Midterm Review

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Contents

1 2014-06-03 Channel Estimation

1.1 Problem Statement:

$$X + Z = Y$$

$$X \sim N(0, \sigma^2)$$
$$Z \sim N(0, 1)$$

$$\mathbb{E}[X] = 0$$
$$Var[X] = 0$$

1.2 Solution:

$$f_{X|Y=y}(X) = \frac{f_X(X) * f_{Y=y|X}(Y)}{f_{Y=y}(Y=y)}$$
$$Pdf = \frac{1}{\sigma\sqrt{2\pi}} * e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

$$Var[Y|X] = 1$$
$$Var[Y] = \sigma^2 + 1$$
$$Var[X] = \sigma^2$$
$$Var[Z] = 1$$

$$f_{X|Y=y}(X) = \frac{\sqrt{\sigma^2+1}}{\sigma\sqrt{2\pi}} e^{\frac{-(y-x)^2}{2} - \frac{x^2}{2\sigma^2} + \frac{y^2}{2(\sigma^2+1)}}$$
$$\frac{d}{dx} f_{X|Y=y}(X) = \frac{\sigma^2}{\sigma^2+1} Y$$

1.3 Multiple Copies of Y

for the Same Realization of X:
It's more effective because there is an averaging effect in the variance of the noise:

$$Var[Y|X] = \frac{1}{n}$$
$$Var[Y] = \sigma^2 + \frac{1}{n}$$
$$Var[X] = \sigma^2$$
$$Var[Z] = \frac{1}{n}$$

This assumes we are working with Gaussians.

2 Another L[X—Y]

$$min\mathbb{E}[(\alpha Y + \beta - X)^2]$$

$$\begin{aligned}
\frac{d}{d\alpha}\mathbb{E}[(\alpha Y + \beta - X)^2] &= \mathbb{E}[2(\alpha Y + \beta - X)(Y)] = 0 \\
&= \mathbb{E}[\alpha Y^2 + \beta Y - XY] = 0 \\
&= \mathbb{E}[\alpha Y^2] = \mathbb{E}[XY - \beta Y] \\
&= \alpha = \frac{\mathbb{E}[XY - \beta Y]}{\mathbb{E}[Y^2]} \\
&= \alpha = \frac{Cov[XY]}{Var[Y]}
\end{aligned}$$

This assumes either $\beta = 0$ or $\mathbb{E}[Y] = 0$. The former we will soon show, and the latter is a result of Y being 0-mean, which is a result of X and Z being 0-mean.

$$\begin{aligned}
\frac{d}{d\beta}\mathbb{E}[(\alpha Y + \beta - X)^2] &= \mathbb{E}[2(\alpha Y + \beta - X)] = 0 \\
&= \mathbb{E}[\beta] = \mathbb{E}[X - \alpha Y] \\
&= \beta = \mathbb{E}[X] - \alpha\mathbb{E}[Y] \\
&= \beta = 0
\end{aligned}$$

$$L[X|Y] = \alpha Y + \beta = \frac{Cov[XY]}{Var[Y]} Y$$

This assumes we are using a linear estimator for random variables that aren't necessarily Gaussian, thus showing that for Gaussians the optimal estimator **is** the linear the estimator.

3 Notes from Kalman Filter Wikipedia

4 2014-06-05

4.1 Q1

$$x[n] = a * x[n-1]$$
$$x[0] \sim N(0, a^{2n})$$

The variance increases so that the probability of being larger remains the same.

4.2 Q2

$$\begin{aligned} x[n] &= a * x[n-1] + w[n-1] \\ w[n-1] &\sim N(0, \sigma_w^2) \\ x[n] &\sim N(0, \sigma_w^2 \Sigma_{i=0}^{n-1} a^{2i} + a^{2n}) \end{aligned}$$

4.3 Q3

$$\begin{aligned} x[n] &= a * x[n-1] \\ y[n] &= c * x[n] \\ \hat{x}[n] &= 1/c * y[n] \end{aligned}$$

In this case, the observation is noiseless.

4.4 Q4

$$\begin{aligned} x[n] &= a * x[n-1] + w[n-1] \\ y[n] &= c * x[n] \\ \hat{x}[n] &= 1 \overset{*}{_{\text{c}}} y[n] \end{aligned}$$

W[n] doesn't matter because it's incorporated into the state, and we're trying to guess the state.

4.5 Q5

Estimate x[n] using memory.
If the observations are noiseless, then memory doesn't matter since we have perfect observation anyway. If observations are noisy, over time the noise $\rightarrow 0$.

If you add noise to the observation, use the $L[X|Y]$ shown above.

5 Notes from EE126 Appendix A

6 Proofs about L[X—Y]

6.1 L[X—Y,Z] = L[X—Y] + L[X—Z]

6.2 L[X—Y,Z] = L[X—Y] + L[X—Z-L[Z—Y]]

7 2014-06-09 2014-06-15 Kalman Filter

7.1 Problem Setup:

$$\begin{aligned} X[n] &= AX[n-1] + W[n-1] \\ Y[n] &= CX[n] + V[n] \end{aligned}$$

$$\begin{aligned} X &\sim N(0, A^{2n} + \sigma_W^2 \Sigma_{i=0}^{n-1} A^{2i} \\ Y &\sim N(0, C^2(A^{2n} + \sigma_W^2 \Sigma_{i=0}^{n-1} A^{2i}) + \sigma_V^2 \end{aligned}$$

$$\begin{aligned} X(0) &\sim N(0, 1) \\ W[n] &\sim N(0, \Sigma_W) \\ V[n] &\sim N(0, \Sigma_V) \end{aligned}$$

7.2 Goal:

$$\begin{aligned} \mathbb{E}[X[n+1]|Y^n] &= \hat{X}[n+1] \\ Y^n &= (Y[0] \dots Y[n]) \\ \mathbb{E}[X[n+1]|Y^n] &= \sqcup \mathbb{E}[X[n]|Y^{n-1}] + \sqcup (Y[n] - \mathbb{E}[Y[n]|Y^{n-1}] \end{aligned}$$

7.3 Equations:

$$L[X|Y] = \mathbb{E}[X] + \frac{cov(X, Y)}{cov(Y)}(Y - \mathbb{E}[Y]) \tag{1}$$

$$L[X|Y, Z] = L[X|Y] + L[X|Z - L[Z|Y]] \tag{2}$$

$$cov(AX, CY) = Acov(X, Y)C' \tag{3}$$

$$if V, W \perp cov(V + W) = cov(V) + cov(W) \tag{4}$$

$$\mathbb{E}[X[n+1]|Y^n] = \mathbb{E}[X[n+1]|Y^{n-1}] + \mathbb{E}[X[n+1]|Y[n] - \mathbb{E}[Y[n]|Y^{n-1}]$$

7.4 $\mathbb{E}[X[n+1]|Y^{n-1}]$

$$\begin{aligned}
&(1)\quad \mathbb{E}[AX[n]+W[n]|Y^{n-1}] = \mathbb{E}[AX[n]|Y^{n-1}] + \mathbb{E}[W[n]|Y^{n-1}] \\
&(2)\quad = A\hat{X}[n] + \mathbb{E}[W[n]] \\
&(3)\quad = A\hat{X}[n]
\end{aligned}$$

7.5 $\mathbb{E}[Y[n]|Y^{n-1}]$

$$\begin{aligned}
&(4)\quad \mathbb{E}[CX[n]+V[n]|Y^{n-1}] = C\mathbb{E}[X[n]|Y^{n-1}] + \mathbb{E}[V[n]|Y^{n-1}] \\
&(5)\quad = C\hat{X}[n]
\end{aligned}$$

7.6 $\mathbb{E}[X[n+1]|Y[n]-C\hat{X}[n]]$

$$\begin{aligned}
&(6)\quad \mathbb{E}[X[n+1]|Y[n]-C\hat{X}[n]] = \mathbb{E}[AX[n]+W[n]|Y[n]-C\hat{X}[n]] \\
&(7)\quad = \mathbb{E}[AX[n]|Y[n]-C\hat{X}[n]] \\
&(8)\quad = \mathbb{E}[AX[n]-A\hat{X}[n]|Y[n]-C\hat{X}[n]]
\end{aligned}$$

Lemma: $Y^{n-1} \perp Y[n] - \mathbb{E}[Y[n]|Y^{n-1}]$

Strong Induction!

