Collection of Proofs

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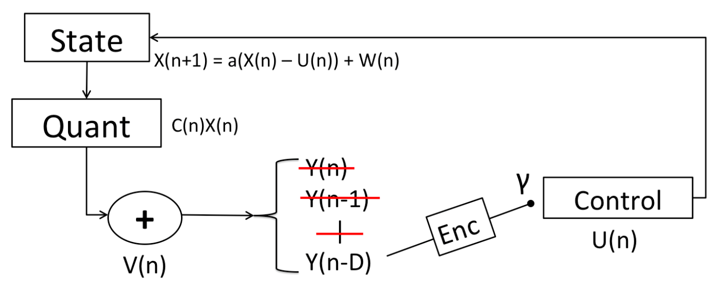
System:



$$X(n+1) = a(X(n) - U(n)) + W(n)$$

$$Y(n) = \begin{cases} Q(X(n)) + V(n) & : n \equiv 0 \ (\text{mod D} + 1) \\ 0 & : \text{else} \end{cases}$$

$$Q(X(n)) = C(n)X(n)$$

$$U(n) = \begin{cases} L[X(n) \mid Y(n-D)] & : n \equiv 0 \ (\text{mod D} + 1) \\ 0 & : \text{else} \end{cases}$$

Contents

New Cost Function – Penalize Large Control Power

Optimal Control – No V(n), $\gamma = 1$, No Delay

$$\min \ \mathbb{E}[x^2(n+1)] + \sum_{k=1}^{n} u^2(k)$$

$\min \ \mathbb{E}[a(x(n) - u(n)) + w(n))^2] + \sum_{k=1}^{n} u^2(k)$
$= \min \ \mathbb{E}[(a(x(n) - \alpha(n)y(n)) + w(n))^2] + \sum_{k=1}^{n} \alpha^2(k)y^2(k)$
$= \min \ \mathbb{E}[(a(x(n) - \alpha(n)c(n)x(n)) + w(n))^2] + \sum_{k=1}^{n} \alpha^2(k)c^2(k)x^2(k)$
$= \min \ \mathbb{E}[a^2(1 - \alpha c(n))^2 x^2(n)] + \sigma_w^2 + \sum_{k=1}^{n} \alpha^2(k)y^2(k)$
$\frac{d}{d\alpha(n)}\mathbb{E}[a^2(1-\alpha(n)c(n))^2x^2(n)]+\sigma_w^2+\sum_{k=1}^{n} \alpha^2(k)y^2(k) = \mathbb{E}[2a^2(1-\alpha(n)c(n))(-c(n))x^2(n)]+ 2\alpha(n)y^2(n) = 0$
$- a^2(\mu_c\sigma_{x(n)}^2 - \alpha(n)(\mu_c^2 + \sigma_c^2)\sigma_{x(n)}^2) + \alpha(n)y^2(n) = 0$
$\alpha(n)(a^2(\mu_c^2 + \sigma_c^2)\sigma_{x(n)}^2 + y^2(n)) = a^2\mu_c\sigma_{x(n)}^2$

$$\alpha(n) = \frac{a^2\mu_c\sigma_{x(n)}^2}{a^2(\mu_c^2 + \sigma_c^2)\sigma_{x(n)}^2 + y^2(n)}$$

This is easy to implement, and my next step is to implement this. My worry is that this isn't a closed solution like the previous α calculations. As time passes, α will change every time a new control is implemented–which is fine, but we should note the new complication. Does this answer make sense? Should I go ahead and implement this?

A Bound – No V(n), $\gamma = 1$, No Delay

$$\alpha(n) = \frac{a^2\mu_c\sigma^2_{x(n)}}{a^2(\mu_c^2+\sigma_c^2)\sigma^2_{x(n)} + y^2(n)}$$

$\mathbb{E}[x^2(n+1)]$
$= \mathbb{E}[(ax(n) - a\alpha y(n) + w(n))^2]$
$= \mathbb{E}[(ax(n) - a\alpha c(n)x(n) + w(n))^2]$
$= \mathbb{E}[(a(1-\alpha c(n))x(n) + w(n))^2]$
$a^2(1 - 2\alpha\mu_c + \alpha^2(\mu_c^2+\sigma_c^2)) < 1$

$$a^2\left(1 - \frac{2a^2\mu_c^2\sigma^2_{x(n)}}{a^2(\mu_c^2+\sigma_c^2)\sigma^2_{x(n)} + y^2(n)} + \frac{a^4\mu_c^2\sigma^4_{x(n)}(\mu_c^2+\sigma_c^2)}{(a^2(\mu_c^2+\sigma_c^2)\sigma^2_{x(n)} + y^2(n))^2}\right) < 1$$

As this is similar to original cost function with V(n), I don't believe this is easily simplifiable. This should just be calculated in code/implementation.

