0.1 The category of cones

Given $A, B, V, W \in \mathcal{C}$, cones over $A, B \in \mathcal{C}$ form a category $Cone(\mathcal{C}, A, B) or Cone_{\mathcal{C}}(A, B)$.

A **cone** from V to A and B consists of:

$$\begin{tikzcd}
& V \arrow[dl, "f"'] \arrow[dr, "g"] & \\
A & & B
\end{tikzcd}$$

A **cone morphism** from (V, f, g) to (W, h, k) is a $\mathcal{C}$ morphism $V \xrightarrow{r} W$ such that the two triangles, from V through W to A and to B, commute.

$$\begin{tikzcd}
V \arrow[r, "r"] \arrow[d, "f"'] \arrow[dr, "g"] & W \arrow[d, "k"] \arrow[dl, "h"'] \\
A & B
\end{tikzcd}$$

f and $r;h$ are equal, as do g and $r;k$, i.e. $r;h = f$ and $r;k = g$.

If a diagram commutes, any two paths from one object to another are equal.

Given $r : V \to W$ and $s : W \to X$, composition is the same as composition in $\mathcal{C}$, with the added check for commutativity of the triangles.

$$\begin{tikzcd}
V \arrow[r, "r"] \arrow[dr, "f"'] \arrow[drr, "g"] & W \arrow[r, "s"] \arrow[d, "h"'] \arrow[dr, "k"] & X \arrow[dl, "l"'] \arrow[d, "m"] \\
& A & B
\end{tikzcd}$$

i.e. $r;s;l = f$ and $r;s;m = g$.

$f = r;h \land h = s;l$
$\therefore f = r;s;l$

$g = r;k \land k = s;m$
$\therefore g = r;s;l$

0.2 Product object (categorical product) in a category of cones

A product of A and B is a "terminal cone" over A and B; i.e. a terminal object in $Cone_{\mathcal{C}}(A, B)$.

Given **projections** $\Pi_1 = (a, b) \mapsto a$ and $\Pi_2 = (a, b) \mapsto b$

$$\begin{tikzcd}
& A \times B \arrow[dl, "\Pi_1"'] \arrow[dr, "\Pi_2"] & \\
A & & B
\end{tikzcd}$$

Where Π_1 and Π_2 are called the left and right projections respectively.

To show that there is a unique morphism to $A \times B$, we must prove that such morphism exists, and is unique.

For example, in **Set**:

We start by defining a unique a morphism from V to $A \times B$.

$$\begin{tikzcd}
V \arrow[r, dashed, "s"] \arrow[d, "f"'] \arrow[dr, "g"] & A \times B \arrow[dl, "\Pi_1"'] \arrow[d, "\Pi_2"] \\
A & B
\end{tikzcd}$$

Given $x \in V$, we can define $h = x \mapsto (fx, gx)$, proving such morphism indeed exists.

We also know $\Pi_1(s(x)) = f(x)$ and $\Pi_2(s(x)) = g(x)$ because s is a cone morphism and paths much commute.

Hence, we know s a unique morphism. i.e. $s;\Pi_1 = f = h;\Pi_1$ and $s;\Pi_2 = g = h;\Pi_2$, $\therefore$ $s = h$

0.3 The sum of sets

The sum of sets A and B is defined by a union:

$A + B = \{\textbf{inl}\, x \mid x \in A\} \cup \{\textbf{inr}\, y \mid y \in B\}$

where **inl** and **inr** are injective: $\forall x, y.\textbf{inl}\, x \neq \textbf{inr}\, y$

Thus $\quad \textbf{inl}\, x = (0, x) \quad$ and $\quad \textbf{inr}\, y = (1, y)$

note that this holds, even for $A + A$ or $\mathbb{N} + \mathbb{N}$.

e.g. $\textbf{inl}\, 3 \neq \textbf{inr}\, 3 \in \mathbb{N} + \mathbb{N}$

Note: the set of functions $A \rightarrow B$ can be written in notation B^A. This is because the number of elements in B^A is m^n where $m = size(B), n = size(A)$.

0.4 The category of cocones

Let $A, B \in C$, a **cocone** over A, B is the dual notion of a **cone**, and defined by the following diagram:

$$\begin{tikzcd}
 & V & \\
A \arrow[ur, "f"] & & B \arrow[ul, "g"']
\end{tikzcd}$$

So a morphism from cocone (V, f, g) to cocone (W, h, k) is R and defined by the following diagram:

$$\begin{tikzcd}
V \arrow[r, "r"] & W \\
A \arrow[u, "f"] \arrow[ur, "h"] & B \arrow[ul, "g"'] \arrow[u, "k"']
\end{tikzcd}$$

Composition is defined by the dual concept of composition in cones.