Base Case: $cov(Y[0], Y[1] - \mathbb{E}[Y[1]|Y[0]]) = 0$

$$\begin{aligned}
&(1)\quad \mathbb{E}[y[0](cax[0] + cw[0] + v[1] - \frac{ac^2\Sigma_{x[0]}}{\Sigma_{y[0]}}y[0])] \\
&(2)\quad = \mathbb{E}[y[0](cax[0] - \frac{ac^2\Sigma_{x[0]}}{\Sigma_{y[0]}}y[0])] \\
&(3)\quad = \mathbb{E}[(cx[0]+v[0])cax[0] - \frac{ac^2\Sigma_{x[0]}}{\Sigma_{y[0]}}y^2[0]] \\
&(4)\quad = \mathbb{E}[c^2ax^2[0] - \frac{ac^2\Sigma_{x[0]}}{\Sigma_{y[0]}}y^2[0]] \\
&(5)\quad = c^2a\Sigma_{x[0]} - \frac{ac^2\Sigma_{x[0]}}{\Sigma_{y[0]}}\Sigma_{y[0]} \\
&(6)\quad = c^2a\Sigma_{x[0]} - ac^2\Sigma_{x[0]} = 0
\end{aligned}$$

Inductive Hypothesis: $cov(Y[n-1], Y[n] - \mathbb{E}[Y[n]|Y[n-1]]) = 0 \wedge \cdots \wedge cov(Y[0], Y[n] - \mathbb{E}[Y[n]|Y[0]]) = 0$

Inductive Step: $cov(Y[n], Y[n+1] - \mathbb{E}[Y[n+1]|Y[n]]) = 0$

$$\mathbb{E}[y[n](cax[n] - \frac{c^2a\Sigma_{x[n]}}{\Sigma_{y[n]}}y[n])] = c^2a\Sigma_{x[n]} - \frac{c^2a\Sigma_{x[n]}}{\Sigma_{y[n]}}\Sigma_{y[n]} = 0$$

In addition,

$$\begin{aligned}
&(7)\quad \forall t \le n, \quad cov(y[t], y[n+1] - \mathbb{E}[y[n+1]|y[t]]) = 0 \\
&(8)\quad = \mathbb{E}[y[t](ca^{n+1-t}x[t] - \frac{c^2a^{n+1-t}\Sigma_{x[t]}}{\Sigma_{y[t]}}y[t])] \\
&(9)\quad = \mathbb{E}[c^2a^{n+1-t}x^2[t] - \frac{c^2a^{n+1-t}\Sigma_{x[t]}}{\Sigma_{y[t]}}y^2[t]] \\
&(10)\quad = c^2a^{n+1-t}\Sigma_{x[t]} - \frac{c^2a^{n+1-t}\Sigma_{x[t]}}{\Sigma_{y[t]}}\Sigma_{y[t]} = 0
\end{aligned}$$

If $t = n+1$,$cov(y[n+1], y[n+1] - \mathbb{E}[y[n+1]|y[n+1]]) = cov(y[n+1], y[n+1] - y[n+1]) = cov(y[n+1], 0) = 0$.
The answer is trivial.

$$cov(Y^{n-1}, Y[n] - \mathbb{E}[Y[n]|Y^{n-1}])$$

$$= \mathbb{E}\left[[Y[0] \dots Y[n-1]]\left[Y[n] - \frac{cov(Y[n], Y^{n-1})}{cov(Y^{n-1})}\begin{bmatrix} Y[0] \\ \vdots \\ Y[n-1]\end{bmatrix}\right]\right]$$

We have proved for $\forall t < n$ this $= 0$, therefore the answer is the 0 vector and we prove the Lemma. Since $\hat{X}[n] = \mathbb{E}[X[n]|Y^{n-1}]$, it is the projection of X onto Y^{n-1}. If $Y^{n-1} \perp \tilde{Y}, \hat{X}[n] \perp \tilde{Y}$. We can add if inside the cov() since it's equivalent to adding 0.

$$cov(AX[n] - A\hat{X}[n], CX[n] - C\hat{X}[n]) = Acov(X[n] - \hat{X}[n])C'$$

$$S_n = cov(X[n] - \hat{X}[n])$$

$$cov(Y[n] - C\hat{X}[n]) = cov(CX[n] + V[n] - C\hat{X}[n])$$

$$= cov(C(X[n] - \hat{X}[n])) + cov(V[n]) = CS_nC' + \sigma_v^2$$

$$K_n = \frac{AS_nC'}{CS_nC' + \sigma_v^2}$$

$$\hat{X}[n+1] = \mathbb{E}[X[n+1]|Y^n]$$

$$\boxed{= A\hat{X}[n] + \frac{Acov(X[n] - \hat{X}[n])C'}{Ccov(X[n] - \hat{X}[n])C' + \sigma_v^2}\left(Y[n] - C\hat{X}[n]\right)}$$

8 2014-06-16 Underlying X1 and X2

8.1 Problem Setup

$\begin{bmatrix} x_1(n+1) \\ x_2(n+1) \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} * \begin{bmatrix} x_1(n) \\ x_2(n) \end{bmatrix}$

Y(n) = $[1 \quad 1] * \begin{bmatrix} x_1(n) \\ x_2(n) \end{bmatrix}$

Calculate $x_1(n), x_2(n)$ from y(n)

Calculate $x_1(n), x_2(n)$ from y(n), y(n-1)

Variations: Y(n) = $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} * \begin{bmatrix} x_1(n) \\ x_2(n) \end{bmatrix}$

Y(n) = $[0 \quad 1] * \begin{bmatrix} x_1(n) \\ x_2(n) \end{bmatrix}$

8.2 Solution

$$X[n] = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} * \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}^{-1} * \begin{bmatrix} y(n-1) \\ y(n) \end{bmatrix}$$

$$X[n] = \begin{bmatrix} 6 & -2 \\ -6 & 3 \end{bmatrix} \begin{bmatrix} y(n-1) \\ y(n) \end{bmatrix}$$

In this case we showed that when dealing with X_1 and X_2, system error matters. This is because instead of trying to estimate the fluctuating line, we are now trying to estimate the "straight line" where the system "should" go.

9 2014-06-18 Conditions of Observability

$$\begin{bmatrix} C \\ \vdots \\ CA^{\lfloor \frac{m}{n} \rfloor} \end{bmatrix} * \begin{bmatrix} x_1(n - \lfloor \frac{m}{n} \rfloor) \\ \vdots \\ x_n(n - \lfloor \frac{m}{n} \rfloor) \end{bmatrix} = \begin{bmatrix} y(n - \lfloor \frac{m}{n} \rfloor) \\ \vdots \\ y(n) \end{bmatrix}$$

Check that $\begin{bmatrix} C \\ \vdots \\ CA^{\lfloor \frac{m}{n} \rfloor} \end{bmatrix}$ is full rank, then delete $\lceil \frac{m}{n} \rceil * n - m = -m mod n$ lines and solve for x.

