Math 212 Portfolio

Lafe Wessel

Draft date: April 22, 2021

Changelog: *List the changes you've made since the last draft, with special attention paid to problems that have received significant revisions since the last draft (i.e., more than fixing typos). If there is any additional information you'd like me to consider as I review this submission, please say so now.*

1. Reworked Conjecture VIII

Instructions: Each of the problems below is/will be presented as a conjecture. Each conjecture asks you to prove or disprove the conjecture, possibly along with some additional directions.

- If the conjecture is true, your job is to write a complete proof for the proposition. If there are multiple parts, you should consider each part in turn.
- If it is false, you should provide a counterexample plus make reasonable modifications to the stated conjecture so that a new proposition is true. Then, write a complete proof of your new proposition. You may want to run your new proposition by me before trying to write a proof–this is allowed and encouraged!

Academic Honesty Policy: The portfolio is an independent project in which no outside resources or collaboration is allowed. You may not ask other professors or discuss the problems with anyone besides me. You should not discuss even which problem you chose. Violation of this policy is grounds for failing the course. The point is that you need to be confident and competent in writing proofs for future courses.

Conjecture I. *Let A and B be subsets of some universe $\mathcal{U}$. Then:*

1. $A \setminus (A \cap \overline{B}) = A \cap B$
2. $\overline{(\overline{A} \cup B)} \cap A = A \setminus B$
3. $(A \cup B) \setminus A = B \setminus A$
4. $(A \cup B) \setminus B = A \setminus (A \cap B)$

All four statements of Conjecture I will be proved to be equal through the set equivalence method: demonstrating that the left side is a subset of the right side and that the right side is a subset of the left side.

Conjecture I.1.

Proof. Let $x \in A \setminus (A \cap \overline{B})$. Thus, x must be an element of A and not an element of $A \cap \overline{B}$, that is $x \in A$ and $x \notin A \cap \overline{B}$. If x is not in $A \cap \overline{B}$, then x must not be in $\overline{B}$, meaning $x \in B$. Thus, $x \in A \cap B$, and, furthermore, $A \setminus (A \cap \overline{B}) \subseteq A \cap B$.

Let $x \in A \cap B$. For this to be true, x must be in both A and B; $x \in A$ and $x \in B$. Thus, x cannot be in the complement of B, $\overline{B}$. Since $x \notin \overline{B}$, $x \notin A \cap \overline{B}$. Because $x \in A$ and $x \notin A \cap \overline{B}$, $x \in A \setminus (A \cap \overline{B})$. Thus, $A \cap B \subseteq A \setminus (A \cap \overline{B})$.

Therefore, since $A \setminus (A \cap \overline{B}) \subseteq A \cap B$ and $A \cap B \subseteq A \setminus (A \cap \overline{B})$, $A \setminus (A \cap \overline{B}) = A \cap B$. $\square$

Conjecture I.2

Proof. Let $x \in \overline{(\overline{A} \cup B)} \cap A$. Thus, $x \in A$ and $x \in \overline{(\overline{A} \cup B)}$. This means that x is not in the complement of $\overline{(\overline{A} \cup B)}$, which is $\overline{A} \cup B$. Thus, $x \notin \overline{A} \cup B$. Consequently, $x \notin \overline{A}$ and $x \notin B$. If $x \notin \overline{A}$, then $x \in A$. Since $x \in A$ and $x \notin B$, $x \in A \setminus B$. Thus, if $x \in A \setminus B$ and $x \in \overline{(\overline{A} \cup B)} \cap A$, $\overline{(\overline{A} \cup B)} \cap A \subseteq A \setminus B$.

Let $x \in A \setminus B$. If this is true, then $x \in A$ and $x \notin B$. It follows that $x \notin \overline{A}$ and, consequently, that $x \notin \overline{A} \cup B$. If $x \notin \overline{A} \cup B$, then x must be in the complement, being $\overline{(\overline{A} \cup B)}$. Since $x \in A$ and $x \in \overline{(\overline{A} \cup B)}$, then $x \in \overline{(\overline{A} \cup B)} \cap A$. Thus, $A \setminus B \subseteq \overline{(\overline{A} \cup B)} \cap A$.

