Proof Checker Notes

Yu-Yang Lin

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1 Syntax Grammar

$$
\begin{array}{rl}
\text{(types)} & \tau ::= \mathrm{bool} \mid \tau \rightarrow \tau \mid \mathrm{nat} \mid \mathrm{list}\ \tau \\
\text{(hypotheses)} & \mathrm{A}\ ,\mathrm{B} ::= \top \mid \bot \mid \mathrm{A} \wedge \mathrm{B} \mid \mathrm{A} \vee \mathrm{B} \mid \mathrm{A} \supset \mathrm{B} \mid \forall\, x : \tau\ .\ \mathrm{A} \mid \exists\, x : \tau\ .\ \mathrm{A} \mid \mathrm{t} = \mathrm{t} : \tau \\
\text{(terms)} & \mathrm{e}\ ,\mathrm{t} ::= x \mid \mathrm{t\ t} \mid \mathrm{true} \mid \mathrm{false} \mid [\ ] \mid \mathrm{t} :: \mathrm{t} \mid \mathrm{zero} \mid \mathrm{suc(\ t\ )} \\
\text{(term context)} & \psi ::= .\ \mid \psi, x : \tau
\end{array}
$$

$$
\begin{array}{ccl}
\psi & \vdash & \mathrm{t} : \tau \\
\psi & \vdash & \mathrm{A\ prop}
\end{array}
$$

2 Specification rules for terms and propositional hypotheses

Note: functions are included as term types, but not directly as term constructors. Instead, function terms are added into the term context (ψ) manually. This simplifies the checker since function type inference is not required.

2.1 Terms

Natural Numbers:

$$\frac{}{\psi \vdash \mathrm{zero} : \mathrm{nat}}\ \text{(nat-zero)} \qquad \frac{\psi \vdash \mathrm{t} : \mathrm{nat}}{\psi \vdash \mathrm{suc(\ t\ )} : \mathrm{nat}}\ \text{(nat-suc-n)}$$

Booleans:

$$\frac{}{\psi \vdash \mathrm{true} : \mathrm{bool}}\ \text{(bool-true)} \qquad \frac{}{\psi \vdash \mathrm{false} : \mathrm{bool}}\ \text{(bool-false)}$$

Lists:

$$\frac{}{\psi \vdash [\ ] : \mathrm{list\ t}}\ \text{(list-nil)} \qquad \frac{\psi \vdash \mathrm{t}' : \mathrm{t} \qquad \psi \vdash \mathrm{t}'' : \mathrm{list\ t}}{\psi \vdash \mathrm{t}' :: \mathrm{t}'' : \mathrm{list\ t}}\ \text{(list-cons)}$$

Variables:

$$\frac{x : \tau \in \psi}{\psi \vdash x : \tau}\ \text{(var)}$$

Application:

$$\frac{\psi \vdash \mathrm{t} : \tau \rightarrow \tau' \qquad \psi \vdash \mathrm{t}' : \tau}{\psi \vdash \mathrm{t\ t}' : \tau'}\ \text{(app)}$$

2.2 Propositions

Truth and Falsity Propositions:

$$\frac{}{\psi \vdash \top \text{ prop}} \quad (\top\text{-prop}) \qquad \frac{}{\psi \vdash \bot \text{ prop}} \quad (\bot\text{-prop})$$

Binary Relation Propositions:

$$\frac{\psi \vdash \text{A prop} \qquad \psi \vdash \text{B prop}}{\psi \vdash \text{A} \wedge \text{B prop}} \quad (\wedge\text{-prop})$$

$$\frac{\psi \vdash \text{A prop} \qquad \psi \vdash \text{B prop}}{\psi \vdash \text{A} \vee \text{B prop}} \quad (\vee\text{-prop})$$

$$\frac{\psi \vdash \text{A prop} \qquad \psi \vdash \text{B prop}}{\psi \vdash \text{A} \supset \text{B prop}} \quad (\supset\text{-prop})$$

$$\frac{\psi \vdash \text{t}: \tau \qquad \psi \vdash \text{t}': \tau}{\psi \vdash (\text{t} = \text{t}' : \tau) \text{ prop}} \quad (\text{eq-prop})$$

