# Robust Optimization

Annkathrin Krämmer, Sabina Przioda, Christian Kreipl, Johannes Milz

Technische Universität München

May 27, 2015





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- Our work is divided into 4 parts
  - Model of a Car
  - Mathematical problem formulation
  - Implementation
  - Visualization

# Motivation

- steering a car
- minimize fuel consumption
- constraints
  - avoid crashes
  - dynamics

**Problem**

Dynamics change considerably, e.g. for different weather.

→ Robust Optimization





Motivation

- steering a car
- minimize fuel consumption
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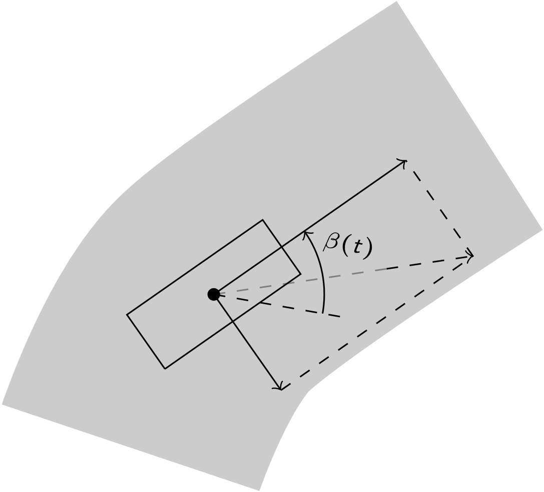
Problem
Dynamics change considerably, e.g. for different weather.
→ Robust Optimization

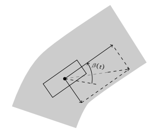
- Our goal is to steer a car such that the fuel consumption is minimal
- Contraints: Avoid Crashes
  - our car is not allowed to leave the road
  - hit other cars or construction sites
- Dynamics of the car
  - motion of the car
    - slipping behavior
    - acceleration behavior
- Problem: Changing weather has a major impact on the slipping behavior
  - we cannot drive fast on icy roads
  - Rain causes aquaplaning
- Way out: Robust Optimization
  - This Optimization method takes changing parameters such as weather into account.
  - Car is steered save for changing conditions such as weather, different roads...

# Model of Car







- Our car is described as a pointmass, for simplicity
- Road is visualized in gray
- Car as a rectangle
- steering angle is denoted by $\beta(t)$
  - We applied Newton's mechanic
  - The motion/ direction of motion is described with
    - The steering angle $\beta(t)$
    - The acceleration force pushing the car forward
    - The central forces
    - If the speed of the car is to high, it slides and may leave the road.

# Dynamical System

$$\dot{x}(t) = G(x(t), u(t), p) \qquad x(0) = x_0,$$

where

$x \in C^1([0, t_f], \mathbb{R}^{n_x})$ is the state variable,
$u \in C^1([0, t_f], \mathbb{R}^{n_u})$ is the control variable,
$p \in \mathbb{R}^{n_p}$ is the uncertain parameter vector.





Dynamical System

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- Motion of the car is described as a dynamical system, i.e., with a ODE and an initial value
- $t_f$ is the final time
- our states: path, velocity, steering angle
- control: brake, motor torque (gas petal), steering angle velocity
- parameter: weather, rolling friction coefficients, uncertain sea level due to wrong NAVI data

# Optimal Control Problem

$$
\begin{aligned}
& \min_{x(\cdot)\in\mathbb{R}^{n_x},\, u(\cdot)\in\mathbb{R}^{n_u}} f_0(x,u) && \\
\text{s.t.}\quad & \dot{x}(t) = G(x(t),u(t),p) && \forall t \in [0,t_f], \\
& x(0) = x_0, && \\
& f_i(x(t),u(t)) \le 0 && \text{for } i = 1,\dots,n_f, \\
& && \forall t \in [0,t_f].
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# Discretization

- Why?





2015-05-25

Discretization

- Why?

