

$$(X_1, X_2, X_3) \sim f(x_1, x_2, x_3) = \begin{cases} 6, & \text{if } 0 < x_1 < x_2 < x_3 \\ 0, & \text{o.w.} \end{cases}$$

(1) Please derive $E(X_1) = ?$

$$f_1(x_1) = \int_{x_1}^1 \int_{x_2}^1 6 \, dx_3 \, dx_2 = \int_{x_1}^1 6(1 - x_2) \, dx_2 = 3(1 - 2x_1 + x_1^2)$$

$$\begin{aligned} E(X_1) &= \int_0^1 x_1 f_1(x_1) \, dx_1 = \int_0^1 x_1 \cdot 3(1 - 2x_1 + x_1^2) \, dx_1 = 3 \int_0^1 (x_1 - 2x_1^2 + x_1^3) \, dx_1 \\ &= 3 \left[\frac{x_1^2}{2} - \frac{2x_1^3}{3} + \frac{x_1^4}{4} \right]_0^1 = 3 \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right) = 3 \cdot \frac{1}{12} = \frac{1}{4} \end{aligned}$$

(2) Please generate data with sample size $n=100$.

$$\begin{aligned}f_{X_1}(x_1) &= \int_{x_2=x_1}^1 \int_{x_3=x_2}^1 6 \, dx_3 \, dx_2 \\&= \int_{x_2=x_1}^1 6(1-x_2) \, dx_2 \\&= 6 \left[x_2 - \frac{x_2^2}{2} \right]_{x_1}^1 = 6 \left(\frac{1}{2} - x_1 + \frac{x_1^2}{2} \right) \\&= 3 - 6x_1 + 3x_1^2 \\&= 3(1-x_1)^2, \quad 0 < x_1 < 1,\end{aligned}$$

$$\begin{aligned}f_{X_1, X_2}(x_1, x_2) &= \int_{x_3=x_2}^1 f_{X_1, X_2, X_3}(x_1, x_2, x_3) \, dx_3 \\&= \int_{x_3=x_2}^1 6 \, dx_3 \\&= 6(1-x_2), \quad 0 < x_1 < x_2 < 1\end{aligned}$$

$$f_{X_2|X_1}(x_2|x_1) = \frac{f_{X_1, X_2}(x_1, x_2)}{f_{X_1}(x_1)}$$

$$\begin{aligned}f_{X_2|X_1}(x_2|x_1) &= \frac{6(1-x_2)}{3(1-x_1)^2} \\&= \frac{2(1-x_2)}{(1-x_1)^2}, \quad x_1 < x_2 < 1\end{aligned}$$

$$f_{X_3|X_1, X_2}(x_3|x_1, x_2) = \frac{f_{X_1, X_2, X_3}(x_1, x_2, x_3)}{f_{X_1, X_2}(x_1, x_2)}$$

$$f_{X_3|X_1, X_2}(x_3|x_1, x_2) = \frac{6}{6(1-x_2)} = \frac{1}{1-x_2}, \quad x_2 < x_3 < 1$$

$$F_{X_1}(x) = \int_0^x 3(1-t)^2 dt = 3 \left[t - t^2 + \frac{t^3}{3} \right]_0^x = 3x - 3x^2 + x^3$$

