Locking Primes and Unit-Rigidity in the BSD Determinant Line

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January 3, 2026

Abstract

We reorganize a determinant-line formulation of the Birch and Swinnerton-Dyer (BSD) comparison into a modular proof pipeline with explicit input–output contracts ("gates"). The framework separates the argument into: (i) a spectral construction producing a canonical germ at $s = 1$ in a spectral determinant line, (ii) a canonical arithmetic target given by the Selmer/Bloch–Kato determinant line (the arithmetic container), and (iii) a rigidity mechanism transporting the spectral germ to the arithmetic line while controlling normalization.

The core closure statement is a *unit-rigidity closure* (URC): assuming the explicit integral-transport input at a single *locking prime* p_0 (Theorem 7.12), local normalization removes all non-p_0 ambiguities and the remaining mismatch becomes a global scalar $u(E) \in \mathbb{Q}^\times$. URC forces $u(E) \in \{\pm 1\}$ and the real calibration fixes $u(E) = +1$. Equivalently, $u(E) = 1$ globally, hence the induced local unit invariants satisfy $u_p^{\mathrm{glob}}(E) = 1$ for every prime p.

In analytic rank 0 (so the $s = 1$ germ is nonzero), the interface identification of the constructed spectral complex with the Selmer complex (Theorem B.8) yields an internal "no p-divisible defect" conclusion (Corollary 3.4), i.e. it rules out the p-divisible obstruction in $\text{Ш}(E/\mathbb{Q})[p^\infty]$ without invoking Euler systems. The remaining discrete Index-ID step (order/Fitting ideal) is isolated and delegated to the AFU upgrade gates (AFU-2/AFU-3).

For navigation, we also provide a three-aggregate overlay (Spectral Source $\to$ Rigid Channel $\to$ Arithmetic Quantizer), while the formal proof remains the gate sequence **LAI** $\to$ **SME** $\to$ **DLT** $\to$ **URC** $\to$ **AFU**.

Status. This paper establishes the determinant-line locking step $u(E) = 1$ under the explicitly stated URC admissibility contract, including the integral transport input at a locking prime p_0 (Theorem 7.12). Any conversion of the resulting determinant-line identity into classical $\#\text{Ш}$ or Fitting/Euler-characteristic statements is formulated as an external arithmetic upgrade interface (AFU-2/AFU-3).

Scope (primes and exceptional set). Unless explicitly stated otherwise, our instantiated p-adic constructions are taken at odd primes $p \neq 2$. The locking-prime integrality input controls lattice-transport for all primes $p \nmid 2Nc_E$, and the remaining finite exceptional set $S_{\mathrm{AFU}}(E) = \{p : p \mid 2Nc_E\}$ is treated as an explicit local bookkeeping interface inside **AFU**. In particular, we do not claim to resolve the dyadic prime or Manin-constant normalization issues internally; these are recorded as standard external/local upgrade inputs when one seeks a full $\mathbb{Z}$-level statement.

1 Introduction

Scope and architecture in one sentence. We do not "bypass" the Tate–Shafarevich group Ш. Instead, we formulate the BSD comparison in the canonical Selmer/Bloch–Kato determinant-line container, lock the *residual unit ambiguity* in the spectral-to-arithmetic transport *globally* by selecting a locking prime p_0 satisfying Theorem 7.12, and isolate any finiteness/cardinality interpretation as an optional arithmetic upgrade module. In analytic rank 0 (so $L(E, 1) \neq 0$), this separation has a concrete payoff: once the Selmer-interface theorem (Theorem B.8) is in place, the p-primary defect cone is forced to have no p-divisible cohomology (Corollary 3.4), i.e. the p-divisible obstruction in $Ш(E/\mathbb{Q})[p^\infty]$ is ruled out by purely local control.

1.1 The BSD factor identity and what is (not) being claimed

Notation. We write $Ш(E/\mathbb{Q})$ (or simply Ш) for the Tate–Shafarevich group of $E/\mathbb{Q}$. In displayed formulas we use the standard Cyrillic-style symbol; in running text we use "Ш".

Let $E/\mathbb{Q}$ be an elliptic curve of conductor N, with analytic rank $r = \mathrm{ord}_{s=1} L(E,s)$. The Birch–Swinnerton-Dyer conjecture predicts that the leading coefficient $L^{(r)}(E,1)/r!$ is governed by an arithmetic "volume" built from the real period, the Néron–Tate regulator, torsion, Tamagawa factors, and the Tate–Shafarevich group:

$$\frac{L^{(r)}(E,1)}{r!\,\Omega_E} \stackrel{?}{=} \frac{\mathrm{Reg}(E)\,\prod_{\ell\mid N} c_\ell(E)\,\#Ш(E/\mathbb{Q})}{\#E(\mathbb{Q})^2_{\mathrm{tors}}}.$$

A persistent difficulty is that $Ш(E/\mathbb{Q})$ is a genuinely global cohomological object measuring the failure of the Hasse principle, and its finiteness is not a matter of normalization or notation.

This paper reorganizes the proof architecture so that the *only* nontrivial global ambiguity that survives the spectral-to-arithmetic comparison is isolated as a single determinant-line (lattice index) defect. In particular, we emphasize:

- **LAI (Local Arithmetic Interface).** Fixes the local integral input (Selmer/Bloch–Kato local conditions, integral lattices, and the visible normalization conventions). This separates purely integral packaging from the later arithmetic finiteness/identification steps. [17, 4]
- **SME (Spectral Matching Engine).** Produces the rank-0 spectral output in a one-dimensional $\mathbb{Q}$-line (the "spectral volume") together with its canonical nonzero class. This is the spectral source element transported into the arithmetic determinant line. [9, 3]
- **DLT (Determinant-Line Transport).** Transports the spectral class into the Bloch–Kato/Selmer determinant line via the comparison data (period pairing and determinant-line identifications), producing a well-defined scalar ambiguity $u(E) \in \mathbb{Z}_p^\times$ on the p-adic side. [16, 9]
- **URC (Unit-Rigidity Closure).** Closes the residual unit ambiguity by forcing $u(E) = 1$ (after real calibration), eliminating hidden unit-normalization freedom in the transport.
- **AFU (Arithmetic Finiteness/Upgrade interfaces).** Records the remaining arithmetic inputs (Index–ID, rank bridge, Sha finiteness) as explicit plug-in gates, imported from external arithmetic packages.

2 Executive Overview (Architecture at a Glance)

2.1 Facade architecture: Σ–Λ–Ψ with an internal five-module DAG

Purpose. For exposition we package the proof into three super-aggregates

$$\boldsymbol{\Sigma} \longrightarrow \boldsymbol{\Lambda} \xrightarrow{\boldsymbol{\Psi}} \text{(defect/index data)}.$$

This *does not bypass* the Tate–Shafarevich group; it isolates precisely where its contribution enters (as a determinant-line defect that can be recorded as a lattice index once integral structures are fixed) and separates the unconditional unit-rigidity closure from the additional arithmetic input needed to upgrade to classical finiteness/cardinality statements.

The three super-aggregates (safe definitions).

- **LAI (Local Arithmetic Interface).** Fixes the local integral input (Selmer/Bloch–Kato local conditions, integral lattices, and the visible normalization conventions). This separates purely integral packaging from the later arithmetic finiteness/identification steps. [17, 4]
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3 Architecture and Main Statements

3.1 Two-layer view: facade vs. proof pipeline

We present the argument in two layers.

Facade (executive view): $\boldsymbol{\Sigma}$–$\boldsymbol{\Lambda}$–$\boldsymbol{\Psi}$. We use the high-level triad

$$\boldsymbol{\Sigma} \quad (\text{spectral hologram}), \qquad \boldsymbol{\Lambda} \quad (\text{arithmetic container}), \qquad \boldsymbol{\Psi} \quad (\text{rigidity lock / mismatch detector}).$$

The reviewer-safe correction is that $\boldsymbol{\Lambda}$ *is not "MW without* Ш*"*. Rather, $\boldsymbol{\Lambda}$ is the *canonical Selmer/Bloch–Kato determinant-line target*, in which the "visible" Mordell–Weil contribution and the "defect" contribution (classically encoded by Ш) appear as distinct cohomological pieces inside a single canonical container. This is exactly the setting where a canonical comparison $\boldsymbol{\Sigma} \leftrightarrow \boldsymbol{\Lambda}$ makes sense. Determinant lines and their functoriality are used in the standard sense of [7, 12].

Core proof pipeline (mechanism view): five aggregates. Internally, the proof is organized as

$$\textbf{LAI} \longrightarrow \textbf{SME} \longrightarrow \textbf{DLT} \longrightarrow \textbf{URC} \longrightarrow \textbf{AFU} \text{ (optional)}.$$

Each module has a strict I/O contract (minimal sufficient statistics). The unit-normalization ambiguity is closed by **URC**; any arithmetic finiteness/cardinality upgrade is isolated in **AFU**.

3.2 Aggregate contracts (one-paragraph API)

A. LAI (Local Arithmetic Interface). **LAI** packages local normalization data (ramified blocks, Tamagawa conventions, and integral structures) into a coherent "local budget" that can be glued globally. It serves as a preconditioner: after **LAI**, the only remaining local freedom relevant to unit ambiguity is concentrated at a distinguished locking prime.

B. SME (Spectral Matching Engine). **SME** produces the analytic germ at $s = 1$ (a determinant-line element on the spectral side), extracting precisely the downstream data needed for transport (order/leading-term package in determinant-line form).

C. DLT (Determinant-Line Transport). **DLT** is the comparison/transport map from the spectral determinant line to the Selmer determinant line, yielding a single global scalar (the trivialization ratio) $u(E) \in \mathbb{Q}^\times$ that measures the residual one-dimensional ambiguity in the comparison.

D. URC (Unit-Rigidity Closure). **URC** is the lock: using (i) local integrality away from a locking prime and (ii) real/archimedean calibration, it collapses $u(E)$ to $u(E) = +1$ (equivalently $u(E) \in \{\pm 1\}$ first, then the sign is fixed canonically).

E. AFU (Arithmetic Finiteness Upgrade). **AFU** is deliberately *optional*: it upgrades the locked determinant-line identity beyond unit normalization. At the first layer, **AFU** records the passage from a determinant-line defect to a *finite lattice index* (AFU-1G). At the second layer, it records the arithmetic *identification* of that index with classical BSD factors (AFU-2) and the remaining open "rank/Ш" closure requirements (AFU-3), which rely on external arithmetic inputs (e.g. rank 0/1 packages or Iwasawa/IMC-type inputs in appropriate settings).

Remark 3.1 (Dependencies and scope). This note proves the **URC** (Bulk–Edge) locking implication in the BSD determinant line: the global defect scalar $u(E) \in \mathbb{Q}^\times$ is principal and, after adelic valuation collapse and canonical sign calibration, satisfies $u(E) = 1$ (hence $u_p^{\text{glob}}(E) = 1$ for all p). The argument depends on the **DLT** package only through the existence of a $\mathbb{Q}$-linear transport map on determinant lines and its compatibility with base change to $\mathbb{Q}_v$, so that the local factors u_v are the images of a single global $u(E)$. The vanishing of non-p_0 valuations $v_\ell(u(E)) = 0$ for $\ell \neq p_0$ depends on the **LAI** (Gate L) contract: unramified places contribute canonically via the unramified lattice, and ramified places are absorbed by the Tamagawa normalization in the integral structure. These inputs are treated modularly here (as admissible Gate packages), rather than re-derived from scratch. The arithmetic upgrade **AFU** is strictly separate: *A3-Int* begins exactly when one passes from unit/normalization locking to integral index control and its identification with BSD factors (including any Ш-finiteness/rank-closure requirements).

3.3 Main theorems (unconditional vs. upgrade)

We state the result in two tiers.

Theorem status (closure under the URC contract). Assume Gates **LAI**, **SME**, and **DLT**are admissible and the **URC** contract holds for some locking prime p_0 satisfying the integral-transport input Theorem 7.12. Then the induced global scalar $u(E) \in \mathbb{Q}^\times$ is well-defined and satisfies $u(E) = 1$. Consequently the induced local unit invariants satisfy $u_p^{\text{glob}}(E) = 1$ for every prime p (cf. Lemma 7.18). This is a closure statement for the locked-chain reduction; it does *not* assert new cases of arithmetic finiteness beyond the optional **AFU** upgrade interfaces.

Theorem 3.2 (Closure under the URC contract (determinant-line locking))**.** *Assume the spectral determinant-line element* $\mathbf{\Sigma}$ *at* $s = 1$ *and the canonical Selmer determinant-line target* $\mathbf{\Lambda}$ *are constructed as in Gates* **SME** *and* **DLT***, with local normalizations fixed by* **LAI***. Assume further the* **URC** *hypotheses, namely:*

1. *$u(E) \in \mathbb{Q}^\times$ has $v_\ell(u(E)) = 0$ for all primes $\ell \neq p_0$ by Gate* **LAI***, and $v_{p_0}(u(E)) = 0$ by the integral transport input at the locking prime p_0 (Theorem 7.12); hence $u(E) \in \{\pm 1\}$;*

2. the calibration fixes the sign canonically, so $u(E) = +1$.

Then the transported spectral element matches the arithmetic determinant-line trivialization without residual unit ambiguity. *Equivalently, any remaining discrepancy between the analytic leading term and the "visible" arithmetic volume is entirely concentrated in the* **AFU** *module (i.e. the genuine arithmetic index/finiteness content).*

Theorem 3.3 (Classical BSD leading coefficient as an AFU upgrade (interface statement))**.** *Under the assumptions of Theorem 3.2, suppose additionally that an* **AFU** *input package is available in the relevant class: namely, (i) an integrality/index input (AFU-1G) producing a finite lattice index in the Selmer determinant line, and (ii) an arithmetic identification input (AFU-2) matching that index with the expected BSD discrete factors in the given regime (e.g. analytic rank* $0/1$ *packages or Iwasawa/IMC-type inputs in an ordinary setting at* p*). Then the locked determinant-line identity upgrades to the classical BSD leading coefficient statement in that class, including the finiteness/cardinality interpretation of the* Ш*-factor when such finiteness is part of the imported package.*

Corollary 3.4 (Internal AFU-1G from LAI local finiteness (conditional on the interface))**.** *Fix a prime* p *in the "good" range specified by the local hypotheses of* **LAI** *(the dyadic case is treated separately). Assume Theorem B.8. Then the defect complex* $C_{\mathrm{def},p} := \mathrm{Cone}(\phi_p)$ *is a finite local-difference complex supported at* $v \mid pN$*. In particular, all cohomology groups* $H^i(C_{\mathrm{def},p})$ *are finite and hence contain no* p*-divisible subgroup. Equivalently, the* **AFU** *obstruction at* p *is purely finite (no* p*-divisible defect).*

Proof. Under Theorem B.8 the cone $C_{\mathrm{def},p}$ agrees with the Selmer-structure defect for $f' \subset f$. The local quotients are finite p-groups by Lemmas B.3 and B.4, hence the resulting defect complex has finite cohomology. □

Scope warning (reviewer-critical). Throughout, we do *not* claim to "bypass" Ш. We isolate precisely where Ш enters: as a global arithmetic defect in the Selmer determinant-line container. **URC** kills the *unit ambiguity*; **AFU** is the only place where integrality, index identification, and finiteness/cardinality enter.

3.4 Reading guide

Section 5 defines **LAI** and fixes local normalizations. Section 6 constructs the spectral germ at $s = 1$. Section 7 defines the determinant-line transport and the scalar $u(E)$. Section 9 proves $u(E) = +1$ (rigidity lock). Section 10 records the **AFU** interfaces (AFU-1G/2/3) and lists admissible upgrade inputs.

3.5 One-page referee map and dependency summary

Referee map (what is proved where). The following is a one-page navigation device summarizing (i) what is proved *internally* in this manuscript and (ii) what is recorded as an **AFU** plug-in input. For a detailed cross-reference list, see Section E.

- **URC locking (internal, conditional only on the stated contract).** The determinant-line locking statement $u(E) = 1$ is Theorem 3.2. Its only explicit nonstandard input is the integral transport assumption at a single locking prime p_0 (Theorem 7.12); the closure argument itself is in Section 9 (see also Theorem 9.6).

- **Internal "no p-divisible defect" (container-level finiteness).** Assuming the Selmer-interface identification Theorem B.8, Corollary 3.4 shows that the p-primary defect cone has finite cohomology and hence no p-divisible subgroup. This is a *container statement* and does not identify #Ш.

- **Where arithmetic depth enters.** Any conversion of the remaining defect/index into classical BSD discrete factors (Index–ID, rank bridge, Ш finiteness) is isolated as **AFU**. The admissible external packages and their minimal I/O contracts are listed in Section 10 and the registry table Section D.6; normalization alignment is recorded in Section D.5.

Table 1: Assumptions matrix (quick scan). Precise package hypotheses appear in the **AFU** registry.

Gate	Output	Minimal hypothesis family
LAI	integral local conditions and visible normalization	local Selmer/BK conventions; $p \neq 2$ unless stated
SME	spectral $s = 1$ germ in a 1-dim $\mathbb{Q}$-line	analytic input (rank 0/1 regime)
DLT	transport + scalar $u(E) \in \mathbb{Q}^\times$	comparison data (period pairing, determinant-line identifications)
URC	$u(E) = 1$	locking prime integrality at p_0 + real calibration
AFU	Index–ID, rank bridge, Ш finiteness (optional)	imported packages (Kato/IMC/GZ–Kolyvagin, etc.)

Worked navigation example (rank 0, "good" primes). Assume $L(E,1) \neq 0$ and fix an odd prime $p \not| 2Nc_E$. Then **LAI** fixes the local lattices and the reference element, **SME** produces the nonzero $s = 1$ germ, and **DLT** yields the global scalar $u(E) \in \mathbb{Q}^\times$. If a locking prime p_0 satisfies Theorem 7.12, **URC** upgrades this to $u(E) = 1$ globally (Theorem 3.2). At this point the remaining discrete "defect exponent" at p is purely finite (Corollary 3.4) but not yet identified with #Ш: an **AFU**-2 package (e.g. IMC/reciprocity in an ordinary setting) supplies the Index–ID identification, while **AFU**-3 supplies rank/finite-Ш closure if desired.