For consistency, I would not use several slides to present this content. I am not a fan of that. What is your opinion?
What do we approximate?
I suggest not to ask questions in the slide, just answer them and mention the questions in your talk (consistency)

We should try to have the same layout for our graphics. I thought that LaTeX does this one its own....

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Discretization
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# Discretization

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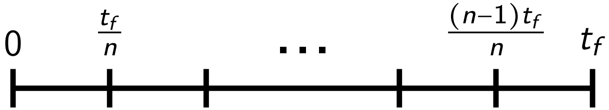
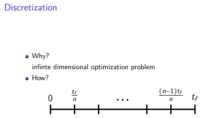
  infinte dimensional optimization problem
- How?





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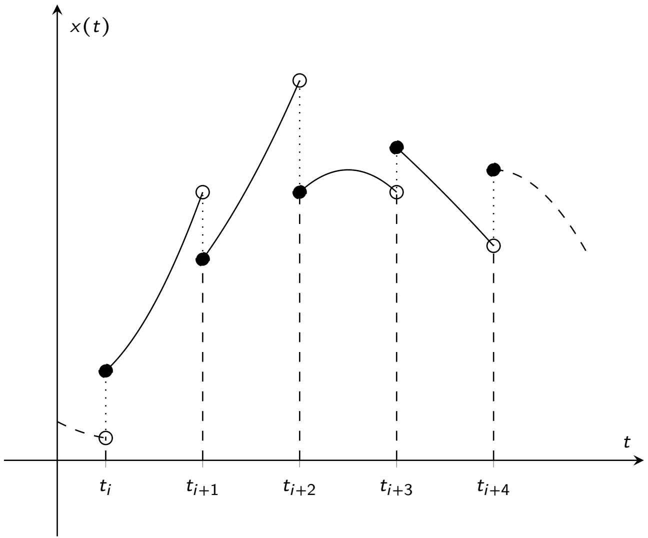






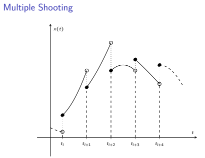


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- The *t* is above the axis
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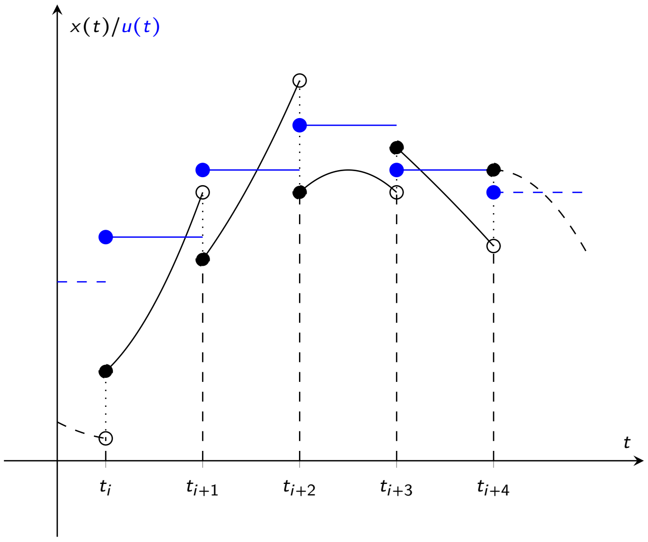






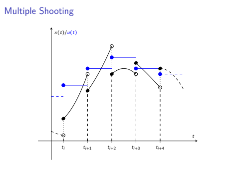

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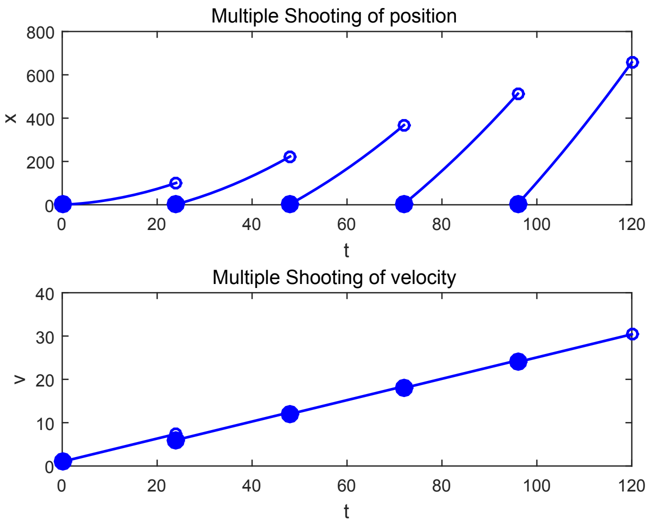






