Project report towards completion of the course

FSG3114 Computational Fluid Dynamics

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1 The 2D lid-driven cavity

1.1 Task 1

Consider the case of two 2×2 matrices M and Y, we have:

$$\begin{cases} M = \begin{bmatrix} m_{11} & 0 \\ 0 & m_{22} \end{bmatrix} \\ Y = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \end{cases} \tag{7}$$

$$\begin{cases} K_{MY} = M \otimes Y = \begin{bmatrix} m_{11}y_{11} & m_{11}y_{12} & 0 & 0 \\ m_{11}y_{21} & m_{11}y_{22} & 0 & 0 \\ 0 & 0 & m_{22}y_{11} & m_{22}y_{12} \\ 0 & 0 & m_{22}y_{21} & m_{22}y_{22} \end{bmatrix} \end{cases}$$

For the simple symmetric case:

$$K_{MY} = M \otimes Y = \begin{bmatrix} y_{11}m_{11} & 0 & y_{12}m_{11} & 0 \\ 0 & y_{11}m_{22} & 0 & y_{12}m_{22} \\ y_{21}m_{11} & 0 & y_{22}m_{11} & 0 \\ 0 & y_{21}m_{22} & 0 & y_{22}m_{22} \end{bmatrix} \tag{8}$$

One can treat any matrix $K_{MY} \in \mathbf{R}^{n \times n}$ as $K_{MY} \in \mathbf{R}^{2n}$. This reshaping is common when treating with matrices, i.e: Perro Frobenius norm. As one can see below for a 2×2 toy case example:

$$\gamma = \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix} \rightarrow \gamma = (\gamma_{11}, \gamma_{21}, \gamma_{12}, \gamma_{22}) \tag{10}$$

In the project, we have used this combination to implement the 2D Laplace operator, which looks like:

$$\nabla^2 \gamma_{i,j} = \frac{\gamma_{i+1,j} - 2\gamma_{i,j} + \gamma_{i-1,j}}{\Delta x^2} + \frac{\gamma_{i,j+1} - 2\gamma_{i,j} + \gamma_{i,j-1}}{\Delta y^2}$$

In this case, if the matrices M and Y are substituted respectively by I and D^2, the first lines of the Kronecker product read:

$$K_{I \otimes D} = \begin{bmatrix} -1 & 1 & \dots & \\ 1 & -2 & 1 & \dots \end{bmatrix}$$

$$K_{D\otimes I} = \begin{bmatrix} -1 & \dots & 1 & \dots & & \\ \dots & \dots & & & & \\ 1 & \dots & -2 & \dots & 1 & \dots \end{bmatrix} \tag{12}$$

If the 2D field has been stored in an array as introduced earlier, we will have:

$$\gamma = (\gamma_{11}, \gamma_{21}, \dots, \gamma_{Ny1}, \gamma_{12}, \dots) \tag{13}$$

Then, one obtains for $K\gamma$:

$$K_{I\otimes D}\gamma = \begin{bmatrix} -\gamma_{11} + \gamma_{21} \\ \gamma_{21} - \gamma_{22} + \gamma_{23} \end{bmatrix}$$

$$K_{D\otimes I}\gamma = \begin{bmatrix} -\gamma_{11} + \gamma_{12} \\ \gamma_{12} - \gamma_{22} + \gamma_{13} \end{bmatrix} \tag{14}$$

Obtaining the first steps described at (1.1).

1.2 Task 2

We can obtain information about the stability of the Navier-Stokes equation using as a model the advection-diffusion equation, which is linear – then it is possible to apply the von Neumann stability analysis.

