3. Sets and Subsets

3.0 Sets and Subsets

- The singleton set has a single element

Common mathematical sets

Name	Set	Symbol	Examples of elements
Natural numbers	N is the set of natural numbers, which includes all integers greater than or equal to 0.	$\mathbb{N}$	0, 1, 2, 3, ...
Integers	Z is the set of all integers.	$\mathbb{Z}$	..., -2, -1, 0, 1, 2, ...
Rational numbers	Q is the set of rational numbers, which includes all real numbers that can be expressed as a/b, where a and b are integers and b ≠ 0.	$\mathbb{Q}$	0, 1/2, 5.23, -5/3
Real numbers	R is the set of real numbers.	$\mathbb{R}$	0, 1/2, 5.23, -5/3, π, $\sqrt{2}$

Real numbers

π, *e*, $\sqrt{2}$

Rational numbers

$^{22}/_{7}$, $^{19}/_{16}$, 1.414

Integers

-1, 0, 42, 707, 531717, 80085

Natural numbers

[0], 1, 2, 3, 4, 5

3.1 Sets

- A set is a collection of objects
- Objects in a set are called elements
- Sets can contain elements of different varieties
- A set is defined by indicating which elements belong to it
- Equality between sets uses the equals sign "=", not the equivalance sign "≡".

Roster Notation

- Roster notation is a way to define a set by listing its elements

Roster notation

S = { 2, 4, 6, 10 }	4 four member set
E = { 1, 3, 5, ... , 99 }	odd numbers, 1 to 99
N = { 3, 6, 9, 12, ... }	multiples of 3, infinitely

- Ellipses (...) can be used to denote long sequences of numbers in a set
- A finite set can be numbered 1 through n for some positive integer n
- An infinite set is not finite
- Two sets are equal if they have exactly the same elements
- Order of elements is unimportant when defining a set

Example Sets

D = { 3, 4, 5 }
N = { 5, 3, 4 }
U = { 5, 3, 4, 6 }

D = N
D ≠ U

6 ∈ U is true   6 ∈ N is false
5 ∉ U is false   7 ∉ N is true

Empty Sets

- The empty set, also known as the null set, it contains no elements, and is represented by the symbol ⌀ or { }.

Empty Sets

D = { }
E = ⌀

i.e.
a ∉ ⌀
"Any element *a* is not a part of the empty set."

Cardinality

- The cardinality of a finite set is the number of distinct elements.
- Cardinality of empty set |⌀| is zero.
- Cardinality of an infinite set is infinity.

Cardinality

S = { 0, 1, 2, 3, 4, 5 } | S | = 6
P = ∅ | P | = 0
L = { 1, 2, 3, 5, 8, 13, ... } | L | = ∞

Positive and Negative Sets

Set Notation	Description
R+	Set of all positive real numbers
Z+	Set of all positive integers
x > 0	x is positive
R-	Set of all negative real numbers
Z-	Set of all negative integers
x < 0	x is negative
0	Neither positive nor negative
x ≧ 0	x is non-negative
x ≦ 0	x is non-positive
Natural numbers	Either defined as non-negative integers or positive integers (excluding 0)

Set Builder Notation

- Set builder notation is a method used to define a set by specifying that the set includes all elements in a larger set that also satisfy certain conditions.

Set builder notation

$A = \{ x \in S : P(x) \}$ "all x in S such that P(x)"

$V = \{ m \mid m \text{ is a vegitarian meal} \}$ "The set V includes all vegetarian meals. "

$D = \{ x \in \mathbb{R} : |x| < 1 \}$
"The set D is all real numbers between -1 and 1."

- S is the larger set from which the elements in A are taken
- P(x) is some condition for membership in A
- The colon symbol ":" is read "such that"
- "Such that" is also represented by the pipe symbol "|"

Universal Set

- The universal set contains all elements mentioned in a particular context.
- It is usually denoted by the variable *U*.

