8. Induction and Recursion

8.1 Sequences

Sequences are Functions

- An sequence is a function that maps natural numbers (or a subset of them) to a set.
- The domain of a sequence is a set of consecutive integers.

Example

- Sequences can represent various types of data, such as a student's GPA over the years they attend college.

$$g(1) = 2.51, \;\; g(2) = 2.77, \;\; g(3) = 3.21, \;\; g(4) = 3.33$$

Notation

- The entire sequence is represented by $\{g_k\}$.
- The value g_k is called a term of the sequence, and k is the index of g_k.
- A single term is represented by g_k (without the curly braces).

$$g_1 = 2.51, \;\; g_2 = 2.77, \;\; g_3 = 3.21, \;\; g_4 = 3.33$$

- Sequences can be written with just the terms if the function name and indices are understood or not important.

$$gpaCollection = [2.51, 2.77, 3.21, 3.33]$$

- Sequences can also start with any integer, not just 0 or 1.

$$\text{Sequence } \{m_k\} \text{ is a collection of Star Wars films}$$

$$\begin{aligned} m_{-3} &= \text{The Phantom Menace} \\ m_{-2} &= \text{Attack of the Clones} \\ m_{-1} &= \text{Revenge of the Sith} \\ m_0 &= \text{A new Hope} \\ m_1 &= \text{The Empire Strikes Back} \\ m_2 &= \text{Return of the Jedi} \end{aligned}$$

Finite and Infinite

- A finite sequence $\{a_k\}$ has:
    1. A finite domain.

2. An initial index m and final index n, such that $n \geq m$.
3. An initial term a_m and final term a_n.
4. A bounded set: $a_m,\ a_{m+1},\ \cdots,\ a_n$.

- An infinite sequence $\{b_k\}$ has:
  1. An infinite domain.
  2. An initial index m, where $\{b_k\}$ is defined for all indices k such that $k \geq m$.
  3. Indices that approach infinity in the positive direction ($i.e.\ \ k \geq m$).
  4. A set with no upper or lower bound: $b_m,\ b_{m+1},\ b_{m+2},\ \cdots$

Explicit Formula

- Sequences can be expressed as an explicit formula that shows how a_k depends on k.

Example

Triangular numbers are numbers that can be arranged in the shape of a triangle.

The n^{th} triangular number can be calculated using the formula:

$$T_n = \frac{n(n+1)}{2} \text{ for } n \geq 2$$

$$= [3, 6, 10, 15, 21, \ldots]$$

Increasing and Decreasing

- For every pair of consecutive indices $k \text{ and } k+1$, a sequence is:

Property	Condition	Example
Increasing	$a_k < a_{k+1}$	$f(x) = x^3$
Non-decreasing	$a_k \leq a_{k+1}$	$f(x) = \lfloor \frac{x}{2} \rfloor$
Decreasing	$a_k > a_{k+1}$	$f(x) = \frac{1}{x}, \text{for } x \in \mathbb{Z}^+$
Non-increasing	$a_k \geq a_{k+1}$	$f(x) = -x - \|x-3\| + 6$

- Increasing sequences are always non-decreasing.
- Decreasing sequences are alays non-increasing.
- A horizontal line is neither increasing nor decreasing.
- A horzontal line is is both non-decreasing and non-increasing.

Geometric Sequences

- A geometric sequence is a sequence of real numbers where every term after the first is found by multiplying the previous term by a common ratio (r).

Explicit Formula

$$a_n = a_i \cdot r^n$$

- a_i = initial term
- r = common ratio

Finite geometric sequence

$$a_n = (1000) \cdot (1.005)^n$$
$$a_i = 1000,\ r = 1.005$$

$$\text{let } m = 0,\ n = 3$$
$$\{a_k\} = [\,1000, 1005, 1010, 1015.10\,]$$

Infinite geometric sequence

$$a_n = (1) \cdot (\frac{1}{2})^n$$
$$a_i = 1, r = \frac{1}{2}$$

$$\{a_k\} = [\,1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16},\ \dots\,]$$

Arithmetic Sequence

- An arithmetic sequence is a sequence of real numbers where each term is found by adding a fixed number called the common difference to the previous term.

Explicit Formula

$$a_n = a_i + d \cdot n$$
$$\text{for } n \geq 0$$

- a_i = initial term
- d = common difference

Infinite Arithmetic Sequence

$$a_n = 0 + 5 \cdot n$$
$$\{a_n\} = [\,0, 5, 10, 15,\ \cdots\,]$$

Finite Arithmetic Sequence

$$a_n = \pi - \frac{\pi}{3} \cdot n$$

$$\text{let } m = 0,\ n = 3$$
$$\{a_n\} = [\,\pi, \frac{2\pi}{3}, \frac{\pi}{3}, 0\,]$$

8.2 Recurrence Relations

Recurrence Relation

- A rule that defines a term a_n as a function of previous terms in the sequence is called a recurrence relation.

