Proof Checker Notes

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1 Syntax Grammar

$$
\begin{array}{rl}
\text{(types)} & \tau ::= \text{bool} \mid \tau \to \tau \mid \text{nat} \mid \text{list } \tau \\
\text{(hypotheses)} & \text{A , B} ::= \top \mid \bot \mid \text{A} \wedge \text{B} \mid \text{A} \vee \text{B} \mid \text{A} \supset \text{B} \mid \forall\, x : \tau \,.\, \text{A} \mid \exists\, x : \tau \,.\, \text{A} \mid \text{t} = \text{t} : \tau \\
\text{(terms)} & \text{e , t} ::= x \mid \text{t t} \mid \text{true} \mid \text{false} \mid [\,] \mid \text{t} :: \text{t} \mid \text{zero} \mid \text{suc(t)} \\
\text{(term context)} & \psi ::= . \mid \psi, x : \tau
\end{array}
$$

$$
\begin{array}{lll}
\psi & \vdash & \text{t} : \tau \\
\psi & \vdash & \text{A prop}
\end{array}
$$

2 Specification rules for terms and propositional hypotheses

Note: functions are included as term types, but not directly as term constructors. Instead, function terms are added into the term context (ψ) manually. This simplifies the checker since function type inference is not required.

2.1 Terms

Natural Numbers:

$$\frac{}{\psi \vdash \text{zero} : \text{nat}} \quad \text{(nat-zero)} \qquad \frac{\psi \vdash \text{t} : \text{nat}}{\psi \vdash \text{suc(t)} : \text{nat}} \quad \text{(nat-suc-n)}$$

Booleans:

$$\frac{}{\psi \vdash \text{true} : \text{bool}} \quad \text{(bool-true)} \qquad \frac{}{\psi \vdash \text{false} : \text{bool}} \quad \text{(bool-false)}$$

Lists:

$$\frac{}{\psi \vdash [\,] : \text{list t}} \quad \text{(list-nil)} \qquad \frac{\psi \vdash \text{t}' : \text{t} \qquad \psi \vdash \text{t}'' : \text{list t}}{\psi \vdash \text{t}' :: \text{t}'' : \text{list t}} \quad \text{(list-cons)}$$

Variables:

$$\frac{x : \tau \in \psi}{\psi \vdash x : \tau} \quad \text{(var)}$$

Application:

$$\frac{\psi \vdash \text{t} : \tau \to \tau' \qquad \psi \vdash \text{t}' : \tau}{\psi \vdash \text{t t}' : \tau'} \quad \text{(app)}$$

2.2 Propositions

Truth and Falsity Propositions:

$$\frac{}{\psi \vdash \top \text{ prop}} \quad (\top\text{-prop}) \qquad \frac{}{\psi \vdash \bot \text{ prop}} \quad (\bot\text{-prop})$$

Binary Relation Propositions:

$$\frac{\psi \vdash \text{A prop} \qquad \psi \vdash \text{B prop}}{\psi \vdash \text{A} \wedge \text{B prop}} \quad (\wedge\text{-prop})$$

$$\frac{\psi \vdash \text{A prop} \qquad \psi \vdash \text{B prop}}{\psi \vdash \text{A} \vee \text{B prop}} \quad (\vee\text{-prop})$$

$$\frac{\psi \vdash \text{A prop} \qquad \psi \vdash \text{B prop}}{\psi \vdash \text{A} \supset \text{B prop}} \quad (\supset\text{-prop})$$

$$\frac{\psi \vdash \text{t}: \tau \qquad \psi \vdash \text{t}': \tau}{\psi \vdash (\text{t} = \text{t}' : \tau) \text{ prop}} \quad (\text{eq-prop})$$

Quantifier Propositions:

$$\frac{\psi, x : \tau \vdash \text{A prop}}{\psi \vdash \forall x : \tau.\ \text{A prop}} \quad (\forall\text{-prop})$$

$$\frac{\psi, x : \tau \vdash \text{A prop}}{\psi \vdash \exists x : \tau.\ \text{A prop}} \quad (\exists\text{-prop})$$

3 Implementation rules for type inference and checking

3.1 Syntax grammar

(infer) e ::= x | e v | true | false | zero | suc(e)
(check) v ::= v :: v | nil | e