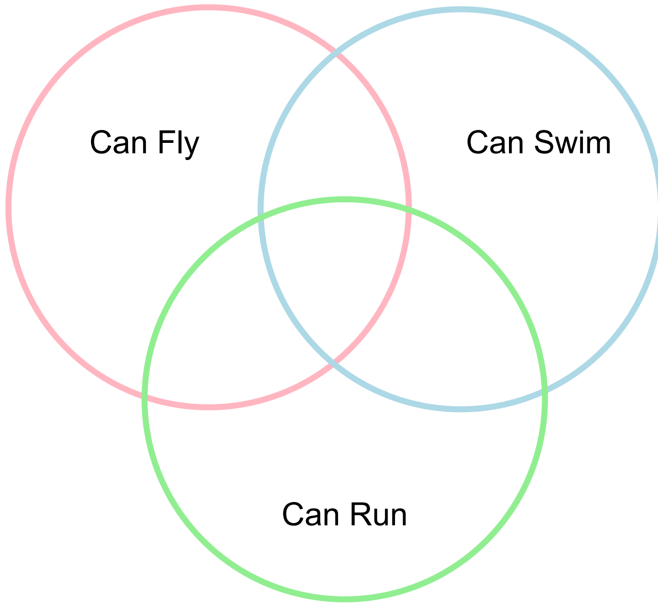
Example

In a discussion about the grades of students at a school, the universal set would be the set of all students at the school

Venn Diagrams

- A rectangle represents the universal set U
- Oval shapes are used to denote sets within U
- Elements are drawn inside the oval if they are in the set represented by the oval

Animal Kingdom



Subset Relationships

- A is <u>subset</u> of B if every element in A is also an element of B
  (A ⊆ B)
- A is not a subset of B if there is an element of A that is not an element of B
  (A ⊈ B)
- Two sets are equal if and only if each is a subset of the other
  (A = B if and only if A ⊆ B and B ⊆ A)
- A is a <u>proper subset</u> of B if A ⊆ B and there is an element of B that is not an element of A (A ⊂ B)

> <u>Note</u>
>
> For every set A:
> ∅ ⊆ A ⊆ U
>
> The empty set is a subset of A,
> and A is a subset of the universal set.

3.2 Set of sets

- A set can have elements that are sets themselves.
- The empty set ∅ is not the same as { ∅ }.

<u>Example</u>

A = { { 1, 2 }, ∅, { 1, 2, 3 }, { 1 } }
B = { 2 , ∅, { 1, 2, 3 }, { 1 } }

<u>In set A</u>
{ 1, 2 } ∈ A
1 ∉ A

<u>In set B</u>
2 ∈ B   { 2 } ⊆ B
1 ∉ B   { 1 } ⊈ B

Power Set

- The <u>power set</u> of a set A, denoted P(A), is the set of all subsets of A.

<u>Example</u>
A = { 1, 2, 3 }

P(A) = { ∅, { 1 }, { 2 }, { 3 }, { 1, 2 }, { 1, 3 }, { 2, 3 }, { 1, 2, 3 } }

Cardinality of a Power Set

Theorem

The cardinality of a power set is
$|P(A)| = 2^n$
where A is a finite set of cardinality n.

Example I
A = { ❤, ☾, ★, ♣ }
$|A| = 4$

P(A) = { ⌀, { ❤ }, { ☾ }, ..., { ❤, ☾, ★, ♣ } }
$|P(A)| = 2^4 = 16$

Example II
B = ⌀
$|B| = 0$

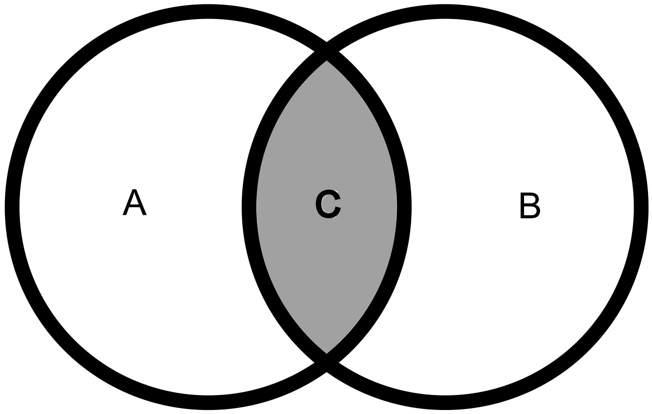
P(B) = { ⌀ }
$|P(B)| = 2^0 = 1$

3.3 Union and intersection

- Standard set operations allow for the creation of new sets from existing sets.
- These operations involve the combination of sets in various ways to define new sets.

