A formulation of perturbation theory in Hamiltonian Mechanics based on dynamical system theory

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The goal of this article is to formulate an approach to perturbation theory in classical mechanics employing the already well-known theory on dynamical systems. The central realization when connecting these two theories is that Hamilton's equations have the structure of a dynamical system, and therefore one can apply the theory of said systems to any Hamiltonian system. This enables us to identify, for instance, fixed points in the phase space (which are associated with equilibrium points in the physical system), determine their stability and study the system's behaviour in the neighbourhood of a fixed point by analyzing the behaviour of its linearization, which is remarkably simpler to study.

***Index Terms*—Perturbation theory, Hamiltonian mechanics, fixed point, linearization, asymptotic and Lyapunov stability**

I. The motivation for formulating a classical perturbation theory

In the current itinerary of most Physical Sciences degrees, it is not uncommon to find that students have long become acquainted with perturbation theory in nonrelativistic quantum mechanics, and nonetheless they are still clueless about how perturbations are introduced and studied in classical mechanics. This directly defies the maxim of most Physics degrees of presenting the classical concepts first and only then teaching their quantum counterparts. It is for this reason I am compelled to explore an approach to perturbation theory in the context of classical mechanics, in particular in the Hamiltonian formulation. It should be noted that one can study perturbation theory in any other formulation of classical mechanics (i.e: Newtonian, Lagrangian, Hamilton-Jacobi, etc.), but throughout this article it shall become evident that the Hamiltonian formulation is the one that requires the least effort, and it is also the one that is in direct contact with quantum mechanics and statistical physics, which is yet another incentive for studying perturbation theory in the Hamiltonian formulation.

Let us begin this article by reviewing the theory of dynamical systems, formulated directly on a differentiable manifold, which shall be the space comprised by all the possible values of the dynamical system, to wit: $\mathcal{M} := \{c_i\}_{i=1}^{N}$. Later in this article, we shall apply the theory reviewed here directly to Hamiltonian systems, where the variables shall be, of course, the generalized positions and momenta. It is also noteworthy that, according to the Darboux theorem, every $2n-$dimensional symplectic manifold is locally isomorphic to a phase space of equal dimension, and since all the considerations made throughout this article shall be local in nature, we shall not be bothered when applying the theory being reviewed here to a Hamiltonian system's phase space, since in virtue of the Darboux theorem the two manifolds under consideration shall be locally isomorphic anyway.

II. Introduction to the theory of dynamical systems

The central problem in the theory of dynamical systems if as follows: given N variables, whose evolution in time is given by a system of first order ODEs (which, in general, need not be linear and could be coupled), we wish to know the solution to said system, which shall give the evolution in time of the N variables. Mathematically, the problem is therefore reduced to solving the N-dimensional Cauchy problem:

$$\begin{cases} \frac{dc_1}{dt} = f_1(c_1, \dots, c_N; t); & c_1(0) = C_1 \\ \vdots & \\ \frac{dc_N}{dt} = f_N(c_1, \dots, c_N; t); & c_N(0) = C_N \end{cases} \tag{1}$$

Or, in reduced notation:

$$\frac{d}{dt}\vec{c} = \vec{F}(c_1, \dots, c_N; t); \qquad \vec{c}(0) = \vec{C} \tag{2}$$

where time is considered a parameter. It should be noted, however, that the breadth of the theory of dynamical systems is far richer than just solving an ODE system, which in general need not even have an analytic solution to begin with. In fact, there is plenty to be known about the system without the need to ever solve it explicitly, and it is this qualitative information that we seek to apply to the perturbation theory of mechanical systems. Of course, we shall almost always be able to solve the system using basic algorithms such as Runge-Kutta 4, but, as is often the case, analytical approximate methods allow us to extract information from the system which allows us to better understand it, whilst using a computer simply solves the issue for that particular case, but barely teaches us anything about the system under consideration. In any case, for the purpose of this article, we shall be particularly interested in studying the behaviour of the system near equilibrium points. However, let us first review the most common concepts appearing in the theory of dynamical systems.

Def. let M be a real, differentiable manifold of some finite dimension N. Let (φ, U) be a chart such that it maps a

region $U \subset M$ with the coordinates $\{x^\alpha\}_{\alpha=1}^N$.

Def. let $\mathbf{F} \in \mathfrak{X}(M)$ be a vector space defined over the manifold, which we shall refer to as "dynamical field." In the theory of dynamical systems, this vector field is known, and the problem is reduced to finding its integral curves. The sole purpose of this definition is to reformulate expression (2) in the language of differentiable manifolds, so it can be readily applied to Hamiltonian systems, which are described by a symplectic manifold.

