$(X_1, X_2, X_3) \sim f(x_1, x_2, x_3) = \begin{cases} 6, if\ 0 < x_1 < x_2 < x_3 \\ 0, o.w. \end{cases}$

(1) Please derive $E(X_1)$=?

$$f_1(x_1) = \int_{x_1}^{1} \int_{x_2}^{1} 6\, dx_3\, dx_2 = \int_{x_1}^{1} 6(1 - x_2)\, dx_2 = 3(1 - 2x_1 + x_1^2)$$

$$E(X_1) = \int_0^1 x_1 f_1(x_1)\, dx_1 = \int_0^1 x_1 \cdot 3(1 - 2x_1 + x_1^2)\, dx_1 = 3\int_0^1 (x_1 - 2x_1^2 + x_1^3)\, dx_1$$

$$= 3\left[\frac{x_1^2}{2} - \frac{2x_1^3}{3} + \frac{x_1^4}{4}\right]_0^1 = 3\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) = 3 \cdot \frac{1}{12} = \frac{1}{4}$$

(2) Please generate data with sample size n=100.

$$
\begin{aligned}
f_{X_1}(x_1) &= \int_{x_2=x_1}^{1} \int_{x_3=x_2}^{1} 6\, dx_3\, dx_2 \\
&= \int_{x_2=x_1}^{1} 6(1-x_2)\, dx_2 \\
&= 6\left[x_2 - \tfrac{x_2^2}{2}\right]_{x_1}^{1} = 6\left(\tfrac{1}{2} - x_1 + \tfrac{x_1^2}{2}\right) \\
&= 3 - 6x_1 + 3x_1^2 \\
&= 3(1-x_1)^2, \qquad 0 < x_1 < 1,
\end{aligned}
$$

$$
\begin{aligned}
f_{X_1,X_2}(x_1,x_2) &= \int_{x_3=x_2}^{1} f_{X_1,X_2,X_3}(x_1,x_2,x_3)\, dx_3 \\
&= \int_{x_3=x_2}^{1} 6\, dx_3 \\
&= 6(1-x_2), \quad 0 < x_1 < x_2 < 1
\end{aligned}
$$

$$
f_{X_2|X_1}(x_2|x_1) = \frac{f_{X_1,X_2}(x_1,x_2)}{f_{X_1}(x_1)}
$$

$$
\begin{aligned}
f_{X_2|X_1}(x_2|x_1) &= \frac{6(1-x_2)}{3(1-x_1)^2} \\
&= \frac{2(1-x_2)}{(1-x_1)^2}, \quad x_1 < x_2 < 1
\end{aligned}
$$

$$
f_{X_3|X_1,X_2}(x_3|x_1,x_2) = \frac{f_{X_1,X_2,X_3}(x_1,x_2,x_3)}{f_{X_1,X_2}(x_1,x_2)}
$$

$$
f_{X_3|X_1,X_2}(x_3|x_1,x_2) = \frac{6}{6(1-x_2)} = \frac{1}{1-x_2}, \quad x_2 < x_3 < 1
$$

$$F_{X_1}(x) = \int_0^x 3(1-t)^2 dt = 3\left[t - t^2 + \frac{t^3}{3}\right]_0^x = 3x - 3x^2 + x^3$$

解 x1 的根:

$$3x_1 - 3x_1^2 + x_1^3 = U_1, \quad U_1 \sim \text{Uniform}(0,1)$$

$$F_{X_2|X_1}(x_2|x_1) = \int_{x_1}^{x_2} \frac{2(1-t)}{(1-x_1)^2} dt = \frac{(1-x_1)^2 - (1-x_2)^2}{(1-x_1)^2} = 1 - \left(\frac{1-x_2}{1-x_1}\right)^2$$

