$(X_1, X_2, X_3) \sim f(x_1, x_2, x_3) = \begin{cases} 6, if\ 0 < x_1 < x_2 < x_3 \\ 0, o.w. \end{cases}$

(1) Please derive $E(X_1)$=?

$$f_1(x_1) = \int_{x_1}^{1} \int_{x_2}^{1} 6\, dx_3\, dx_2 = \int_{x_1}^{1} 6(1 - x_2)\, dx_2 = 3(1 - 2x_1 + x_1^2)$$

$$E(X_1) = \int_0^1 x_1 f_1(x_1)\, dx_1 = \int_0^1 x_1 \cdot 3(1 - 2x_1 + x_1^2)\, dx_1 = 3\int_0^1 (x_1 - 2x_1^2 + x_1^3)\, dx_1$$

$$= 3\left[\frac{x_1^2}{2} - \frac{2x_1^3}{3} + \frac{x_1^4}{4}\right]_0^1 = 3\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) = 3 \cdot \frac{1}{12} = \frac{1}{4}$$

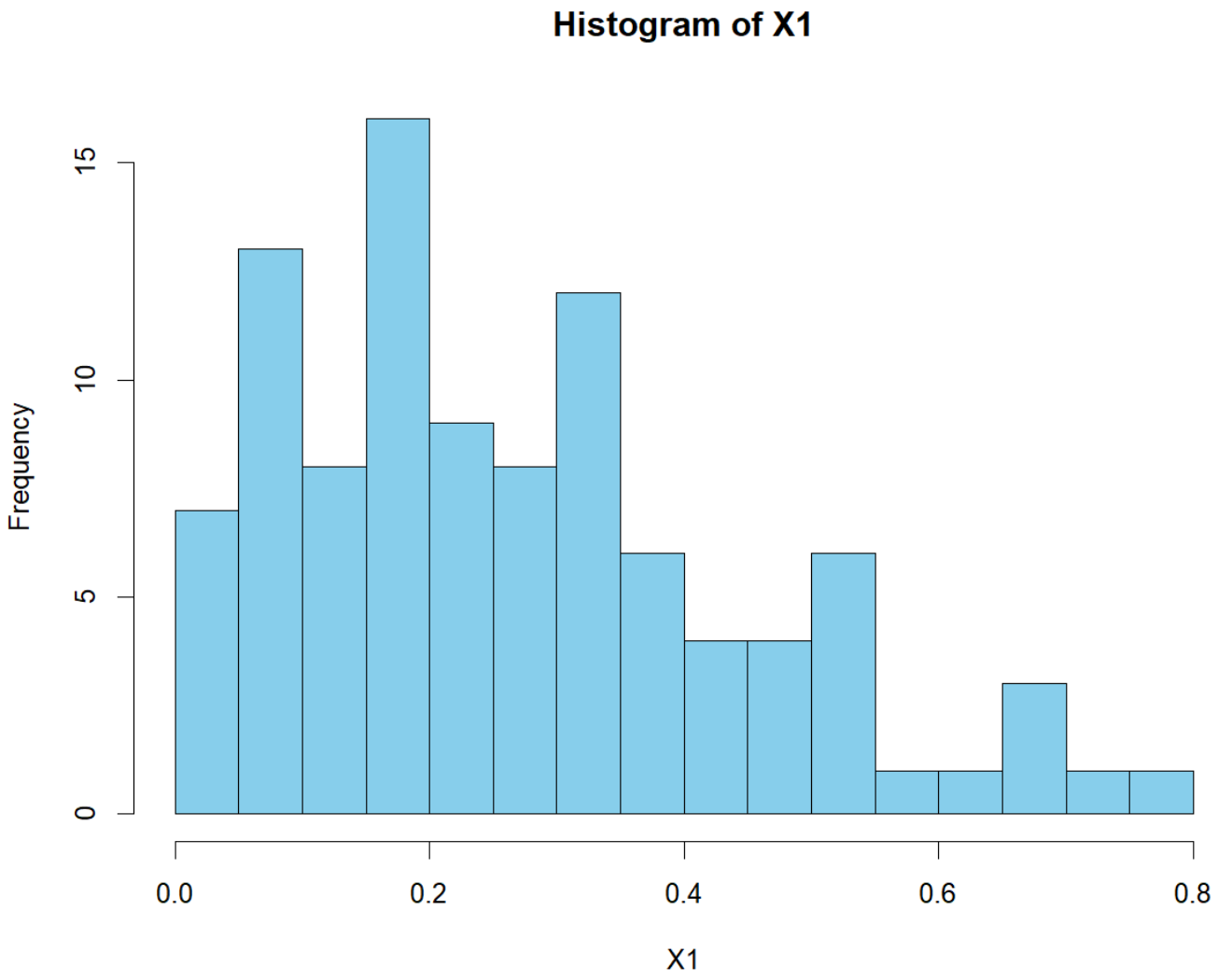
(2) Please generate data with sample size n=100.