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} + a\frac{\partial u}{\partial y} = \frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{15}$$

The discretization procedure that we have applied in the project can be considered somehow equivalent to a standard FTCS scheme (forward in time, backward in space), which for (**??**) reads:

$$\begin{aligned} u_{n+1,i} - u_{n,i}\Delta t = -a\left(\frac{u_{n,i+1,j} - u_{n,i-1,j}}{2\Delta x} + \frac{u_{n,i,j+1} - u_{n,i,j-1}}{\Delta y}\right) \\ + \frac{1}{Re}\left(\frac{u_{n,i+1,j} - 2u_{n,i,j} + u_{n,i-1,j}}{\Delta x^2} + \frac{u_{n,i,j+1} - 2u_{n,i,j} + u_{n,i,j-1}}{\Delta y^2}\right) \end{aligned} \tag{16}$$

To perform the von Neumann analysis, we are supposed to substitute:

$$u_{n,i,j} \to u_{n,k,q} e^{ik\Delta x i + jq\Delta y j} \tag{17}$$

If we assume the same grid spacing $\Delta x = \Delta y$ and the same domain size $L_x = L_y$ – which is, by the way, the case of the lid-driven cavity – we will have $k \equiv q$ and we can write:

$$u_{n,i\pm1,j} = u_{n,k} e^{ik\Delta x(i+j)} e^{ik\Delta x} \tag{18}$$

From now on, the complex unit i is denoted i since no ambiguities are possible. After some passages, one can reach the expression for the amplification factor:

$$G_k = |1 - 2i\sigma \sin(k\Delta x) + 4\beta(\cos(k\Delta x) - 2)| \tag{19}$$

Where:

$$\sigma = a\Delta t \quad \text{and} \quad \beta = \frac{\Delta t}{Re\Delta x^2} \tag{20}$$

It is worth to write the stability condition $G_k < 1$ for two different particular cases:

$$k\Delta x \to \pi \quad \text{and} \quad k\Delta x \to 0 \tag{21}$$

In the first case:

$$G_k = |1 + 8\beta| < 1 \Rightarrow \beta < \frac{1}{4} \tag{22}$$

In the second case, after expanding $\sin X \approx X$ and $\cos X \approx 1/2 - X^2$, it turns out:

$$G_k \approx |1 - 2i\sigma k\Delta x - 2\beta(k\Delta x)| < 1 \tag{23}$$

Then, ignoring the term of order $(k\Delta x)^4$:

$$-\beta + \sigma^2 < 0 \tag{24}$$

These two conditions can be written as an explicit condition for the time interval, given the other parameters. In this case, they read:

$$\Delta t < \frac{1}{4Re\Delta x^2} \tag{25a}$$

$$\Delta t < \frac{1}{a^2 Re} \tag{25b}$$

Considering their expression, it is possible to state that 25a is going to be more restrictive for low Reynolds number, while 25b for higher ones. To evaluate the maximum Δt allowed, it is needed to give an estimate for the advection velocity a. The most precautionary is the maximum component of the velocity field u: in our case, the boundary condition on the top of domain: $a = 1$.

1.3 Task 3

The given BCs are the following taking into account the adopted grid, staggered, are:

$$U_{i+1/2,1/2} = U_{BC} \tag{1}$$

$$V_{1/2,j+1/2} = V_{BC} \tag{2}$$

The value at the ghost cells, outside the domain, need to be chosen such that the averaging operation gives the proper results. It is important to mention that this ghost cells are well known as dummy cells, here we will refer to them as ghost:

$$U_{i+1/2,1/2} = U_{i+1/2,1} + U_{i+1/2,0}\Delta y \Rightarrow U_{\text{ghost},i+1/2,0} = 2\Delta y U_{BC} - U_{i+1/2,1/2} \tag{3}$$

$$V_{\text{ghost},0,j+1/2} = 2\Delta x V_{BC} - V_{1/2,j+1/2} \tag{4}$$

Dummy cells for the pressure:

The general form of the Laplace operator in discrete form has been already reported in (11). For the pressure and at $i = 1$, $j = 2, \ldots, Ny - 1$ we get:

$$\nabla^2 P_{1,2} = P_{2,2} - 2P_{1,2} + P_{0,2}\frac{\Delta x^2}{\Delta y^2} + P_{1,3} - 2P_{1,2} + P_{1,1}\frac{\Delta y^2}{\Delta x^2} \tag{5}$$

$$\nabla^2 P_{1,Ny-1} = P_{2,Ny-1} - 2P_{1,Ny-1} + P_{0,Ny-1}\frac{\Delta x^2}{\Delta y^2} + P_{1,Ny} - 2P_{1,Ny-1} + P_{1,Ny-2}\frac{\Delta y^2}{\Delta x^2} \tag{6}$$