$\begin{bmatrix} C \\ \vdots \\ CA^{\lfloor \frac{m}{n} \rfloor} \end{bmatrix}$ must be full rank, or span X.

10 2014-06-19 Various Proofs

10.1 Best Control

$$\begin{aligned}
&min\mathbb{E}[||x(n+1)||^2] \\
=&min\mathbb{E}[||ax(n) + w(n) + u(n)||^2] \\
=&min\mathbb{E}[||ax(n) + w(n) + \alpha y(n) + \beta||^2] \\
=&min\mathbb{E}[||ax(n) + w(n) + \alpha cx(n) + \alpha v(n) + \beta||^2]
\end{aligned}$$

$$\begin{aligned}
\frac{d}{d\alpha}\mathbb{E}[] &= \mathbb{E}[2(ax(n) + w(n) + \alpha cx(n) + \alpha v(n) + \beta)(cx(n) + v(n))] = 0 \\
&= \mathbb{E}[acx^2(n) + \alpha(c^2x^2(n) + v^2(n))] = 0 \\
&= \alpha\mathbb{E}[c^2x^2(n) + v^2(n)] = -\mathbb{E}[acx^2(n)] \\
&= \alpha = \frac{-acVar(x)}{Var(Y)}
\end{aligned}$$

$$\begin{aligned}
\frac{d}{d\beta}\mathbb{E}[] &= \mathbb{E}[2(ax(n) + w(n) + \alpha cx(n) + \alpha v(n) + \beta)] = 0 \\
&= \mathbb{E}[\beta] = 0
\end{aligned}$$

$$u(n) = \frac{-acVar(X)}{Var(Y)}Y(n) = -L[X|Y]$$

10.2 Error of Control Problem

$$\begin{aligned}
&\mathbb{E}[||x(n) + u(n)||^2] \\
&= \mathbb{E}[||x(n) - \frac{ac\Sigma_X}{\Sigma_Y}(cx(n) + v(n))||^2] \\
&= \mathbb{E}[||(1 - \frac{ac\Sigma_X}{\Sigma_Y})x(n) - \frac{ac\Sigma_X}{\Sigma_Y}v(n)||^2] \\
&\rightarrow \boxed{(1 - \frac{ac\Sigma_X}{\Sigma_Y})^2\Sigma_X + (\frac{ac\Sigma_X}{\Sigma_Y})^2\Sigma_V}
\end{aligned}$$

10.3 Without System Error, Estimation Error = 0

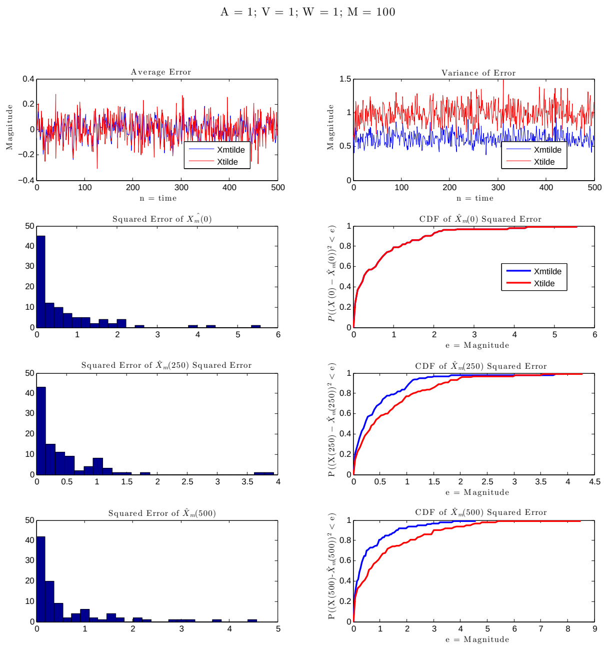
In this system, we assume $a \leq 1$.

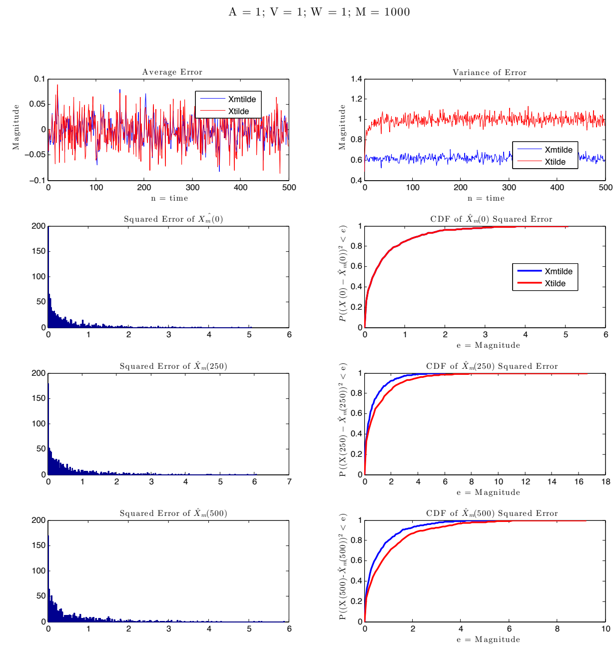
$$\Sigma_n = cov(x(n) - \hat{x}(n))$$
$$S_n = A^2\Sigma_{n-1} + \Sigma_W = A^2\Sigma_{n-1}$$
$$\Sigma_n = (1 - KnC)Sn = A^2(1 - KnC)\Sigma_{n-1}$$
$$\rightarrow 0 < \left(1 - \frac{c^2 S_n}{c^2 S_n + \Sigma_V}\right) < 1$$
$$\left(1 - \frac{c^2 S_n}{c^2 S_n + \Sigma_V}\right) < 1 \rightarrow \frac{c^2 S_n}{c^2 S_n + \Sigma_V} > 0$$

It is easy to see the error coefficient is greater than 0, because 1 - fraction > 0. By definition $c \neq 0$, since the signal must have *some* power. By definition Σ_n starts out > 0, thus $S_n > 0$, thus $\left(1 - \frac{c^2 S_n}{c^2 S_n + \Sigma_V}\right) < 1$ and $n \rightarrow \infty \implies \Sigma_n \rightarrow 0$.

11 2014-06-29 Kalman Filter with Multiplicative Noise

12 Results of Simulations





These two figures represent a Kalman Filter with Additive Noise estimating a system with only additive noise. Xtilde represents a memoryless estimate, while Xmtilde represents the Kalman Filter's estimate.