Original Cost Function: Minimize Mean Square State

Optimal Control – No V(n), $\gamma = 1$, Delay

$$\min\ \mathbb{E}[x^2(n+1)]$$

$\min\ \mathbb{E}[(a(x(n) - \alpha y(n-D)) + w(n))^2]$
$= \min\ \mathbb{E}[(a(x(n) - \alpha c(n-D)x(n-D)) + w(n))^2]$
$= \min\ \mathbb{E}[(a(a^D x(n-D) + \sum_{i=1}^{D} a^{i-1}w(n-i) - \alpha c(n-D)x(n-D)) + w(n))^2]$
$= \min\ \mathbb{E}[(a(a^D - \alpha c(n-D)x(n-D) + \sum_{i=0}^{D} a^i w(n-i))^2]$
$\frac{d}{d\alpha} \downarrow = \mathbb{E}[2a^2(a^D - \alpha c(n-D))x^2(n-D)(-c(n-D))] = 0$
$\mathbb{E}[c(n-D)(a^D - \alpha c(n-D))x^2(n-D)] = 0$
$a^D\mu_c\sigma^2_{x(n-D)} = \alpha(\mu_c^2+\sigma_c^2)\sigma^2_{x(n-D)}$

$$\alpha = \frac{a^D\mu_c}{\mu_c^2+\sigma_c^2}$$

The addition of delay simply means our control has to project into the future (by a scaling factor) for when it will be applied.

A Bound – No V(n), $\gamma = 1$, Delay

$$\alpha = \frac{a^D \mu_c}{\mu_c^2 + \sigma_c^2}$$

$\mathbb{E}[(a(x(n) - \alpha y(n-D)) + w(n))^2]$
$= \mathbb{E}[(ax(n) - a\alpha y(n-D) + w(n))^2]$
$= \mathbb{E}[(a^{D+1}x(n-D) - a\alpha y(n-D) + \sum_{i=0}^{D} a^i w(n-i))^2]$
$= \mathbb{E}[(a^{D+1}x(n-D) - a\alpha c(n-D)x(n-D) + \sum_{i=0}^{D} a^i w(n-i))^2]$
$= \mathbb{E}[(a^{D+1} - a\alpha c(n-D))^2 x^2(n-D)] + \mathbb{E}[\sum_{i=0}^{D} a^{2i} w^2(n-i)]$
$a^{2(D+1)} - 2a^{D+2}\alpha\mu_c + a^2\alpha^2(\mu_c^2 + \sigma_c^2) < 1$
$a^{2(D+1)} - \frac{2a^{2(D+1)}\mu_c^2}{\mu_c^2+\sigma_c^2} + \frac{a^{2(D+1)}\mu_c^2}{\mu_c^2+\sigma_c^2} < 1$
$a^{2(D+1)} - \frac{a^{2(D+1)}\mu_c^2}{\mu_c^2+\sigma_c^2} < 1 \;\rightarrow\; a^{2(D+1)}\left(1 - \frac{\mu_c^2}{\mu_c^2+\sigma_c^2}\right) < 1 \;\rightarrow\; a^{2(D+1)} < \frac{\mu_c^2+\sigma_c^2}{\sigma_c^2}$

$$a^{2(D+1)} < \frac{\mu_c^2 + \sigma_c^2}{\sigma_c^2} \qquad a < \left(\frac{\mu_c^2 + \sigma_c^2}{\sigma_c^2}\right)^{\frac{1}{2(D+1)}}$$

Optimal Control – No V(n), $\gamma = 1$, No Delay

$$\min\ \mathbb{E}[x^2(n+1)]$$

$$u(n) = \alpha y(n) = \alpha c(n)x(n)$$

$\min\ \mathbb{E}[(a(x(n) - u(n)) + w(n))^2]$
$= \min\ \mathbb{E}[(a(x(n) - \alpha y(n)) + w(n))^2]$
$= \min\ \mathbb{E}[(a(x(n) - \alpha c(n)x(n)) + w(n))^2]$
$= \min\ \mathbb{E}[a^2(1 - \alpha c(n))^2 x^2(n)] + \sigma_w^2$
$\frac{d}{d\alpha}\mathbb{E}[a^2(1 - \alpha c(n))^2 x^2(n)] + \sigma_w^2 = \mathbb{E}[2a^2(1 - \alpha c(n))(-c(n))x^2(n)] = 0$
$\mathbb{E}[c(n)x^2(n) - \alpha c^2(n)x^2(n)] = 0$
$\mu_c\sigma_{x(n)}^2 = \alpha(\mu_c^2 + \sigma_c^2)\sigma_{x(n)}^2$

$$\alpha = \frac{\mu_c}{\mu_c^2 + \sigma_c^2}$$

These are the results we expect, from Gireeja's Noncoherence Paper.