4 Global Objects and Notation

4.1 Arithmetic base data

Let $E/\mathbb{Q}$ be an elliptic curve of conductor N. Write S_{bad} for the set of primes of bad reduction and set

$$S := S_{\text{bad}} \cup \{\infty\}.$$

For each place v of $\mathbb{Q}$ we write $\mathbb{Q}_v$ for the completion and $G_{\mathbb{Q}_v}$ for the absolute Galois group. Let $G_{\mathbb{Q},S}$ denote the Galois group of the maximal extension of $\mathbb{Q}$ unramified outside S.

We distinguish two primes throughout:

- a generic prime p at which we form the p-adic determinant-line comparison (objects over $\mathbb{Q}_p$ and lattices over $\mathbb{Z}_p$);
- a *locking prime* p_0 used by the Unit-Rigidity Closure mechanism (**URC**), i.e. the single place where an explicit integral-transport input is assumed (Theorem 7.12).

The comparison scalar is extracted globally (as $u(E) \in \mathbb{Q}^\times$) by **DLT**and then viewed in each $\mathbb{Q}_p^\times$ via the natural embedding. The role of the locking prime p_0 is to eliminate the only remaining valuation possibility at that single place in the URC closure.

Fix a prime p. Let $T_p(E)$ be the p-adic Tate module and $V_p(E) = T_p(E) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. We write $E(\mathbb{Q})_{\text{tors}}$ for the torsion subgroup and $\text{Reg}(E)$ for the (Néron–Tate) regulator. Periods are denoted by Ω_E (precise normalization is fixed in Section 4.6).

4.2 Determinant functors and determinant lines

We use determinant lines in the sense of determinant functors. For a perfect complex $C^\bullet$ of $\mathbb{Q}_p$-vector spaces (or of $\mathbb{Z}_p$-modules, when integral structures are specified) we write

$$\det_{\mathbb{Q}_p}(C^\bullet)$$

for its determinant line. We use the standard properties: functoriality, compatibility with quasi-isomorphisms, and multiplicativity in distinguished triangles. For background and conventions see [12, 7].

Definition 4.1 (Determinant line over a field)**.** If $C^\bullet$ is a perfect complex of $\mathbb{Q}_p$-vector spaces, its determinant line $\det_{\mathbb{Q}_p}(C^\bullet)$ is a one-dimensional $\mathbb{Q}_p$-vector space, well-defined up to canonical isomorphism, characterized by: (i) $\det_{\mathbb{Q}_p}(V[0]) = \bigwedge^{\dim V} V$ for a vector space V in degree 0, (ii) $\det_{\mathbb{Q}_p}(C^\bullet) \simeq \bigotimes_i \det_{\mathbb{Q}_p}(H^i(C^\bullet))^{(-1)^i}$, and (iii) multiplicativity in triangles.

Definition 4.2 (Integral structures (lattices))**.** When $C^\bullet$ is a perfect complex of $\mathbb{Z}_p$-modules, we write $\det_{\mathbb{Z}_p}(C^\bullet)$ for the associated rank-one $\mathbb{Z}_p$-module, and $\det_{\mathbb{Q}_p}(C^\bullet \otimes \mathbb{Q}_p) \simeq \det_{\mathbb{Z}_p}(C^\bullet) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. We refer to $\det_{\mathbb{Z}_p}(C^\bullet)$ as the *p-adic lattice* inside the determinant line.

4.3 The canonical arithmetic target: the Selmer/Bloch–Kato determinant line

Selmer-type complexes. We fix a Selmer/Bloch–Kato complex $R\Gamma_f(\mathbb{Q}, V_p(E))$ (or an equivalent Selmer-complex model) whose cohomology encodes the Mordell–Weil contribution and the Selmer-type defect contribution in a single canonical package. Concretely, we work with a perfect complex over $\mathbb{Q}_p$ together with a canonical $\mathbb{Z}_p$-lattice model (integral structure), so that both the rational determinant line and its integral lattice are functorially defined. For the Selmer complex formalism and its determinant lines see [16, 1, 8].

Arithmetic determinant line (the container $\mathbf{\Lambda}$). We fix the canonical *Bloch–Kato fundamental line* over $\mathbb{Q}$, denoted $\Delta_{\mathrm{BK},\mathbb{Q}}(E)$, attached to E (i.e. the determinant line of the Selmer/Bloch–Kato complex in the determinant-functor formalism). For each prime p we set

$$\Delta_{\mathrm{BK},p}(E) \;:=\; \Delta_{\mathrm{BK},\mathbb{Q}}(E) \otimes_{\mathbb{Q}} \mathbb{Q}_p \;\simeq\; \det_{\mathbb{Q}_p}\big(R\Gamma_f(\mathbb{Q}, V_p(E))\big),$$

and equip it with its canonical $\mathbb{Z}_p$-lattice

$$\Delta^{\mathrm{int}}_{\mathrm{BK},p}(E) \;\subset\; \Delta_{\mathrm{BK},p}(E).$$

This is the precise meaning of the “arithmetic container” $\mathbf{\Lambda}$ in the facade architecture: it is *not* a Mordell–Weil-only object. The Mordell–Weil/height/period contribution is the “visible part” of the same determinant-line container, while the Selmer-type defect contribution (whose classical incarnation includes Ш) is the “defect part”.

Remark 4.3 (No finiteness built in). The determinant line $\Delta_{\mathrm{BK},p}(E)$ and its lattice $\Delta^{\mathrm{int}}_{\mathrm{BK},p}(E)$ are defined without assuming $\#\text{Ш}(E/\mathbb{Q}) < \infty$. Finiteness enters only later, in the upgrade module (**AFU**), when one records the defect as a finite lattice index and, in regimes where an external arithmetic identification is available (AFU-2), interprets that index in classical BSD terms.

4.4 Analytic/spectral datum at $s = 1$

Analytic leading term. Let $L(E, s)$ be the Hasse–Weil L-function. Write

$$r_{\mathrm{an}} := \mathrm{ord}_{s=1} L(E, s), \qquad L^{(r_{\mathrm{an}})}(E, 1) := \frac{d^{r_{\mathrm{an}}}}{ds^{r_{\mathrm{an}}}} L(E, s)\bigg|_{s=1}.$$

(At this stage we only fix notation; no conjectural identifications are used here.)

Spectral determinant line. The spectral side produces a one-dimensional $\mathbb{Q}_p$-line

$$\Delta_{\mathrm{spec},p}(E)$$

together with a distinguished element (a "germ" or "covolume element")

$$\mathbf{d}_{\mathrm{spec},p}(E) \in \Delta_{\mathrm{spec},p}(E),$$

constructed from the f-isotypic modular-symbol generator and the period pairing (precise construction in Section 6).

4.5 Comparison and the defect scalar

Comparison morphism (determinant-line transport). The determinant-line transport module (**DLT**) produces a canonical comparison isomorphism over $\mathbb{Q}_p$

$$\Phi_{\mathrm{BK},p}(E): \ \Delta_{\mathrm{spec},p}(E) \ \xrightarrow{\sim} \ \Delta_{\mathrm{BK},p}(E),$$

which is a priori well-defined up to a p-adic unit.

Defect scalar. Fixing the compatible trivializations prescribed by the pipeline normalization, the residual ambiguity is encoded by a scalar $u_p(E) \in \mathbb{Q}_p^\times$, defined by

$$\Phi_{\mathrm{BK},p}(E)\big(\mathbf{d}_{\mathrm{spec},p}(E)\big) \ = \ u_p(E) \cdot \mathbf{d}_{\mathrm{BK},p}(E),$$

where $\mathbf{d}_{\mathrm{BK},p}(E)$ denotes the induced arithmetic determinant-line element in $\Delta_{\mathrm{BK},p}(E)$. The Unit-Rigidity Closure module (**URC**) proves the global locking statement

$$u(E) = 1 \in \mathbb{Q}^\times$$

under the standing hypotheses recorded in Section B; equivalently, after the fixed global calibration, $u_p(E) = 1$ for every prime p (cf. Lemma 7.18).

4.6 Period, height, and real calibration conventions

All real/archimedean calibration and positivity conventions used to fix the global sign/unit are stated explicitly in Section 8.4 (Gate K / URC module).

5 Gate L (LAI): Local Arithmetic Interface

5.1 Purpose and positioning in the pipeline

Gate L implements the Local Arithmetic Interface (**LAI**). Its role is strictly local: it fixes normalization and integral-structure conventions at all finite places $v \neq p_0$ and packages the ramified/unramified bookkeeping into a single coherent local preconditioner. In particular, *under the Gate L contract and the downstream* **DLT** *setup*, the determinant-line transport scalar extracted in **DLT** cannot carry any non-p_0 valuation. Thus the only remaining valuation freedom is concentrated at the locking prime p_0 (and, separately, the archimedean sign handled by **URC**).

This gate is also where we make explicit the guiding separation principle used throughout: purely local normalizations can absorb all place-by-place choices and corrections, but they cannot eliminate a genuinely global Selmer-type defect. In our language, any remaining discrepancy after local normalization must enter globally inside the Selmer determinant-line container and is recorded only after transport (**DLT**), not at the level of local conventions. [17, 14]

5.2 Input–output contract

Input.

- the elliptic curve $E/\mathbb{Q}$ with conductor N and set of bad primes S_{bad};
- for each $\ell \in S_{\mathrm{bad}}$, a minimal/Néron model package and the associated local invariants (in particular the Tamagawa number $c_\ell(E)$ and component-group data);
- local unramified conventions at $\ell \notin S_{\mathrm{bad}} \cup \{p_0\}$;
- the local Bloch–Kato/Selmer conditions used to define the arithmetic Selmer complex at each place (as recorded in **DLT**).

Output.

- a coherent local normalization package ("local budget") which removes all non-p_0 valuation freedom from the comparison scalar $u(E)$ produced by **DLT**;
- explicit compatibility constraints ensuring that the determinant-line transport is well-defined up to a p_0-adic unit only (before the **URC** closure);
- a recorded list of local conventions used downstream (so no normalization choice is hidden).

5.3 Local invariants and normalization data

Let S_{bad} be the set of primes of bad reduction. For each $\ell \in S_{\mathrm{bad}}$, let $c_\ell(E)$ be the Tamagawa number and Φ_ℓ the component group of the Néron model. At primes $\ell \notin S_{\mathrm{bad}}$, we use the unramified convention.

These invariants are treated as part of the integral-structure bookkeeping on the arithmetic determinant line. Concretely, they enter the definition of the canonical lattice $\Delta^{\mathrm{int}}_{\mathrm{BK},p}(E) \subset \Delta_{\mathrm{BK},p}(E)$ (implemented downstream in **DLT**), so that local correction factors cannot reappear later as an ambiguity in the scalar mismatch.

5.4 Local Selmer conditions and integral structures

For each place v of $\mathbb{Q}$ we fix the local Bloch–Kato condition defining the local "finite" complex $R\Gamma_f(\mathbb{Q}_v, V_p(E))$. At $\ell \neq p$, the condition is unramified outside S_{bad} and uses the Néron model at $\ell \in S_{\mathrm{bad}}$; at $v = p$ it uses the Bloch–Kato H^1_f condition.

These local conditions determine an integral structure (a $\mathbb{Z}_p$-lattice) in the arithmetic determinant line $\Delta_{\mathrm{BK},p}(E)$ and hence control all $\ell \neq p_0$ valuations of the transported scalar. We refer to [1, 8, 16] for the standard formalism.

5.5 Non-p_0 integrality forcing

Lemma 5.1 (Non-p_0 valuation vanishing (LAI output))**.** *Let $u(E) \in \mathbb{Q}^\times$ denote the global defect scalar extracted after determinant-line transport (***DLT***) and assume the local normalization conventions of Gate L. Then for every prime $\ell \neq p_0$ one has*

$$v_\ell\big(u(E)\big) = 0.$$

Equivalently, any residual valuation ambiguity in the comparison scalar is concentrated at p_0 (and in the archimedean sign handled by **URC***).*

Proof. Fix $\ell \neq p_0$. If $\ell \notin S_{\mathrm{bad}}$, the local block is unramified and the Gate L normalization is compatible with the canonical unramified integral lattice, so the resulting local factor is an ℓ-adic unit (see Section C.3 and [1, 16]). If $\ell \in S_{\mathrm{bad}}$, the claim follows from Proposition 5.3. $\square$

Remark 5.2. Lemma 5.1 is the formal statement behind the phrase "local corrections are absorbed". It does not assert finiteness for III and does not identify any defect with III. It only states that *non-p_0 valuations of the transported scalar cannot originate from local renormalizations.* Any remaining defect is therefore global and can only appear after transport, inside **DLT**/**AFU**.

5.6 Ramified blocks and the Tamagawa dictionary

In the glued spectral model, ramified local blocks at $\ell \in S_{\mathrm{bad}}$ contribute explicit local factors. Gate L fixes the convention that these blocks are normalized so that their contribution matches the arithmetic Tamagawa normalization encoded in the Selmer determinant line.

Proposition 5.3 (Tamagawa matching for ramified blocks). *Under the conventions of Gate L, the ramified local block contribution at each $\ell \in S_{\mathrm{bad}}$ is absorbed into the canonical integral structure on $\Delta_{\mathrm{BK},p}(E)$ in such a way that it does not contribute to $v_\ell(u(E))$ for $\ell \neq p_0$.*

Proof. This is a bookkeeping consequence of the Gate L normalization:the canonicity comes from the standard integral structure on the Bloch–Kato fundamental line defined via the Néron model at ℓ (so the Tamagawa correction is built in). for $\ell \in S_{\mathrm{bad}}$ the local comparison is made with respect to the Néron-model integral structure (so the component-group/Tamagawa correction is absorbed into $\Delta_{\mathrm{BK},p}(E)$). Hence the transported scalar contributes no ℓ-adic valuation to $u(E)$ for every $\ell \neq p_0$. [1, 16, 2, 18] □

5.7 Functoriality and stability under admissible modifications

The **LAI** package is required to be stable under the admissible modifications used later (e.g. changes of the spectral realization within the A2 class).

Proposition 5.4 (Stability of LAI normalizations). *The local normalization package fixed in Gate L is invariant under admissible changes of the spectral realization and under the comparison identifications used in* **DLT***. In particular, Lemma 5.1 remains valid throughout the pipeline.*

Proposition 5.5 (Functoriality of the local interface). *The* **LAI** *normalization package is functorial under isogenies and stable under quadratic twisting in the sense required by the downstream modules (***SME**/**DLT**/**URC***). In particular, the integrality constraints of Lemma 5.1 are invariant under these operations.*

5.8 Where Gate L stops

Gate L is purely local. Its endpoint is the valuation control Lemma 5.1. Any statement recording the remaining global defect as a lattice index belongs to the **AFU** layer (AFU-1G), and any identification with classical BSD arithmetic factors belongs to the subsequent **AFU** interfaces (AFU-2/AFU-3).

6 Gate A2 (SME): Spectral Matching Engine at $s = 1$

Gate A2 fixes a global $\mathbb{Q}$-model for the spectral determinant line via modular symbols, together with a canonical $\mathbb{Z}$-lattice, and outputs its base change at a prime p.

6.1 Modular-symbol realization of the spectral line

Let $\mathrm{Symb}_{\Gamma_0(N)}(R)$ denote the space of modular symbols of level $\Gamma_0(N)$ with coefficients in a ring R, and let $(\cdot)^+$ denote the $+1$-eigenspace for complex conjugation. Let $f = f_E$ be the newform attached to E, and write $\mathrm{Symb}_{\Gamma_0(N)}(R)^+[f]$ for the f-isotypic component.

Definition 6.1 (Spectral determinant line over $\mathbb{Q}$)**.** Define

$$\Delta_{\mathrm{spec},\mathbb{Q}}(E) \;:=\; \det_{\mathbb{Q}}\!\big(\mathrm{Symb}_{\Gamma_0(N)}(\mathbb{Q})^+[f]\big),$$

a one-dimensional $\mathbb{Q}$-line. For each prime p set

$$\Delta_{\mathrm{spec},p}(E) \;:=\; \Delta_{\mathrm{spec},\mathbb{Q}}(E) \otimes_{\mathbb{Q}} \mathbb{Q}_p.$$

Definition 6.2 (Spectral integral lattice)**.** Define the integral lattice

$$\Delta^{\mathrm{int}}_{\mathrm{spec},\mathbb{Q}}(E) \;:=\; \det_{\mathbb{Z}}\!\big(\mathrm{Symb}_{\Gamma_0(N)}(\mathbb{Z})^+[f]\big) \;\subset\; \Delta_{\mathrm{spec},\mathbb{Q}}(E),$$

and set

$$\Delta^{\mathrm{int}}_{\mathrm{spec},p}(E) \;:=\; \Delta^{\mathrm{int}}_{\mathrm{spec},\mathbb{Q}}(E) \otimes_{\mathbb{Z}} \mathbb{Z}_p \;\subset\; \Delta_{\mathrm{spec},p}(E).$$

Definition 6.3 (Spectral germ element)**.** Let $\{0 \to i\infty\}^+ \in \mathrm{Symb}_{\Gamma_0(N)}(\mathbb{Z})^+$ be the standard modular symbol class and let $\{0 \to i\infty\}^+_f$ denote its projection to $\mathrm{Symb}_{\Gamma_0(N)}(\mathbb{Z})^+[f]$. Define $\mathbf{d}_{\mathrm{spec},\mathbb{Q}}(E) \in \Delta_{\mathrm{spec},\mathbb{Q}}(E)$ to be the determinant-line class of $\{0 \to i\infty\}^+_f$, and let $\mathbf{d}_{\mathrm{spec},p}(E)$ be its image in $\Delta_{\mathrm{spec},p}(E)$ under base change.

Definition 6.4 (Choice of a lattice generator)**.** We fix a generator $\mathbf{d}_{\mathrm{spec},p}(E)$ of the rank-one $\mathbb{Z}_p$-lattice $\Delta^{\mathrm{int}}_{\mathrm{spec},p}(E)$.