Quantifier Propositions:

$$\frac{\psi, x : \tau \vdash \text{A prop}}{\psi \vdash \forall x : \tau.\ \text{A prop}} \quad (\forall\text{-prop})$$

$$\frac{\psi, x : \tau \vdash \text{A prop}}{\psi \vdash \exists x : \tau.\ \text{A prop}} \quad (\exists\text{-prop})$$

3 Implementation rules for type inference and checking

3.1 Syntax grammar

$$\begin{array}{rl} \text{(infer)} & \text{e ::= x} \mid \text{e v} \mid \text{true} \mid \text{false} \mid \text{zero} \mid \text{suc(e)} \\ \text{(check)} & \text{v ::= v :: v} \mid \text{nil} \mid \text{e} \end{array}$$

Type Inferece Rule:

$$\bar{\psi} \vdash \bar{t} \Rightarrow \overset{+}{\tau}$$

Type Checking Rule:

$$\bar{\psi} \vdash \bar{t} \Leftarrow \bar{\tau}$$

3.2 Term type inference

Variables:

$$\frac{x : \tau \in \psi}{\psi \vdash x \Rightarrow \tau} \quad (\text{var})$$

Application:

$$\frac{\psi \vdash \text{t} \Rightarrow \tau \to \tau' \qquad \psi \vdash \text{t}' \Leftarrow \tau}{\psi \vdash \text{t t}' \Rightarrow \tau'} \quad (\text{app})$$

Natural Numbers:

$$\frac{}{\psi \vdash \text{zero} \Rightarrow \text{nat}} \text{ (nat-zero)} \qquad \frac{\psi \vdash \text{t} \Leftarrow \text{nat}}{\psi \vdash \text{suc(t)} \Rightarrow \text{nat}} \text{ (nat-suc-n)}$$

Booleans:

$$\frac{}{\psi \vdash \text{true} \Rightarrow \text{bool}} \text{ (bool-true)} \qquad \frac{}{\psi \vdash \text{false} \Rightarrow \text{bool}} \text{ (bool-false)}$$

3.3 Term type checking

Lists:

$$\frac{}{\psi \vdash [\,] \Leftarrow \text{list t}} \text{ (list-nil)} \qquad \frac{\psi \vdash \text{t}' \Leftarrow \text{t} \qquad \psi \vdash \text{t}'' \Leftarrow \text{list t}}{\psi \vdash \text{t}' :: \text{t}'' \Leftarrow \text{list t}} \text{ (list-cons)}$$

Inference Case:

$$\frac{\psi \vdash \text{t} \Rightarrow \tau' \qquad \tau = \tau'}{\psi \vdash \text{t} \Leftarrow \tau} \text{ (app)}$$

3.4 Propositions type checking

Truth and Falsity Propositions:

$$\frac{}{\psi \vdash \top \text{ prop}} \text{ ($\top$-prop)} \qquad \frac{}{\psi \vdash \bot \text{ prop}} \text{ ($\bot$-prop)}$$

Binary Relation Propositions:

$$\frac{\psi \vdash \text{A prop} \qquad \psi \vdash \text{B prop}}{\psi \vdash \text{A} \wedge \text{B prop}} \text{ ($\wedge$-prop)}$$

$$\frac{\psi \vdash \text{A prop} \qquad \psi \vdash \text{B prop}}{\psi \vdash \text{A} \vee \text{B prop}} \text{ ($\vee$-prop)}$$

$$\frac{\psi \vdash \text{A prop} \qquad \psi \vdash \text{B prop}}{\psi \vdash \text{A} \supset \text{B prop}} \text{ ($\supset$-prop)}$$

$$\frac{\psi \vdash \text{t} \Leftarrow \tau \qquad \psi \vdash \text{t}' \Leftarrow \tau}{\psi \vdash (\text{t} = \text{t}' \Leftarrow \tau) \text{ prop}} \text{ (eq-prop)}$$