Then one can see that it is needed to impose the value of the pressure for the cells $(0, j)$, $(i, 0)$, $(Nx + 1, j)$, and $(i, Ny + 1)$. Since we are imposing the boundary condition $\partial P/\partial n = 0$:

$$P_{1,j} - P_{0,j}\frac{\Delta x}{\Delta x} = 0 \Rightarrow P_{1,j} = P_{0,j} \tag{7}$$

$$P_{Nx+1,j} - P_{Nx,j}\frac{\Delta x}{\Delta x} = 0 \Rightarrow P_{Nx+1,j} = P_{Nx,j} \tag{8}$$

$$P_{i,1} - P_{i,0}\frac{\Delta y}{\Delta y} = 0 \Rightarrow P_{i,1} = P_{i,0} \tag{9}$$

$$P_{i,Ny+1} - P_{i,Ny}\frac{\Delta y}{\Delta y} = 0 \Rightarrow P_{i,Ny+1} = P_{i,Ny} \tag{10}$$

By substitution of (10) in (6).

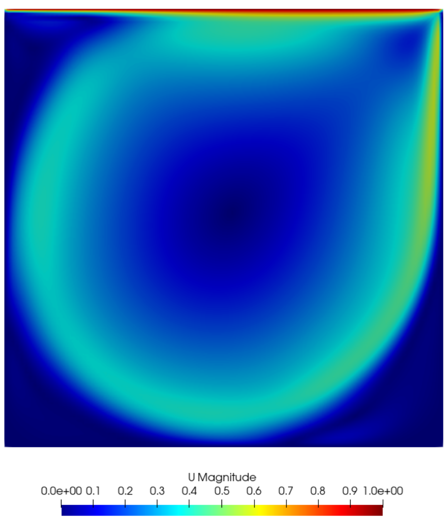
$$\nabla^2 P_{1,2} = P_{2,2} + P_{1,2}\frac{\Delta x^2}{\Delta y^2} + P_{1,3} - 2P_{1,2} + P_{1,1}\frac{\Delta y^2}{\Delta x^2} \tag{11}$$

$$\nabla^2 P_{1,Ny-1} = P_{2,Ny-1} + P_{1,Ny-1}\frac{\Delta x^2}{\Delta y^2} + P_{1,Ny} - 2P_{1,Ny-1} + P_{1,Ny-2}\frac{\Delta y^2}{\Delta x^2} \tag{12}$$

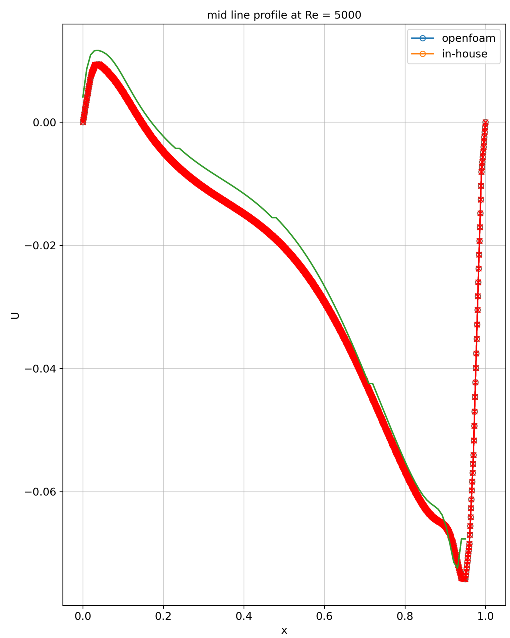
1.4 Task 4

In figures below one can see the evolution of the velocity along the simulation. It takes around $t = 3, 24, 100$ and $t > 100$ timeunits for $Re = 25, 250, 5000$ and 10000. As the Re increases the time to have steady solutions increases noticeably.