Therefore, since $\overline{(\overline{A} \cup B)} \cap A \subseteq A \setminus B$ and $A \setminus B \subseteq \overline{(\overline{A} \cup B)} \cap A$, $\overline{(\overline{A} \cup B)} \cap A = A \setminus B$. $\square$

Conjecture I.3

Proof. Let $x \in (A \cup B) \setminus A$. For x to be an element, it must be in $A \cup B$ and not in A. Thus, $x \in A \cup B$ and $x \notin A$. Since $x \notin A$, x must be a part of B for $A \cup B$ to be true: $x \in B$. If $x \notin A$ and $x \in B$, then $x \in B \setminus A$. If $x \in (A \cup B) \setminus A$ and $x \in B \setminus A$, then $(A \cup B) \setminus A \subseteq B \setminus A$.

Let $x \in B \setminus A$. Thus, x must be in B and not in A. Accordingly, $x \in B$ and $x \notin A$. It follows that $x \in A \cup B$, and, consequently, that $x \in (A \cup B) \setminus A$ since $x \notin A$. Thus, because $x \in B \setminus A$ and $x \in (A \cup B) \setminus A$, $B \setminus A \subseteq (A \cup B) \setminus A$.

Therefore, since $(A \cup B) \setminus A \subseteq B \setminus A$ and $B \setminus A \subseteq (A \cup B) \setminus A$, $(A \cup B) \setminus A = B \setminus A$. $\square$

Conjecture I.4

Proof. Let $x \in (A \cup B) \setminus B$. It follows that $x \in A \cup B$ and $x \notin B$. This means that $x \in A$ since $x \in A \cup B$ is true. Thus, $x \notin A \cap B$. Because $x \in A$ and $x \notin A \cap B$, $x \in A \setminus (A \cap B)$ is true. Thus, since $x \in (A \cup B) \setminus B$ and $x \in A \setminus (A \cap B)$, $(A \cup B) \setminus B \subseteq A \setminus (A \cap B)$.

Let $x \in A \setminus (A \cap B)$. Thus, $x \in A$ and $x \notin A \cap B$, meaning that $x \notin B$. If follows that $x \in A \cup B$ and, since $x \notin B$, $x \in (A \cup B) \setminus B$ are both true. Thus, if $x \in A \setminus (A \cap B)$ and $x \in (A \cup B) \setminus B$ are true, then $A \setminus (A \cap B) \subseteq (A \cup B) \setminus B$ is true.

Therefore, since $(A \cup B) \setminus B \subseteq A \setminus (A \cap B)$ and $A \setminus (A \cap B) \subseteq (A \cup B) \setminus B$ are true, it must be true that $(A \cup B) \setminus B = A \setminus (A \cap B)$. $\square$

Conjecture II. *Define* $f : \mathbb{N} \setminus \{0\} \to \mathbb{Z}$ *as follows: for each* $n \in \mathbb{N} \setminus \{0\}$,

$$f(n) = \frac{1 + (-1)^n(2n-1)}{4}.$$

Then f *is a bijection.*

Conjecture II will be proved to be a bijection by first showing that it is an injection and subsequently that it is a surjection. Proving the function injective will be done by demonstrating that $f(x) = f(y) \to x = y$, and proving the function surjective will be done by demonstrating that every value of the codomain is reflected by a value of the domain.

We notice that when n is even, the result is positive, and when n is odd, the result is negative. This is because $(-1)^1 = -1, (-1)^3 = -1, ...$ and $(-1)^2 = 1, (-1)^4 = 1, ...$ Thus, we must evaluate the function both when n is even and when n is odd.

Proof. First, we show that f is an injection.

When n is even, we show that $f(x) = f(y) \to x = y$, and we know that $(-1)^n = 1$.

$$f(x) = \frac{1 + (1)(2x-1)}{4} \tag{1}$$

$$f(y) = \frac{1 + (1)(2y-1)}{4} \tag{2}$$

$$\frac{1 + (1)(2x-1)}{4} = \frac{1 + (1)(2y-1)}{4} \tag{3}$$

$$1 + (1)(2x-1) = 1 + (1)(2y-1) \tag{4}$$

$$(1)(2x-1) = (1)(2y-1) \tag{5}$$

$$2x - 1 = 2y - 1 \tag{6}$$

$$2x = 2y \tag{7}$$

$$x = y \tag{8}$$

When n is odd, we show that $f(x) = f(y) \to x = y$, and we know that $(-1)^n = -1$.