The set intersection operation

- The intersection of sets A and B is denoted as A ∩ B and read as "A intersect B".
- It returns a set of all elements that are common to both A and B.
- The result of the intersection operation is always a subset of both A and B, even if the returned set is empty.



- C is the set of elements that A and B share.

- If A and B have no common elements, then their intersection is an empty set.

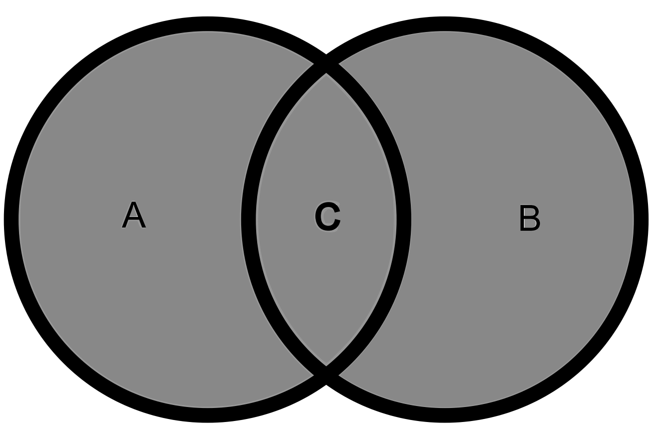
- The intersection operation can be used on infinite sets.

$$\begin{aligned} A &= \{ \text{ x} \in \text{Z x is an integer multiple of } 2 \text{ } \} \\ B &= \{ \text{ x} \in \text{Z x is an integer multiple of } 3 \text{ } \} \\ A \cap B &= \{ \text{ x} \in \text{Z x is an integer multiple of } 6 \text{ } \} \end{aligned}$$

The set union operation

- A union B, denoted A ∪ B, is the set of all elements that are elements of A or B.

- A ∪ B is read "A union B".



- The definition for union uses the inclusive or, meaning that if an element is an element of both A and B, then it is also an element of $A \cup B$.

Example

Consider the kind professor.

If a student receives an A on either of their first two midterms,

they can skip the final exam.

Sets A and B are defined as:

$A = \{ x: \text{student } x \text{ received an A on midterm 1} \}$

$B = \{ x: \text{student } x \text{ received an A on midterm 2} \}$

$A \cup B = \{ x: \text{student } x \text{ received an A on midterm 1 or midterm 2} \}$

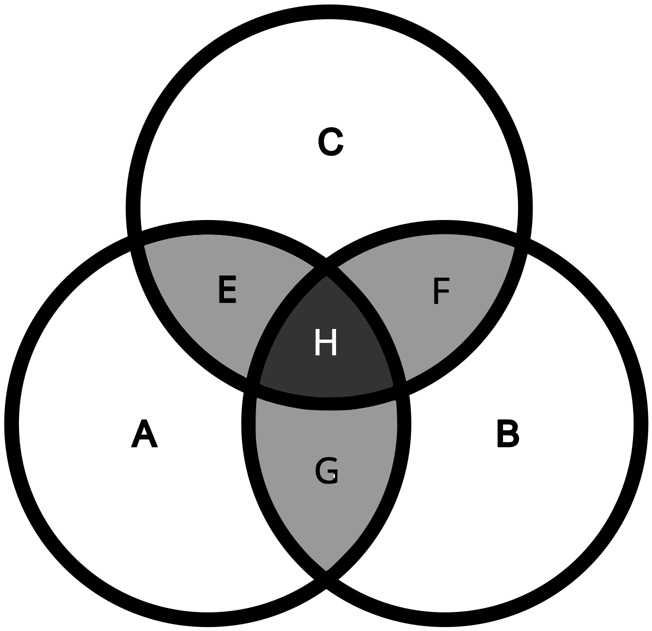
$A \cup B = \{ x: \text{student } x \text{ may skip the final exam} \}$

- Set operations can be combined to define more sets, such as $A \cup (B \cap C)$ which is the union of A and $B \cap C$.
- Parentheses are crucial because $(A \cup B) \cap C$ differs from $A \cup (B \cap C)$.