1.5 Task 5



(a) Velocity field



(b) Mid line horizontal velocity comparison.

Figure 1: Case D)

Fig. 13a depicts both results, one can observe how both results resemble. However, it is important to mention that slight errors arise, one can observe how the first peak is not well captured and probably this effect expands into the mid sections. On the other hand, the lower peak at the end is well captured.

1.6 Task 6

In Fig. 8b ona can see the case with moving wall leveraged. The one with $U_s = -1, U_t = 1$ shows a complete symmetry since the cavity experiences the same velocity in the upper but opposite direction.

1.7 Cases

1.7.1 Case A

1.7.2 Case B

1.7.3 Case C

1.7.4 Case D: Moving Walls

2 Rayleigh-Bernard Problem

2.1 Task 1

The non-dimensional parameters Prandtl number (Pr) and Rayleigh number (Ra) are defined as follows: The Prandtl number (Pr) is a measure that tells us how heat moves through a fluid compared to how it moves momentum (like how it flows or moves). A low Pr number means heat moves much faster than momentum (like in water). A high Pr number means momentum moves faster than heat (like in oil). This number helps us understand how fluids behave when they're heated or cooled.

- **Prandtl number (Pr)**:

$$\begin{aligned} Pr &= \frac{\text{viscous diffusion rate}}{\text{thermal diffusion rate}} \\ &= \frac{\nu cp}{\alpha} \end{aligned}$$

  The Prandtl number represents the ratio of momentum diffusivity (kinematic viscosity ν) to thermal diffusivity (α) and c_p is the specific heat capacity of the fluid at constant pressure. Physically, it indicates the relative importance of momentum diffusivity to thermal diffusivity in a fluid. A high Prandtl number implies that momentum diffuses slowly compared to heat, typical of fluids with low viscosity.

- **Rayleigh number (Ra)**:

$$\begin{aligned} Ra &= \frac{\text{buoyancy force}}{\text{viscous force}} \\ &= \frac{g\rho\beta\Delta T L^3}{\nu^2} \end{aligned}$$

  The Rayleigh number represents the ratio of buoyancy forces to viscous forces within a fluid. It depends on parameters such as gravitational acceleration (g), coefficient of thermal expansion (β), temperature difference (ΔT), characteristic length scale (L), and kinematic viscosity

(ν). Physically, it characterizes the relative contributions of buoyancy-driven flow to viscous effects. A high Rayleigh number indicates that buoyancy forces dominate over viscous forces, leading to convective flow.

2.2 Case A

Run the code for $Ra = 200, 2000, 60000$ $\theta_T = 1$ and $\theta_B = 0$.

2.3 Case B

Run the code for $Ra = 1715$ $\theta_T = 0$ and $\theta_B = 1$.

2.4 Task 2 & 3

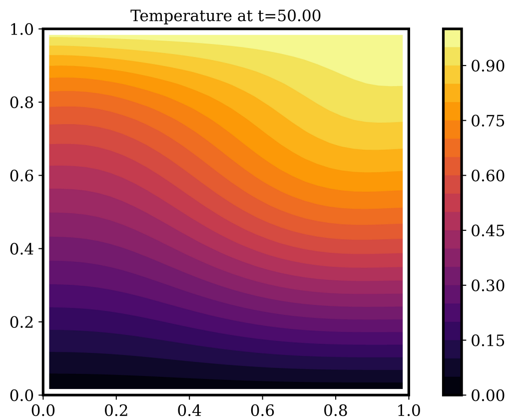
Considering Figures on page 15, one can conclude that the critical Ra is between 1700 and 1715, theoretically a value of 1708.