Optimal Control – V(n), $\gamma = 1$, Delay

$$\min\ \mathbb{E}[x^2(n+1)]$$

$\min\ \mathbb{E}[(ax(n) - a\alpha y(n-D) + w(n))^2]$
$= \min\ \mathbb{E}[(a^{D+1}x(n-D) + \sum_{i=0}^{D} a^i w(n-i) - a\alpha y(n-D))^2]$
$= \min\ \mathbb{E}[(a^{D+1}x(n-D) - a\alpha c(n-D)x(n-D) - a\alpha v(n-D) + \sum_{i=0}^{D} a^i w(n-i))^2]$
$\frac{d}{d\alpha} \downarrow = \mathbb{E}[-2a^{D+2}c(n-D)x^2(n-D) + 2\alpha a^2 c^2(n-D)x^2(n-D) + 2a^2\alpha v^2(n-D)] = 0$

$-a^{D+2}\mu_c\sigma^2_{x(n-D)} + \alpha a^2(\mu_c^2+\sigma_c^2)\sigma^2_{x(n-D)} + \alpha a^2\sigma_v^2 = 0$
$\alpha((\mu_c^2+\sigma_c^2)\sigma^2_{x(n-D)} + \sigma_v^2) = a^D\mu_c\sigma^2_{x(n-D)}$

$$\alpha(n) = \frac{a^D\mu_c\sigma^2_{x(n-D)}}{(\mu_c^2+\sigma_c^2)\sigma^2_{x(n-D)} + \sigma_v^2}$$

A Bound – V(n), $\gamma = 1$, Delay

$$\alpha(n) = \frac{a^D\mu_c\sigma^2_{x(n-D)}}{(\mu_c^2+\sigma_c^2)\sigma^2_{x(n-D)} + \sigma_v^2}$$

$\mathbb{E}[a^{2(D+1)}x^2(n-D) - 2a^{D+2}\alpha(n)c(n-D)x^2(n-D) + a^2\alpha^2(n)c^2(n-D)x^2(n-D)]$
$a^{2(D+1)} - 2a^{D+2}\alpha(n)\mu_c + a^2\alpha^2(n)(\mu_c^2+\sigma_c^2) < 1$
$a^{2(D+1)} - \frac{2a^{2(D+1)}\mu_c\sigma^2_{x(n-D)}}{(\mu_c^2+\sigma_c^2)\sigma^2_{x(n-D)}+\sigma_v^2} + \frac{a^{2(D+1)}\mu_c^2\sigma^4_{x(n-D)}(\mu_c^2+\sigma_c^2)}{((\mu_c^2+\sigma_c^2)\sigma^2_{x(n-D)}+\sigma_v^2)^2} < 1$

$$a^{2(D+1)}\left(1 - \frac{2\mu_c^2\sigma^2_{x(n-D)}}{(\mu_c^2+\sigma_c^2)\sigma^2_{x(n-D)} + \sigma_v^2} + \frac{\mu_c^2\sigma^4_{x(n-D)}(\mu_c^2+\sigma_c^2)}{((\mu_c^2+\sigma_c^2)\sigma^2_{x(n-D)} + \sigma_v^2)^2}\right) < 1$$

This is really ugly. I started looking into simplifying it, but I don't think it can be simplified further. It would be best to directly calculate everything out in implementation/code.

Optimal Control – V(n) $\gamma = 1$, No Delay

$$\min\ \mathbb{E}[x^2(n+1)]$$

$\min\ \mathbb{E}[(a(1-\alpha c(n))x(n) - a\alpha v(n) + w(n))^2]$
$= \min\ \mathbb{E}[a^2(1-\alpha c(n))^2x^2(n) + \mathbb{E}[a^2\alpha^2v^2(n)] + \sigma_w^2$
$\frac{d}{d\alpha}\mathbb{E}[a^2(1-\alpha c(n))^2x^2(n) + \mathbb{E}[a^2\alpha^2v^2(n)] + \sigma_w^2 = \mathbb{E}[-2a^2(1-\alpha c(n))c(n)x^2(n)] + 2a^2\alpha\sigma_v^2 = 0$
$= -\mu_c\sigma^2_{x(n)} + \alpha(\mu_c^2+\sigma_c^2)\sigma^2_{x(n)} + \alpha\sigma_v^2 = 0$

$$\alpha(n) = \frac{\mu_c\sigma^2_{x(n)}}{(\mu_c^2+\sigma_c^2)\sigma^2_{x(n)} + \sigma_v^2}$$

