Robust Optimization

Annkathrin Krämmer, Sabina Przioda, Christian Kreipl,
Johannes Milz

Technische Universität München

May 27, 2015

2015-05-25

Robust Optimization

Annkathrin Krämmer, Sabina Przioda, Christian Kreipl,
Johannes Milz

Technische Universität München

May 27, 2015

Contents

Contents
1. Motivation
2. Model of Car
3. Optimal Control Problem
4. Discretization
5. Robust Optimization Problem
6. Visualization

- Our work is divided into 4 parts
  - Model of a Car
  - Mathematical problem formulation
  - Implementation
  - Visualization

Motivation I

- steering a car
- minimize fuel consumption
- constraints
    - avoid crashes
    - dynamics

Problem

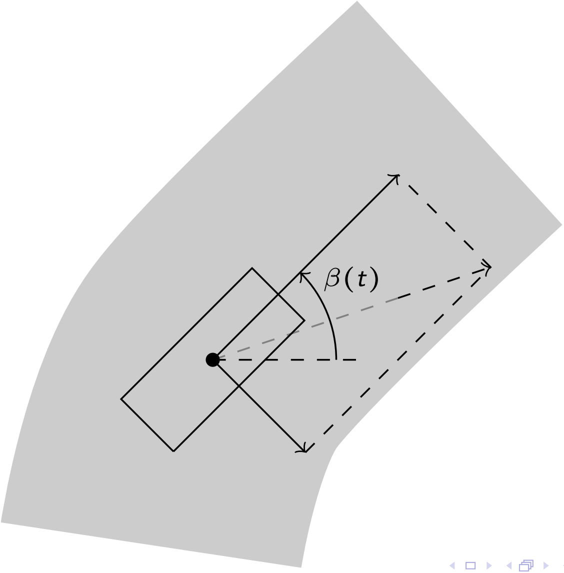
Dynamics change considerably, e.g. for different weather.

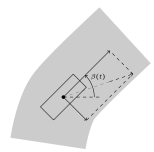
→ Robust Optimization

Motivation I
steering a car
minimize fuel consumption
constraints
avoid crashes
dynamics
Problem
Dynamics change considerably, e.g. for different weather.
→ Robust Optimization

- Our goal is to steer a car such that the fuel consumption is minimal
- Contraints: Avoid Crashes
    - our car is not allowed to leave the road
    - hit other cars or construction sites
- Dynamics of the car
    - motion of the car
        - slipping behavior
        - acceleration behavior
- Problem: Changing weather has a major impact on the slipping behavior
    - we cannot drive fast on icy roads
    - Rain causes aquaplaning
- Way out: Robust Optimization
    - This Optimization method takes changing parameters such as weather into account.
    - Car is steered save for changing conditions such as weather, different roads...

Model of Car





- Our car is described as a pointmass, for simplicity
- Road is visualized in gray
- Car as a rectangle
- steering angle is denoted by $\beta(t)$
  - We applied Newton's mechanic
  - The motion/ direction of motion is described with
    - The steering angle $\beta(t)$
    - The acceleration force pushing the car forward
    - The central forces
    - If the speed of the car is to high, it slides and may leave the road.

Dynamical System

$$\dot{x}(t) = G(x(t), u(t), p) \qquad\qquad x(0) = x_0,$$

where

$$x \in C^1([0, t_f], \mathbb{R}^n) \text{ is the state variable,}$$
$$u \in C^1([0, t_f], \mathbb{R}^m) \text{ is the control variable,}$$
$$p \in \mathbb{R}^{n_p} \text{ is the uncertain parameter vector.}$$

Dynamical System

$\dot{x}(t) = G(x(t), u(t), p) \qquad x(0) = x_0,$

where

$x \in C^1([0, t_f], \mathbb{R}^n)$ is the state variable,
$u \in C^1([0, t_f], \mathbb{R}^m)$ is the control variable,
$p \in \mathbb{R}^{n_p}$ is the uncertain parameter vector.