$$f(x) = \frac{1 + (-1)(2x - 1)}{4} \tag{9}$$

$$f(y) = \frac{1 + (-1)(2y - 1)}{4} \tag{10}$$

$$\frac{1 + (-1)(2x - 1)}{4} = \frac{1 + (-1)(2y - 1)}{4} \tag{11}$$

$$1 + (-1)(2x - 1) = 1 + (-1)(2y - 1) \tag{12}$$

$$(-1)(2x - 1) = (-1)(2y - 1) \tag{13}$$

$$-2x + 1 = -2y + 1 \tag{14}$$

$$-2x = -2y \tag{15}$$

$$x = y \tag{16}$$

Therefore, since $f(x) = f(y) \to x = y$ both when n is even and odd, we know that the function $f(n) = \frac{1+(-1)^n(2n-1)}{4}$ is an injection.

Next, we show that f is a surjection.

We must solve for y in $f(x) = y$ to demonstrate that the codomain has at least one corresponding value of the domain in order for $f(n)$ to be surjective.

When n is even, we know that $(-1)^n = 1$. Solving for x, we find that $x = 2y$. Thus, since the function $2y = x$ is linear, having all possible values of x and y as its domain and codomain, $f(n)$ is surjective when n is even.

When n is odd, we know that $(-1)^n = -1$. Solving for x, we find that $x = -2y + 1$. Thus, since the function $-2y + 1 = x$ is linear, having all possible values of x and y as its domain and codomain, $f(n)$ is surjective when n is odd.

Since $f(n)$ is surjective both when n is even and when it is odd, we know that the function is surjective.

Therefore, since $f(n)$ has been demonstrated to be both injective and surjective, the function is a bijection. $\square$

Example III. *For the following example, choose two of the four problems to do. Exactly one of your choices should be a combinatorial proof.*

1. *(Combinatorial) For $n \geq 1$,*
$$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}.$$
2. *(Combinatorial) For $0 \leq k \leq n$,*
$$\sum_{m=k}^{n} \binom{m}{k} = \binom{n+1}{k+1}.$$
3. *Consider the alphabet $\{a, b, c, d, e, f\}$ and make words without repetition of letters allowed.*
   (a) *How many six-letter words are there?*
   (b) *How many words begin with d or e?*
   (c) *How many words end in b or a?*
   (d) *How many words begin with d or e and end in b or a?*
   (e) *How many have first letter neither d nor e and last letter neither b nor a?*
4. *We wish to improve upon the ogre's distribution of 43 cupcakes to 12 baby mice by ensuring that every baby mouse gets at least two cupcakes. How many ways are there to accomplish this?*

Proof. Proof 1

The summation

$$\sum_{k=0}^{n} k \binom{n}{k} = n2^{n-1}$$

when $n \geq 1$, shall be demonstrated to be true as a way of counting sets. The cardinality of the power set of a set with cardinality n is 2^n. This number is the total number of ways to include or exclude any elements from the set. Thus, 2^{n-1} is the number of ways to choose any number of elements from a set with cardinality $n-1$. Multiplying this by n is the ways to arrange that set for each element in n. Thus, the right-hand side of the equation can be thought of as counting the ways to choose elements from a set for each way that we can remove one element.

The sum of each row of Pascal's Triangle is equal to 2^{depth}. Thus,

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n,$$

since each element of Pascal's Triangle is equal to $\binom{n}{k}$ for a given n and k when $n \geq 1$ and $n \geq k \geq 0$. Multiplying each $\binom{n}{k}$ by k counts the ways to choose k from a set of n elements for each element being chosen, k. (i.e. For each of k sweaters, how many ways are there to choose k sweaters from a wardrobe of n sweaters?) This keeps the symmetry of Pascal's Triangle and when summed across a row, will equal $n \cdot 2^{n-1}$. Thus, the left-hand side of the equation can be thought of as counting the ways that we can choose k elements from n for each element that we choose, and summing them until we are choosing the entire set.