That is to say, given the dynamical field $\mathbf{F}$, we aim to obtain the curves $\gamma \equiv x^\alpha : I \subset \mathbb{R} \to M$ such that their tangent vector in every point of their parametrization is precisely the dynamical field evaluated on the same point:

$$\frac{d}{dt}x^\alpha = F^\alpha\Big|_{x^\alpha(t)} \tag{3}$$

In particular, the selection of an initial condition shall fix a single integral curve of the congruence, at least in a neighbourhood of the point where the initial condition is taken, in virtue of the Piccard-Lindëlof theorem. This shall not pose a problem, since we shall only be concerned with the system's behaviour in the neighbourhood of a point anyway.

Def. let $p \in U \subset M$ be a point in the region of the phase space under consideration, such that the dynamical field is zero on that point. Then, we shall refer to said point as a "fixed point", and it will be denoted as p_*. Note that, since the dynamical field is a vector field, the nature of a fixed point does not depend on the choice of coordinates we make, since $\mathbf{F}(p_*) = 0$ is an intrinsic statement. An alternative definition for a fixed point is: "a point where the dynamical system's time evolution is trivial":

$$\mathbf{F}(p_*) = 0 \tag{4}$$

These terms provide us with the language necessary in order to formulate the Hartman-Grobman theorem on a differentiable manifold. Let us note that a formal introduction to this theorem would inevitably require a plethora of other definitions, such as the concept of topologic conjugation, as well as a dissertation on why, although topological onjugation is strictly a homomorphism and not an isomorphism between the dynamical system and its linearization, we are still allowed to conclude the original system behaves like the linearized one in a neighbourhood of a fixed point. For our purposes, we shall formulate the Hartman-Grobman theorem in a more relaxed manner, directly in geometric language. Furthermore, we shall admit the most common case in which the dynamical matrix is diagonalizable, although more strict versions of this theorem do not make reference to the matrix's spectrum, and it is not even a requirement for the theorem to hold that said matrix is diagonalizable.

Theorem (Hartman-Grobman) Let there be a dynamical system with variables $\{x^\alpha\}_{\alpha=1}^N$ such that its evolution is given by some smooth application $F : \mathbb{R}^n \to \mathbb{R}^n$. Let $p_* \in U$ be a fixed point of this system, such that $F(p_*) = 0$, and let $M^\alpha_{\ \beta} = \partial_\beta F^\alpha$ be the Jacobian matrix of the system, evaluated in p_*, which we shall call "dynamical matrix." Then, the all the eigenvalues of M have a nonzero real part, the system behaves qualitatively like its linearization in a neighbourhood of the fixed point, $E(p_*)$.

$$\frac{d}{dt}x^\alpha \approx M^\alpha_{\ \beta}\varepsilon^\beta \tag{5}$$

Dem.

Let x^α_* be the coordinates of the fixed point in the chart selected to map $U \subset M$. We can write the coordinates of the points in a neighbourhood of the fixed point as follows: $x^\alpha = x^\alpha_* + \varepsilon^\alpha$, where ε^α is sufficiently small. Let us note there is no need to define a metric (not even a norm) to conclude a point is in a neighbourhood of the considered fixed point, given that manifolds are, by definition, topologies, and as such they do have the concept of open subset. Nonetheless, without defining a metric or norm, one cannot conclude exactly where the neighbourhood of the fixed point being considered finishes. This should not be too bothering, since we are attempting to formulate an inherently approximate theory for the study of dynamical systems, and the results will always be accompanied by a given error, small as it may be, which shall be dependent on ε.

According to expression (3), near a fixed point we shall have:

$$\frac{d}{dt}(x^\alpha_* + \varepsilon^\alpha) = F^\alpha(x_* + \varepsilon)$$

Nonetheless, since we admit ε is sufficiently small, we can expand the dynamical field to first order in the perturbation parameter. Furthermore, since the fixed point does not change in time, the expression can be left as:

$$\frac{d}{dt}\varepsilon^\alpha = F^\alpha(x_*) + \partial_\beta F^\alpha(x)|_{x_*}\varepsilon^\beta + \mathcal{O}(\varepsilon^2)$$

By definition of a fixed point, the dynamical field is zero on it, and so we finally obtain the linearization for the problem:

$$\frac{d}{dt}\varepsilon^\alpha = \partial_\beta F^\alpha|_{x_*}\varepsilon^\beta \tag{6}$$

It should be noted that we have not made an explicit mention to the condition of the Jacobian matrix's eigenvalues having a nonzero real part. This is due to the fact that this condition comes from a topological consideration, where the original system and its linearization are only comparable if there are no sub-exponential growth directions, since otherwise the condition of topological conjugation between them is not met. This concludes the demonstration■

A. *Classification of fixed pointas*

In virtue of the Hartman-Grobman theorem, it is now in order to classify fixed points of a given dynamical system according to the eigenvalues of its dynamical matrix.

