Optimum Linear Estimation of Stochastic Signals in the Presence of Multiplicative Noise

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Abstract

This paper considers optimum (MMSE) linear recursive estimation of stochastic signals in the presence of multiplicative noise in addition to measurement noise. Often problems associated with phenomena such as fading or reflection of the transmitted signal at an ionospheric layer, and also situations involving sampling, gating, or amplitude modulation, can be cast into such formulation. The different kinds of estimation problems treated include one-stage prediction, filtering, and smoothing. Algorithms are presented for discrete time as well as for continuous time estimation.

Manuscript received May 25, 1970.

This work was supported by the U.S. Air Force Office of Scientific Research under Contract F44620-68-C-0023.

I. Introduction

The problem of optimum linear estimation of a stochastic signal from noisy measurements has been extensively treated in the literature [1]-[4]. Optimum linear recursive estimation schemes have been particularly attractive because of their computational advantages and simplicity of structure [2]. Different criteria of optimality, such as minimum mean-square error, maximum likelihood, and least-square error, have been used [3] depending upon the problems of interest.

Kalman Filter

This paper considers optimum linear recursive estimation of nonstationary stochastic signals in the presence of multiplicative noise in addition to measurement noise. Often problems associated with phenomena such as fading or reflection of the transmitted signal at an ionospheric layer, and also certain situations involving sampling, gating, or amplitude modulation, can be cast into the above formulation [5], [6].

The problem of estimation of signals corrupted by multiplicative noise under the restrictive assumptions of stationarity and semi-infinite interval of observation has been solved earlier [6]. Nahi [7] has treated in the different context of uncertain observation (hypothesis testing) a problem mathematically similar to a special case of one of the problems considered in this paper.

The estimation algorithms presented in this paper are optimum in the linear minimum mean-square error sense. The different estimation problems treated include one-stage prediction, filtering, and smoothing. Algorithms are presented for discrete time as well as for continuous time estimation.

II. Discrete Time Estimation

Problem Statement

The signal to be processed at the kth instant of time is

$$Z(k) = U(k)C(k)X(k) + V(k) \tag{1}$$

where

1) $X(k)$, an $n \times 1$ vector stochastic signal, is the evolution of the linear difference equation

$$X(k+1) = A(k+1,k)X(k) + B(k+1,k)W(k); \tag{2}$$

$W(k)$ is an $r \times 1$ white sequence with zero mean and $E[W(k)W'(\ell)] = Q(k)\delta_{k\ell}$;
$X(0)$ is an $n \times 1$ vector random variable with zero mean and covariance S_0;
$A(k+1, k)$ is an $n \times n$ matrix and $B(k+1,k)$ is an $n \times r$ matrix.

2) $V(k)$ is an $m \times 1$ vector white sequence with zero mean and $E[V(r)V'(\ell)] = R(k)\delta_{k\ell}$.

3) $U(k)$, the multiplicative noise, is a scalar white sequence with nonzero mean $m(k)$ and covariance $N(k)$.

4) $C(k)$ is an $m \times n$ matrix.
5) Processes $U(k)$, $V(k)$, $W(k)$ and the random variable $X(0)$ are statistically independent of each other.

The problem is to obtain an estimate $\hat{X}(k/\ell)$ of the signal $X(k)$ as a linear combination of the observations $Z(0)$, $Z(1), \ldots, Z(\ell - 1), Z(\ell)$ such that

$$E\{[X(k) - \hat{X}(k/\ell)]' \, [X(k) - \hat{X}(k/\ell)]\}$$
$$= E\,[\tilde{X}'(k/\ell)\tilde{X}(k/\ell)] \qquad (3)$$

is minimum.

Note that $\ell = k - 1$ is one-stage prediction, $\ell = k$ is filtering, and $\ell > k$ is smoothing.