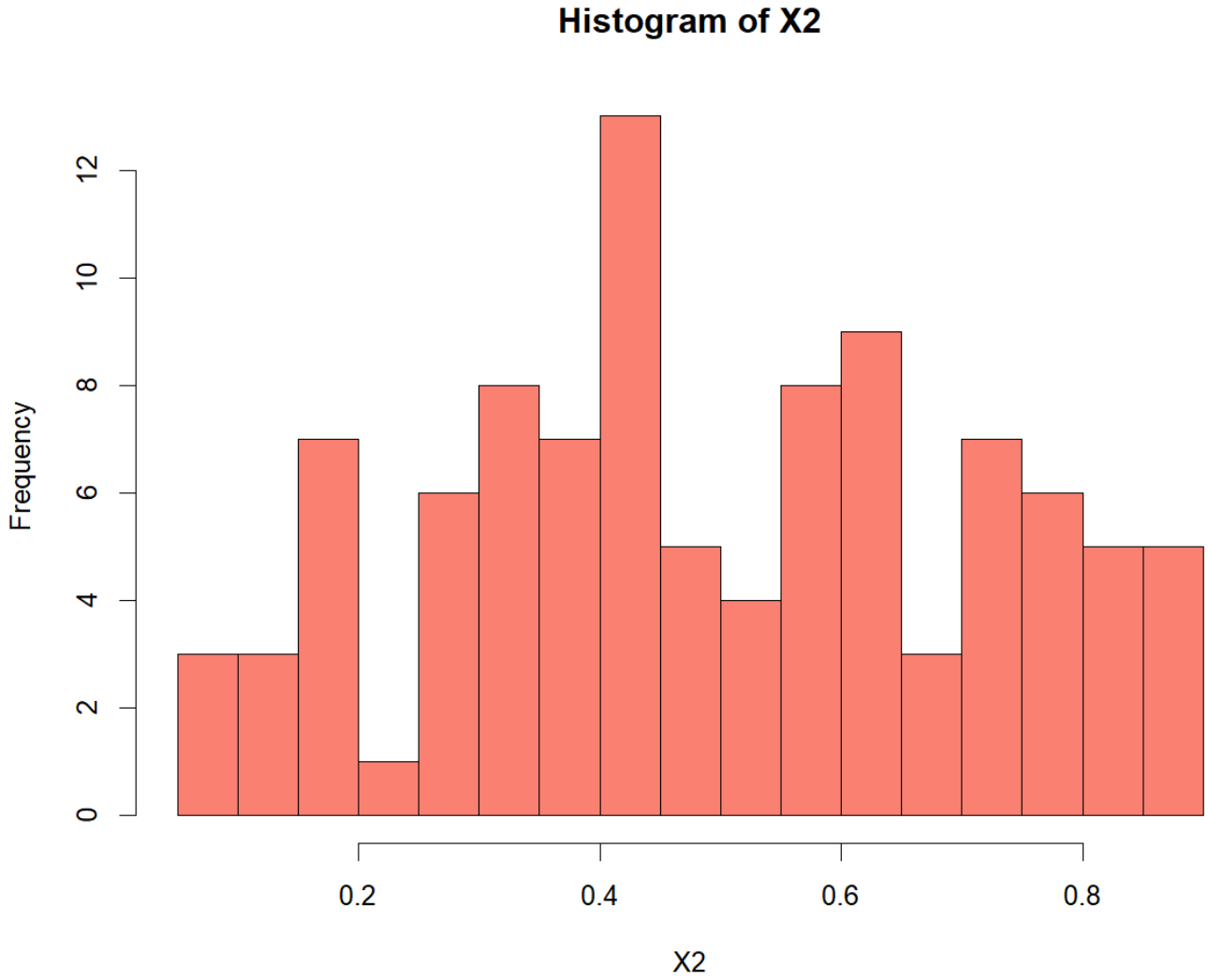
$$X(U) = \{x \in [0,1] : x^3 - 3x^2 + 3x - U = 0,\ U \sim \text{Uniform}(0,1)\}$$

