Studying the black body as an ideal Bose-Einstein ultrarelativistic spin-1 gas

Germán Cousillas Martínez[1]
[1]germacou@ucm.es

Oftentimes, the statistical study of the black body is carried out considering it a photon gas in thermodynamic equilibrium with its environment. Furthermore, its study is conducted applying Boltzmann statistics to the gas, from where Planck's well known results are derived naturally, as well as Wien's law, Stefan-Boltzmann's law, etc. Nonetheless, for a more thorough investigation of the photon gas, it is interesting to apply Bose-Einstein statistics. This allows us to take into account photons are indeed indistinguishable particles, as well as exploring whether a black body can be subject to phase transitions and have a condensed phase. We find in this study the black body is indeed in a highly condensed phase at room temperature, and it is precisely this condensation which gives place to the simple polynomial dependence of the system's energy and pressure with temperature, as well as the independence of pressure from the system's volume.

***Index Terms*—Black body, photon gas, Bose-Einstein statistics, phase transition**

I. Basic assumptions when modeling the system

A photon gas can be modeled according to the following descriptive hypotheses.

1) It is a Bose-Einstein ideal gas
2) The bosons have spin 1
3) The bosons are ultrarelativistic

Therefore, the system's Hamiltonian is of the form:

$$H = \sum_{i=1}^{N} |\vec{p}_i| c + \varphi(\vec{r}_i) \tag{1}$$

where $\varphi(\vec{r}_i)$ is a box potential keeping the bosons confined in a finite region of space of volume V. Naturally, the partition function corresponding to the system will be given by:

$$\ln Q = -\sum_{k=1}^{N} \ln(1 - z e^{-\beta \epsilon_k}) \tag{2}$$

where ϵ_k are the energy levels of an ultrarrelativistic particle's Hamiltonian when subjected to a box potential.

II. Energy of the black body: confirmation of Stefann-Boltzmann's law

Let us begin the study of our system by calculating the Hamiltonian's expectation value. It is of special interest to check whether the system's energy's dependence with temperature is, as is well known, $E \propto T^4$.

The expectation value of the Hamiltonian is given by:

$$E \equiv \langle H \rangle = \sum_{k=1}^{\infty} \frac{\epsilon_k}{z^{-1} e^{\beta \epsilon_k} - 1} \tag{3}$$

In order to calculate this sum, one must employ the continuum approximation. It can be easily demonstrated that a Hamiltonian of the form:

$$H = \alpha |\vec{p}|^s; \quad \alpha > 0; \ s \in \mathbb{N} \tag{4}$$

induces a density of states given by:

$$g(\epsilon) = \frac{L^d C_d}{s h^d \alpha^{d/s}} \epsilon^{d/s - 1} \tag{5}$$

where d is the system's dimension (in this case, $d = 3$ and $C_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$.

Therefore, the approximated calculation of the system's internal energy is as follows:

$$E \approx \frac{L^3 C_3}{h^3 c^3} \int_0^{\infty} \epsilon^2 \frac{\epsilon}{z^{-1} e^{\beta \epsilon} - 1} d\epsilon \tag{6}$$

Performing the change of variables $x = \beta \epsilon$ so as to make the integral dimensionless, we find:

$$E \approx \frac{L^3 C_3}{h^3 c^3} \frac{1}{\beta^4} \int_0^{\infty} \frac{x^3}{z^{-1} e^x - 1} dx \tag{7}$$

where the integral is, of course, a Bose function with $n = 4$ (with a prefactor of $\Gamma(4)$). Hence:

$$\boxed{E \approx \frac{24\pi V}{c^3 h^3} k_B^4 T^4 g_4(z)} \tag{8}$$

We immediately notice the correct dependence on temperature is recovered, although there is an aditional factor $g_4(z)$, which depends on temperature implicitly as well. If the system were able to have a condensed state, then $g_4(z) \rightarrow \zeta(4)$ and the right dependence with temperature for the photon gas would be recovered, which already hints us to the fact that the photon gas can, indeed, present a condensed phase. Allow us to study this more in depth.