Quantifier Propositions:

$$\frac{\psi, x \Leftarrow \tau \vdash \text{A prop}}{\psi \vdash \forall x \Leftarrow \tau.\ \text{A prop}} \text{ ($\forall$-prop)}$$

$$\frac{\psi, x \Leftarrow \tau \vdash \text{A prop}}{\psi \vdash \exists x \Leftarrow \tau.\ \text{A prop}} \text{ ($\exists$-prop)}$$

3.5 Function signatures

$$\begin{array}{lcl}
\texttt{infer_term} & : & \psi \rightarrow \text{t} \rightarrow \tau \ \texttt{option} \\
\texttt{check_term} & : & \psi \rightarrow \text{t} \rightarrow \tau \rightarrow \texttt{unit option} \\
\texttt{check_prop} & : & \psi \rightarrow \text{A} \rightarrow \texttt{unit option}
\end{array}$$

```
val infer_term  :  ctx -> term -> tp option
val check_term  :  ctx -> term -> tp -> unit option
val check_prop  :  ctx -> prop -> unit option
```

4 Well-formedness of proofs

4.1 Syntax grammar

$$
\begin{array}{rrcl}
\text{(proofs)} & \text{p , q} & ::= & \text{by H} \\
 & & | & \text{(p , q)} \\
 & & | & \text{let (H}',\text{H}'') = \text{H in p} \\
 & & | & \text{(p , q) either} \\
 & & | & \text{match [H] : (A} \lor \text{B) with (} \\
 & & & \qquad\qquad | \ \text{A [H}']\text{: p -> C} \\
 & & & \qquad\qquad | \ \text{B [H}'']\text{: q -> C)} \\
\text{(hypotheses context)} & \Gamma & ::= & \cdot \\
 & & | & \Gamma \text{ , H : A} \\
 & & | & \text{Assume A [H] , p}
\end{array}
$$

$$
\begin{array}{rl}
\psi;\Gamma & \vdash \text{p : A} \\
\psi & \vdash \Gamma
\end{array}
$$

4.2 Rules

Truth and Falsity:

$$\frac{}{\psi;\Gamma \vdash \top : \text{C}} \ (\top\text{R}) \qquad \frac{}{\psi;\Gamma,\text{ H} : \bot \vdash \text{Absurd H : C}} \ (\bot\text{L})$$

Conjunction:

$$\frac{\psi;\Gamma,\text{ H : A} \land \text{B , H}' \text{ : A , H}'' \text{ : B} \vdash \text{p : C}}{\psi;\Gamma,\text{ H : A} \land \text{B} \vdash \text{let (H}',\text{H}'') = \text{H in p : C}} \ (\land\text{L})$$

$$\frac{\psi;\Gamma \vdash \text{p : A} \qquad \psi;\Gamma \vdash \text{q : B}}{\psi;\Gamma \vdash \text{(p , q) : A} \land \text{B}} \ (\land\text{R})$$

Disjunction:

$$\frac{\psi;\Gamma\text{ , H : A} \lor \text{B , H}' \text{ : A} \vdash \text{p : C} \qquad \psi;\Gamma\text{ , H : A} \lor \text{B , H}'' \text{ : B} \vdash \text{q : C}}{\psi;\Gamma,\text{ H : A} \lor \text{B} \vdash \text{match [H] with(A [H}'\text{]: p } | \text{ B [H}''\text{]: q) : C}} \ (\lor\text{L})$$

$$\frac{\psi;\Gamma \vdash \text{p : A}}{\psi;\Gamma \vdash \text{Left p : A} \lor \text{B}} \ (\lor R_1)$$

$$\frac{\psi;\Gamma \vdash \text{q : B}}{\psi;\Gamma \vdash \text{Right q : A} \lor \text{B}} \ (\lor R_2)$$

