Noisy Observations Simulation

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June 6, 2014

Setup

$$\begin{aligned} \textbf{System} \quad x(n) &= ax(n-1) + w(n-1) \\ y(n) &= cx(n) + v(n) \end{aligned}$$

$$\begin{aligned} x(0) &\sim N(0,1) \\ w(n-1) &\sim N(0,\sigma_w^2) \\ v(n) &\sim N(0,\sigma_v^2) \end{aligned}$$

$$\textbf{Distribution}: \quad \frac{N(cx[n],\sigma_v^2)N(0,a^{2n}+\sigma_w^2\sum_{i=0}^{n-1}a^{2i})}{N(0,c^2a^{2n}+c^2\sigma_w^2\sum_{i=0}^{n-1}a^{2i}+\sigma_v^2)}$$

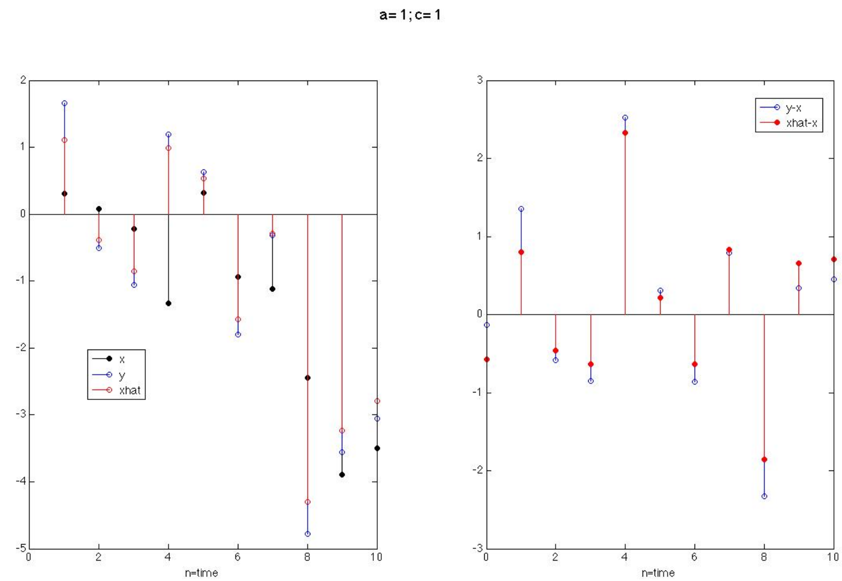
$$\hat{x}[n] = \frac{yc(a^{2n}+\sigma_w^2\sum_{i=0}^{n-1}a^{2i})}{c^2(a^{2n}+\sigma_w^2\sum_{i=0}^{n-1}a^{2i})+\sigma_v^2} = \frac{yc\sigma_x^2}{\sigma_y^2}$$

Code (MatLab)

