Information Theory and Related Fileds

Lecture 1: Information-Theoretic Quantities

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Outline

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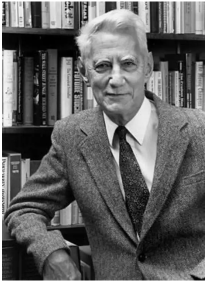
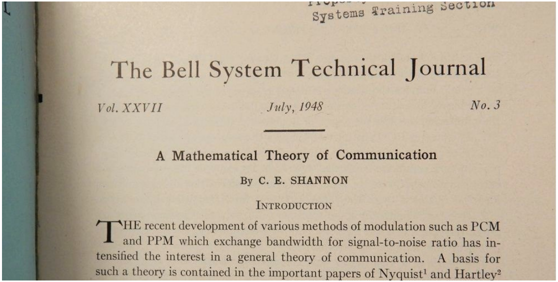
Birth of Information Theory

Information theory was essentially established

- by Claude Shannon in 1948
- in the paper "A Mathematical Theory of Communication"
- via introducing (information) entropy (borrowed from a similar notion in thermodynamics)





Systems Training Section

The Bell System Technical Journal

Vol. XXVII July, 1948 No. 3

A Mathematical Theory of Communication

By C. E. SHANNON

INTRODUCTION

THE recent development of various methods of modulation such as PCM and PPM which exchange bandwidth for signal-to-noise ratio has intensified the interest in a general theory of communication. A basis for such a theory is contained in the important papers of Nyquist[1] and Hartley[2]

Communication System

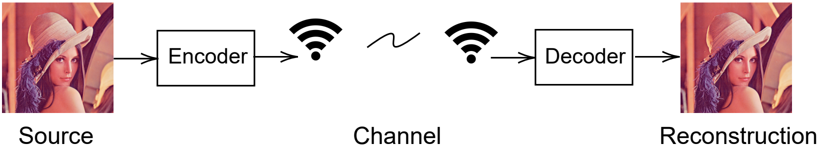


Figure: Practical Communication

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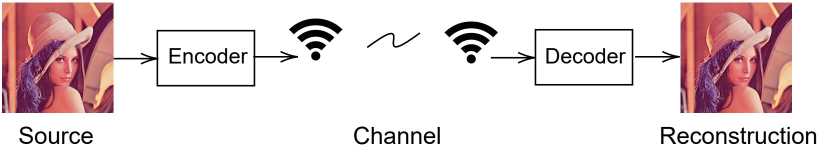

Figure: Practical Communication

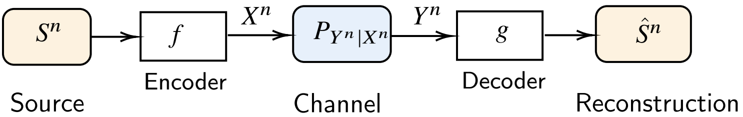

Figure: Mathematical Model, where notation $Z^n := (Z_1, Z_2, ..., Z_n)$ denotes a random vector

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- Memoryless channel $P_{Y|X}$: A channel with product conditional distribution $P_{Y|X}^{\otimes n}$
  - In other words, component Y_i is generated only by X_i via $P_{Y|X}$, i.e.,

$$\begin{array}{ccccc} X_1 & \longrightarrow & P_{Y|X} & \longrightarrow & Y_1 \\ ... & & ... & & ... \\ X_n & \longrightarrow & P_{Y|X} & \longrightarrow & Y_n \end{array}$$

  - If the inputs $X_1, X_2, ...$ are i.i.d., then the outputs $Y_1, Y_2, ...$ are also i.i.d.

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- So, for source P_{S^n}, channel $P_{Y^n|X^n}$, encoder f, and decoder g, the joint distribution induced by them is

$$P_{S^n X^n Y^n \hat{S}^n} = P_{S^n} P_{X^n|S^n} P_{Y^n|X^n} P_{\hat{S}^n|Y^n}$$

where $P_{X^n|S^n=s^n} = \delta_{f(s^n)}$ and $P_{\hat{S}^n|Y^n=y^n} = \delta_{g(y^n)}$ with δ_z denoting the Dirac measure at z

Goal of Information Theory

Question: Given a pair of source and channel, can the source be transmitted asymptotically losslessly over the channel in the sense that $\mathbb{P}(S^n \neq \hat{S}^n) \to 0$ (as $n \to \infty$)?

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- Capacity of a memoryless channel $P_{Y|X}$ is

$$C(P_{Y|X}) := \max_{P_X} I(X;Y)$$

Solution

Theorem ([Shannon'48])

Consider memoryless source S and memoryless channel $P_{Y|X}$. There is a sequence of encoder-decoder pairs (f_n, g_n) such that $\mathbb{P}(S^n \neq \hat{S}^n) \to 0$ (as $n \to \infty$) if $H(S) < C(P_{Y|X})$, and only if $H(S) \leq C(P_{Y|X})$.

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- To be proven in the first three lectures.

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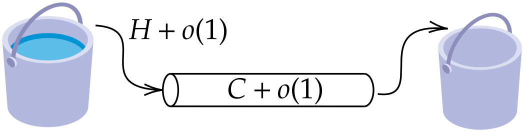
- To be proven in the first three lectures.
- Analogy: Water of volume $nH + o(n)$ can be transfered within time n via a pipe of capacity $C + o(1)$, if $H < C$ and only if $H \leq C$.



Outline

Entropy

Let X, Y be discrete random variables (r.v.'s). We adopt convention $0 \log 0 = 0$.

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Definition

Recall that entropy of discrete X is

$$H(X) := \sum_{x} P_X(x) \log \frac{1}{P_X(x)} = \mathbb{E}_{P_X}\left[\log \frac{1}{P_X(X)}\right]$$

Joint entropy of X and Y is

$$H(X,Y) := \sum_{x,y} P_{XY}(x,y) \log \frac{1}{P_{XY}(x,y)}$$

Conditional entropy of Y given X is

$$H(Y|X) := \sum_{x,y} P_{XY}(x,y) \log \frac{1}{P_{Y|X}(y|x)} = \sum_{x} P_X(x) H(Y|X=x)$$

Facts and Examples

Fact

1) Nonnegativity: $H(X)$, $H(X,Y)$, $H(Y|X) \geq 0$ *(Moreover,* $H(Y|X) = 0$ *iff* $Y = g(X)$ *for some function)*
2) Chain rule: $H(X,Y) = H(X) + H(Y|X)$

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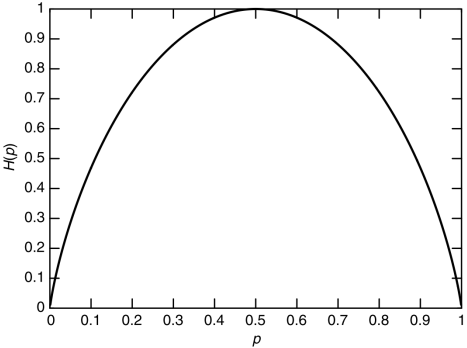
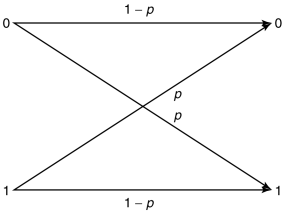
- Binary Source: For $X \sim \mathrm{Bern}(p)$, $H(X) = H_2(p) := p \log \frac{1}{p} + (1-p) \log \frac{1}{1-p}$, where H_2 is called binary entropy function



Facts and Examples (cont.)

- Binary Symmetric Channel $\mathrm{BSC}(p)$: $Y = X \oplus W$ with $W \sim \mathrm{Bern}(p)$ independent of X. For such channel, $H(Y|X) = H_2(p)$



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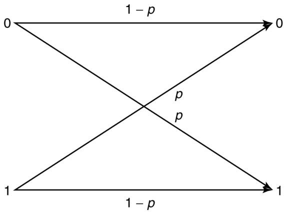

- Doubly Symmetric Binary Source (DSBS): $X \sim \mathrm{Bern}(\frac{1}{2})$ and Y is the output of $\mathrm{BSC}(p)$ with input X.

$$P_{XY} = \begin{array}{ccc} X\backslash Y & 0 & 1 \\ 0 & \frac{1-p}{2} & \frac{p}{2} \\ 1 & \frac{p}{2} & \frac{1-p}{2} \end{array}.$$

For such source, $H(X,Y) = 1 + H_2(p)$.

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Definition

$\iota(A) = \log \frac{1}{P(A)}$ is called the self-information of A.

Fact

It holds that

$$H(X) = \sum_x P_X(x) \cdot \iota_X(x)$$

where $\iota_X(x) = \log \frac{1}{P_X(x)}$ *is the self-information of outcome* $X = x$.

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- That is, the entropy of X is equal to the expected self-information of X. In other words, it is the average amount of information that we gain when measuring X.

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- That is, the entropy of X is equal to the expected self-information of X. In other words, it is the average amount of information that we gain when measuring X.
- Another way to define entropy via axioms was given by Shannon in [Shannon'48] (omitted)

Mutual Information

Definition

Mutual information (MI) between X and Y is defined as

$$I(X;Y) = \sum_{x,y} P_{XY}(x,y) \log \frac{P_{XY}(x,y)}{P_X(x)P_Y(y)}$$

Conditional mutual information between X and Y given W is defined as

$$I(X;Y|W) = \sum_{x,y,w} P_{XYW}(x,y,w) \log \frac{P_{XY|W}(x,y|w)}{P_{X|W}(x|w)P_{Y|W}(y|w)}$$

Facts

Fact

It holds that

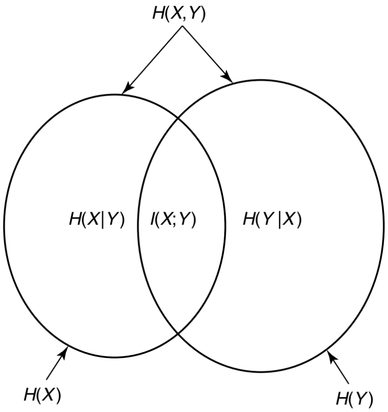
$$\begin{aligned} I(X;Y) = I(Y;X) &= H(X) + H(Y) - H(X,Y) \\ &= H(X) - H(X|Y) \\ &= H(Y) - H(Y|X) \end{aligned}$$

and

$$I(X;g(X)) = H(g(X)).$$

The equalities above still hold if we put the condition "$|W$" in each quantity.

Venn Diagram



Relative Entropy

Definition (Relative Entropy for Discrete Distributions)

For a distribution P and a nonnegative measure μ, the relative entropy [a.k.a. Kullback–Leibler (KL) divergence] of P with respect to (w.r.t.) μ is defined as

$$D(P\|\mu) := \sum_x P(x) \log\left(\frac{P(x)}{\mu(x)}\right).$$

In particular, $D(P\|Q) := \sum_x P(x) \log\left(\frac{P(x)}{Q(x)}\right)$.

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In particular, $D(P\|Q) := \sum_x P(x) \log \left(\frac{P(x)}{Q(x)} \right)$.

Fact (Entropy and MI are special cases of relative entropy)

1) If μ is the counting measure, denoted by $\mathrm{Count}(\mathcal{X})$, then

$$D(P\| \mathrm{Count}(\mathcal{X})) = \sum_x P(x) \log P(x) = -H(X).$$

2) $D(P\| \mathrm{Unif}(\mathcal{X})) = \log |\mathcal{X}| - H(X)$.
3) It holds that $D(P_{XY}\|P_X \otimes P_Y) = I(X;Y)$.

Conditional Relative Entropy

Definition (Conditional Relative Entropy)

For probability measure P_X, transition probability measure $P_{Y|X}$, and transition measure $\mu_{Y|X}$, the relative entropy of $P_{Y|X}$ w.r.t. $\mu_{Y|X}$ conditioned on P_X is defined as

$$D(P_{Y|X}\|\mu_{Y|X}|P_X) := D(P_X P_{Y|X}\|P_X \mu_{Y|X}) = \mathbb{E}_{P_X}\left[D(P_{Y|X}\|\mu_{Y|X})\right].$$

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Fact (Conditional entropy and MI are special cases of conditional relative entropy)

1) If $\mu_{Y|X=x}$ is the counting measure $\mathrm{Count}(\mathcal{Y})$ for every x, then

$$D(P_{Y|X} \| \mathrm{Count}(\mathcal{Y}) | P_X) = \sum_{x,y} P_{XY}(x,y) \log P_{Y|X}(y|x) = -H(Y|X).$$

2) $D(P_{Y|X} \| \mathrm{Unif}(\mathcal{Y}) | P_X) = \log|\mathcal{Y}| - H(Y|X)$.
3) It holds that $D(P_{Y|XW} \| P_{Y|W} | P_{XW}) = I(X;Y|W)$.

Outline

Chain Rule

Theorem (Chain rule)

$$D(P_{X^n} \| Q_{X^n}) = \sum_{i=1}^{n} D(P_{X_i|X^{i-1}} \| Q_{X_i|X^{i-1}} | P_{X^{i-1}}),$$

which still holds with substitution $Q \leftarrow \mu$ (arbitrary nonnegative measure). In particular,

$$H(X^n) = \sum_{i=1}^{n} H(X_i|X^{i-1}) \ \text{and}\ I(X^n; Y) = \sum_{i=1}^{n} I(X_i; Y|X^{i-1}).$$

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Proof: By the disintegration theorem, $P_{X^n} = \prod_{i=1}^n P_{X_i|X^{i-1}}$ and $Q_{X^n} = \prod_{i=1}^n Q_{X_i|X^{i-1}}$. Then,

$$\begin{aligned} D(P_{X^n} \| Q_{X^n}) = \mathbb{E}_P \left[\log \left(\frac{P(X^n)}{Q(X^n)} \right) \right] &= \sum_{i=1}^{n} \mathbb{E}_P \left[\log \left(\frac{P(X_i|X^{i-1})}{Q(X_i|X^{i-1})} \right) \right] \\ &= \sum_{i=1}^{n} D(P_{X_i|X^{i-1}} \| Q_{X_i|X^{i-1}} | P_{X^{i-1}}). \end{aligned}$$

Nonnegativity

Theorem (Nonnegativity)

For probability measures P, Q*, it holds that* $D(P\|Q) \geq 0$ *with equality iff* $P = Q$*. In particular,* $I(X;Y) \geq 0$ *with quality iff* X *and* Y *are independent.*

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Proof:

$$\begin{aligned}
D(P\|Q) &= \sum_x Q(x)\frac{P(x)}{Q(x)} \log\left(\frac{P(x)}{Q(x)}\right) \\
&= \mathbb{E}_Q\left[\varphi\left(\frac{P(X)}{Q(X)}\right)\right] && \varphi(t) = t\log t \\
&\geq \varphi\left(\mathbb{E}_Q\left[\frac{P(X)}{Q(X)}\right]\right) && \text{by convexity of } \varphi \text{ and Jensen's inequality} \\
&= \varphi(1) = 0
\end{aligned}$$

Consequence 1: Conditioning Reducing Entropy

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Proof: $I(X;Y) = H(X) - H(X|Y) \geq 0$

Consequence 2: Joint Relative Entropy is Larger

Theorem (Joint relative entropy is larger than marginal ones)

$$D(P_{XY}\|Q_{XY}) \geq D(P_Y\|Q_Y) \quad (and\ D(P_{XY}\|Q_{XY}) \geq D(P_X\|Q_X)),$$

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which still holds with $Q \leftarrow \mu$.

Proof:

$$\begin{aligned} D(P_{XY}\|Q_{XY}) &= D(P_Y\|Q_Y) + D(P_{X|Y}\|Q_{X|Y}|P_Y) && \text{by chain rule} \\ &\geq D(P_Y\|Q_Y) && \text{by nonnegativity} \end{aligned}$$

Two Special Cases

Corollary (Data processing inequality (DPI))

Given a channel $P_{Y|X}$, it holds that

$$D(P_X \| Q_X) \geq D(P_Y \| Q_Y),$$

where $P_X \to P_{Y|X} \to P_Y$ and $Q_X \to P_{Y|X} \to Q_Y$.

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Corollary (Conditioning increases relative entropy)

Given a distribution P_X, it holds that

$$D(P_{Y|X} \| Q_{Y|X} | P_X) \geq D(P_Y \| Q_Y), \tag{1}$$

where $P_X \to P_{Y|X} \to P_Y$ and $P_X \to Q_{Y|X} \to Q_Y$.

Consequence 3: Convexity

- If further, $P_X = \mathrm{Bern}(\lambda)$, then (1) implies

$$(1-\lambda)D(P_0\|Q_0) + \lambda D(P_1\|Q_1) \geq D((1-\lambda)P_0 + \lambda P_1\|(1-\lambda)Q_0 + \lambda Q_1)$$

where $P_i := P_{Y|X=i}$, $Q_i := Q_{Y|X=i}$, $i = 0, 1$.

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4) $I(X;Y) = D(P_{XY}\|P_X \otimes P_Y) = D(P_{Y|X}\|P_Y|P_X)$. Given P_X, $P_{Y|X} \mapsto P_Y$ is linear.

Consequence 4: Superadditivity

Theorem (Superadditivity of Relative Entropy)

$$D(P_{X^n} \| Q_X^{\otimes n}) \geq \sum_{i=1}^{n} D(P_{X_i} \| Q_X),$$

which still holds with $Q \leftarrow \mu$.

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Corollary (Subadditivity of Entropy)

$H(X^n) \leq \sum_{i=1}^{n} H(X_i)$ *with equality iff the* X_i*'s are independent.*

Consequence 5: Maximum Entropy

Theorem

$$H(X) \leq \log |\mathcal{X}|$$

where $|\mathcal{X}|$ *denotes the number of elements in the range of* X*, with equality iff* X *is uniformly distributed over* $\mathcal{X}$*.*

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where $|\mathcal{X}|$ denotes the number of elements in the range of X, with equality iff X is uniformly distributed over $\mathcal{X}$.

Proof: $D(P \| \mathrm{Unif}(\mathcal{X})) = \log |\mathcal{X}| - H(X) \ge 0.$

Consequence 6: DPI for Mutual Information

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If $X \leftrightarrow Y \leftrightarrow Z$, then $I(X;Y) \geq I(X;Z)$ and $I(X;Y) \geq I(X;Y|Z)$. In particular, if $Z = g(Y)$, we have $I(X;Y) \geq I(X;g(Y))$ and $I(X;Y) \geq I(X;Y|g(Y))$.

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Proof:

$$\begin{aligned} I(X;Y) &= H(X) - H(X|Y) \\ &= H(X) - H(X|Y,Z) \qquad \text{by conditional independence } P_{X|YZ} = P_{X|Y} \\ &= I(X;Y,Z) \\ &= I(X;Z) + I(X;Y|Z) \end{aligned}$$

By nonnegativity of mutual information, we obtain the desired results.

Summary: Properties of D

$$D(P\|\,\mathrm{Count}(\mathcal{X})) = -H(X).$$
$$D(P\|\,\mathrm{Unif}(\mathcal{X})) = \log|\mathcal{X}| - H(X)$$
$$D(P_{XY}\|P_X\otimes P_Y) = I(X;Y)$$
$$D(P_{X^n}\|Q_{X^n})= \sum_{i=1}^{n} D(P_{X_i|X^{i-1}}\|Q_{X_i|X^{i-1}}|P_{X^{i-1}})$$
$$D(P\|Q)\geq 0 \text{ with equality iff } P = Q$$
$$D(P_{XY}\|Q_{XY}) \geq \max\{D(P_Y\|Q_Y)\, D(P_X\|Q_X)\}$$
$$D(P_X\|Q_X) \geq D(P_Y\|Q_Y), \text{ if } P_X \to P_{Y|X} \to P_Y \text{ and } Q_X \to P_{Y|X} \to Q_Y$$
$$D(P_{Y|X}\|Q_{Y|X}|P_X) \geq D(P_Y\|Q_Y), \text{ if } P_X \to P_{Y|X} \to P_Y \text{ and } P_X \to Q_{Y|X} \to Q_Y$$
$$D(P_{X^n}\|Q_X^{\otimes n}) \geq \sum_{i=1}^{n} D(P_{X_i}\|Q_X)$$
$$(P,Q) \mapsto D(P\|Q) \text{ is convex.}$$

Summary: Properties of H

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\begin{aligned}
H(X) &\geq 0 \\
H(X|Y) &\leq H(X) \text{ with equality iff } X \perp Y \\
H(g(X)|X) &= 0 \\
H(X,Y) &= H(X) + H(Y|X) \\
H(X^n) &= \sum_{i=1}^{n} H(X_i|X^{i-1}) \\
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I(X;Y) &= I(Y;X) \\
I(X;g(X)) &= H(g(X)) \\
I(X^n;Y) &= \sum_{i=1}^{n} I(X_i;Y|X^{i-1}), \\
I(X;Y) &\geq \max\{I(X;Z),\, I(X;Y|Z)\} \text{ if } X \leftrightarrow Y \leftrightarrow Z \\
P_X &\mapsto I(X;Y) \text{ is concave} \\
P_{Y|X} &\mapsto I(X;Y) \text{ is convex.}
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Outline

Relative Entropy in Abstract Spaces

- We now consider arbitrary measurable space $(\mathcal{X}, \Sigma_{\mathcal{X}})$.
- We say P is absolutely continuous w.r.t. μ, written as $P \ll \mu$, if for any measurable A, $P(A) = 0$ whenever $\mu(A) = 0$.

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Definition

For a probability measure P and a nonnegative (σ-finite) measure μ on $(\mathcal{X}, \Sigma_{\mathcal{X}})$ such that $P \ll \mu$, the relative entropy of P w.r.t. μ is

$$D(P\|\mu) := \int \log\left(\frac{\mathrm{d}P}{\mathrm{d}\mu}\right) \mathrm{d}P$$

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- If μ is the Lebesgue measure, then $h(X) := -D(P\|\mu) = -\int q(x) \log q(x) \mathrm{d}x$ is the differential entropy of $X \sim P$.

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- If μ is the Lebesgue measure, then $h(X) := -D(P\|\mu) = -\int q(x)\log q(x)\mathrm{d}x$ is the differential entropy of $X \sim P$.
- We define the mutual information via $I(X;Y) := D(P_{XY}\|P_X \otimes P_Y)$

Alternative Definition via Discretization

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It holds that

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Proof of "$\leq$" part is based on discretization of $\frac{\mathrm{d}P_X}{\mathrm{d}Q_X}$ [Van Erven–Harremos, 2014]

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Thank you for your attention!