Gravitational radiation and Jefimenko-type solutions in the weak field limit

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The present article aims to study the radiative phenomenon of the gravitational field taking as a role model the theory of Classical Electrodynamics. Parting from Maxwell's equations in order to establish the founding equations of GEM, the possible consequences of this theory are explored, as well as the limits of its validity.

***Index Terms*—Radiation, gravitational field, Jefimenko equations, gravitoelectromagnetism**

I. Maxwell's equations and Jefimenko's formal solution

Maxwell's equations in the presence of some charge density ρ and a charged current $\vec{j}$ take the most general possible form:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho_c}{\varepsilon_0} \tag{1}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{2}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{3}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}_c + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} \tag{4}$$

Where the sub-index c denotes the signaled magnitudes are electromagnetic in nature. The formal solution to these equations in integral form is given by the so-called *Jefimenko equations* (cf. [1]).

$$\begin{aligned}\vec{E}(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\int \Bigg[\frac{\rho_c(\vec{r}\,',t_r)\,\vec{R}}{R^3} + \frac{\vec{R}}{R^2 c}\frac{\partial \rho_c(\vec{r}\,',t_r)}{\partial t} \\ - \frac{1}{Rc^2}\frac{\partial \vec{J}_c(\vec{r}\,',t_r)}{\partial t}\Bigg] d^3\vec{r}\,'.\end{aligned} \tag{5}$$

$$\begin{aligned}\vec{B}(\vec{r},t) = \frac{\mu_0}{4\pi}\int \Bigg[\frac{\vec{J}_c(\vec{r}\,',t_r)\times\vec{R}}{R^3} \\ + \frac{1}{R^2 c}\frac{\partial \vec{J}_c(\vec{r}\,',t_r)}{\partial t}\times\vec{R}\Bigg] d^3\vec{r}\,'.\end{aligned} \tag{6}$$

where

- $\vec{R} = \vec{r} - \vec{r}\,'$
- $\vec{r}$ is the point where the fields are being measured
- $\vec{r}\,'$ is a the position of an element of volume of the source
- $t_r = t - R/c$ is the retarded time

II. Gravitoelectromagnetism and an analogy with Maxwell's equations

The gravitational field generated by a body is obtained solving Einstein's field equations, which lay the foundations for General Relativity. Nonetheless, under certain conditions, the behaviour of said field is formally identical to that of the electromagnetic field.

These conditions are as follows: (1) the mass-energy distribution under consideration must give place to a sufficiently weak gravitational field, and (2) the region of spacetime around the distribution must be sufficiently plane. If either of these conditions is met (one can easily intuit they are often equivalent), then it is proceeding to use the equations of Gravitoelectromagnetism (GEM), which can be derived as a limiting case from Einstein's equations, to describe the gravitational field under consideration. The equations are as follows:

$$\vec{\nabla} \cdot \vec{E}_g = -4\pi G \rho_g \tag{7}$$

$$\vec{\nabla} \cdot \vec{B}_g = 0 \tag{8}$$

$$\vec{\nabla} \times \vec{E}_g = -\frac{\partial \vec{B}_g}{\partial t} \tag{9}$$

$$\vec{\nabla} \times \vec{B}_g = -\frac{4\pi G}{c^2}\vec{J}_g + \frac{1}{c^2}\frac{\partial \vec{E}_g}{\partial t} \tag{10}$$

where:

- $\vec{E}_g$ is the *gravitoelectric field* (what we ordinarily call *gravitational field* in the Newtonian theory)
- $\vec{B}_g$ is the *gravitomagnetic field*
- ρ_g is the mass density in the considered region (for the sake of simplicity, only massive sources shall be considerd)
- $\vec{J}_g$ is the mass current density present in the considered region

It can be easily checked that the first of these equations reduces to the better-known Poisson equation for a gravitational field source in the classical Newtonian approach, just by considering the electric field to be the gradient of some scalar potential, $\vec{E}_g = -\vec{\nabla}\varphi$. Nonetheless, it is important to note that, as is the case in electromagnetism, this only holds for a static solution, since in general $\vec{E}_g = -\vec{\nabla}\varphi - \partial_t \vec{A}$.

It is interesting to note that a direct consequence of the validity of these equations is they explicitly acknowledge the speed at which the gravitational field propagates is no other than c, which further emphasizes the point that c should not be regarded as *the speed of light*, but rather as *the speed of causality in a plane spacetime*. Another rather remarkable consequence is that all the phenomena which are present in electromagnetism (e.g: field induction) are also necessarily present in this limit, as the equations which dictate the behaviour of the fields are formally identical in both cases. Therefore, GEM should predict phenomena such as the following: in classical electromagnetism, if one positions two wire loops facing eachother and has a varying current pass through one of them, the loop will give place to a varying magnetic field, which at the same time will trigger a current being observed in the second loop. Therefore, if one were to substitute the loops for, say, asteroid belts, where one of them initially rotates in a non-uniform manner and the other is at rest, then it would be expectable to observe the second asteroid belt eventually begins to rotate, in a completely similar fashion to how the second wire loop experienced an induced current. That is to say, it appears to be possible to induce mass currents just as one induces currents in electromagnetism.

Quite naturally, it is also worthwhile to explore compact sources such as a star. If it is static, then it is evident from the GEM equations that the case is completely analogous to that of a charged dielectric sphere giving place to an electric field. Nonetheless, if the star were rotating, it would give place to a gravitomagnetic field in the very same way as a charged dielectric sphere gives place to a magnetic field when it is rotating. It is rather remarkable to note that, in the latter case, the *agent* giving place to this magnetic field would be the magnetic dipole moment induced by the rotation, whilst in the former, the GEM analogue would simply be the sphere's angular momentum.