Type Inferece Rule:

$$\bar{\psi} \vdash \bar{t} \Rightarrow \overset{+}{\tau}$$

Type Checking Rule:

$$\bar{\psi} \vdash \bar{t} \Leftarrow \bar{\tau}$$

3.2 Term type inference

Variables:

$$\frac{x : \tau \in \psi}{\psi \vdash x \Rightarrow \tau} \quad (\text{var})$$

Application:

$$\frac{\psi \vdash \text{t} \Rightarrow \tau \to \tau' \qquad \psi \vdash \text{t}' \Leftarrow \tau}{\psi \vdash \text{t t}' \Rightarrow \tau'} \quad (\text{app})$$

Natural Numbers:

$$\frac{}{\psi \vdash \text{zero} \Rightarrow \text{nat}} \quad \text{(nat-zero)} \qquad \frac{\psi \vdash \text{t} \Leftarrow \text{nat}}{\psi \vdash \text{suc(t)} \Rightarrow \text{nat}} \quad \text{(nat-suc-n)}$$

Booleans:

$$\frac{}{\psi \vdash \text{true} \Rightarrow \text{bool}} \quad \text{(bool-true)} \qquad \frac{}{\psi \vdash \text{false} \Rightarrow \text{bool}} \quad \text{(bool-false)}$$

3.3 Term type checking

Lists:

$$\frac{}{\psi \vdash [\,] \Leftarrow \text{list t}} \quad \text{(list-nil)} \qquad \frac{\psi \vdash \text{t}' \Leftarrow \text{t} \qquad \psi \vdash \text{t}'' \Leftarrow \text{list t}}{\psi \vdash \text{t}' :: \text{t}'' \Leftarrow \text{list t}} \quad \text{(list-cons)}$$

Inference Case:

$$\frac{\psi \vdash \text{t} \Rightarrow \tau' \qquad \tau = \tau'}{\psi \vdash \text{t} \Leftarrow \tau} \quad \text{(app)}$$

3.4 Propositions type checking

Truth and Falsity Propositions:

$$\frac{}{\psi \vdash \top \text{ prop}} \quad (\top\text{-prop}) \qquad \frac{}{\psi \vdash \bot \text{ prop}} \quad (\bot\text{-prop})$$

Binary Relation Propositions:

$$\frac{\psi \vdash \text{A prop} \qquad \psi \vdash \text{B prop}}{\psi \vdash \text{A} \wedge \text{B prop}} \quad (\wedge\text{-prop})$$

$$\frac{\psi \vdash \text{A prop} \qquad \psi \vdash \text{B prop}}{\psi \vdash \text{A} \vee \text{B prop}} \quad (\vee\text{-prop})$$

$$\frac{\psi \vdash \text{A prop} \qquad \psi \vdash \text{B prop}}{\psi \vdash \text{A} \supset \text{B prop}} \quad (\supset\text{-prop})$$

$$\frac{\psi \vdash \text{t} \Leftarrow \tau \qquad \psi \vdash \text{t}' \Leftarrow \tau}{\psi \vdash (\text{t} = \text{t}' \Leftarrow \tau) \text{ prop}} \quad \text{(eq-prop)}$$

Quantifier Propositions:

$$\frac{\psi, x \Leftarrow \tau \vdash \text{A prop}}{\psi \vdash \forall x \Leftarrow \tau.\ \text{A prop}} \quad (\forall\text{-prop})$$

$$\frac{\psi, x \Leftarrow \tau \vdash \text{A prop}}{\psi \vdash \exists x \Leftarrow \tau.\ \text{A prop}} \quad (\exists\text{-prop})$$

3.5 Function signatures

```
infer_term  :  ψ → t → τ option
check_term  :  ψ → t → τ → unit option
check_prop  :  ψ → A → unit option

val infer_term  :  ctx -> term -> tp option
val check_term  :  ctx -> term -> tp -> unit option
val check_prop  :  ctx -> prop -> unit option
```

4 Well-formedness of proofs

4.1 Syntax grammar

$$
\begin{array}{rrcl}
\text{(proofs)} & \mathrm{p}\,,\,\mathrm{q} & ::= & \mathrm{by\ H} \\
& & \mid & (\mathrm{p}\,,\,\mathrm{q}) \\
& & \mid & \texttt{let}\ (\mathrm{H}',\mathrm{H}'') = \mathrm{H}\ \texttt{in}\ \mathrm{p} \\
& & \mid & (\mathrm{p}\,,\,\mathrm{q})\ \texttt{either} \\
& & \mid & \texttt{match [H] : (A} \vee \texttt{B) with (} \\
& & & \qquad \texttt{| A [H'] : p -> C} \\
& & & \qquad \texttt{| B [H''] : q -> C)} \\
\text{(hypotheses context)} & \Gamma & ::= & \cdot \\
& & \mid & \Gamma\,,\,\mathrm{H} : \mathrm{A} \\
& & \mid & \mathrm{Assume\ A\ [\ H\ ]\ ,\ p}
\end{array}
$$

$$
\begin{array}{rl}
\psi;\Gamma & \vdash \mathrm{p} : \mathrm{A} \\
\psi & \vdash \Gamma
\end{array}
$$