Solution

The necessary and sufficient conditions to be satisfied by the optimum estimate $\hat{X}(k/\ell)$ can easily be shown to be

$$E\,[\tilde{X}(k/\ell)Z'(j)] = 0 \qquad (4)$$

for $j = 0,1,2, \ldots, (\ell - 1), \ell$. Instead of working with the observations $Z(j)$ it is convenient to work with a white sequence, called innovations [8], obtained by a causal, linear, and causally invertible operation on the observations $Z(j)$. In terms of the innovation process $\nu(j)$, the necessary and sufficient conditions (4) can be restated as

$$E\,[\tilde{X}(k/\ell)\nu'(j)] = 0 \qquad (5)$$

for $j = 0,1,2, \ldots, (\ell - 1), \ell$.

The innovation process for the problem at hand is given by

$$\nu(j) = Z(j) - M(j)C(j)\hat{X}(j/j - 1). \qquad (6)$$

A theorem, useful for the derivation of the estimator equations, about the innovations process will now be stated and proved.

Theorem 1: The innovations process $\nu(k)$ as in (6) is a white sequence with zero mean and covariance

$$E\,[\nu(k)\nu'(\ell)] = R_\nu(k)\delta_{k\ell}$$ Var Y tilde

with

$$R_\nu(k) = M^2(k)C(k)P(k)C'(k)$$
$$+ N(k)C(k)S(k)C'(k) + R(k) \qquad (7)$$

where

$$P(k) = E\,[\tilde{X}(k/k - 1)\tilde{X}'(k/k - 1)]$$
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and

$$S(k) = E\,[X(k)X'(k)].$$

Proof: That the innovations process has zero mean follows directly by taking expectations on both sides of (6). To prove that the innovations process is white with the indicated covariance, let $k > \ell$. Then

$$E\,[\nu(k)\nu'(\ell)] = E\{[U(k)C(k)X(k) + V(k)$$
$$- M(k)C(k)\hat{X}(k/k - 1)]\,\nu'(\ell)\}$$
[using (1) in (6)]
$$= E\{[M(k)C(k)\tilde{X}(k/k - 1) + \big(U(k)$$
$$- M(k)\big)\,C(k)X(k) + V(k)]\,\nu'(\ell)\}$$
[adding and subtracting $M(k)C(k)X(k)$].

After multiplying out the terms in the order in which they appear, and then taking expectations, it can be seen that the first term is zero because of (5) and the third term is zero because $V(k)$ is independent of $U(\ell)$, $V(\ell)$, and $X(\ell)$.

Consider the second term:

$$E\{[\,\big(U(k) - M(k)\big)\; C(k)X(k)]\,\nu'(\ell)\}$$
$$= E\{[U(k)C(k)X(k)]\;[M(\ell)C(\ell)\tilde{X}(\ell/\ell - 1)$$
$$+ \big(U(\ell) - M(\ell)\big)\; C(\ell)X(\ell) + V(\ell)]'$$
$$- \;[M(k)C(k)X(k)]\;[M(\ell)C(\ell)\tilde{X}(\ell/\ell - 1)$$
$$+ \big(U(\ell) - M(\ell)\big)\; C(\ell)X(\ell) + V(\ell)]'\}$$
$$= 0.$$

It can be similarly shown that for $k < \ell$

$$E\,[\nu(k)\nu'(\ell)] = 0.$$

For $k = \ell$, we have

$$E\,[\nu(k)\nu'(k)] = E\{[M(k)C(k)\tilde{X}(k/k - 1)$$
$$+ \big(U(k) - M(k)\big)C(k)X(k) + V(k)]$$
$$\cdot\; [M(k)C(k)\tilde{X}(k/k - 1) + \big(U(k) - M(k)\big)$$
$$\cdot\; C(k)X(k) + V(k)]'\}$$
$$= M^2(k)C(k)P(k)C'(k)$$
$$+ N(k)C(k)S(k)C'(k) + R(k)$$

where the cross terms involving $V(k)$ or $\big(U(k) - M(k)\big)C(k)X(k)$ are zero because $V(k)$ is future noise for $X(k/k - 1)$ or because $E\,[U(k) - M(k)]$ equals zero. Therefore,

$$E\,[\nu(k)\nu'(\ell)] = R_\nu(k)\delta_{k\ell}.$$

Hence the theorem.

