Collection of Proofs

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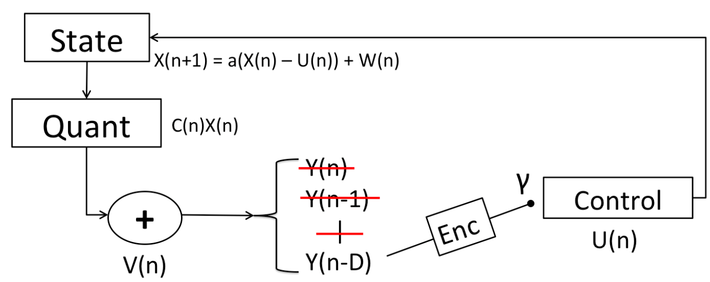
System:



$$X(n+1) = a(X(n) - U(n)) + W(n)$$

$$Y(n) = \begin{cases} Q(X(n)) + V(n) & : n \equiv 0 \pmod{\mathrm{D}+1} \\ 0 & : \text{else} \end{cases}$$

$$Q(X(n)) = C(n)X(n)$$

$$U(n) = \begin{cases} L[X(n) \mid Y(n-D)] & : n \equiv 0 \pmod{\mathrm{D}+1} \\ 0 & : \text{else} \end{cases}$$

Contents

New Cost Function – Penalize Large Control Power

Optimal Control – No V(n), $\gamma = 1$, No Delay

$$\min\ \mathbb{E}[x^2(n+1)] + \sum_{k=1}^{n} u^2(k)$$

$\min\ \mathbb{E}[a(x(n) - u(n)) + w(n))^2] + \sum_{k=1}^{n} u^2(k)$
$= \min\ \mathbb{E}[(a(x(n) - \alpha(n)y(n)) + w(n))^2] + \sum_{k=1}^{n} \alpha^2(k)y^2(k)$
$= \min\ \mathbb{E}[(a(x(n) - \alpha(n)c(n)x(n)) + w(n))^2] + \sum_{k=1}^{n} \alpha^2(k)c^2(k)x^2(k)$
$= \min\ \mathbb{E}[a^2(1 - \alpha c(n))^2 x^2(n)] + \sigma_w^2 + \sum_{k=1}^{n} \alpha^2(k)y^2(k)$
$\frac{d}{d\alpha(n)}\mathbb{E}[a^2(1-\alpha(n)c(n))^2x^2(n)]+\sigma_w^2+\sum_{k=1}^{n} \alpha^2(k)y^2(k) = \mathbb{E}[2a^2(1-\alpha(n)c(n))(-c(n))x^2(n)]+ 2\alpha(n)y^2(n) = 0$
$-a^2(\mu_c\sigma^2_{x(n)} - \alpha(n)(\mu_c^2 + \sigma_c^2)\sigma^2_{x(n)}) + \alpha(n)y^2(n) = 0$
$\alpha(n)(a^2(\mu_c^2 + \sigma_c^2)\sigma^2_{x(n)} + y^2(n)) = a^2\mu_c\sigma^2_{x(n)}$

$$\alpha(n) = \frac{a^2\mu_c\sigma^2_{x(n)}}{a^2(\mu_c^2 + \sigma_c^2)\sigma^2_{x(n)} + y^2(n)}$$

This is easy to implement, and my next step is to implement this. My worry is that this isn't a closed solution like the previous α calculations. As time passes, α will change every time a new control is implemented–which is fine, but we should note the new complication. Does this answer make sense? Should I go ahead and implement this?

A Bound – No V(n), $\gamma = 1$, No Delay

$$\alpha(n) = \frac{a^2\mu_c\sigma^2_{x(n)}}{a^2(\mu_c^2+\sigma_c^2)\sigma^2_{x(n)} + y^2(n)}$$

$\mathbb{E}[x^2(n+1)]$
$= \mathbb{E}[(ax(n) - a\alpha y(n) + w(n))^2]$
$= \mathbb{E}[(ax(n) - a\alpha c(n)x(n) + w(n))^2]$
$= \mathbb{E}[(a(1-\alpha c(n))x(n) + w(n))^2]$
$a^2(1 - 2\alpha\mu_c + \alpha^2(\mu_c^2+\sigma_c^2)) < 1$

$$a^2\left(1 - \frac{2a^2\mu_c^2\sigma^2_{x(n)}}{a^2(\mu_c^2+\sigma_c^2)\sigma^2_{x(n)} + y^2(n)} + \frac{a^4\mu_c^2\sigma^4_{x(n)}(\mu_c^2+\sigma_c^2)}{(a^2(\mu_c^2+\sigma_c^2)\sigma^2_{x(n)} + y^2(n))^2}\right) < 1$$

As this is similar to original cost function with V(n), I don't believe this is easily simplifiable. This should just be calculated in code/implementation.