These are the results we expect from the LLSE theorem: $\alpha = \frac{cov(X,Y)}{var(Y)} = \frac{\mathbb{E}[XY]}{\mathbb{E}[Y^2]} \leftarrow$ assuming $\mathbb{E}[X] = \mathbb{E}[Y] = 0$

With No Delay, Memoryless Control is Optimal - Full Noise

Setup:

$$
\begin{aligned}
x(n+1) &= a(x(n) - u(n)) \\
y(n) &= c(n)x(n) + v(n) \\
u(n) &= \alpha(n)y(n) + \beta(n)y(n-1)
\end{aligned}
$$

$\min \mathbb{E}[x^2(n+1)] = \min a^2 \mathbb{E}[(x(n) - u(n))^2]$
$= \min a^2\mathbb{E}[(x(n) - \alpha(n)y(n) - \beta(n)y(n-1))^2]$
$\frac{d}{d\beta(n)} a^2 \mathbb{E}[(x(n) - \alpha(n)y(n) - \beta(n)y(n-1))^2] =$
$a^2\, 2\, \mathbb{E}[(-y(n-1))(x(n) - \alpha(n)y(n) - \beta(n)y(n-1))] = 0$
$\mathbb{E}[x(n)y(n-1) - \alpha(n)y(n)y(n-1) - \beta(n)y^2(n-1)] = 0$
$\beta(n)\mathbb{E}[y^2(n-1)] = \mathbb{E}[x(n)y(n-1) - \alpha(n)(c(n)x(n) + v(n))y(n-1)]$
$\beta(n)\mathbb{E}[y^2(n-1)] = \mathbb{E}[(1 - \alpha(n)c(n))x(n)y(n-1)]$
$\beta(n)\mathbb{E}[y^2(n-1)] = \mathbb{E}[1 - \alpha(n)c(n)]\ \mathbb{E}[x(n)y(n-1)]$

From here, we can see that $\mathbb{E}[x(n)y(n-1)] = 0$ or is uncorrelated. This is because when there is no delay, the content of the previous observation has already been killed by the current timestep, so the current timestep state is uncorrelated with all previous observations. In other words, there is no need for memory because from the perspective of the controller, it has succeeded in bringing the state to 0. I will now prove $\mathbb{E}[x(n)y(n-1)] = 0$, just to be sure.

$\mathbb{E}[x(n)y(n-1)] = \mathbb{E}[ax(n-1)y(n-1) - au(n-1)y(n-1)]$
$= a\mathbb{E}[x(n-1)y(n-1) - \alpha(n-1)y^2(n-1) - \beta(n-1)y(n-2)y(n-1)]$
$= a\left(\mathbb{E}[x(n-1)y(n-1)] - \alpha(n-1)\mathbb{E}[y^2(n-1)] - \beta(n-1)\mathbb{E}[y(n-2)y(n-1)]\right)$

Assume $\beta(n-1) = 0$ and $\alpha(n-1) = \frac{\mathbb{E}[x(n-1)y(n-1)]}{\mathbb{E}[y^2(n-1)]}$. These are valid assumptions based on how the control is initialized. Then:

$$
\mathbb{E}[x(n)y(n-1)] = a\left(\mathbb{E}[x(n-1)y(n-1)] - \frac{\mathbb{E}[x(n-1)y(n-1)]}{\mathbb{E}[y^2(n-1)]}\ \mathbb{E}[y^2(n-1)] - 0\right) = 0
$$

Therefore, $\beta(n)\ \mathbb{E}[y^2(n-1)] = \mathbb{E}[1 - \alpha(n)c(n)] \cdot 0 = 0 \implies \forall n\ \beta(n) = 0$.

Some notes:

1: Cost function, replace x(n+1) with definition

2: Replace u(n)

6: v(n) is independent of other variables and 0-mean, so it goes away

7: $(1-\alpha(n)c(n)$ is independent of $x(n)y(n-1)$. We assume $\alpha(n)$ is a constant $\alpha(n) \in \mathbb{R}$, thus it is independent of all RVs. $x(n)$ is dependent on $c(n-1)$ but independent of $c(n)$. In addition, $c(n)$ is independent of previous timesteps, including $y(n-1)$.

Also note, this can be modeled as an HMM, in which case the memoryless property makes sense. The state is all innovation.

It is interesting to note that in the Gaussian case, the LLSE is the MMSE and is therefore the optimal control. In this case, multiplicative noise means the observation is no longer Gaussian, yet the memoryless LLSE remains optimal. This makes sense–there is no reason multiplicative noise would change anything, since control means the state is killed at each timestep. With delay, this is no longer necessarily true–memory will become important.