To estimate the wavelength of the instability, we can consider the value of the velocity U on the "plane" $y = 0$. It turns out:

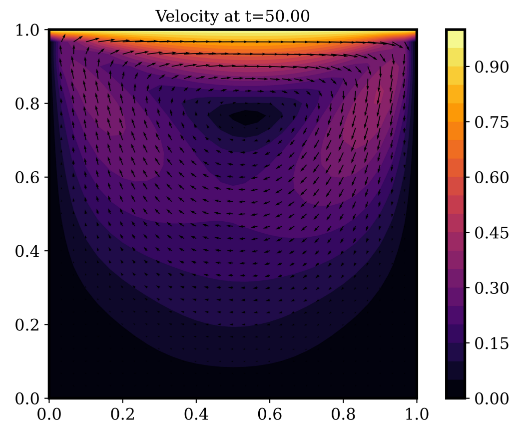
$$\lambda \approx 2 \Rightarrow k_c = \frac{2\pi}{\lambda_c} \approx 3.1 \quad (11)$$

That is in good agreement with the theoretical value of 3.12.

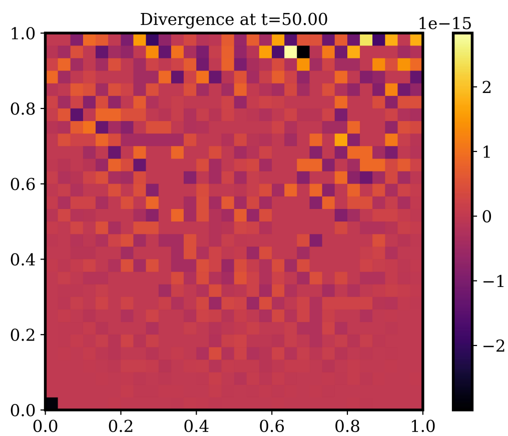
To obtain a better description, it would be needed to adopt periodic boundary conditions for both the velocity and the pressure.