4.2 Rules

Truth and Falsity:

$$\frac{}{\psi;\Gamma \vdash \top : \mathrm{C}}\ (\top\mathrm{R}) \qquad \frac{}{\psi;\Gamma,\,\mathrm{H} : \bot \vdash \mathrm{Absurd\ H} : \mathrm{C}}\ (\bot\mathrm{L})$$

Conjunction:

$$\frac{\psi;\Gamma,\,\mathrm{H} : \mathrm{A} \wedge \mathrm{B}\,,\,\mathrm{H}' : \mathrm{A}\,,\,\mathrm{H}'' : \mathrm{B} \vdash \mathrm{p} : \mathrm{C}}{\psi;\Gamma,\,\mathrm{H} : \mathrm{A} \wedge \mathrm{B} \vdash \texttt{let}\ (\mathrm{H}',\mathrm{H}'')\ =\ \mathrm{H}\ \texttt{in}\ \mathrm{p} : \mathrm{C}}\ (\wedge\mathrm{L})$$

$$\frac{\psi;\Gamma \vdash \mathrm{p} : \mathrm{A} \qquad \psi;\Gamma \vdash \mathrm{q} : \mathrm{B}}{\psi;\Gamma \vdash (\mathrm{p}\,,\,\mathrm{q}) : \mathrm{A} \wedge \mathrm{B}}\ (\wedge\mathrm{R})$$

Disjunction:

$$\frac{\psi;\Gamma\,,\,\mathrm{H} : \mathrm{A} \vee \mathrm{B}\,,\,\mathrm{H}' : \mathrm{A} \vdash \mathrm{p} : \mathrm{C} \qquad \psi;\Gamma\,,\,\mathrm{H} : \mathrm{A} \vee \mathrm{B}\,,\,\mathrm{H}'' : \mathrm{B} \vdash \mathrm{q} : \mathrm{C}}{\psi;\Gamma,\,\mathrm{H} : \mathrm{A} \vee \mathrm{B} \vdash \texttt{match [H] with(}\ \mathrm{A}\ \texttt{[H']}\ \texttt{:}\ \mathrm{p}\ \mid\ \mathrm{B}\ \texttt{[H'']}\ \texttt{:}\ \mathrm{q}\ \texttt{)}\ : \mathrm{C}}\ (\vee\mathrm{L})$$

$$\frac{\psi;\Gamma \vdash \mathrm{p} : \mathrm{A}}{\psi;\Gamma \vdash \texttt{Left}\ \mathrm{p} : \mathrm{A} \vee \mathrm{B}}\ (\vee R_1)$$

$$\frac{\psi;\Gamma \vdash \mathrm{q} : \mathrm{B}}{\psi;\Gamma \vdash \texttt{Right}\ \mathrm{q} : \mathrm{A} \vee \mathrm{B}}\ (\vee R_2)$$

Implication:

$$\frac{\psi;\Gamma,\,\mathrm{H} : \mathrm{A} \supset \mathrm{B} \vdash \mathrm{p} : \mathrm{A} \qquad \psi;\Gamma,\,\mathrm{H} : \mathrm{A} \supset \mathrm{B}\,,\,\mathrm{H}' : \mathrm{B} \vdash \mathrm{q} : \mathrm{C}}{\psi;\Gamma,\,\mathrm{H} : \mathrm{A} \supset \mathrm{B} \vdash (\mathrm{p},\,\mathrm{B}\ [\ \mathrm{H}'\ ]\ \mathrm{via\ H},\,\mathrm{q}) : \mathrm{C}}\ (\supset\mathrm{L})$$

$$\frac{\psi;\Gamma\,,\,\mathrm{H} : \mathrm{A} \vdash \mathrm{p} : \mathrm{B}}{\psi;\Gamma \vdash (\ \mathrm{Assume\ A\ [\ H\ ]\ ,\ p}\ ) : \mathrm{A} \supset \mathrm{B}}\ (\supset\mathrm{R})$$