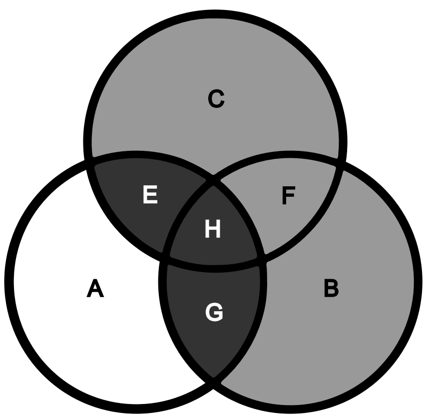
Example: A ∩ b ∩ C

$A \cap B = G \cup H$
$B \cap C = F \cup H$
$C \cap A = E \cup H$

$A \cap B \cap C = H$



Note: - Only set H is a set of A ∩ b ∩ C. - Sets E, F, and G are not subsets of the set H. #### Example: A ∩ (B ∪ C)



The intersection of A with the union of B and C is the union of subsets H, E, and G.

A ∩ (B ∪ C) = H ∪ E ∪ G

Note: - Only the union of sets E, H, and G is a set of A ∩ (B ∪ C). - Sets C, B, and F are not subsets of the set E ∪ H ∪ G. ### Intersection and Union Sequence Notation

- The intersection, or union, of a sequence of sets A_1 to A_n can be shown with a special notation.
- The "*i*=1" below the operator sign and "*n*" above it to indicate that the operation applies to all sets with integer indices *i* from 1 through *n*.

Intersection Sequence

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n = \{x : x \in A_i \text{ for all } i \text{ such that } 1 \leq i \leq n\}$$

Union Sequence

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n = \{x : x \in A_i \text{ for some } i \text{ such that } 1 \leq i \leq n\}$$

Example: String Length

Consider the following set of 50 words:

1 - 10	***11 - 20***	***21 - 30***	***31 - 40***	***41 - 50***
Ballons	Dripping	Meatballs	Plums	Spread
Balls	Eggplant	Melons	Poles	Swollen
Banana	Friction	Milkshake	Pound	Taco
Beaver	Head	Mounds	Rack	Throb
Buns	Holes	Nuts	Salami	Tool
Bush	Hooters	Package	Sausage	Trim
Cans	Hose	Peaches	Screw	Wad
Cherry	Jugs	Pickle	Shaft	Weiner
Cream	Juicy	Pipe	Snatch	Wet
Cucumber	Knockers	Piston	Split	Wood

- The longest words in this list have 9 characters, they are: meatballs and milkshake.
- The shortest words in this list have 3 characters, they are: wad and wet.

Define A to be the set of all listed words.
A = { Ballons, Balls, ..., Wet, Wood }

Define A_j to be the subset of all listed words with length j.
A_1 = ∅
A_2 = ∅

A_3 = { Wad, Wet }
...
A_9 = { Meatballs, Milkshake }

- There are no words with length 1, so A_1 is an empty set.

We could write the intersection of all words in this list as:

$$\bigcap_{i=3}^{9} A_i = \{\text{ Wad, Wet }\} \cap \cdots \cap \{\text{ Meatballs, Milkshake }\} = \emptyset$$

- i = 3 signifies that the first set is A_3
- n = 9 signifies that the last set is A_9
- Since no two sets share any elements, the intersection of all subsets of A is an empty set.

For whatever strange reason, we could write a sequential union of word sets instead:

$$\begin{aligned}\bigcup_{i=7}^{8} A_i &= \{\text{ Ballons, ... , Swollen }\} \cap \{\text{ Cucumber, ... , Knockers }\} \\ &= \{\text{ Ballons, Hooters, Package, Peaches,} \\ &\qquad \text{Sausage, Swollen, Cucumber, Dripping,} \\ &\qquad \text{Eggplant, Friction, Knockers }\}\end{aligned}$$

For the sake of completion, the sequential union of all subsets of set A would be set A.

$$\begin{aligned}\bigcup_{i=1}^{9} A_i &= \{\text{ Wad, Wet }\} \cup \cdots \cup \{\text{ Meatballs, Milkshake }\} \\ &= \{\text{ Wad, ... , Milkshake }\} = A\end{aligned}$$