Original Cost Function: Minimize Mean Square State

Optimal Control – No V(n), $\gamma = 1$, Delay

$$\min\ \mathbb{E}[x^2(n+1)]$$

$\min\ \mathbb{E}[(a(x(n) - \alpha y(n-D)) + w(n))^2]$
$= \min\ \mathbb{E}[(a(x(n) - \alpha c(n-D)x(n-D)) + w(n))^2]$
$= \min\ \mathbb{E}[(a(a^D x(n-D) + \sum_{i=1}^{D} a^{i-1}w(n-i) - \alpha c(n-D)x(n-D)) + w(n))^2]$
$= \min\ \mathbb{E}[(a(a^D - \alpha c(n-D)x(n-D) + \sum_{i=0}^{D} a^i w(n-i))^2]$
$\frac{d}{d\alpha} \downarrow = \mathbb{E}[2a^2(a^D - \alpha c(n-D))x^2(n-D)(-c(n-D))] = 0$
$\mathbb{E}[c(n-D)(a^D - \alpha c(n-D))x^2(n-D)] = 0$
$a^D\mu_c\sigma^2_{x(n-D)} = \alpha(\mu_c^2+\sigma_c^2)\sigma^2_{x(n-D)}$

$$\alpha = \frac{a^D\mu_c}{\mu_c^2+\sigma_c^2}$$

The addition of delay simply means our control has to project into the future (by a scaling factor) for when it will be applied.

A Bound – No V(n), $\gamma = 1$, Delay

$$\alpha = \frac{a^D \mu_c}{\mu_c^2 + \sigma_c^2}$$

$\mathbb{E}[(a(x(n) - \alpha y(n-D)) + w(n))^2]$
$= \mathbb{E}[(ax(n) - a\alpha y(n-D) + w(n))^2]$
$= \mathbb{E}[(a^{D+1}x(n-D) - a\alpha y(n-D) + \sum_{i=0}^{D} a^i w(n-i))^2]$
$= \mathbb{E}[(a^{D+1}x(n-D) - a\alpha c(n-D)x(n-D) + \sum_{i=0}^{D} a^i w(n-i))^2]$
$= \mathbb{E}[(a^{D+1} - a\alpha c(n-D))^2 x^2(n-D)] + \mathbb{E}[\sum_{i=0}^{D} a^{2i} w^2(n-i)]$
$a^{2(D+1)} - 2a^{D+2}\alpha\mu_c + a^2\alpha^2(\mu_c^2 + \sigma_c^2) < 1$
$a^{2(D+1)} - \frac{2a^{2(D+1)}\mu_c^2}{\mu_c^2+\sigma_c^2} + \frac{a^{2(D+1)}\mu_c^2}{\mu_c^2+\sigma_c^2} < 1$
$a^{2(D+1)} - \frac{a^{2(D+1)}\mu_c^2}{\mu_c^2+\sigma_c^2} < 1 \;\rightarrow\; a^{2(D+1)}\left(1 - \frac{\mu_c^2}{\mu_c^2+\sigma_c^2}\right) < 1 \;\rightarrow\; a^{2(D+1)} < \frac{\mu_c^2+\sigma_c^2}{\sigma_c^2}$

$$a^{2(D+1)} < \frac{\mu_c^2 + \sigma_c^2}{\sigma_c^2} \qquad a < \left(\frac{\mu_c^2 + \sigma_c^2}{\sigma_c^2}\right)^{\frac{1}{2(D+1)}}$$

Optimal Control – No V(n), $\gamma = 1$, No Delay

$$\min \mathbb{E}[x^2(n+1)]$$

$$u(n) = \alpha y(n) = \alpha c(n) x(n)$$

$\min \mathbb{E}[(a(x(n) - u(n)) + w(n))^2]$
$= \min \mathbb{E}[(a(x(n) - \alpha y(n)) + w(n))^2]$
$= \min \mathbb{E}[(a(x(n) - \alpha c(n)x(n)) + w(n))^2]$
$= \min \mathbb{E}[a^2(1 - \alpha c(n))^2 x^2(n)] + \sigma_w^2$
$\frac{d}{d\alpha}\mathbb{E}[a^2(1 - \alpha c(n))^2 x^2(n)] + \sigma_w^2 = \mathbb{E}[2a^2(1 - \alpha c(n))(-c(n))x^2(n)] = 0$
$\mathbb{E}[c(n)x^2(n) - \alpha c^2(n)x^2(n)] = 0$
$\mu_c \sigma_{x(n)}^2 = \alpha(\mu_c^2 + \sigma_c^2)\sigma_{x(n)}^2$

$$\alpha = \frac{\mu_c}{\mu_c^2 + \sigma_c^2}$$

These are the results we expect, from Gireeja's Noncoherence Paper.