Arithmetic Sequence

- Arithmetic sequences can be defined by the following recurrence relation:
  - $a_0 = a$ (initial value)
  - $a_n = d + a_{n-1}$ for $n \geq 1$ (recurrence relation)
- Initial value = a. Common difference = d.

Geometric Sequence

$$a_n = r \cdot a_{n-1}$$
$$\text{for } n \geq 1$$

- a_0 = initial value
- r = common ratio

Fibonacci Sequence

- Leonardo Fibonacci used a recurrence relation to define the Fibonacci sequence, which models rabbit population growth.

Rules:

- The rabbit colony begins with one pair of newborn rabbits.
- Rabbits must be at least one month old before they can reproduce.
- Every pair of reproducing rabbits gives birth to a new pair (one male and one female) over the course of a month.

$$\begin{aligned} &f_0 = 0 \\ &f_1 = 1 \\ &f_n = f_{n-1} + f_{n-2} \quad \text{for } n \geq 2 \end{aligned}$$

$$\{f_n\} = [\, 0, 1, 1, 2, 3, 5, 8, 13, 21, \cdots \,]$$

Number of Initial Values

- The number of initial values required depends on which previous terms are used to define the n^{th} term.
- For example, in the Fibonacci sequence, f_n depends on the previous two terms.
  I.e. f_0 and f_1 are required to define f_2

Dynamical Systems

- A dynamical system is a system that changes over time, with its state determined by a set of well-defined rules that depend on the past states of the system.
- In a discrete time dynamical system, time is divided into discrete time intervals, and the state during one interval is a function of the state in previous time intervals.
- The history of the system is defined by a sequence of states, indexed by non-negative integers.

Fibonacci's Rabbit Colony

- Fibonacci's rabbit colony is an example of a discrete time dynamical system.
- Each time interval is a month.
- The state of the system is the number of pairs of rabbits.
- Dynamical systems often give rise to recurrence relations that describe how the system changes over time.

8.3 Summations

Summation Notation

- Summation notation is used to express the sum of terms in a numerical sequence.
- The capital letter sigma Σ is used to denote that the terms are to be added together.

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + \cdots + a_{n-1} + a_n$$

- n is the upper limit
- i is called the index
- m is the lower limit
- Use parentheses to indicate that all the terms are included in the summation.

$$\sum_{j=1}^{n}(j+1) \neq \sum_{j=1}^{n} j+1$$

$$\sum_{j=1}^{n} j+1 = \left(\sum_{j=1}^{n} j\right) + 1$$

Example

- Consider the sequence:

$$a_n = n^3, \quad \text{for } n = 1, 2, 3, 4$$

$$\begin{aligned} &= 1^3 + 2^3 + 3^3 + 4^3 \\ &= 1 + 8 + 27 + 64 \\ &= 100 \end{aligned}$$

- Summation notation:

$$\sum_{j=1}^{4} j^3 = 1^3 + 2^3 + 3^3 + 4^3 = 100$$

Extracting Final Term

- It is often useful to extract or insert a final term into a summation.

$$\begin{aligned}\sum_{k=m}^{n} a_k &= \underbrace{a_m + a_{m+1} + a_{m+2} + \cdots + a_{n-1}}_{\text{terms } m \text{ to } n-1} \quad + \underbrace{a_n}_{\text{final term}} \quad \text{for } n > m \\ &= \sum_{k=m}^{n-1} a_k + a_n\end{aligned}$$

Example

$$\sum_{k=0}^{n} 10^k = \sum_{k=0}^{n-1} 10^k + 10^n$$

Variable Limits

- Lower or upper limits are denoted by variables.
- Limits are require variables for evaluation of the sum, provided as values.
- A limit can be either finite or infinite.

$$\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k = 2$$

- Variables can be subsituted.

Example

$$\sum_{k=1}^{n} \sqrt{k-1}$$

$$\begin{gathered}\text{let } j = k - 1 \\ \begin{matrix} \text{when } k = 1 & \text{when } k = n \\ \underbrace{j = 1 - 1 = 0}_{\text{New Lower Limit}} & \underbrace{j = n - 1}_{\text{New Upper Limit}} \end{matrix}\end{gathered}$$

$$\sum_{k=1}^{n} \sqrt{k-1} = \sum_{j=0}^{n-1} \sqrt{j}$$

$$\begin{aligned}\sum_{k=1}^{10} \sqrt{k-1} &= \sum_{j=0}^{9} \sqrt{j} \\ &= 19.306\end{aligned}$$

Closed Form

- A closed form expresses a summation as an algebraic formula.