This proof answers the question how many ways can I choose one element, then for each of 2 options I choose 2 from the set, then for 3 options I choose 3 from the set, all the way until for each n options I choose n elements from the set containing n elements. Or, I could count the total ways to choose from a set of $n-1$ elements for each of n elements, being $n2^{n-1}$. $\square$

Proof. Proof 3

(a) There are 6 options for the first letter, 5 for the second, 4 for the third... for a total of $6! = 720$ options for 6-letter words.

(b) There are 2 options for the first letter, 5 for the second, 4 for the third... for a total of $2 \cdot 5! = 240$ words that start with d or e.

(c) This is essentially (b) but backwards. There are 2 options for the last letter, 5 options for the penultimate... for a total of $2 \cdot 5! = 240$ words that end in a or b.

(d) To find the total, we must add the results from (b) and (c), but subtract out the number that get overcounted through PIE. Thus, we must subtract the number that start with d or e and end in b or a. We have 2 options for the first and last letters, and 4 for the second, 3 for the third... for a total of $2 \cdot 2 \cdot 4! = 96$ words. Next, we subtract 96 from the sum of (a) and (b), $240 + 240 = 480$. Therefore, we have $480 - 96 = 384$ words that both begin with d or e and end with a or b.

(e) This question asks for the complement of (d): in (d), we found how many words both begin with d or e and end with b or a, whereas now we must find the words that do not fall into that category. Since we know how many words we have in total from (a), we can subtract the words that do not fit the criteria from the total. Thus, we have $720 - 384 = 336$ words that neither begin with d or e nor end with a or b.

□

Theorem IV. *Consider the recurrence relation $a_n = sa_{n-1} + d$ where $s \neq 1$. Prove that*

$$a_n = \left(a_0 + \frac{d}{s-1}\right) s^n - \frac{d}{s-1}$$

is a solution. Then use this theorem to solve for a closed formula for the recurrence $a_n = 5a_{n-1}+3$ where $a_0 = 1$.

Proof. To show that the shown solution is actually a solution to the shown recurrence relation, we will substitute a_{n-1} into the original recurrence relation.

$$a_{n-1} = (a_0 + \frac{d}{s-1})s^{n-1} - \frac{d}{s-1}$$

This can then be substituted into $a_n = sa_{n-1} + d$.

$$\begin{aligned}
a_n &= s((a_0 + \frac{d}{s-1})s^{n-1} - \frac{d}{s-1}) + d \\
a_n &= s((a_0 + \frac{d}{s-1})s^{n-1}) + \frac{-sd}{s-1} + d \\
a_n &= s \cdot s^{n-1}(a_0 + \frac{d}{s-1}) + \frac{-sd}{s-1} + d \\
a_n &= s^n(a_0 + \frac{d}{s-1}) + \frac{-sd}{s-1} + d \\
a_n &= s^n(a_0 + \frac{d}{s-1}) + \frac{-sd}{s-1} + \frac{d(s-1)}{s-1} \\
a_n &= s^n(a_0 + \frac{d}{s-1}) + \frac{-sd}{s-1} + \frac{sd-d}{s-1} \\
a_n &= s^n(a_0 + \frac{d}{s-1}) + \frac{-sd+sd-d}{s-1} \\
a_n &= s^n(a_0 + \frac{d}{s-1}) + \frac{-d}{s-1} \\
a_n &= (a_0 + \frac{d}{s-1})s^n - \frac{d}{s-1}
\end{aligned}$$

Thus, the solution can be used to find a closed-form solution to recurrence relations of the form $a_n = sa_{n-1} + d$ where $s \neq 1$. □

Proof. To solve the recurrence relation $a_n = 5a_{n-1} + 3$ where $a_0 = 1$, we will find the closed-form equation using the formula $a_n = \left(a_0 + \frac{d}{s-1}\right) s^n - \frac{d}{s-1}$, since the relation follows the form $a_n = sa_{n-1} + d$. Using this formula and previously known values, we know that $a_0 = 1$, $d = 3$, and $s = 5$. Plugging these in and solving, we see that

$$a_n = (1 + \frac{3}{5-1}) \cdot 5^n - \frac{3}{5-1}$$
$$a_n = (1 + \frac{3}{4})5^n - \frac{3}{4}$$
$$a_n = \frac{7}{4}5^n - \frac{3}{4}$$

□

Conjecture V. *For all $n \geq 1$,*

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n \cdot (n+1)} = \frac{n}{n+1}$$

Proof. (By induction.) Let $P(n)$ be the statement that for all $n \geq 1$, $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n \cdot (n+1)} = \frac{n}{n+1}$.