- Motion of the car is described as a dynamical system, i.e., with a ODE and an initial value
- t_f is the final time of our ride
- our states: path, velocity, steering angle
- control: brake, motor torque (gas petal), steering angle velocity
- parameter: weather, rolling friction coefficients, uncertain sea level due to wrong NAVI data

Optimal Control Problem

$$
\begin{aligned}
&\min_{x(\cdot)\in\mathbb{R}^{n_x},\,u(\cdot)\in\mathbb{R}^{n_u}} f_0(x,u) && \\
\text{s.t.}\quad & \dot{x}(t) = G(x(t),u(t),p) && \forall t\in[0,t_f], \\
& x(0) = x_0, && \\
& f_i(x(t),u(t)) \le 0 && \text{for } i=1,\dots,n_f, \\
& && \forall t\in[0,t_f].
\end{aligned}
$$

Optimal Control Problem

$$
\begin{aligned}
&\min_{x(\cdot)\in\mathbb{R}^{n_x},\,u(\cdot)\in\mathbb{R}^{n_u}} f_0(x,u) && \\
\text{s.t.}\quad & \dot{x}(t) = G(x(t),u(t),p) && \forall t\in[0,t_f], \\
& x(0) = x_0, && \\
& f_i(x(t),u(t)) \le 0 && \text{for } i=1,\dots,n_f, \\
& && \forall t\in[0,t_f].
\end{aligned}
$$

Discretization

- infinte dimensional optimization problem

Discretization

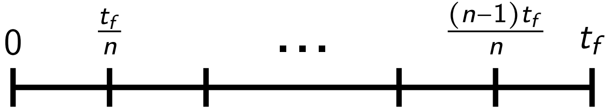
- infinte dimensional optimization problem

For consistency, I would not use several slides to present this content. I am not a fan of that. What is your opinion?
What do we approximate?
I suggest not to ask questions in the slide, just answer them and mention the questions in your talk (consistency)

We should try to have the same layout for our graphics. I thought that LaTeX does this one its own....

Discretization

- infinte dimensional optimization problem



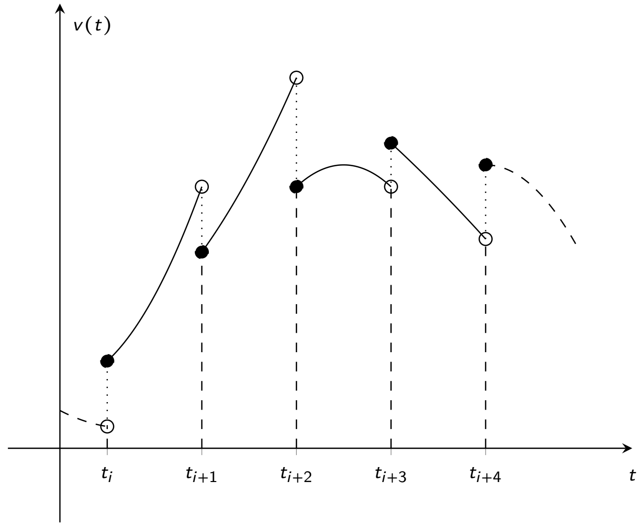
2015-05-25

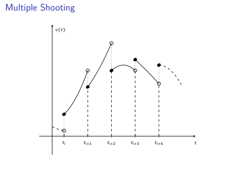
Discretization
- infinte dimensional optimization problem

For consistency, I would not use several slides to present this content. I am not a fan of that. What is your opinion?
What do we approximate?
I suggest not to ask questions in the slide, just answer them and mention the questions in your talk (consistency)

We should try to have the same layout for our graphics. I thought that LaTeX does this one its own....