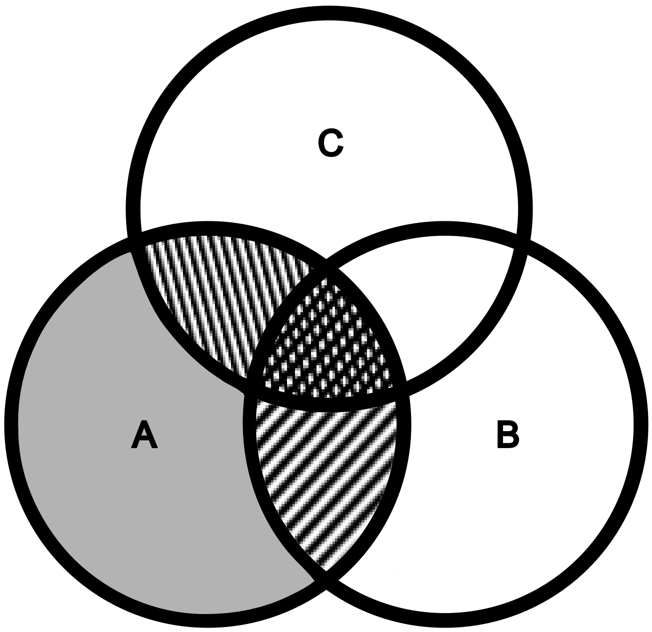
3.4.0 Difference and Symmetric Difference

Difference of Sets

- The difference between two sets A and B is the set of elements that are in A but not in B.
- The difference operator is the minus symbol "-".

- The difference operation is not commutative unless A ≡ B.
- If A ≡ B the difference is ⌀.

Example: (A - B) - C



Note:

- Striped regions are areas removed by application of the difference operator.
- The remaining subset of A contains all elements in common with B and C removed.

Symmetric Difference of Sets

- The symmetric difference of sets A and B contains only the elements that belong exclusively to A or B, but not both.

- Its operator is the xor symbol "⨁".

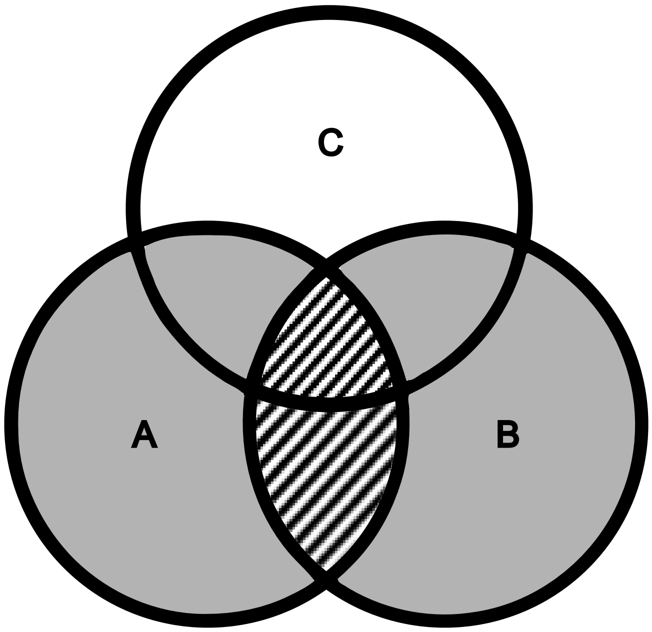
- It is commutative, that is: A ⨁ B ≡ B ⨁ A.

- Logical Equivalence:

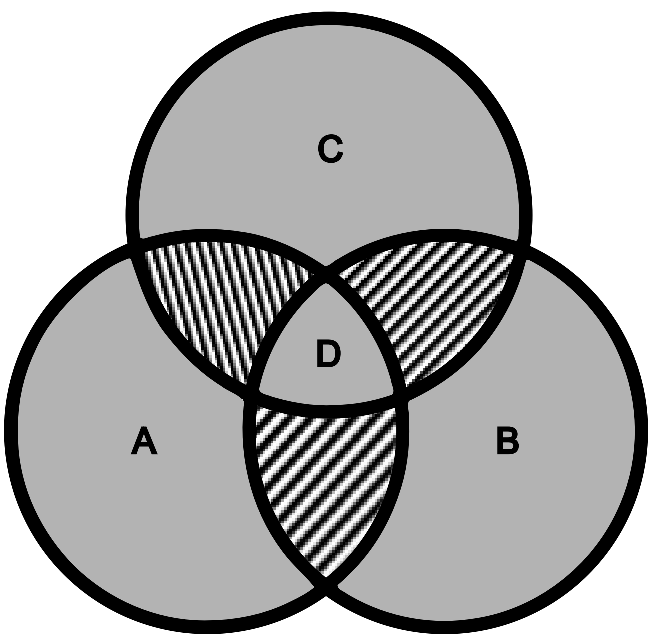
  A ⨁ B ≡ (A - B) ∪ (B - A)

Example: (A ⊕ B) ⊕ C

First step:



Solution:



Note:

- Since the set D was removed by the A ⊕ B operation, it could then be replaced by the symmetric difference operation of that resultant set with set C.

3.4.1 Set Complement Operation

- Recall that U is the symbol for the universal set.

- The <u>complement</u> of set A, denoted $\overline{A}$, is the set of all elements in U not in A.

- The complement of a set requires a well defined U.

- Logical equivalence:

  $\overline{A} \equiv U - A$

Example: Odd Compliments Even

Let the universal set be the set of all integers,
and define set A as the set of all even integers.

$U = \mathbb{Z}$
$A = \{ x \in \mathbb{Z}: x \text{ is even} \}$

∴ The compliment of set A is the set of all odd integers.

$\overline{A} \equiv U - A \equiv \{ x \in \mathbb{Z}: x \text{ is odd} \}$

3.5 Set Identities

- The operators of set theory - intersection, union, and complement - are all defined based on logical operations.

Intersection

$x \in A \cap B \leftrightarrow (x \in A) \wedge (x \in B)$

Union

$x \in A \cup B \leftrightarrow (x \in A) \vee (x \in B)$

Element of

$x \in \overline{A} \leftrightarrow \neg(x \in A)$

- True and false correspond with the universal and empty sets.

Something exists in the universal set

$x \in U \leftrightarrow T$

Nothing exists in the empty set

$x \in \varnothing \leftrightarrow F$

- A set identity is an equation involving sets that is always true.
- It doesn't depend on the contents of the sets in the equation.
- It's similar to an equivalence in logic, which holds regardless of the truth values of individual variables.

De Morgan's Set Identity

$x \in \overline{A \cap B}$	$\longleftrightarrow \neg(x \in A \cap B)$	Definition of complement
	$\longleftrightarrow \neg(x \in A \wedge x \in B)$	Definition of intersection
	$\equiv \neg(x \in A) \vee \neg(x \in B)$	De Morgan's law for proposition
	$\longleftrightarrow (x \in \overline{A}) \vee \neg(x \in \overline{B})$	Definition of complement
	$\longleftrightarrow (x \in \overline{A} \cup \overline{B})$	Definition of union
$\overline{A \cap B} = \overline{A} \cup \overline{B}$		De Morgan's set identity

Name	Identities	
Idempotent laws	$A \cup A = A$	$A \cap A = A$
Associative laws	$(A \cup B) \cup C = A \cup (B \cup C)$	$(A \cap B) \cap C = A \cap (B \cap C)$
Commutative laws	$A \cup B = B \cup A$	$A \cap B = B \cap A$
Distributive laws	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Identity laws	$A \cup \varnothing = A$	$A \cap U = A$
Domination laws	$A \cap \varnothing = \varnothing$	$A \cup U = U$
Double complement law	$\overline{\overline{A}} = A$	
Complement laws	$A \cap \overline{A} = \varnothing$ $\overline{U} = \varnothing$	$A \cup \overline{A} = U$ $\overline{\varnothing} = U$
De Morgan's laws	$\overline{A \cup B} = \overline{A} \cap \overline{B}$	$\overline{A \cap B} = \overline{A} \cup \overline{B}$
Absorption laws	$A \cup (A \cap B) = A$	$A \cap (A \cup B) = A$

3.6.0 Cartesian Products

Ordered Pair

- An ordered pair is written as (x, y).
- The first entry of the ordered pair is x, and the second entry is y.
- The use of parentheses for an ordered pair indicates that the order of entries is significant.
- Two ordered pairs (x, y) and (u, w) are equal if and only if $x = u$ and $y = w$.
- For example, $(x, y) \neq (y, x)$ unless $x = y$.