We therefore obtain a category $Cocone_C(A, B)$ of cones over A, B.

0.4.1 Coproducts and initial cones in Set

The **coproduct** (sum defined in section 4.3) $A + B$ is an "initial cone" in $Cocone_{Set}(A, B)$.

To show $A + B$ is initial in $Cocone_{Set}(A, B)$, we must show there is a unique morphism $R : A + B \to V$.

$$\begin{tikzcd}
V & A + B \arrow[l, dashed, "R"'] \\
A \arrow[u, "f"] \arrow[ur, "\mathbf{inl}"] & B \arrow[ul, "g"'] \arrow[u, "\mathbf{inr}"']
\end{tikzcd}$$

To do this, we define a function s given $x, y \in A + B$:

$$s = \begin{cases} \mathbf{inl}\ x \mapsto fx \\ \mathbf{inr}\ y \mapsto gy \end{cases}$$

We know s is equal to R, and hence a unique morphism, because:

- $\mathbf{inl}; s$ and f commute
- $\mathbf{inr}; s$ and g commute

Given the diagram commutes, because it's a cone morphism, the morphisms are all equal there is a unique function.

0.4.2 The dual notion

A coproduct in C for A, B is a product in C^{op} for A, B.

0.5 Generalising from binary to arbitrary

0.5.1 Generalised categorical products

Generalised products are denoted by the **categorical product** of the I indexed family A_i where $i \in I$, and I is a set, countable or uncountable, that indexes the family. These are n-ary products if I is of size n.

This is given by $\prod_{i \in I}(A_i)$, where in composition, commutativity of all triangles formed commute.

$\forall i \in I \,.\, f_i = r; \Pi_i$ in the following diagram:

$$\begin{tikzcd}
V \arrow[rr, dashed, "s"] \arrow[d, "f_0"'] \arrow[drr, "f_i"] & & A \times B \arrow[d, "\Pi_i"] \arrow[dll, "\Pi_0"'] \\
A_0 & \dots & A_i
\end{tikzcd}$$

In **Set**: A way to define general product is:

$$\prod_{i \in I}(A_i) \stackrel{\text{def}}{=} \text{the set of functions } p \text{ with domain } I \text{ such that } \forall i \in I \,.\, p_i \in A_i$$

Here, the i^{th} projection would be $p \mapsto p\ i$, i.e. the i^{th} element of p, or applying p to i.

0.5.2 Generalised sum of sets

The **categorical coproduct** of $(A_i)_{i \in I}$ is given by $\sum_{i \in I} A_i = \{(i, a) | i \in I \wedge a \in A_i\}$

Hence, the categorical product is a big tuple/function, and the categorical coproduct is a big disjoint-union/sum.

Another notation used for coproducts is $\coprod_{i \in I}(A_i)$

0.5.3 Example: product and sum of the empty family

In **Set**, the Π of the empty family ($\varnothing \rightarrow A$) is the singleton set, so the set of cardinality of the product is 1.

$\prod_{\varnothing} = \{f_{\varnothing} : \varnothing \to \varnothing\} = \{()\} = 1$

The sum of the empty family is the empty set.

$\sum \varnothing = \varnothing$

0.5.4 Common patterns with monoids

The patterns observed with the product and sum of the empty set are analogous with different monoids.

For instance, for lists:

- $Sum\ [] = 0 \quad (\mathbb{N}, 0, +)$
- $Product\ [] = 1 \quad (\mathbb{N}, 1, \times)$
- $Max\ [] = 0 \quad (\mathbb{N}, 0, max)$
- $Min\ []$ undefined (infinity if analogous to division by zero, assymptotes, etc.)
- $And\ [] = True \quad (\mathbb{B}, True, \wedge)$
- $Or\ [] = False \quad (\mathbb{B}, False, \vee)$

For these monoids, it is a common pattern that vacuous arguments return the identity as a convention. Note that *Min* is an example of a non-monoid, and thus, can't return the identity, making the function undefined on vacuous arguments.

Thus, if there is a monoid $(M, e, *)$, and we want an n-ary application of $*$ on the elements M, the application must end with e.

Hence, every n-ary application following a monoidal structure returns the identity by convention.