New Cost Function: Penalizing Large Control Power

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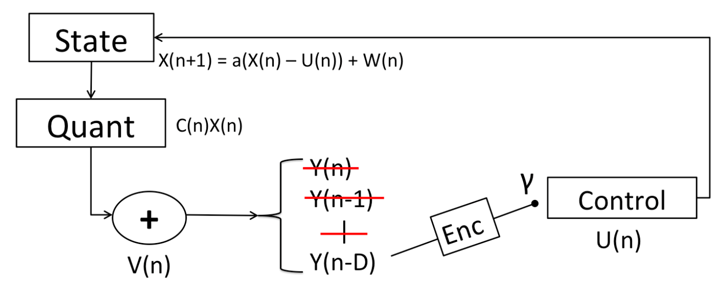
System:



$$X(n+1) = a(X(n) - U(n)) + W(n)$$

$$Y(n) = \begin{cases} Q(X(n)) + V(n) & : n \equiv 0 \pmod{\mathrm{D}+1} \\ 0 & : \text{else} \end{cases}$$

$$Q(X(n)) = C(n)X(n)$$

$$U(n) = \begin{cases} L[X(n) \mid Y(n-D)] & : n \equiv 0 \pmod{\mathrm{D}+1} \\ 0 & : \text{else} \end{cases}$$

New Cost Function – No V(n)

$$\min \mathbb{E}[x^2(n+1)] + \sum_{k=1}^{n} u^2(k)$$

$\min \mathbb{E}[a(x(n) - u(n)) + w(n))^2] + \sum_{k=1}^{n} u^2(k)$
$= \min \mathbb{E}[(a(x(n) - \alpha y(n)) + w(n))^2] + \sum_{k=1}^{n} \alpha^2 y^2(k)$
$= \min \mathbb{E}[(a(x(n) - \alpha c(n)x(n)) + w(n))^2] + \sum_{k=1}^{n} \alpha^2 c^2(k) x^2(k)$
$= \min \mathbb{E}[a^2(1 - \alpha c(n))^2 x^2(n)] + \sigma_w^2 + \sum_{k=1}^{n} \alpha^2 u^2(k)$
$\frac{d}{d\alpha}\mathbb{E}[a^2(1-\alpha c(n))^2 x^2(n)] + \sigma_w^2 + \sum_{k=1}^{n} \alpha^2 u^2(k) = \mathbb{E}[2a^2(1-\alpha c(n))(-c(n))x^2(n)] + \sum_{k=1}^{n} 2\alpha u^2(k) = 0$
$-a^2(\mu_c \sigma_{x(n)}^2 - \alpha(\mu_c^2 + \sigma_c^2)\sigma_{x(n)}^2) + \sum_{k=1}^{n} \alpha u^2(k) = 0$
$\alpha(a^2(\mu_c^2 + \sigma_c^2)\sigma_{x(n)}^2 + \sum_{k=1}^{n} u^2(k)) = a^2 \mu_c \sigma_{x(n)}^2$

$$\alpha(n) = \frac{a^2 \mu_c \sigma_{x(n)}^2}{a^2(\mu_c^2 + \sigma_c^2)\sigma_{x(n)}^2 + \sum_{k=1}^{n} u^2(k)}$$

This is easy to implement, and my next step is to implement this. My worry is that this isn't a closed solution like the previous α calculations. As time passes, α will change every time a new control is implemented–which is fine, but we should note the new complication. Does this answer make sense? Should I go ahead and implement this?

Original Cost Function – No V(n)

$$\min \mathbb{E}[x^2(n+1)]$$

$$u(n) = \alpha y(n) = \alpha c(n)x(n)$$

$\min \mathbb{E}[(a(x(n) - u(n)) + w(n))^2]$
$= \min \mathbb{E}[(a(x(n) - \alpha y(n)) + w(n))^2]$
$= \min \mathbb{E}[(a(x(n) - \alpha c(n)x(n)) + w(n))^2]$
$= \min \mathbb{E}[a^2(1 - \alpha c(n))^2 x^2(n)] + \sigma_w^2$
$\frac{d}{d\alpha}\mathbb{E}[a^2(1 - \alpha c(n))^2 x^2(n)] + \sigma_w^2 = \mathbb{E}[2a^2(1 - \alpha c(n))(-c(n))x^2(n)] = 0$
$\mathbb{E}[c(n)x^2(n) - \alpha c^2(n)x^2(n)] = 0$
$\mu_c \sigma_{x(n)}^2 = \alpha(\mu_c^2 + \sigma_c^2)\sigma_{x(n)}^2$

$$\alpha = \frac{\mu_c}{\mu_c^2 + \sigma_c^2}$$

These are the results we expect, from Gireeja's Noncoherence Paper.

Original Cost Function

$$\min \mathbb{E}[x^2(n+1)]$$

$\min \mathbb{E}[(a(1 - \alpha c(n))x(n) - a\alpha v(n) + w(n))^2]$
$= \min \mathbb{E}[a^2(1 - \alpha c(n))^2 x^2(n) + \mathbb{E}[a^2\alpha^2 v^2(n)] + \sigma_w^2$
$\frac{d}{d\alpha}\mathbb{E}[a^2(1-\alpha c(n))^2 x^2(n) + \mathbb{E}[a^2\alpha^2 v^2(n)] + \sigma_w^2 = \mathbb{E}[-2a^2(1-\alpha c(n))c(n)x^2(n)] + 2a^2\alpha\sigma_v^2 = 0$
$= -\mu_c\sigma_{x(n)}^2 + \alpha(\mu_c^2 + \sigma_c^2)\sigma_{x(n)}^2 + \alpha\sigma_v^2 = 0$

$$\alpha(n) = \frac{\mu_c \sigma_{x(n)}^2}{(\mu_c^2 + \sigma_c^2)\sigma_{x(n)}^2 + \sigma_v^2}$$

These are the results we expect from the LLSE theorem: $\alpha = \frac{cov(X,Y)}{var(Y)} = \frac{\mathbb{E}[XY]}{\mathbb{E}[Y^2]} \leftarrow$ assuming $\mathbb{E}[X] = \mathbb{E}[Y] = 0$