

Robust Optimization

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Johannes Milz

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Contents

- 1 Motivation
- 2 Model of Car
- 3 Optimal Control Problem
- 4 Discretization
- 5 Robust Optimization Problem
- 6 Visualization

2015-05-26

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- 1 Motivation
- 2 Model of Car
- 3 Optimal Control Problem
- 4 Discretization
- 5 Robust Optimization Problem
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- Our work is divided into 4 parts
 - Model of a Car
 - Mathematical problem formulation
 - Implementation
 - Visualization

Motivation

- steering a car
- minimize fuel consumption
- constraints
 - avoid crashes
 - dynamics

Problem

Dynamics change considerably, e.g. for different weather.

→ Robust Optimization

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Robust Optimization

Motivation

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- steering a car
- minimize fuel consumption
- constraints
 - avoid crashes
 - dynamics

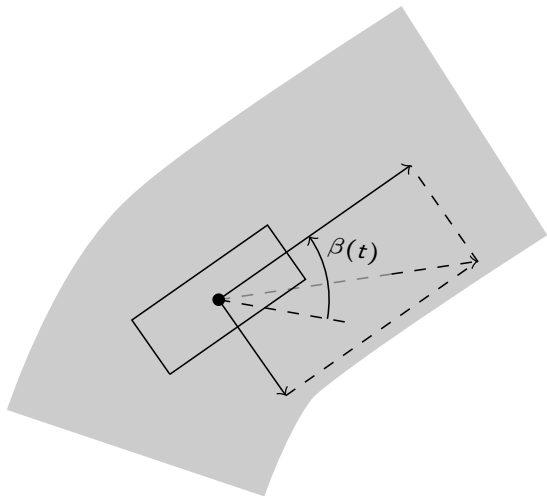
Problem

Dynamics change considerably, e.g. for different weather.

→ Robust Optimization

- Our goal is to steer a car such that the fuel consumption is minimal
- Constraints: Avoid Crashes
 - our car is not allowed to leave the road
 - hit other cars or construction sites
- Dynamics of the car
 - motion of the car
 - slipping behavior
 - acceleration behavior
- Problem: Changing weather has a major impact on the slipping behavior
 - we cannot drive fast on icy roads
 - Rain causes aquaplaning
- Way out: Robust Optimization
 - This Optimization method takes changing parameters such as weather into account.
 - Car is steered save for changing conditions such as weather, different roads...

Model of Car



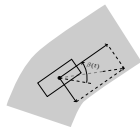
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Robust Optimization

Model of Car

Model of Car

Model of Car



- Our car is described as a pointmass, for simplicity
- Road is visualized in gray
- Car as a rectangle
- steering angle is denoted by $\beta(t)$
 - We applied Newton's mechanic
 - The motion/ direction of motion is described with
 - The steering angle $\beta(t)$
 - The acceleration force pushing the car forward
 - The central forces
 - If the speed of the car is too high, it slides and may leave the road.

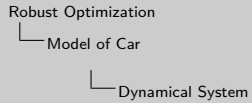
Dynamical System

$$\dot{x}(t) = G(x(t), u(t), p) \qquad x(0) = x_0,$$

where

$x \in C^1([0, t_f], \mathbb{R}^{n_x})$ is the state variable,
 $u \in C^1([0, t_f], \mathbb{R}^{n_u})$ is the control variable,
 $p \in \mathbb{R}^{n_p}$ is the uncertain parameter vector.

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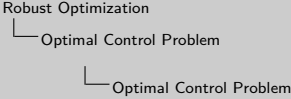
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- Motion of the car is described as a dynamical system, i.e., with a ODE and an initial value
- t_f is the final time
- our states: path, velocity, steering angle
- control: brake, motor torque (gas pedal), steering angle velocity
- parameter: weather, rolling friction coefficients, uncertain sea level due to wrong NAVI data

Optimal Control Problem

$$\begin{aligned} \min_{x(\cdot) \in \mathbb{R}^{n_x}, u(\cdot) \in \mathbb{R}^{n_u}} & f_0(x, u) \\ \text{s.t.} \quad & \dot{x}(t) = G(x(t), u(t), p) \quad \forall t \in [0, t_f], \\ & x(0) = x_0, \\ & f_i(x(t), u(t)) \leq 0 \quad \text{for } i = 1, \dots, n_f, \\ & \quad \quad \quad \forall t \in [0, t_f]. \end{aligned}$$

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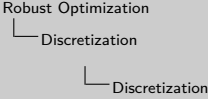
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Discretization

- infinte dimensional optimization problem

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Discretization

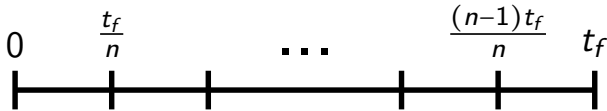
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What do we approximate?
I suggest not to ask questions in the slide, just answer them and mention the questions in your talk (consistency)

We should try to have the same layout for our graphics. I thought that LaTeX does this one its own....