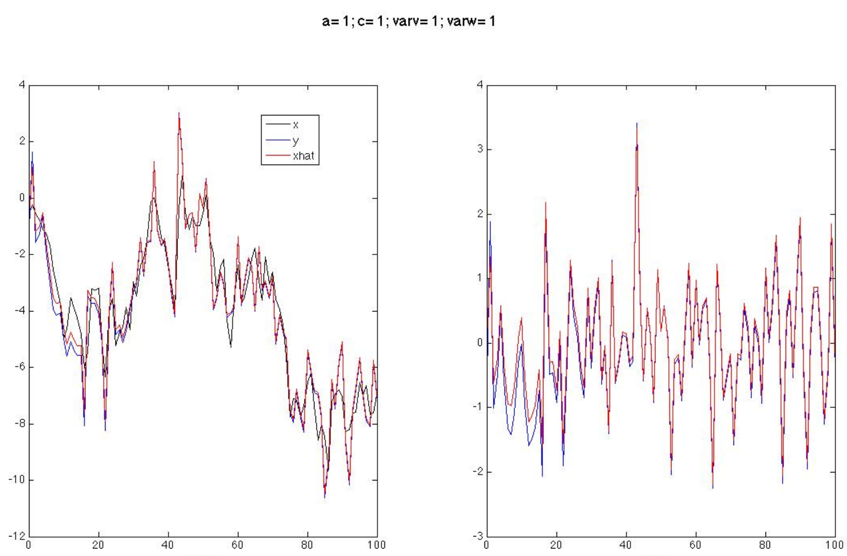
```
clc;
clear all;
set(0,'DefaultAxesFontSize', 14);

a = 1; %Test 2,0.5,10
c = 1;
varv = 1;
varw = 1;

%initialize
x = normrnd(0,1,[1,1]); %x[0]
y = c*x+normrnd(0,varv);
xhat = y*c*1/(c^2+varv);
varx = 1;
```

```
%time passing
for n=1:10
    w = normrnd(0,varw);
    x = [x a*x(1,n)+w];
    v = normrnd(0,varv);
    y = [y c*x(1,(n+1))+v];
    varx = a^2*varx + varw;
    vary = c^2*varx + varv;
    xhat = [xhat y(1,n+1)*c*varx/vary];
end

%hold all
subplot(1,2,1), stem(0:10,x,'k','filled')
subplot(1,2,1), hold on, stem(0:10,y,'b')
subplot(1,2,1), hold on, stem(0:10,xhat,'r')
xlabel('n=time')
legend1 = legend('x','y','xhat','Location','Best');

subplot(1,2,2), stem(0:10,y/c-x,'b')
subplot(1,2,2), hold on, stem(0:10,xhat-x,'r','filled')
xlabel('n=time')
legend2 = legend('y/c-x','xhat-x');

suptitle(['a= ' num2str(a) '; c= ' num2str(c) '; varv= ' num2str(varv) '; varw= ' num2str(varw)])
```

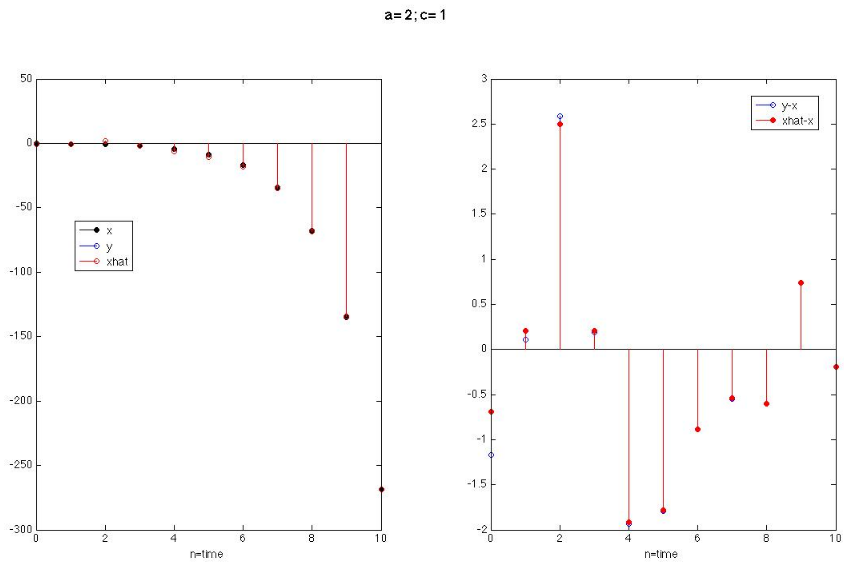
The initial setup is a=1, c=1, varw = 1, varv = 1.