III. Possibility of a phase transition to a condensed state

Just as interesting as studying the system's energy is to determine whether it can experience Bose-Einstein condensation.

By definition, Bose-Einstein condensation takes place when the capacity of the excited energy levels becomes finite, which leads to them being saturated and the rest of bosons being accomodated by the fundamental level. The mean number of particles in the k-th level is given by Bose-Einstein statistics to be:

$$\langle n_k \rangle = \frac{1}{z^{-1}e^{\beta\epsilon_k} - 1} \tag{9}$$

If we now take the origin of energy to be in the fundamental state, then $\epsilon_0 = 0$ and hence the occupation of the fundamental level is given by:

$$\langle n_0 \rangle = \frac{z}{1-z} \tag{10}$$

The mean number of particles in the excited levels, on the other hand, is the sum of all occupation numbers of the individual excited states. Let us study whether it is finite.

$$\langle N_{exc} \rangle = \sum_{k=1}^{\infty} \frac{1}{z^{-1}e^{\beta\epsilon_k} - 1} \tag{11}$$

Employing the continuum approximation once more:

$$\langle N_{exc} \rangle \approx \frac{L^3 C_3}{c^3 h^3} \int_0^{\infty} \frac{1}{z^{-1}e^{\beta\epsilon} - 1} d\epsilon \tag{12}$$

We now apply the same change of variables as we did with the energy, $x = \beta\epsilon$, and we finally obtain:

$$\langle N_{exc} \rangle \approx \frac{4\pi V}{(ch\beta)^3} \int_0^{\infty} \frac{x^2}{z^{-1}e^{x} - 1} dx \tag{13}$$

Which we immediately recognize to be the Bose function with $n = 3$. Therefore:

$$\boxed{\langle N_{exc} \rangle \approx \frac{8\pi V}{(ch\beta)^3} g_3(z)} \tag{14}$$

According to the present model, the black body can experience Bose-Einstein condensation, due to the fact that $g_3(z) \in [0, \zeta(3)]; \forall z \in [0,1]$. Particularly, the condition for the phase transition to take place is for all the bosons to be in the excited levels as we approach $z \to 1$. Mathematically:

$$\boxed{\langle N \rangle = \frac{8\pi V}{(ch\beta_c)^3} \zeta(3)} \tag{15}$$

where $\beta_c = \frac{1}{k_B T_c}$. It is now immediate to isolate the critical temperature of the system (T_c):

$$\boxed{T_c = \frac{hc}{2k_B} \left(\frac{\langle N \rangle}{V}\right)^{1/3} \left(\frac{1}{\pi\zeta(3)}\right)^{1/3}} \tag{16}$$

For the typical values $\langle N \rangle = 10^{23}$ y $V = 1m^3$, we obtain the system's critical temperature is:

$$T_c \sim 2 \cdot 10^5 (K) \tag{17}$$

We thus observe that, according to this model, **black bodies are in a highly condensed phase at room temperature**.

With these results, it is immediate to obtain the occupation numbers of the excited and fundamental levels. In order to calculate the (relative) population of the excited states as a whole, it is sufficient with evaluating the quotient $\langle N_{exc} \rangle / \langle N \rangle$. For temperatures below T_c, as is often the case, the system will be in a condensed phase, which leads to the dependence of the different thermodynamic coordinates and potentials on temperature to simplify greatly. In particular:

$$\boxed{\langle n_{exc} \rangle = \left(\frac{T}{T_c}\right)^3} \tag{18}$$

And naturally, the fundamental level's population will be given by:

$$\boxed{\langle n_0 \rangle = 1 - \left(\frac{T}{T_c}\right)^3} \tag{19}$$

For instance, for a black body at room temperature, we find that the fraction of photons in the excited states is:

$$\langle n_{exc} \rangle (25K) \approx 2 \cdot 10^{-12}$$

That is to say, in a system of 10^{23} photons, only 10^{11} of them would be found in the excited levels.

As is natural, if the system's temperature is over T_c, virtually all of the photons would be in the excited levels and almost none would be found in the fundamental level. Furthermore, the system would no longer be a Bose-Einstein condensate and the dependences with temperature would complicate, due to the fact that the Bose-Einstein functions (which depend on the system's fugacity and therefore, although implicitly, on its temperature), would no longer converge to their corresponding Riemann zetas.

