0.1 Definition

A **category** consists of the following:

- A **class** of objects obC
  if obC is a set, this is a *small* category
- For any objects $A, B \in C$
  a set $hom_C(A,B)$ of morphisms $f : A \to B$
  i.e. a homset from A to B
- For each object A
  a morphism $id_A : A \to A$
  i.e. an identity morphism for every object
- For any objects A, B, C and morphisms $f : A \to B$ and $g : B \to C$
  the **composition** is $f;_{A,B,C}\, g : A \to C$
  (also written $g \circ f$, the subscripts for (;) are required but sometimes shorthanded)
  such that:
  - For any objects A, B and $f : A \to B$
    $id_A; f = f = f; id_B$ [i.e. composing the identity with f equals f]
  - For any objects A, B, C, D and morphisms $f : A \to B$ and $g : B \to C$ and $h : C \to D$
    $f; (g; h) = (f; g); h$ [i.e. composition is associative]

0.2 Examples

0.2.1 The category of sets, Set

- An object is a set
- A morphism $A \to B$ is a function, the homset contains all function definitions as well as undefinable ones
- The identity id_A is the identity function $x \mapsto x$
- The composite $A \xrightarrow{f} B \xrightarrow{g} C$ is the composition of a function [$x \mapsto g(f(x))$]

0.2.2 The category of matrices, Mat

- An object is a natural number $n \in \mathbb{N}$
- The morphisms $m \to n \in hom(m, n)$ are all $m \times n$ matrices (m rows, n cols). i.e. every morphism is a collection of field elements (t_i^j) where i runs from 0 to $m-1$ and j from 0 to $n-1$.

- The identity id_n is the identity matrix of $n \times n$ (morphism $n \to n$) such that it acts as the identity for matrix multiplication

- The composite of $(s_i^j) : m \to n \in hom(m, n)$ and $(t_j^k) : n \to p \in hom(p, m)$ is matrix multiplication where the product $(s_i^j; t_j^k) : m \to p \in hom(p, m)$ is

  $(s_i^j \mid {}^{i<m}_{j<n})\ (t_j^k \mid {}^{j<n}_{k<p}) = (\sum_{j<n} s_i^j t_j^k \mid {}^{i<m}_{k<p})$

0.2.3 The category of towns in Britain

- An object is a town in Britain

- The morphisms $A \to B \in hom(A, B)$ are all routes from A to B. i.e. a finite sequence (list) of adjacent towns.

- The identity id_A is a route from a town A to itself which consists of of a single element A.

  i.e. the identity always is a list of length 1.

- The composite of $f : A \to B$ and $g : B \to A$ is the concatenations of the routes f and g.

  e.g. if $f = [A, A_1, A_2, A_3, B]$ and $g = [B, B_1, B_2, B_3, C]$ then $f; g = [A, A_1, A_2, A_3, B, B_1, B_2, B_3, C]$

0.3 Homework Examples

0.3.1 The category of groups and group homomorphisms, Grp

- An object is a group

- The morphism $h : G \to H \in hom(G, H)$ is a group homomorphism.

  i.e. a function that preserves the algebraic structure from group $(G, *)$ in group $(H, \bullet)$:

  given $a * b = c$ we have $h(a) \bullet h(b) = h(c)$

  i.e. $h(a * b) = h(a) \bullet h(b)$

- The identity id_G is a homomorphism that maps G to itself.

  More specifically, given $G = (X, e, *)$, $x \in X$:

  $id_G : G \to G$

  $id_G = x \mapsto x$

- The composite of morphisms $h : G \to H$ and $k : H \to K$ is $h; k : G \to K$ or $k \circ h : G \to K$

  i.e. a homomorphism from G to K

A **group** (G) is a set with an operation ($\bullet : G^2 \to G$) that combines two elements in G to form a third element in G. In computer science, we generally include element e to remove existentials. For $(G, \bullet)$ to be a group, it must satisfy:

- **Closure** : $(a, b \in G) \Rightarrow (a \bullet b \in G)$.
- **Associativity** : $\forall a, b, c \in G \;.\; (a \bullet b) \bullet c = a \bullet (b \bullet c)$
- **Identity** : $\forall a \in G \;.\; \exists e \in G \;.\; e \bullet a = a \bullet e = a$
- **Inverse Element** : $\forall a \in G \;.\; \exists b \in G \;.\; a \bullet b = b \bullet a = e$

  $*$ note: a group without inverse functions is a **monoid**.

  i.e. for morphism $\mathbb{N} \to \mathbb{N}$, the set of all functions is a monoid while the set of all bijections is a group.

0.3.2 The category of vector spaces and linear transformations, K-Vect

- An object is a vector space (a scalable collection of vectors) over a fixed field K
- The morphism $V \to W \in hom(V, W)$ is a **K-linear map**, transformations between two linear subspaces that preserve addition and scalar multiplication
- The identity id_V is an endomorphism on V (a morphism from vector space V to itself)

  More specifically, given $v \in V$:

  $id_V = v \mapsto v$
- The composite of morphisms $f : V \to W$ and $g : W \to Y$ is $f; g : V \to Y$

0.3.3 The category of posets (partially ordered sets) and monotone functions

- An object is a poset
- For $(S, \leq)$ and $(T, \leq)$, the morphism $f : S \to T \in hom(S, T)$ is a **order-preserving** or **monotone** function.

  A monotone function is one such that:

  $\forall x, y \in S.(x \leq y) \Rightarrow f(x) \leq f(y)$
- The identity id_S is a morphism from S to itself
- The composite of $f : S \to T$ and $g : T \to U$ is $f; g : S \to U$ or $(g \circ f) : S \to U$, which is also monotone

Posets $(S, \leq)$ formalise the intuitive concept of ordering, sequencing, or arrangement of the elements of a set.

Posets consist of a set (S) and a binary relation ($\leq$) that indicates a partial order.

A **partial order** is a binary relation between elements of a set ($a, b, c \in S$) that is:

- **reflexive**: $a \leq a$
- **antisymmetric**: $(a \leq b) \wedge (b \leq a) \Rightarrow a = b$
- **transitive**: $(a \leq b) \wedge (b \leq c) \Rightarrow a \leq c$

0.3.4 The category of set relations, Rel

- An object is a set
- The morphism $f : A \to B \in hom(A, B)$ is a relation.

  A relation is between A and B is defined by the crossproduct of both sets, i.e. a set of tuples.

  $(R \subseteq A \times B)$ and $\{(a, b) \mid (a, b) \in R \wedge a \in A \wedge b \in B\}$
- The identity morphism $id_A : A \to A$ is the identity relation $\{(a, a) \in A^2 \mid a \in A\}$
- The composite of $R : A \to B$ and $S : B \to C$ is $R; S : A \to C$ given by:

  $S \circ R = \{(a, c) \in A \times C \mid \exists b \in B.(a, b) \in R \wedge (b, c) \in S\}$