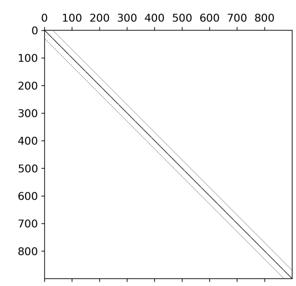
(a) Temperature field



(b) Velocity field

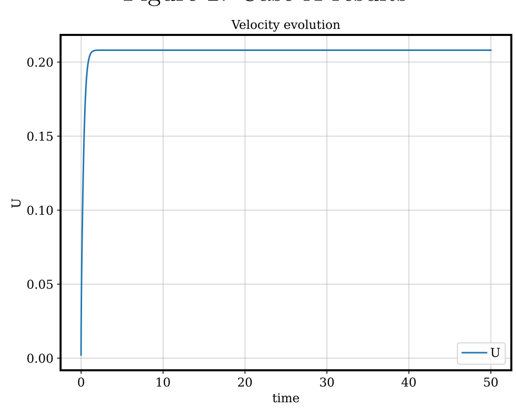


(c) Divergence Field

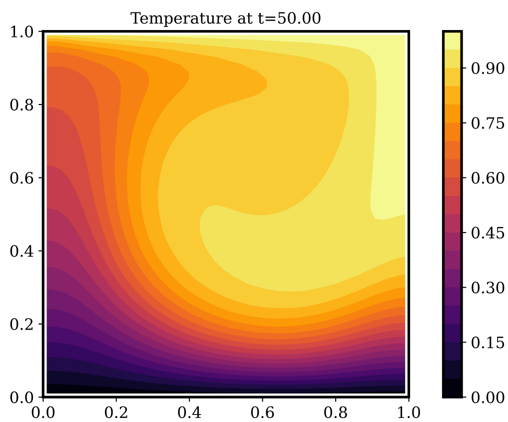


(d) Pressure distribution matrix sparsity

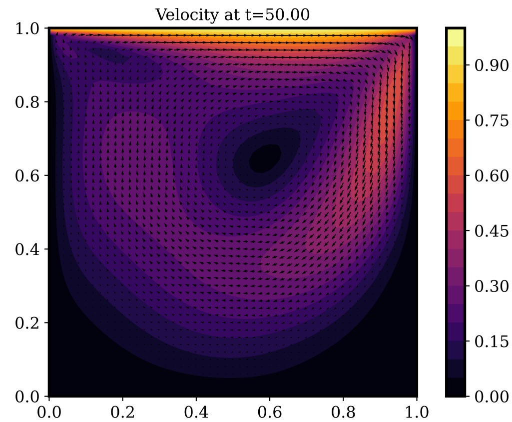
Figure 2: Case A results



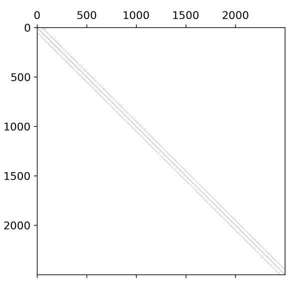
(a) Convergence



(a) Temperature field

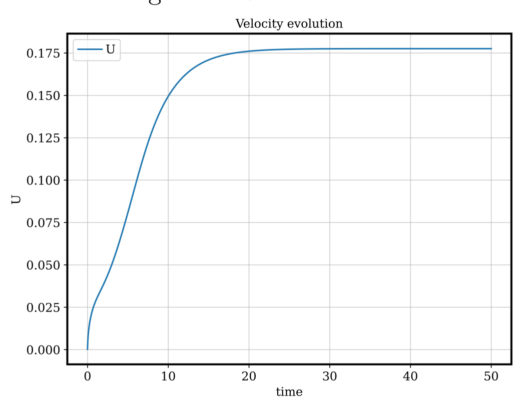


(b) Velocity field