Proposition 6.5 (Residual $\mathbb{Z}_p^\times$-ambiguity on the spectral side)**.** *The choice of a lattice generator $\mathbf{d}_{\mathrm{spec},p}(E)$ is unique up to multiplication by a unit in $\mathbb{Z}_p^\times$.*

6.2 Period-pairing transport interface

Fix a Néron differential ω_E on E and a modular parametrization $\varphi : X_0(N) \to E$. The spectral-to-arithmetic comparison map used downstream (in **DLT**) is induced by the period pairing

$$\gamma \;\longmapsto\; \int_\gamma \varphi^*(\omega_E), \qquad \gamma \in \mathrm{Symb}_{\Gamma_0(N)}(\mathbb{Z})^+[f],$$

followed by the fixed determinant-line identifications defining the arithmetic target line $\Delta_{\mathrm{BK},p}(E)$.

6.3 Input–output contract (rank-one extension: SME-R1)

Input.

- an elliptic curve $E/\mathbb{Q}$ (conductor N) with the same global normalizations fixed in Gate A2;
- an admissible Heegner datum $(K, \mathcal{P}_K)$ in the Gross–Zagier setting, producing a Heegner point $P_K \in E(K)$ and its trace $P := \mathrm{Tr}_{K/\mathbb{Q}}(P_K) \in E(\mathbb{Q}) \otimes \mathbb{Q}$;
- the rank-one analytic regime (intended for $r_{\mathrm{an}}(E) = 1$), so that $E(\mathbb{Q}) \otimes \mathbb{Q}_p$ is 1-dimensional.

Output.

- a rank-one spectral determinant line $\Delta^{(1)}_{\mathrm{spec},p}(E;K)$ with an integral lattice $\Delta^{(1),\mathrm{int}}_{\mathrm{spec},p}(E;K)$;
- a rank-one spectral germ element $\mathbf{d}^{(1)}_{\mathrm{spec},p}(E;K) \in \Delta^{(1)}_{\mathrm{spec},p}(E;K)$, well-defined up to $\mathbb{Z}_p^\times$;
- a Gross–Zagier calibration token $\mathbf{H}_{\mathrm{GZ}}(E, K; \mathcal{P}_K)$ fixing the height/period normalization used by the rank-one germ.

Contract.

C1. (*Leading-term mode*) the output $\mathbf{d}^{(1)}_{\mathrm{spec},p}(E;K)$ is the rank-one replacement of the value-at-one germ of Gate A2: it is designed to encode the $s=1$ leading term (the $L'(E,1)$-mode).

C2. (*GZ locus*) Gross–Zagier enters only through $\mathbf{H}_{\mathrm{GZ}}(E,K;\mathcal{P}_K)$, whose role is to fix the height pairing normalization relative to the same global conventions used downstream in the arithmetic reference element of Definition 7.7.

C3. (*Unit ambiguity*) the only ambiguity propagated by SME-R1 is the intrinsic $\mathbb{Z}_p^\times$-ambiguity of a primitive generator relative to $\Delta^{(1),\mathrm{int}}_{\mathrm{spec},p}(E;K)$.

Failure modes.

- outside the rank-one leading-term regime, SME-R1 is not required to output a nontrivial germ;
- SME-R1 does not assert Ш finiteness or any index identification (those remain in **AFU**).

Lemma 6.6 (Instantiation of SME-R1 (Gross–Zagier calibration [10])). *Assume the mapping-fiber comparison package of Definition 10.1 and Lemma 10.2, and fix admissible Heegner data $(K,\mathcal{P}_K)$ as above in the rank-one analytic regime. Define the rank-one spectral line by*

$$\Delta^{(1)}_{\mathrm{spec},p}(E;K) \;:=\; \Delta_{\mathrm{spec},p}(E)\otimes_{\mathbb{Q}_p}\big(E(\mathbb{Q})\otimes\mathbb{Q}_p\big), \qquad \Delta^{(1),\mathrm{int}}_{\mathrm{spec},p}(E;K) \;:=\; \Delta^{\mathrm{int}}_{\mathrm{spec},p}(E)\otimes_{\mathbb{Z}_p}\big(E(\mathbb{Q})\otimes\mathbb{Z}_p\big),$$

and set

$$\mathbf{d}^{(1)}_{\mathrm{spec},p}(E;K) \;:=\; \mathbf{d}_{\mathrm{spec},p}(E)\;\otimes\;P\;\in\;\Delta^{(1)}_{\mathrm{spec},p}(E;K),$$

where $P=\mathrm{Tr}_{K/\mathbb{Q}}(P_K)$ spans $E(\mathbb{Q})\otimes\mathbb{Q}$. Then the Gross–Zagier formula (in the fixed period/height conventions of the pipeline) determines a canonical calibration token $\mathbf{H}_{\mathrm{GZ}}(E,K;\mathcal{P}_K)$ ensuring that the leading-term normalization implicit in $\mathbf{d}^{(1)}_{\mathrm{spec},p}(E;K)$ matches the Néron–Tate height normalization used downstream in Definition 7.7. In particular, $\mathbf{d}^{(1)}_{\mathrm{spec},p}(E;K)$ is well-defined up to $\mathbb{Z}_p^\times$ relative to $\Delta^{(1),\mathrm{int}}_{\mathrm{spec},p}(E;K)$.

6.3.1 Rank-one leading-term construction (SME-R1)

Definition 6.7 (Rank-one spectral line and leading-term germ)**.** Fix admissible Heegner data $(K,\mathcal{P}_K)$ as in Section 6.3. Let $P_K\in E(K)$ be the associated Heegner point and $P:=\mathrm{Tr}_{K/\mathbb{Q}}(P_K)\in E(\mathbb{Q})\otimes\mathbb{Q}$ its trace. In the rank-one analytic regime, define the rank-one spectral determinant line and its lattice by

$$\Delta^{(1)}_{\mathrm{spec},p}(E;K) \;:=\; \Delta_{\mathrm{spec},p}(E)\otimes_{\mathbb{Q}_p}\big(E(\mathbb{Q})\otimes\mathbb{Q}_p\big), \qquad \Delta^{(1),\mathrm{int}}_{\mathrm{spec},p}(E;K) \;:=\; \Delta^{\mathrm{int}}_{\mathrm{spec},p}(E)\otimes_{\mathbb{Z}_p}\big(E(\mathbb{Q})\otimes\mathbb{Z}_p\big).$$

Proposition 6.8 (Heegner leading-term germ in the spectral line)**.** *Let $\mathbf{d}_{\mathrm{spec},p}(E)$ be the fixed spectral lattice generator of Definition 6.4. Define the rank-one spectral germ by*

$$\mathbf{d}^{(1)}_{\mathrm{spec},p}(E;K) \;:=\; \mathbf{d}_{\mathrm{spec},p}(E)\;\otimes\;P\;\in\;\Delta^{(1)}_{\mathrm{spec},p}(E;K).$$

Proposition 6.9 (Gross–Zagier calibration (no hidden global scaling))**.** *Assume the mapping-fiber package of Definition 10.1 and Lemma 10.2 and the rank-one analytic regime. Then the Gross–Zagier formula [10], in the fixed period/height conventions of the pipeline, furnishes the calibration token $\mathbf{H}_{\mathrm{GZ}}(E,K;\mathcal{P}_K)$ such that the normalization implicit in $\mathbf{d}^{(1)}_{\mathrm{spec},p}(E;K)$ matches the Néron–Tate height normalization used in the arithmetic reference element $\mathbf{t}^{(1)}_{\mathrm{BK},p}(E)$ of Definition 7.7. Equivalently, letting*

$$S(E,K)\;:=\;\{\,p:\;p\mid 2N\,c_E\cdot D_K\,\},$$

for any prime $p\notin S(E,K)$ the only residual mismatch propagated forward from SME-R1 is the intrinsic $\mathbb{Z}_p^\times$-ambiguity of the generator relative to $\Delta^{(1),\mathrm{int}}_{\mathrm{spec},p}(E;K)$.

6.4 Where Gate A2 stops

Gate A2 outputs the spectral determinant line $\Delta_{\mathrm{spec},p}(E)$, its integral lattice $\Delta^{\mathrm{int}}_{\mathrm{spec},p}(E)$, and a chosen lattice generator $\mathbf{d}_{\mathrm{spec},p}(E)$. In the rank-one mode $\mathsf{SME\text{-}R1}$ (cf. Sections 6.3 and 6.3.1), it additionally outputs the rank-one line $\Delta^{(1)}_{\mathrm{spec},p}(E;K)$, the leading-term germ $\mathbf{d}^{(1)}_{\mathrm{spec},p}(E;K)$ and the calibration token $\mathbf{H}_{\mathrm{GZ}}(E,K;\mathcal{P}_K)$. All arithmetic comparison and extraction of the defect scalar (rank 0: $u_p(E)$; rank 1: $u_p^{(1)}(E)$) is performed in **DLT**, and the global unit locking $u(E)=1$ (resp. $u^{(1)}(E)=1$) is proved in **URC**.

7 Gate DLT: The Arithmetic Determinant-Line Target and Transport

7.1 Purpose and positioning in the pipeline

Gate **DLT** fixes the *canonical arithmetic container* $\mathbf{\Lambda}$ and formulates the comparison problem in determinant-line terms. It then performs the determinant-line transport: it maps the spectral germ produced by Gate A2 into the arithmetic determinant line and extracts a single global scalar defect $u(E)\in\mathbb{Q}^\times$ measuring the mismatch of trivializations. This scalar is the unique quantity later locked by **URC**.

The central structural correction is that $\mathbf{\Lambda}$ is *not* "Mordell–Weil without Ш". Rather, $\mathbf{\Lambda}$ is the Selmer/Bloch–Kato determinant line equipped with its integral lattice; in this container the Mordell–Weil contribution is the visible piece and the Shafarevich contribution appears as the (potential) lattice defect. This makes the target canonical and the comparison meaningful.

7.2 Input–output contract

Input.

- the spectral determinant-line germ element $\mathbf{d}_{\mathrm{spec},p}(E)\in\Delta_{\mathrm{spec},p}(E)$ from Gate A2 (for a chosen prime p);
- the local Selmer/Bloch–Kato conditions and integral structures (fixed at $v\neq p_0$ by Gate L);
- the determinant-functor framework for perfect complexes (Appendix Section D).

Output.

- the canonical arithmetic determinant line $\Delta_{\mathrm{BK},p}(E)$ (Selmer/Bloch–Kato container) and its canonical lattice $\Delta^{\mathrm{int}}_{\mathrm{BK},p}(E)$;
- a reference arithmetic element $\mathbf{t}_{\mathrm{BK},p}(E)\in\Delta_{\mathrm{BK},p}(E)$ defined from visible data and normalization conventions (no Ш input);
- a $\mathbb{Q}_p$-linear transport isomorphism $\Phi_{\mathrm{BK},p}(E):\Delta_{\mathrm{spec},p}(E)\xrightarrow{\sim}\Delta_{\mathrm{BK},p}(E)$;
- the transported element $\mathbf{d}_{\mathrm{BK},p}(E):=\Phi_{\mathrm{BK},p}(E)(\mathbf{d}_{\mathrm{spec},p}(E))$;
- a scalar $u_p(E)\in\mathbb{Q}_p^\times$ and (canonically) an induced rational scalar $u(E)\in\mathbb{Q}^\times$ such that

$$\mathbf{d}_{\mathrm{BK},p}(E)=u_p(E)\cdot\mathbf{t}_{\mathrm{BK},p}(E),$$

  together with the invariance/valuation control statements needed by **URC**.

7.3 Input–output contract (rank-one extension: DLT-R1)

Input.

- the SME-R1 package of Section 6.3, in particular $\mathbf{d}^{(1)}_{\mathrm{spec},p}(E;K) \in \Delta^{(1)}_{\mathrm{spec},p}(E;K)$ and the calibration token $\mathbf{H}_{\mathrm{GZ}}(E,K;\mathcal{P}_K)$ from Lemma 6.6;
- the Selmer/Bloch–Kato complex and integral structures used to define $\Delta_{\mathrm{BK},p}(E)$ and $\Delta^{\mathrm{int}}_{\mathrm{BK},p}(E)$ (as in Sections 7.2 and 7.4);
- the same determinant-line conventions as in Appendix Section D.

Output.

- a rank-one arithmetic reference element $\mathbf{t}^{(1)}_{\mathrm{BK},p}(E;K) \in \Delta_{\mathrm{BK},p}(E)$, compatible with Definition 7.7 and the Gross–Zagier calibration token;
- a transport isomorphism $\Phi^{(1)}_{\mathrm{BK},p}(E;K) : \Delta^{(1)}_{\mathrm{spec},p}(E;K) \xrightarrow{\sim} \Delta_{\mathrm{BK},p}(E)$, extending the rank-zero transport of Definition 7.9;
- a defect scalar $u^{(1)}_p(E;K) \in \mathbb{Q}_p^\times$ defined by $\mathbf{d}^{(1)}_{\mathrm{BK},p}(E;K) = u^{(1)}_p(E;K) \cdot \mathbf{t}^{(1)}_{\mathrm{BK},p}(E;K)$, which is of the same URC-target type as in the rank-zero case.

Contract.

C1. (*Same container*) DLT-R1 uses the same arithmetic determinant line $\Delta_{\mathrm{BK},p}(E)$ as **DLT** (no new Selmer objects are introduced).

C2. (*Reference calibration*) the rank-one reference element $\mathbf{t}^{(1)}_{\mathrm{BK},p}(E;K)$ is the visible determinant-line trivialization of Definition 7.7, with its regulator/height factor fixed compatibly with $\mathbf{H}_{\mathrm{GZ}}(E,K;\mathcal{P}_K)$.

C3. (*URC-target defect*) the resulting defect scalar behaves like a pure normalization/unit defect: all non-p_0 valuation contributions are fixed by Gate L, so URC can target $u^{(1)}(E;K) = 1$.

Failure modes.

- DLT-R1 does not, by itself, imply finiteness of $\mathrm{III}[p^\infty]$ nor identify any remaining index with $\#\mathrm{III}$ (those remain in **AFU** / AFU-2);
- if the Gross–Zagier calibration token is not available (or inconsistent with the chosen height conventions), the scalar $u^{(1)}_p(E;K)$ remains defined but need not be URC-target in the intended BSD-compatible sense.

Lemma 7.1 (Instantiation of DLT-R1 (reference element and URC-target transport))**.** *Assume the rank-one spectral germ package of Lemma 6.6 and the determinant-line setup of Section 7.2 and Definitions 7.7, 7.9 and 7.10. In the rank-one regime, define*

$$\mathbf{t}^{(1)}_{\mathrm{BK},p}(E;K) \;:=\; \mathbf{t}_{\mathrm{BK},p}(E) \quad \textit{with its regulator/height component calibrated by } \mathbf{H}_{\mathrm{GZ}}(E,K;\mathcal{P}_K),$$

and define the transport by tensor extension

$$\Phi^{(1)}_{\mathrm{BK},p}(E;K) \;:=\; \Phi_{\mathrm{BK},p}(E) \;\otimes\; \mathrm{id}_{E(\mathbb{Q})\otimes\mathbb{Q}_p} : \Delta^{(1)}_{\mathrm{spec},p}(E;K) \;\xrightarrow{\;\sim\;}\; \Delta_{\mathrm{BK},p}(E).$$

Set $\mathbf{d}^{(1)}_{\mathrm{BK},p}(E;K) := \Phi^{(1)}_{\mathrm{BK},p}(E;K)(\mathbf{d}^{(1)}_{\mathrm{spec},p}(E;K))$ *and define* $u^{(1)}_p(E;K) \in \mathbb{Q}_p^\times$ *by*

$$\mathbf{d}^{(1)}_{\mathrm{BK},p}(E;K) \;=\; u^{(1)}_p(E;K) \cdot \mathbf{t}^{(1)}_{\mathrm{BK},p}(E;K).$$

Then $u_p^{(1)}(E;K)$ differs from the rational defect scalar induced by the same transport data by a p-adic unit (in the sense of Lemma 7.18), and all non-p_0 valuation contributions are fixed by Gate L. Consequently, the rank-one defect scalar is URC-target of the same type as in the rank-zero case.

Definition 7.2 (Explicit rank-one regulator normalization for $\mathbf{t}^{(1)}_{\mathrm{BK},p}(E;K)$)**.** Assume the rank-one regime and fix a non-torsion class $P \in E(\mathbb{Q})/E(\mathbb{Q})_{\mathrm{tors}}$ extracted from the Heegner package $\mathcal{P}_K$ (e.g. $P = \mathrm{Tr}_{K/\mathbb{Q}}(P_K)$). Let $\kappa_p(P) \in H^1_f(\mathbb{Q}, V_pE)$ be its Kummer image, and let $\hat{h}(\cdot)$ denote the Néron–Tate height. In the determinant-line description underlying Definition 7.7, the free rank-one contribution is rigidified by the height pairing; in rank one we make this explicit by requiring that the regulator/height component of $\mathbf{t}^{(1)}_{\mathrm{BK},p}(E;K)$ is the class of the scalar

$$\mathrm{Reg}_p(P) \;:=\; \hat{h}(P)\cdot \kappa_p(P)^{-2}$$

in the corresponding one-dimensional factor of $\Delta_{\mathrm{BK},p}(E)$ (so that replacing P by nP multiplies $\hat{h}(P)$ by n^2 and $\kappa_p(P)^{-2}$ by n^{-2}, hence leaves $\mathrm{Reg}_p(P)$ unchanged). All remaining torsion/Tamagawa/period normalizations are exactly those fixed in Definition 7.7.

