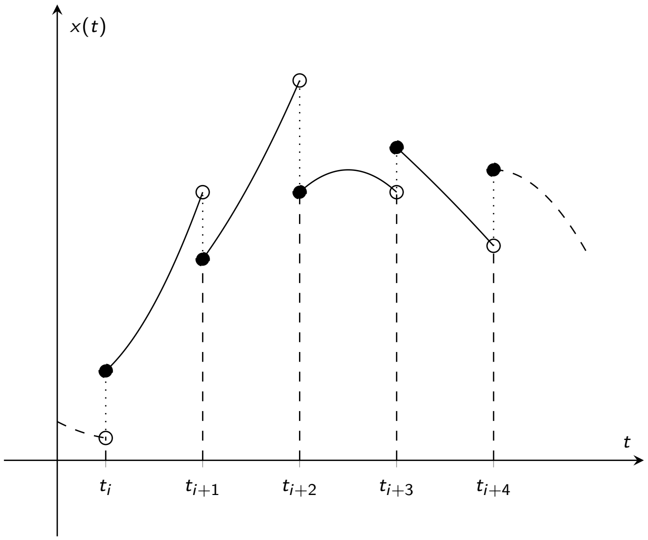

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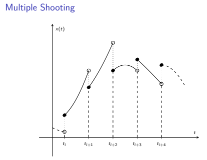

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- The *t* is above the axis
- It might be good to add another blue y -axis!
- The same for the control vector. Is the control between the state by accident?
- I would not present MATLAB graphics (consistency) - just another tikz graphic with continuous state.
- Will the solution output a continuous control? (might be a question of the audience.
- I would omit the plots for Mwh and R*Fb. Why are the vertical lines not dashed? (consistency)

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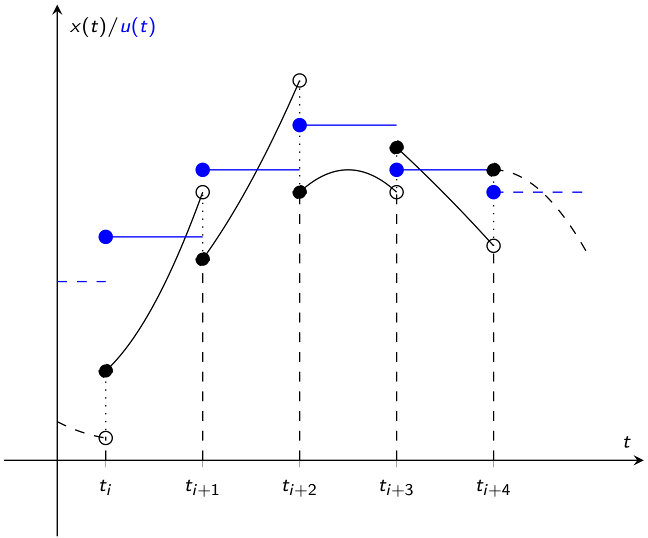

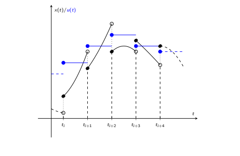

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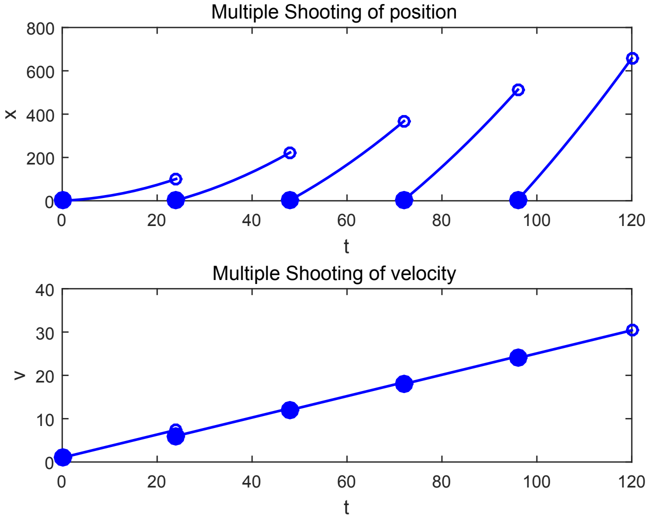

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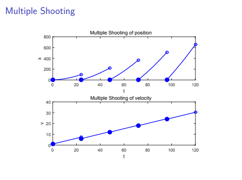