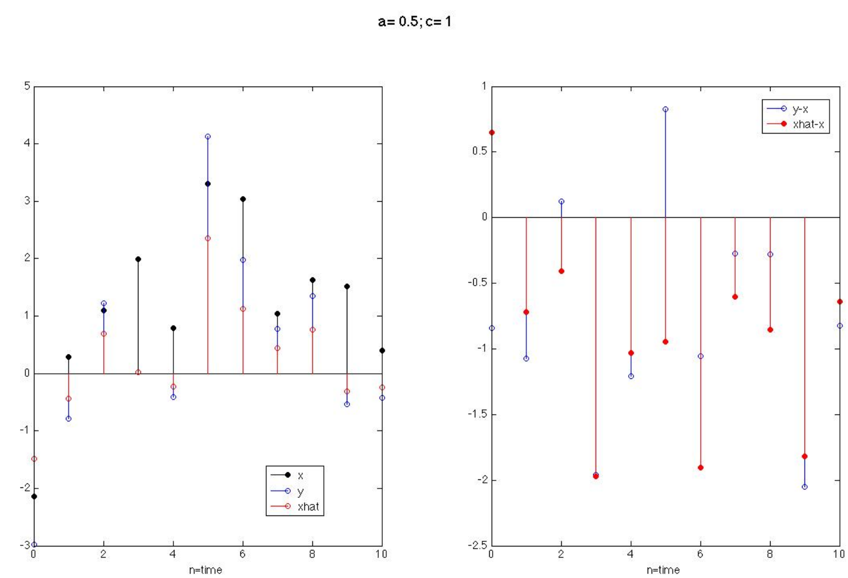
In the beginning, our calculated xhat performs better in estimating x than the naive approach (y=x). Near the end, it appeared that y did better than xhat, but that was probably due to noise. A test over a longer period of time (n=100 instead of n=10) indicates that y converges toward xhat. This is because the noise in the observation approaches the expected value of 0.

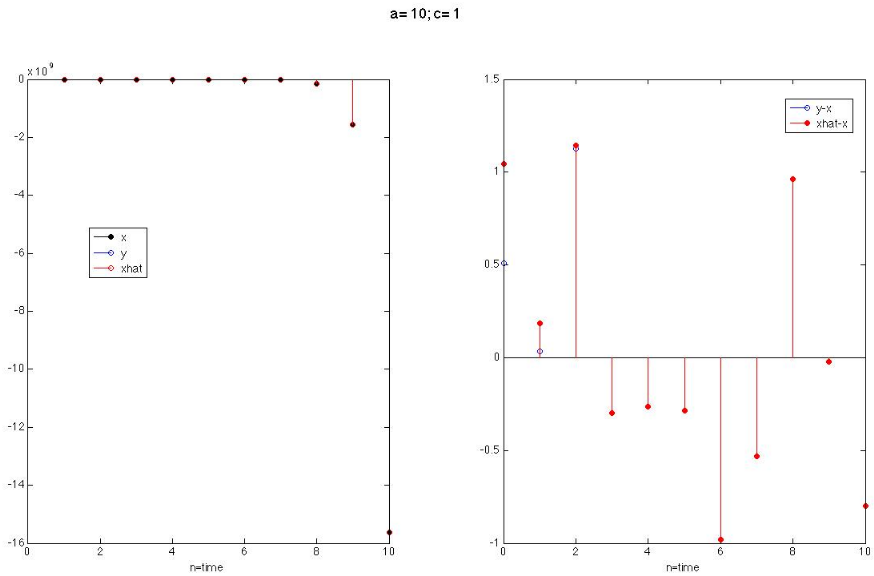


1 Varying A



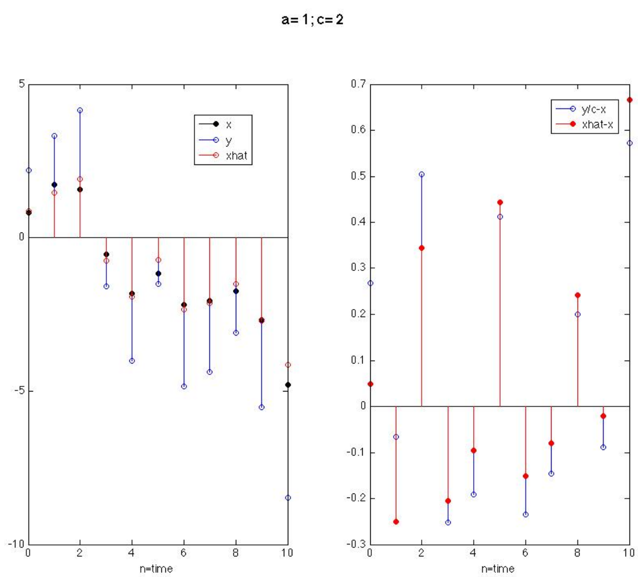
When a is a fraction, there is greater resolution in the plot; since the y limits are smaller, we can see zoomed in the error of the observation and estimate. In addition, the noise has a greater effect on the system.