The fact that the critical temperature is so high for typical values leads to an important conclusion: at room temperature, we have $z \simeq 1$, and therefore:

$$\boxed{\mu = 0} \tag{20}$$

This is also a well-known result: black bodies have a null chemical potential.

IV. Energy per particle

Let us now study the mean energy per particle of the system. Computing the quotient E/N, we immediately get:

$$\boxed{E = 3\frac{g_4(z)}{\zeta(3)} \langle N \rangle k_B T \left(\frac{T}{T_c}\right)^3} \tag{21}$$

V. HEAT CAPACITY OF THE BLACK BODY

We know draw our attention towards the model's heat capacity. Let us part from the mangitude's thermodynamic definition:

$$C_V \equiv \left(\frac{dE}{dT}\right)_V \tag{22}$$

Beware of the fact that, since fugacity z is a function of temperature, we must make an explicit distinction between the condensed and the non-condensed phases. For the latter, we have.

$$E = \frac{24\pi V}{c^3 h^3}\zeta(4)k_B^4 T^4 \tag{23}$$

and so the heat capacity of the black body in the condensed phaase is given by:

$$\boxed{C_V = \frac{96\pi V}{c^3 h^3}\zeta(4)k_B^4 T^3 \quad T < T_C} \tag{24}$$

VI. EQUATION OF STATE

We shall now study the equation of state of the present model. The system's pressure is given by the expression:

$$P = \frac{k_B T}{V}\ln Q \tag{25}$$

Substituting the expression of $\ln Q$, we get:

$$P = -\frac{k_B T}{V}\sum_{k=0}^{\infty}\ln(1 + ze^{-\beta\epsilon_k}) \tag{26}$$

Once more applying the continuum approximation in order to perform this summation, we have:

$$P = -k_B T\frac{C_3}{c^3 h^3}\int_0^{\infty}\ln(1 - ze^{-\beta\epsilon})\epsilon^2 d\epsilon \tag{27}$$

where the aforementioned density of states has been employed. This integral can be calculated by parts, where the first term becomes null. Therefore, we obtain the equality:

$$\int_0^{\infty}\ln(1 - ze^{-\beta\epsilon})\epsilon^2 d\epsilon = -\frac{\beta}{3}\int_0^{\infty}\epsilon^3\frac{1}{z^{-1}e^{\beta\epsilon} - 1}d\epsilon$$

Again, carrying out the usual change of variables and identifying the result as a Bose function, we have:

$$\boxed{P = \frac{8\pi}{3}\frac{g_4(z)}{c^3 h^3}k_B^4 T^4} \tag{28}$$

Notice the radiation pressure depends on the fourth power of temperature, which is a well-known result of the theory of black bodies. Nonetheless, we once more end up with an additional Bose function which distorts the pressure's dependence on temperature. In order to obtain the equation of state, we shall now divide the expression by the number of particles, and so we obtain:

$$\boxed{PV = \frac{1}{3}\frac{g_4(z)}{\zeta(3)}\langle N\rangle k_B T\left(\frac{T}{T_c}\right)^3} \tag{29}$$

It is interesting to note from this equation of state that the identity

$$\boxed{PV = \frac{1}{3}E} \tag{30}$$

holds. This last result was expected, since the system is ideal.

VII. BEHAVIOUR OF THE PHOTON GAS IN THE NON-CONDENSED PHASE

Up until now, our study has revolver around the condensed phase of the model, given that it is the usual state in which one finds the black body. Nonetheless, it is pertinent to ask what happens at temperatures higher than T_c, when the system is no longer condensed. It is here where we begin to add on to the predictions of Boltzmann statistics, which determine $z \to 1$ for any temperature.