Using hypotheses:

$$\frac{}{\psi;\Gamma,\,[\ \mathrm{H}\ ] : \mathrm{A} \vdash \mathrm{by\ H} : \mathrm{A}}\ (\mathrm{by})$$

$$\frac{\psi; \Gamma \vdash \text{p} : \text{A}}{\psi; \Gamma \vdash \text{p Therefore A} : \text{A}} \quad \text{(therefore)}$$

4.3 Function signature

```
check_proof :  ψ → Γ → P → A → unit option
```

5 Quantifiers in proofs

5.1 Rules

Existentials:

$$\frac{\psi; \Gamma \vdash \text{t} : \tau \qquad \psi; \Gamma \vdash \text{p} : [\text{x} \mapsto \text{t}]\ \text{A}}{\psi; \Gamma \vdash \text{Choose t ; p} : \exists x : \tau.\text{A}} \quad (\exists\ \text{R})$$

$$\frac{\psi, y : \tau; \Gamma, \text{H} : \exists x : \tau.A, \text{H'}: [x \mapsto y]A \vdash \text{p} : \text{C}}{\psi; \Gamma, \text{H} : \exists x : \tau.\ \text{A} \vdash \texttt{let}\ (y,\text{H'}) = \text{H}\ \texttt{in}\ \text{p} : \text{C}} \quad (\exists\ \text{L})$$

Universals:

$$\frac{\psi, y : \tau; \Gamma, \vdash \text{p} : [x \mapsto y]\ \text{A}}{\psi; \Gamma \vdash \text{Assume } y : \tau\ .\ \text{p} : \forall x : \tau.A} \quad (\forall\ \text{R})$$

$$\frac{\psi; \Gamma \vdash \text{t} : \tau \qquad \psi; \Gamma\ , \text{H} : \forall x : \tau\ .\ \text{A}\ , \text{H'} : [\text{x} \mapsto \text{t}]\ \text{A} \vdash \text{p} : \text{C}}{\psi; \Gamma, \text{H} : \forall x : \tau.\ \text{A} \vdash \texttt{let}\ \text{H'} = \text{H with t}\ \texttt{in}\ \text{p} : \text{C}} \quad (\forall\ \text{L})$$

5.2 Substituting terms into variables

```
subs = [x ↦ z]
```

```
subs_term  :  x → t → t → t
subs_prop  :  x → t → A → [x] → A
```

```
val subs_term  :  var -> term -> term -> term
val subs_prop  :  var -> term -> prop -> var list -> prop
```

6 α-equivalence

6.1 Terms

Variables:

$$\frac{}{x \overset{\alpha}{=} x} \quad (\text{var}\overset{\alpha}{=})$$

Booleans:

$$\frac{}{\text{true} \overset{\alpha}{=} \text{true}} \quad (\text{bool-true}\overset{\alpha}{=}) \qquad \frac{}{\text{false} \overset{\alpha}{=} \text{false}} \quad (\text{bool-false}\overset{\alpha}{=})$$

Natural Numbers:

$$\frac{}{\text{zero} \overset{\alpha}{=} \text{zero}} \quad (\text{nat-zero}\overset{\alpha}{=}) \qquad \frac{\text{t} \overset{\alpha}{=} \text{t}'}{\text{suc(}\ \text{t}\ \text{)} \overset{\alpha}{=} \text{suc(}\ \text{t}'\ \text{)}} \quad (\text{nat-suc-n}\overset{\alpha}{=})$$

Lists:

$$\frac{}{[] \overset{\alpha}{=} []} \quad (\text{list-nil}\overset{\alpha}{=}) \qquad \frac{\text{e} \overset{\alpha}{=} \text{e}' \qquad \text{v} \overset{\alpha}{=} \text{v}'}{\text{e::v} \overset{\alpha}{=} \text{e}'::\text{v}'} \quad (\text{list-cons}\overset{\alpha}{=})$$

Application:

$$\frac{\text{e} \overset{\alpha}{=} \text{e}' \qquad \text{v} \overset{\alpha}{=} \text{v}'}{\text{e v} \overset{\alpha}{=} \text{e}'\ \text{v}'} \quad (\text{var}\overset{\alpha}{=})$$