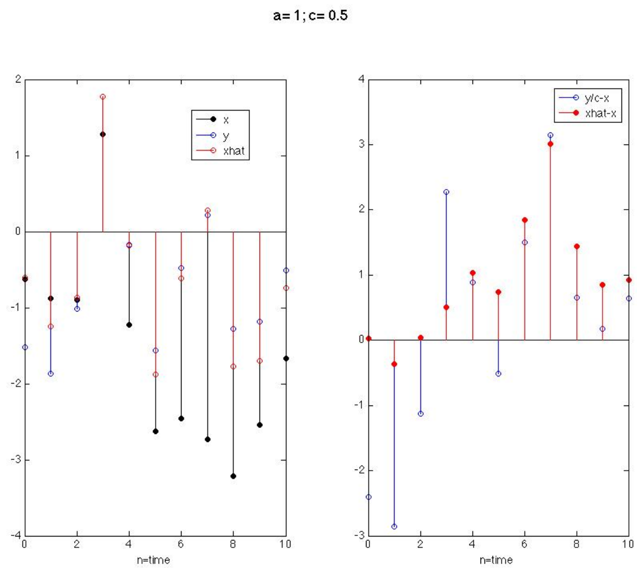


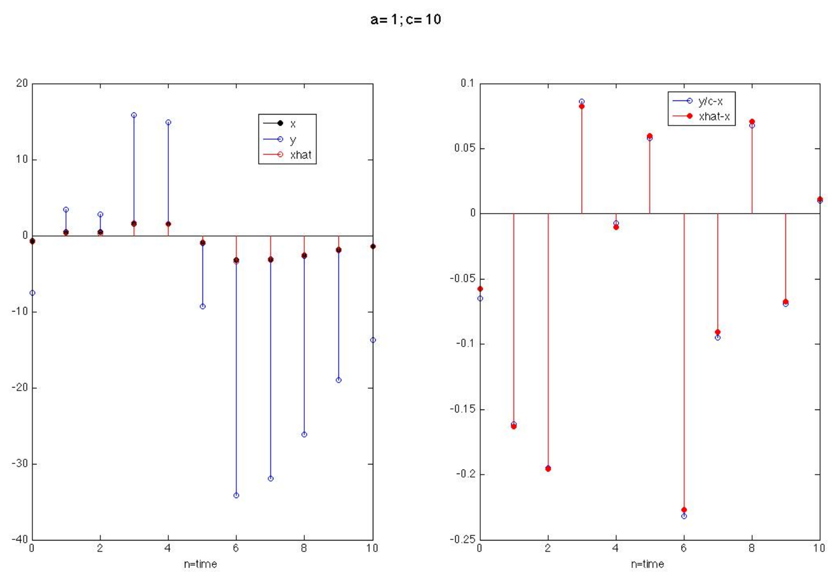


Because a is so large, it's hard to see any difference between x and xhat. What is most important is that even though magnitude of x is in the order of 10^9, the error remains $[-1.5, 1.5]$. Is there anything special about this interval?