One-Stage Prediction ($\ell = k - 1$)

As recursive algorithms are sought, $\hat{X}(k/k-1)$ will be obtained as a linear combination of $\hat{X}(k-1/k-2)$ and the new observation $Z(k-1)$. The optimum algorithms are presented in the following theorem.

Theorem 2:

a) The optimum linear one-stage prediction estimate $\hat{X}(k/k-1)$ is given by

$$\hat{X}(k/k-1) = [A(k,k-1) - M(k-1)K(k,k-1)C(k-1)] \cdot \hat{X}(k-1/k-2) + K(k,k-1)Z(k-1) \tag{8}$$

$$\hat{X}(0) - E[X(0)] = 0.$$

b) $K(k,k-1)$ is an $n \times m$ matrix given by the relation

$$K(k,k-1) = M(k-1)A(k,k-1) \cdot P(k-1)C'(k-1)R_v^{-1}(k-1). \tag{9}$$

c) $P(k)$, the variance of the one-stage prediction error $\tilde{X}(k/k-1)$, is given by the recursive relation

$$\begin{aligned} P(k) = {} & [A(k,k-1) - M(k-1)K(k,k-1)C(k-1)] \\ & \cdot P(k-1)A'(k,k-1) \\ & + B(k,k-1)Q(k-1)B'(k,k-1) \end{aligned} \tag{10}$$

$$P(0) = S(0) = S_0.$$

Proof:

1) The estimate $\hat{X}(k/k-1)$ can be written as a linear combination of $v(\ell)$, $\ell = 0,1,2,\ldots,(k-1)$:

$$\hat{X}(k/k-1) = \sum_{\ell=0}^{k-1} F(k,\ell)v(\ell) \tag{11}$$

where $F(k,\ell)$ is an $n \times m$ matrix such that $X(k/k-1)$ as in (11) satisfies (4) and (5).

Use of (11) in (5) yields

$$F(k,j) = E[X(k)v'(j)]R_v^{-1}(j) \tag{12}$$

$$j = 0,1,2,\ldots,(k-1).$$

Substituting for $F(k,\ell)$ from (12) in (11) and noting that $W(k-1)$ and $v(\ell)$, $\ell = 0,1,\ldots,(k-1)$ are independent, yields a modified form of (11) as

$$\begin{aligned} \hat{X}(k/k-1) &= A(k,k-1)\sum_{\ell=0}^{k-1} E[X(k-1)v'(\ell)]\, R^{-1}(\ell)v(\ell) \\ &= A(k,k-1)\sum_{\ell=0}^{k-2} E[X(k-1)v'(\ell)]\, R^{-1}(\ell)v(\ell) \\ &\quad + A(k,k-1)E[X(k-1)v'(k-1)] \\ &\quad \cdot R_v^{-1}(k-1)v(k-1) \\ &= A(k,k-1)\hat{X}(k-1/k-2) \\ &\quad + K(k,k-1)v(k-1) \end{aligned} \tag{13}$$

where we let

$$A(k,k-1)E[X(k-1)v'(k-1)]R_v^{-1}(k-1) = K(k,k-1).$$

2) But $K(k,k-1)$ must be such that it satisfies (9). This follows from the fact that

$$E[X(k-1)v'(k-1)] = M(k-1)P(k-1)C'(k-1).$$

3) Therefore, (13) is indeed the same equation as (8) when substitution for $v(k-1)$ in terms of $Z(k-1)$ and $\hat{X}(k-1/k-2)$ is made.

This completes the proof of Theorem 2 a) and b).

4) The proof of Theorem 2 c) is as follows:

$$\boxed{\begin{aligned} P(k) &= E[\tilde{X}(k/k-1)\tilde{X}'(k/k-1)] \\ &= E[\tilde{X}(k/k-1)X'(k)] \end{aligned}}$$

[using (5) and (11)]

$$\begin{aligned} &= E\{\tilde{X}(k/k-1)\,[A(k,k-1)X(k-1) \\ &\quad + B(k,k-1)W(k-1)]'\} \end{aligned}$$

[using (2)]

$$\begin{aligned} &= \{[A(k,k-1) - M(k-1)K(k,k-1)C(k-1)] \\ &\quad \cdot P(k-1)A'(k,k-1)\} \\ &\quad + \{B(k-1)Q(k-1)B'(k-1)\}. \end{aligned}$$

The proof of Theorem 2 is now complete.