6.2 Propositions

Truth and Falsity:

$$\frac{}{\top \overset{\alpha}{=} \top} \quad (\top \overset{\alpha}{=}) \qquad \frac{}{\bot \overset{\alpha}{=} \bot} \quad (\bot \overset{\alpha}{=})$$

Binary Relations:

$$\frac{\text{A} \overset{\alpha}{=} \text{A}' \qquad \text{B} \overset{\alpha}{=} \text{B}'}{\text{A} \wedge \text{B} \overset{\alpha}{=} \text{A}' \wedge \text{B}'} \quad (\wedge \overset{\alpha}{=})$$

$$\frac{\text{A} \overset{\alpha}{=} \text{A}' \qquad \text{B} \overset{\alpha}{=} \text{B}'}{\text{A} \vee \text{B} \overset{\alpha}{=} \text{A}' \vee \text{B}'} \quad (\vee \overset{\alpha}{=})$$

$$\frac{\text{A} \overset{\alpha}{=} \text{A}' \qquad \text{B} \overset{\alpha}{=} \text{B}'}{\text{A} \supset \text{B} \overset{\alpha}{=} \text{A}' \supset \text{B}'} \quad (\supset \overset{\alpha}{=})$$

Equality:

$$\frac{\text{t}_1 \overset{\alpha}{=} \text{t}_1' \qquad \text{t}_2 \overset{\alpha}{=} \text{t}_2' \qquad \tau \overset{\alpha}{=} \tau'}{(\text{t}_1 = \text{t}_2 : \tau) \overset{\alpha}{=} (\text{t}_1' = \text{t}_2' : \tau')} \quad (= \overset{\alpha}{=})$$

Quantifiers:

$$\frac{И\, z\, .\, (x\ z)\ \text{B} \overset{\alpha}{=} (x\ z)\ \text{B}' \qquad \tau \overset{\alpha}{=} \tau'}{\exists x : \tau.\ \text{B} \overset{\alpha}{=} \exists y : \tau'.\ \text{B}'} \quad (\exists \overset{\alpha}{=})$$

$$\frac{И\, z\, .\, (x\ z)\ \text{B} \overset{\alpha}{=} (x\ z)\ \text{B}' \qquad \tau \overset{\alpha}{=} \tau'}{\forall x : \tau.\ \text{B} \overset{\alpha}{=} \forall y : \tau'.\ \text{B}'} \quad (\forall \overset{\alpha}{=})$$

6.3 Swapping variable names

```
swap = (x z)
```

```
swap_term  :  x → z → t → t
swap_prop  :  x → z → A → A
```

```
val swap_term  :  var -> var -> term -> term
val swap_prop  :  var -> var -> prop -> prop
```

7 Induction in proofs

7.1 Rules through predicates

Natural Numbers:

$$\frac{\psi;\Gamma \vdash \text{p} : \text{P(zero)} \qquad \psi\ ,\ n : \text{nat}\ ;\ \Gamma\ ,\ \text{H} : \text{P}(n) \vdash \text{q} : \text{P(suc}(n))}{\psi;\Gamma \vdash \text{(Induction on nat: case zero : p ; case suc}(n) : \text{[H] , q)} : (\forall\ m : \text{nat}\ .\ \text{P}(m))} \quad \text{(induction-nat)}$$

Lists:

$$\frac{\psi;\Gamma \vdash \text{p} : \text{P([])} \qquad \psi\ ,\ x : \tau\ ,\ xs : \text{list}\ \tau\ ;\ \Gamma\ ,\ \text{H} : \text{P}(xs) \vdash \text{q} : \text{P}(x :: xs)}{\psi;\Gamma \vdash \text{(Induction on list: case [] : p ; case } (x :: xs) : \text{[H] , q)} : (\forall\ ys : \text{list}\ \tau\ .\ \text{P}(ys))} \quad \text{(induction-list)}$$

Booleans:

$$\frac{\psi;\Gamma \vdash \text{p} : \text{P(true)} \qquad \psi;\Gamma \vdash \text{q} : \text{P(false)}}{\psi;\Gamma \vdash \text{(Induction on bool: case true : p ; case false : q)} : (\forall\ b : \text{bool}\ .\ \text{P(}\ b\ \text{))}} \quad \text{(induction-bool)}$$