- The estimation error variance is bounded

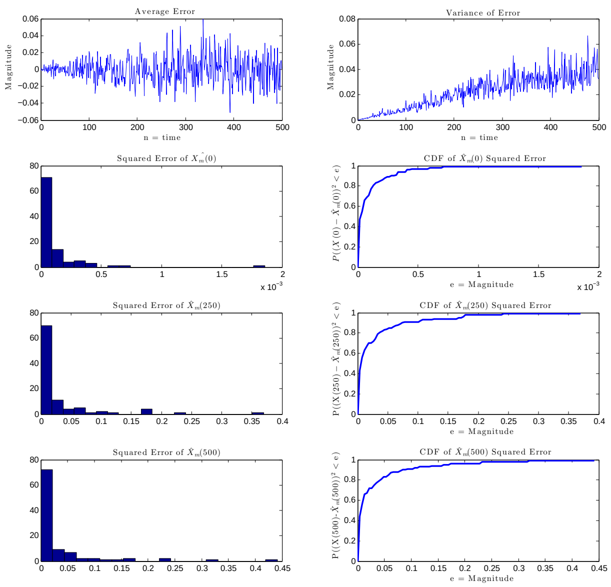
- The Kalman Filter performs better than the optimal memoryless estimator; over many trials it's clearer the error variance is lower

- If $0 < A < 1$, $Xmtilde \to Xtilde$. We showed earlier that the Kalman Filter is only intended for an estimation system, not a control system, and as the value of the state converges to 0 the estimation system behaves like the control problem. Since the value of the state is so close to 0 at each timestep, memory provides no additional benefit/utility to estimation.

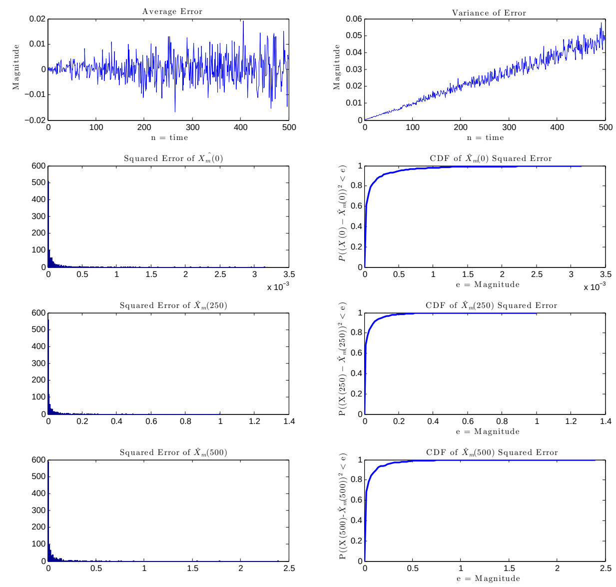
- The CDF of estimation error is affected by C, V, and W

Note: CDF of $\hat{X}_m(n)$ Squared Error means the plots are of the estimation error.