Implication:

$$\frac{\psi;\Gamma,\text{ H : A} \supset \text{B} \vdash \text{p : A} \qquad \psi;\Gamma,\text{ H : A} \supset \text{B , H}' \text{ : B} \vdash \text{q : C}}{\psi;\Gamma,\text{ H : A} \supset \text{B} \vdash \text{(p, B [H}' \text{] via H, q) : C}} \ (\supset\text{L})$$

$$\frac{\psi;\Gamma\text{ , H : A} \vdash \text{p : B}}{\psi;\Gamma \vdash \text{(Assume A [H] , p) : A} \supset \text{B}} \ (\supset\text{R})$$

Using hypotheses:

$$\frac{}{\psi;\Gamma,\text{ [H] : A} \vdash \text{by H : A}} \ (\text{by})$$

$$\frac{\psi;\Gamma \vdash \mathrm{p} : \mathrm{A}}{\psi;\Gamma \vdash \mathrm{p\ Therefore\ A} : \mathrm{A}} \quad \text{(therefore)}$$

4.3 Function signature

$$\texttt{check\_proof} \ : \ \psi \rightarrow \Gamma \rightarrow \texttt{P} \rightarrow \texttt{A} \rightarrow \texttt{unit option}$$

5 Quantifiers in proofs

5.1 Rules

Existentials:

$$\frac{\psi;\Gamma \vdash \mathrm{t} : \tau \qquad \psi;\Gamma \vdash \mathrm{p} : [\mathrm{x} \mapsto \mathrm{t}]\ \mathrm{A}}{\psi;\Gamma \vdash \mathrm{Choose\ t\ ;\ p} : \exists x : \tau.\mathrm{A}} \quad (\exists\ \mathrm{R})$$

$$\frac{\psi, y : \tau;\Gamma, \mathrm{H} : \exists x : \tau.A, \mathrm{H'}\text{:}\ [x \mapsto y]A \vdash \mathrm{p} : \mathrm{C}}{\psi;\Gamma, \mathrm{H} : \exists x : \tau.\ \mathrm{A} \vdash \mathrm{let}\ (y,\mathrm{H'}) = \mathrm{H\ in\ p} : \mathrm{C}} \quad (\exists\ \mathrm{L})$$

Universals:

$$\frac{\psi, y : \tau;\Gamma, \vdash \mathrm{p} : [x \mapsto y]\ \mathrm{A}}{\psi;\Gamma \vdash \mathrm{Assume}\ y : \tau\ .\ \mathrm{p} : \forall x : \tau.A} \quad (\forall\ \mathrm{R})$$

$$\frac{\psi;\Gamma \vdash \mathrm{t} : \tau \qquad \psi;\Gamma\ , \mathrm{H} : \forall x : \tau\ .\ \mathrm{A}\ , \mathrm{H'} : [\mathrm{x} \mapsto \mathrm{t}]\ \mathrm{A} \vdash \mathrm{p} : \mathrm{C}}{\psi;\Gamma, \mathrm{H} : \forall x : \tau.\ \mathrm{A} \vdash \mathrm{let\ H'} = \mathrm{H\ with\ t\ in\ p} : \mathrm{C}} \quad (\forall\ \mathrm{L})$$

5.2 Substituting terms into variables

$$\texttt{subs} = [x \mapsto z]$$

$$\begin{array}{lcl} \texttt{subs\_term} & : & x \rightarrow \texttt{t} \rightarrow \texttt{t} \rightarrow \texttt{t} \\ \texttt{subs\_prop} & : & x \rightarrow \texttt{t} \rightarrow \texttt{A} \rightarrow [x] \rightarrow \texttt{A} \end{array}$$

```
val subs_term  :  var -> term -> term -> term
val subs_prop  :  var -> term -> prop -> var list -> prop
```

6 α-equivalence

6.1 Terms

Variables:

$$\frac{}{x \overset{\alpha}{=} x} \quad (\mathrm{var}\overset{\alpha}{=})$$

Booleans:

$$\frac{}{\mathrm{true} \overset{\alpha}{=} \mathrm{true}} \quad (\text{bool-true}\overset{\alpha}{=}) \qquad \frac{}{\mathrm{false} \overset{\alpha}{=} \mathrm{false}} \quad (\text{bool-false}\overset{\alpha}{=})$$