Discretization

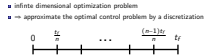
- infinite dimensional optimization problem
- \Rightarrow approximate the optimal control problem by a discretization



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Robust Optimization
└─ Discretization
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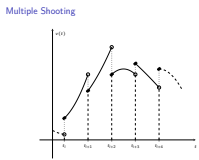
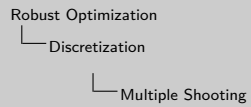
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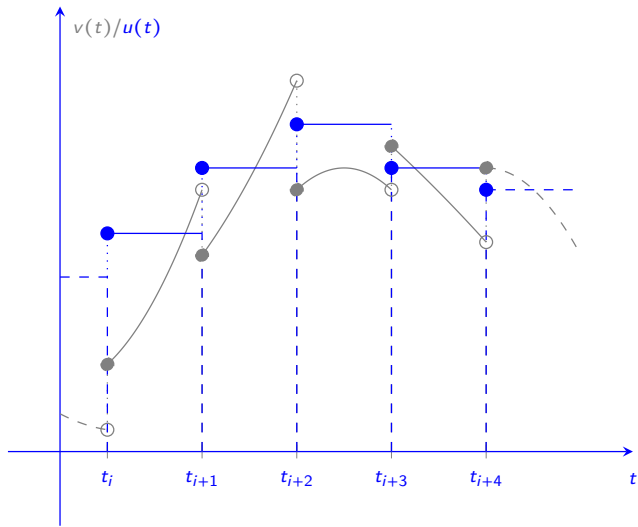
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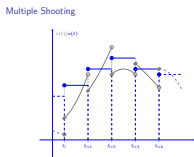
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Multiple Shooting



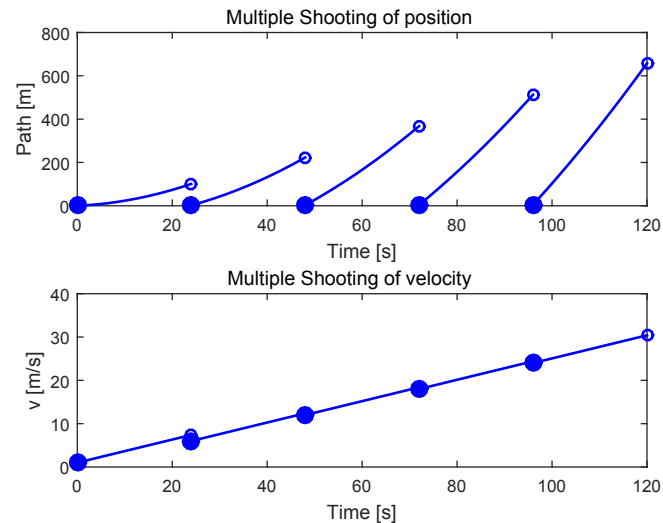
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Robust Optimization
└─ Discretization
 └─ Multiple Shooting

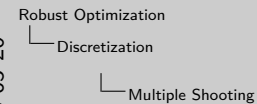


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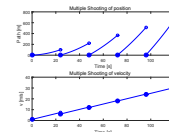
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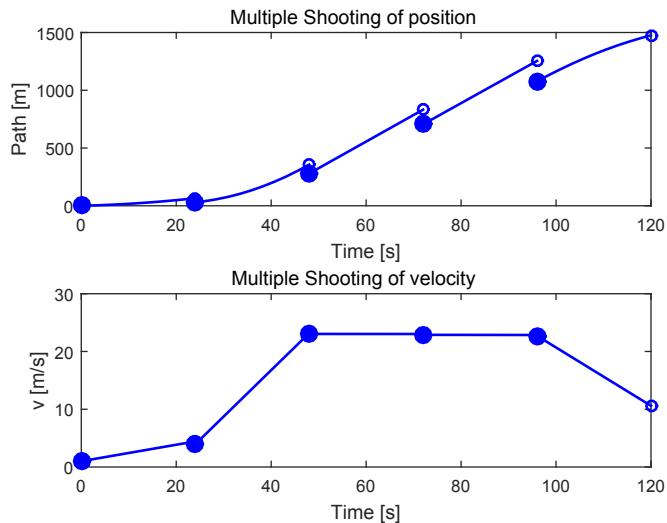