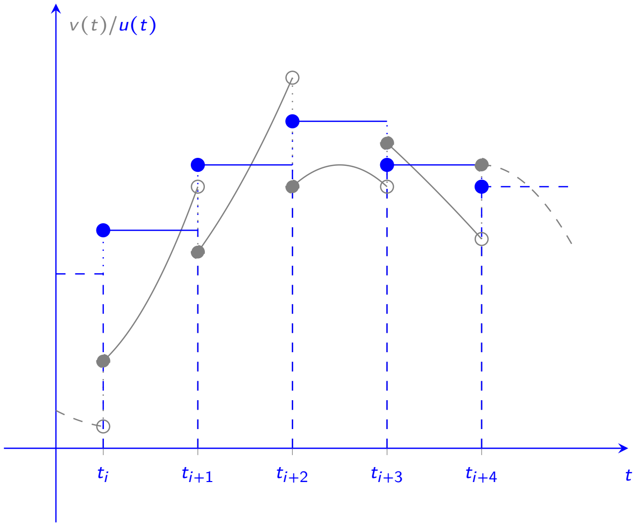
Multiple Shooting

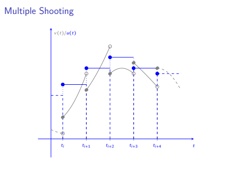




- It might be confusing to plot the state vector as a 1D vector. Maybe, it would be better to plot the velocity. I will mention that this is a state
- The *t* is above the axis
- It might be good to add another blue y -axis!
- The same for the control vector. Is the control between the state by accident?
- I would not present MATLAB graphics (consistency) - just another tikz graphic with continuous state.
- Will the solution output a continuous control? (might be a question of the audience.
- I would omit the plots for Mwh and R*Fb. Why are the vertical lines not dashed? (consistency)

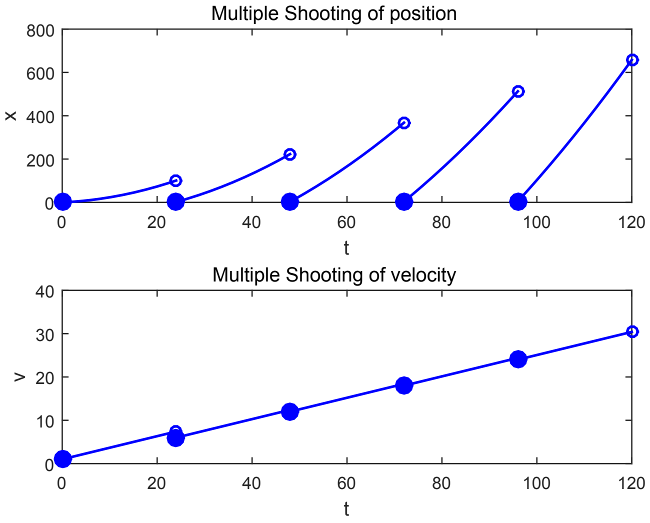
Multiple Shooting



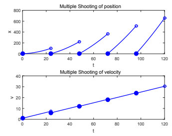


- It might be confusing to plot the state vector as a 1D vector. Maybe, it would be better to plot the velocity. I will mention that this is a state
- The t is above the axis
- It might be good to add another blue y -axis!
- The same for the control vector. Is the control between the state by accident?
- I would not present MATLAB graphics (consistency) - just another tikz graphic with continuous state.
- Will the solution output a continuous control? (might be a question of the audience.
- I would omit the plots for Mwh and R*Fb. Why are the vertical lines not dashed? (consistency)