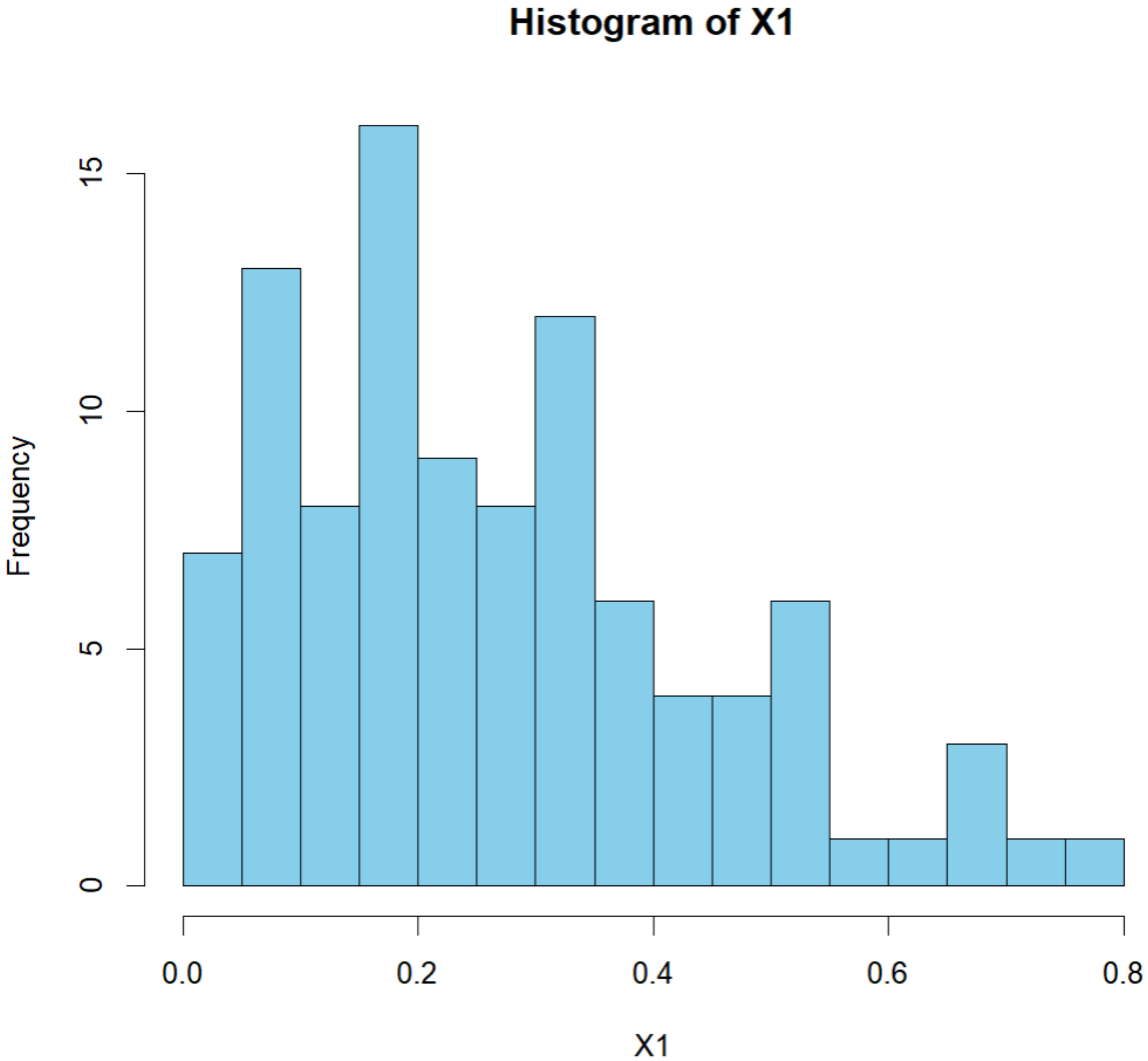
解 x2 的根:

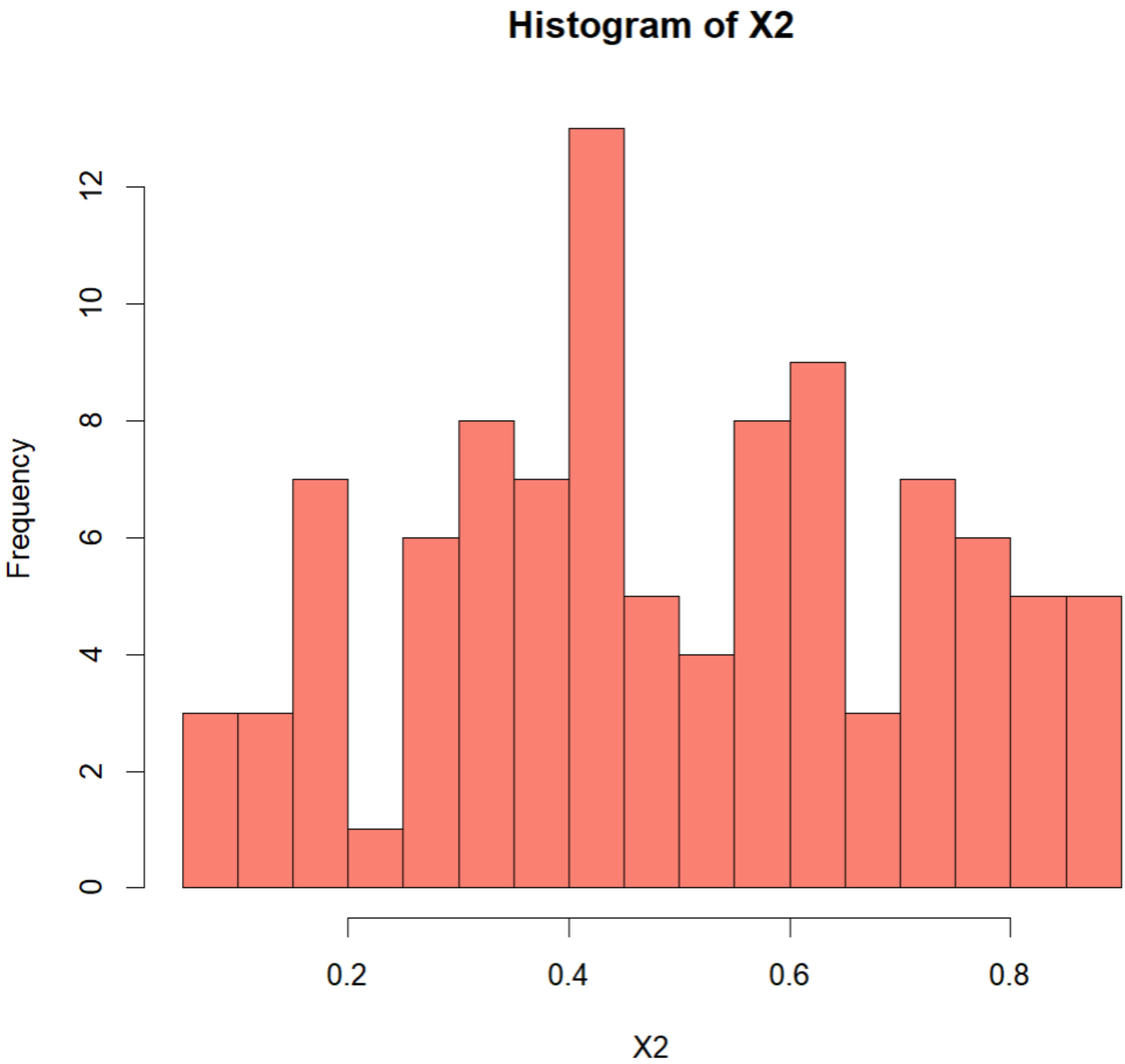
$$1 - \left(\frac{1-x_2}{1-x_1}\right)^2 - U = 0, \quad U \sim \text{Uniform}(0,1),\ x_1 < x_2 < 1$$

從 pdf 可知 x3:

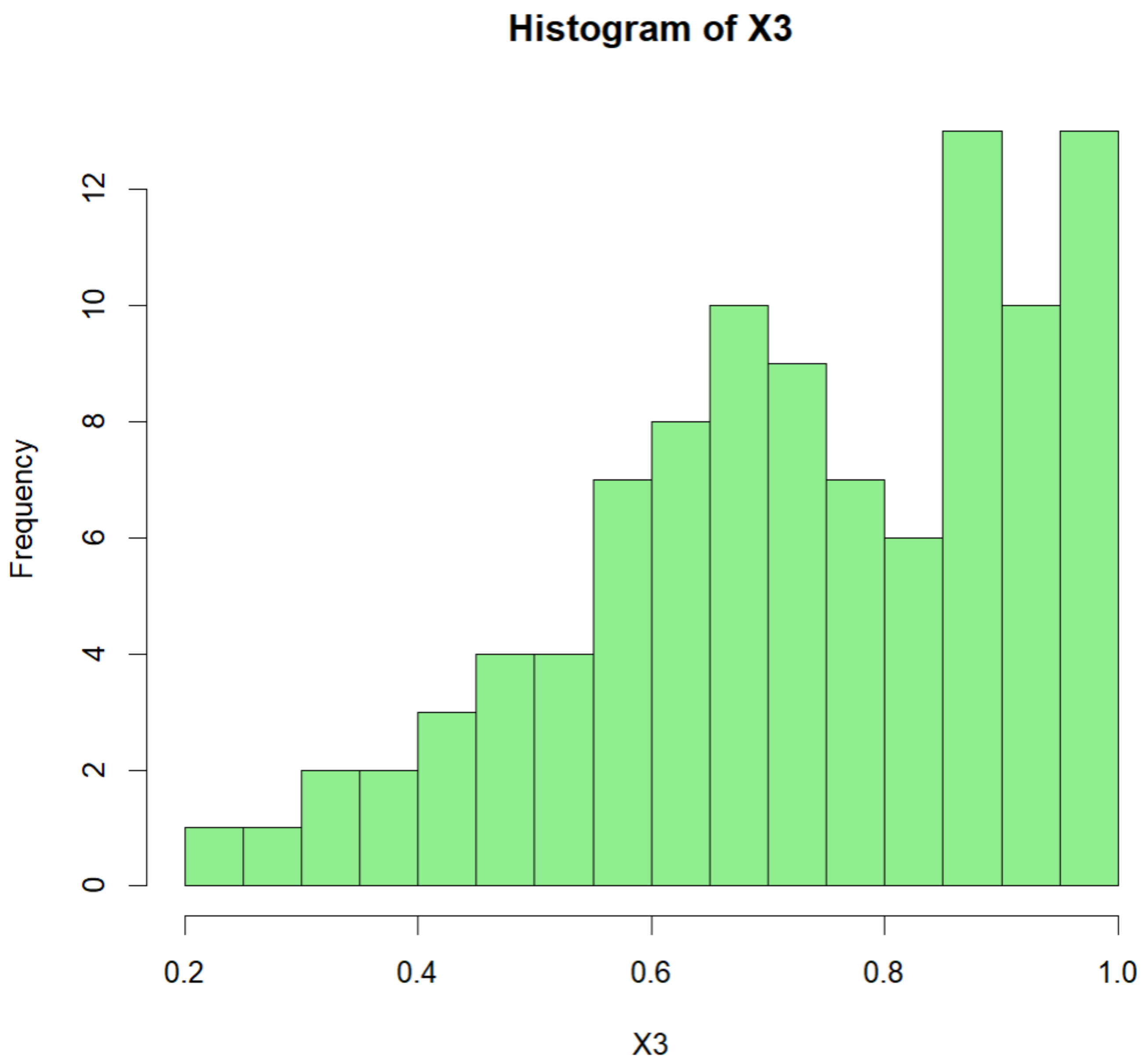
$$X_3 \mid X_1 = x_1, X_2 = x_2 \sim \text{Uniform}(x_2, 1)$$

```
> #使結果可重現
> set.seed(123)
>
> #樣本大小
> n = 100
>
> X1 = c()
>
> X2 = c()
>
> X3 = c()
>
> for(i in 1:n){
+   #產生x1
+   u1 = runif(1,0,1)
+
+   F1 <- function(x) 3*x- 3*x^2 + x^3 - u1
+
+   x1 = uniroot(F1, c(0, 1))$root
+
+   #產生x2
+   u2 = runif(1,0,1)
+
+   F2_1 <- function(x) 1-((1-x)/(1-x1))^2 - u2
+
+   x2_1 = uniroot(F2_1, c(x1, 1))$root
+
+   #產生x3
+   x3_12 = runif(1,x2_1,1)
+
+   X1[i] = x1
+   X2[i] = x2_1
+   X3[i] = x3_12
+ }
>
> hist(X1, breaks = 20, main = "Histogram of X1", col = "skyblue", xlab = "X1")
> hist(X2, breaks = 20, main = "Histogram of X2", col = "salmon", xlab = "X2")
> hist(X3, breaks = 20, main = "Histogram of X3", col = "lightgreen", xlab = "X3")
```


Histogram of X1
Frequency
0
5
10
15
0.0
0.2
0.4
0.6
0.8
X1



Histogram of X2
Frequency
0
2
4
6
8
10
12
0.2
0.4
0.6
0.8
X2


Histogram of X3
Frequency
0
2
4
6
8
10
12
0.2
0.4
0.6
0.8
1.0
X3

(3) Please calculate your generated sample mean of X_1.

```
> mean(x1)
[1] 0.2671452
```

程式碼網址: https://github.com/Lai-jun-yan/Mathematical_Statistics