2 Varying C

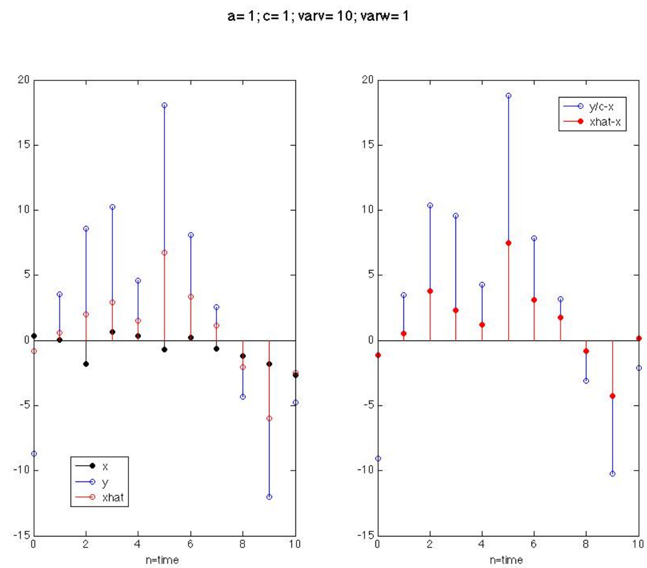




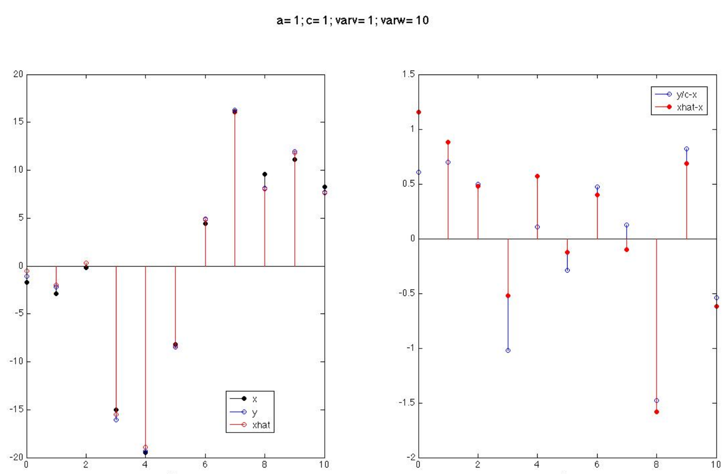


When c increases the error in estimate decreases **significantly** (the same factor?). This might have something to do with the scaling decreasing the effect of noise in the system. It's interesting because c shouldn't affect the SNR at all.

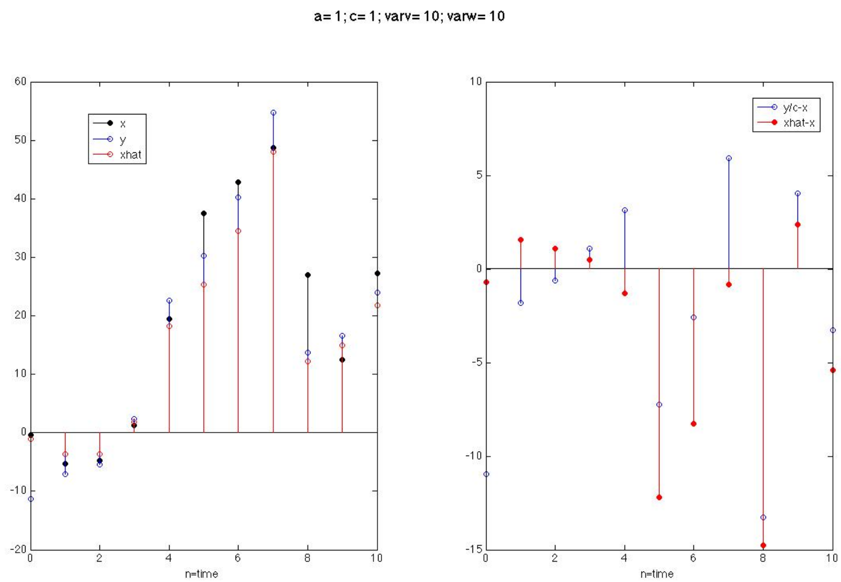
3 Varying Variance of Noise



Up until now, xhat and y/c were comparable in accuracy of estimating x. In this plot, I increased the noise of the observation so that it would affect y, and at every timestep xhat does better (thus proving xhat is a better estimator when noise can be significant.)



In this plot, I increased the noise of the system. In this case, xhat and y are equally accurate at estimating x, with estimation error in the interval $[-1.5, 1.5]$. There really is no benefit to using xhat instead of y/c.



In this plot, I increased the noise of both the system and the observation. Again, xhat isn't *that* much better than y/c. This shows xhat is only useful when the noise of observation outweighs noise of the system.