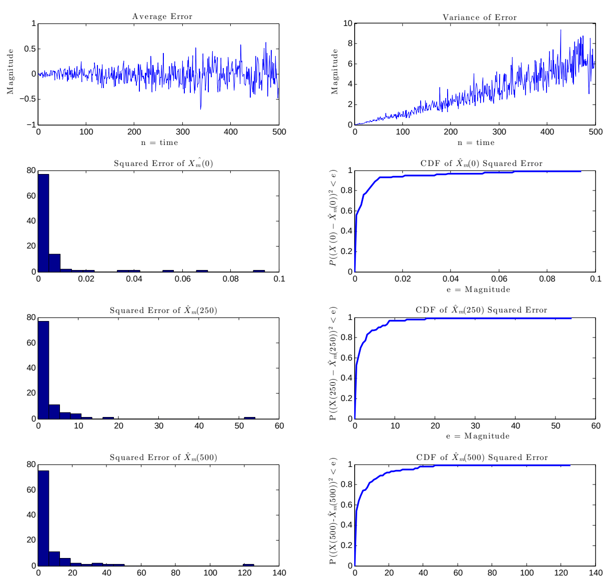
A = 1; V = 0.01; W = 1; M = 100



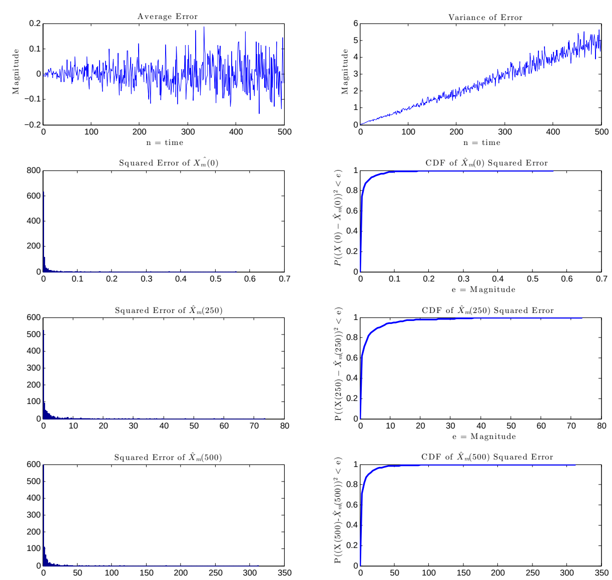
A = 1; V = 0.01; W = 1; M = 1000



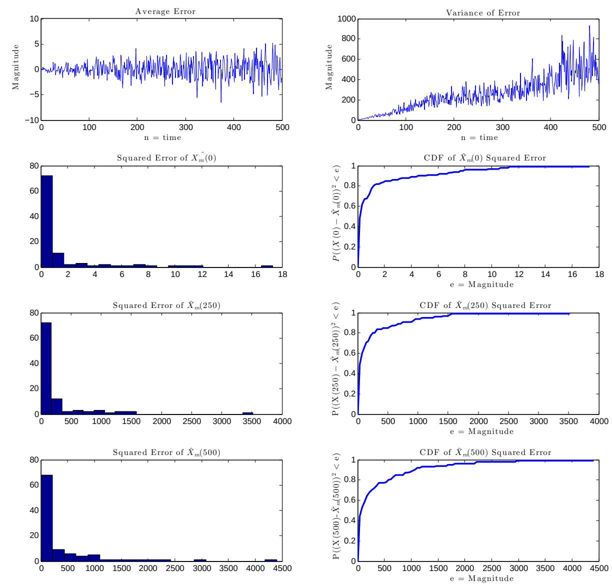
A = 1; V = 0.1; W = 1; M = 100



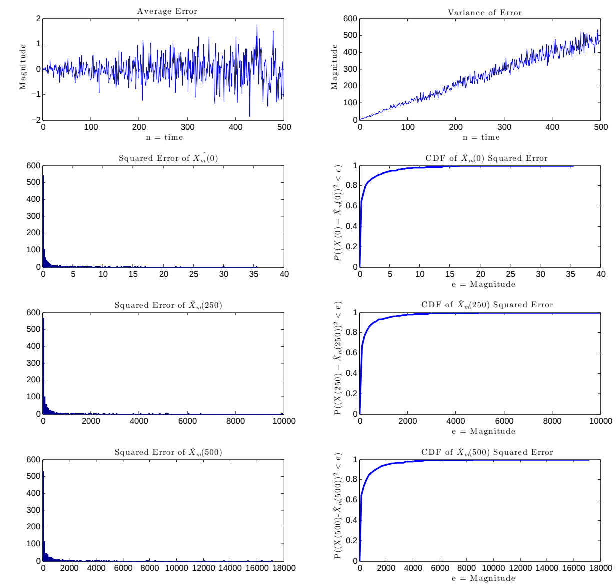
A = 1; V = 0.1; W = 1; M = 1000



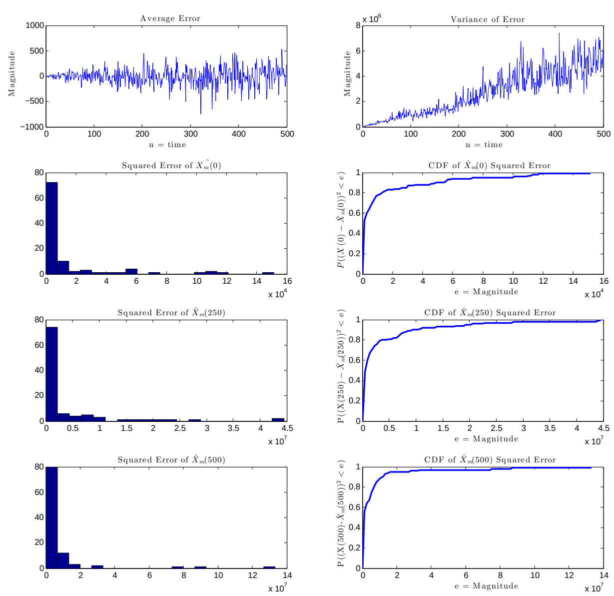
A = 1; V = 1; W = 1; M = 100



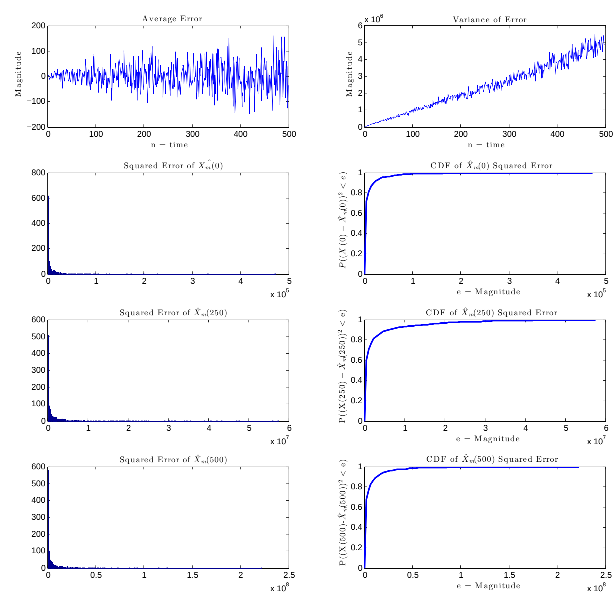
A = 1; V = 1; W = 1; M = 1000



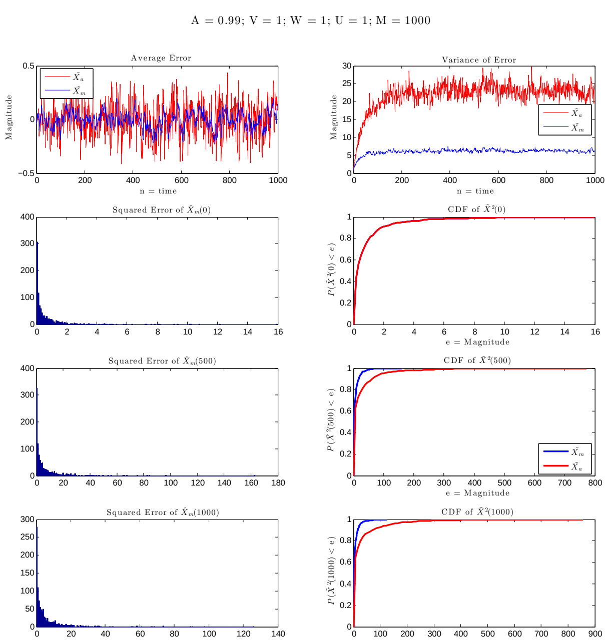
A = 1; V = 100; W = 1; M = 100

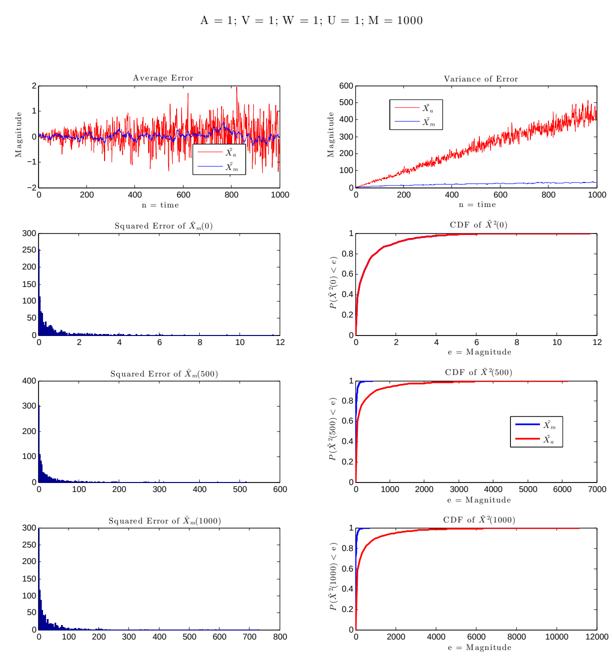


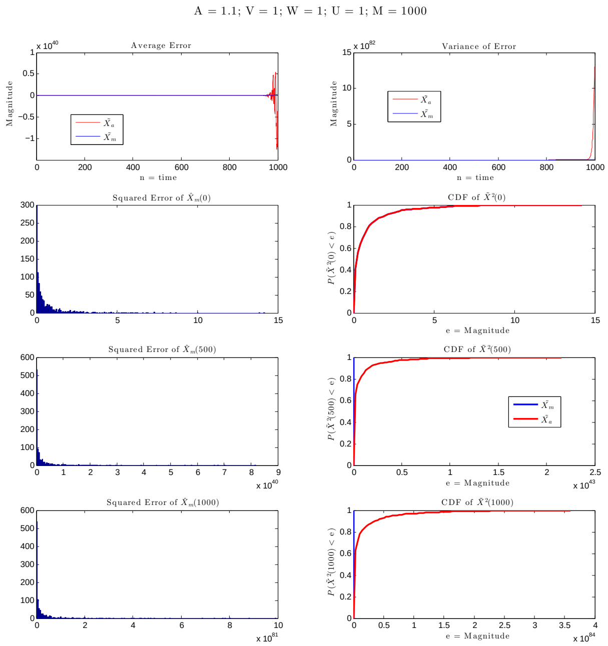
A = 1; V = 100; W = 1; M = 1000

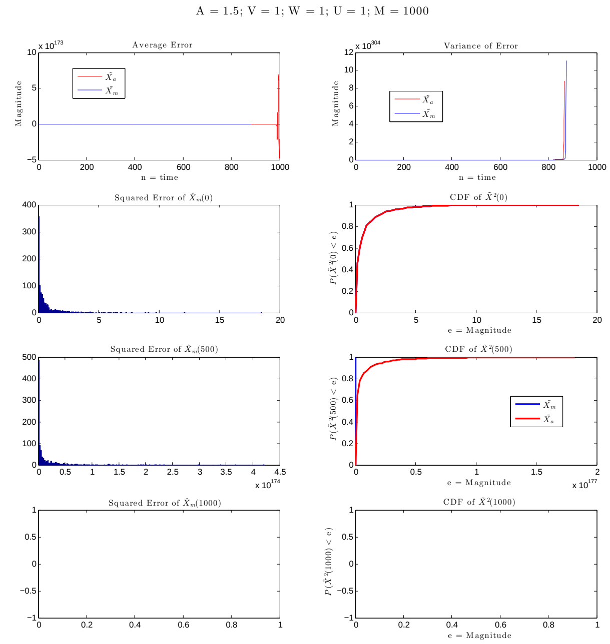


These figures represent a Kalman Filter for Additive Noise with a system that has state multiplicative noise and additive noise but no observation noise. The estimation error variance steadily increases because the multiplicative noise causes the signal to increase, thus increasing the error from the multiplicative noise. In this setup, the multiplicative error was $(1 + r(n)) = (1 + normrnd(0, varv))$. The four figures represent ascending levels of V (or $\frac{1}{p}$), with the estimation error variance increasing as V increases.
Note: CDF of $\hat{X}_m(n)$ Squared Error means the plots are of the estimation error.