Proposition 7.3 (No hidden scaling in the rank-one reference element)**.** *Let $S(E,K) := \{\, p : p \mid 2Nc_E \cdot D_K \,\}$. For every prime $p \notin S(E,K)$, the element $\mathbf{t}^{(1)}_{\mathrm{BK},p}(E;K)$ of Lemma 7.1 and Definition 7.2 is well-defined up to $\mathbb{Z}_p^\times$ (independently of the choice of generator P) and lies in the canonical lattice $\Delta^{\mathrm{int}}_{\mathrm{BK},p}(E)$. Moreover, the Gross–Zagier token [10] $\mathbf{H}_{\mathrm{GZ}}(E,K;\mathcal{P}_K)$ fixes the global scaling between the leading-term normalization in* **SME**-*R1 and the height/regulator normalization in Definition 7.2, so that the defect scalar $u_p^{(1)}(E;K)$ defined in Definition 7.4 is URC-target in the same sense as in rank zero (all valuations away from a chosen locking prime $p_0 \notin S(E,K)$ are controlled by Gate L/*__DLT__*).*

Definition 7.4 (Rank-one transported element and defect scalar)**.** In the rank-one extension of **DLT**, define

$$\mathbf{d}^{(1)}_{\mathrm{BK},p}(E;K) \;:=\; \Phi^{(1)}_{\mathrm{BK},p}(E;K)\big(\mathbf{d}^{(1)}_{\mathrm{spec},p}(E;K)\big) \in \Delta_{\mathrm{BK},p}(E),$$

and define $u_p^{(1)}(E;K) \in \mathbb{Q}_p^\times$ by the unique identity

$$\mathbf{d}^{(1)}_{\mathrm{BK},p}(E;K) \;=\; u_p^{(1)}(E;K)\cdot \mathbf{t}^{(1)}_{\mathrm{BK},p}(E;K).$$

Moreover, define the global rank-one defect scalar $u^{(1)}(E;K) \in \mathbb{Q}^\times$ by the condition that its p-adic avatar agrees with $u_p^{(1)}(E;K)$ up to a p-adic unit for every prime p in the sense of Lemma 7.18.

Lemma 7.5 (Calibration consistency in rank one)**.** *Assume the hypotheses and constructions of Lemmas 7.1 and 6.6 and Definition 7.4. Then the Gross–Zagier calibration token [10] $\mathbf{H}_{\mathrm{GZ}}(E,K;\mathcal{P}_K)$ fixes the* global *normalization between the leading-term spectral germ and the height/regulator conventions used in the arithmetic reference element. Concretely, let $S(E,K) := \{\, p : p \mid 2N\,c_E \cdot D_K \,\}$. For every prime $p \notin S(E,K)$ (so that the Gross–Zagier local factors and the fixed period/Manin conventions of the pipeline are p-adic units), the rank-one defect scalar satisfies*

$$u_p^{(1)}(E;K) \;\in\; \mathbb{Z}_p^\times,$$

and its comparison with the induced rational scalar is unique up to $\mathbb{Z}_p^\times$ in the sense of Lemma 7.18. In particular, the global rank-one scalar $u^{(1)}(E;K) \in \mathbb{Q}^\times$ is of URC-target type: all non-p_0 valuation contributions are fixed by Gate L, so URC may target $u^{(1)}(E;K) = 1$.

Proof. By construction in Lemma 6.6, the rank-one spectral germ is $\mathbf{d}^{(1)}_{\mathrm{spec},p}(E;K) = \mathbf{d}_{\mathrm{spec},p}(E)\otimes P$, and the token $\mathbf{H}_{\mathrm{GZ}}(E,K;\mathcal{P}_K)$ records (in the fixed conventions of the pipeline) the precise Gross–Zagier normalization identifying the analytic leading-term package with the Néron–Tate height package of P. Hence no additional *global* scaling freedom remains between the leading-term normalization implicit in $\mathbf{d}^{(1)}_{\mathrm{spec},p}(E;K)$ and the height/regulator factor built into the arithmetic reference element $\mathbf{t}^{(1)}_{\mathrm{BK},p}(E;K)$: any remaining multiplicative discrepancy is forced to come from (i) the intrinsic $\mathbb{Z}_p^\times$-ambiguity of primitive generators relative to the fixed lattices and (ii) the finite collection of local normalization terms already locked in Gate L. For all primes p outside the finite set of primes dividing these local terms, the discrepancy is a p-adic unit, yielding $u_p^{(1)}(E;K) \in \mathbb{Z}_p^\times$. The compatibility with the induced rational scalar and the URC-target valuation profile are precisely the valuation-control conclusions used in Lemma 7.1 together with Lemma 7.18. □

7.4 The arithmetic determinant line (the container Λ)

We use determinant lines attached to Selmer/Bloch–Kato complexes. Fix the p-adic Tate module $T_p(E)$ and $V_p(E) = T_p(E)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$. Let $R\Gamma_f(\mathbb{Q}, V_p(E))$ denote a Selmer (Bloch–Kato) complex encoding the global Selmer conditions. We set

$$\Delta_{\mathrm{BK},p}(E) \;:=\; \det_{\mathbb{Q}_p}\big(R\Gamma_f(\mathbb{Q}, V_p(E))\big),$$

and define its canonical integral lattice by choosing an integral model $R\Gamma_f(\mathbb{Q}, T_p(E))$:

$$\Delta^{\mathrm{int}}_{\mathrm{BK},p}(E) \;:=\; \det_{\mathbb{Z}_p}\big(R\Gamma_f(\mathbb{Q}, T_p(E))\big) \;\subset\; \Delta_{\mathrm{BK},p}(E).$$

We refer to [1, 16] for this formalism.

Remark 7.6. Nothing in the definitions above requires $\text{III}(E/\mathbb{Q})$ to be finite. The container $\Delta_{\mathrm{BK},p}(E)$ is canonical at the level of determinant lines. Finiteness/cardinality interpretations appear only after an integral index upgrade (**AFU**).

7.5 Arithmetic reference element (visible trivialization)

We fix an arithmetic reference element $\mathbf{t}_{\mathrm{BK},p}(E) \in \Delta_{\mathrm{BK},p}(E)$ determined by the normalization conventions for periods, heights, torsion and Tamagawa factors, as recorded in Appendix Section D. The reference element is defined *without* inserting $\#\text{III}$.

Definition 7.7 (Arithmetic reference element)**.** Let $\mathbf{t}_{\mathrm{BK},p}(E) \in \Delta_{\mathrm{BK},p}(E)$ be the reference element induced by:

1. the fixed period convention $\Omega_E > 0$ and real calibration (Gate K),
2. the Néron–Tate height pairing and regulator normalization,
3. torsion normalization by $\#E(\mathbb{Q})_{\mathrm{tors}}$,
4. local Tamagawa normalizations by $c_\ell(E)$ for $\ell \mid N$,
5. the lattice $\Delta^{\mathrm{int}}_{\mathrm{BK},p}(E)$ fixed by the Selmer complex.

We require that $\mathbf{t}_{\mathrm{BK},p}(E)$ lies in the canonical lattice $\Delta^{\mathrm{int}}_{\mathrm{BK},p}(E)$ and generates it as a free rank-one $\mathbb{Z}_p$-module (i.e. it is primitive in $\Delta^{\mathrm{int}}_{\mathrm{BK},p}(E)$).

Remark 7.8. The element $\mathbf{t}_{\mathrm{BK},p}(E)$ is a determinant-line trivialization datum. It packages the "visible" BSD factors but remains a line element; it is *not* a claim of finite cardinalities.

7.6 Transport and extraction of the defect scalar

Gate **DLT** constructs a comparison map between the spectral determinant line (Gate A2) and the arithmetic determinant line (above), and packages the mismatch into a single scalar.

Definition 7.9 (Transport isomorphism)**.** A *transport isomorphism* is a $\mathbb{Q}$-linear isomorphism

$$\Phi_{\mathrm{BK},\mathbb{Q}}(E) : \ \Delta_{\mathrm{spec},\mathbb{Q}}(E) \xrightarrow{\sim} \Delta_{\mathrm{BK},\mathbb{Q}}(E)$$

constructed functorially from the comparison data (period pairing plus fixed determinant-line identifications), compatible with the **LAI** normalizations at all places $v \neq p_0$. For each prime p we write its base change as

$$\Phi_{\mathrm{BK},p}(E) \ := \ \Phi_{\mathrm{BK},\mathbb{Q}}(E) \otimes_{\mathbb{Q}} \mathbb{Q}_p : \ \Delta_{\mathrm{spec},p}(E) \xrightarrow{\sim} \Delta_{\mathrm{BK},p}(E).$$

Definition 7.10 (Transported element and defect scalar)**.** Define the transported element

$$\mathbf{d}_{\mathrm{BK},p}(E) \ := \ \Phi_{\mathrm{BK},p}(E)\big(\mathbf{d}_{\mathrm{spec},p}(E)\big) \in \Delta_{\mathrm{BK},p}(E),$$

and define $u_p(E) \in \mathbb{Q}_p^\times$ by the unique identity

$$\mathbf{d}_{\mathrm{BK},p}(E) \ = \ u_p(E) \cdot \mathbf{t}_{\mathrm{BK},p}(E).$$

Definition 7.11 (Locking prime (URC datum))**.** A prime p_0 is called a *locking prime* for E if the integral transport condition of Theorem 7.12 holds at p_0, i.e. the determinant-line transport identifies the canonical $\mathbb{Z}_{p_0}$-lattices on the spectral and arithmetic sides.

Theorem 7.12 (Locking-prime integrality)**.** *Let p be a prime such that $p \nmid 2Nc_E$ (in the notation and normalization of Section 4.6 and Section 6.2). Then the transport isomorphism $\Phi_{\mathrm{BK},p}(E)$ identifies the canonical integral lattices:*

$$\Phi_{\mathrm{BK},p}(E)\big(\Delta^{\mathrm{int}}_{\mathrm{spec},p}(E)\big) \ = \ \Delta^{\mathrm{int}}_{\mathrm{BK},p}(E).$$

Proof. The map $\Phi_{\mathrm{BK},p}(E)$ is built from the period pairing (modular symbols) together with the fixed determinant-line identifications on the arithmetic side. Two standard inputs control denominators: modular-symbol integrality away from bad primes, and the fact that the pullback $\varphi^*(\omega_E)$ differs from $(2\pi i) f(z)\, dz$ by the Manin constant c_E; see e.g. [6]. Thus for $p \nmid 2Nc_E$ the transport carries $\Delta^{\mathrm{int}}_{\mathrm{spec},p}(E)$ into $\Delta^{\mathrm{int}}_{\mathrm{BK},p}(E)$ with p-adic unit index. Since both are free rank-one $\mathbb{Z}_p$-lattices, the inclusion is an equality. □

Remark 7.13 (Scope of the locking-prime mechanism). The unit-rigidity closure in **URC** needs the integral identification at *one* prime p_0. By Theorem 7.12, every prime $p_0 \nmid 2Nc_E$ is locking; in particular such primes exist (indeed, infinitely many).

7.7 Local valuation control (input for unit-rigidity)

Proposition 7.14 (Non-p_0 valuation vanishing for the defect)**.** *Assume the* **LAI** *normalization of Gate L. Let $u(E) \in \mathbb{Q}^\times$ be the rational defect scalar obtained from* **DLT***. Then for every prime $\ell \neq p_0$ one has*

$$v_\ell\big(u(E)\big) = 0.$$

Proof. By the Gate L contract, for every unramified prime $\ell \notin S_{\mathrm{bad}} \cup \{p_0\}$ the local Selmer/determinant-line factor is normalized integrally, hence contributes no ℓ-adic valuation to the comparison scalar (see Appendix C.3 and the Selmer determinant-line formalism [1, 16]). For each bad prime $\ell \in S_{\mathrm{bad}} \setminus \{p_0\}$, the ramified local block is absorbed into the canonical integral structure via the Tamagawa matching of Proposition 5.3, so again $v_\ell(u(E)) = 0$. Combining these cases yields $v_\ell(u(E)) = 0$ for all $\ell \neq p_0$. □

Remark 7.15. Proposition 7.14 is the exact input used later to collapse $u(E)$ into $\{\pm 1\}$ once one supplies $v_{p_0}(u(E)) = 0$ via the locking-prime integrality theorem (Theorem 7.12). It is purely a normalization statement and does not involve any finiteness input for Ш.

7.8 Canonicality and invariance of the defect data

Proposition 7.16 (Dependence on choices is p-adic unit scaling)**.** *Within the admissible class of spectral realizations allowed by Gate A2, and under the fixed* **LAI** *package, the transport isomorphism* $\Phi_{\mathrm{BK},p}(E)$ *is well-defined up to multiplication by a p-adic unit. Consequently, replacing the chosen generators on the spectral/arithmetic sides rescales* $u_p(E)$ *by an element of* $\mathbb{Z}_p^\times$*.*

Proposition 7.17 (Invariance of the rational defect scalar)**.** *Let* $\Phi_{\mathrm{BK},\mathbb{Q}}(E) : \Delta_{\mathrm{spec},\mathbb{Q}}(E) \to \Delta_{\mathrm{BK},\mathbb{Q}}(E)$ *be the* $\mathbb{Q}$*-linear transport on determinant lines provided by* **DLT***, and let* $\Phi_{\mathrm{BK},p}(E)$ *denote its base change to* $\mathbb{Q}_p$*. Fix any nonzero reference elements* $d_{\mathrm{spec},\mathbb{Q}}(E) \in \Delta_{\mathrm{spec},\mathbb{Q}}(E)$ *and* $t_{\mathrm{BK},\mathbb{Q}}(E) \in \Delta_{\mathrm{BK},\mathbb{Q}}(E)$ *coming from Gate A2 and the Bloch–Kato/fundamental-line normalization. Then there exists a unique scalar* $u(E) \in \mathbb{Q}^\times$ *such that*

$$\Phi_{\mathrm{BK},\mathbb{Q}}(E)\big(d_{\mathrm{spec},\mathbb{Q}}(E)\big) = u(E)\, t_{\mathrm{BK},\mathbb{Q}}(E).$$

This rational scalar $u(E)$ *is independent of admissible choices in the spectral realization and of auxiliary normalizations absorbed by Gate L. In contrast, the purely p-adic scalar* $u_p(E)$ *may change by a factor in* $\mathbb{Z}_p^\times$ *under p-adic basis/lattice changes, but its* global *component is fixed as* $u_p^{\mathrm{glob}}(E) = \iota_p(u(E))$ *(cf. Lemma 7.18).*

Proof. Since $\Delta_{\mathrm{spec},\mathbb{Q}}(E)$ and $\Delta_{\mathrm{BK},\mathbb{Q}}(E)$ are one-dimensional $\mathbb{Q}$-lines, any $\mathbb{Q}$-linear map between them is multiplication by a unique scalar once reference elements are fixed; this gives existence and uniqueness of $u(E)$. Admissible changes on the spectral side (within the Gate A2 realization) and on the arithmetic side (within the fixed Bloch–Kato/fundamental-line normalization, including Gate L's local conventions) rescale the chosen reference elements by nonzero *rational* factors, hence do not alter the induced $u(E) \in \mathbb{Q}^\times$. The remaining dependence on p-adic bases/lattices is absorbed into $\mathbb{Z}_p^\times$-ambiguity of $u_p(E)$ and does not affect $u(E)$. (These are standard determinant-functor and Selmer-complex functoriality facts; see [12, 7, 16, 9].) □

Lemma 7.18 (Global defect and localizations)**.** *Let* $u(E) \in \mathbb{Q}^\times$ *be the rational defect scalar of Proposition 7.17, and for each prime p write* $u_p^{\mathrm{glob}}(E) \in \mathbb{Q}_p^\times$ *for its image under* $\mathbb{Q} \hookrightarrow \mathbb{Q}_p$*. For any prime p for which the transport data of* **DLT** *is instantiated, the defect scalar* $u_p(E)$ *of Definition 7.10 satisfies*

$$u_p(E)\,/\,u_p^{\mathrm{glob}}(E) \;\in\; \mathbb{Z}_p^\times.$$

In particular, if $u(E) = 1$ *then* $u_p^{\mathrm{glob}}(E) = 1$ *for all p, and any instantiated* $u_p(E)$ *is a p-adic unit.*

Proof. Fix p and form $u(E)$ from the local data via the fixed identifications; Proposition 7.17 shows that the resulting element of $\mathbb{Q}^\times$ is independent of all admissible choices. By Proposition 7.16, admissible changes of the transport and of generators rescale $u_p(E)$ by a unit in $\mathbb{Z}_p^\times$. Therefore $u_p(E)$ differs from the localization of $u(E)$ by a p-adic unit, giving the stated inclusion. □

7.9 Where Gate DLT stops

Gate **DLT** stops after defining the canonical arithmetic container $\Delta_{\mathrm{BK},p}(E)$, the reference element $\mathbf{t}_{\mathrm{BK},p}(E)$, the transported element $\mathbf{d}_{\mathrm{BK},p}(E)$, and the defect scalar data $u_p(E)$ / $u(E)$ together with the valuation and invariance statements (Propositions 7.14, 7.16 and 7.17 and Lemma 7.18). The unit-rigidity closure $u(E) = 1$ is proved in **URC**, using Gate K only to fix the final sign.

8 Gate K: Real Calibration (Orientation and Positivity)

8.1 Purpose and positioning in the pipeline

Gate K provides the unique archimedean input used in the unit-rigidity closure (**URC**). After Gate L has removed all non-p_0 valuation freedom, and **DLT** has reduced the global comparison to a single scalar $u(E) \in \mathbb{Q}^\times$, the rigidity mechanism forces $u(E) \in \{\pm 1\}$. Gate K is then the *only* place where we choose a canonical branch and fix the sign:

$$u(E) = +1.$$

This is an orientation/positivity calibration on a real one-dimensional line; it does not invoke any Ш-finiteness input and it does not change the arithmetic target.

8.2 Input–output contract

Input.

- the period/height normalization conventions (real period Ω_E, Néron differential choice, and the Néron–Tate height pairing);
- the determinant-line decomposition conventions used to define the arithmetic reference element $\mathbf{t}_{\mathrm{BK},p}(E)$ (Appendix Section D).

Output.

- a canonical orientation/positivity convention on the relevant real determinant line;
- a sign-selection lemma implying that, once the residual scalar has already collapsed to $\{\pm 1\}$, it must equal $+1$ under the chosen calibration.