Natural Numbers:

$$\frac{}{\mathrm{zero} \overset{\alpha}{=} \mathrm{zero}}\ (\text{nat-zero}\overset{\alpha}{=}) \qquad \frac{\mathrm{t} \overset{\alpha}{=} \mathrm{t}'}{\mathrm{suc(\ t\ )} \overset{\alpha}{=} \mathrm{suc(\ t'\ )}}\ (\text{nat-suc-n}\overset{\alpha}{=})$$

Lists:

$$\frac{}{[] \overset{\alpha}{=} []}\ (\text{list-nil}\overset{\alpha}{=}) \qquad \frac{\mathrm{e} \overset{\alpha}{=} \mathrm{e}' \qquad \mathrm{v} \overset{\alpha}{=} \mathrm{v}'}{\mathrm{e::v} \overset{\alpha}{=} \mathrm{e'::v'}}\ (\text{list-cons}\overset{\alpha}{=})$$

Application:

$$\frac{\mathrm{e} \overset{\alpha}{=} \mathrm{e}' \qquad \mathrm{v} \overset{\alpha}{=} \mathrm{v}'}{\mathrm{e\ v} \overset{\alpha}{=} \mathrm{e'\ v'}}\ (\text{var}\overset{\alpha}{=})$$

6.2 Propositions

Truth and Falsity:

$$\frac{}{\top \overset{\alpha}{=} \top}\ (\top \overset{\alpha}{=}) \qquad \frac{}{\bot \overset{\alpha}{=} \bot}\ (\bot \overset{\alpha}{=})$$

Binary Relations:

$$\frac{\mathrm{A} \overset{\alpha}{=} \mathrm{A}' \qquad \mathrm{B} \overset{\alpha}{=} \mathrm{B}'}{\mathrm{A} \wedge \mathrm{B} \overset{\alpha}{=} \mathrm{A}' \wedge \mathrm{B}'}\ (\wedge \overset{\alpha}{=})$$

$$\frac{\mathrm{A} \overset{\alpha}{=} \mathrm{A}' \qquad \mathrm{B} \overset{\alpha}{=} \mathrm{B}'}{\mathrm{A} \vee \mathrm{B} \overset{\alpha}{=} \mathrm{A}' \vee \mathrm{B}'}\ (\vee \overset{\alpha}{=})$$

$$\frac{\mathrm{A} \overset{\alpha}{=} \mathrm{A}' \qquad \mathrm{B} \overset{\alpha}{=} \mathrm{B}'}{\mathrm{A} \supset \mathrm{B} \overset{\alpha}{=} \mathrm{A}' \supset \mathrm{B}'}\ (\supset \overset{\alpha}{=})$$

Equality:

$$\frac{\mathrm{t}_1 \overset{\alpha}{=} \mathrm{t}_1' \qquad \mathrm{t}_2 \overset{\alpha}{=} \mathrm{t}_2' \qquad \tau \overset{\alpha}{=} \tau'}{(\mathrm{t}_1 = \mathrm{t}_2 : \tau) \overset{\alpha}{=} (\mathrm{t}_1{}' = \mathrm{t}_2{}' : \tau')}\ (= \overset{\alpha}{=})$$

Quantifiers:

$$\frac{\text{И}\, z\, .\ (x\ z)\ \mathrm{B} \overset{\alpha}{=} (x\ z)\ \mathrm{B}' \qquad \tau \overset{\alpha}{=} \tau'}{\exists x : \tau.\ \mathrm{B} \overset{\alpha}{=} \exists y : \tau'.\ \mathrm{B}'}\ (\exists \overset{\alpha}{=})$$

$$\frac{\text{И}\, z\, .\ (x\ z)\ \mathrm{B} \overset{\alpha}{=} (x\ z)\ \mathrm{B}' \qquad \tau \overset{\alpha}{=} \tau'}{\forall x : \tau.\ \mathrm{B} \overset{\alpha}{=} \forall y : \tau'.\ \mathrm{B}'}\ (\forall \overset{\alpha}{=})$$