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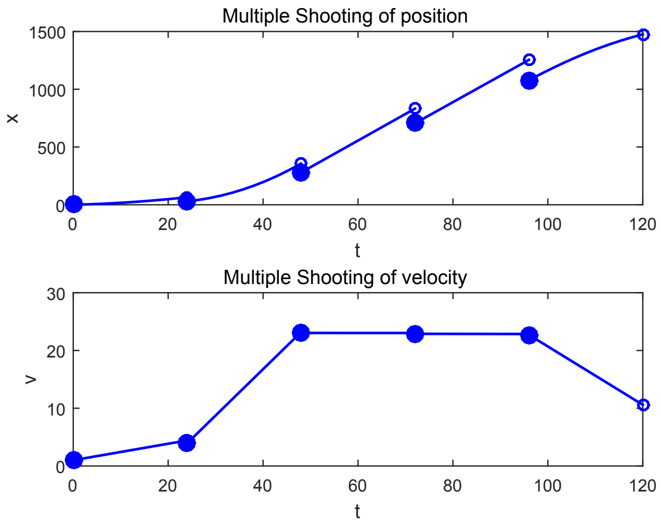

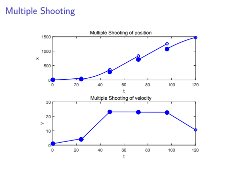

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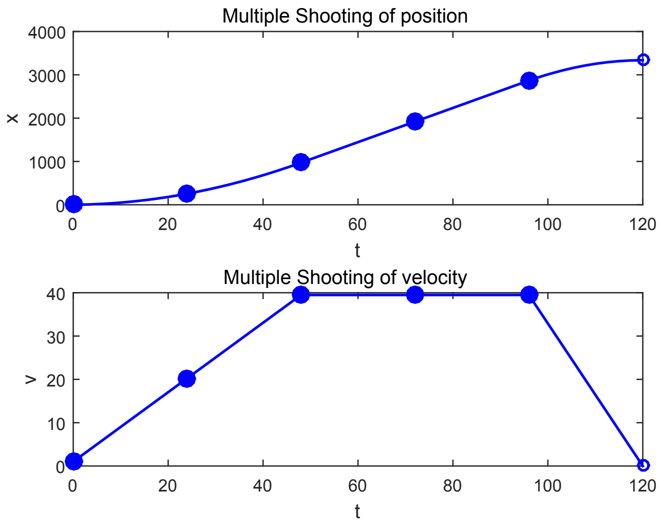

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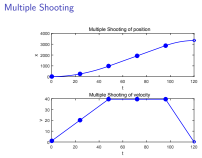

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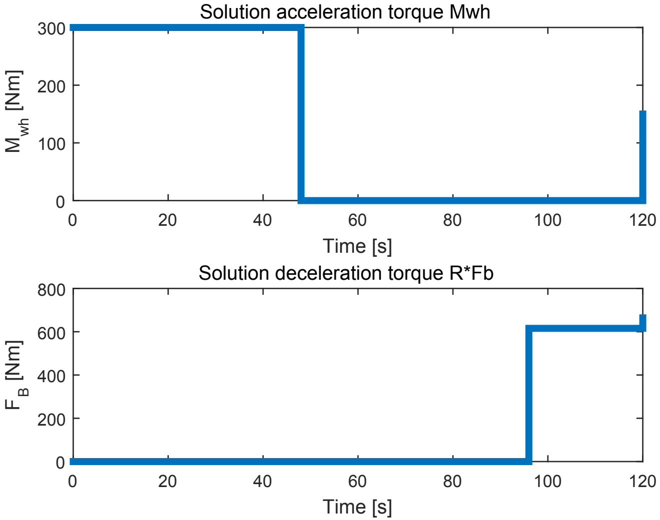

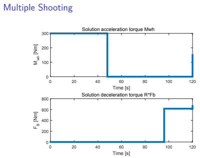

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Problem Formulation

Discrete Optimization Problem

$$\min_{x \in \mathbb{R}^{n_x}, u \in \mathbb{R}^{n_u}} f_0(x, u)$$

$$\begin{aligned} \text{s.t.} \quad & f_i(x,u) \leq 0 && \text{for } i = 1, \ldots, n_f \\ & g_j(x,u,p) = 0 && \text{for } j = 1, \ldots, n_x \end{aligned}$$

with $p \in \mathbb{R}^{n_p}$ uncertain parameter vector

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I think this might not be clear: Both, the weather and height are $\in \mathbb{R}^{n_p}$ or not necessarily the same dimension?