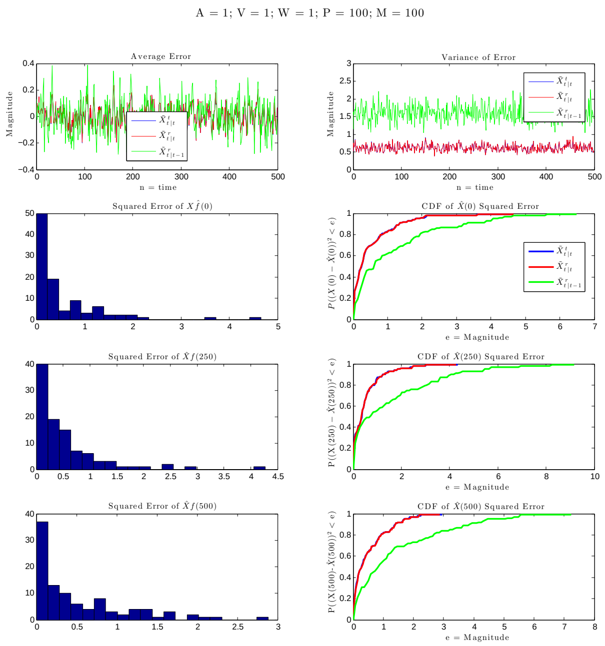


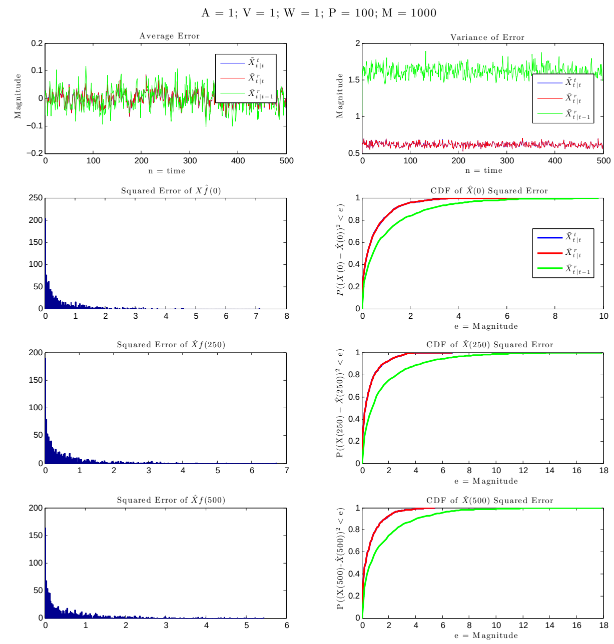


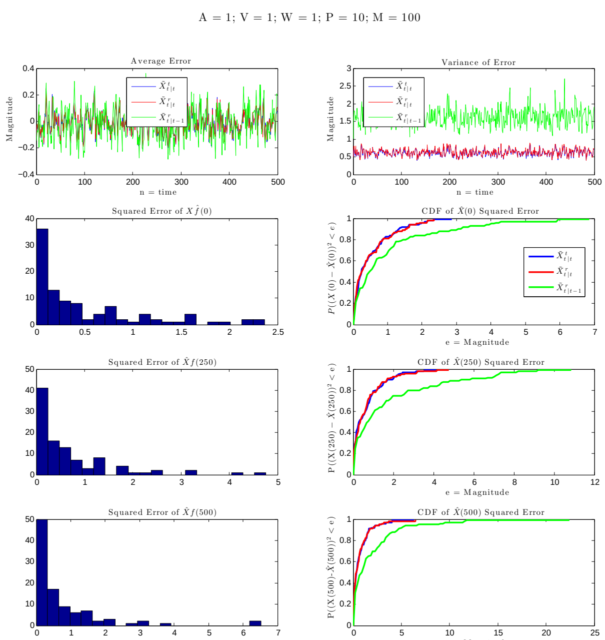


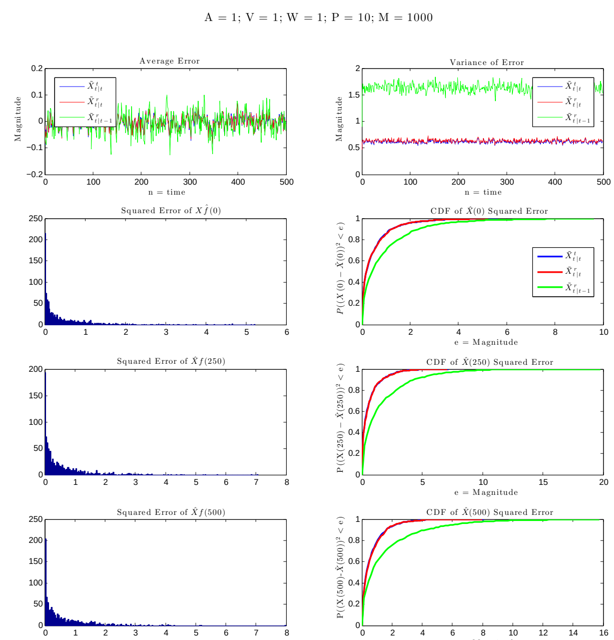
This is the implementation of a Kalman Filter for Multiplicative Noise (Rajasekaran) with a system that has both multiplicative and additive noise. $\tilde{X}_a$ represents the estimation error for a Kalman Filter for only additive noise, while $\tilde{X}_m$ represents the new Kalman Filter. It is apparent the new Kalman Filter is doing much better; both have bounded error variances, but $\tilde{X}_m$ is significantly lower.

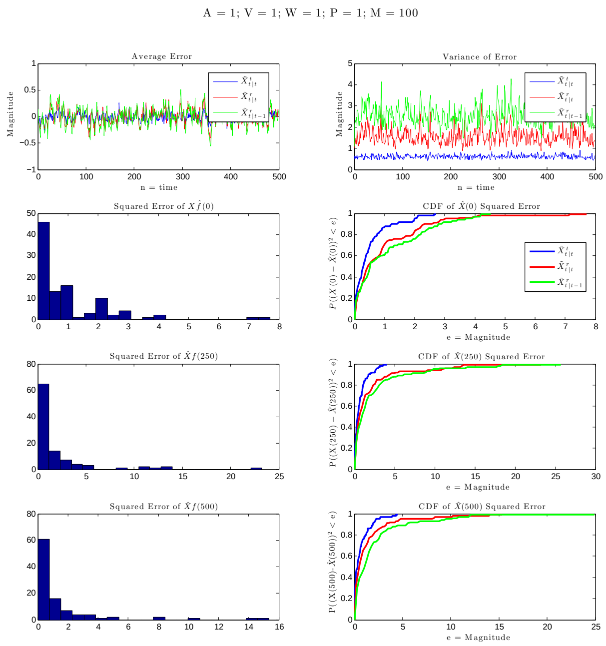
In the upper left corner, A = 0.99. Since A < 1, the state converges toward 0 and the estimation error variance for X_a is bounded. That being said, Var($\tilde{X}_m$) is still better (bounded at a lower value) than Var($\tilde{X}_a$). In the upper right, A = 1. Var($\tilde{X}_a$) matches the behavior we would expect, which is increasing linearly similar to the previous "Ignoring Multiplicative Noise with A = 1" plots above. Var($\tilde{X}_m$) is bounded, again as we would expect since we're using a Kalman Filter intended for multiplicative noise. In the lower left, A = 1.1. In this case, you can see the estimation error variance explodes, with Var($\tilde{X}_a$) increasing up to 10^82. Var($\tilde{X}_m$), on the other hand, remains at its low bounded value. In the lower right, A = 1.5. The system is growing too fast over n = 1000 timesteps, and Matlab can't handle the numbers because they're getting too large. Both estimation error variances appear large because of value truncation/roundoff error done in Matlab. This simply means values $A > 1.5$ aren't testable in Matlab, even if Rajasekaran's Kalman Filter applies.