6.3 Swapping variable names

$$\mathtt{swap} = (x\ z)$$

$$\begin{aligned} \mathtt{swap\_term} &: x \to z \to \mathtt{t} \to \mathtt{t} \\ \mathtt{swap\_prop} &: x \to z \to \mathtt{A} \to \mathtt{A} \end{aligned}$$

```
val swap_term  :  var -> var -> term -> term
val swap_prop  :  var -> var -> prop -> prop
```

7 Induction in proofs

7.1 Rules through predicates

Natural Numbers:

$$\frac{\psi;\Gamma \vdash \mathrm{p} : \mathrm{P(zero)} \qquad \psi\ ,\, n : \mathrm{nat}\ ;\Gamma\ ,\mathrm{H} : \mathrm{P}(n) \vdash \mathrm{q} : \mathrm{P(suc}(n))}{\psi;\Gamma \vdash \text{(Induction on nat: case zero : p ; case suc}(n)\text{) : [H] , q) : }(\forall\ m : \mathrm{nat}\ .\ \mathrm{P}(m))} \quad \text{(induction-nat)}$$

Lists:

$$\frac{\psi;\Gamma \vdash \mathrm{p} : \mathrm{P}(\,[\,]\,) \qquad \psi\ ,\, x : \tau\ ,\, xs : \mathrm{list}\ \tau\ ;\Gamma\ ,\mathrm{H} : \mathrm{P}(xs) \vdash \mathrm{q} : \mathrm{P}(x :: xs)}{\psi;\Gamma \vdash \text{(Induction on list: case [] : p ; case }(x :: xs)\text{ : [H] , q) : }(\forall\ ys : \mathrm{list}\ \tau\ .\ \mathrm{P}(ys))} \quad \text{(induction-list)}$$

Booleans:

$$\frac{\psi;\Gamma \vdash \mathrm{p} : \mathrm{P(\ true\ )} \qquad \psi;\Gamma \vdash \mathrm{q} : \mathrm{P(\ false\ )}}{\psi;\Gamma \vdash \text{(Induction on bool: case true : p ; case false : q) : }(\forall\ b : \mathrm{bool}\ .\ \mathrm{P}(\ b\ ))} \quad \text{(induction-bool)}$$

7.2 Rules through substitution

Natural Numbers:

$$\frac{\psi;\Gamma \vdash \mathrm{p} : [m \mapsto \mathrm{zero}]\ \mathrm{C} \qquad \psi\ ,\, n : \mathrm{nat}\ ;\Gamma\ ,\mathrm{H} : [m \mapsto \mathrm{zero}]\ \mathrm{C} \vdash \mathrm{q} : [\mathrm{m} \mapsto \mathrm{suc}(n)]\ \mathrm{C}}{\psi;\Gamma \vdash \text{(Ind-Nat: zero : p ; suc}(n)\text{ : [H] , q) : }(\forall\ m : \mathrm{nat}\ .\ \mathrm{C})} \quad \text{(induction-nat)}$$

Lists:

$$\frac{\psi;\Gamma \vdash \mathrm{p} : [ys \mapsto [\,]\,]\ \mathrm{C} \qquad \psi\ ,\, x : \tau\ ,\, xs : \mathrm{list}\ \tau\ ;\Gamma\ ,\mathrm{H} : [ys \mapsto xs\ ]\ \mathrm{C} \vdash \mathrm{q} : [ys \mapsto x :: xs\ ]\ \mathrm{C}}{\psi;\Gamma \vdash \text{(Ind-List: [] : p ; }(x :: xs)\text{ : [H] , q) : }(\forall\ ys : \mathrm{list}\ \tau\ .\ \mathrm{C})} \quad \text{(induction-list)}$$

Booleans:

$$\frac{\psi;\Gamma \vdash \mathrm{p} : [b \mapsto \mathrm{true}\ ]\ \mathrm{C} \qquad \psi;\Gamma \vdash \mathrm{q} : [b \mapsto \mathrm{false}\ ]\ \mathrm{C}}{\psi;\Gamma \vdash \text{(Ind-Bool: true : p ; false : q) : }(\forall\ b : \mathrm{bool}\ .\ \mathrm{C})} \quad \text{(induction-bool)}$$