Sets of Ordered Pairs

- Sets of ordered pairs may be written in Roster notation: { (a, a), (a, b), (b, a), (b, b) }
- The order in an unordered set does not matter, {x, y} is equal to {y, x}.

A Cartesian Product is a Set of Ordered Pairs

- The Cartesian product of two sets A and B, denoted A x B, is the set of all ordered pairs in which the first entry is in A and the second entry is in B.

  A x B = { (a, b) : a ∈ A and b ∈ B }

- A x B is not equal to B x A, unless A = B or either A or B is empty.

- If A and B are finite sets, |A x B| = |A|·|B|.

Example A x B

Define sets

A = { 1, 2 }

B = { a, b, c }

A x B

	a	b	c
1	(1, a)	(1, b)	(1, c)
2	(2, a)	(2, b)	(2, c)

A x B = {
(1, a), (1, b), (1, c),
(2, a), (2, b), (2, c)
}

| A x B | = 6

B x A

	1	2
a	(a, 1)	(a, 2)
b	(b, 1)	(b, 2)
c	(c, 1)	(c, 2)

B x A = {
(a, 1), (a, 2),
(b, 1), (b, 2)
(c, 1), (c, 2)
}

| B x A | = |B|·|A|

N-Fold Cartesian Product

- The Cartesian product of a set A with itself can be represented as A × A or A^2.
- The Cartesian product can also be extended to more than two sets. For example: A × B × C.
- Generally, the Cartesian product of *n* sets, A_1, A_2, ..., A_n is:
  $A_1 \times A_2 \times \ldots \times A_n = \{ (a_1, a_2, \ldots, a_n) : a_i \in A_i \text{ for all } i \text{ such that } 1 \leq i \leq n \}$

Example: Binary Numbers

Define Binary Pair
A = {0, 1}
A^n is the set of all ordered n-tuples with entries of 0 or 1.

3-Bit Binary Number
$\{0, 1\}^3$ = {
(0, 0, 0), (0, 0, 1), (0, 1, 0), (0, 1, 1),
(1, 0, 0), (1, 0, 1), (1, 1, 0), (1, 1, 1)
}

There are 8 elements in the set, which can be used
to represent base-10 numbers, 0 to 7.

3.6.1 Strings

Strings, Length, Index

- A string is a sequence of symbols, frequently alphabetic characters.
- The set of characters used in a set of strings is called the alphabet.
- When the elements of an n-tuple are symbols, it can be converted to a string.
  (a, b, c, 1, 2, 3) = "abc123"
- The length of a string is the number of characters in the string.
  "Hello" is of length 5.

- The index, of a character represents numerical the position of that character in a string.
  "Hello"[0] = "H"
  "Hello"[4] = "o"
  "Hello"[1] = "e"

- Depending on context, an index can start from 0 or 1, and there is a long-standing debate as to which implementation is considered best practice.

- Anyone who starts an index from 1 should be shot out of a cannon into the sun.

Binary Strings

- A binary string consists of only 0s and 1s, so the alphabet of a binary string is {0, 1}.
- A bit is a binary digit, and is the smallest unit of information in a standard computer.
- A light switch can represent a bit, where the state of being on or off corresponds with 1 and 0.
- The base-2 numbering system, also known as the binary system, uses bits to represent numercal values.

$$\begin{aligned} 10101_2 &= (1)2^4 + (0)2^3 + (1)2^2 + (0)2^1 + (1)2^0 \\ &= 16 + 0 + 4 + 0 + 1 \\ &= 21 \end{aligned}$$

- This can be taken advantage of to store alphabetic strings as binary.

01000110	01101001	01111010	01111010	01000010	01110101	01111010	0111101
70	105	122	122	66	117	122	122
F	i	z	z	B	u	z	z

- This table uses the ASCII character set to convert numbers to characters.

Empty Strings

- An empty string is sybolized by λ (Lambda) and is similar to an empty set.