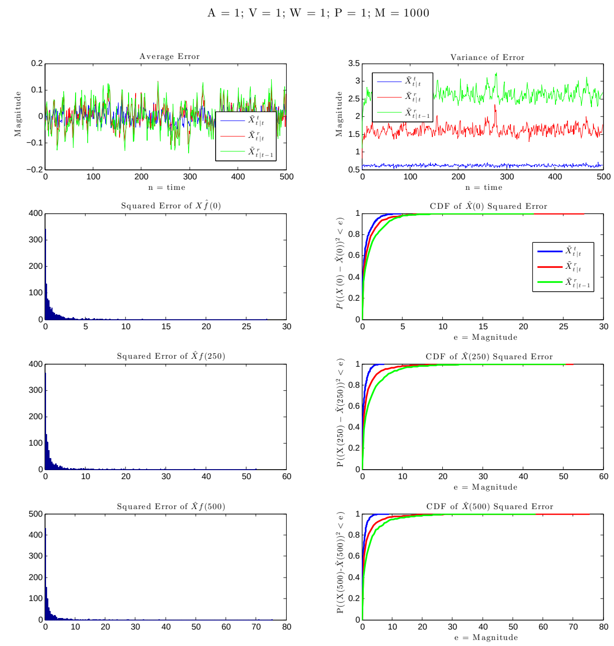

A = 1; V = 1; W = 1; P = 100; M = 100
Average Error
Variance of Error
Squared Error of $X\hat{f}(0)$
CDF of $\hat{X}(0)$ Squared Error
Squared Error of $\hat{X}f(250)$
CDF of $\hat{X}(250)$ Squared Error
Squared Error of $\hat{X}f(500)$
CDF of $\hat{X}(500)$ Squared Error
Magnitude
n = time
e = Magnitude

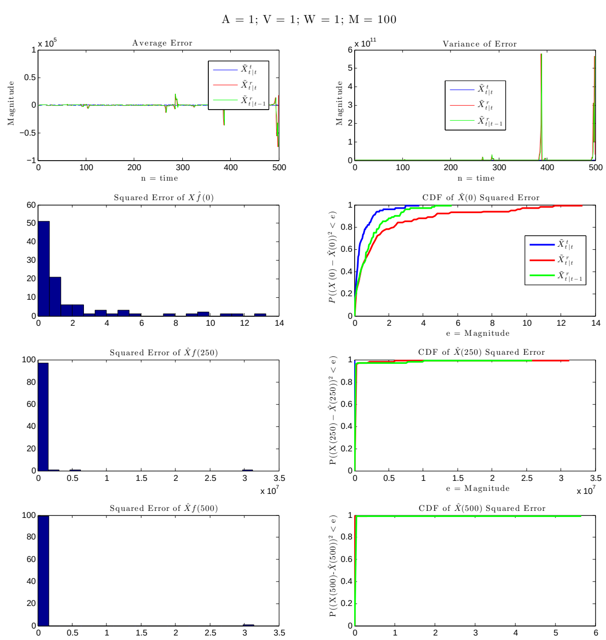

A = 1; V = 1; W = 1; P = 100; M = 1000
Average Error
Variance of Error
Squared Error of $X\hat{f}(0)$
CDF of $\hat{X}(0)$ Squared Error
Squared Error of $\hat{X}f(250)$
CDF of $\hat{X}(250)$ Squared Error
Squared Error of $\hat{X}f(500)$
CDF of $\hat{X}(500)$ Squared Error
Magnitude
n = time
e = Magnitude


A = 1; V = 1; W = 1; P = 10; M = 100
Average Error
Variance of Error
Squared Error of $X\hat{f}(0)$
CDF of $\hat{X}(0)$ Squared Error
Squared Error of $\hat{X}f(250)$
CDF of $\hat{X}(250)$ Squared Error
Squared Error of $\hat{X}f(500)$
CDF of $\hat{X}(500)$ Squared Error
Magnitude
n = time
e = Magnitude

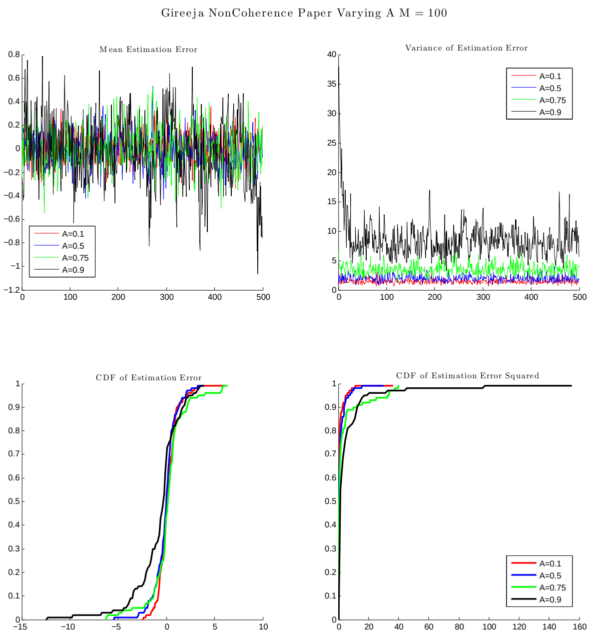

A = 1; V = 1; W = 1; P = 10; M = 1000
Average Error
Variance of Error
Squared Error of $X\hat{f}(0)$
CDF of $\hat{X}(0)$ Squared Error
Squared Error of $\hat{X}f(250)$
CDF of $\hat{X}(250)$ Squared Error
Squared Error of $\hat{X}f(500)$
CDF of $\hat{X}(500)$ Squared Error
Magnitude
n = time
e = Magnitude