Multiple Shooting

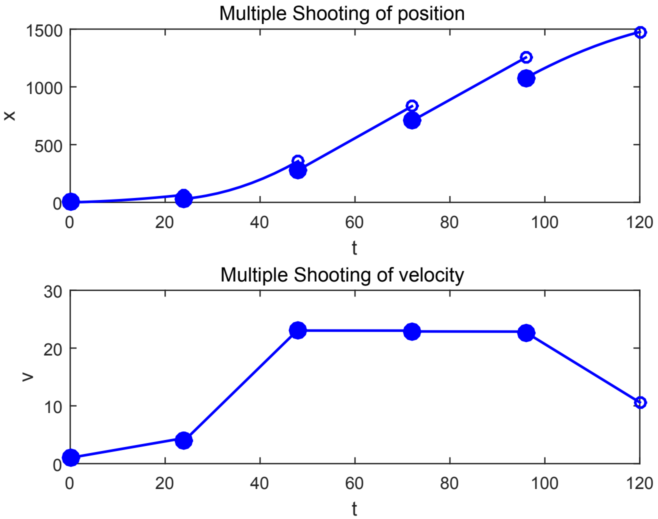


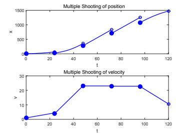
Multiple Shooting



- It might be confusing to plot the state vector as a 1D vector. Maybe, it would be better to plot the velocity. I will mention that this is a state
- The *t* is above the axis
- It might be good to add another blue y -axis!
- The same for the control vector. Is the control between the state by accident?
- I would not present MATLAB graphics (consistency) - just another tikz graphic with continuous state.
- Will the solution output a continuous control? (might be a question of the audience.
- I would omit the plots for Mwh and R*Fb. Why are the vertical lines not dashed? (consistency)

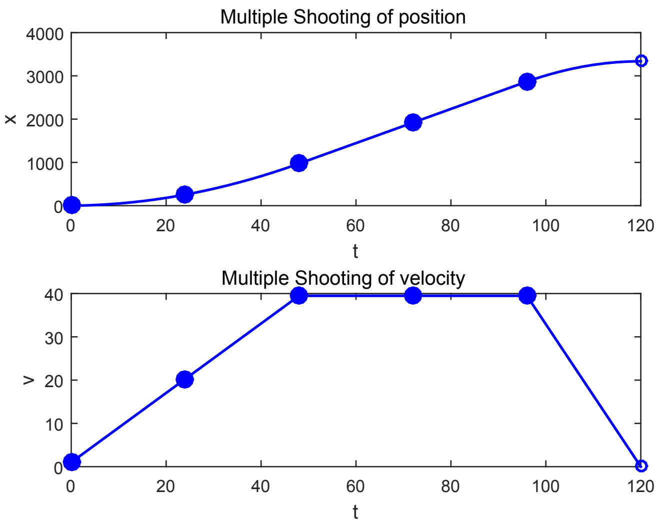
Multiple Shooting

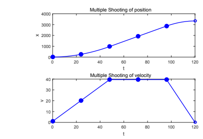




- It might be confusing to plot the state vector as a 1D vector. Maybe, it would be better to plot the velocity. I will mention that this is a state
- The t is above the axis
- It might be good to add another blue y -axis!
- The same for the control vector. Is the control between the state by accident?
- I would not present MATLAB graphics (consistency) - just another tikz graphic with continuous state.
- Will the solution output a continuous control? (might be a question of the audience.
- I would omit the plots for Mwh and R*Fb. Why are the vertical lines not dashed? (consistency)

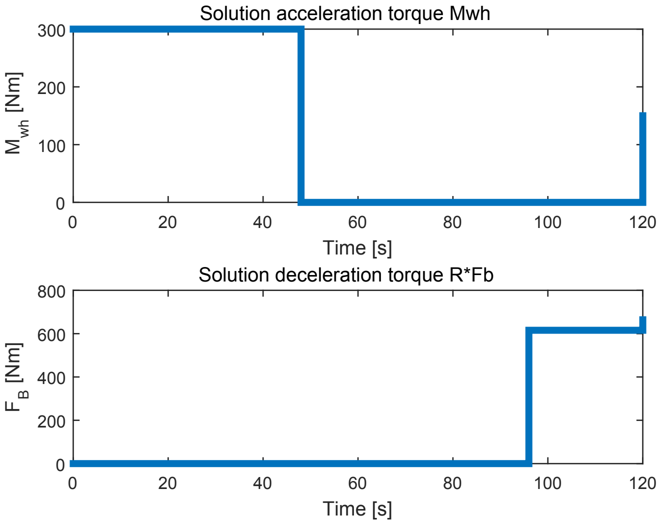
Multiple Shooting

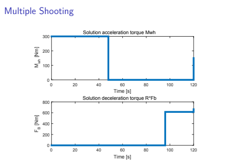




- It might be confusing to plot the state vector as a 1D vector. Maybe, it would be better to plot the velocity. I will mention that this is a state
- The t is above the axis
- It might be good to add another blue y -axis!
- The same for the control vector. Is the control between the state by accident?
- I would not present MATLAB graphics (consistency) - just another tikz graphic with continuous state.
- Will the solution output a continuous control? (might be a question of the audience.
- I would omit the plots for Mwh and R*Fb. Why are the vertical lines not dashed? (consistency)