Case A: Real spectrum

Def. a fixed point will be a **node** if the dynamical matrix has a definite sign on said point.

Def. a fixed point shall be a **saddle point** if the dynamical matrix does not have a definite sign on said point.

Case B: Complex spectrum

Def. a fixed point will be a **focus** if the dynamical matrix has at least a pair of conjugate eigenvalues.

Def. a fixed point will be a **vortex** if all the eigenvalues of the dynamical matrix are pure imaginary numbers.

B. Stability of the different fixed points

Up until now, we have stated the most general aspects of the theory of dynamical systems in the language of differentiable manifolds, which will be of the uttermost interest in order to later apply said theory directly to the phase space in Hamiltonian mechanics. However, there is still some work to do before that: in perturbation theories, just as relevant as identifying equilibrium points is determining whether these are stable or not. Therefore, it is now proceeding to study the stability of the different fixed points of our classification.

In order to do this, we shall apply the Hartman-Grobman theorem to the system under consideration in the neighbourhood of a given fixed point. As per our statement of the theorem, this procedure shall only allow us to study those fixed points for which the dynamical matrix's spectrum's real part is nonzero, so vortices (which usually correspond to systems such as the harmonic oscillator) are out of our consideration. However, for the rest of fixed points, one can in principle always apply the Hartman-Grobman theorem, and therefore if the linearization of the system appears to be stable around some fixed point, then the theorem guarantees that the original system will also be stable around it. Let us now formally integrate expression (6), which yields:

$$\varepsilon^\alpha(t) = e^{Mt}\varepsilon_0^\alpha \tag{7}$$

where $\varepsilon_0^\alpha = \varepsilon^\alpha(0)$. If, as is the supposition being made throughout this article, the dynamical matrix is diagonalizable, then there exists an eigenbasis in which it acquires a diagonal form, and we can therefore decompose: ε_0^α in this basis $B := \{v_i^\alpha\}_{i=1}^N$, resulting in:

$$\varepsilon^\alpha(t) = \sum_{i=1}^{N} c_i e^{\lambda_i t} v_i^\alpha \tag{8}$$

The advantage of this expression is it immediately enables us to formulate quite naturally a stability criterion for a fixed point.

Theorem (stability criterion): let p_* be a fixed point such that the dynamical matrix on said point fulfills the conditions of the Hartman-Grobman theorem. Then, we shall say the fixed point is:

1) Stable in the direction v_i^α if $Re(\lambda_i) < 0$
2) Unstable in the direction v_i^α if $Re(\lambda_i) > 0$

In general, we will refer to a fixed point as "stable" if it is stable for all directions (i.e: all the eigenvalues of the dynamical matrix have a negative real part). According to this criterion, the imaginary part of the dynamical matrix's spectrum does not appear to play any significant role in determining the stability of a fixed point, so it is proceeding to ask ourselves whether there is any qualitative differences in the behaviour of a system in the neighbourhood of a fixed point when a given eigenvalue of the dynamical matrix has a nonzero imaginary part and when it does not. In order to study this case and to be able to graphically represent it, let us consider the simplest scenario: a dynamical system with only two variables. The question under consideration is then reduced to the study the difference between a node and a focus. Of course, this distinction is present regardless of whether the fixed points are stable or not, but it is generally the case in which they are that is the most interesting, so for the sake of this argument we shall admit both the node and the focus are stable.

