4. Functions

4.0 Functions

- A function maps elements from one set to another.
- Many functions are mathematical functions that map numbers to numbers.
- Discrete mathematics deals with functions that map between other kinds of sets, such as binary strings or a set of tasks.
- E.g mapping assignments of people to teams or guests to hotel rooms.

4.1 Definition of Function

- A function f maps elements of a set X to elements of a set Y.
- f is a subset of $X \times Y$ such that for every $x \in X$, there is exactly one $y \in Y$ for which $(x, y) \in f$.

Function Notation

f: X → Y

- $f: X \to Y$ represents the fact that f is a function from X to Y.
- $(x, y) \in f$ if f maps x to y.

f(x) = y

- $f(x) = y$ is an alternate notation to show that f maps x to y (or $(x, y) \in f$).

- A function from X to Y can be viewed as a subset of $X \times Y$.

- It is possible that X and Y are the same set, in which case f is a subset of $X \times X$.

Domain and Target (Co-Domain)

Given: $f(x) = y$

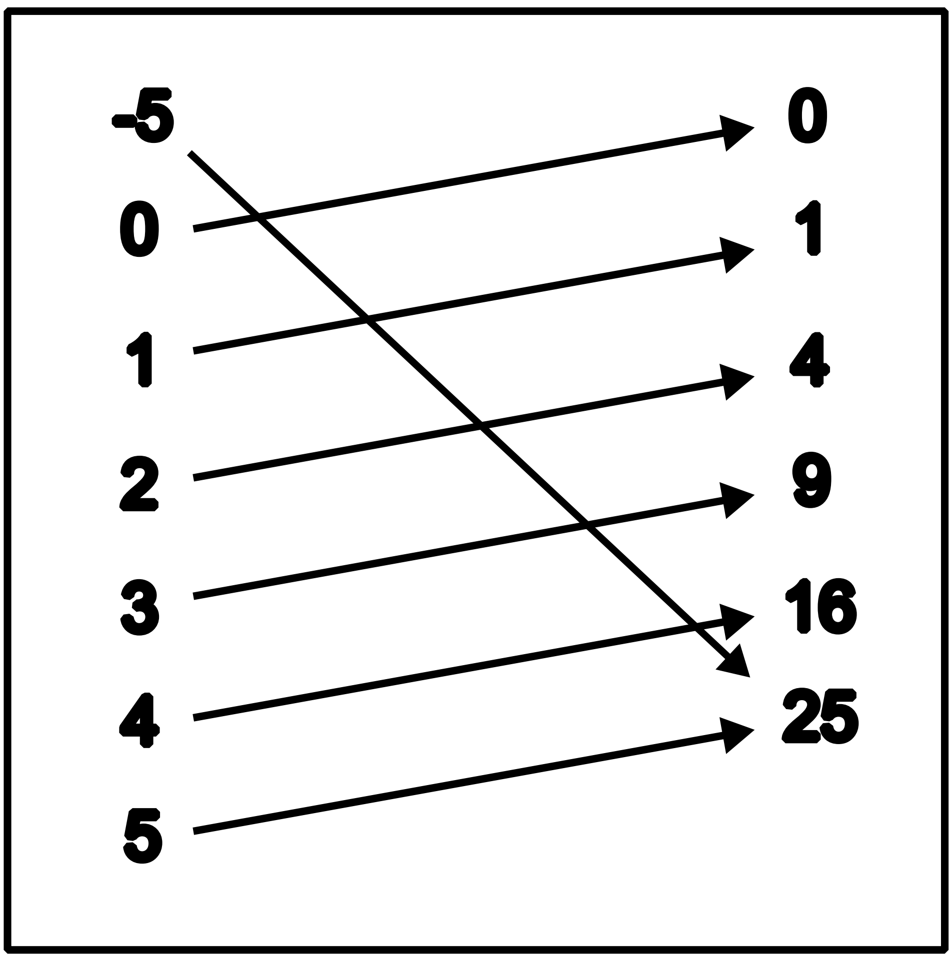
- X is called the domain of f.
- Y is the target (or co-domain) of f.

Well Defined Functions

- If f maps an element of the domain to zero or multiple elements of the target, it is not well-defined.

Arrow Diagrams

- A function f that maps X to Y and has a finite set X can be specified by listing the pairs (x, y) in f.
- An arrow diagram can also be used to represent a function with a finite domain.
- In an arrow diagram, the domain X is listed on the left and the target Y is listed on the right.



- There is an arrow from x ∈ X to y ∈ Y if and only if (x, y) ∈ f.
- Since f is a function, each x ∈ X has exactly one y ∈ Y such that (x, y) ∈ f.

- In the arrow diagram for a function, there is exactly one arrow pointing out of every element in the domain.

Range

- The range of function f: X → Y is the set of all possible output values (y) that can be obtained by inputting values (x) from X into the function.

- It can be expressed in set notation as:
  Range of f = { y: (x, y) ∈ f, for some x ∈ X }

- The range is a subset of the target set Y, but it may not necessarily be equal to Y.

- In an arrow diagram, the range is represented by the set of elements in the target set Y that have arrows pointing to them.

Must Specifiy a Domain and Target

- A mathematical function f is defined by how it acts on an input x.
- The definition must specify the domain and target of f.
- Example: abs: R → R, where abs(x) = $|x|$.
- The function abs maps every real number to a real number.
- However, abs does not map any number to a negative number.

Strings

- Functions can have domain and target sets that consist of strings.

  E.g.
  f: $\{0, 1\}^n \rightarrow \{0, 1\}^{(n + 1)}$

- f is a function that takes a binary string of length n, and returns a binary string of length n + 1.

Example: Parity Bits

```
/** Appends an even parity bit to a binary string */
function addParityBit(binaryString: string): string {
  let sum = 0;

  // calculate the sum of all 1's in the binary string
  for (let i = 0; i < binaryString.length; i++) {
    if (binaryString[i] === '1') {
      sum++;
    }
  }

  // determine the parity based on the sum
  const parityBit = (sum % 2 === 0) ? '0' : '1';

  // return the binary string with the parity bit appended
  return binaryString + parityBit;
}
```

- This function takes an n-bit string as input, and outputs an (n+1)-bit string.

- Generally, for any $x \in \{0, 1\}^n$, $f(x) = x\{0, 1\}^1$.

  E.g.
  addParityBit("1001") = "1001**0**"
  addParityBit("1011") = "1011**1**"

Function equality

- Two functions, f and g, are equal if they have the same domain and target and $f(x) = g(x)$ for every element x in the domain.
- The notation $f = g$ is used to indicate that functions f and g are equal.

4.2 Ceiling and Floor Functions

- The floor and ceiling functions map real numbers onto integers
- Floor function rounds real numbers down to the nearest integer
- Ceiling function rounds real numbers up to the nearest integer

Ceiling Function

- The ceiling function maps a real number to the nearest integer in the upward direction.
- It is denoted:
  ceiling: **R** → **Z**,
  where ceiling(x) = the smallest integer y such that x ≤ y.

Examples

- ceiling(π) = 4
- ceiling(5.5) = 6
- ceiling(-2.5) = -2
- ceiling(-4) = -4.

Floor Function

- The floor function maps a real number to the nearest integer in the downward direction.

- It is denoted:

  floor: **R** → **Z**,
  where floor(x) = the largest integer y such that y ≤ x.

Examples

- floor(π) = 3
- floor(5.5) = 5
- floor(-2.5) = -3
- floor(-4) = -4.

Ceiling and Floor Notation

$$\text{floor}(x) = \lfloor x \rfloor$$

$$\text{ceiling}(x) = \lceil x \rceil$$

4.3 Properties of functions

One-to-one or Injective function

- A one-to-one function has a single target element for each element in the domain.
- If $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$
- An injective function f maps different elements in X to different elements in Y.
- In an arrow diagram, a one-to-one function no target element will have two arrows pointed at itself.

Onto or Surjective function

- An onto function has a target set that is fully covered by the function.
- Range of f is equal to target Y
- For every $y \in Y$, there is an $x \in X$ such that $f(x) = y$.
- In an arrow diagram, an onto function every target element will have an arrow pointed at itself.

Bijective Functions

- A bijective function is both one-to-one and onto.
- A bijective function is also known as a bijection.
- A bijection can also be referred to as a one-to-one correspondence.

Size Implications

- Function with finite domain and target can reveal information about their relative sizes.
- One-to-one or onto property of function used to infer this information.

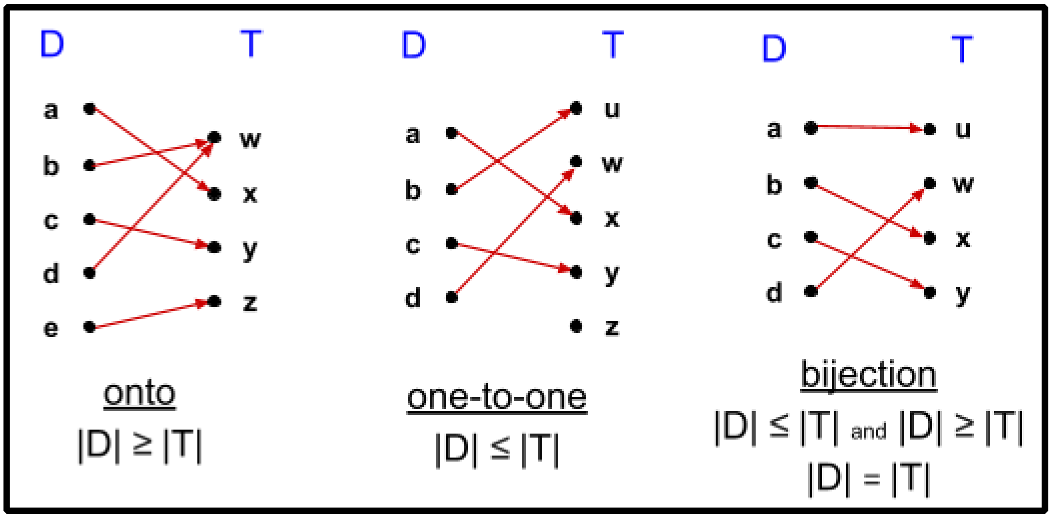
Consider the function $f: D \rightarrow T$

1. Onto:
    - $|D| \geq |T|$.
    - This means that for every element in the target, there is at least one element in the domain.
2. One-to-one:

- |D| ≤ |T|.
- This means that every element in the domain maps to a unique element in the target.

3. Bijection:

- |D| ≤ |T| and |D| ≥ |T|.
- This implies that |D| = |T|.



[Credit: Zybooks]

Counting Elements with Bijection Between sets

- To count elements in a set, define a bijection between that set and another set with a known size.

4.4 The Inverse of a Function

A function f has an inverse if and only if f is a bijection.

- A bijection function f: X → Y has an inverse denoted by f^{-1}.

- To obtain the inverse of f, exchange the first and second entries in each pair in f.

- Reversing pairs in a function doesn't always give a well-defined function.

- Some functions don't have an inverse.

- Function f: X → Y has an inverse only if reversing pairs in f results in a well-defined function from Y to X.

- f^{-1} is a well-defined function if every element in Y is mapped to exactly one element in X.

Inverse Function Notation

<u>Define</u>
$f(x) = y$ to be a bijective function.

<u>Implies</u>
$f^{-1}(y) = x$

Read: "f inverse of y equals x."

Inverse Composition Identity

For every element $x \in X$,
$f^{-1}(f(x)) = x$.

$f(f^{-1}(x)) = x$.

- The composition of f and f^{-1} is an identity function.

Restricting Domain to Assert Bijective Definition

- The function $f(x) = x^2$ is not one-to-one, because $f(x) = f(-x)$ for any real number x.
- If the domain is restricted to positive real numbers, then $f(x) = x^2$ is a bijection.

- Since each positive real number has a unique square root, the inverse of $f(x) = x^2$ can exist.

<u>Restrict domain</u> to all positive real numbers
$f: \mathbb{R}^+ \to \mathbb{R}^+$
$f(x) = x^2$

Since $f(x)$ is bijective, f^{-1} is well-defined.
$f^{-1}(y) = \sqrt{y}$

4.5 Composition of functions

- <u>Composition</u> refers to applying a function to the result of another function.
- The composition of two functions f and g is denoted as: g o f.
- It is read "g of f".
- The composition of functions creates a new function.

<u>Define</u>
$f: X \to Y$ and $g: Y \to Z$

<u>The Composition of g with f</u>
$\{ (g \circ f): X \to Z \mid \text{for all } x \in X \}$
$(g \circ f)(x) = g(f(x))$

- Function order matters; $f \circ g \neq g \circ f$.

<u>Define</u>
$f: R+ \to R+, f(x) = x^3$
$g: R+ \to R+, g(x) = x + 2$

<u>Expand Compositions</u>
$(f \circ g)(x) = f(g(x)) = (x + 2)^3$

$(g \circ f)(x) = g(f(x)) = x^3 + 2$

- More than two functions can be composed, and composition itself is associative.

f o g o h = (f o g) o h = f o (g o h) = f(g(h(x)))

Identity Function

- The identity function maps each element of a set to itself, and always maps a set to itself.

The identity function on A
I_A: A → A, is defined as I_A(a) = a, for all a ∈ A.

- When a function f from set A to set B has an inverse, the composition of f with its inverse yields the identity function.

4.6 Logarithms and exponents

- The exponential function $\exp_b$: $\mathbb{R} \rightarrow \mathbb{R}^+$ is defined as:

$$\exp_b(x) = b^x$$
$$\text{where b is a positive real number and b} \neq 1$$

- In the expression b^x, the parameter b is called the "base" of the function, and the input x is called the "exponent".

Properties of Exponents

- Consider the following equalities:

$$b^x b^y = b^{x+y}$$

$$(b^x)^y = b^{xy}$$

$$\frac{b^x}{b^y} = b^{x-y}$$

$$(bc)^x = b^x c^x$$

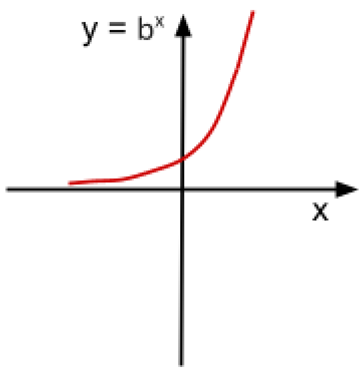
- These equalities are always true for any positive real numbers b, c, and any real numbers x, and y.

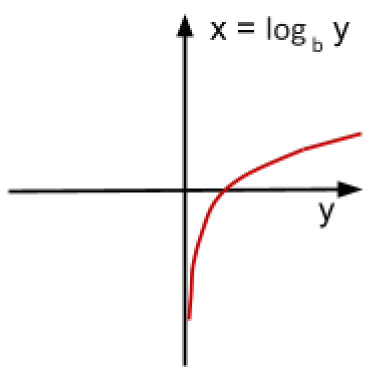
Logarithm Function

- Exponential function is one-to-one and onto, so it has an inverse.
- Logarithm function is the inverse of the exponential function.
- For real number b > 0 and b ≠ 1, $\log_b$: $\mathbb{R}^+ \to \mathbb{R}$ is defined as:

$$b^x = y \quad \Leftrightarrow \quad \log_b(y) = x$$

- The parameter b is called the base of the logarithm in the expression $\log_b(y)$.
- A graph of b^x and $\log_b(y)$:





Properties of Logarithms

- For any positive numbers b, c, x, and y, such that b ≠ 1 and c ≠ 1, the following equalities are always true:

$$\log_b(xy) = \log_b x + \log_b y$$

$$\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$$

$$\log_b(x^y) = y \log_b x$$

$$\log_c x = \frac{\log_b x}{\log_b c}$$

Strictly Increasing or Decreasing

- Function f is strictly increasing if $x_1 < x_2$, then $f(x_1) < f(x_2)$.
  I.e. Always has a postive slope.

- Function f is strictly decreasing if $x_1 < x_2$, then $f(x_1) > f(x_2)$.
  I.e. Always has a negative slope.

- If $b > 1$, then $f(x) = b^x$ and $f(x) = \log_b(x)$ are strictly increasing.

- The fact that both functions are strictly increasing can help in approximating their values.

<u>Approximate $\log_3(420)$</u>

$3^6 = 729$
$3^5 = 243$

$\log_3(243) = 5$
$\log_3(729) = 6$

$\therefore 5 < \log_3(420) < 6$

Population Growth

- Population growth is modeled using an exponential function, where the population p of lizards on an island at time t is described by liz(t) = p•bt, for some number b.

- The value of b determines the rate at which the lizard population increases.

- Once the function liz(t) is developed, questions such as the time it takes to reach a certain population can be answered.

- The population is modeled by the following equation:

$$n = p \cdot b^t \Leftrightarrow \frac{n}{p} = b^t$$

- The number of days for the population to reach n would be approximately $\log_b(n/p)$.

Divide-and-conquer

- Divide-and-conquer solves problems by dividing them into two groups, solving each group separately, and combining the solutions.
- Sorting a list of numbers can be done by dividing the list in half, sorting each half, and merging the two sorted lists.
- The logarithm function is important for analyzing these algorithms.

Ingrid's chocolates:

- Ingrid has n chocolates.

Ingrid has $n = 2^k$ chocolates

- If Ingrid's bag has a number of chocolates that is a power of two ($n = 2^k$ for non-negative integer k), she can divide them evenly with each friend she meets.
- In this case, Ingrid starts with 2^k chocolates.
- Each friend takes half of the chocolates, $2^{(k-1)}$, and the exponent is reduced by 1 each time.
- This continues until each friend has just one chocolate, $2^0 = 1$.
- The number of friends Ingrid can meet is $k = \log_2(n)$.

Ingrid has 19 chocolates

- If n is not a power of 2, then $\log_2(n)$ is not an integer
- The number of encounters with friends must be a whole number
- Ingrid will have to divide her pile of chocolates unevenly at some point
- If she keeps the larger half, then she can divide her chocolates in half $\lceil \log_2 n \rceil$ times.
- If she keeps the smaller half, then she can divide her chocolates in half $\lfloor \log_2 n \rfloor$ times.

Dividing piles and the logarithm function

Let n and b be positive integers with b > 1.

Consider a process in which in each step, n is replaced with $\lfloor n/b \rfloor$, until n < b.

The process lasts for $\lfloor \log_b n \rfloor$ steps.

If instead in each step, n is replaced with $\lceil n/b \rceil$, until n = 1. The process lasts for $\lceil \log_b n \rceil$ steps.