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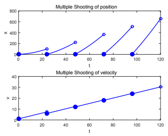

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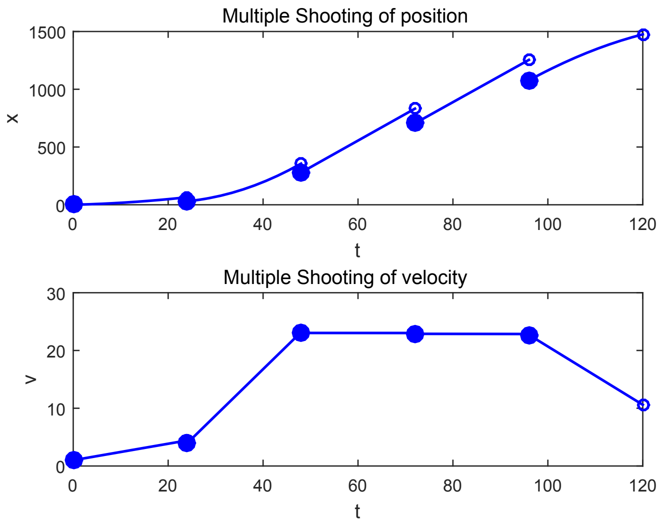






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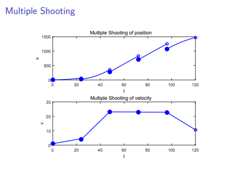


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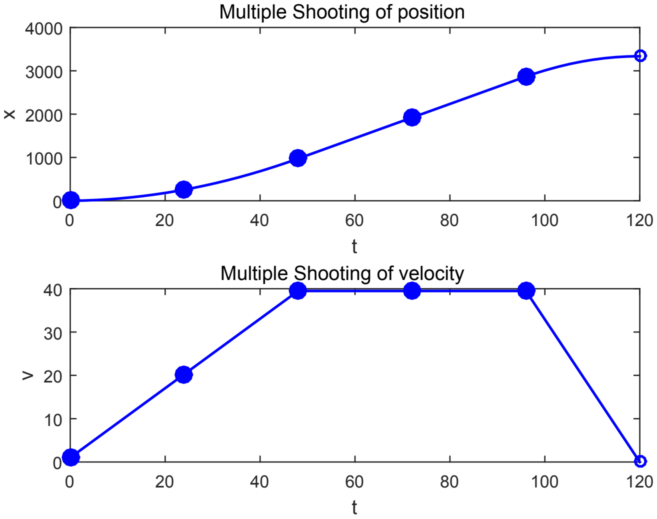






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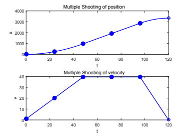


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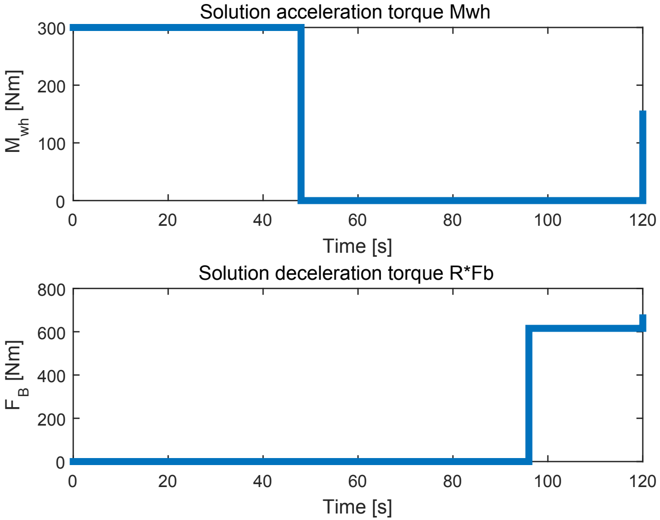