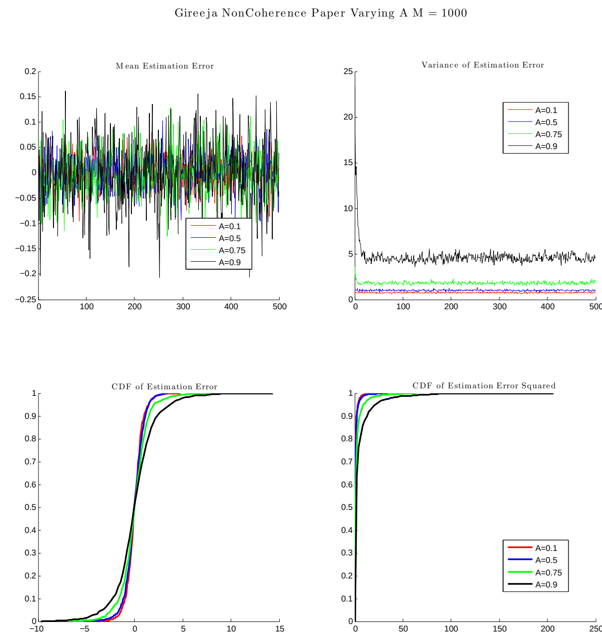


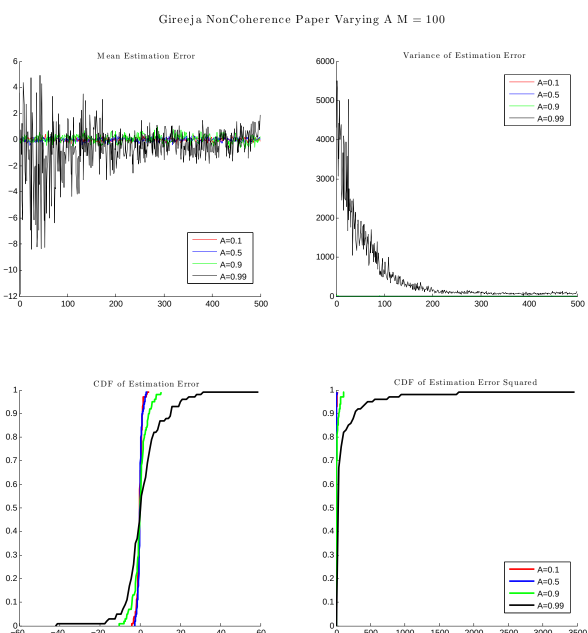




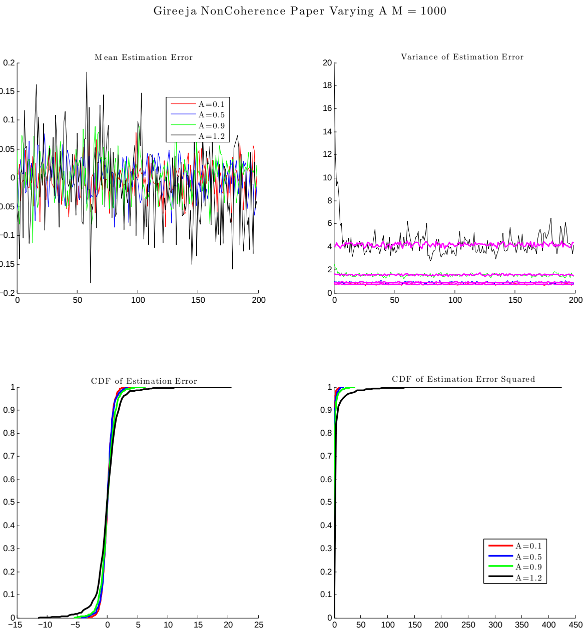
These figures are the implementation of Schenato's paper, with multiplicative noise but no packet drops. Multiplicative (quantization) noise happens over the channel between the transmitter and the receiver. $\tilde{X}^t_{t|t}$ represents the transmitter's estimate of the state, using a Kalman Filter for additive noise (because so far there has only been additive noise.) $\tilde{X}^r_{t|t-1}$ represents a prediction of the state post-quantization noise, and $\tilde{X}^r_{t|t}$ represents the estimation of the state post-quantization noise. When the power is extremely small (thus the multiplicative noise has large variance), $\tilde{X}^r_{t|t-1}$ does better than $\tilde{X}^r_{t|t}$. This is because the multiplicative noise has such a huge effect, the prediction with the moderate growth A is closer to the real state than the totally inaccurate noisy signal.

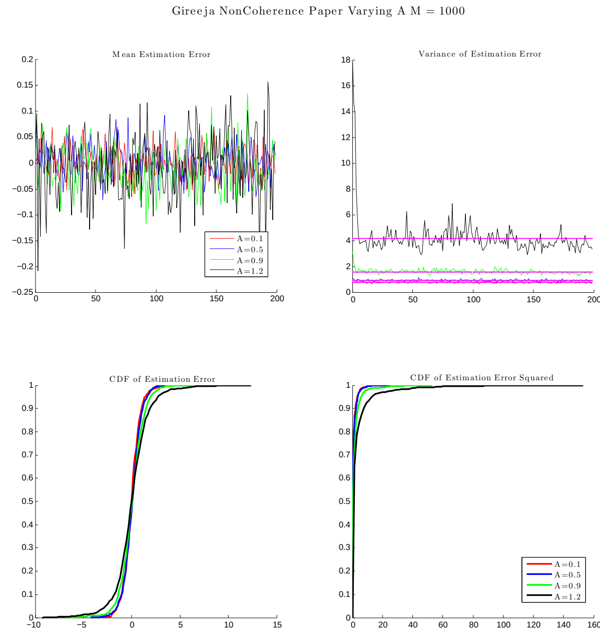




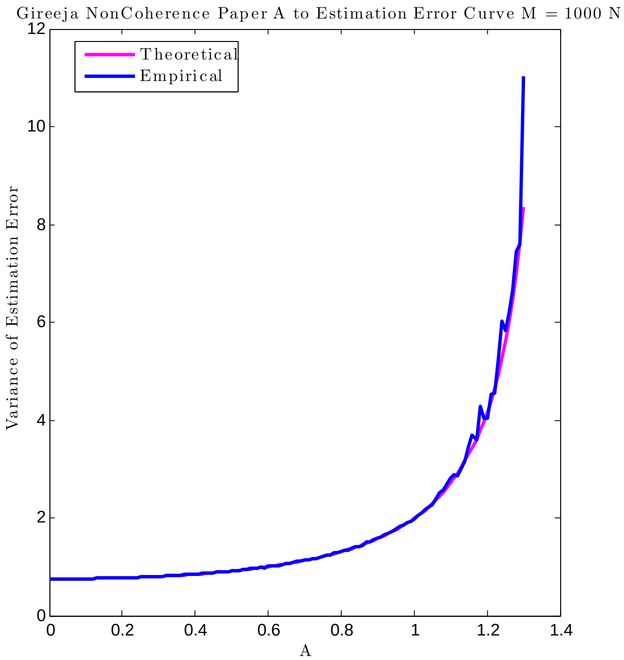


These figures are the implementation of Gireeja's Non-Coherence paper for estimation, not control. The figures above show that A affects the final bounded value of the estimation error variance. The figure to the left shows why A must be < 1; when A is close to 1 it takes longer to converge towards its asymptotic estimation error variance, and when A ≥ 1 the system will explode.





The figure on the left shows that my theoretical calculation is a reasonable estimate of the estimation error variance. It represents mean(theoretical variance calculation), or the average theoretical variance over M = 1000 trials. The theoretical variance must still be averaged because it involves the realization of Y, or the realization of the multiplicative noise, meaning there is still an element of randomness in the theoretical variance calculation. Since the variance doesn't increase over time, I chose to also plot the mean over time.
The figure on the right represents mean(mean(theoretical variance)), or the average theoretical variance over both M trials and n timesteps. The variance is bounded, meaning it asymptotes to a specific number.



In this figure, blue represents the asymptotic empirical variance for levels of A 0:0.01:1.3, while magenta represents the asymptotic theoretical variance. These values are mean(mean()), or averaged over both trials and timesteps. The curve appears roughly exponential, or at the very least when $A > 1$ the curve rises sharply. Based on Gireeja's paper, for these values the curve should asymptote at $\sqrt{2} = 1.414$, which is reflected in the curve. Since this curve was generated over n = 250 timesteps and M = 1000 trials, the spikes in the blue curve should *not* be attributed to randomness. Multiple runs could confirm whether those spikes are significant or not.