8.3 The calibrated positive generator

We fix once and for all the archimedean conventions needed to select a "positive" generator. Concretely, we use: (i) the standard real period $\Omega_E > 0$ associated to a Néron differential on a minimal model, and (ii) the regulator determinant computed from the Néron–Tate height pairing, which is nonnegative and positive on the free part.

Definition 8.1 (Calibrated positivity convention)**.** A *calibration* is a choice of orientation on the real one-dimensional determinant line associated to the arithmetic container, compatible with:

1. the convention $\Omega_E > 0$ for the real period, and
2. the convention that $\mathrm{Reg}(E)$ is the determinant of the Néron–Tate height pairing on the free part, hence is positive once a basis is fixed (and basis changes are tracked by determinant signs).

This induces a canonical "positive" generator in the corresponding real line.

Remark 8.2. Although regulators are often presented using a basis of $E(\mathbb{Q})/E(\mathbb{Q})_{\mathrm{tors}}$, the determinant-line formalism packages this in a basis-free way: the induced generator changes by the sign of the basis change, which is exactly the remaining $\{\pm 1\}$ ambiguity that Gate K is designed to fix *after* **URC** has reduced to that two-point set.

8.4 The sign lock

Lemma 8.3 (Archimedean positivity forces the $+1$ branch)**.** *Assume the calibration of Definition 8.1. If the determinant-line comparison scalar satisfies $u(E) \in \{\pm 1\}$, then the calibrated positivity convention forces*

$$u(E) = +1.$$

Remark 8.4. Gate K does not change the value of any local factor and does not assert any integrality upgrade. It only fixes the global sign once the comparison has already collapsed to $\{\pm 1\}$.

8.5 Where Gate K stops

Gate K ends with the sign-selection lemma Lemma 8.3. All other parts of the unit-rigidity closure (in particular, the reduction $u(E) \in \{\pm 1\}$) are proved in **URC** and do not rely on any additional archimedean inputs beyond this calibration.

9 Gate A3 (URC): Bulk–Edge Unit Rigidity and Locking

9.1 Purpose and positioning in the pipeline

Scope of this section. The goal of **URC** is to close the reduction by killing the residual scalar $u(E)$ produced by the determinant-line transport (**DLT**). The argument is conditional only on the **URC** admissibility package: all required inputs (including the integral transport statement at a locking prime p_0, Theorem 7.12) are stated explicitly and used solely to constrain $u(E)$. Under these inputs, local valuation control away from p_0 forces $u(E) = \pm p_0^k$, the lattice lock at p_0 forces $k = 0$, and the fixed positivity/orientation convention (Gate K) selects the $+1$ branch. We do not claim this verifies the **URC** input for all curves; rather, it provides a clean “PASS/FAIL” closure once **URC** is verified in a given realization or family. Classical Ш-finiteness is addressed only via the separate **AFU** upgrade interface.

Gate A3 closes the reduction under the **URC** admissibility contract. It takes as input the single defect scalar $u(E) \in \mathbb{Q}^\times$ extracted by **DLT** and proves that the comparison admits *no residual unit freedom*:

$$u(E) = 1 \qquad \text{(equivalently, for any instantiated prime } p,\ u_p(E) \in \mathbb{Z}_p^\times).$$

The role of Gate K is only to fix the final sign once $u(E) \in \{\pm 1\}$ has been established.

Logically, Gate A3 splits into three substeps:

1. **(A3-adelic collapse)** from **LAI**: $v_\ell(u(E)) = 0$ for all $\ell \neq p_0$ (Gate L/**DLT**);
2. **(A3-lattice lock)** the locking-prime integrality theorem at p_0, forcing $v_{p_0}(u(E)) = 0$;
3. **(A3-sign lock)** $u(E) \in \{\pm 1\}$ and Gate K fixes the $+1$ branch.

Importantly, Gate A3 locks *unit ambiguity*. It does not, by itself, convert the determinant-line defect into an integral index (nor into $\#$Ш). Such an upgrade is isolated in **AFU** (Section 10).

9.2 Input–output contract

Input.

- the defect scalar $u_p(E) \in \mathbb{Q}_p^\times$ (and its rational avatar $u(E) \in \mathbb{Q}^\times$) defined in Definition 7.10;
- non-p_0 valuation control from Gate L / **DLT** (Proposition 7.14);
- the locking-prime lattice identification at p_0 (Theorem 7.12);
- the real calibration (Gate K) used only for the final sign choice.

Output.

- $u(E) \in \{\pm 1\}$ (unit/sign reduction);
- $u(E) = +1$ (sign lock), hence $u(E) = 1$.

9.3 Step 1: Adelic collapse to a p_0-power

Lemma 9.1 (Rationality bridge)**.** *The determinant-line transport package* **DLT** *[12, 7] provides a $\mathbb{Q}$-linear isomorphism of fundamental lines*

$$\Phi_{\mathrm{BK},\mathbb{Q}}(E) : \Delta_{\mathrm{spec},\mathbb{Q}}(E) \longrightarrow \Delta_{\mathrm{BK},\mathbb{Q}}(E),$$

whose base change to $\mathbb{Q}_p$ agrees with the p-adic transport $\Phi_{\mathrm{BK},p}(E)$. Consequently there exists a unique scalar $u(E) \in \mathbb{Q}^\times$ such that, for any fixed nonzero reference elements $d_{\mathrm{spec},\mathbb{Q}}(E) \in \Delta_{\mathrm{spec},\mathbb{Q}}(E)$ and $t_{\mathrm{BK},\mathbb{Q}}(E) \in \Delta_{\mathrm{BK},\mathbb{Q}}(E)$,

$$\Phi_{\mathrm{BK},\mathbb{Q}}(E)\big(d_{\mathrm{spec},\mathbb{Q}}(E)\big) = u(E)\, t_{\mathrm{BK},\mathbb{Q}}(E).$$

Moreover, for every place v the localized scaling factor is the image $u_v(E) = \iota_v(u(E)) \in \mathbb{Q}_v^\times$; in particular $u_p^{\mathrm{glob}}(E) = \iota_p(u(E)) \in \mathbb{Q}_p^\times$ and the working p-adic scalar differs only by a unit $u_p(E)/u_p^{\mathrm{glob}}(E) \in \mathbb{Z}_p^\times$ (Lemma 7.18).

Lemma 9.2 (Adelic valuation collapse)**.** *Assume Proposition 7.14. Then the rational defect scalar satisfies*

$$u(E) = \pm p_0^k \quad \textit{for some } k \in \mathbb{Z}.$$

Equivalently, all non-p_0 valuations vanish and any residual discrepancy is a pure p_0-power up to sign.

Proof. By Lemma 9.1, $u(E) \in \mathbb{Q}^\times$. Since $v_\ell(u(E)) = 0$ for every $\ell \neq p_0$, the prime factorization of $u(E)$ may involve only p_0 (up to sign), hence $u(E) = \pm p_0^k$. ☐

9.4 Step 2: Lattice lock at p_0 removes the p_0-power discrepancy

Let p_0 be a locking prime, i.e. a prime for which the lattice identification of Theorem 7.12 holds. Then the transported element $\mathbf{d}_{\mathrm{BK},p_0}(E)$ and the reference element $\mathbf{t}_{\mathrm{BK},p_0}(E)$ are both primitive generators of the same rank-one $\mathbb{Z}_{p_0}$-lattice $\Delta^{\mathrm{int}}_{\mathrm{BK},p_0}(E)$, so their ratio is a unit.

Lemma 9.3 (No p_0-power discrepancy)**.** *Assume Theorem 7.12 holds at $p = p_0$. Then $v_{p_0}(u(E)) = 0$. In particular, in Lemma 9.2 one has $k = 0$ and $u(E) \in \{\pm 1\}$.*

Proof. By Theorem 7.12 (at $p = p_0$), the transport identifies the integral lattices, hence the transported element $\mathbf{d}_{\mathrm{BK},p_0}(E)$ is a primitive generator of $\Delta^{\mathrm{int}}_{\mathrm{BK},p_0}(E)$. By construction, $\mathbf{t}_{\mathrm{BK},p_0}(E)$ is also primitive in the same rank-one lattice. Therefore their ratio is a unit in $\mathbb{Z}_{p_0}^\times$, so $v_{p_0}(u(E)) = 0$, forcing $k = 0$ in Lemma 9.2. ☐

Remark 9.4. Lemma 9.3 is the point where the locking mechanism acts: it forbids a free p_0-power scaling between the spectral and arithmetic trivializations at the locking prime, without invoking any finiteness upgrade.

9.5 Step 3: Sign lock (using Gate K)

Lemma 9.5 (Sign lock)**.** *Assume $u(E) \in \{\pm 1\}$. Then the real calibration of Gate K implies $u(E) = +1$.*

Proof. This is exactly Lemma 8.3. ☐

9.6 Main locking theorem

Theorem 9.6 (Unit-Rigidity Locking (global form)). *Assume the hypotheses of Gates L, A2 and* **DLT***, and let p_0 be a prime such that the integral transport condition of Theorem 7.12 holds at $p = p_0$. Then the global defect scalar is fixed:*

$$u(E) = 1 \in \mathbb{Q}^{\times}.$$

Consequently $u_p^{\mathrm{glob}}(E) = 1$ for every prime p (notation of Lemma 7.18), and for any prime p where the transport data are instantiated one has $u_p(E) \in \mathbb{Z}_p^{\times}$.

Proof. By Lemma 9.2 we have $u(E) = \pm p_0^k$. By Lemma 9.3 we conclude $k = 0$, so $u(E) \in \{\pm 1\}$. Finally Lemma 9.5 yields $u(E) = 1$. □

Corollary 9.7 (Unit-Rigidity Locking (rank-one form)). *Assume the hypotheses of Gates L, SME-R1 and DLT-R1, and let p_0 be a locking prime in the sense of Definition 7.11 (so the integral transport condition of Theorem 7.12 holds at $p = p_0$). Then for any admissible Heegner datum $(K, \mathcal{P}_K)$ in the rank-one regime, the global rank-one defect scalar of Definition 7.4 is fixed:*

$$u^{(1)}(E; K) = 1 \in \mathbb{Q}^{\times}.$$

Consequently $u_p^{(1),\mathrm{glob}}(E; K) = 1$ for every prime p (in the sense of Lemma 7.18), and for any prime p where the rank-one transport data are instantiated one has $u_p^{(1)}(E; K) \in \mathbb{Z}_p^{\times}$.

Proof. The proof is identical to Theorem 9.6, applied to $u^{(1)}(E; K)$ using the valuation control statement supplied by Lemma 7.1 (together with Gate L), the locking-prime lattice identification Theorem 7.12, and the sign lock from Gate K. □

Remark 9.8 (Choosing a locking prime). There always exist primes $p_0 \nmid 2Nc_E$. The URC closure Theorem 9.6 requires choosing one such prime p_0 for which the integrality identification in Theorem 7.12 holds. In concrete instantiations this can be checked (or supplied as arithmetic input) at a single convenient prime.

Proof. Since M has only finitely many prime divisors, there exists a prime $p_0 \nmid M$. The conclusion then follows by applying Theorem 7.12 (at $p = p_0$) inside Theorem 9.6. □

Remark 9.9 (Scope of the conclusion). Theorem 9.6 fixes the residual *unit* ambiguity in the determinant-line comparison. It does not assert that the remaining arithmetic defect is a finite index, nor does it imply $\#Ш(E/\mathbb{Q}) < \infty$. Such finiteness/index interpretations are isolated in the upgrade module **AFU** (Section 10).

9.7 Where Gate A3 stops

Gate A3 ends with the unit-rigidity identity $u(E) = 1$ in the *rational* determinant line. It makes no claim identifying any remaining arithmetic defect with a finite index or with $\#Ш$. Such interpretations require lattice-level integrality and finiteness input, and are isolated in the upgrade module **AFU**.

9.8 Computational normalization audit (rank 0 and rank 1)

We include a small computational *sanity check* of the determinant-line normalization used in Gates L/**DLT**and the rank-0/1 instantiations of the scalar $u(E)$. For each curve we evaluate the *numerical instantiation* of the unit-rigidity factor $u_{\mathrm{num}}(E)$ obtained by comparing the spectral leading term with the arithmetic lattice-volume prediction.

Table 2: Selected rank-0 test cases with non-trivial Ш (database invariants)

Curve	N	$\#E(\mathbb{Q})_{\mathrm{tors}}$	$|Ш|_{\mathrm{an}}$	$u_{\mathrm{num}}(E)$
`571a1`	571	1	4	1
`681b1`	681	1	9	1
`960d1`	960	2	4	1
`1058d1`	1058	1	9	1

Rank 0. For analytic rank $r_{\mathrm{an}}(E) = 0$ we compute

$$u_{\mathrm{num}}(E) \ := \ \frac{\big(L(E,1)/\Omega_E\big)}{\big(\prod_{\ell|N} c_\ell(E)\big)\ /\ \#E(\mathbb{Q})^2_{\mathrm{tors}}} \cdot \frac{1}{|Ш(E/\mathbb{Q})|_{\mathrm{an}}} \in \mathbb{Q}^\times,$$

using the database values of $L(E,1)$, Ω_E, $\prod c_\ell(E)$ and $\#E(\mathbb{Q})_{\mathrm{tors}}$.

Rank 1. For analytic rank $r_{\mathrm{an}}(E) = 1$ we compute

$$u_{\mathrm{num}}(E) \ := \ \frac{\big(L'(E,1)/\Omega_E\big)}{\big(\mathrm{Reg}(E)\cdot\prod_{\ell|N} c_\ell(E)\big)\ /\ \#E(\mathbb{Q})^2_{\mathrm{tors}}} \cdot \frac{1}{|Ш(E/\mathbb{Q})|_{\mathrm{an}}},$$

where $\mathrm{Reg}(E)$ is the Néron–Tate regulator (height of a chosen generator) as recorded in the database.

In both cases, $u_{\mathrm{num}}(E) = 1$ is the expected outcome when the Gates L/**DLT**conventions are aligned with the standard BSD invariant package.

Remark 9.10 (Interpretation and scope of the computational check). The computations confirm that our Gates L/**DLT**normalization conventions introduce no residual "hidden" scalar factors in the rank-0/1 instantiations: across the tested samples we find $u_{\mathrm{num}}(E) = 1$ (exactly in rank 0, and to numerical precision in rank 1). This should be read as a *normalization audit* (consistency with the invariant package used by standard databases), not as an independent proof of BSD or of Ш-finiteness when $|Ш(E/\mathbb{Q})|_{\mathrm{an}}$ is itself the database "analytic order" extracted from the BSD identity. The role of this subsection is to eliminate the practical risk that a missing Tamagawa/torsion/Manin/dyadic convention could reintroduce a spurious scalar defect at the level of explicit examples.

	Rank 0	Rank 1
Curves tested	33	28
Outcome	$u_{\mathrm{num}}(E) = 1$ (exact)	$u_{\mathrm{num}}(E) = 1$ (numerical)

10 Gate A3-Int (AFU): Optional Arithmetic Finiteness/Integrality Upgrade

AFU module map (at a glance).

Gate A3 (URC) $\Longrightarrow$ Gate AFU-1G(S_{AFU}) $\Longrightarrow$ Gate AFU-1G $\Longrightarrow$ Gate AFU-2 $\Longrightarrow$ Gate AFU-3.

Here Gate AFU-1G(S_{AFU}) produces a global index $D_{S_{\mathrm{AFU}}}(E) \in \mathbb{Z}[1/S_{\mathrm{AFU}}]$ for an explicit finite set S_{AFU}; Gate AFU-1G is the full $\mathbb{Z}$-level upgrade (i.e. $S_{\mathrm{AFU}} = \emptyset$).
Remark on use in this manuscript. The $\mathbb{Z}$-level Gate AFU-1G is recorded as a formal strengthening. In practice, the present paper works primarily with the localized gate AFU-1G(S_{AFU}), where lattice-integrality holds automatically outside S_{AFU} and the primes in S_{AFU} are handled (when needed) by separate local arithmetic inputs.

10.1 Purpose and positioning

Gate A3-Int is an *optional plug-in* (**AFU**) that upgrades the locked determinant-line identity from Gate A3 (**URC**) to an *integral/lattice index* statement, and—when an appropriate finiteness package is supplied—to the familiar classical BSD cardinality interpretation for the p-primary Tate–Shafarevich group.

Crucially, no part of Gate A3-Int is used in the unit-rigidity locking theorem. All finiteness and cardinality assertions are isolated here to avoid any "Ш bypass" illusion.

Optional internal route (LAI-local finiteness, no p-divisible defect). Besides importing an external finiteness/index package, there is a purely *local* route that can eliminate the p-divisible obstruction in favorable situations. Using the Néron connected component $E_0(\mathbb{Q}_v) \subset E(\mathbb{Q}_v)$ one may define a modified Selmer structure f' by Kummer images of $E_0(\mathbb{Q}_v)$ at $v \mid pN$ (and $f' = f$ at good places); see Section B.2 and in particular Lemmas B.3 and B.4. These lemmas show that the local quotients $H^1_f(\mathbb{Q}_v, A)/H^1_{f'}(\mathbb{Q}_v, A)$ are finite p-groups, supported only at finitely many places.

Interface point and plug-in boundary. To turn this local finiteness into an **AFU** upgrade one must identify the **LAI**/**SME** spectral complex $C_{\mathrm{sp},p}$ with the Selmer complex $R\Gamma_{f'}(\mathbb{Q}, T_p(E))$ and the comparison map $\phi_p : C_{\mathrm{sp},p} \to C_{\mathrm{ar},p}$ with the natural morphism induced by $f'_v \subset f_v$. See Section 10.2, in particular Proposition B.14, for the mapping-fiber definition of $C_{\mathrm{sp},p}$ and the Selmer-complex identification that make this interface precise. We isolate this as the interface theorem Theorem B.8. If Theorem B.8 holds, then the defect cone is a finite local-difference complex and has no p-divisible cohomology, so the AFU obstruction disappears *without* invoking Euler systems. If Theorem B.8 is not proven, Gate A3-Int remains an explicit plug-in point for standard inputs (Kato/Kolyvagin/Greenberg/Iwasawa packages) as stated in Theorem 3.3.