In the first place, it is convenient to find a way to calculate z when temperatures are over T_c. When the system leaves the condensed phase, all of the photons will be accomodated in the excited levels, so it will be correct to state that: $\langle N_{exc}\rangle = \langle N\rangle$. With a bit of work, we reach an implicit equation for z as a function of temperature, which can be solved numerically.

$$\boxed{g_3(z) = \left(\frac{T_c}{T}\right)^3\zeta(3)} \tag{31}$$

Naturallty, this identity holds whenever $T > T_c$. This can be proven trivially by noticing $g_3(z) \to \zeta(3)$ when $z \to 1$. This allows us to know the the system's pressure for any temperature. For the aforementioned typical values, the pressure as a function of temperature is given by:

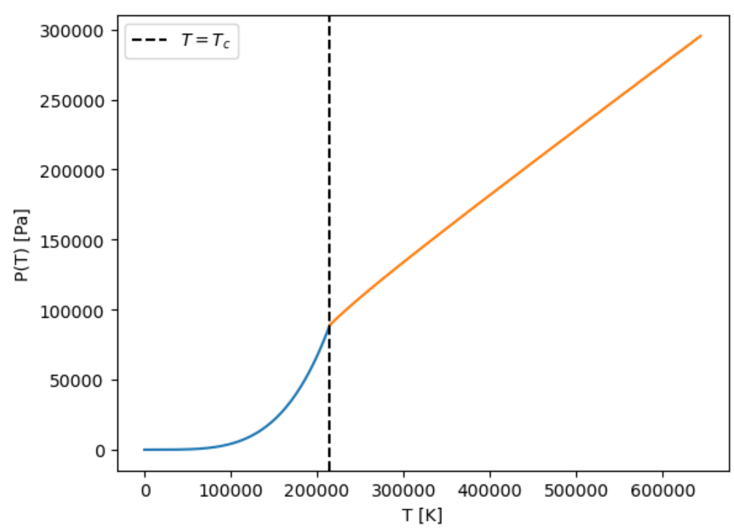

Fig. 1. Pressure of the photon gas as a function of temperature. Notice how it initially goes as $P \sim T^4 P$ but immediately turns to $P \sim T$ for temperatures higher than T_c

The behaviour of pressure as a function of temperature should not be surprising whatsoever: for $T < T_c$, we have

$z \simeq 1$, and therefore $P(T) \sim T^4$. On the other hand, as we surpass the critical temperature, we approach the classical limit, and so $g_n(z) \sim z$ holds. As for equation (31), we then have $g_3(z) \sim g_4(z) \sim z \propto 1/T^3$. Hence, there is only a linear term of T left.

It is noteworthy that, as is showed computationally, there is no discontinuity in the pressure during the phase transition. Nonetheless, the curve $P(T)$ is non-differentiable when $T = T_c$. Therefore, this phase transition appears to be of the first type of the Ehrenfest classification.

Let us now study how the system's internal energy behaves with respect to temperature, keeping volume constant. According to (30), if volume is kept constant, we should obtain the same curve for energy than that of pressure, only divided by a factor of 3.

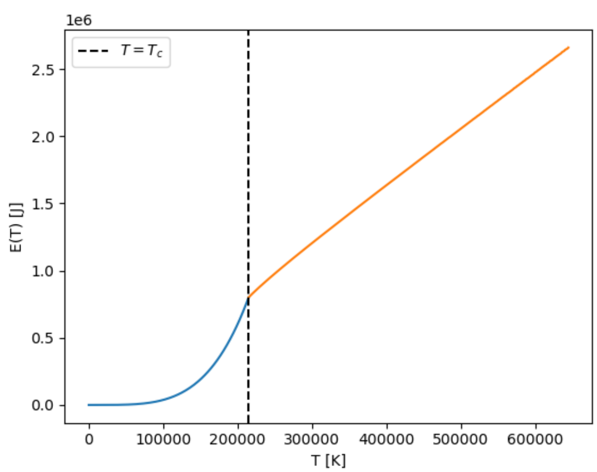

Fig. 2. Energy of the photon gasn as a function of temperature

We shall now study the heat capacity at constant volume of the black body. The derivative is computed numerically, since the dependence of E with temperature is complex when z does not approach unity.