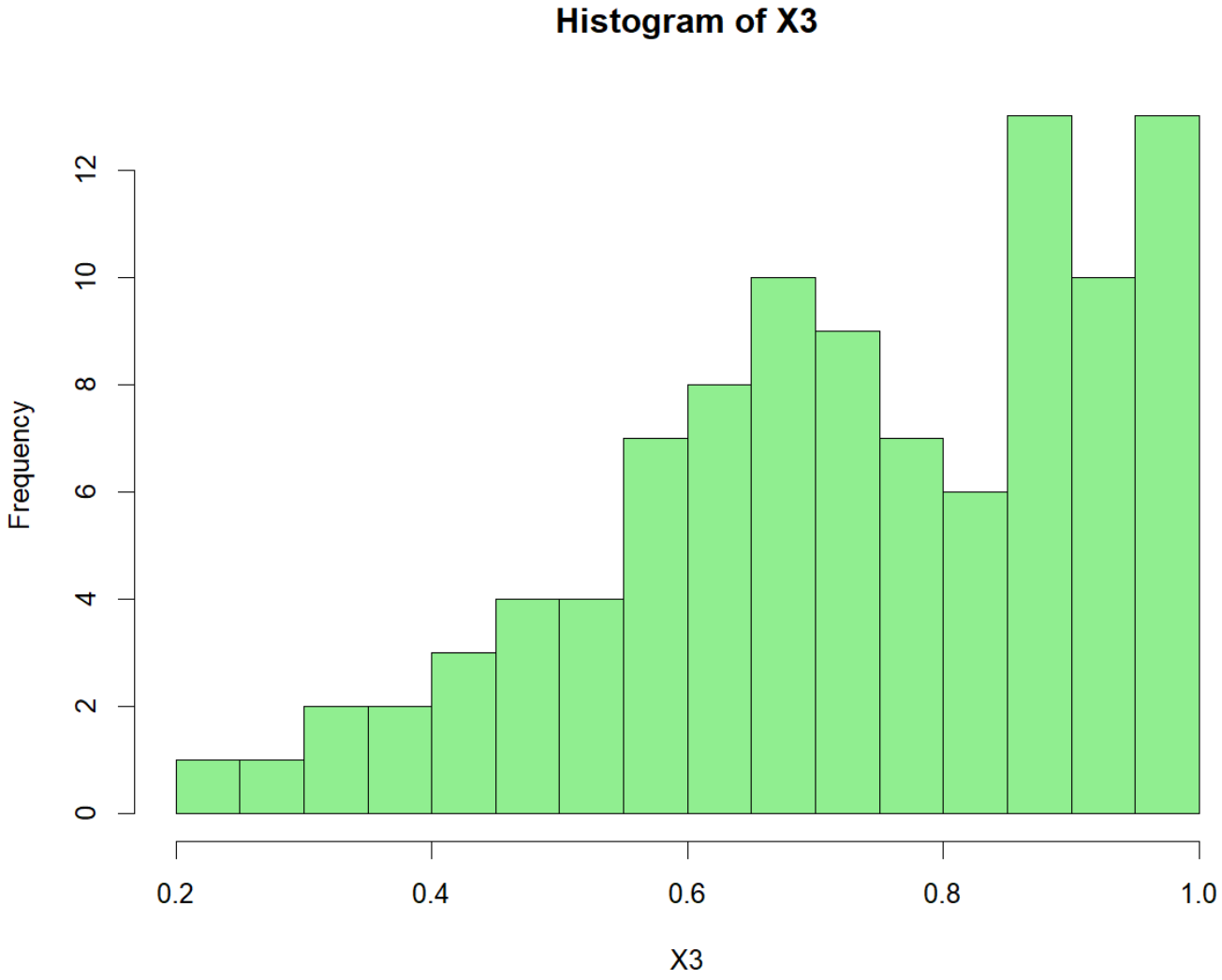
$$X_{2,1}(U) = \left\{x \in [x_1, 1] : \frac{6(x - x_1) - 3(x^2 - x_1^2)}{3 - 6x_1 + 3x_1^2} = U,\ U \sim \text{Uniform}(0,1)\right\}$$

$$X_{3,12} \sim \text{Uniform}(x_2, 1)$$

```
> #樣本大小
> set.seed(123)
> n = 100
>
> X1 = c()
>
> X2 = c()
>
> X3 = c()
>
> for(i in 1:n){
+   #產生x1
+   u1 = runif(1,0,1)
+
+   F1 <- function(x) 3*x- 3*x^2 + x^3 - u1
+
+   x1 = uniroot(F1, c(0, 1))$root
+
+   #產生x2
+   u2 = runif(1,0,1)
+
+   F2_1 <- function(x) (6*(x-x1)-3*(x^2-x1^2))/(3-6*x1+3*x1^2) - u2
+
+   x2_1 = uniroot(F2_1, c(x1, 1))$root
+
+   #產生x3
+   x3_12 = runif(1,x2_1,1)
+
+   X1[i] = x1
+   X2[i] = x2_1
+   X3[i] = x3_12
+ }
>
> hist(X1, breaks = 20, main = "Histogram of X1", col = "skyblue", xlab = "X1")
> hist(X2, breaks = 20, main = "Histogram of X2", col = "salmon", xlab = "X2")
> hist(X3, breaks = 20, main = "Histogram of X3", col = "lightgreen", xlab = "X3")
```


Histogram of X1
Frequency
0
5
10
15
0.0
0.2
0.4
0.6
0.8
X1


Histogram of X2
Frequency
0
2
4
6
8
10
12
0.2
0.4
0.6
0.8
X2


Histogram of X3
Frequency
0
2
4
6
8
10
12
0.2
0.4
0.6
0.8
1.0
X3

(3) Please calculate your generated sample mean of X_1.

```
> mean(X1)
[1] 0.2671452
```

程式碼網址: https://github.com/Lai-jun-yan/Mathematical_Statistics