A priori, it could appear that the behaviour of a system around a focus is just like the one around a node, but with additional oscillations in the parametrization due to the imaginary factor appearing in the exponential. It is possible to see this difference in the behaviour of a system around a stable node and a stable focus graphically. In order to do so, let us consider the following systems:

$$\begin{cases} \dot{x} = -x + xy \\ \dot{y} = -2y \end{cases}$$

$$\begin{cases} \dot{x} = -x - y - x(x^2 + y^2) \\ \dot{y} = x - y - y(x^2 + y^2) \end{cases}$$

Both dynamical systems present a fixed point in $(x_*, y_*) = (0, 0)$. However, it can be easily checked that, for the former system, this point corresponds to a stable node, and for the latter it is a stable focus. In order to convince ourselves of this, let us consider the dynamical matrices of both systems on the fixed point $(0, 0)$:

$$M_1 = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$$

We note that the first system's dynamical matrix already has a diagonal form in the canonical basis, and therefore we can expand the perturbation vector in it directly. The linearization of the first system, given by (8), is therefore:

$$\varepsilon^\alpha = c_1 e^{-t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

In a similar fashion, one can diagonalize the second system's dynamical matrix and find it has eigenvalues: $\lambda_1 = -1 + i$ y $\lambda_2 = -1 - i$, and eigenvectors $\vec{v}_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 1 \\ i \end{pmatrix}$, respectively. It should be noted that the eigenvalues are complex, but their real part is negative, thus yielding the focus stable. Therefore, the linearization of the second system in a neighbourhood of the fixed point is:

$$\varepsilon^{\alpha}(t) = \tilde{c}_1 e^{-(1-i)t} \begin{pmatrix} 1 \\ -i \end{pmatrix} + \tilde{c}_2 e^{-(1+i)t} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

Equivalently, vector notation can be used to denote the curves corresponding to the linearization of the systems:

$$\vec{\varepsilon}(t) = \begin{pmatrix} 3e^{-t} \\ e^{-2t} \end{pmatrix}$$
$$\vec{\tilde{\varepsilon}} = 2e^{-t} \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$$

Where both vectors are now in the canonical basis, as otherwise their comparison would not make sense. The initial conditions for the second system have been carefully chosen so that the curve is real, as otherwise imaginary factors arise and the curve cannot be plotted. Let us now represent the behaviour of both systems in a neighbourhood the fixed point $(0, 0)$ in their respective phase spaces (which are simply $\mathbb{R}^2$ for simplicity, so it makes sense to plot them together).

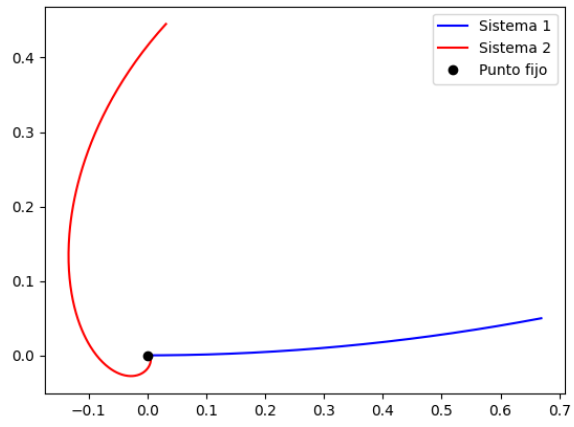

Fig. 1. Behaviour of systems 1 (blue) and 2 (red) in a neighbourhood of their respective fixed points

Indeed, we find that our intuition was on the right: system 2 follows a spiral around its focus, whilst system 1 *falls* to its node exponentially. Strictly, both systems fall to their respective fixed point exponentially, but the behaviour of the second system is modulated by a trigonometric function, which makes its trajectory a spiral. Therefore, we may conclude this is the behavioural difference between (stable) nodes and foci: the former fall to their fixed point directly, whilst the latter do it following a spiral trajectory.

III. Perturbation theory in classical mechanics

Up until now, we have reviewed the theory of dynamical systems, formulated on differentiable manifolds and emphasizing the relevance of the linearization of said systems. It is now in order to ask how this theory can be of any use in order to build a perturbation theory in classical mechanics, and the fact is all the work is already done.

Theorem: every mechanical system with n degrees of freedom admitting a description in the Hamiltonian formalism is a $2n-$dimensional dynamical system. Furthermore, the variables of said system coincide with the canonical variables (or Darboux pairs) used for describing the mechanical system.