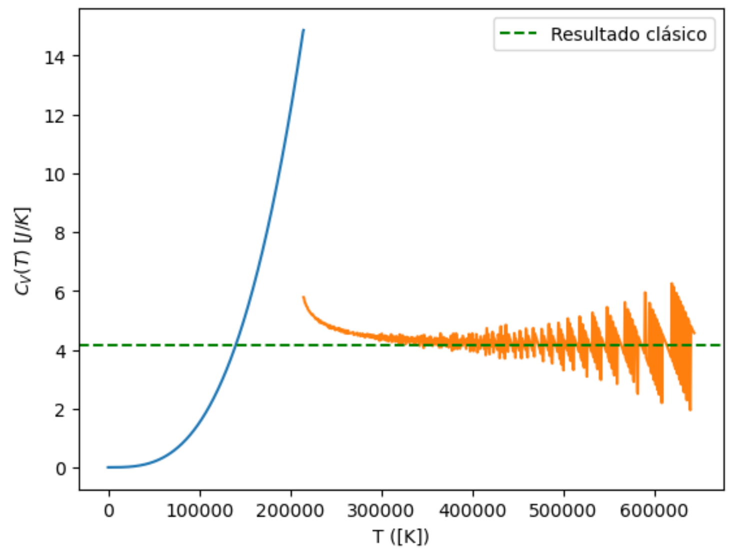

Fig. 3. Heat capacity at a constant volume of the photon gas. Notice how for temperatures over T_c the system stops behaving as a black body ($C_V \sim T^3$) and immediately behaves as an ideal, cllassical ultrarelativistic gas ($C_V \sim T$).

These three calculations support the fact that, for temperatures higher than T_c, the photon gas behaves as a classical, ideal and ultrarelativistic gas, as one would expect. For temperatures below T_c, the behaviour is that of a black body from its usual theory.

VIII. Additional considerations regarding the model

The present model aims to approach the photon gas (black body) making an emphasis on the fact that photons are spin-1 bosons, even if they are not strictly ultrearelativistic (because they travel at the speed of light, but one can approach this speed arbitrarily).

In addition, when studying the photon gas using Boltzmann statistics, one introduces a factor 2 by hand to account for the two transversal modes of photons. Said factor has not been taken into account in this work, although it might be necessary to recover the exact same results as those of the classical theory of black bodies.

A. Bounds

Note the results that have been obtained throughout the whole article depend on the Bose functions for $n = 3$ and $n = 4$, both of which are bounded between 0 and their corresponding zeta function. The most interesting coordnate to study is pressure, given that it depends exclusively on temperature for $T < T_c$. This result is well-known for black bodies: pressure depends solely on the fourth power of temperature. Therefore, pressure in the condensed phase is given by:

$$\boxed{P(T) = \frac{8\pi\zeta(4)}{3c^3h^3} k_B^4 T^4 = \gamma T^4} \tag{32}$$

con $\gamma \simeq 4.19 \cdot 10^{-17}(Pa \cdot K^{-4})$.

We also note the system's pressure is greatest in the condensed phase. Once it leaves said phase, temperature will depend on $g_4(z) \leq \zeta(4)$ and so it will be lower.

Something similar happens to the system's energy, which is also dependent on the fourth power of temperature, but this potential is also dependent on volume (equivalently, on the number of particles). Therefore, we have:

$$\boxed{E(V,T) = \frac{24\pi V\zeta(4)}{c^3h^3}k_B^4T^4 = \kappa VT^4} \tag{33}$$

con $\kappa = 3.77 \cdot 10^{-16}(J/(m^3K^4))$.

IX. Conclusion

The present model aims to study the black body as an ideal Bose-Einstein gas with ultrarelaticistic spin-1 particles. Even if the model correctly predicts the dependence of pressure and energy with temperature and volume, it also predicts black bodies are a Bose condensate at room temperature. *A fortiori*, it is preciselt the fact that the system is condensed what allows it to have such simple dependences on temperature.

If the model were correct and photon gases really are virtually always in a condensed phase, this would explain why even the hottest stars (O, B), whose temperatures reach $40,000K$, behave as black bodies. This approximation would be specially good for the inner regions of less massive stars and in the outer regions of more massive stars, all of which have radiation as their main energy transport mechanism.