Arithmetic Closed Form

- A series with a constant difference (d) between consecutive terms.

$$\sum_{k=0}^{n-1}(a+kd) = an + \frac{d(n-1)n}{2}$$
$$\text{for any integer } n \geq 1$$

Example

$$\sum_{k=0}^{44}(2+k\times 5) = 2(45) + \frac{5(45-1)45}{2}$$

$$= 90 + 5\times 45 \times \frac{44}{2}$$

$$= 5\times 22\times 45\times +90$$

Example

$$\sum_{k=0}^{5}(3k+1) = 1+4+7+10+13+16 \qquad = 51$$
$$n-1=5$$
$$n=6$$
$$\sum_{k=0}^{5}(3k+1) = (1)(6) + \frac{3\cdot(6-1)\cdot 6}{2} \qquad an + \frac{d(n-1)n}{2}$$
$$= 6 + \frac{90}{2}$$
$$= 51$$

Geometric Closed Form

- A series with an initial term (a) and constant ratio (r) between consecutive terms.

$$\sum_{k=0}^{n-1} a\cdot r^k = \frac{a(r^n-1)}{r-1}$$

$$\text{for any real number } r \neq 1 \text{ and any integer } n \geq 1$$

Example

$$\sum_{k=0}^{5} 4\left(\frac{1}{2}\right)^k = 4 + 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \qquad = \frac{63}{8}$$

$$\begin{aligned} n - 1 &= 5 \\ n &= 6 \end{aligned}$$

$$\begin{aligned} \sum_{k=0}^{5} 4\left(\frac{1}{2}\right)^k &= \frac{4[(\frac{1}{2})^6 - 1]}{\frac{1}{2} - 1} \qquad\qquad \frac{a(r^n - 1)}{r - 1} \\ &= \frac{4[\frac{1}{64} - 1]}{-\frac{1}{2}} \\ &= 2 \cdot 4[1 - \frac{1}{64}] = 8 \cdot \frac{63}{64} \\ &= \frac{63}{8} \end{aligned}$$

Exponential Growth

- A bacterial colony grows at a rate of 4% every hour.
- Consider a colony with a population of 1000.
- How large would the colony be after 24 hours?

$$\begin{aligned} \sum_{k=0}^{24} 1000(1.04)^k &= 1000 \cdot \frac{(1.04)^{25} - 1}{1.04 - 1} \\ &= \text{~}41,646 \end{aligned}$$

8.4 Mathematical induction

Mathematical Induction

- A proof technique for proving statements about elements in a sequence.
- Establishes that a statement parameterized by n is true for all positive integers n.

Components of Inductive Proof

- Using an inductive proof we can verify the truth of statements such as:

  The sum of first n Fibonacci numbers is equal to the $(n+2)^{th}$ Fibonacci number minus 1

Base Case

- The base case proves the theorem for the first value in the sequence.
- Typically assigns $n = 1$ to the summation index, but can assign higher integers, e.g. $n = 2$.

Fibonacci Sequence

$$\{a_n\} = 0, 1, 1, 2, 3, 5, 8, 13, \cdots$$
$$\sum_{k=2}^{n} [a_{n-2} + a_{n-1}] = a_{n+2} - 1$$

$$\sum_{k=2}^{2} [a_0 + a_1] \qquad a_4 - 1$$

$$\sum_{k=2}^{2} [0 + 1] = 1 \qquad 2 - 1 = 1$$

Inductive Step

- Assumes the theorem is true for k, and proves it for $k+1$.
- Example: If you have three wishes on day k, you can get three wishes for day $k+1$.

Principle of Mathematical Induction

- If the base case and inductive step are both true, the theorem holds for all positive integers.

Principle of mathematical induction.
Let $S(n)$ be a statement parameterized by a positive integer n.
Then $S(n)$ is true for all positive integers n, if:

1. $S(1)$ is true (the base case).
2. For all $k \in \mathbb{Z}^+$, $S(k)$ implies $S(k+1)$ (the inductive step).

Inductive Step and Inductive Hypothesis

- The inductive step states that an infinite sequence of implications are true.
- This implies that the truth of $S(k)$ guarantees the truth of $S(k+1)$.
- The inductive hypothesis is the assumption that $S(k)$ is true.
- The inductive hypothesis serves as the basis for proving the implication between $S(k)$ and $S(k+1)$.
- It assumes the truth of $S(k)$ and allows for the deduction or inference of $S(k+1)$ being true as well.

Inductive Step: For all $k \in \mathbb{Z}^+$,

$$\begin{aligned} S(k) \text{ implies } S(k+1) \quad \longleftrightarrow \quad & S(1) \text{ implies } S(2) \\ & \wedge\, S(2) \text{ implies } S(3) \\ & \wedge\, S(3) \text{ implies } S(4) \\ & \wedge \cdots \end{aligned}$$