Multiple Shooting





- It might be confusing to plot the state vector as a 1D vector. Maybe, it would be better to plot the velocity. I will mention that this is a state
- The t is above the axis
- It might be good to add another blue y -axis!
- The same for the control vector. Is the control between the state by accident?
- I would not present MATLAB graphics (consistency) - just another tikz graphic with continuous state.
- Will the solution output a continuous control? (might be a question of the audience.
- I would omit the plots for Mwh and R*Fb. Why are the vertical lines not dashed? (consistency)

Problem Formulation

Discrete Optimization Problem

$$
\begin{aligned}
&\min_{x \in \mathbb{R}^{n_x}, u \in \mathbb{R}^{n_u}} f_0(x, u) \\
\text{s.t.} \quad & f_i(x, u) \leq 0 && \text{for } i = 1, \dots, n_f \\
& g_j(x, u, p) = 0 && \text{for } j = 1, \dots, n_x
\end{aligned}
$$

with $p \in \mathbb{R}^{n_p}$ uncertain parameter vector

Problem Formulation

Discrete Optimization Problem

$$
\begin{aligned}
&\min_{x \in \mathbb{R}^{n_x}, u \in \mathbb{R}^{n_u}} f_0(x, u) \\
\text{s.t.} \quad & f_i(x, u) \leq 0 && \text{for } i = 1, \dots, n_f \\
& g_j(x, u, p) = 0 && \text{for } j = 1, \dots, n_x
\end{aligned}
$$

with $p \in \mathbb{R}^{n_p}$ uncertain parameter vector

Problem Formulation

Discrete Optimization Problem

$$
\begin{aligned}
&\min_{x\in\mathbb{R}^{n_x},u\in\mathbb{R}^{n_u}} f_0(x,u) && \\
\text{s.t.}\quad & f_i(x,u) \le 0 && \text{for } i = 1,\dots,n_f \\
& g_j(x,u,p) = 0 && \text{for } j = 1,\dots,n_x
\end{aligned}
$$

$$\text{with } p \in \mathbb{P}_{box} = \left\{ p \in \mathbb{R}^{n_p} \middle| \, p_{lower} \le p \le p_{upper} \right\}$$

Problem Formulation
Discrete Optimization Problem
$$\min_{x\in\mathbb{R}^{n_x},u\in\mathbb{R}^{n_u}} f_0(x,u)$$
s.t. $f_i(x,u) \le 0$ for $i = 1,\dots,n_f$
$g_j(x,u,p) = 0$ for $j = 1,\dots,n_x$
with $p \in \mathbb{P}_{box} = \{ p \in \mathbb{R}^{n_p} | \, p_{lower} \le p \le p_{upper} \}$

Problem Formulation

Discrete Optimization Problem

$$
\begin{aligned}
& \min_{x\in\mathbb{R}^{n_x},\, u\in\mathbb{R}^{n_u}} && f_0(x,u) && \\
\text{s.t.} \quad & && f_i(x,u) \le 0 && \text{for } i = 1,\dots,n_f \\
& && g_j(x,u,p) = 0 && \text{for } j = 1,\dots,n_x
\end{aligned}
$$

$$
\text{with } p \in \mathbb{P}_{box} = \left\{ p \in \mathbb{R}^{n_p} \,\middle|\, p_{lower} \le p \le p_{upper} \right\}
$$

$$
\begin{aligned}
\rightsquigarrow \Phi_i(u) := & \max_{x\in\mathbb{R}^{n_x},\, p\in\mathbb{R}^{n_p}} && f_i(x,u) && \text{for every } i = 0,\dots,n_f \\
\text{s.t.} \quad & && g(x,u,p) = 0 && \\
& && p \in \mathbb{P}_{box} &&
\end{aligned}
$$

