Math 212 Portfolio

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Changelog: *List the changes you've made since the last draft, with special attention paid to problems that have received significant revisions since the last draft (i.e., more than fixing typos). If there is any additional information you'd like me to consider as I review this submission, please say so now.*

1. Reworked Conjectures II
2. Added Conjecture III

Instructions: Each of the problems below is/will be presented as a conjecture. Each conjecture asks you to prove or disprove the conjecture, possibly along with some additional directions.

- If the conjecture is true, your job is to write a complete proof for the proposition. If there are multiple parts, you should consider each part in turn.
- If it is false, you should provide a counterexample plus make reasonable modifications to the stated conjecture so that a new proposition is true. Then, write a complete proof of your new proposition. You may want to run your new proposition by me before trying to write a proof–this is allowed and encouraged!

Academic Honesty Policy: The portfolio is an independent project in which no outside resources or collaboration is allowed. You may not ask other professors or discuss the problems with anyone besides me. You should not discuss even which problem you chose. Violation of this policy is grounds for failing the course. The point is that you need to be confident and competent in writing proofs for future courses.

Conjecture I. *Let A and B be subsets of some universe $\mathcal{U}$. Then:*

1. $A \setminus (A \cap \overline{B}) = A \cap B$
2. $\overline{(\overline{A} \cup B)} \cap A = A \setminus B$
3. $(A \cup B) \setminus A = B \setminus A$
4. $(A \cup B) \setminus B = A \setminus (A \cap B)$

All four statements of Conjecture I will be proved to be equal through the set equivalence method: demonstrating that the left side is a subset of the right side and that the right side is a subset of the left side.

Conjecture I.1.

Proof. Let $x \in A \setminus (A \cap \overline{B})$. Thus, x must be an element of A and not an element of $A \cap \overline{B}$, that is $x \in A$ and $x \notin A \cap \overline{B}$. If x is not in $A \cap \overline{B}$, then x must not be in $\overline{B}$, meaning $x \in B$. Thus, $x \in A \cap B$, and, furthermore, $A \setminus (A \cap \overline{B}) \subseteq A \cap B$.

Let $x \in A \cap B$. For this to be true, x must be in both A and B; $x \in A$ and $x \in B$. Thus, x cannot be in the complement of B, $\overline{B}$. Since $x \notin \overline{B}$, $x \notin A \cap \overline{B}$. Because $x \in A$ and $x \notin A \cap \overline{B}$, $x \in A \setminus (A \cap \overline{B})$. Thus, $A \cap B \subseteq A \setminus (A \cap \overline{B})$.

Therefore, since $A \setminus (A \cap \overline{B}) \subseteq A \cap B$ and $A \cap B \subseteq A \setminus (A \cap \overline{B})$, $A \setminus (A \cap \overline{B}) = A \cap B$. □

Conjecture I.2

Proof. Let $x \in \overline{(\overline{A} \cup B)} \cap A$. Thus, $x \in A$ and $x \in \overline{(\overline{A} \cup B)}$. This means that x is not in the complement of $\overline{(\overline{A} \cup B)}$, which is $\overline{A} \cup B$. Thus, $x \notin \overline{A} \cup B$. Consequently, $x \notin \overline{A}$ and $x \notin B$. If $x \notin \overline{A}$, then $x \in A$. Since $x \in A$ and $x \notin B$, $x \in A \setminus B$. Thus, if $x \in A \setminus B$ and $x \in \overline{(\overline{A} \cup B)} \cap A$, $\overline{(\overline{A} \cup B)} \cap A \subseteq A \setminus B$.

Let $x \in A \setminus B$. If this is true, then $x \in A$ and $x \notin B$. It follows that $x \notin \overline{A}$ and, consequently, that $x \notin \overline{A} \cup B$. If $x \notin \overline{A} \cup B$, then x must be in the complement, being $\overline{(\overline{A} \cup B)}$. Since $x \in A$ and $x \in \overline{(\overline{A} \cup B)}$, then $x \in \overline{(\overline{A} \cup B)} \cap A$. Thus, $A \setminus B \subseteq \overline{(\overline{A} \cup B)} \cap A$.

Therefore, since $\overline{(\overline{A} \cup B)} \cap A \subseteq A \setminus B$ and $A \setminus B \subseteq \overline{(\overline{A} \cup B)} \cap A$, $\overline{(\overline{A} \cup B)} \cap A = A \setminus B$. □

Conjecture I.3

Proof. Let $x \in (A \cup B) \setminus A$. For x to be an element, it must be in $A \cup B$ and not in A. Thus, $x \in A \cup B$ and $x \notin A$. Since $x \notin A$, x must be a part of B for $A \cup B$ to be true: $x \in B$. If $x \notin A$ and $x \in B$, then $x \in B \setminus A$. If $x \in (A \cup B) \setminus A$ and $x \in B \setminus A$, then $(A \cup B) \setminus A \subseteq B \setminus A$.

Let $x \in B \setminus A$. Thus, x must be in B and not in A. Accordingly, $x \in B$ and $x \notin A$. It follows that $x \in A \cup B$, and, consequently, that $x \in (A \cup B) \setminus A$ since $x \notin A$. Thus, because $x \in B \setminus A$ and $x \in (A \cup B) \setminus A$, $B \setminus A \subseteq (A \cup B) \setminus A$.

Therefore, since $(A \cup B) \setminus A \subseteq B \setminus A$ and $B \setminus A \subseteq (A \cup B) \setminus A$, $(A \cup B) \setminus A = B \setminus A$. $\square$

Conjecture I.4

Proof. Let $x \in (A \cup B) \setminus B$. It follows that $x \in A \cup B$ and $x \notin B$. This means that $x \in A$ since $x \in A \cup B$ is true. Thus, $x \notin A \cap B$. Because $x \in A$ and $x \notin A \cap B$, $x \in A \setminus (A \cap B)$ is true. Thus, since $x \in (A \cup B) \setminus B$ and $x \in A \setminus (A \cap B)$, $(A \cup B) \setminus B \subseteq A \setminus (A \cap B)$.