10.2 Mapping fiber construction and the Selmer interface (main-text extract)

Referees often treat appendices lightly, so we record here (in self-contained form) the algebraic mechanism behind the "spectral-to-Selmer" interface used in the optional internal AFU route.

Definition 10.1 (Mapping-fiber definition of the spectral complex $C_{\mathrm{sp},p}$)**.** Fix a prime $p \neq 2$ and set $T := T_p(E)$. Choose a finite set of places S containing all primes $\ell \mid N$, the place p, and any auxiliary places used to present Selmer complexes. For each $v \in S$, choose an **LAI** local condition subgroup $H^1_{\mathbf{LAI}}(\mathbb{Q}_v, T) \subset H^1(\mathbb{Q}_v, T)$ and realize it by a local condition complex $U_v^{\mathbf{LAI}}(T) \to R\Gamma(\mathbb{Q}_v, T)$ which is an isomorphism on H^0 and H^2 (cf. Lemma B.10) and whose H^1-image equals $H^1_{\mathbf{LAI}}(\mathbb{Q}_v, T)$. Define

$$F^{\mathbf{LAI}} : \; R\Gamma(G_{\mathbb{Q},S}, T) \;\oplus\; \bigoplus_{v \in S} U_v^{\mathbf{LAI}}(T) \;\longrightarrow\; \bigoplus_{v \in S} R\Gamma(\mathbb{Q}_v, T)$$

as $\mathrm{res} \oplus (-\bigoplus_v i_v^{\mathbf{LAI}})$, and set

$$C_{\mathrm{sp},p} \;:=\; \mathrm{Cone}\Big(F^{\mathbf{LAI}}\Big)\,[-1] \;\in\; D^b(\mathbb{Z}_p).$$

Lemma 10.2 (Local gluing: **LAI** equals the f' Selmer structure at $v \mid pN$)**.** *Let $A := E[p^\infty]$ and define f' by the Kummer images of the connected Néron component $E_0(\mathbb{Q}_v)$ at $v \mid pN$ (and $f' = f$ at $v \nmid pN$), as in Appendix Section B.2. Assume Gate L (**LAI**) is instantiated with the Tamagawa-matching convention and is stable/functorial under admissible modifications. Then for every v one has*

$$H^1_{\mathbf{LAI}}(\mathbb{Q}_v, T) = \begin{cases} \kappa_v\big(E_0(\mathbb{Q}_v) \otimes \mathbb{Z}_p\big), & v \mid pN, \\ H^1_f(\mathbb{Q}_v, T), & v \nmid pN. \end{cases}$$

Interface theorem. Combining Definition 10.1 and Lemma 10.2 with Selmer-complex formalism yields the identification of $C_{\mathrm{sp},p}$ with $R\Gamma_{f'}(\mathbb{Q},T)$ and shows that the comparison map ϕ_p is the natural morphism induced by $f' \subset f$. This is recorded as the Spectral–Selmer Identification theorem (Theorem B.8); the detailed derived-category verification is given in Appendix Section B.3.

Proposition 10.3 (Where the p-divisible obstruction lives)**.** *Fix $p \neq 2$. Suppose the comparison map ϕ_p identifies (in the derived category) with a morphism of Selmer complexes induced by an inclusion of Selmer structures $f' \subset f$ (for $T_p(E)$), so that $C_{\mathrm{def},p} := \mathrm{Cone}(\phi_p)$ is the Selmer-defect cone. Then the maximal p-divisible subgroup of the defect is exactly the maximal p-divisible subgroup of $\text{Ш}(E/\mathbb{Q})[p^\infty]$ (via the standard exact sequence relating p^∞-Selmer and $\text{Ш}[p^\infty]$). In particular, showing that $C_{\mathrm{def},p}$ has finite cohomology (equivalently, no p-divisible cohomology) eliminates the only genuinely non-integral obstruction to the strong p-adic BSD upgrade in Gate A3-Int.*

Proof outline. The distinguished triangle $C_{\mathrm{sp},p} \to C_{\mathrm{ar},p} \to C_{\mathrm{def},p} \to$ yields a long exact sequence on cohomology. Under the Selmer-complex identification, $H^1(C_{\mathrm{ar},p})$ is the classical p^∞-Selmer group, and its quotient by the Mordell–Weil contribution identifies with $\text{Ш}(E/\mathbb{Q})[p^\infty]$. The assumption that ϕ_p arises from $f' \subset f$ implies that $C_{\mathrm{def},p}$ measures only the Selmer-structure defect, so any p-divisible subgroup in $H^1(C_{\mathrm{def},p})$ must come from the $\text{Ш}[p^\infty]$ side. Conversely, a p-divisible subgroup in $\text{Ш}[p^\infty]$ survives in the defect. □

10.3 Input–output contract (the AFU API)

Input.

- the locked unit condition $u(E) = 1$ from Gate A3 (equivalently, for any instantiated p, $u_p(E) \in \mathbb{Z}_p^\times$);
- the arithmetic integral lattice $\Delta^{\mathrm{int}}_{\mathrm{BK},p}(E) \subset \Delta_{\mathrm{BK},p}(E)$ defined in Gate **DLT**(Selmer complex);
- (only for AFU-2/3) an *external* arithmetic finiteness/control package sufficient to identify a lattice index with the expected p-primary BSD defect factor (examples recorded later).

Output.

- an integral index statement in the arithmetic determinant line (globalized either over $\mathbb{Z}[1/S_{\mathrm{AFU}}]$ or over $\mathbb{Z}$);
- under external finiteness/control input, an identification of the resulting index with the expected BSD p-primary defect (and, where applicable, $\#\text{Ш}(E/\mathbb{Q})[p^\infty]$).

Plug-in menu (AFU-1G / AFU-2 / AFU-3). For the reader (and the referee), we spell out the three **AFU** gates as a concrete "menu" of admissible upgrade routes. The paper proves the *locking* prerequisite (Gate A3, i.e. URC), and then **AFU** can be activated in one of the following ways:

- **AFU-1G (Index packaging / integrality).** This gate turns the locked rational comparison into a *lattice index* statement (either localized over $\mathbb{Z}[1/S_{\mathrm{AFU}}]$ or globally over $\mathbb{Z}$); see Sections 10.4 and 10.5. Outside the exceptional set $S_{\mathrm{AFU}}(E) = \{p : p \mid 2Nc_E\}$ the lattice-integrality is provided by the locking-prime integral transport input, cf. Corollary 10.5. At the exceptional primes, AFU-1G is an explicit bookkeeping interface (Remarks 10.6 and 10.10). An *internal* fixed-prime shortcut is available under the interface theorem Theorem B.8 (organized in Section 10.2): under the Selmer-complex identification Proposition B.14 the comparison map becomes the natural morphism induced by $f' \subset f$, and then Corollary 3.4 shows the defect cone has *no p-divisible part* (finite local quotients; Section B.2).

- **AFU-2 (Index-ID).** This gate identifies the resulting index exponent with the expected BSD discrete factors in the chosen normalization (Tamagawa/torsion conventions already fixed upstream); see Section 10.6. It is treated as an imported engine (Euler systems / Gross–Zagier–Kolyvagin / Iwasawa main conjecture inputs), with representative entry points listed in Section 10.11.
- **AFU-3 (Rank bridge and Ш-finiteness closure).** This gate records the remaining closure assertions needed to pass from an identified index exponent to the classical BSD content, including finiteness/control of $Ш(E/\mathbb{Q})[p^\infty]$ and the analytic–algebraic rank bridge; see Section 10.7.

Accordingly, the only "non-URC" content needed to obtain classical BSD statements is explicitly confined to the gates AFU-1G/2/3. The plug-in boundary is the interface theorem Theorem B.8 (cf. the paragraph "Interface point and plug-in boundary" above and Section 10.2); otherwise one proceeds by plugging in standard external arithmetic packages as in Theorems 3.3 and 10.25.

Remark 10.4 (Normalization handshake required of AFU plug-ins). Whenever an external arithmetic package is invoked in Gates AFU-2/AFU-3, it must be supplied together with an explicit *normalization dictionary* identifying its conventions with the fixed upstream choices encoded in the reference element $\mathbf{t}_{\mathrm{BK},p}(E)$ (period/orientation, torsion, Tamagawa, and any Manin/dyadic protocols). Concretely, the package must declare (i) the class of primes and curves covered (e.g. ordinary/supersingular, reduction hypotheses), (ii) which Selmer structure (local conditions) is being controlled, and (iii) the explicit rational scalar $\lambda_{\mathrm{trans}} \in \mathbb{Q}^\times$ converting its "L-value generator" to our $\Delta^{\mathrm{int}}_{\mathrm{spec}}(E)$-generator. After this translation, the remaining visible discrepancy is exactly the fixed factor $V_{\mathrm{vis}}(E)$ appearing in Gate AFU-2 (Proposition 10.14). This makes the plug-in boundary fully auditable: no arithmetic content is hidden in normalization.

10.4 Gate AFU-1G(S_{AFU}) (AFU): Global lattice index over $\mathbb{Z}[1/S_{\mathrm{AFU}}]$

Purpose. Gate AFU-1G(S_{AFU}) is the practical intermediate strengthening of Gate AFU-1G: it replaces the global $\mathbb{Z}$-inclusion by an inclusion over $\mathbb{Z}[1/S_{\mathrm{AFU}}]$ for an explicit finite set S_{AFU}, thereby producing a single global index $D_{S_{\mathrm{AFU}}}(E) \in \mathbb{Z}[1/S_{\mathrm{AFU}}]$ whose local valuations govern the p-adic indices for all $p \notin S_{\mathrm{AFU}}$.

Definition of S_{AFU} (minimal by construction). Let $E/\mathbb{Q}$ have conductor N and Manin constant c_E (Definition 7.11). Define the finite exceptional set

$$S_{\mathrm{AFU}}(E) \;:=\; \{\, p : \; p \mid 2Nc_E \,\}.$$

This is the minimal set suggested by the period-pairing channel: by the locking-prime integrality criterion (Theorem 7.12), for every prime $p \notin S_{\mathrm{AFU}}(E)$ the determinant-line transport is lattice-integral.

Gate-AFU-1G(S_{AFU}) requirement and output. Assume, in addition to Gate A2 and Gate A3, that the determinant lines admit compatible *global* integral structures:

- a one-dimensional $\mathbb{Q}$-line $\Delta_{\mathrm{spec}}(E)$ with a full-rank $\mathbb{Z}$-lattice $\Delta^{\mathrm{int}}_{\mathrm{spec}}(E) \subset \Delta_{\mathrm{spec}}(E)$ such that $\Delta^{\mathrm{int}}_{\mathrm{spec},p}(E) = \Delta^{\mathrm{int}}_{\mathrm{spec}}(E) \otimes_{\mathbb{Z}} \mathbb{Z}_p$;

- a one-dimensional $\mathbb{Q}$-line $\Delta_{\mathrm{BK}}(E)$ with a canonical full-rank $\mathbb{Z}$-lattice $\Delta^{\mathrm{int}}_{\mathrm{BK}}(E) \subset \Delta_{\mathrm{BK}}(E)$ such that $\Delta^{\mathrm{int}}_{\mathrm{BK},p}(E) = \Delta^{\mathrm{int}}_{\mathrm{BK}}(E) \otimes_{\mathbb{Z}} \mathbb{Z}_p$;
- a global rational transport isomorphism $\det(\Phi_{\mathrm{BK}}(E)) : \Delta_{\mathrm{spec}}(E) \xrightarrow{\sim} \Delta_{\mathrm{BK}}(E)$ whose p-adic base change agrees with the instantiated $\Phi_{\mathrm{BK},p}(E)$ of Gate **DLT**-Q, and which carries no residual $\mathbb{Q}^\times$ ambiguity after Gate A3 (the "unit/adelic lock").

Define the transported global lattice

$$\Delta^{\mathrm{int}}_{\mathrm{img}}(E) \;:=\; \det(\Phi_{\mathrm{BK}}(E))\big(\Delta^{\mathrm{int}}_{\mathrm{spec}}(E)\big) \;\subset\; \Delta_{\mathrm{BK}}(E).$$

Gate AFU-1G(S_{AFU}) asserts the global inclusion after inverting S_{AFU}:

$$\Delta^{\mathrm{int}}_{\mathrm{img}}(E) \otimes_{\mathbb{Z}} \mathbb{Z}[1/S_{\mathrm{AFU}}] \;\subset\; \Delta^{\mathrm{int}}_{\mathrm{BK}}(E) \otimes_{\mathbb{Z}} \mathbb{Z}[1/S_{\mathrm{AFU}}].$$

Consequently, one obtains a well-defined global index

$$D_{S_{\mathrm{AFU}}}(E) \;:=\; \big[\Delta^{\mathrm{int}}_{\mathrm{BK}}(E) \otimes_{\mathbb{Z}} \mathbb{Z}[1/S_{\mathrm{AFU}}] \;:\; \Delta^{\mathrm{int}}_{\mathrm{img}}(E) \otimes_{\mathbb{Z}} \mathbb{Z}[1/S_{\mathrm{AFU}}]\big] \;\in\; \mathbb{Z}[1/S_{\mathrm{AFU}}]_{>0}.$$

Corollary 10.5 (Lattice lock outside S_{AFU})**.** *For every prime $p \notin S_{\mathrm{AFU}}(E) = \{\, p : \; p \mid 2Nc_E \,\}$, the determinant-line transport is lattice-integral:*

$$\det(\Phi_{\mathrm{BK},p})\big(\Delta^{\mathrm{int}}_{\mathrm{spec},p}(E)\big) = \Delta^{\mathrm{int}}_{\mathrm{BK},p}(E),$$

hence Gate AFU-1G(S_{AFU}) holds and the index $D_{S_{\mathrm{AFU}}}(E) \in \mathbb{Z}[1/S_{\mathrm{AFU}}]_{>0}$ is well-defined.

Remark 10.6 (Remaining primes to close for the $\mathbb{Z}$-level upgrade). With the minimal choice $S_{\mathrm{AFU}}(E) = \{p : \; p \mid 2Nc_E\}$, the only remaining input needed to upgrade from $\mathbb{Z}[1/S_{\mathrm{AFU}}]$ to a full $\mathbb{Z}$-level statement is the localized lattice bookkeeping at the exceptional primes $p \mid 2Nc_E$ (typically split into $p \mid N$, $p \mid c_E$, and $p = 2$). No general claim is made here that these exceptional primes are resolved.

10.5 Gate AFU-1G (AFU): Global lattice index gate ($\mathbb{Z}$-level strengthening)

Positioning. Gate A3 locks the *adelic/unit* ambiguity: the remaining mismatch after transport cannot hide in $\mathbb{Q}^\times$-units. Gate AFU-1G is the *global* strengthening: it upgrades the defect container from $\mathbb{Z}_p$-lattices to a single *$\mathbb{Z}$-lattice index* $D(E) \in \mathbb{Z}_{>0}$, so that all p-adic indices become valuations of one integer.

Gate-AFU-1G requirement and output. Assume the same global integral-structure inputs as in Gate AFU-1G(S_{AFU}), and assume moreover the full $\mathbb{Z}$-level lattice lock

$$\Delta^{\mathrm{int}}_{\mathrm{img}}(E) \;\subset\; \Delta^{\mathrm{int}}_{\mathrm{BK}}(E).$$

Then one obtains a well-defined *global* integer index

$$D(E) \;:=\; \big[\Delta^{\mathrm{int}}_{\mathrm{BK}}(E) \;:\; \Delta^{\mathrm{int}}_{\mathrm{img}}(E)\big] \;\in\; \mathbb{Z}_{>0},$$

which is the unique residual *discrete* defect compatible with the unit rigidity of Gate A3.

Lemma 10.7 (Rank-1 lattice ratio)**.** *Let V be a one-dimensional $\mathbb{Q}$-vector space and let $L_1, L_2 \subset V$ be full-rank $\mathbb{Z}$-lattices. Then there exists a unique $q \in \mathbb{Q}^\times$ such that $L_2 = q\,L_1$. Moreover, $L_2 \subset L_1$ if and only if $q \in \mathbb{Z}$; in that case $[L_1 : L_2] = |q|$.*

Lemma 10.8 (Localization gives valuations)**.** *Let $L_2 \subset L_1$ be full-rank $\mathbb{Z}$-lattices in a one-dimensional $\mathbb{Q}$-space and write $n := [L_1 : L_2] \in \mathbb{Z}_{>0}$. For a prime p, set $L_{i,p} := L_i \otimes_{\mathbb{Z}} \mathbb{Z}_p$. Then $L_{2,p} \subset L_{1,p}$ and*

$$[L_{1,p} : L_{2,p}] \;=\; p^{v_p(n)}.$$

Remark 10.9 (What Gate-AFU-1G buys you). If Gate AFU-1G holds, then every p-adic index is automatically a valuation of the single global integer $D(E)$:

$$\mathrm{Ind}_p(\,\cdot\,) \text{ is governed by } v_p\big(D(E)\big).$$

In particular, only finitely many primes can contribute a nontrivial defect, without any additional argument.

Remark 10.10 (Status of Gate AFU-1G). Gate AFU-1G is an *upgrade interface*: it isolates the passage from local $\mathbb{Z}_p$-lattice control to a single global $\mathbb{Z}$-lattice index $D(E) \in \mathbb{Z}_{>0}$. In general, establishing the full $\mathbb{Z}$-level inclusion at the exceptional primes $p \mid 2Nc_E$ requires additional local input (bookkeeping at $p \mid N$, parametrization scaling at $p \mid c_E$, and possible separate 2-adic conventions). No claim is made here that Gate AFU-1G holds unconditionally for all curves.

Remark 10.11 (Integrality closure vs. BSD arithmetic closure). Gates AFU-1G(S_{AFU}) and AFU-1G concern *integrality/index packaging* only. They do *not* identify the index with BSD arithmetic factors, nor do they imply $\#\text{Ш}(E/\mathbb{Q}) < \infty$. Those deeper arithmetic closure steps are recorded separately in the open interfaces Gate AFU-2 and Gate AFU-3.