Later, you write $pin\mathbb{P}_{box}$

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\end{aligned}
$$

$$\text{with } p \in \mathbb{P}_{box} = \{ p \in \mathbb{R}^{n_p} | \, p_{lower} \leq p \leq p_{upper} \}$$

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Example:

Problem Formulation
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Worst Case Formulation

EXAMPLE?

$$
\begin{aligned}
\Phi_i(u) := \quad & \max_{x \in \mathbb{R}^{n_x}, p \in \mathbb{R}^{n_p}} f_i(x, u) \\
\text{s.t.} \quad & g(x, u, p) = 0 \\
& p \in \mathbb{P}_{box}
\end{aligned}
$$

Robust Counterpart

$$
\begin{aligned}
& \min_{u \in \mathbb{R}^{n_u}} \Phi_0(u) \\
\text{s.t.} \quad & \Phi_i(u) \leq 0 \qquad \text{for } i = 1, \ldots, n_f.
\end{aligned}
$$

$\Rightarrow$ bilevel structure!

Worst Case Formulation
EXAMPLE?
$\Phi_i(u) := \max_{x \in \mathbb{R}^{n_x}, p \in \mathbb{R}^{n_p}} f_i(x, u)$ s.t. $g(x, u, p) = 0$, $p \in \mathbb{P}_{box}$
Robust Counterpart
$\min_{u \in \mathbb{R}^{n_u}} \Phi_0(u)$ s.t. $\Phi_i(u) \leq 0$ for $i = 1, \ldots, n_f$.
$\Rightarrow$ bilevel structure!

I would not write the remark down.

Approximation Technique

Linearization

$$\tilde{\Phi}_i(u) := \max_{(x-\bar{x})\in\mathbb{R}^{n_x},(p-\bar{p})\in\mathbb{R}^{n_p}} f_i(\bar{x},u) + \frac{\partial f_i}{\partial x}(\bar{x},u)(x-\bar{x})$$

$$\text{s.t.} \quad \frac{\partial g}{\partial x}(\bar{x},u,\bar{p})(x-\bar{x}) + \frac{\partial g}{\partial p}(\bar{x},u,\bar{p})(p-\bar{p}) = 0$$

$$p-\bar{p} \text{ s.th. } p \in \mathbb{P}_{box}$$

$$\min_{u\in\mathbb{R}^{n_u},\bar{x}\in\mathbb{R}^{n_x}} \tilde{\Phi}_0(u)$$

$$\text{s.t.} \quad \tilde{\Phi}_i(u) \le 0 \qquad \text{for } i = 1,\dots,n_f$$

$$g(\bar{x},u,\bar{p}) = 0$$

$\Rightarrow$ Standard Optimization Problem (SQP or else)

Approximation Technique
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s.t. $\frac{\partial g}{\partial x}(\bar{x},u,\bar{p})(x-\bar{x}) + \frac{\partial g}{\partial p}(\bar{x},u,\bar{p})(p-\bar{p}) = 0$
$p-\bar{p}$ s.th. $p \in \mathbb{P}_{box}$
$\min_{u\in\mathbb{R}^{n_u},\bar{x}\in\mathbb{R}^{n_x}} \tilde{\Phi}_0(u)$
s.t. $\tilde{\Phi}_i(u) \le 0$ for $i = 1,\dots,n_f$
$g(\bar{x},u,\bar{p}) = 0$
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Might not be clear what s.th. means I would not write this commend down. (5 times 5 rule ;)

Visualization

Visualization

References

- Aharon Ben-Tal, Laurent El Ghaoui, and Arkadi Nemirovski, *Robust optimization*, Princeton University Press, 2009.
- Moritz Diehl, Hans Georg Bock, Holger Diedam, and Pierre-Brice Wieber, *Fast direct multiple shooting algorithms for optimal robot control*, Fast Motions in Biomechanics and Robotics (2005).
- Moritz Diehl, Hans Georg Bock, and Ekaterina Kostina, *An approximation technique for robust nonlinear optimization*, Math. Program., Ser. B 107, 213Ŭ230 (2006).
- Matthias Gerdts, *Optimale steuerung*, Universität Würzburg, WiSe 2009/2010.

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