Worst Case Formulation

$$
\begin{aligned}
\Phi_i(u) := \quad & \max_{x \in \mathbb{R}^{n_x}, p \in \mathbb{R}^{n_p}} f_i(x, u) && \text{for every } i = 0, \dots, n_f \\
\text{s.t.} \quad & g(x, u, p) = 0 \\
& p \in \mathbb{P}_{box}
\end{aligned}
$$

Robust Counterpart

$$
\begin{aligned}
& \min_{u \in \mathbb{R}^{n_u}} \Phi_0(u) \\
\text{s.t.} \quad & \Phi_i(u) \le 0 && \text{for } i = 1, \dots, n_f.
\end{aligned}
$$

$\Rightarrow$ bilevel structure!

Worst Case Formulation

$\Phi_i(u) := \max_{x \in \mathbb{R}^{n_x}, p \in \mathbb{R}^{n_p}} f_i(x, u)$ for every $i = 0, \dots, n_f$
s.t. $g(x, u, p) = 0$
$p \in \mathbb{P}_{box}$

Robust Counterpart

$\min_{u \in \mathbb{R}^{n_u}} \Phi_0(u)$
s.t. $\Phi_i(u) \le 0$ for $i = 1, \dots, n_f$.

$\Rightarrow$ bilevel structure!

Worst Case Formulation

$$
\begin{aligned}
\Phi_i(u) :=\quad & \max_{x\in\mathbb{R}^{n_x},\, p\in\mathbb{R}^{n_p}} f_i(x,u) \qquad && \text{for every } i = 0, \dots, n_f \\
\text{s.t.}\quad & g(x,u,p) = 0 \\
& p \in \mathbb{P}_{box}
\end{aligned}
$$

Robust Counterpart

$$
\begin{aligned}
& \min_{u\in\mathbb{R}^{n_u}} \Phi_0(u) \\
\text{s.t.}\quad & \Phi_i(u) \le 0 \qquad && \text{for } i = 1, \dots, n_f.
\end{aligned}
$$

$\Rightarrow$ bilevel structure!

Worst Case Formulation

$\Phi_i(u) := \max_{x\in\mathbb{R}^{n_x}, p\in\mathbb{R}^{n_p}} f_i(x,u)$ for every $i = 0, \dots, n_f$ s.t. $g(x,u,p) = 0$, $p \in \mathbb{P}_{box}$

Robust Counterpart

$\min_{u\in\mathbb{R}^{n_u}} \Phi_0(u)$ s.t. $\Phi_i(u) \le 0$ for $i = 1, \dots, n_f$.

$\Rightarrow$ bilevel structure!

Worst Case Formulation

$$
\begin{aligned}
\Phi_i(u) := \quad & \max_{x \in \mathbb{R}^{n_x}, p \in \mathbb{R}^{n_p}} f_i(x,u) && \text{for every } i = 0, \dots, n_f \\
\text{s.t.} \quad & g(x,u,p) = 0 \\
& p \in \mathbb{P}_{box}
\end{aligned}
$$

Robust Counterpart

$$
\begin{aligned}
& \min_{u \in \mathbb{R}^{n_u}} \tilde{\Phi}_0(u) \\
\text{s.t.} \quad & \tilde{\Phi}_i(u) \le 0 && \text{for } i = 1, \dots, n_f.
\end{aligned}
$$

~~$\Rightarrow$ bilevel structure!~~

Worst Case Formulation

Robust Counterpart

~~$\Rightarrow$ bilevel structure!~~

Approximation Technique

Linearization

at a point $(\bar{x}, u, \bar{p})$ with $g(\bar{x}, u, \bar{p}) = 0$ and $\bar{p} = \frac{p_{lower} + p_{upper}}{2}$:

$$
\begin{aligned}
\tilde{\Phi}_i(u) := \quad & \max_{(x-\bar{x}) \in \mathbb{R}^{n_x}, (p-\bar{p}) \in \mathbb{R}^{n_p}} f_i(\bar{x}, u) + \frac{\partial f_i}{\partial x}(\bar{x}, u)(x - \bar{x}) \\
\text{s.t.} \quad & \frac{\partial g}{\partial x}(\bar{x}, u, \bar{p})(x - \bar{x}) + \frac{\partial g}{\partial p}(\bar{x}, u, \bar{p})(p - \bar{p}) = 0 \\
& p \in \mathbb{P}_{box}
\end{aligned}
$$