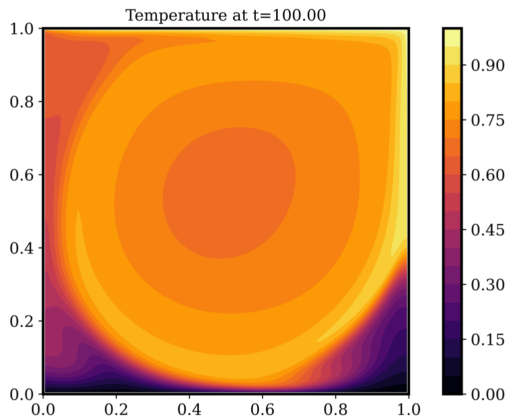


(c) Pressure distribution matrix sparsity

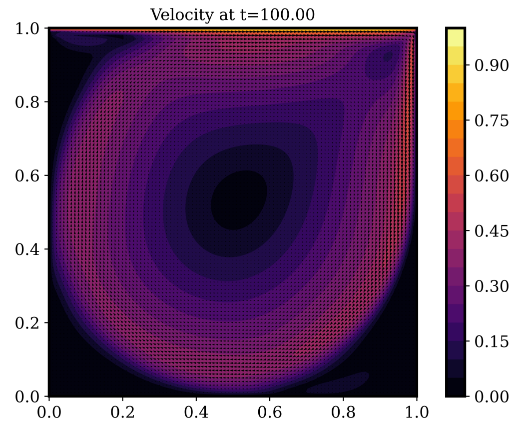
Figure 4: Case B results



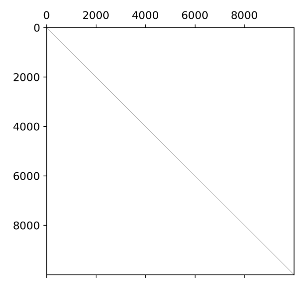
(a) Convergence



(a) Temperature field

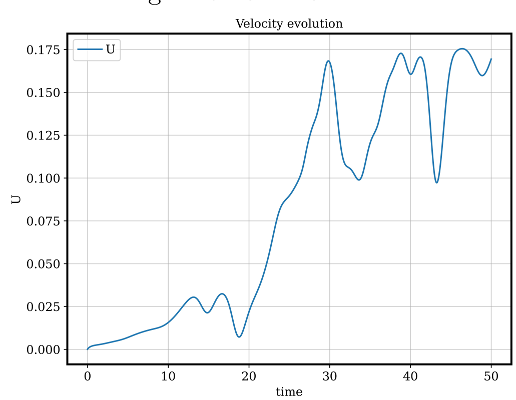


(b) Velocity field

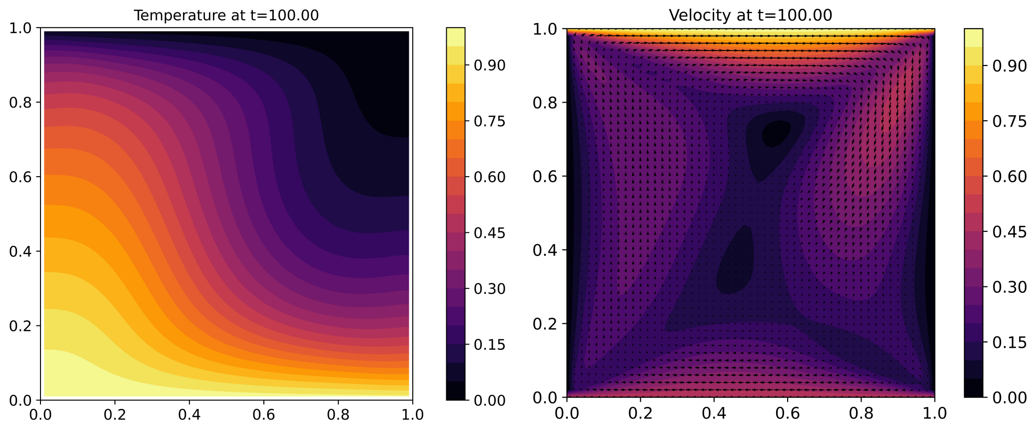


(c) Pressure distribution matrix sparsity

Figure 6: Case C results



(a) Convergence



(a) Temperature field (b) Velocity field.