Dem: it is enough to realize that Hamilton's equations already have the form of a dynamical system:

$$\dot{\xi}^{\alpha} = \Omega^{\alpha\beta}\partial_{\beta}H \tag{9}$$

We can further expand this expression to obtain Hamilton's equations:

$$\begin{cases} \dot{q}_1 = \frac{\partial H}{\partial p_1} \\ \vdots \\ \dot{q}_n = \frac{\partial H}{\partial p_n} \\ p_1 = -\frac{\partial H}{\partial q_1} \\ \vdots \\ p_n = -\frac{\partial H}{\partial q_n} \end{cases}$$

where, in general, the partial derivatives of the Hamiltonian with respect to some variable depend on every variable being used. Therefore, since we have explicitly found the dynamical field for the system, $\mathbf{F} = \iota_{\Delta_H}\Omega$ with $\Delta H = \partial H$, the proof is finished■

Hence, the conceptual relationship between dynamical and Hamiltonian systems is finished, and we can now apply the whole dynamical system theory to classical mechanics employing the Hamiltonian formalism. Of course, fixed points now have a clear physical interpretation: it is those states of the mechanical system for which it does not evolve any further. That is to say, fixed points are equilibrium states! Using our theory, we can (1) identify equilibrium points in the phase space, (2) determine their stability and (3) study the behaviour of the system in a neighbourhood of these points. It is precisely this last point that motivates the whole article: the linearization of a dynamical system is precisely what we call the perturbation of a mechanical system.

An additional advantage to this approach is that it works even for those systems which do not admit a Lagrangian or Newtonian description (i.e: for those systems whose Hessian matrix $\frac{\partial^2 H}{\partial p_\alpha \partial p_\beta}$ is degenerate). Furthermore, since this approach exploits the structure of the phase space, it works regardless of whether the system is relativistic or not, as this only changes the Hamiltonian.

A. Application: circular orbits in a central potential

Let us now illustrate the applications of this perturbation theory to the problem of a body subject to a central potential. Since the movement is restricted to a plane, let us adopt polar coordinates and write:

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + V(r)$$

Since the azimuthal variable θ is cyclic, p_θ is a constant of motion over physical trajectories, and we can write a reduced Hamiltonian for the system:

$$H_{eff} = \frac{p_r^2}{2m} + V_{eff}(r)$$

where $V_{eff}(r) = \frac{\alpha_\theta^2}{2m} + V(r)$ y $\alpha_\theta = p_\theta =$ constant Let us now write Hamilton's equations for the system under consideration:

$$\begin{cases} \dot{r} = \frac{p_r}{m} \\ \dot{p}_r = -\frac{dV_{ef}}{dr} \end{cases}$$

Therefore, the fixed points shall be:

$$(r_*, p_{r*}) = (r_0, 0)\ tq.\ V'_{ef}(r_0) = 0$$

This result is well-known: the equilibrium states of a particle subjected to a central potential are those which lie at the extrema of the effective potential. It is noteworthy that, since these orbits are essentially the roots of the derivative of the effective potential, they shall always be circular (since $r = r_0$ is constant).

Let us now study the stability of the equilibrium points we have just found. Firstly, let us build the dynamical matrix of the system on the calculated fixed point:

$$J = \begin{pmatrix} 0 & 1/m \\ -V''_{ef}(r_0) & 0 \end{pmatrix}$$

Diagonalizing, we see the matrix has two eigenvalues, which can either be real or pure imaginary conjugate numbers:

$$\lambda_\pm = \pm\frac{1}{\sqrt{m}}\sqrt{-V''_{ef}(r_0)}$$

Observation: if $V''_{eff}(r_0) < 0$, then one of the eigenvalues is positive and the other one is negative. Therefore, the fixed point will be a saddle point and hence it shall be unstable (meaning there exists at least one direction in the phase space where this fixed point is not stable). On the other hand, if $V''_{eff}(r_0) > 0$, the spectrum of the dynamical matrix turns out to be purely imaginary (the fixed point is a vortex) and the Hartman-Grobman theorem no longer applies.

B. A brief digression on the different types of equilibria

If we wish to delve deeper into the theory, we shall see the fixed point we have found for the application is stable in a Lypanuov sense. Up until now, we have deemed "stable" those fixed points to which the system fell exponentially in the parametrization. This behaviour is often referred to as "asymptotic stability." Nonetheless, more often than not this condition is too strong, and one is compelled to formulate weaker equilibrium conditions, such as that of Lyapunov. This can be expressed as follows:

Def.: let $\vec{x}_*$ be the coordinates of a fixed point of a dynamical system, and let $\vec{x}(0)$ be the state of the dynamical system in the parametrization's origin. Then, we shall say $\vec{x}_*$ is a stable equilibrium point in a Lypaunov sense if $\forall\varepsilon > 0, \exists\delta > 0 tq. ||\vec{x}_0 - \vec{x}_*|| < \delta \Longrightarrow ||\vec{x}(t) - \vec{x}_*|| < \epsilon\ \forall t > 0.$

Intuitively, the Lyapunov stability condition can be understood as follows: "if the system starts (in the parametrization) in a state sufficiently close to the atrractor, then it will stay reasonably close to it during its evolution". Therefore, $V''_{ef}(r_0) > 0$ denotes a stable equilibrium point, but the stability type is not asymptotic, but rather Lyapunov.