Let $x \in A \setminus (A \cap B)$. Thus, $x \in A$ and $x \notin A \cap B$, meaning that $x \notin B$. If follows that $x \in A \cup B$ and, since $x \notin B$, $x \in (A \cup B) \setminus B$ are both true. Thus, if $x \in A \setminus (A \cap B)$ and $x \in (A \cup B) \setminus B$ are true, then $A \setminus (A \cap B) \subseteq (A \cup B) \setminus B$ is true.

Therefore, since $(A \cup B) \setminus B \subseteq A \setminus (A \cap B)$ and $A \setminus (A \cap B) \subseteq (A \cup B) \setminus B$ are true, it must be true that $(A \cup B) \setminus B = A \setminus (A \cap B)$. $\square$

Conjecture II. *Define* $f : \mathbb{N} \setminus \{0\} \to \mathbb{Z}$ *as follows: for each* $n \in \mathbb{N} \setminus \{0\}$,

$$f(n) = \frac{1 + (-1)^n(2n - 1)}{4}.$$

Then f *is a bijection.*

Conjecture II will be proved to be a bijection by first showing that it is an injection and subsequently that it is a surjection. Proving the function injective will be done by demonstrating that $f(x) = f(y) \to x = y$, and proving the function surjective will be done by demonstrating that every value of the codomain is reflected by a value of the domain.

We notice that when n is even, the result is positive, and when n is odd, the result is negative. This is because $(-1)^1 = -1, (-1)^3 = -1, ...$ and $(-1)^2 = 1, (-1)^4 = 1, ...$ Thus, we must evaluate the function both when n is even and when n is odd.

Proof. First, we show that f is an injection.

When n is even, we show that $f(x) = f(y) \to x = y$, and we know that $(-1)^n = 1$.

$$f(x) = \frac{1 + (1)(2x - 1)}{4} \tag{1}$$

$$f(y) = \frac{1 + (1)(2y - 1)}{4} \tag{2}$$

$$\frac{1 + (1)(2x - 1)}{4} = \frac{1 + (1)(2y - 1)}{4} \tag{3}$$

$$1 + (1)(2x - 1) = 1 + (1)(2y - 1) \tag{4}$$

$$(1)(2x - 1) = (1)(2y - 1) \tag{5}$$

$$2x - 1 = 2y - 1 \tag{6}$$

$$2x = 2y \tag{7}$$

$$x = y \tag{8}$$

When n is odd, we show that $f(x) = f(y) \to x = y$, and we know that $(-1)^n = -1$.

$$f(x) = \frac{1 + (-1)(2x - 1)}{4} \tag{9}$$

$$f(y) = \frac{1 + (-1)(2y - 1)}{4} \tag{10}$$

$$\frac{1 + (-1)(2x - 1)}{4} = \frac{1 + (-1)(2y - 1)}{4} \tag{11}$$

$$1 + (-1)(2x - 1) = 1 + (-1)(2y - 1) \tag{12}$$

$$(-1)(2x - 1) = (-1)(2y - 1) \tag{13}$$

$$-2x + 1 = -2y + 1 \tag{14}$$

$$-2x = -2y \tag{15}$$

$$x = y \tag{16}$$

Therefore, since $f(x) = f(y) \to x = y$ both when n is even and odd, we know that the function $f(n) = \frac{1+(-1)^n(2n-1)}{4}$ is an injection.

Next, we show that f is a surjection.

We must solve for y in $f(x) = y$ to demonstrate that the codomain has at least one corresponding value of the domain in order for $f(n)$ to be surjective.

When n is even, we know that $(-1)^n = 1$. Solving for x, we find that $x = 2y$. Thus, since the function $2y = x$ is linear, having all possible values of x and y as its domain and codomain, $f(n)$ is surjective when n is even.

When n is odd, we know that $(-1)^n = -1$. Solving for x, we find that $x = -2y + 1$. Thus, since the function $-2y + 1 = x$ is linear, having all possible values of x and y as its domain and codomain, $f(n)$ is surjective when n is odd.

Since $f(n)$ is surjective both when n is even and when it is odd, we know that the function is surjective.

Therefore, since $f(n)$ has been demonstrated to be both injective and surjective, the function is a bijection. $\square$

Example III. *For the following example, choose two of the four problems to do. Exactly one of your choices should be a combinatorial proof.*

1. *(Combinatorial) For* $n \geq 1$,
$$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}.$$
2. *(Combinatorial) For* $0 \leq k \leq n$,
$$\sum_{m=k}^{n} \binom{m}{k} = \binom{n+1}{k+1}$$
3. *Consider the alphabet* $\{a, b, c, d, e, f\}$ *and make words without repetition of letters allowed.*
   (a) *How many six-letter words are there?*
   (b) *How many words begin with* d *or* e*?*
   (c) *How many words end in* b *or* a*?*
   (d) *How many words begin with* d *or* e *and end in* b *or* a*?*
   (e) *How many have first letter neither* d *nor* e *and last letter nether* b *nor* a*?*
4. *We wish to improve upon the ogre's distribution of 43 cupcakes to 12 baby mice by ensuring that every baby mouse gets at least two cupcakes. How many ways are there to accomplish this?*

Proof. □

Evaluation: ____________

Opening: ____________

Logical Correctness: ____________

Reasons: ____________

Use of Notation: ____________

Clarity and Writing: ____________

LaTeX Formatting: ____________

Stating the Conclusion: ____________

Other Comments: