6. Binary Relationships

6.0 Introduction

- A binary relation is a way of expressing a relationship between two sets.
- The term "binary" is used to describe a relation that is a subset of the Cartesian product of two sets.

6.1 Binary Relationships

A binary relation between two sets *A* and *B* is a subset *R* of *A x B*.

For $a \in A$ and $b \in B$
$(a, b) \in R$ is denoted by aRb

University Enrollment

- Suppose we have a university with a set of students S and a set of classes C.
- A relation E can be defined between $s \in S$ and $c \in C$ to indicate whether a student s is enrolled in a class c.
- sEc means that student s is enrolled in class c.
- It is possible for a student to be enrolled in more than one class, and for a class to have more than one student enrolled in it.
- If a student s is not currently enrolled in any classes, then there is no class c such that sEc.

Infinite Sets

- Relations can be defined on infinite sets by specifying a rule or condition that determines whether two elements in the set are related to each other.

For example

- The relation C between $\mathbb{R}$ and $\mathbb{Z}$ can be defined as:

$$xCy \text{ if } |x - y| \leq 1$$

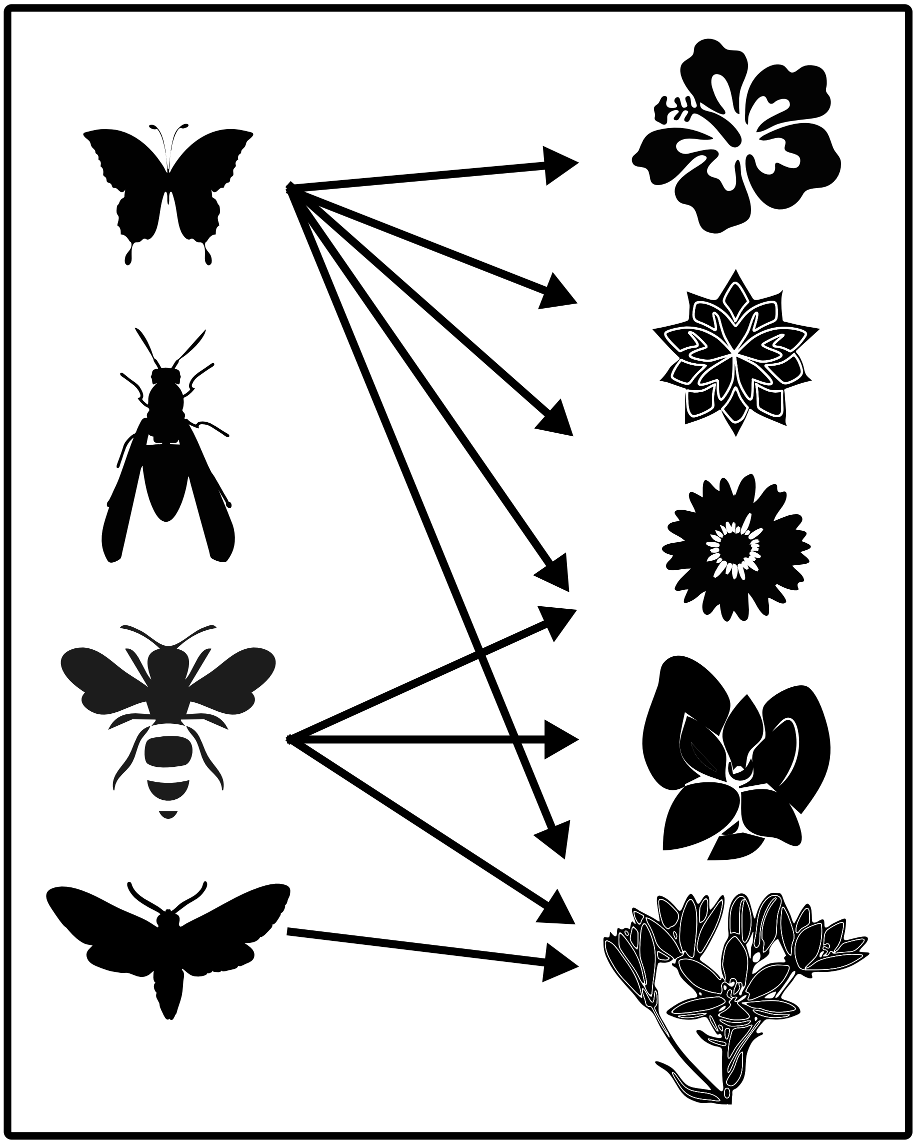
- The relationship xCy exists if the distance between real number x and integer y is at most 1.

Binary Relations with Arrow Diagrams

- If two finite sets, A and B, have a binary relation R between them, R can be represented as a list of ordered pairs or in an arrow diagram.
- In the arrow diagram representation, the elements of A are listed on the left and the elements of B are listed on the right.
- An arrow is drawn from each element a ∈ A to its related element b ∈ B, indicating the relation R between them where aRb.

<u>Example</u>
let I = { Butterfly, Wasp, Bee, Moth }
F = { Hawaiian, Lotus, Daisy, Orchid, Star }
Relation: iPf if insect i ∈ I pollinates flower f ∈ F

P = {
(Butterfly, Hawaiian), (Butterfly, Lotus), (Butterfly, Daisy),
(Butterfly, Orchid), (Butterfly, Star),
(Bee, Daisy), (Bee, Orchid), (Bee, Star),
(Moth, Star)
}



Matrix Representation

- In mathematics, a binary relation between two sets A and B can be represented using a matrix.
- This matrix representation is a 2-dimensional array of numbers, with |A| rows and |B| columns.
- Each row of the matrix corresponds to an element of A, while each column corresponds to an element of B.
- A relationship between $a \in A$ and $b \in B$ is represented by a value of 1 in the row of a and the column of b, otherwise the cell value will be 0.

Example
let I = { Butterfly, Wasp, Bee, Moth }
F = { Hawaiian, Lotus, Daisy, Orchid, Star }
Relation: iPf if insect i ∈ I pollinates flower f ∈ F

P = {
(Butterfly, Hawaiian), (Butterfly, Lotus), (Butterfly, Daisy),
(Butterfly, Orchid), (Butterfly, Star),

(Bee, Daisy), (Bee, Orchid), (Bee, Star),
(Moth, Star)
}

$$
\begin{array}{r|ccccc|}
 & \text{Hawaiian} & \text{Lotus} & \text{Orchid} & \text{Daisy} & \text{Star} \\
\text{Butterfly} & 1 & 1 & 1 & 1 & 1 \\
\text{Wasp} & 0 & 0 & 0 & 0 & 0 \\
\text{Bee} & 0 & 0 & 1 & 1 & 1 \\
\text{Moth} & 0 & 0 & 0 & 0 & 1
\end{array}
$$

Manufacturing Requirements

- The requirements for a manufacturing plant that needs to fulfill a stream of orders can be represented by a binary relation between the set of all orders and the set of machines in the plant.
- Each order is related to a machine if that order requires the use of that machine.
- This binary relation can be used to find an efficient schedule for the orders, ensuring that machines are not needed for more than one order at the same time.
- By representing the requirements as a relation, it is easier to analyze the dependencies between orders and machines and optimize the production process accordingly.

Binary Relation on a Set

- A binary relation on a set A is a subset of AxA, consisting of ordered pairs of elements from A.
- A and B can be the same set.
- The set A is called the domain of the binary relation.
- Binary relations on the same set can be useful in expressing properties of elements within the set, such as divisibility or partial ordering.

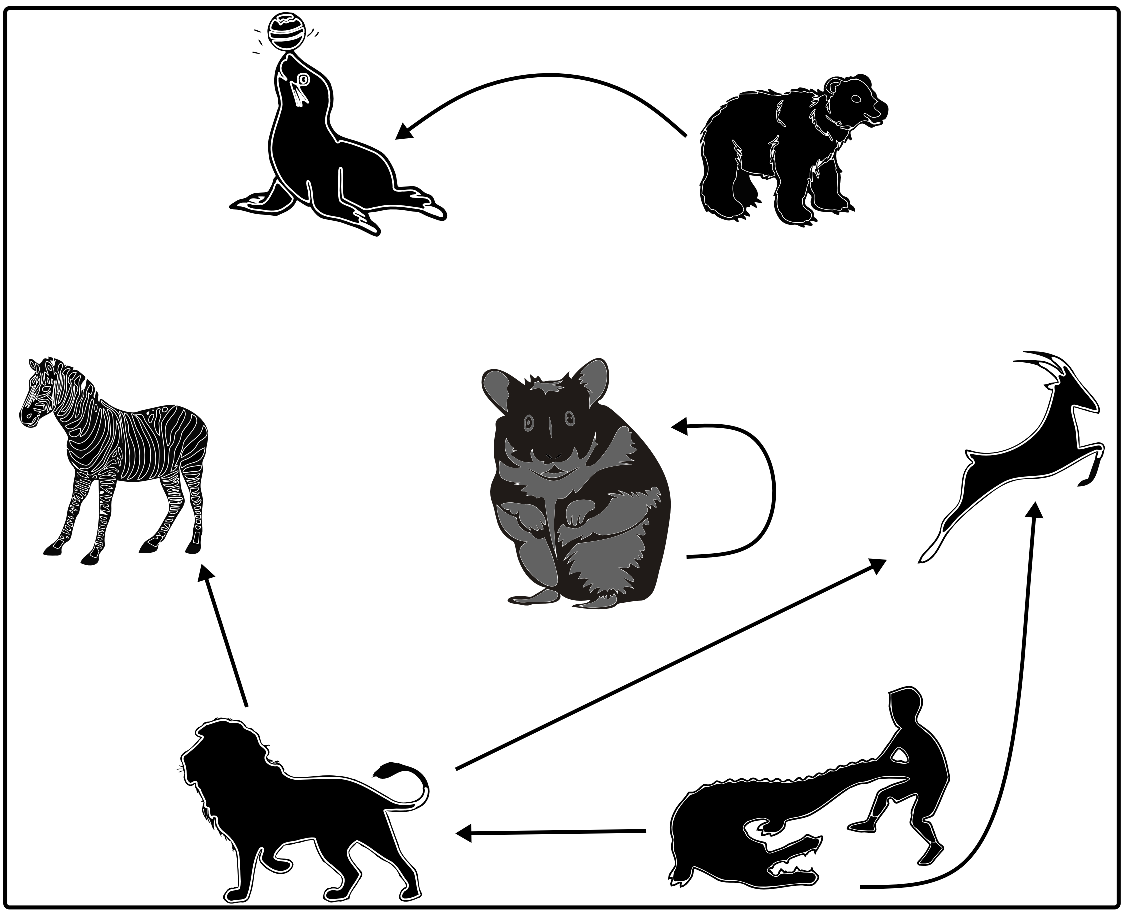
Finite Set Relationships with Arrow Diagrams

- An arrow diagram for a relation R on a finite set A shows the elements of A with arrows connecting related elements.
- A self-loop is used to show an element related to itself.

The Hamster King

In the jungle, the mighty jungle, the hamster belongs to a set.

let
A = { Lion, Zebra, Gazelle, Crocodile, Hamster, Polar Bear, Seal }
P = {
(Lion, Zebra), (Lion, Gazelle),
(Crocodile, Gazelle), (Crocodile, Lion),
(Hamster, Hamster),
(Polar Bear, Seal)
}



6.2 Properties

Overview

- A binary relationship R on set A can have named properties that describe its behavior.

1. Reflexivity
    - R is reflexive if every element in the set is related to itself.
    - Every element must have a self reference.

$\text{For all } x \in A, (x, x) \in R$

2. Anti-Reflexivity (Irreflexivity)
    - R is anti-reflexive (irreflexive) if no element in the set is related to itself.
    - No element may have a self reference.

    $$\text{For all } x \in A, (x, x) \notin R$$

3. Symmetry
    - R is symmetric if the relationship between any two distinct elements in the set is bidirectional.
    - All relationships between elements must be bi-directional.
    - Self references are allowed.

    $$\text{If } (x, y) \in R, \text{ then } (y, x) \in R$$

4. Antisymmetry
    - R is antisymmetric if no two distinct elements share a bidirectional relationship.
    - All relationships are uni-directional.
    - Self References are allowed.

    $$\text{For all distinct } x \in A \text{ and } y \in A,$$
    $$\text{if } (x, y) \in R \text{ and } (y, x) \in R, \text{ then x = y}$$

5. Asymmetry
    - R is asymmetric if no two distinct elements share a bidirectional relationship.
    - All relationships are uni-directional.
    - Self References are <u>not</u> allowed.

    $$\text{For all distinct } x \in A \text{ and } y \in A,$$
    $$\text{if } (x, y) \in R \text{ then } (y, x) \notin R$$

6. Transitivity
    - R is transitive if the relationship between any two elements in the set can be extended to a third element.
    - Any three element connected in a series must have a connection from the first element to the third element.

    $$\text{If } (x, y) \in R \text{ and } (y, z) \in R, \text{ then } (x, z) \in R$$

7. Connectedness
    - R is connected if every pair of distinct elements in the set are related directly or indirectly.
    - Every element can be reached from every other element through a path of relationships.

    $$\text{For all } x \in A \text{ and } y \in A \text{ where } x \neq y,$$
    $$\exists a_1, a_2, \ldots, a_n \in A \text{ such that } (x, a_1), (a_1, a_2), \ldots, (a_n, y) \in R$$

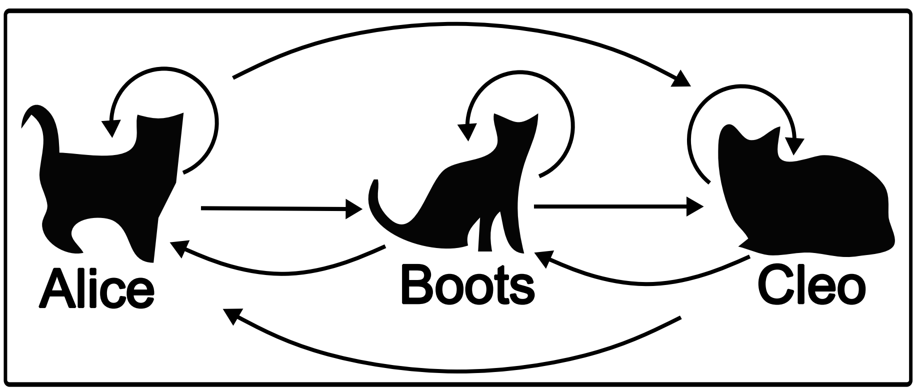
Reflexive and Anti-Reflexive

- A binary relation R on set A is reflexive if every element in A is related to itself (xRx for every x ∈ A).
- Conversely, if there exists an x in A such that xRx is not true, then R is not reflexive.
- A binary relation R on set A is anti-reflexive if no element in A is related to itself (not xRx for every x ∈ A).
- Another term for anti-reflexive is irreflexive.
- If some elements in A are related to themselves and some are not, then R is neither reflexive nor anti-reflexive.

Example of a Reflexive Relationship

The set of cats C where x has the same biological mother as y.

let
C = { Alice, Boots, Cleo }
R = {
(Alice, Alice), (Alice, Boots), (Alice, Cleo),
(Boots, Boots), (Boots, Alice), (Boots, Cleo),
(Cleo, Cleo), (Cleo, Alice), (Cleo, Boots)
}

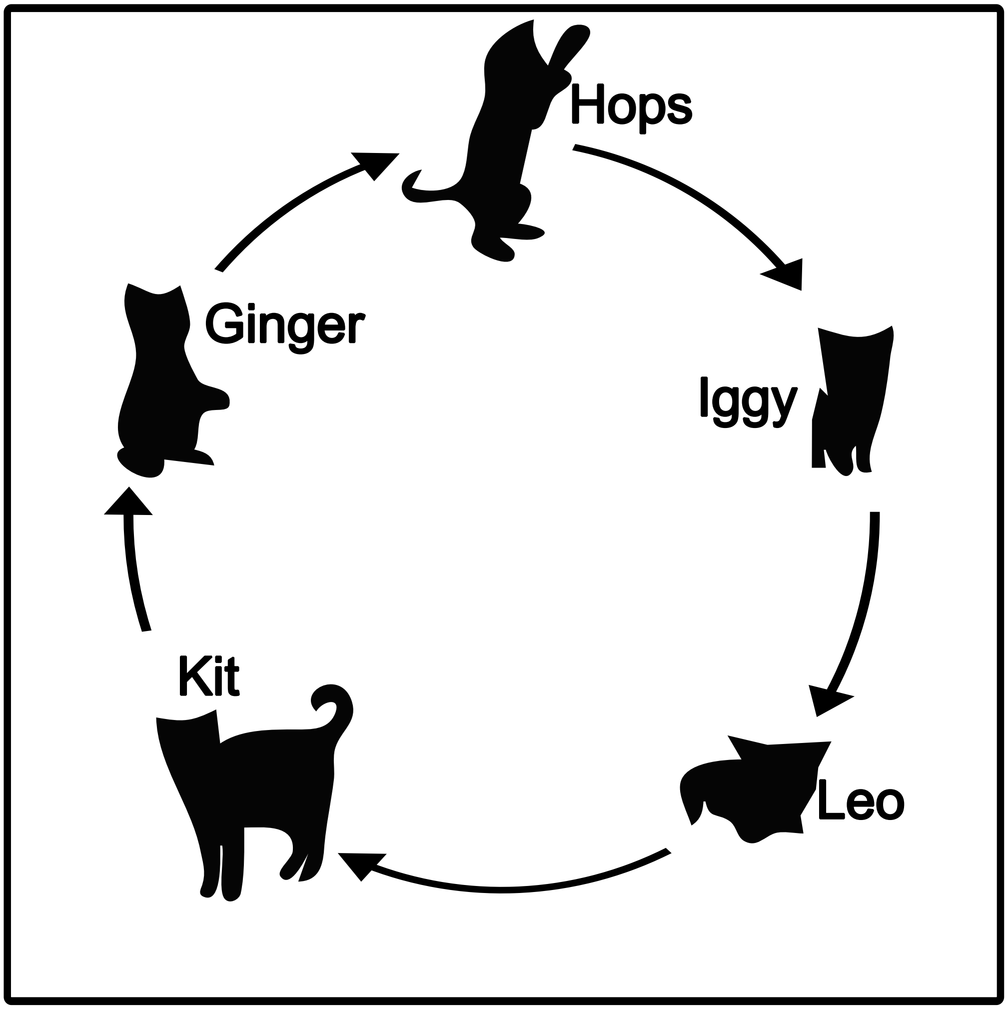


- Three maternal sisters all share the same mother with one another, and also with themselves.
- All elements have a direct relationship with itself.

Example of an Anti-Reflexive Relationship

The set of cats C where x is hunting y.

Let
C = { Hops, Iggy, Leo, Kit, Ginger }
H = {
(Hops, Iggy), (Iggy, Leo), (Leo, Kit), (Kit, Ginger), (Ginger, Hops)
}



- These cats are playing with eachother.
- There are no elements with direct a relationship to itself.

Symmetry

- A relation R on set A is symmetric if for every pair of elements x ∈ A and y ∈ A, either both xRy and yRx are true, or neither xRy nor yRx is true.

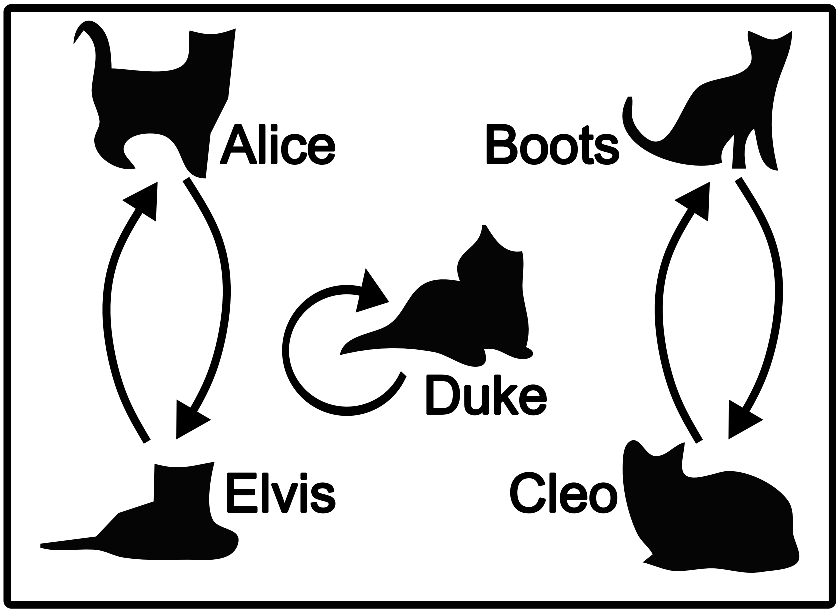
  I.e. for every pair of elements x and y, if x is related to y, then y is related to x.
- If there exists a pair of elements x and y where x is related to y but y is not related to x, then the relation is not symmetric.
- The criteria for a relation to be symmetric is universal, meaning that it must hold for every pair of elements in the domain.

Example of a Symmetric Relationship

The set of cats C where x is grooming y, and y is grooming x.

Let
C = { Alice, Boots, Duke, Elvis, Cleo }
G = {

(Alice, Elvis), (Elvis, Alice),
(Duke, Duke),
(Boots, Cleo), (Cleo, Boots)
}



- All cats are grooming the cat that it is being groomed by.
- All nodes with a relationship to another node have the same relationship from that node back to itself.

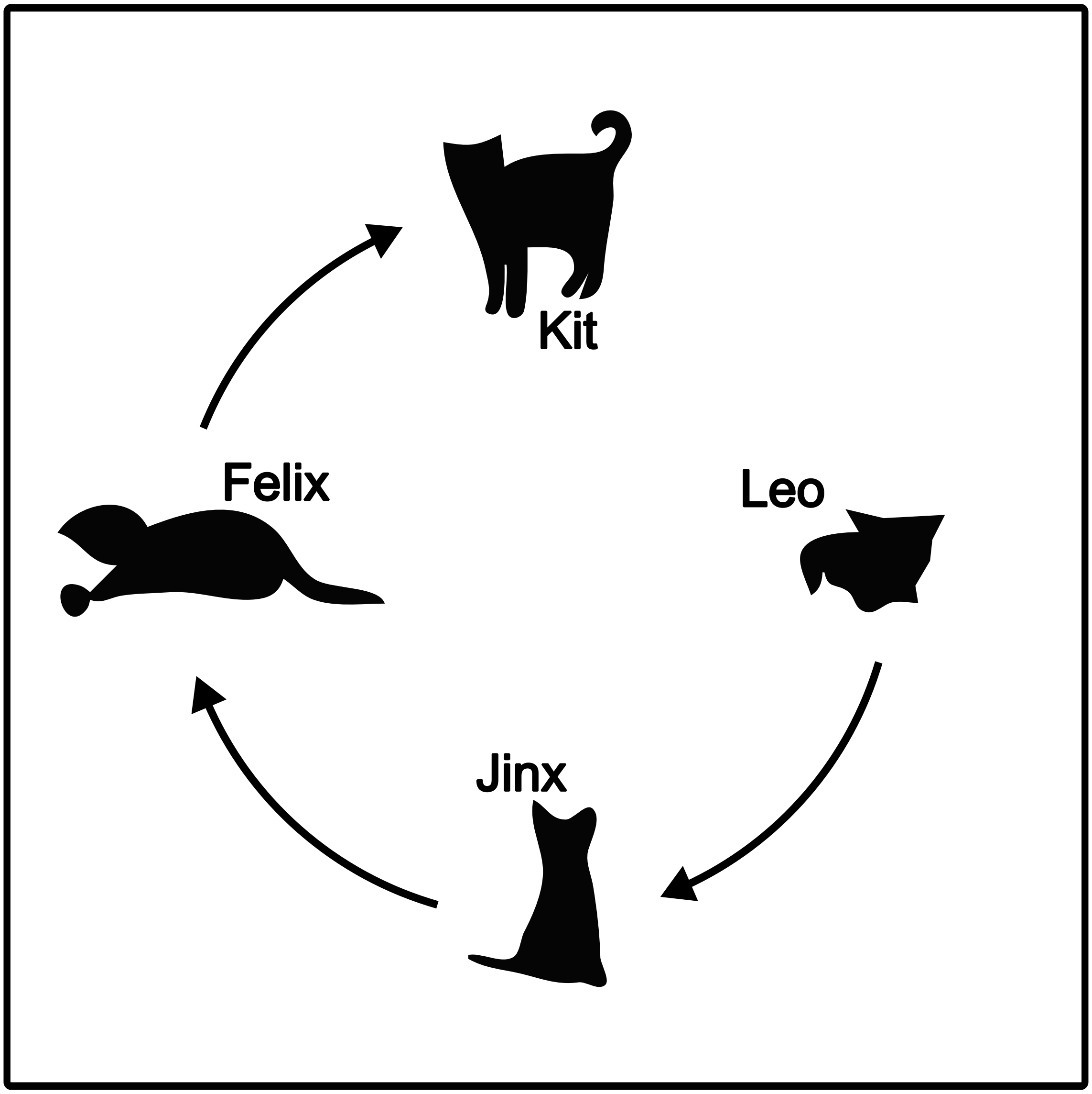
Anti-Symmetric

- A relation is anti-symmetric if for every pair of elements x and y, xRy is true and yRx is not true, where y ≠ x.
- To prove anti-symmetry, show that xRy and yRx being true implies x = y for any distinct elements x and y in the domain.

Example of an Anti-Symmetric Relationship

The set of cats C where x is hiding from y.

Let
C = { Kit, Leo, Jinx, Felix }
F = {
(Leo, Jinx),
(Jinx, Felix),
(Felix, Kit)
}



- Leo hides from Jinx, who in turn is hiding from Felix, who in turn is hiding from Kit.
- All relationships are uni-directional.
- No two nodes have the relationship xRy and yRx at the same time.

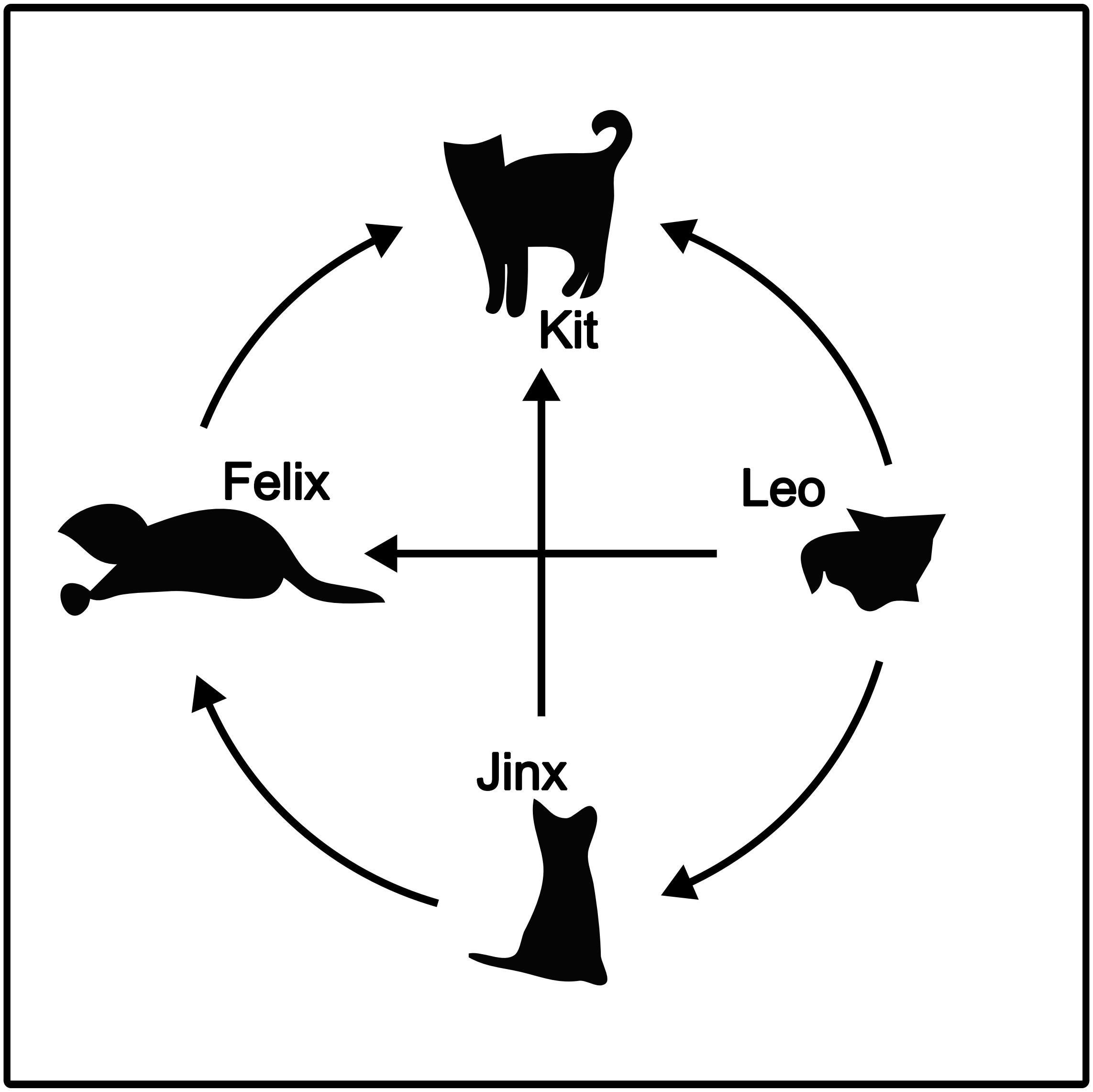
Transitive

- A binary relationship R on a set A is transitive if xRy and yRz implies xRz for every x, y, and z in A.
- To prove transitivity, it suffices to show that if xRy and yRz, then xRz holds for all x, y, and z in A.
- A binary relationship is not transitive if there exists a triple x, y, and z in the domain where xRy and yRz, but not xRz.
- The transitivity criteria are universal, and therefore must hold for every triple of elements in the domain.

Example of a Transitive Relationship

The set of cats C where x is fatter than y.

Let
C = { Kit, Leo, Jinx, Felix }
F = {
(Leo, Jinx), (Leo, Felix), (Leo, Kit)
(Jinx, Felix), (Jinx, Kit)
(Felix, Kit)
}



- Chonkyness is a hierarchical relationship.
- Every set of three related elements in a series has a relationship from the first element to the third element.
  I.e.
  Since the subset { (Leo, Jinx), (Jinx, Felix) } ∈ R exists, { (Leo, Felix) } ∈ R must exist.

Practice Identifying Transitive Relationships

- The domain for each relation is $\mathbb{R}^+$, the set of all positive real numbers.

Explain why the following relationships are either transitive or non-transitive.

1. xRy if $y = x^2$
2. xRy if $y \geqq x^2$
3. xRy if $\lfloor x \rfloor \leqq \lfloor y \rfloor$

▶ Hint

Solutions

1. The relationship is non-transitive because there exists three elements that are related, but where the first and last elements are not related.

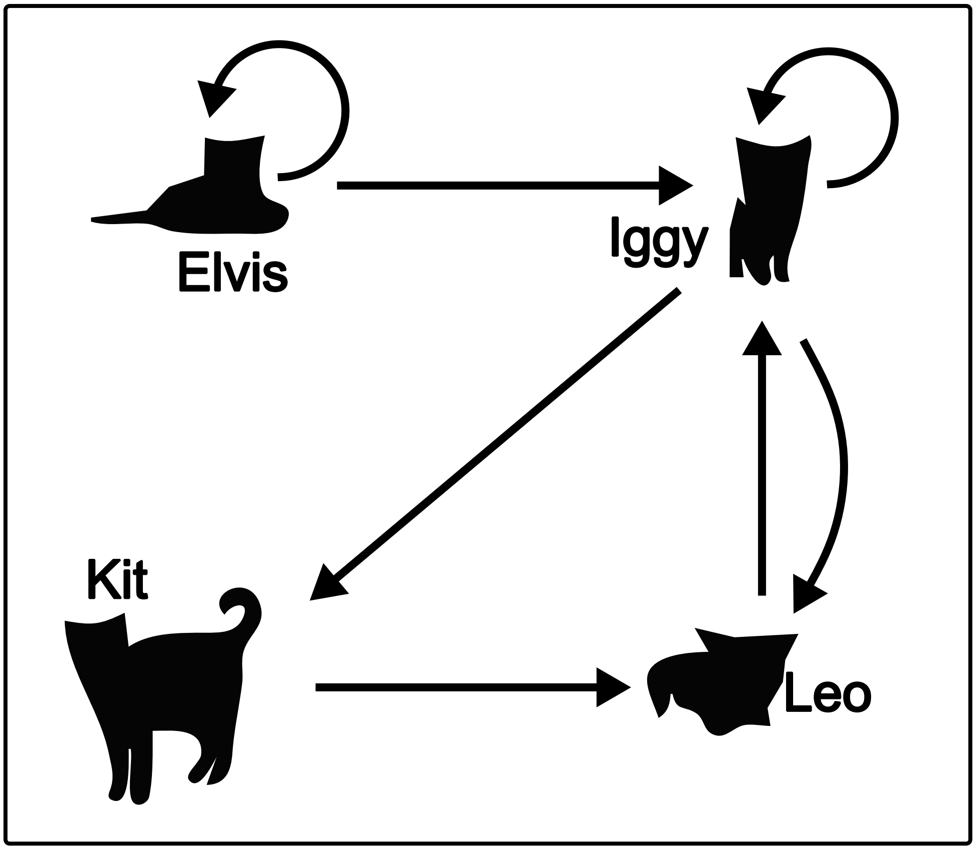
   E.g.
   The subset { (2, 4), (4, 16) } ∈ R exists, but the element (2, 16) does not.
2. The relationship is transitive because if $y \geqq x^2$ and $z \geqq y^2$, then $z \geqq x^4$.

   E.g.
   The subset { (2, 4), (4, 16), (2, 16) } exists.
3. The relationship is transitive because if $\lfloor x \rfloor \leqq \lfloor y \rfloor$ and $\lfloor y \rfloor \leqq \lfloor z \rfloor$, then $\lfloor x \rfloor \leqq \lfloor z \rfloor$.

   E.g.
   The subset { (0.1, 0.2), (0.2, 0.3), (0.1, 0.3) } ∈ R exists.

Practice Identifying Transitive Relationships II (Visually)

The relation R on the set { Elvis, Iggy, Kit, Leo } is defined by this arrow diagram:



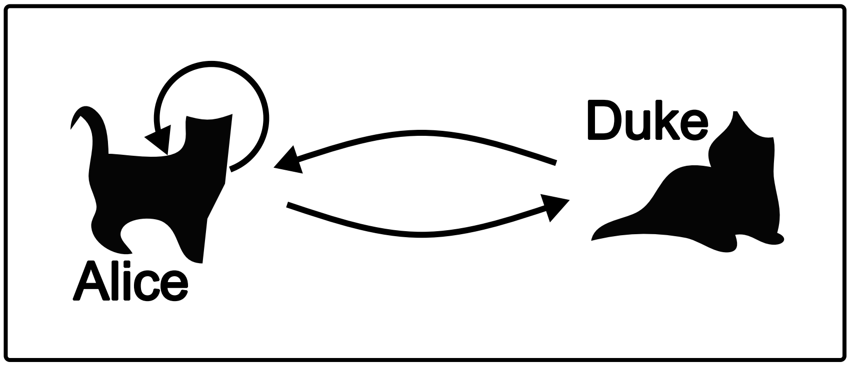
1. Is the relation R reflexive, anti-reflexive, or neither?
2. Is the relation R symmetric, anti-symmetric, or neither?
3. Is the relation R transitive?

Solutions

1. The relation R is neither reflexive, nor anti-reflexive.
   - R is not reflexive, the relationship on Kit and Leo is not self-referencing.
   - { (Kit, Kit), (Leo, Leo) } is not a subset of R.
   - R is not anti-reflexive, the relationship on Elivs and Iggy is self-referencing.
   - { (Elvis, Elvis), (Iggy, Iggy) } is a subset of R.
2. The relation R is neither symmetric, nor anti-symmetric
   - R is not symmetric, the relationship between Kit and Leo is uni-directional.
   - (Kit, Leo) $\in$ R, and (Leo, Kit) $\notin$ R
3. The relation R is not transitive.
   - R is not transitive, Elvis and Kit are not related.
   - The sequence { (Elvis, Iggy), (Iggy, Kit) } is a subset of R, but (Elvis, Kit) $\notin$ R.

Practice Identifying Transitive Relationships III (Trick Q.)

The relation R on the set { Alice, Duke } is defined by this arrow diagram:



1. Is the relation R transitive?

Solution

1. The relation R is not transitive.
    - the sequence { (Duke, Alice), (Alice, Duke) } ⊂ R, but (Duke, Duke) ∉ R.

Proving and Disproving Properties

- Each property of a binary relation is a universal condition that must hold for all elements in the domain to establish that the relation has the property.
- To show that a relation does not have a property, it is sufficient to find one counter-example - specific elements in the domain that do not satisfy the condition.

Example Using 4-Bit Strings

Consider a relation R with the domain of all 4-bit strings, defined such that two strings are related if they share the same first bit or last bit (or both).

E.g. 1010 is related to 0110 because they share the same last bit.

1. is R reflexive?
2. is R symmetric?
3. is R transitive?

Solutions

1. R is reflexive.
    - $\forall x(xRx)$
    - All bits of a binary string are equal to themselves, therefore the first bit of a binary string must be the same as the first bit of itself.
2. R is symmetric.
    - $\forall x \forall y(xRy \rightarrow yRx)$
    - If the firrt bit of x matches the first bit of y, then the first bit of y must match the first bit of x.
3. R is not transitive.

- It is not true that $\forall x \forall y \forall [(xRy \wedge xRz) \to xRz]$
- Counter-example:
  { (1001, 1000), (1000, 0000) } ⊂ R, but (1001, 0000) ∉ R

Example Using Integer Multiples

The relation R is defined on the set of positive integers such that xRy holds if x evenly divides y.

- $\forall x \forall y (x \mid y \to xRy)$
- When x divides y evenly there must be a positive integer n such that $y = xn$.

1. is R reflexive, anti-reflexive, or neither?
2. is R symmetric, anti-symmetric, or neither?
3. is R transitive?

▶ Hint

Solutions

1. R is reflexive.
   - For any x, $x = x \cdot 1$, so xDx will always hold.
2. R is anti-symmetric.
   - If $x \mid y$, and $y \mid x$, then $x = y$ and the relationship is self-referencing.
   - If $x \mid y$, and $y \nmid x$, then $x \neq y$ and the relationship is uni-directional.

$$
\begin{aligned}
&xDy \to (x \cdot n = y) \\
&yDx \to (y \cdot m = x) \\
\\
&xDy \to ((y \cdot m) \cdot n = y) \\
&xDy \to (m \cdot n = 1)
\end{aligned}
$$

$$
\text{Since } m \text{ and } n \text{ are integers, } m = n = 1 \\
\therefore (xDy \wedge yDx) \to (x = y)
$$

$$
\text{So R is anti-symmetric.} \blacksquare
$$

3. R is transitive.
   - If $x \mid y$, and $y \mid z$, then $x \mid z$.

$$
\begin{aligned}
xDy &\to (x \cdot n = y) \\
yDz &\to (y \cdot m = z) \\
xDz &\to (x \cdot k = z) \\
\\
xDz &\to (x \cdot k = y \cdot m) \\
xDz &\to (x \cdot k = (x \cdot n) \cdot m) \\
xDz &\to (k = n \cdot m)
\end{aligned}
$$

$$\text{Since } m \text{ and } n \text{ are integers, } k \text{ must be an integer.}$$

$$\therefore x \text{ divides } z,\ k \text{ times.} \blacksquare$$

Example Using Relatively Prime Integers

Relation R is defined on the set of positive integers, where xRy holds if x and y are relatively prime.

- Two positive integers are considered relatively prime if the only positive integer that evenly divides both of them is 1.
- 4 and 6 are not relatively prime, both are divible by 2.
- 4 and 9 are relatively prime, 1 is the only number that divides both evenly.

1. Which pair is not relatively prime?
   A) x = 3 and y = 3
   B) x = 99 and y = 100
   C) x = 47 and y = 48
2. Which pair is relatively prime?
   A) x = 11 and y = 121
   B) x = 12 and y = 64
   C) x = 22 and y = 26
3. is R reflexive, anti-reflexive, or neither?
4. is R symmetric, anti-symmetric, or neither?
5. find a set of 3 numbers that show R is not transitive.

▶ Hint

Solutions

1. A
   - (3, 3) is not relatively prime, both are divisible by 3.
2. C
   - (22, 26) is relatively prime, they do not share prime factors.
3. R is neither reflexive, nor anti-reflexive.
   - 1 is divisible by only 1, therefore it is a relative prime of itself.
   - Every number x is divisible by x, therefore:
     - no positive integer greater than 1 can be its own relative prime,
     - xRx does not exist.
4. R is symmetric.

- Every pair of relatively prime numbers shares the same relationship, so each is bi-directional.

5. { 9, 25, 42 }
    - 9 and 25, and 25 and 42 are relative primes, but 9 and 42 are not.

6.3 Directed Graphs, Paths, and Cycles

- A vertex is a node in a graph, which can be connected to other vertices via edges.
- Vertices are represented as labeled dots.
- An edge is a connection between two vertices in a graph used to indicate a relationship.
- Directed edges are represented as arrows connecting two vertices.
- A directed graph (digraph) is an ordered pair of sets (V, E), where V is a set of vertices and E is a subset of $V \times V$ representing directed edges.
- The initial vertex is the tail of the arrow, while the final vertex is the head.
- An edge with the same head and tail is called a self-loop.
- Vertex in-degree refers to the number of incoming edges.
- Vertex out-degree refers to the number of outgoing edges.

$$in - degree(u) = |v|(v, u) \in E|$$

$$out - degree(u) = |v|(u, v) \in E|$$

Examples of Directed Graphs

- Representing networks, dependencies, and relationships in a system.
- Modeling the Internet, with the vertex set as URLs and directed edges as hyperlinks.
- Flight paths, with the vertex set as airports, and directed edges as flights between airports.

Walks

- A walk in a directed graph is a sequence of alternating vertices and edges.
- The length of a walk is the number of edges a walk has.
- Number of edges is equal to number of vertices minus one.
  $|V| - 1 = |E|$
- A walk can be simplified to the set of verticies, because the information an edge provides is redundant.
- An open walk is a set of verticies where the first and last verticies are not the same.
  $v_0 \neq v_i$
- A closed walk is a set of verticies where the first and last vertices are the same.

Definition

A walk in a directed graph, labeled G, is a finite sequence of vertices and edges, where V is the set of vertices and E is the set of directed edges.

$$G = \langle v_0, e_1, v_1, e_2, v_2, \cdots, v_{k-1}, e_k, v_k \rangle$$
$$\text{where for each } 1 \le i \le k,$$
$$e_i = (v_{i-1}, v_i) \in E$$

Circuits, Cycles, Paths, Trails

- A trail is a walk where each edge appears only once.
  e.g. $\langle a, b, c, a, d \rangle$
- A path is a walk where every vertex is unique, though there is an exception for the endpoints v_0 and v_i.
  e.g. $\langle a, b, c, d \rangle$ or $\langle a, b, c, d, a \rangle$
- A circuit is a trail that forms a loop, connecting back to its starting point.
  e.g. $\langle a, b, c, a, d, a \rangle$
- A cycle is a circuit with a minimum length of 1, where all vertices are unique, except for the endpoints v_0 and v_i, which are identical.
  e.g. $\langle a, b, c, d, a \rangle$

6.4 Composition of Relations

- The composition of two relations on a set involves finding common elements and creating a new relation.
- Directed graphs and binary relations have a one-to-one correspondence.
- A walk of length k from vertex a to vertex b in a directed graph G implies something about the binary relation corresponding to G.
- A composition of relations R and S on set A is another relation on A, denoted $S \circ R$.
- Imagine R and S as functions, where R is evaluated before S, producing the expression $S \circ R$.

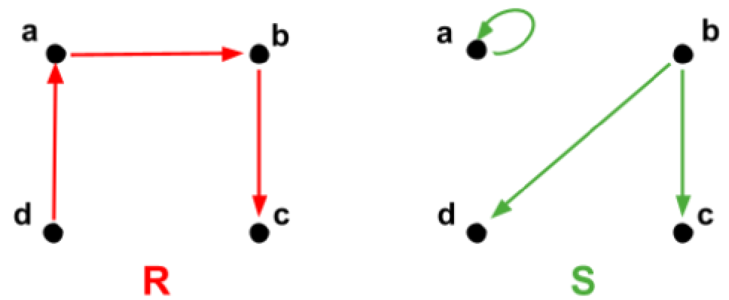
Definition

The composition of relations R and S on set A is a new relation $S \circ R$ such that $(a, c) \in S \circ R$ if and only if there exists $b \in A$ such that $(a, b) \in R$ and $(b, c) \in S$

Example

PARTICIPATION ACTIVITY | 6.4.2: Composition of relations.

Consider the following two relations on the set {a, b, c, d}:



1) Is (a, d) in S o R?

- (selected) Yes
- No

Correct

Yes. (a, b) ∊ R and (b, d) ∊ S.

2) Is (b, d) in S o R?

- Yes
- (selected) No

Correct

There is no x ∊ A such that (b, x) ∊ R and (x, d) ∊ S.

3) Is (b, c) in S o R?

- Yes
- (selected) No

Correct

There is no x ∊ A such that (b, x) ∊ R and (x, c) ∊ S.

4) Is (d, a) in S o R?

- (selected) Yes
- No

Correct

Yes. (d, a) ∊ R and (a, a) ∊ S.

6.5 Graph Powers / Transitive Closure

- A relation on a set can be self-composed (composed with itself).
- The relationship $x(P \circ P)z$ means that x is the grandparent of z.
- The directed graph of $P \circ P$ represents all two-step walks in P.
- A walk with k steps in R can be defined generally:

  R^1 as R
  R^k as $R \circ R^{k-1}$, for $k \geq 2$

  allowing R to be self-composed k times.
- The edge set E of a directed graph G represents a relation, with E^k being E self-composed k times.
- G^k is a directed graph with edge set E^k, which expresses walk relationships between vertices.
- G^k is called the k^{th} power of G.

The Graph Power Theorem

- For any two vertices u and v in a directed graph G, there exists an edge from u to v in G raised to the power of k if and only if there exists a walk of length k from u to v in G.

Transitive Closure

- The transitive closure of G, denoted G^+, expresses accessibility by walks of any positive length on a directed graph G.

Definition

The transitive closure of G is the union of G^k for all $k \geq 1$.

$$G^+ = \bigcup_{k=1}^{n} G^k$$

- G^+ is called the transitive closure of G.
- (u, v) is an edge in G^+ if there exists a walk of any positive length from vertex u to vertex v in G.
- If there are n elements in G, then G^+ is the union of G^1 to G^n
- The transitive closure of R on a finite domain with n elements, denoted R^+, is the union of R^k for all $k \geq 1$

$$R^+ = \bigcup_{k=1}^{n} R^k$$

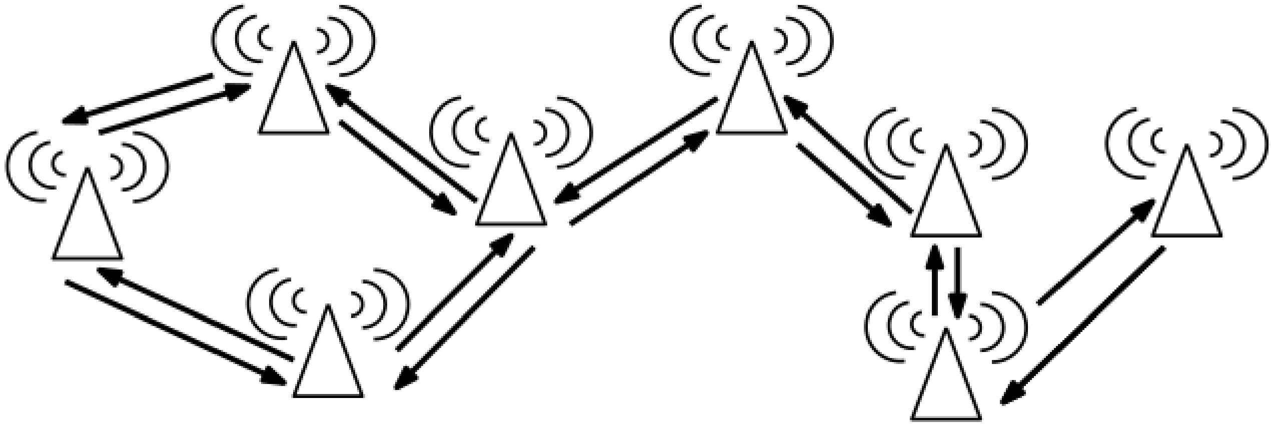
- R^+ is the smallest relation that is both transitive and includes all pairs from R.
- Any relation that is transitive and contains all pairs from R must also contain all pairs in R^+.
- To compute the transitive closure, find all paths between vertices in G of any length.
- The transitive closure can be represented as the union of all subgraphs G^k up to the maximum path length.

Procedure to find the transitive closure of a relation R

Repeat the following step until no pair is added to R:

- If there are three elements $x, y, z \in A$ such that $(x, y) \in R$, $(y, z) \in R$ and $(x, z) \notin R$, then add (x, z) to R.

Example of a Sensor Network



Credit: ZyBooks

A sensor network is a collection of sensors distributed over a geographical area, typically used in industrial and military applications to monitor processes, machinery, or people. The sensors are small, low-cost devices with limited power that fades with use. Each sensor can only communicate with nearby sensors within its range, which may depend on its remaining power. The communication graph for the network has a vertex set corresponding to the sensors with a directed edge from sensor x to sensor y if x can send a message directly to y. Messages can also be transmitted through the network along a path of connected sensors. One of the goals in designing sensor networks is to maintain connectivity for as long as possible.

To test connectivity, we need to answer questions like "Is there a path from sensor x to sensor y along the directed edges of the communication graph?" We can answer this question for all pairs of sensors by computing the transitive closure G^+ of the communication graph G. G^+ includes all paths of any length between sensors and provides a complete picture of connectivity in the network.

Proof of the Graph Power Theorem

[Todo]