$$3x_1 - 3x_1^2 + x_1^3 = U_1, \quad U_1 \sim \text{Uniform}(0, 1)$$

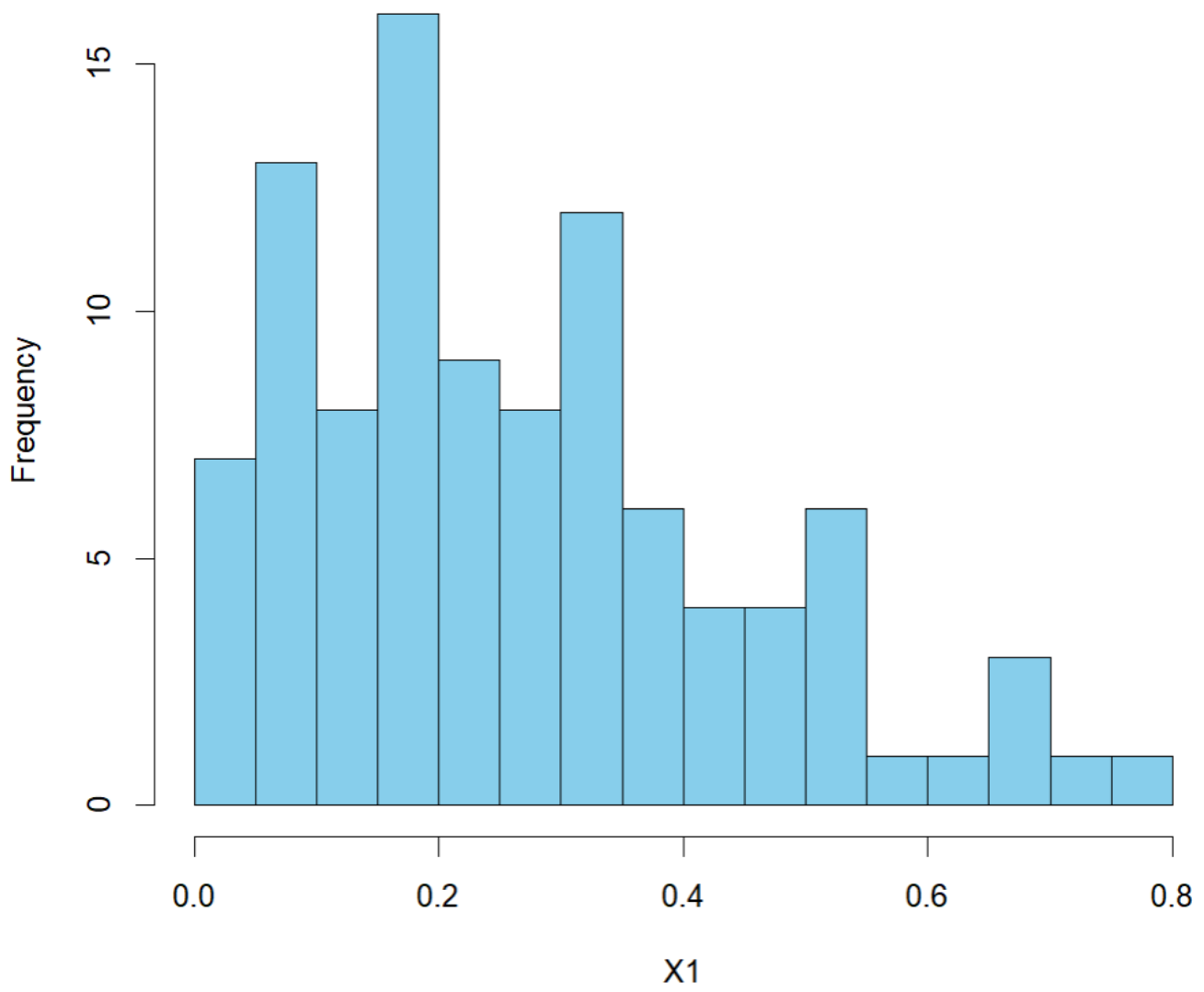
$$F_{X_2|X_1}(x_2|x_1) = \int_{x_1}^{x_2} \frac{2(1-t)}{(1-x_1)^2} dt = \frac{(1-x_1)^2 - (1-x_2)^2}{(1-x_1)^2} = 1 - \left(\frac{1-x_2}{1-x_1} \right)^2$$

$$1 - \left(\frac{1-x_2}{1-x_1} \right)^2 - U = 0, \quad U \sim \text{Uniform}(0, 1), \quad x_1 < x_2 < 1$$