Base Case: For $n = 1$.

$$\begin{aligned} P(1) = \frac{1}{1 \cdot (1+1)} &= \frac{1}{1+1} \\ \frac{1}{2} &= \frac{1}{2} \end{aligned}$$

Since $P(1)$ is true, we can move on to the Inductive Hypothesis.

Inductive Hypothesis: Assume $P(k)$ is true for all $1 \leq k < n$.

Inductive Step: Consider $k + 1$.

$$P(k+1) = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{k \cdot (k+1)} + \frac{1}{(k+1) \cdot (k+1+1)}.$$

This can be reduced to

$$\frac{k}{k+1} + \frac{1}{(k+1) \cdot (k+2)},$$

which can be further reduced

$$\begin{aligned} &= \frac{k}{k+1} \cdot \frac{k+2}{k+2} + \frac{1}{(k+1) \cdot (k+2)} \\ &= \frac{k(k+2)}{(k+1)(k+2)} + \frac{1}{(k+1)(k+2)} \\ &= \frac{k^2 + 2k + 1}{(k+1)(k+2)} \\ &= \frac{(k+1)^2}{(k+1)(k+2)} \\ &= \frac{k+1}{k+2} \\ &= \frac{k+1}{(k+1)+1}. \end{aligned}$$

Therefore, since $P(k)$ is true for all $1 \leq k < n$, by strong induction we can say that $P(n)$ is true for all values of $n \geq 1$. □

For Theorem VI, we will use the following definition.

Definition 1. Let $a, b, \in \mathbb{Z}$ and $m \in \mathbb{N}$ with $m > 1$. We say that a is *congruent to b modulo m* if $m|(a-b)$. We write $a \equiv b \mod m$.

Thus, e.g., $11 \equiv 3 \mod 4$, since $4|11-3$, but $9 \not\equiv 3 \mod 4$ since $4 \nmid 9-3$.

Theorem VI. *Suppose $a, b \in \mathbb{Z}$ and $m \in \mathbb{N}$ with $m > 1$ such that $a \equiv b \mod m$. Then $a^2 \equiv b^2 \mod m$.*

Proof. Let $c \equiv d \mod m$ for $c, d \in \mathbb{Z}$. The expression $a \equiv b \mod m$ can be rearranged to $\frac{a-b}{n} = q_1$ for $q_1 \in \mathbb{Z}$. This can be further rearranged to $a = b + nq_1$. Applying this to $c \equiv d \mod m$, we get $c = d + nq_2$ for $q_2 \in \mathbb{Z}$. Multiplying these together, we get $ac = (b + nq_1)(d + nq_2)$, expanding results in $ac = bd + bnq_2 + dnq_1 + q_1q_2n^2$. Factoring n gives $ac = bd + n(bq_2 + dq_1 + q_1q_2n)$, which demonstrates that if $a \equiv b \mod m$ and $c \equiv d \mod m$, then $ac \equiv bd \mod m$. If $a = c$ and $b = d$, then $a^2 \equiv b^2 \mod m$. Thus, if $a \equiv b \mod m$, then $a^2 \equiv b^2 \mod m$. $\square$

Conjecture VII. *If a, b, and c are integers, then $ab + ac$ is even.*

Proof. First, we must establish if the sums and products of even and odd integers have even or odd results.

Let $x, y \in \mathbb{Z}$ and are even. Thus, $x = 2k$ and $y = 2n$. Adding these together, we get $x + y = 2k + 2n = 2(k + n)$, meaning that the sum of 2 even integers is even. Multiplying, we get $xy = (2k)(2n) = 2(2kn)$, meaning that the product of 2 even integers is even.

Let $x, y \in \mathbb{Z}$ and are odd. Thus, $x = 2k + 1$ and $y = 2n + 1$. Adding these together, we get $x + y = 2k + 2n + 2 = 2(k + n + 1)$, meaning that the sum of 2 odd integers is even. Multiplying, we get $xy = (2k + 1)(2n + 1) = 2(2kn + k + n) + 1$, meaning that the product of 2 odd integers is odd.