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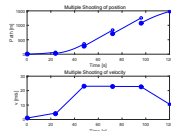
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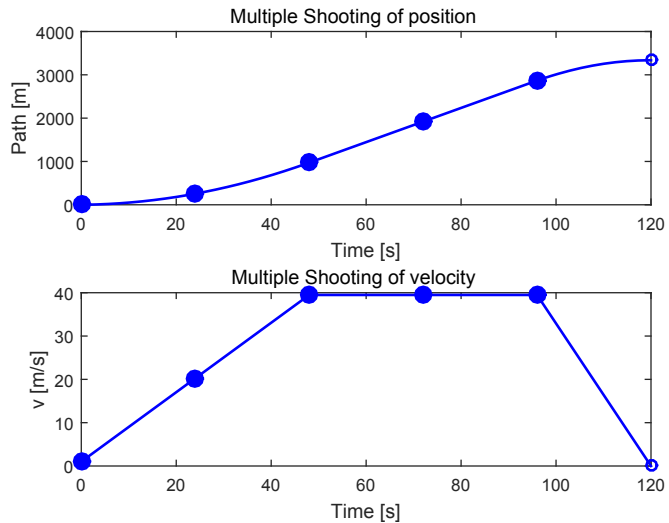
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└─ Discretization
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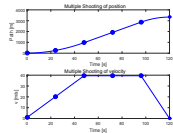
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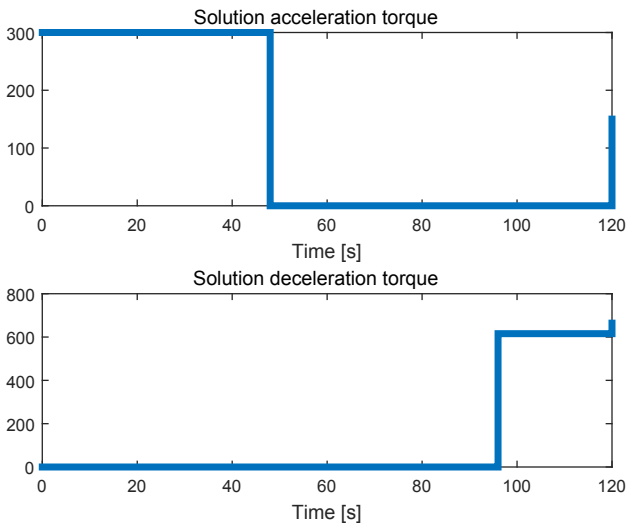
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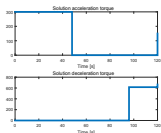
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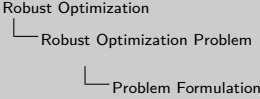
Problem Formulation

Discrete Optimization Problem

$$\begin{aligned} & \min_{x \in \mathbb{R}^{n_x}, u \in \mathbb{R}^{n_u}} f_0(x, u) \\ \text{s.t.} \quad & f_i(x, u) \leq 0 \quad \text{for } i = 1, \dots, n_f \\ & g_j(x, u, p) = 0 \quad \text{for } j = 1, \dots, n_g \end{aligned}$$

with $p \in \mathbb{R}^{n_p}$ uncertain parameter vector

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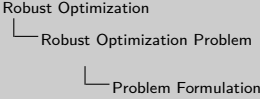
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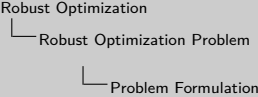
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Worst Case Formulation

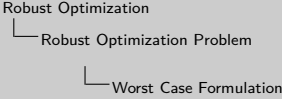
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Robust Counterpart

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⇒ bilevel structure!