7.2 Rules through substitution

Natural Numbers:

$$\frac{\psi;\Gamma \vdash \text{p} : [\text{m} \mapsto \text{zero}]\ \text{C} \qquad \psi\ ,\ n : \text{nat}\ ;\ \Gamma\ ,\ \text{H} : [\text{m} \mapsto \text{zero}]\ \text{C} \vdash \text{q} : [\text{m} \mapsto \text{suc}(n)]\ \text{C}}{\psi;\Gamma \vdash \text{(Ind-Nat: zero : p ; suc}(n) : \text{[H] , q)} : (\forall\ m : \text{nat}\ .\ \text{C})} \quad \text{(induction-nat)}$$

Lists:

$$\frac{\psi;\Gamma \vdash \text{p} : [ys \mapsto [\,]\,]\ \text{C} \qquad \psi\ ,\ x : \tau\ ,\ xs : \text{list}\ \tau\ ;\ \Gamma\ ,\ \text{H} : [ys \mapsto xs\,]\ \text{C} \vdash \text{q} : [ys \mapsto x :: xs\,]\ \text{C}}{\psi;\Gamma \vdash \text{(Ind-List: [] : p ; } (x :: xs) : \text{[H] , q)} : (\forall\ ys : \text{list}\ \tau\ .\ \text{C})} \quad \text{(induction-list)}$$

Booleans:

$$\frac{\psi;\Gamma \vdash \text{p} : [b \mapsto \text{true}\,]\ \text{C} \qquad \psi;\Gamma \vdash \text{q} : [b \mapsto \text{false}\,]\ \text{C}}{\psi;\Gamma \vdash \text{(Ind-Bool: true : p ; false : q)} : (\forall\ b : \text{bool}\ .\ \text{C})} \quad \text{(induction-bool)}$$

8 Equality in proofs

8.1 Abstract congruence closure [1, p. 4–7]

8.1.1 Definition

Rewrite-Rules:

D-rule :	$f(c_0, ...c_k) \to c$	*where f is a term constructor and c_i are constants in K*
C-rule :	$c \to d$	*where c and d are constants in K*

Sets:

$$D : \{\textit{D-rule}\}$$
$$C : \{\textit{C-rule}\}$$

$$E : \{(t = t) : \tau\}$$
$$K : \{x | x \notin E\}$$
$$R : D \cup C$$

Closure Construction:

$$
\begin{aligned}
&\mathit{state} : (K, E, R)\\
&\texttt{state\_transition} : \mathit{state} \to \mathit{state}\ \texttt{option}\\
&\texttt{construct\_closure}\ \sigma = \texttt{construct\_closure}(\texttt{state\_transition}\ \sigma)\\
&[\![\texttt{construct\_closure}]\!] : (\emptyset, E, \emptyset) \mapsto (K, \emptyset, R)\\
&\qquad\qquad \textit{where } R \textit{ is the (abstract) congruence closure for } E
\end{aligned}
$$

8.1.2 Sate transition rules

$$\text{Extension:} \quad \frac{(K, E[t], R) \qquad t = f(c_0, ..., c_k) \qquad c_i \notin K}{(K \cup \{c\}, E[c], R \cup \{t \to c \ : D\})} \quad \text{(Ext)}$$

$$\text{Simplification:} \quad \frac{(K, E[t], R \cup \{t \to c : D\})}{(K, E[c], R \cup \{t \to c : D\})} \quad \text{(Sim)}$$

$$\text{Orientation:} \quad \frac{(K \cup \{c\}, E \cup \{t = c\}, R\})}{(K \cup \{c\}, E, R \cup \{t \to c : D\})} \quad \text{(Ori)}$$

$$\text{Deletion:} \quad \frac{(K, E \cup \{t = t\}, R\})}{(K, E, R\})} \quad \text{(Del)}$$

$$\text{Deduction:} \quad \frac{(K, E, R \cup \{t \to c : D, t \to d : D\})}{(K, E \cup \{c = d\}, R \cup \{t \to d : D\})} \quad \text{(Ded)}$$

$$\text{Collapse:} \quad \frac{(K, E, R \cup \{s[c] \to c' : D, c \to d : C\})}{(K, E, R \cup \{s[d] \to c' : D, c \to d : C\})} \quad \text{(Col)}$$

$$\text{Composition:} \quad \frac{(K, E, R \cup \{t \to c : D, c \to d : C\})}{(K, E, R \cup \{t \to d : D, c \to d : C\})} \quad \text{(Com)}$$

References

[1] Leo Bachmair, Ashish Tiwari, and Laurent Vigneron. Abstract congruence closure. *J. Autom. Reasoning*, 31(2):129–168, 2003.