Let $x, y \in \mathbb{Z}$, where x is even, and y is odd. Thus, $x = 2k$ and $y = 2n + 1$. Adding these together, we get $x + y = 2k + 2n + 1 = 2(k + n) + 1$, meaning that the sum of an odd and even integer is odd. Multiplying, we get $xy = (2k)(2n + 1) = 2(2kn + k)$, meaning that the product of an even and odd integer is even.

From these proofs, we can construct a truth table. For brevity's sake, 1 signifies 'odd' and 0 signifies 'even'.

a	b	c	ab	ac	ab+ac
1	1	1	1	1	0
1	1	0	1	0	1
1	0	1	0	1	1
1	0	0	0	0	0
0	1	1	0	0	0
0	1	0	0	0	0
0	0	1	0	0	0
0	0	0	0	0	0

Therefore, since there are instances in the truth table where $ab + ac$ is odd, the conjecture is false. The conjecture should be amended to state "For any integers a, b, c, if a is even, then $ab + ac$ is even." This is known to be true by the truth table shown. □

Conjecture VIII. *If r is any real number and ξ is irrational, then $r+\xi$ is irrational or $-r+\xi$ is irrational.*[1]

Proof. To prove if any real number $r\pm$ any irrational(ξ) is irrational, we must investigate when r is rational and irrational since the real numbers encompass both the rational and irrational numbers. The following equation expresses this: $\xi \pm r = k$, where ξ is irrational, r is real, and k is irrational.

To (partially) prove by contradiction, suppose that k is rational. If r and k are rational, then $k+(-r)$ is rational because a rational plus a rational is rational. Thus $(\xi+r)-r=\xi$ is rational, showing the contradiction. Thus, if r is rational, then k is irrational.

To prove the second part by contradiction, assume that k is rational. Thus, we have two equations: $-r+\xi=\frac{a}{b}$ and $r+\xi=\frac{c}{d}$ where $a,b,c,d\in\mathbb{Z}$. If we add these equations together, we get $-r+\xi+r+\xi=\frac{a}{b}+\frac{c}{d}$. Reducing, we have $2\xi=\frac{ad+bc}{bd}$, which becomes $\xi=\frac{ad+bc}{2bd}$. Consequently, we are attempting to define an irrational number as a fraction, which violates the definition of an irrational number. Thus, k is irrational when r is irrational.

Therefore, k is irrational when r is rational or irrational, meaning that the conjecture is true. $\square$

[1] You may assume without proof that the sum of two rational numbers is rational.

Conjecture IX. *Every graph has at least two vertices of the same degree.*

[Hint: consider two cases–one in which your graph has a vertex of degree 0, and one in which it does not.]

Proof. This proof implies that there are at least two vertices in the graph, so for the sake of the argument, we shall ignore graphs with only one vertex. However, if the proof is found to be true, then I would suggest a rewording of the original statement to exclude graphs with fewer than two vertices.

Consider a graph with n vertices. A given vertex can be incident to 0 edges, 1 edge, all the way to all other edges, being $n-1$. Consequently, the degree of every vertex is in the set $\{0, 1, 2, ..., n-1\}$, since this set starts at 0 and ends at $n-1$, there are n elements within it. Let there be a vertex with degree 0, this means that it is not connected to any other vertices, meaning that no vertices can be of degree $n-1$. Likewise, if there is a vertex of degree $n-1$, then there cannot be a vertex of degree 0. Consequently, if there are any vertices of either degree 0 or $n-1$, then the set of possible vertex degrees decreases by one. This means the cardinality of the degree set is now $n-1$, meaning that for n vertices, at least two will have the same degree. Consider a graph where no vertices are of degree 0 or $n-1$, this means that 2 elements have been removed from the set of degrees, which now has the cardinality $n-2$. Again, this means that at least 2 vertices will have the same cardinality. Therefore, if a graph has at least two vertices, then two of those vertices will have the same degree. $\square$

Evaluation: ________

Opening: ________

Logical Correctness: ________

Reasons: ________

Use of Notation: ________

Clarity and Writing: ________

LaTeX Formatting: ________

Stating the Conclusion: ________

Other Comments: