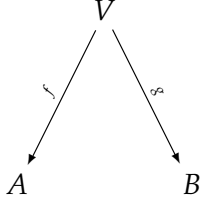


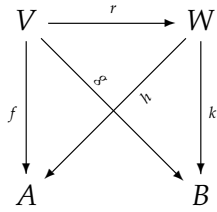
0.1 The category of cones

Given $A, B, V, W \in \mathcal{C}$, cones over $A, B \in \mathcal{C}$ form a category $\text{Cone}(\mathcal{C}, A, B)$ or $\text{Cone}_{\mathcal{C}}(A, B)$.

A **cone** from V to A and B consists of:



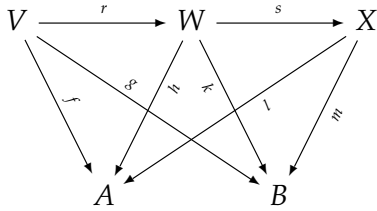
A **cone morphism** from (V, f, g) to (W, h, k) is a \mathcal{C} morphism $V \xrightarrow{r} W$ such that the two triangles, from V through W to A and to B , commute.



f and $r;h$ are equal, as do g and $r;k$, i.e. $r;h = f$ and $r;k = g$.

If a diagram commutes, any two paths from one object to another are equal.

Given $r : V \rightarrow W$ and $s : W \rightarrow X$, composition is the same as composition in \mathcal{C} , with the added check for commutativity of the triangles.



i.e. $r;s;l = f$ and $r;s;m = g$.

$$f = r;h \wedge h = s;l$$

$$\therefore f = r;s;l$$

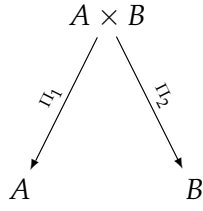
$$g = r;k \wedge k = s;m$$

$$\therefore g = r;s;l$$

0.2 Product object (categorical product) in a category of cones

A product of A and B is a “terminal cone” over A and B ; i.e. a terminal object in $\text{Cone}_{\mathcal{C}}(A, B)$.

Given **projections** $\Pi_1 = (a, b) \mapsto a$ and $\Pi_2 = (a, b) \mapsto b$

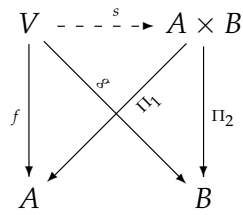


Where Π_1 and Π_2 are called the left and right projections respectively.

To show that there is a unique morphism to $A \times B$, we must prove that such morphism exists, and is unique.

For example, in **Set**:

We start by defining a unique a morphism from V to $A \times B$.



Given $x \in V$, we can define $h = x \mapsto (fx, gx)$, proving such morphism indeed exists.

We also know $\Pi_1(s(x)) = f(x)$ and $\Pi_2(s(x)) = g(x)$ because s is a cone morphism and paths much commute.

Hence, we know s a unique morphism. i.e. $s; \Pi_1 = f = h; \Pi_1$ and $s; \Pi_2 = g = h; \Pi_2$, $\therefore s = h$

0.3 The sum of sets

The sum of sets A and B is defined by a union:

$$A + B = \{\mathbf{inl} \ x \mid x \in A\} \cup \{\mathbf{inr} \ y \mid y \in B\}$$

where \mathbf{inl} and \mathbf{inr} are injective: $\forall x, y. \mathbf{inl} \ x \neq \mathbf{inr} \ y$

Thus $\mathbf{inl} \ x = (0, x)$ and $\mathbf{inr} \ y = (1, y)$

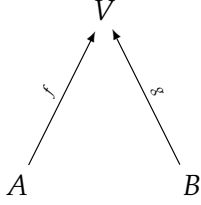
note that this holds, even for $A + A$ or $\mathbb{N} + \mathbb{N}$.

e.g. $\mathbf{inl} \ 3 \neq \mathbf{inr} \ 3 \in \mathbb{N} + \mathbb{N}$

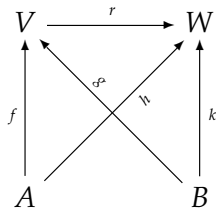
Note: the set of functions $A \rightarrow B$ can be written in notation B^A . This is because the number of elements in B^A is m^n where $m = \text{size}(B)$, $n = \text{size}(A)$.

0.4 The category of cocones

Let $A, B \in C$, a **cocone** over A, B is the dual notion of a **cone**, and defined by the following diagram:



So a morphism from cocone (V, f, g) to cocone (W, h, k) is R and defined by the following diagram:



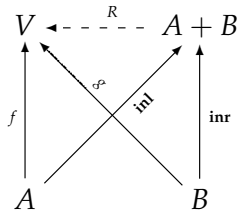
Composition is defined by the dual concept of composition in cones.

We therefore obtain a category $Cocone_C(A, B)$ of cones over A, B .

0.4.1 Coproducts and initial cones in Set

The **coproduct** (sum defined in section 4.3) $A + B$ is an “initial cone” in $Cocone_{Set}(A, B)$.

To show $A + B$ is initial in $Cocone_{Set}(A, B)$, we must show there is a unique morphism $R : A + B \rightarrow V$.



To do this, we define a function s given $x, y \in A + B$:

$$s = \begin{cases} \text{inl } x \mapsto fx \\ \text{inr } y \mapsto gy \end{cases}$$

We know s is equal to R , and hence a unique morphism, because:

- $\text{inl}; s$ and f commute
- $\text{inr}; s$ and g commute

Given the diagram commutes, because it's a cone morphism, the morphisms are all equal there is a unique function.

0.4.2 The dual notion

A coproduct in C for A, B is a product in C^{op} for A, B .

0.5 Generalising from binary to arbitrary

0.5.1 Generalised categorical products

Generalised products are denoted by the **categorical product** of the I indexed family A_i where $i \in I$, and I is a set, countable or uncountable, that indexes the family. These are n -ary products if I is of size n .

This is given by $\prod_{i \in I} (A_i)$, where in composition, commutativity of all triangles formed commute.

$\forall i \in I . f_i = r; \Pi_i$ in the following diagram:

$$\begin{array}{ccc}
 V & \xrightarrow{\quad s \quad} & A \times B \\
 \downarrow f_0 & \swarrow \scriptstyle f_i & \searrow \scriptstyle \Pi_0 \\
 A_0 & \dots & A_i \\
 & & \downarrow \scriptstyle \Pi_i
 \end{array}$$

In **Set**: A way to define general product is:

$\prod_{i \in I} (A_i) \stackrel{\text{def}}{=} \text{the set of functions } p \text{ with domain } I \text{ such that } \forall i \in I . p_i \in A_i$

Here, the i^{th} projection would be $p \mapsto p_i$, i.e. the i^{th} element of p , or applying p to i .

0.5.2 Generalised sum of sets

The **categorical coproduct** of $(A_i)_{i \in I}$ is given by $\sum_{i \in I} A_i = \{(i, a) | i \in I \wedge a \in A_i\}$

Hence, the categorical product is a big tuple/function, and the categorical coproduct is a big disjoint-union/sum.

Another notation used for coproducts is $\coprod_{i \in I} (A_i)$

0.5.3 Example: product and sum of the empty family

In **Set**, the Π of the empty family ($\emptyset \rightarrow A$) is the singleton set, so the set of cardinality of the product is 1.

$$\prod_{\emptyset} = \{f_{\emptyset} : \emptyset \rightarrow \emptyset\} = \{()\} = 1$$

The sum of the empty family is the empty set.

$$\sum \emptyset = \emptyset$$

0.5.4 Common patterns with monoids

The patterns observed with the product and sum of the empty set are analogous with different monoids.

For instance, for lists:

- $Sum [] = 0$ $(\mathbb{N}, 0, +)$
- $Product [] = 1$ $(\mathbb{N}, 1, \times)$
- $Max [] = 0$ $(\mathbb{N}, 0, max)$
- $Min []$ undefined (infinity if analogous to division by zero, asymptotes, etc.)
- $And [] = True$ $(\mathbb{B}, True, \wedge)$
- $Or [] = False$ $(\mathbb{B}, False, \vee)$

For these monoids, it is a common pattern that vacuous arguments return the identity as a convention. Note that Min is an example of a non-monoid, and thus, can't return the identity, making the function undefined on vacuous arguments.

Thus, if there is a monoid $(M, e, *)$, and we want an n-ary application of $*$ on the elements M , the application must end with e .

Hence, every n-ary application following a monoidal structure returns the identity by convention.