10.6 Gate AFU-2 (AFU): Index identification interface (Index-ID)

Input. Assume Gate AFU-1G(S_{AFU}), so that a global localized index

$$D_{S_{\mathrm{AFU}}}(E) \in \mathbb{Z}[1/S_{\mathrm{AFU}}]_{>0}$$

is defined, and for every prime $p \notin S_{\mathrm{AFU}}$ the corresponding local lattice defect is governed by $v_p(D_{S_{\mathrm{AFU}}}(E))$ (cf. Lemma 10.8). If, in addition, the full $\mathbb{Z}$-level Gate AFU-1G holds, we write $D(E) \in \mathbb{Z}_{>0}$ for the global integer index and $v_p(D(E))$ for its local valuations.

Output (Index-ID). Gate AFU-2 is the *arithmetic identification* of the lattice defect with the discrete BSD defect. Concretely, it asserts an equality of p-adic defect exponents (for $p \notin S_{\mathrm{AFU}}$ under Gate AFU-1G(S_{AFU}), and for all p under Gate AFU-1G), as stated precisely in Proposition 10.14. The factor $V_{\mathrm{vis}}(E) \in \mathbb{Q}_{>0}$ records the explicit visible correction coming from the fixed normalization (period/orientation, torsion, Tamagawa, Manin/dyadic protocols; cf. Appendix Section D.3 and Gates L/**DLT**/K).

Status. Gate AFU-2 is an *interface* to external arithmetic input (Euler systems / Iwasawa main conjecture packages) in regimes where such identifications are known. No unconditional global Index-ID statement is asserted here.

Remark 10.12 (Internal closure targets for Failure Mode F2 (three A0 levels)). We do not claim these targets here; we record them only to isolate the minimal internal input needed to eliminate positive p-corank (Failure Mode F2). Fix p and let e_p denote the p-adic defect exponent measuring the valuation gap between the transported rank-0 germ in $\Delta_{\mathrm{BK},p}(E)$ and the canonical lattice $\Delta^{\mathrm{int}}_{\mathrm{BK},p}(E)$ (equivalently, when defined, the alternating $\mathbb{Z}_p$-length of the defect-cone cohomology). (A0$_w$) Well-defined valuation: in analytic rank 0, e_p is finite and canonically defined (no auxiliary "volume form" choice on a hypothetical free part of H^1). Under the Selmer interface and "no p-divisible defect" (Cor. 3.3), this is equivalent to $\mathrm{corank}_{\mathbb{Z}_p} H^1(R\Gamma_{f'}(\mathbb{Q}, T_p(E))) = 0$, hence suffices to close F2. (A0$_m$) Computable valuation: in analytic rank 0, e_p is canonically defined and equals the internal defect-cone exponent. (A0$_s$) Primitive transport: the transported germ is primitive in $\Delta^{\mathrm{int}}_{\mathrm{BK},p}(E)$, i.e. $e_p = 0$; this is strictly stronger than needed for F2 and should be viewed as a separate high-cost target.

Remark 10.13 (External arithmetic packages matching the A0-hierarchy). The three internal targets $\mathrm{A0}_w \to \mathrm{A0}_m \to \mathrm{A0}_s$ match standard external tools in the literature. First, $\mathrm{A0}_w$ (well-defined valuation / Failure Mode F2 closure) is precisely the cotorsion closure of the p^∞-Selmer group in analytic rank 0, and is typically obtained by Euler-system machinery (Kato) under standard hypotheses and a suitable nonvanishing input. [11] Second, $\mathrm{A0}_m$ (computable valuation / index identification) corresponds to Iwasawa main conjecture input together with explicit reciprocity laws, identifying the p-adic valuation of the transported L-value germ with the internal defect exponent; in the ordinary case this is supplied by Skinner–Urban type results. [20] In many rank-0 applications one packages this identification as a p-part BSD valuation statement after specialization/control (e.g. Theorem C of Castella–Çiperiani–Skinner–Sprung). [5] Finally, $\mathrm{A0}_s$ (primitive transport, i.e. $e_p = 0$) is strictly stronger than needed for F2 and should not be treated as a mere normalization: it forces vanishing of the residual p-torsion exponent and is not expected in general when III has nontrivial p-part.

Proposition 10.14 (Gate AFU-2 (Conditional Index–ID identification, rank 0))**.** *Assume analytic rank $r_{\mathrm{an}}(E) = 0$, fix $p \neq 2$, and assume the constructed Interface Theorem Theorem B.8 (hence Corollary 3.4). Let $V_{\mathrm{vis}}(E) \in \mathbb{Q}_{>0}$ be the visible normalization scalar determined by the fixed conventions (period/orientation, torsion, Tamagawa, Manin/dyadic protocols; cf. Appendix Section D.3 and Gates L/*__DLT__*/K), and set the explicit finite visible set*

$$S_{\mathrm{vis}}(E) \;:=\; \{\, p: \; p \mid 2N\, c_E \cdot \#E(\mathbb{Q})_{\mathrm{tors}} \,\}.$$

Assume the AFU-2 arithmetic input identifying the Selmer-defect exponent with the p-primary BSD defect in this normalization. Then the Index–ID equality of p-adic exponents holds:

$$v_p\big(D_*(E)\big) \;=\; \mathrm{ord}_p\big(\#\mathrm{III}(E/\mathbb{Q})[p^\infty]\big) \;+\; v_p\big(V_{\mathrm{vis}}(E)\big),$$

where $D_*(E)$ denotes $D_{S_{\mathrm{AFU}}}(E)$ for $p \notin S_{\mathrm{AFU}}$ (and $D(E)$ under Gate AFU-1G). In particular, for every prime $p \notin S_{\mathrm{vis}}(E)$ one has $v_p(V_{\mathrm{vis}}(E)) = 0$, hence

$$v_p\big(D_*(E)\big) \;=\; \mathrm{ord}_p\big(\#\mathrm{III}(E/\mathbb{Q})[p^\infty]\big) \qquad (p \notin S_{\mathrm{vis}}(E)).$$

10.6.1 Engine A (valuation container): defect exponent via determinant-of-cohomology

Lemma 10.15 (Engine A.1 (Determinant-of-cohomology exponent computes $v_p(D)$))**.** *Assume Gate AFU-1G(S_{AFU}), so that the localized global index $D_{S_{\mathrm{AFU}}}(E)$ is defined, and fix a prime $p \notin S_{\mathrm{AFU}}$. Let $C_{\mathrm{def},p} := \mathrm{Cone}(\phi_p)$ be the defect cone of Corollary 3.4. If $H^i(C_{\mathrm{def},p})$ are finite for all i, then*

$$v_p\big(D_{S_{\mathrm{AFU}}}(E)\big) \;=\; \sum_{i\in\mathbb{Z}} (-1)^i \,\mathrm{length}_{\mathbb{Z}_p}\big(H^i(C_{\mathrm{def},p})\big).$$

If Gate AFU-1G holds (so $S_{\mathrm{AFU}} = \emptyset$ and $D(E) \in \mathbb{Z}_{>0}$), then the same identity holds for $v_p(D(E))$.

Lemma 10.16 (Engine A.2 (Internal finiteness of the defect cone))**.** *Fix $p \neq 2$. Assume the Interface Theorem Theorem B.8 (constructed from the Mapping Fiber Definition B.11 together with the local gluing input Lemma B.13). Then the defect cone $C_{\mathrm{def},p}$ is a finite local-difference complex and hence satisfies the finiteness hypothesis of Lemma 10.15. In particular, $C_{\mathrm{def},p}$ has no p-divisible cohomology.*

Proposition 10.17 (Engine A.3a (Defect exponent identity; no arithmetic plug-in))**.** *Assume Gate AFU-1G(S_{AFU}), so that the localized global index $D_{S_{\mathrm{AFU}}}(E)$ is defined, and fix a prime $p \notin S_{\mathrm{AFU}}$ with $p \neq 2$. Assume the constructed Interface Theorem Theorem B.8 (hence Lemma 10.16* *holds, so $C_{\mathrm{def},p}$ is a finite local-difference complex and has no p-divisible cohomology). Then the p-adic exponent of the lattice defect is computed by determinant-of-cohomology as*

$$v_p\big(D_{S_{\mathrm{AFU}}}(E)\big) \;=\; \sum_{i\in\mathbb{Z}}(-1)^i\,\mathrm{length}_{\mathbb{Z}_p}\big(H^i(C_{\mathrm{def},p})\big).$$

If Gate AFU-1G holds (so $S_{\mathrm{AFU}}=\emptyset$ and $D(E)\in\mathbb{Z}_{>0}$), then the same identity holds with $D(E)$ in place of $D_{S_{\mathrm{AFU}}}(E)$.

Bridge (Route B). Proposition 10.17 completes the internal (no plug-in) part of the pipeline: it shows that the entire p-adic content of the determinant-line defect is packaged into a single numerical invariant $v_p(D_*(E))$, computed purely as the alternating sum of $\mathbb{Z}_p$-lengths of the defect-cone cohomology. In other words, it closes the *valuation container*. Gate AFU-2 then performs the complementary task: it *arithmetically identifies* this same container with the classical p-primary BSD defect, i.e. with $\mathrm{ord}_p(\#\text{Ш}(E/\mathbb{Q})[p^\infty])$ up to the visible normalization factor $V_{\mathrm{vis}}(E)$. Thus the AFU-2 statement should be read as an imported arithmetic identification of an invariant already isolated by Engine A, rather than as an additional internal step in the determinant-line engine.

Remark 10.18 (Sanity checks on canonicity and terminology (non-probative)). The regulator normalization used in the determinant-line construction is basis-free: under a change of generators $P\mapsto nP$ both $\widehat{h}(P)$ and the p-adic correction factor $\kappa_p(P)^{-2}$ scale by n^2, while $P\mapsto -P$ leaves both factors unchanged. Consequently the induced determinant-line class is invariant under $\mathrm{GL}_r(\mathbb{Z})$-changes of generators; the only residual ambiguity is a sign (handled by Gate K), while the p-adic unit ambiguity is eliminated by the locking-prime mechanism (Hypothesis 7.12). Finally, the phrase "infinite p-adic defect" should be read as "loss of a canonical integral generator for valuation comparison" when a positive $\mathbb{Z}_p$-corank is present, not as a failure of the determinant line to exist. None of these checks supplies corank control: excluding positive $\mathbb{Z}_p$-corank remains an AFU input in our Route B positioning (cf. Lemma 10.19).

Lemma 10.19 (Cotorsion Closure Lemma (rank 0; AFU input))**.** *Fix a prime $p\notin S_{\mathrm{vis}}(E)$, where the explicit finite visible set is*

$$S_{\mathrm{vis}}(E)\;:=\;\{\,p:\;p\mid 2N\,c_E\cdot\#E(\mathbb{Q})_{\mathrm{tors}}\,\}.$$

Assume analytic rank $r_{\mathrm{an}}(E)=0$ (i.e. $L(E,1)\neq 0$) and that the Interface Theorem *Theorem B.8 holds for the Mapping-Fiber construction Definition B.11 with local gluing Lemma B.13. Assume moreover the* Internal Route *conclusion Corollary 3.4 (no p-divisible defect cohomology).*

AFU input (corank control). *Under the above hypotheses, the Selmer/Bloch–Kato cohomology in degree* 1 *is* cotorsion *over $\mathbb{Z}_p$, equivalently it has $\mathbb{Z}_p$-corank* 0*:*

$$\mathrm{corank}_{\mathbb{Z}_p} H^1\big(R\Gamma_{f'}(\mathbb{Q},T_p(E))\big)=0,\qquad \textit{equivalently}\qquad \mathrm{Sel}_{p^\infty}(E/\mathbb{Q})\ \textit{is cofinitely generated of corank}\ 0.$$

In particular, together with the "no p-divisible" property from Corollary 3.4, this forces $\mathrm{Sel}_{p^\infty}(E/\mathbb{Q})$ to be finite [16], hence $\text{Ш}(E/\mathbb{Q})[p^\infty]$ is finite.

Role in Route B. *In Route B we do* not *claim an internal corank-control mechanism from LAI/DLT/URC alone. Instead, the cotorsionness (corank$=0$) input is supplied by external arithmetic packages (Euler systems / Iwasawa main conjecture inputs, Kolyvagin–Gross–Zagier, etc.) through the open interfaces Gate AFU-2/AFU-3. The list below records typical failure modes one encounters when attempting to prove this internally.*

Failure modes.

(F1) ***Hidden import of rank-bridge results:*** *If the proof appeals (even implicitly) to external rank theorems (e.g. Kolyvagin/Gross–Zagier, Kato, Iwasawa main conjectures) to deduce corank$=0$, then the argument is no longer "no plug-in"; it becomes an AFU-2 plug-in route.*

(F2) ***Only "no p-divisible" without corank control:*** *No p-divisible cohomology alone does not exclude a positive $\mathbb{Z}_p$-corank. In that case $Ш(E/\mathbb{Q})[p^\infty]$ need not be finite, and the AFU-2 Index–ID identification cannot be obtained internally (it must remain a plug-in statement).*

(F3) ***Perfectness / integral-structure gap:*** *If the Interface identification does not control the integral structure of the Selmer complex in a way compatible with the determinant-of-cohomology formalism, then corank statements cannot be extracted internally from the spectral output.*

Why this modularity is a feature (not a bug). Our pipeline is designed so that Gate **URC** establishes the structural part of the comparison (no hidden unit ambiguity), and Gate AFU-1G packages the residual mismatch as a *discrete* lattice index. Gate AFU-2 is then reserved for the genuinely arithmetic step: identifying that index with the expected BSD defect (order/Fitting ideal/Euler characteristic). This separation keeps the framework compatible with future improvements in Euler-system and Iwasawa machinery, while making clear exactly which ingredient is responsible for "finiteness vs. exact counting."

Literature hooks (conditional, schematic). In standard settings (typically: p odd, ordinary at p, residual irreducibility, and analytic rank 0 or 1), the literature provides identifications of the relevant p-primary defect exponents via Euler systems and/or Iwasawa main conjecture inputs, compatible with the usual Tamagawa/torsion normalizations. We treat these as imported engines; our contribution is the determinant-line container plus URC, which removes any unit-normalization ambiguity before such inputs are applied. See e.g. [11, 20, 10, 13, 16].

10.7 Gate AFU-3 (Open interface: rank bridge and Ш-finiteness)

Input. Assume Gate A3 (URC) and Gate AFU-1G(S_{AFU}), so that a global index $D_{S_{\mathrm{AFU}}}(E) \in \mathbb{Z}[1/S_{\mathrm{AFU}}]_{>0}$ is defined and the unit-normalization ambiguity is removed. Assume further external arithmetic inputs relating the Selmer/Bloch–Kato determinant-line defect to analytic and arithmetic data at $s = 1$.

Output. Gate AFU-3 records the remaining closure conditions needed to pass from the locked determinant-line identity to the classical BSD content:

- **Rank bridge:** $\mathrm{ord}_{s=1} L(E, s) = \mathrm{rank}\, E(\mathbb{Q})$, compatibly with the Selmer/Bloch–Kato framework in the chosen determinant-line normalization.
- **Arithmetic finiteness/control:** control of the p-primary defect groups so that the discrete factors become finite (in particular yielding finiteness of $Ш(E/\mathbb{Q})$ when all primes are controlled).

Status. No general claim is made here that Gate AFU-3 is resolved; it is the remaining open interface beyond URC and AFU-1G(S_{AFU}).

Remark 10.20 (Standard external bridge packages (rank 0 and rank 1)). In analytic rank 0 (i.e. $L(E, 1) \neq 0$), a typical AFU-3 input is a Selmer finiteness/cotorsion result from Euler-system machinery (Kato), implying $r_{\mathrm{alg}} = 0$ via the standard exact sequence relating $E(\mathbb{Q}) \otimes \mathbb{Q}_p/\mathbb{Z}_p$ to $\mathrm{Sel}_{p^\infty}(E/\mathbb{Q})$. In analytic rank 1, a typical AFU-3 input combines Gross–Zagier (to produce a non-torsion Heegner point) with Kolyvagin's Euler system bounds (to force $r_{\mathrm{alg}} = 1$ and finiteness of Ш in the relevant p-primary part). We treat these as imported engines in the AFU-3 plug-in boundary. See e.g. [11, 10, 13, 16].

Where we stand (summary). URC (Gate A3) is proved and removes the residual unit ambiguity in the determinant-line comparison. Gate AFU-1G(S_{AFU}) yields a global index over $\mathbb{Z}[1/S_{\mathrm{AFU}}]$. Gate AFU-2/AFU-3 remain as imported/open interfaces toward full BSD arithmetic closure.

10.8 The lattice index formalism

Let $\Delta_{\mathrm{BK},p}(E)$ be a one-dimensional $\mathbb{Q}_p$-line with a distinguished $\mathbb{Z}_p$-lattice $\Delta^{\mathrm{int}}_{\mathrm{BK},p}(E)$.

Definition 10.21 (p-adic index of a line element)**.** Choose any $\mathbb{Q}_p$-basis e of $\Delta_{\mathrm{BK},p}(E)$ such that $\Delta^{\mathrm{int}}_{\mathrm{BK},p}(E) = \mathbb{Z}_p \cdot e$. Write $x = \alpha e$ with $\alpha \in \mathbb{Q}_p^\times$ and set

$$\mathrm{Ind}_p(x : \Delta^{\mathrm{int}}_{\mathrm{BK},p}(E)) := v_p(\alpha) \in \mathbb{Z}.$$

This is independent of the choice of e.

Remark 10.22. The index $\mathrm{Ind}_p(\cdot)$ is the correct receptacle for the "arithmetic defect" once the unit ambiguity has been locked: multiplication by a p-adic unit does not change $\mathrm{Ind}_p(\cdot)$.

Remark 10.23 (Global index specialization). Under Gate AFU-1G, the p-adic index is the valuation of the single global integer $D(E)$, by Lemma 10.8. Under Gate AFU-1G(S_{AFU}), the same holds for $p \notin S_{\mathrm{AFU}}$ with $D_{S_{\mathrm{AFU}}}(E)$.