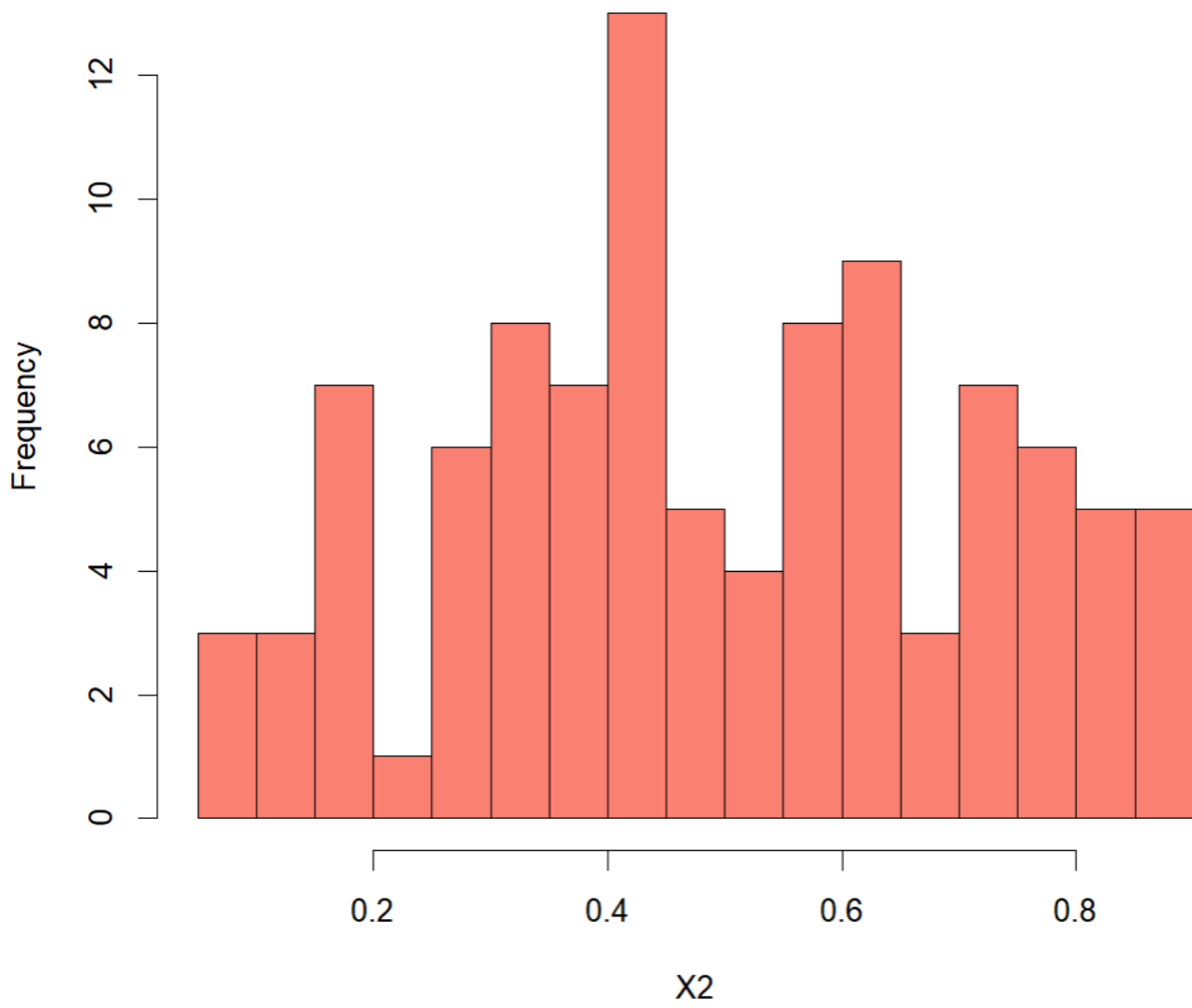
$$X_3 \mid X_1 = x_1, X_2 = x_2 \sim \text{Uniform}(x_2, 1)$$

```
> #使結果可重現
> set.seed(123)
>
> #樣本大小
> n = 100
>
> x1 = c()
>
> x2 = c()
>
> x3 = c()
>
> for(i in 1:n){
+   #產生x1
+   u1 = runif(1,0,1)
+
+   F1 <- function(x) 3*x- 3*x^2 + x^3 - u1
+
+   x1 = uniroot(F1, c(0, 1))$root
+
+   #產生x2
+   u2 = runif(1,0,1)
+
+   F2_1 <- function(x) 1-((1-x)/(1-x1))^2 - u2
+
+   x2_1 = uniroot(F2_1, c(x1, 1))$root
+
+   #產生x3
+   x3_12 = runif(1,x2_1,1)
+
+   x1[i] = x1
+   x2[i] = x2_1
+   x3[i] = x3_12
+ }
>
> hist(x1, breaks = 20, main = "Histogram of x1", col = "skyblue", xlab = "x1")
> hist(x2, breaks = 20, main = "Histogram of x2", col = "salmon", xlab = "x2")
> hist(x3, breaks = 20, main = "Histogram of x3", col = "lightgreen", xlab = "x3")
```

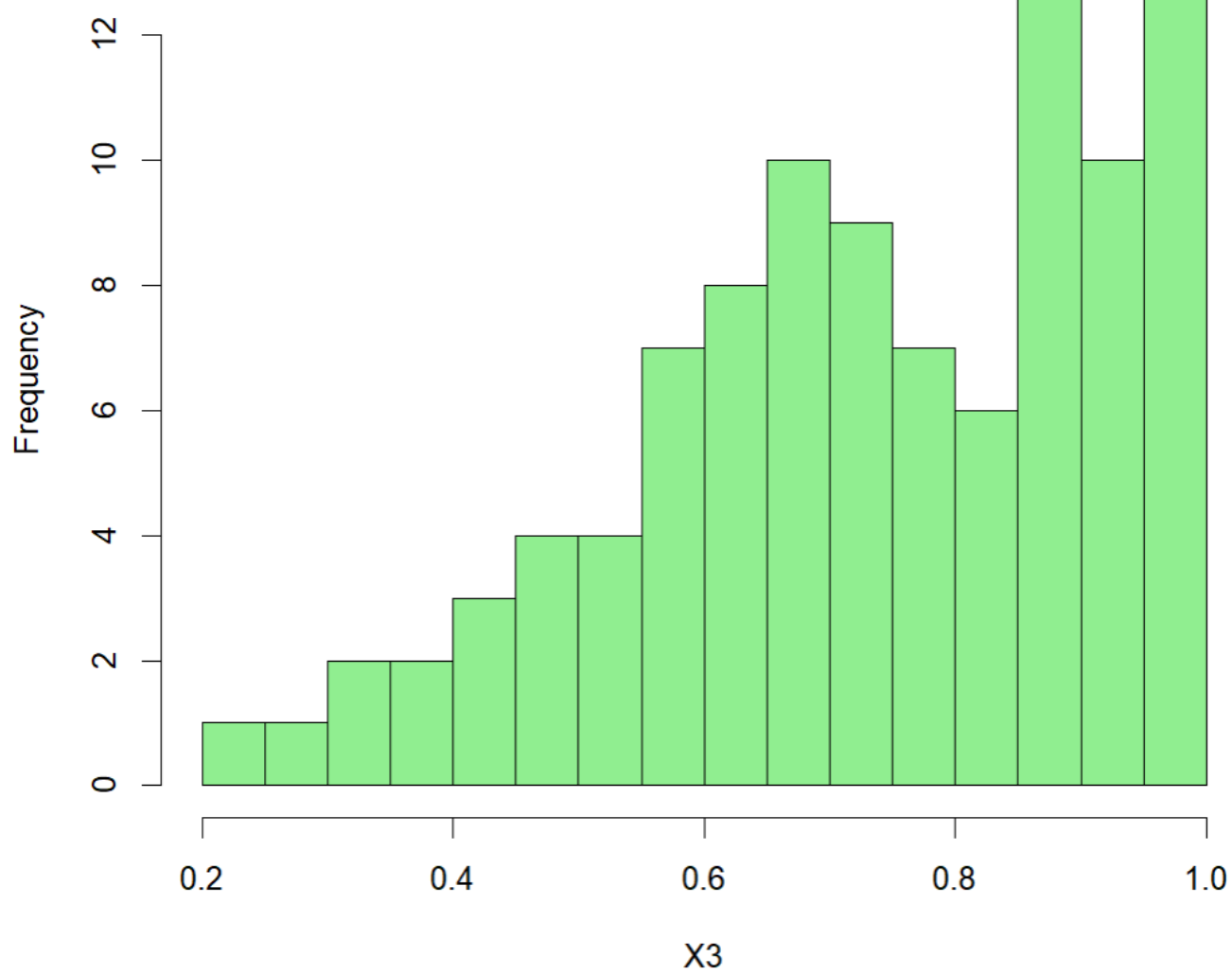
Histogram of X1



Histogram of X2



Histogram of X3



(3) Please calculate your generated sample mean of X_1 .

```
> mean(X1)
[1] 0.2671452
```

程式碼網址: https://github.com/Lai-jun-yan/Mathematical_Statistics