Motivated by this example, it is worth to establish a distinction between asymptotically stable and Lyapunov-stable equilibrium points. Armed with our newly acquired intuition on the topic, we see that the first group eventually reaches the fixed point in a finite time, after which the system stops evolving. This could be, for instance, the case for an overdamped oscillator. On the other hand, the systems belonging to the second group, which are far more common in Physics, stay in a bounded region of the phase space (the attractor), orbiting the fixed point. This is the case, for instance, of a harmonic oscillator or a body describing circular orbits in a central potential.

IV. Conclusions and comments

Trhoughout this article, we have laid the foundations for the treatment of dynamical systems employing the language of differentiable manifolds. After covering this theory rather exhaustively, chiefly focusing on the perturbative treatment of systems in the neighbourhood of a fixed point and always admitting the conditions for the Hartman-Grobman theorem were being met, we have established the central connection of the present dissertation, which is no other than the realization that Hamilton's equations have a dynamical system-like structure, and therefore we can apply the whole apparatus of this theory to mechanical systems in this formulation, thus obtaining a natural perturbation theory.

It is now in order to make a brief commentary on the strenghts and weaknesses of the proposed approach. On the one hand, it is noteworthy that, even if the Hartman-Grobman theorem's conditions are not met, this theory still allows us to at least identify equilibrium points in the system. This is still remarkably useful, as the stability of said points can

be studied afterwards employing some other method, perhaps computational, but at least one knows *where* the system's equilibrium points will be. In fact, the proposed method allows for a systematic way of finding the equilibrum points of a given system in a simple manner, as the problem is reduced to solving a system of n algebraic equations:

$$F^{\alpha}(x,t) = 0$$

which can always be solved with numerical methods like the fixed point method or, if need be, something more sophisticated, like the Gauss-Newton method.

On the other hand, we find that the proposed approach is deemed insufficient in several cases with very much physical interest. In particular, in the application of this theory to the study of a particle subjected to a central potential, our approach predicts the instability of those equilibrium points for which $V''_{eff} < 0$, but it is not capable of predicting that those equilibrium points for which $V''_{eff} > 0$ are indeed stable. This is due to the fact that the definitions of stability that have been given only apply to asymptotic stability, which implies that the system necessarily converges to a fixed point for long enough times. However this behaviour is not the one observed in most physical systems, where one usually observes an orbit-like behaviour around an attractor in the phase space, without ever fully reaching the fixed point at its centre. This can be solved rather clumsily if we adopt more relxed stability criteria, such as that of Lyapunov, but our perturbation theory still does not provide a criterion for deciding whether a vortex shall be Lyapunov-stable or unstable.

It is proceeding to finish this article by mentioning some possible areas of interest to which this method could be expanded. Firstly, it is worth noting that this theory appears to be applicable to quantum systems built through canonical quantization of their classical counterparts A possible test to determine whether this is the case could be considering the Hydrogen atom and attempting to recover the semiclassical Bohr radius as the equilibrium solution. Perhaps it would be necessary take into account the Bohr-Sommerfeld prescription of semiclassical angular momentum quantization.

It would also be interesting to expand this treatment to many-particle systems, as this would allow for the formulation of a perturbation theory in the phase space in Statistical Physics. However, the problem with this idea is, as is usually the case in this discipline, the elevated number of degrees of freedom the system would have: since we need one coordinate for every degree of freedom, finding equilibrium points using this approach would be a pointless task, even with a computer. Of course, the source of this difficulty is we would be attempting to make an individualistic treatment of a many-particle system, which is not a statistical approach to begin with, and it is precisely the introduction of statistical distributions that makes this problem tractable.

It can be therefore concluded that the proposed approach is useful as a perturbation theory when the mechanical system has a tractable number of degrees of freedom. The evident advantages of this approach are (1) that it allows to identify equilibrium points simply by solving an algebraic system of equations and (2) if the conditions for the Hartman-Grobman theorem are met, then it also enables the study of the stability of said equilibrium points and even the behaviour of the system in a neighbourhood of said points, without ever needing to actually solve Hamilton's equations.