8 Equality in proofs

8.1 Abstract congruence closure [1, p. 4–7]

8.1.1 Definition

Rewrite-Rules:

$$\begin{array}{rrl} \textit{D-rule}: & f(c_0, ...c_k) \to c & \textit{where } f \textit{ is a term constructor and } c_i \textit{ are constants in } K \\ \textit{C-rule}: & c \to d & \textit{where } c \textit{ and } d \textit{ are constants in } K \end{array}$$

Sets:

$$D : \{\textit{D-rule}\}$$
$$C : \{\textit{C-rule}\}$$

$$E : \{(t = t) : \tau\}$$
$$K : \{x \mid x \notin E\}$$
$$R : D \cup C$$

Closure Construction:

$$state : (K, E, R)$$

$$\texttt{build\_acc}: \quad \frac{(\emptyset, E, \emptyset)}{(K, \emptyset, R)} \quad \textit{where R is the abstract congruence closure (ACC) of E}$$

8.1.2 Sate transition rules

$$\text{Extension:} \quad \frac{(K, E[t], R) \qquad t = f(c_0, ..., c_k) \qquad c_i \in K \wedge c \notin K}{(K \cup \{c\}, E[c], R \cup \{t \to c \ : D\})} \quad \text{(Ext)}$$

$$\text{Simplification:} \quad \frac{(K, E[t], R \cup \{t \to c : D\})}{(K, E[c], R \cup \{t \to c : D\})} \quad \text{(Sim)}$$

$$\text{Orientation1:} \quad \frac{(K \cup \{c\}, E \cup \{t = c\}, R\})}{(K \cup \{c\}, E, R \cup \{t \to c : D\})} \quad \text{(Ori1)}$$

$$\text{Orientation2:} \quad \frac{(K \cup \{c, d\}, E \cup \{c = d\}, R\}) \qquad c < d}{(K \cup \{c, d\}, E, R \cup \{c \to d : C\})} \quad \text{(Ori2)}$$

$$\text{Orientation3:} \quad \frac{(K \cup \{c, d\}, E \cup \{c = d\}, R\}) \qquad d < c}{(K \cup \{c, d\}, E, R \cup \{d \to c : C\})} \quad \text{(Ori2)}$$

$$\text{Deletion:} \quad \frac{(K, E \cup \{t = t\}, R\})}{(K, E, R\})} \quad \text{(Del)}$$

$$\text{Deduction1:} \quad \frac{(K, E, R \cup \{t \to c : D, t \to d : D\})}{(K, E \cup \{c = d\}, R \cup \{t \to d : D\})} \quad \text{(Ded1)}$$

$$\text{Deduction2:} \quad \frac{(K, E, R \cup \{c \to c' : C, c \to d : C\})}{(K, E \cup \{c' = d\}, R \cup \{c \to d : C\})} \quad \text{(Ded2)}$$

$$\text{Collapse:} \quad \frac{(K, E, R \cup \{s[c] \to c' : D, c \to d : C\})}{(K, E, R \cup \{s[d] \to c' : D, c \to d : C\})} \quad \text{(Col)}$$

$$\text{Composition:} \quad \frac{(K, E, R \cup \{t \to c : D, c \to d : C\})}{(K, E, R \cup \{t \to d : D, c \to d : C\})} \quad \text{(Com)}$$

8.2 Equality through congruence

$$\frac{\psi \,;\, \Gamma \,\vdash\, \mathrm{H}_i : (t_i = t_i') : \tau_i \qquad \forall i \in \{1...n\} \,.\, \overrightarrow{t_i = t_i'} \,\models\, t = t'}{\psi \,;\, \Gamma \,\vdash\, \text{By Equality} \,(\, \mathrm{H}_0 \,, ... \,, \mathrm{H}_n \,) : (t = t' : \tau)} \quad \text{(eq)}$$

References

[1] Leo Bachmair, Ashish Tiwari, and Laurent Vigneron. Abstract congruence closure. *J. Autom. Reasoning*, 31(2):129–168, 2003.