Approximation Technique

Linearization

at a point $(\bar{x}, u, \bar{p})$ with $g(\bar{x}, u, \bar{p}) = 0$ and $\bar{p} = \frac{p_{lower} + p_{upper}}{2}$:

$$
\begin{aligned}
\tilde{\Phi}_i(u) := \quad & \max_{(x-\bar{x}) \in \mathbb{R}^{n_x}, (p-\bar{p}) \in \mathbb{R}^{n_p}} f_i(\bar{x}, u) + \frac{\partial f_i}{\partial x}(\bar{x}, u)(x - \bar{x}) \\
\text{s.t.} \quad & \frac{\partial g}{\partial x}(\bar{x}, u, \bar{p})(x - \bar{x}) + \frac{\partial g}{\partial p}(\bar{x}, u, \bar{p})(p - \bar{p}) = 0 \\
& p \in \mathbb{P}_{box}
\end{aligned}
$$

With a unique optimal solution $\tilde{\Phi}_i(u)$:

Standard Optimization Problem

$$
\begin{aligned}
& \min_{u \in \mathbb{R}^{n_u}, \bar{x} \in \mathbb{R}^{n_x}} \tilde{\Phi}_0(u) \\
\text{s.t.} \quad & \tilde{\Phi}_i(u) \leq 0 \qquad \text{for } i = 1, \ldots, n_f \\
& g(\bar{x}, u, \bar{p}) = 0
\end{aligned}
$$

Approximation Technique
Linearization
at a point $(\bar{x}, u, \bar{p})$ with $g(\bar{x}, u, \bar{p}) = 0$ and $\bar{p} = \frac{p_{lower} + p_{upper}}{2}$:
With a unique optimal solution $\tilde{\Phi}_i(u)$:
Standard Optimization Problem

Visualization

References I

- Ben-Tal, A., El Ghaoui, L., and Nemirovski, A. (2009).
  Robust Optimization.
  Princeton University Press.
- Diehl, M., Bock, H. G., Diedam, H., and Wieber, P.-B. (2005).

  Fast direct multiple shooting algorithms for optimal robot control.
  Fast Motions in Biomechanics and Robotics.
- Diehl, M., Bock, H. G., and Kostina, E. (2006).
  An approximation technique for robust nonlinear optimization.
  Math. Program., Ser. B 107, pages 213–230.
- Gerdts, M. (WiSe 2009/2010).
  Optimale Steuerung.
  Universität Würzburg.

References I

- Ben-Tal, A., El Ghaoui, L., and Nemirovski, A. (2009).
  Robust Optimization.
  Princeton University Press.
- Diehl, M., Bock, H. G., Diedam, H., and Wieber, P.-B. (2005).

  Fast direct multiple shooting algorithms for optimal robot control.
  Fast Motions in Biomechanics and Robotics.
- Diehl, M., Bock, H. G., and Kostina, E. (2006).
  An approximation technique for robust nonlinear optimization.
  Math. Program., Ser. B 107, pages 213–230.
- Gerdts, M. (WiSe 2009/2010).
  Optimale Steuerung.
  Universität Würzburg.

References II

- Mitschke, M. and Wallentowitz, H. (2014).
  Dynamik der Kraftfahrzeuge.
  Springer.
- Tipler, P. A. and Mosca, G. (2015).
  Physik für Wissenschaftler und Ingenieure.
  SpringerSpektrum.

References II

- Mitschke, M. and Wallentowitz, H. (2014).
  Dynamik der Kraftfahrzeuge.
  Springer.
- Tipler, P. A. and Mosca, G. (2015).
  Physik für Wissenschaftler und Ingenieure.
  SpringerSpektrum.