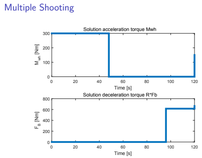


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# Problem Formulation

## Discrete Optimization Problem

$$
\begin{aligned}
& \min_{x \in \mathbb{R}^{n_x}, u \in \mathbb{R}^{n_u}} f_0(x, u) & \\
\text{s.t.} \quad & f_i(x, u) \leq 0 & \text{for } i = 1, \ldots, n_f \\
& g_j(x, u, p) = 0 & \text{for } j = 1, \ldots, n_x
\end{aligned}
$$

with $p \in \mathbb{R}^{n_p}$ uncertain parameter vector





Problem Formulation
Discrete Optimization Problem
$\min_{x \in \mathbb{R}^{n_x}, u \in \mathbb{R}^{n_u}} f_0(x, u)$ s.t. $f_i(x, u) \leq 0$ for $i = 1, \ldots, n_f$; $g_j(x, u, p) = 0$ for $j = 1, \ldots, n_x$
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# Problem Formulation

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with $p \in \mathbb{P}_{box} = \left\{ p \in \mathbb{R}^{n_p} \middle| \, p_{lower} \le p \le p_{upper} \right\}$





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\rightsquigarrow \Phi_i(u) := \quad & \max_{x \in \mathbb{R}^{n_x}, p \in \mathbb{R}^{n_p}} f_i(x,u) \qquad \text{for every } i = 0, \ldots, n_f \\
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$\Rightarrow$ bilevel structure!





Worst Case Formulation

$\Phi_i(u) := \max_{x\in\mathbb{R}^{n_x}, p\in\mathbb{R}^{n_p}} f_i(x,u)$ for every $i = 0,\ldots,n_f$
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$p \in \mathbb{P}_{box}$

Robust Counterpart

$\min_{u\in\mathbb{R}^{n_u}} \Phi_0(u)$
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## Robust Counterpart

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~~$\Longrightarrow$ bilevel structure!~~





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# Approximation Technique

## Linearization

at a point $(\bar{x}, u, \bar{p})$ with $g(\bar{x}, u, \bar{p}) = 0$ and $\bar{p} = \frac{p_{lower} + p_{upper}}{2}$:

$$
\begin{aligned}
\tilde{\Phi}_i(u) := \quad & \max_{(x-\bar{x}) \in \mathbb{R}^{n_x}, (p-\bar{p}) \in \mathbb{R}^{n_p}} f_i(\bar{x}, u) + \frac{\partial f_i}{\partial x}(\bar{x}, u)(x - \bar{x}) \\
\text{s.t.} \quad & \frac{\partial g}{\partial x}(\bar{x}, u, \bar{p})(x - \bar{x}) + \frac{\partial g}{\partial p}(\bar{x}, u, \bar{p})(p - \bar{p}) = 0 \\
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Approximation Technique
Linearization
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With a unique optimal solution $\tilde{\Phi}_i(u)$:

## Standard Optimization Problem

$$
\begin{aligned}
& \min_{u \in \mathbb{R}^{n_u}, \bar{x} \in \mathbb{R}^{n_x}} \tilde{\Phi}_0(u) \\
\text{s.t.} \quad & \tilde{\Phi}_i(u) \leq 0 \qquad \text{for } i = 1, \ldots, n_f \\
& g(\bar{x}, u, \bar{p}) = 0
\end{aligned}
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Approximation Technique
Linearization
at a point $(\bar{x}, u, \bar{p})$ with $g(\bar{x}, u, \bar{p}) = 0$ and $\bar{p} = \frac{p_{lower} + p_{upper}}{2}$:
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Standard Optimization Problem

# Visualization

# References