III. General solution of the GEM equations

Employing the argument that, if the GEM equations are formally identical to Maxwell's equations then their solution must also be formally identical to the Jefimenko equations, it is reasonable to propose to solve the GEM equations without carrying out any further calculations. Instead, one can simply consider the following substitutions in the Jefimenko equations:

$$\frac{\rho_c}{\varepsilon_0} \rightarrow -4\pi G \rho_g$$
$$\mu_0 \vec{J}_c \rightarrow -\frac{4\pi G}{c^2}\vec{J}_g$$

Taking into account said substitutions, as well as the fact that both electromagnetic and mass densities (current densities as well) are related by a constant factor in the corresponding equations, and therefore there is no substantual change in the Jefimenko equations (there would be, if the functional forms were different, but they are proportional), the solution to the GEM equations is then given by:

$$\vec{E}_g(\vec{r},t) = -G\int\left(\frac{\rho_g(\vec{r}\,',t_r)\vec{R}}{R^3} + \frac{\vec{R}}{R^2 c}\frac{\partial\rho_g(\vec{r}\,',t_r)}{\partial t} - \frac{1}{Rc^2}\frac{\partial\vec{J}_g(\vec{r}\,',t_r)}{\partial t}\right)d^3\vec{r}\,' \tag{11}$$

$$\vec{B}_g = -\frac{G}{c^2}\int\left(\frac{\vec{J}_g(\vec{r}\,',t_r)\times\vec{R}}{R^3} + \frac{1}{R^2 c}\frac{\partial\vec{J}_g(\vec{r}\,',t_r)}{\partial t}\times\vec{R}\right)d^3\vec{r}\,' \tag{12}$$

IV. Gravitational radiation

After finding a formal solution for the GEM equations and making explicit the similarity between GEM and electromagnetism, it is reasonable to expect the gravitoelectric field ($\vec{E}_g$) to behave in the same fashion as the electric field, and the same applies to the gravitomagnetic field ($\vec{B}_g$). Nonetheless, the analogy cannot be pushed arbitrarily far, since electromagnetism allows for two charges to repel eachother, whilst two *gravitational charges* (i.e: massive particles) can only experience an attractive force (this statement needs to be carefully analized, since we have not taken into consideration agents such as pressure, which can contribute to matter being repulsive. Nonetheless, for the sake of this article, we shall consider the strong energy conditions holds, thus making matter attractive).

Let us now take a step back and look at the GEM equations. Since Maxwell's equations allow for wave solutions, it is reasonable to think GEM also does, and if so, these would be a semiclassical analogue of gravitational waves. It is a well-known result of classical electrodynamics that the electomagnetic field emitted by an accelerated charge can be separated into two contributions: a velocity field and an acceleration or radiation field. Since GEM admits the fields are sufficiently weak, it is reasonable to think that, if this separation were indeed possible in GEM, then one would only observe the radiation field, which would fall with the inverse of the distance. Since the GEM equations are linear, let us consider a monocrhomatic harmonic source we can later integrate to give place to some arbitrary (but sufficiently regular) function of space and time:

$$\rho_g(\vec{r}\,',t_r) = \rho_g(\vec{r}\,')e^{i\omega t_r}$$
$$\vec{J}_g(\vec{r}\,',t_r) = \vec{J}_g(\vec{r}\,')e^{i\omega t_r}$$

where t_r is retarded time. Introducing these sources in equations (11) and (12), we have:

$$\mathbf{E_g} = -G\int \boldsymbol{\rho_g}\hat{R}\frac{e^{-ikR}}{R^2}d^3\vec{r}\,' - ikG\int\left(\boldsymbol{\rho_g}\hat{R} - \frac{\vec{J}_g}{c}\right)\frac{e^{-ikR}}{R}d^3\vec{r}\,'$$

$$\mathbf{B_g} = -\frac{G}{c^2}\int \mathbf{J_g}\times\hat{R}\frac{e^{-ikR}}{R^2}d^3\vec{r}\,' - \frac{G}{c^2}ik\int \mathbf{J_g}\times\hat{R}\frac{e^{-ikR}}{R}d^3\vec{r}\,'$$

where the symbols in bold letters are phasors. It is now evident that GEM does indeed allow for velocity and radiation fields, in a completely analogous fashion to the electromagnetic case. Let us notice in particular that the first term of $\vec{E}_g$ is precisely the gravitational field predicted by Newton's universal law of gravitation, although it is now evaluated, as would be expected, in retarded time.

V. Impossibility of an isotropic gravitational radiation emitter

We may finish this article considering one final characteristic of GEM, which is yet another parallelism with electrodynamics: the theory does not allow for isotropic radiation emitters (i.e: GEM waves with spherical symmetry). It is immediate to show that a semiclassical gravitational wave cannot propagate in the vacuum since it is not a solution of (11)-(12).

Following this train of thought, it would be interesting to consider doing further research on the apparent almost-complete equivalence between GEM and electromagnetism. For instance, one could consider the Poynting vector, the averaged norm of which would allow to quantify the power by unit area being emitted by a massive source in the form of gravitational waves. Naturally, detecting this radiation could be quite challenging in practice, since a quick calculation of the order of magnitude of the Poynting vector yields:

$$\vec{S}_g \propto \vec{E}_g \times \vec{B}_g \Longrightarrow ||\vec{S}_g|| \propto \frac{G^2}{c^2} \simeq 10^{-38}$$

References

[1] J. D. Jackson, *Classical Electrodynamics*, 2nd ed. John Wiley and Sons, 1999.