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Robust Optimization Problem

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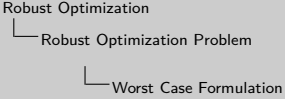
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$$\begin{aligned} & \min_{u \in \mathbb{R}^{n_u}} \tilde{\Phi}_0(u) \\ \text{s.t.} \quad & \tilde{\Phi}_i(u) \leq 0 \quad \text{for } i = 1, \dots, n_f. \end{aligned}$$

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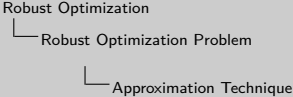
Approximation Technique

Linearization

at a point (\bar{x}, u, \bar{p}) with $g(\bar{x}, u, \bar{p}) = 0$ and $\bar{p} = \frac{p_{lower} + p_{upper}}{2}$:

$$\begin{aligned} \tilde{\Phi}_i(u) := & \max_{(x-\bar{x}) \in \mathbb{R}^{n_x}, (p-\bar{p}) \in \mathbb{R}^{n_p}} f_i(\bar{x}, u) + \frac{\partial f_i}{\partial x}(\bar{x}, u)(x - \bar{x}) \\ \text{s.t.} \quad & \frac{\partial g}{\partial x}(\bar{x}, u, \bar{p})(x - \bar{x}) + \frac{\partial g}{\partial p}(\bar{x}, u, \bar{p})(p - \bar{p}) = 0 \\ & p \in \mathbb{P}_{box} \end{aligned}$$

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at a point (\bar{x}, u, \bar{p}) with $g(\bar{x}, u, \bar{p}) = 0$ and $\bar{p} = \frac{p_{lower} + p_{upper}}{2}$:

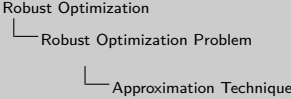
$$\begin{aligned} \tilde{\Phi}_i(u) := & \max_{(x-\bar{x}) \in \mathbb{R}^{n_x}, (p-\bar{p}) \in \mathbb{R}^{n_p}} f_i(\bar{x}, u) + \frac{\partial f_i}{\partial x}(\bar{x}, u)(x - \bar{x}) \\ \text{s.t.} \quad & \frac{\partial g}{\partial x}(\bar{x}, u, \bar{p})(x - \bar{x}) + \frac{\partial g}{\partial p}(\bar{x}, u, \bar{p})(p - \bar{p}) = 0 \\ & p \in \mathbb{P}_{box} \end{aligned}$$

With a unique optimal solution $\tilde{\Phi}_i(u)$:

Standard Optimization Problem

$$\begin{aligned} & \min_{u \in \mathbb{R}^{n_u}, \bar{x} \in \mathbb{R}^{n_x}} \tilde{\Phi}_0(u) \\ \text{s.t.} \quad & \tilde{\Phi}_i(u) \leq 0 \quad \text{for } i = 1, \dots, n_f \\ & g(\bar{x}, u, \bar{p}) = 0 \end{aligned}$$

2015-05-26



Approximation Technique

Linearization

at a point (\bar{x}, u, \bar{p}) with $g(\bar{x}, u, \bar{p}) = 0$ and $\bar{p} = \frac{p_{lower} + p_{upper}}{2}$:

$$\begin{aligned} \tilde{\Phi}_i(u) := & \max_{(x-\bar{x}) \in \mathbb{R}^{n_x}, (p-\bar{p}) \in \mathbb{R}^{n_p}} f_i(\bar{x}, u) + \frac{\partial f_i}{\partial x}(\bar{x}, u)(x - \bar{x}) \\ \text{s.t.} \quad & \frac{\partial g}{\partial x}(\bar{x}, u, \bar{p})(x - \bar{x}) + \frac{\partial g}{\partial p}(\bar{x}, u, \bar{p})(p - \bar{p}) = 0 \\ & p \in \mathbb{P}_{box} \end{aligned}$$

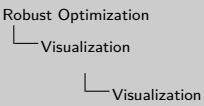
With a unique optimal solution $\tilde{\Phi}_i(u)$:

Standard Optimization Problem

$$\begin{aligned} & \min_{u \in \mathbb{R}^{n_u}, \bar{x} \in \mathbb{R}^{n_x}} \tilde{\Phi}_0(u) \\ \text{s.t.} \quad & \tilde{\Phi}_i(u) \leq 0 \quad \text{for } i = 1, \dots, n_f \\ & g(\bar{x}, u, \bar{p}) = 0 \end{aligned}$$

Visualization

2015-05-26



Visualization

References

2015-05-26

- Robust Optimization
 - Visualization
 - References

References