Figure 8: Moving bottom wall $U_s = -1$ case at Re = 100

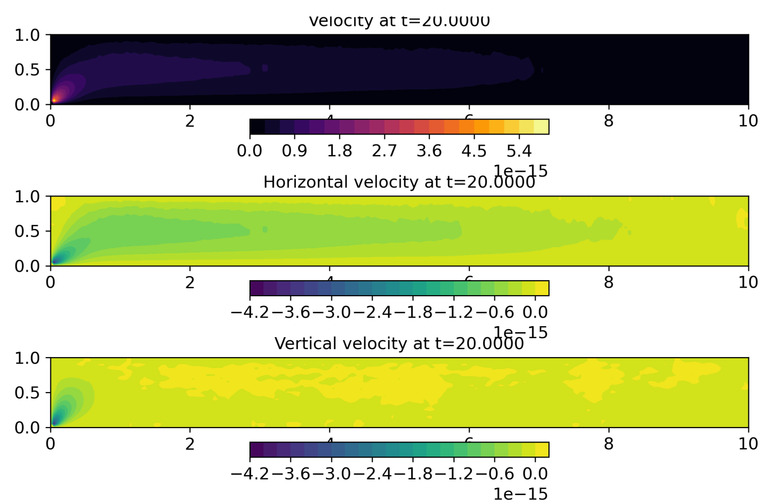


Figure 9: Temperature and velocity fields for Ra =200

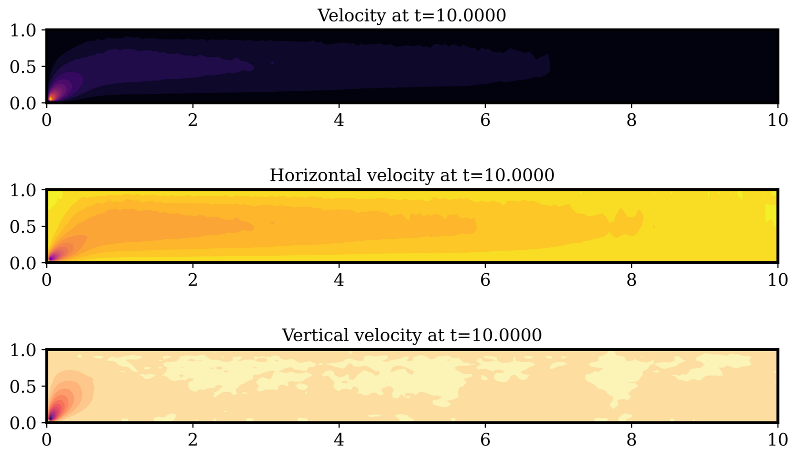


Figure 10: Temperature and velocity fields for Ra =2000

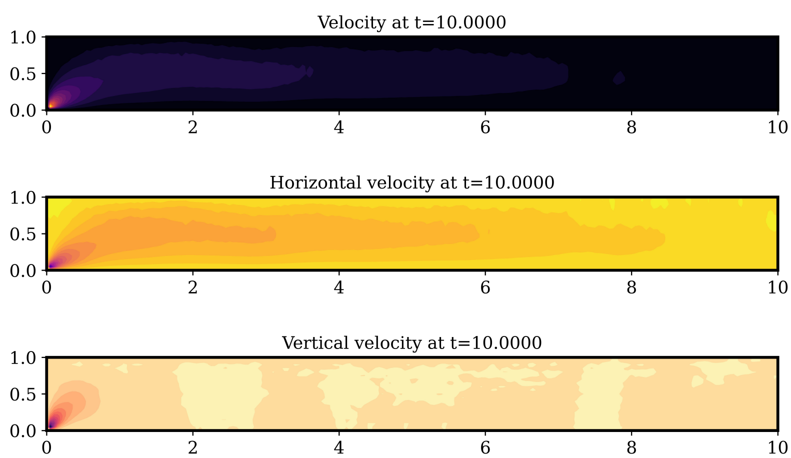
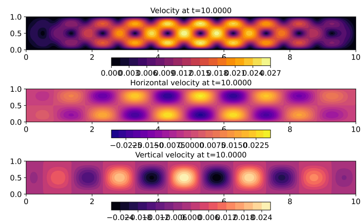
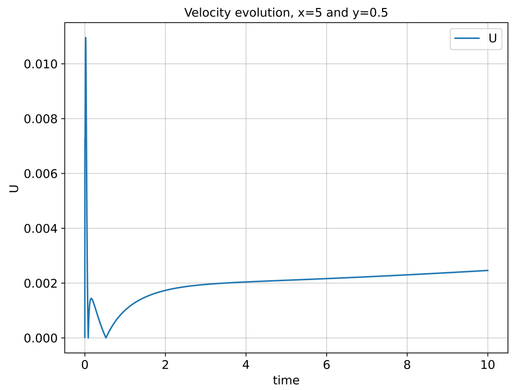


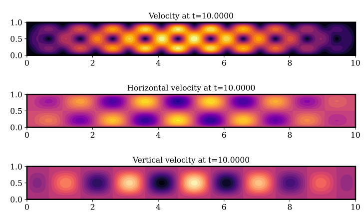
Figure 11: Temperature and velocity fields for Ra =6000



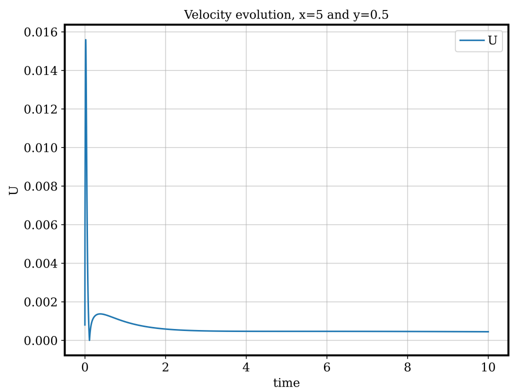
(a) Temperature and velocity fields for Ra =1715



(b) Velocity field evolution



(a) Temperature and velocity fields for Ra =1710



(b) Velocity field evolution