Optimal Control – V(n), $\gamma = 1$, Delay

$$\min \mathbb{E}[x^2(n+1)]$$

$\min \mathbb{E}[(ax(n) - a\alpha y(n-D) + w(n))^2]$
$= \min \mathbb{E}[(a^{D+1}x(n-D) + \sum_{i=0}^{D} a^i w(n-i) - a\alpha y(n-D))^2]$
$= \min \mathbb{E}[(a^{D+1}x(n-D) - a\alpha c(n-D)x(n-D) - a\alpha v(n-D) + \sum_{i=0}^{D} a^i w(n-i))^2]$
$\frac{d}{d\alpha} \downarrow = \mathbb{E}[-2a^{D+2}c(n-D)x^2(n-D) + 2\alpha a^2 c^2(n-D)x^2(n-D) + 2a^2\alpha v^2(n-D)] = 0$

$- a^{D+2}\mu_c\sigma^2_{x(n-D)} + \alpha a^2(\mu_c^2 + \sigma_c^2)\sigma^2_{x(n-D)} + \alpha a^2\sigma_v^2 = 0$
$\alpha((\mu_c^2 + \sigma_c^2)\sigma^2_{x(n-D)} + \sigma_v^2) = a^D\mu_c\sigma^2_{x(n-D)}$

$$\alpha(n) = \frac{a^D\mu_c\sigma^2_{x(n-D)}}{(\mu_c^2 + \sigma_c^2)\sigma^2_{x(n-D)} + \sigma_v^2}$$

A Bound – V(n), $\gamma = 1$, Delay

$$\alpha(n) = \frac{a^D\mu_c\sigma^2_{x(n-D)}}{(\mu_c^2 + \sigma_c^2)\sigma^2_{x(n-D)} + \sigma_v^2}$$

$\mathbb{E}[a^{2(D+1)}x^2(n-D) - 2a^{D+2}\alpha c(n-D)x^2(n-D) + a^2\alpha^2c^2(n-D)x^2(n-D)]$
$a^{2(D+1)} - 2a^{D+2}\alpha\mu_c + a^2\alpha^2(\mu_c^2 + \sigma_c^2) < 1$
$a^{2(D+1)} - \frac{2a^{2(D+1)}\mu_c\sigma^2_{x(n-D)}}{(\mu_c^2+\sigma_c^2)\sigma^2_{x(n-D)}+\sigma_v^2} + \frac{a^{2(D+1)}\mu_c^2\sigma^4_{x(n-D)}(\mu_c^2+\sigma_c^2)}{((\mu_c^2+\sigma_c^2)\sigma^2_{x(n-D)}+\sigma_v^2)^2} < 1$

$$a^{2(D+1)}\left(1 - \frac{2\mu_c^2\sigma^2_{x(n-D)}}{(\mu_c^2 + \sigma_c^2)\sigma^2_{x(n-D)} + \sigma_v^2} + \frac{\mu_c^2\sigma^4_{x(n-D)}(\mu_c^2 + \sigma_c^2)}{((\mu_c^2 + \sigma_c^2)\sigma^2_{x(n-D)} + \sigma_v^2)^2}\right) < 1$$

This is really ugly. I started looking into simplifying it, but I don't think it can be simplified further. It would be best to directly calculate everything out in implementation/code.

Optimal Control – V(n) $\gamma = 1$, No Delay

$$\min\ \mathbb{E}[x^2(n+1)]$$

$\min\ \mathbb{E}[(a(1-\alpha c(n))x(n) - a\alpha v(n) + w(n))^2]$
$= \min\ \mathbb{E}[a^2(1-\alpha c(n))^2x^2(n) + \mathbb{E}[a^2\alpha^2v^2(n)] + \sigma_w^2$
$\frac{d}{d\alpha}\mathbb{E}[a^2(1-\alpha c(n))^2x^2(n) + \mathbb{E}[a^2\alpha^2v^2(n)] + \sigma_w^2 = \mathbb{E}[-2a^2(1-\alpha c(n))c(n)x^2(n)] + 2a^2\alpha\sigma_v^2 = 0$
$= -\mu_c\sigma^2_{x(n)} + \alpha(\mu_c^2 + \sigma_c^2)\sigma^2_{x(n)} + \alpha\sigma_v^2 = 0$

$$\alpha(n) = \frac{\mu_c\sigma^2_{x(n)}}{(\mu_c^2 + \sigma_c^2)\sigma^2_{x(n)} + \sigma_v^2}$$

These are the results we expect from the LLSE theorem: $\alpha = \frac{cov(X,Y)}{var(Y)} = \frac{\mathbb{E}[XY]}{\mathbb{E}[Y^2]} \leftarrow$ assuming $\mathbb{E}[X] = \mathbb{E}[Y] = 0$