Filtering ($\ell = k$)

The optimal linear recursive filtering algorithms are presented in the following theorem.

Theorem 3:

a) The optimal linear filtered estimate $\hat{X}(k/k)$ is given as

$$\boxed{\hat{X}(k/k) = \hat{X}(k/k-1) + K_f(k)[Z(k) - M(k)C(k)\hat{X}(k/k-1)].} \tag{14}$$

b) $K_f(k)$ is an $n \times m$ matrix given by

$$K_f(k) = M(k)P(k)C'(k)R_v^{-1}(k). \tag{15}$$

c) The variance of the filtering error $P_f(k)$ is given by

$$\begin{aligned} P_f(k) = {} & A(k,k-1)P_f(k-1)A'(k,k-1) \\ & - K_f(k)R_v(k)K_f'(k) \\ & + B(k,k-1)Q(k-1)B'(k,k-1) \end{aligned} \tag{16}$$

$$P_f(0) = S(0) = S_0.$$

Proof:

1) The estimate $\hat{X}(k/k)$ can be written as a linear combination of $\nu(\ell)$, $\ell = 0,1,2,\ldots,(k-1),k$:

$$\hat{X}(k/k) = \sum_{\ell=0}^{k} H(k,\ell)\nu(\ell) \quad (17)$$

where $H(k, 1)$ is an $n \times m$ matrix such that $\hat{X}(k/k)$ as in (17) satisfies (4) and (5).

Equation (12) is true, also, for $j = k$, with F replaced by H. Splitting the summation in (17) into two terms, one containing a summation up to $(k-1)$ terms and the other involving the kth term only, (17) can be rewritten as

$$\hat{X}(k/k) = \hat{X}(k/k-1) + E[X(k)\nu(k)]R_\nu^{-1}(k)\nu(k). \quad (18)$$

But,

$$\begin{aligned} E[X(k)\nu'(k)] &= E\{[\hat{X}(k/k-1) + \tilde{X}(k/k-1)] \\ &\quad \cdot [M(k)C(k)\hat{X}(k/k-1) \\ &\quad + U(k) - M(k)C(k)X(k) + V(k)]'\} \\ &= M(k)P(k)C'(k) \end{aligned}$$

by using the facts that $V(k)$ is future noise and $E[U(k) - M(k)]$ equals zero.

Denoting $M(k)P(k)C'(k)R_\nu^{-1}(k)$ by $K_f(k)$ and substituting for $\nu(k)$ in (18) yields the filtered estimate $\hat{X}(k/k)$ as in (14).

This completes the proof of Theorem 3 a) and b).

2) The proof of Theorem 3 c) is conveniently derived if the following Lemma is made use of.

Lemma: The one stage prediction estimate $\hat{X}(k/k-1)$ and the filtered estimate $\hat{X}(k-1/k-1)$ are related in the following manner:

$$\hat{X}(k/k-1) = A(k,k-1)\hat{X}(k-1/k-1). \quad (19)$$

This can easily be proved by making use of the obvious relation

$$K(k+1,k) = A(k+1,k)K_f(k) \quad (20)$$

and will not be presented here.

The proof of Theorem 3 c) follows from

$$\begin{aligned} P_f(k) &= E[\tilde{X}(k/k)\tilde{X}'(k/k)] \\ &= E[X(k)X'(k)] - E[\hat{X}(k/k)\hat{X}'(k/k)] \end{aligned}$$

and using the lemma to obtain $E[\hat{X}(k/k)\hat{X}'(k/k)]$.

This completes the proof of Theorem 3.

Smoothing ($\ell > k$)

Only the basic smoothing equation is presented, as specializing to the three different kinds [9] of smoothing should be straightforward.

Theorem 4: The optimum linear smoothed estimate $\hat{X}(k/\ell)$ is given by

$$\hat{X}(k/\ell) = \hat{X}(k/k) + P(k)Y(k) \quad (21)$$

where

$$Y(k) = \sum_{j=k+1}^{\ell} M(j)L'(j,k)C'(j)R_\nu^{-1}(j)\nu(j) \quad (22)$$

and

$$L(j,k) = \prod_{i=k}^{j-1} [A(i+1,i) - M(i)K(i+1)C(i)].$$

Proof: The estimate $\hat{X}(k/\ell)$ can be written as a linear combination of $\nu(j)$, $j = 0,1,\ldots,k,\ldots,(\ell-1),\ell$:

$$\hat{X}(k/\ell) = \sum_{j=0}^{\ell} G(k,j)\nu(j) \quad (23)$$

where $G(k, j)$ is an $n \times m$ matrix such that $\hat{X}(k/\ell)$ as in (23) satisfies (4) and (5).

Equation (12) is true for $j = 0,1,2,\ldots,\ell$ with F replaced by G. Splitting the summation in (23) into two summations, one containing terms up to k inclusive and one containing terms $(k+1),\ldots,\ell$, (23) can be written as

$$\hat{X}(k/\ell) = \hat{X}(k/k) + \sum_{j=k+1}^{\ell} E[X(k)\nu'(j)]R_\nu^{-1}(j)\nu(j). \quad (24)$$

But,

$$\begin{aligned} E[X(k)\nu'(j) &= E\{[\tilde{X}(k/k-1) + \hat{X}(k/k-1)] \\ &\quad \cdot [M(j)C(j)\tilde{X}(j/j-1) \\ &\quad + \left(U(j) - M(j)\right)C(j)X(j) + V(j)]'\} \\ &\qquad \text{for } j = (k+1),\ldots,\ell \\ &= M(j)P(k,j)C'(j) \end{aligned} \quad (25)$$

where

$$\begin{aligned} P(k,j) &= E[\tilde{X}(k/k-1)\tilde{X}'(j/j-1)] \\ &= P(k)L'(j,k) \end{aligned} \quad (26)$$

by writing $\tilde{X}(j/j-1)$ in terms of $\tilde{X}(k/k-1)$, and

$$L(j,k) = \prod_{i=k}^{j-1} [A(i+1,i) - M(k)K(i+1,i)C(i)].$$

Let

$$Y(k) = \sum_{j=k+1}^{\ell} M(j)L'(j, k)C'(j)R_\nu^{-1}(j)\nu(j).$$

Using (25) and (26) and the above relation in (24) yields (21).

This completes the proof of Theorem 4.

III. Continuous Time Estimation

Problem Statement

The signal to be processed at time t is

$$Z(t) = U(t)C(t)X(t) + V(t) \tag{27a}$$

where

1) $\boldsymbol{X}(t)$, an $n \times 1$ vector stochastic signal, is the evolution of the linear differential equation

$$\dot{X}(t) = A(t)X(t) + B(t)W(t) \tag{27b}$$

where $W(t)$ is an $r \times 1$ vector white process with zero mean and $E[W(t)W'(u)] = Q(t)\delta(t-u)$;

$X(0)$ is an $n \times 1$ vector, random variable with zero mean and covariance S_0;

$A(t)$ is an $n \times n$ matrix and $B(t)$ is an $n \times r$ matrix.

2) $V(t)$ is an $m \times 1$ vector white process with zero mean and $E[V(t)V'(u)] = R(t)\delta(t-u)$.
3) $U(t)$, the multiplicative noise, is a scalar white process with nonzero mean $M(t)$ and variance $N(t)$.
4) $C(t)$ is an $m \times n$ matrix.
5) Processes $U(t)$, $V(t)$, $W(t)$ and the random variable $X(0)$ are statistically independent of each other.

The problem is to obtain an estimate $\hat{X}(t/b)$ of the signal $X(t)$ by a linear operation of $Z(u), 0 \leqslant u < b$ such that

$$E\{[X(t) - \hat{X}(t/b)]\,[X(t) - \hat{X}(t/b)]'\} = E[\tilde{X}(t/b)\tilde{X}'(t/b)]$$

is minimum.

Note that $b = t$ is the filtering case and $b > t$ is smoothing.

Solution

The necessary and sufficient conditions to be satisfied by the optimum estimate $\hat{X}(t/b)$ can easily be shown as

$$E[\tilde{X}(t/b)Z'(u)] = 0 \quad 0 \leqslant u < b. \tag{28}$$

In terms of the innovations process $\nu(t)$ for this problem, (28) can be restated as

$$E[\tilde{X}(t/b)\nu'(u)] = 0 \quad 0 \leqslant u < b. \tag{29}$$

A theorem, useful for the derivation of the filtering and smoothing algorithms, about the innovations process will now be stated.

Theorem 5: The innovation process

$$\nu(t) = Z(t) - M(t)C(t)\hat{X}(t/t) \tag{30}$$

is zero mean and white with

$$E[\nu(t)\nu'(u)] = R_\nu(t)\delta(t-u) \tag{31}$$

where

$$R_\nu(t) = R(t) + N(t)C(t)S(t)C'(t)$$

and

$$S(t) = \text{Var}\,[X(t)].$$

Proof: The proof is similar to that of Theorem 1 and, therefore, will not be presented.

Filtering ($b = t$)

The optimum linear filtering algorithms are given in Theorem 6.

Theorem 6:

a) The optimum linear filtered estimate $\hat{X}(t/t)$ is the evolution of the linear differential equation

$$\dot{\hat{X}}(t/t) = [A(t) - M(t)K(t)C(t)]\,\hat{X}(t/t) + K(t)Z(t)$$

$$X(0) = E\,[X(0)] = 0. \tag{32}$$

b) $K(t)$ is an $n \times m$ matrix given by the relation

$$K(t) = M(t)P(t)C'(t)R_\nu^{-1}(t). \tag{33}$$

c) $P(t)$, the variance of the filtering error $\tilde{X}(t/t)$, is the evolution of the differential equation

$$\dot{P}(t) = A(t)P(t) + P(t)A'(t) - M^2(t)P(t)C'(t)R_\nu^{-1}(t)C(t)P(t) - B(t)Q(t)B'(t) \tag{34}$$

$$P(0) = S(0) = S_0.$$

Proof:

1) The optimum linear filtered estimate $\hat{X}(t/t)$ can be written as a linear operation on the innovations as

$$\hat{X}(t/t) = \int_0^t H(t, s)\nu(s)ds \tag{35}$$

where $H(t, s)$ is the impulse response of a linear dynamical system such that $\hat{X}(t/t)$ as in (35) satisfies (29). Use of (29) with $\hat{X}(t/t)$ as in (35) yields

$$H(t, s) = E[X(t)\nu'(s)]\,R_\nu^{-1}(s) \tag{36}$$

for $s < t$.

Therefore, (35) can be rewritten as

$$\hat{X}(t/t) = \int_0^t E[X(t)v'(s)]R_v^{-1}(s)v(s)ds. \tag{37}$$

Differentiating the above equation with respect to t and making use of (27b) and (36), and denoting $E[X(t)v'(t)]R_v^{-1}(t)$ as $K(t)$, one obtains

$$\dot{\hat{X}}(t/t) = A(t)\hat{X}(t/t) + K(t)v(t). \tag{38}$$

2) $K(t)$ is given by

$$\begin{aligned} K(t) &= E[X(t)v'(t)]R_v^{-1}(t) \\ &= E[(\hat{X}(t/t) + \tilde{X}(t/t))v'(t)]R_v^{-1}(t) \\ &= M(t)P(t)C'(t)R_v^{-1}(t). \end{aligned}$$

Now, substituting for $v(t)$ in (38), (32) is easily obtained.

This completes the proof of Theorem 6 a) and b).

3) To prove Theorem 6 c), $P(t)$ is written as

$$\begin{aligned} P(t) &= E[\tilde{X}(t/t)\tilde{X}'(t/t)] \\ &= E[X(t)X'(t)] - E[\hat{X}(t/t)\hat{X}'(t/t)]. \end{aligned}$$

Differentiating the above expression with respect to time yields

$$\dot{P}(t) = \frac{d}{dt}E[X(t)X'(t)] - \frac{d}{dt}E[\hat{X}(t/t)\hat{X}'(t/t)].$$

Substituting for the two terms on the right-hand side of the above expression as

$$\begin{aligned} \frac{d}{dt}E[X(t)X'(t)] &= A(t)E[X(t)X'(t)] + E[X(t)X'(t)]A'(t) \\ &\quad + B(t)Q(t)B'(t) \end{aligned}$$

and

$$\begin{aligned} \frac{d}{dt}E[\hat{X}(t/t)\hat{X}'(t/t)] &= A(t)E[\hat{X}(t/t)\tilde{X}'(t/t) \\ &\quad + E[\hat{X}(t/t)\hat{X}'(t/t)]A'(t) \\ &\quad + K(t)R(t)K'(t) \end{aligned}$$

and substituting for $K(t)$ from (33) yields (34).

This completes the proof of Theorem 6.

Smoothing ($b > t$)

Theorem 7: The optimum linear smoothed estimate $\hat{X}(t/b)$ is given by

$$\hat{X}(t/b) = \hat{X}(t/t) + P(t)Y(t) \tag{39}$$

where

$$Y(t) = \int_t^b M(s)L'(s, t)C'(s)v(s)ds \tag{40}$$

and $L(s, t)$ is the state transition matrix corresponding to $A(t) - M(t)K(t)C(t)$.

Proof: The optimum smoothed estimate (linear) $\hat{X}(t/b)$ can be written as

$$\hat{X}(t/b) = \int_0^b G(t, s)v(s)ds \tag{41}$$

where $G(t, s)$, an $n \times m$ matrix, is such that $\hat{X}(t/b)$ as in (41) satisfies (29).

Use of (29) with $X(t/b)$ as in (41) yields

$$G(t, s) = E[X(t)v'(s)]R_v^{-1}(s) \tag{42}$$

for $0 \leqslant s < b$.

Use Theorem 6 to write

$$\hat{X}(t/b) = \hat{X}(t/t) + \int_t^b E[X(t)v'(s)]R_v^{-1}(s)v(s)ds. \tag{43}$$

But,

$$\begin{aligned} E[X(t)v'(s)] &= E\{[\hat{X}(t/t) + \tilde{X}(t/t)][M(s)C(s)\tilde{X}(s/s) \\ &\quad + (U(s) - M(s))C(s)X(s) + V(s)]'\} \\ &= M(s)E[\tilde{X}(t/t)\tilde{X}'(s/s)]C'(s) \quad s > t \\ &= M(s)P(t)L'(s, t)C'(s). \end{aligned} \tag{44}$$

Denoting

$$\int_t^b M(s)L'(s, t)C'(s)v(s)ds$$

by $Y(t)$ and making use of (44), (43) is easily rewritten as in (39).

This completes the proof of Theorem 7.

Specialization of the basic smoothing equation (39) to the three different kinds of smoothing is straightforward and will not be presented here. The error variance equations are also easily derived for these.

IV. Conclusions

The problem of estimation in the presence of multiplicative noise was treated in detail. Although the model used is an ideal one, the solution should provide the first step in solving actual physical problems which present serious modeling difficulties.

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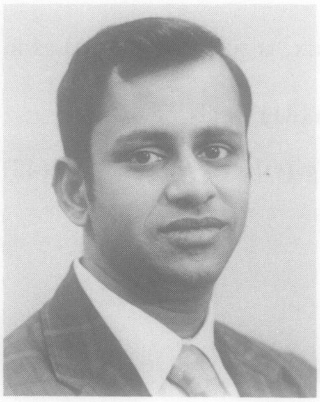

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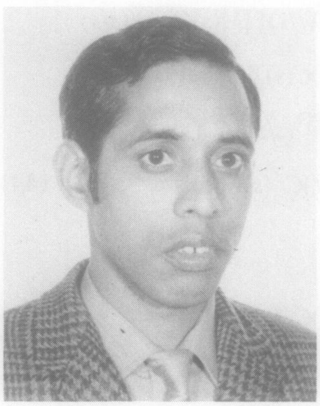

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