- The empty string is the only string of length 0.
- The set of all binary strings of length 0 can be written as: $\{0, 1\}^0 = \{ \lambda \}$.

Concatenation

- Concatenation of two strings s and t is denoted as st.
- To concatenate s and t, we put them together.
- For example, if s = 010 and t = 101, then st = 010101.
- We can also concatenate a string with a single symbol, such as t0 = 1010.
- Concatenating any string x with the empty string λ gives back x.
- So, $x\lambda = x$.

3.7 Partitions

Disjoint Sets

- Sets A and B are disjoint if they have no common elements, and therefore have no overlap.
- The intersection of disjoint sets is the empty set.
  $A \cap B = \varnothing$
- Pairwise disjoint sets are those sets in a sequence that do not have any common elements with one another.
  $A_1, A_2, \ldots, A_n$
  $A_k \cap A_j = \varnothing$ for any k and j in the range from 1 through n where $k \neq j$.

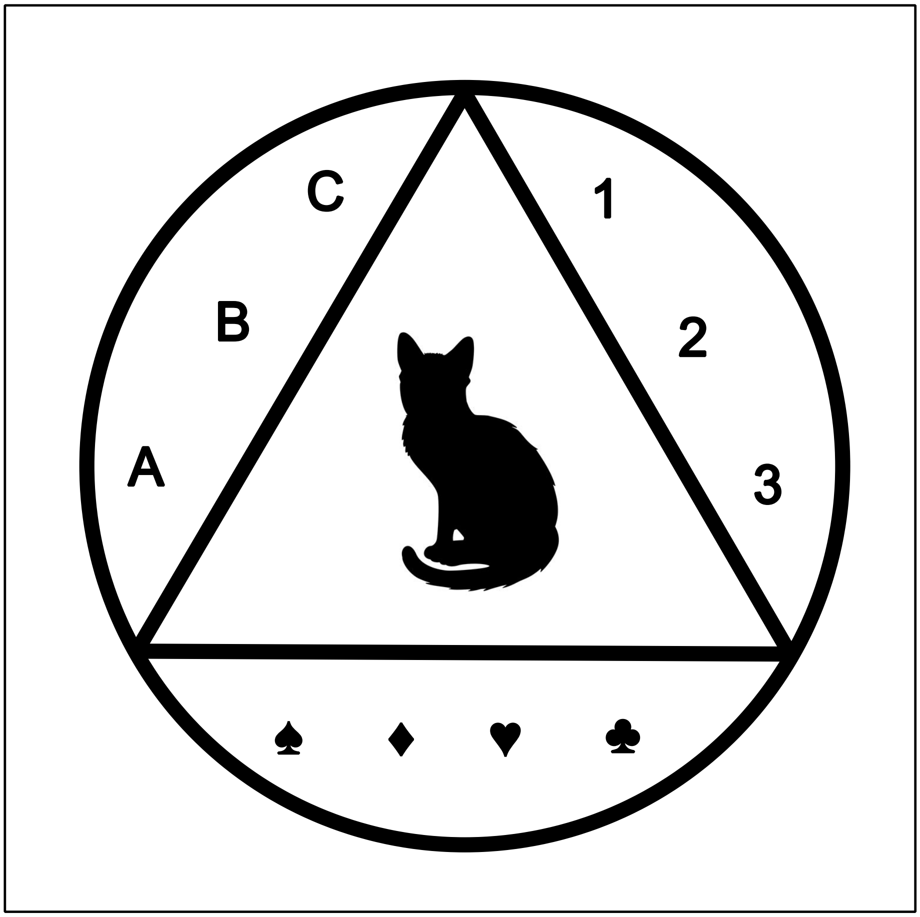
Partitions

- A partition is a collection of non-empty subsets of a non-empty set A.
- Each element of A is in exactly one of the subsets.

Requirements for a partition

- For all k, $A_k \subseteq A$.
- For all k $A_k \neq \varnothing$
- $A_1, A_2, \ldots, A_n$ are pairwise disjoint.
- $A = A_1 \cup A_2 \cup \ldots \cup A_n$

Example



- There are four partitions: letters, numbers, suits, and cat.
- The sum of partition cardinalities is the cardinality of their superset.