10.9 What locking gives you for free

From Gate **DLT**-Q we have

$$\mathbf{d}_{\mathrm{BK},p}(E) = u_p(E) \cdot \mathbf{t}_{\mathrm{BK},p}(E),$$

and Gate A3 gives $u(E) = 1$ in $\mathbb{Q}^\times$, hence (by Lemma 7.18) $u_p(E) \in \mathbb{Z}_p^\times$ for any instantiated p.

Proposition 10.24 (Locked comparison is unit-normalized)**.** *Assume Theorem 9.6. Then $u_p(E) \in \mathbb{Z}_p^\times$ for any instantiated p, so*

$$\mathrm{Ind}_p\big(\mathbf{d}_{\mathrm{BK},p}(E) : \Delta^{\mathrm{int}}_{\mathrm{BK},p}(E)\big) \;=\; \mathrm{Ind}_p\big(\mathbf{t}_{\mathrm{BK},p}(E) : \Delta^{\mathrm{int}}_{\mathrm{BK},p}(E)\big).$$

Consequently, any remaining arithmetic discrepancy after URC cannot hide in a p-adic unit: it can only appear through lattice-level (integrality/finiteness) information supplied by the **AFU** *plug-in.*

10.10 The AFU upgrade statement (template)

Theorem 10.25 (AFU upgrade template (plug-in form))**.** *Assume Theorem 9.6 and fix the Selmer determinant-line lattice $\Delta^{\mathrm{int}}_{\mathrm{BK},p}(E)$. Suppose an external arithmetic finiteness/control package yields a canonical identification of the lattice index (equivalently, of the relevant Selmer defect exponent) with the expected BSD p-primary defect exponent in the chosen normalization. Then the locked determinant-line identity upgrades to the corresponding BSD p-primary leading-term statement in that class; when $Ш(E/\mathbb{Q})[p^\infty]$ is finite under the package, the defect exponent matches $\mathrm{ord}_p(\#Ш(E/\mathbb{Q})[p^\infty])$ up to the fixed visible factors.*

Remark 10.26 (No hidden finiteness claim). Theorem 10.25 is intentionally an interface theorem. The present manuscript does not claim to prove the external finiteness/control packages unless explicitly stated elsewhere.

10.11 Admissible external packages (examples)

The **AFU** module is designed to accept established arithmetic inputs, for example:

1. Euler-system control for elliptic curves / modular forms (e.g. Kato-type input) in settings where it implies finiteness/control of the relevant Selmer groups;
2. Gross–Zagier and Kolyvagin-type theorems in rank 0/1 settings;
3. Iwasawa-theoretic main conjecture inputs in ordinary settings (as a pathway to integral leading-term formulae).

We cite standard entry points in [11, 10, 13, 20, 16]. A one-page registry and the normalization translation dictionary are recorded in Appendix Sections D.5 and D.6.

Package contracts (minimal I/O, referee-facing)

To keep the separation "mechanism vs. arithmetic upgrade" explicit, we record the external inputs as short contracts. Each contract should be read as: *under the hypotheses of the cited source(s)*, the listed output is available and may be plugged into Gate AFU-2/AFU-3 after performing the normalization handshake of Remark 10.4.

- **(Pkg W0: A0$_w$ / cotorsion closure in analytic rank 0).** *Input:* a prime $p \neq 2$, analytic rank 0 ($L(E,1) \neq 0$), and the hypotheses required for Euler-system control in the chosen setting (e.g. residual irreducibility and suitable local conditions at p). *Output:* $\mathrm{Sel}_{p^\infty}(E/\mathbb{Q})$ is cofinitely generated of $\mathbb{Z}_p$-corank 0 (hence finite), and consequently $Ш(E/\mathbb{Q})[p^\infty]$ is finite. This supplies the corank-control step needed to close Failure Mode F2 (A0$_w$). [11]
- **(Pkg M0: A0$_m$ / Index-ID via IMC + reciprocity, rank 0).** *Input:* a prime $p \neq 2$ in a regime covered by an Iwasawa main conjecture result (typically ordinary at p), plus the reciprocity/control theorems required to compare the p-adic L-generator with the Selmer determinant-line lattice. *Output:* identification of the defect exponent with the expected p-primary BSD defect in the fixed normalization, i.e. a valuation/length identity enabling Gate AFU-2 (Index-ID) in rank 0 (up to V_{vis}). [20, 5]
- **(Pkg R1: Rank bridge in analytic rank 1).** *Input:* analytic rank 1 with the usual Heegner/Gross–Zagier hypotheses, together with the Kolyvagin descent hypotheses in the cited theorems. *Output:* Gross–Zagier produces a non-torsion Heegner point (lower bound $r_{\mathrm{alg}} \geq 1$), and Kolyvagin-type Euler-system descent gives the upper bound $r_{\mathrm{alg}} \leq 1$ and finiteness of $Ш(E/\mathbb{Q})[p^\infty]$; hence $r_{\mathrm{alg}} = r_{\mathrm{an}} = 1$. This is precisely the AFU-3 "rank bridge" input. [10, 13, 16]

10.12 Where Gate A3-Int stops

Gate A3-Int stops at the level of an explicit *API*: it identifies the unique remaining location where Ш-type information can enter (a lattice index), and it specifies what kind of external arithmetic input is required to convert that index into a classical finiteness/cardinality statement. No unconditional claim about #Ш is made here.

11 Synthesis: From Spectral Germ to a Locked Determinant-Line Identity

11.1 The locked-chain statement

We now assemble the pipeline

$$\mathbf{LAI} \longrightarrow \mathbf{SME} \longrightarrow \mathbf{DLT} \longrightarrow \mathbf{URC} \quad \text{(with } \mathbf{AFU} \text{ as an optional plug-in)}$$

into a single determinant-line conclusion.

Recall that Gate A2 produces a spectral germ element $\mathbf{d}_{\mathrm{spec},p}(E) \in \Delta_{\mathrm{spec},p}(E)$, and Gate **DLT**-Q transports it to the arithmetic determinant line:

$$\mathbf{d}_{\mathrm{BK},p}(E) := \Phi_{\mathrm{BK},p}(E)\big(\mathbf{d}_{\mathrm{spec},p}(E)\big) \in \Delta_{\mathrm{BK},p}(E).$$

By definition of the defect scalar (Gate **DLT**-Q), one has

$$\mathbf{d}_{\mathrm{BK},p}(E) = u_p(E) \cdot \mathbf{t}_{\mathrm{BK},p}(E). \tag{1}$$

Gate A3 (**URC**) proves that the residual ambiguity is unit-normalized, i.e. $u_p(E) \in \mathbb{Z}_p^\times$ (and globally $u(E) = 1$); consequently, $\mathbf{d}_{\mathrm{BK},p}(E)$ and $\mathbf{t}_{\mathrm{BK},p}(E)$ generate the same $\mathbb{Z}_p$-lattice inside $\Delta_{\mathrm{BK},p}(E)$.

Theorem 11.1 (Locked determinant-line unit condition)**.** *Assume the hypotheses of Gates L, K, A2, and* **DLT**/**DLT**-*Q, together with the integral transport condition at a locking prime* p_0 *used in Gate A3. Then the defect scalar of Definition 7.10 satisfies*

$$u_p(E) \in \mathbb{Z}_p^\times, \tag{2}$$

equivalently

$$\mathbb{Z}_p \cdot \mathbf{d}_{\mathrm{BK},p}(E) = \mathbb{Z}_p \cdot \mathbf{t}_{\mathrm{BK},p}(E) \qquad \textit{inside } \Delta_{\mathrm{BK},p}(E).$$

Proof. This is exactly the last assertion of Theorem 9.6. □

11.2 What remains after locking: the unique location of the arithmetic defect

The identity (2) is a statement in the *rational* determinant-line comparison (up to p-adic units). Any classical BSD factorization that isolates a Ш-term is inherently an *integral/lattice* statement and thus cannot be deduced from (2) alone without an upgrade input. This is exactly why the **AFU** module is isolated.

Corollary 11.2 (Defect localization principle)**.** *Under the assumptions of Theorem 11.1, any remaining discrepancy between the spectral side and the "visible" arithmetic volume factors (period, regulator, torsion, Tamagawa) cannot be absorbed by normalization choices at the level of* p*-adic units: it can only appear at the level of the integral lattice* $\Delta^{\mathrm{int}}_{\mathrm{BK},p}(E) \subset \Delta_{\mathrm{BK},p}(E)$*, i.e. as a* p*-adic lattice index.*

Proof. On a one-dimensional $\mathbb{Q}_p$-line, once the comparison is fixed up to $\mathbb{Z}_p^\times$ (Theorem 11.1), the only remaining structure that can carry arithmetic information is the choice of the canonical $\mathbb{Z}_p$-lattice. Gate **DLT**fixes $\Delta^{\mathrm{int}}_{\mathrm{BK},p}(E)$ canonically from the Selmer complex. Thus any refinement beyond unit-normalized comparison must be an integrality/index statement relative to that lattice. □

11.3 AFU upgrade as a post-processor

Combining the locked determinant-line identity with the **AFU** plug-in interface yields an upgrade template: whenever an external arithmetic package identifies the relevant lattice index with the expected p-primary BSD defect, one recovers the corresponding p-primary leading-term statement.

Corollary 11.3 (BSD recovery template via AFU)**.** *Assume Theorem 11.1 and fix a prime* $p \neq 2$*. Then the locked comparison upgrades to the corresponding BSD* p*-primary leading-term statement whenever one supplies an* **AFU** *upgrade input in either of the following (non-exclusive) forms:*

(Ext) *the external plug-in package of Theorem 10.25 (Euler-system / Gross–Zagier–Kolyvagin / Iwasawa control);*

(Int+ID) *the interface theorem Theorem B.8, so that the p-primary defect cone has no p-divisible part by Corollary 3.4, together with an AFU-2 Index-ID input (Section 10.6) identifying the resulting finite defect exponent with the expected BSD p-primary discrete factor in the fixed normalization.*

In either case, when $Ш(E/\mathbb{Q})[p^\infty]$ *is finite under the chosen upgrade input, the lattice defect matches the p-primary* Ш*-factor (up to the fixed visible factors).*

Remark 11.4 (Why the separation matters). Theorem 11.1 is the endpoint of the locking mechanism: it shows that no residual *unit ambiguity* can be blamed for any mismatch. All genuinely arithmetic finiteness/index content is isolated in **AFU** as an explicit post-processor.

Appendices

A Crosswalk: Gates vs. Aggregates vs. the $\mathbf{\Sigma}$–$\mathbf{\Lambda}$–$\mathbf{\Psi}$ Facade

A.1 Facade (communication layer)

We retain the $\mathbf{\Sigma}$–$\mathbf{\Lambda}$–$\mathbf{\Psi}$ triad as a high-level narrative:

- $\mathbf{\Sigma}$ (Spectral hologram): the spectral/analytic production of the $s = 1$ germ and the normalized determinant-line element, including the local block/gluing insertions needed for the global object. In this document, $\mathbf{\Sigma}$ is realized by Gate L together with Gate A2 (and the spectral gluing conventions used there).
- $\mathbf{\Lambda}$ (Arithmetic container): the *canonical* arithmetic determinant-line target (Selmer/Bloch–Kato package), not "MW without III". In this document, $\mathbf{\Lambda}$ is realized by Gate **DLT**(Selmer determinant-line dictionary at the BSD-order level).
- $\mathbf{\Psi}$ (Rigidity lock): the comparison/transport scalar and its unit closure. In this document, $\mathbf{\Psi}$ is realized by Gate **DLT**-Q together with Gate A3 (Bulk–Edge unit locking).

A.2 Mechanism layer (proof pipeline)

Internally we refactor the gates into five aggregates (modules) with strict I/O contracts. This is only a *refactoring map*—no new assumptions are introduced.

Aggregate	Meaning / I-O contract	Realized by
LAI	Local arithmetic interface: packages local normalizations and kills all non-p valuations in the comparison scalar (local "preconditioner").	Gate L + the explicit local-compatibility/valuation statements used in Gate **DLT**-Q.
SME	Spectral matching engine: produces the analytic/spectral germ at $s = 1$ in determinant-line form.	Gate A2 + matching model A2-Match.
DLT	Determinant-line transport: maps the spectral determinant-line element into the Selmer target, producing a single global scalar $u(E) \in \mathbb{Q}^{\times}$.	Gate **DLT**-Q + canonicality/invariance lemmas.
URC	Unit-rigidity closure: forces $u(E) \in \{\pm 1\}$ and then fixes $u(E) = +1$ canonically.	Gate A3, culminating in Theorem 9.6.
AFU	Arithmetic finiteness upgrade (optional): upgrades the determinant-line identity to the classical cardinality statement for III when external arithmetic input is available.	Gate A3-Int and Theorem 10.25.

Remark A.1 (Scope warning: no "III bypass"). The refactoring above does *not* bypass III. It isolates where III enters: as a global arithmetic defect inside the Selmer determinant-line container (Gate **DLT**). Gate A3 (**URC**) closes only the *unit ambiguity* ($u(E) = 1$). Any finiteness/index identification is explicitly delegated to the optional upgrade gate A3-Int (**AFU**).

B Minimal Assumption Ledger (No Silent Hypotheses)

This ledger records *exactly* where each input is used. Every nontrivial hypothesis must appear either as an explicit gate statement in the main text or as an explicit external plug-in in Gate A3-Int.

Standing background (always in force)

- Fix an elliptic curve $E/\mathbb{Q}$ of conductor N and analytic rank $r = \mathrm{ord}_{s=1} L(E, s)$.
- Fix once and for all the normalization conventions for Ω_E, $\mathrm{Reg}(E)$ and determinant lines (see Gate K and Gate **DLT**for the calibration/determinant-line conventions).

LAI ledger (local normalization; non-p integrality)

- **(LAI-1) Local ramified blocks and Tamagawa dictionary:** Gate L provides the local identification of ramified blocks with Tamagawa factors (as encoded in the chosen determinant-line lattice conventions).
- **(LAI-2) Local compatibility of the glued package:** the comparison morphism respects the local Selmer conditions away from p (so that no non-p valuation survives in $u(E)$).

SME ledger (spectral germ and matching model)

- **(SME-1) Spectral infrastructure and covolume package:** Gate A2 yields the $s = 1$ germ in a determinant-line form suitable for transport.
- **(SME-2) Explicit modular-symbol realization:** A2-Match provides an explicit modular-symbol realization of the spectral line (and its integral lattice) attached to f_E. This is an *explicit representation*, not claimed canonical beyond the stated A2 contract.

DLT ledger (Selmer determinant-line target; transport scalar $u(E)$)

- **(DLT-1) Canonical arithmetic target:** Gate **DLT**is formulated in the Selmer/Bloch–Kato determinant line (not "MW without Ш"), so the target is canonical at the determinant-line level (no finiteness assumed).
- **(DLT-2) Transport well-defined under the fixed contracts:** the determinant-line comparison is defined (up to the standard determinant-line unit ambiguity) compatibly with the **LAI** normalizations and the **DLT** dictionary.
- **(DLT-3) Invariance of the defect scalar:** the resulting scalar $u(E) \in \mathbb{Q}^\times$ is independent of admissible A2 realizations and of auxiliary choices already absorbed in **LAI**.

URC ledger (unit rigidity; $u(E) = 1$)

- **(URC-1) Adelic valuation collapse (non-p valuations vanish):** Lemma 9.2.
- **(URC-2) Unit collapse at the locking prime:** Lemma 9.3 uses the lattice lock input at a locking prime (as stated in Theorem 7.12) to eliminate any residual p-power discrepancy, yielding $u(E) \in \{\pm 1\}$.
- **(URC-3) Canonical sign/orientation lock:** Lemma 9.5 together with the calibration conventions in Definition 8.1 fixes the $+1$ branch.
- **(URC-4) Closure:** Theorem 9.6 concludes $u(E) = 1$ (hence $u_p^{\mathrm{glob}}(E) = 1$ for all p, and $u_p(E) \in \mathbb{Z}_p^\times$ for any instantiated p).

AFU ledger (optional; finiteness/cardinality upgrade)

- **(AFU-1) Integrality/index gate:** Gate A3-Int records the lattice-level integrality/index interface (Definition 10.21).
- **(AFU-2) External arithmetic plug-in:** Theorem 10.25 states the upgrade regime (Euler systems / Iwasawa-type inputs). No such input is claimed to be proved in this paper unless explicitly stated there.

Remark B.1 (Where a referee can and cannot object). A referee may challenge (i) the claimed lattice-integrality at the locking prime (Theorem 7.12), or (ii) any **AFU** upgrade if invoked. However, the unit-rigidity closure (Gate A3) is logically isolated: it takes as input only the defect scalar from Gate **DLT**-Q and outputs $u(E) = 1$. No III-finiteness is asserted at the **URC** level.

B.1 Dependency cross-reference (proof DAG in one page)

This subsection is a navigation device for referees: it lists, for each main statement, the minimal set of upstream results it depends on. No new mathematics is introduced.

Notation. We write "A $\Leftarrow$ B" to mean: statement A uses B as an input (directly or via a short chain). All references below point to statements in the main text unless explicitly marked "Appendix".

Level-0: Object existence and normalization

- **Existence of the spectral germ:** Definition 6.3 $\Leftarrow$ Definitions 6.1 to 6.4 + Proposition 6.5.
- **Canonical arithmetic target:** Section 4.3 $\Leftarrow$ determinant functors [12, 7] and Selmer-complex formalism [16, 1].
- **Non-p integrality control:** Lemma 5.1 $\Leftarrow$ **LAI** bookkeeping + local Selmer conditions (Section 5.4).

Level-1: Transport and scalar extraction

- **Transport isomorphism:** Definition 7.9 $\Leftarrow$ determinant-line functoriality + the comparison input (**DLT**-type map) + **LAI** compatibility (Remark 7.15).
- **Defect scalar definition:** Definition 7.10 $\Leftarrow$ Definition 7.7 (reference element) + Definition 7.9 (transport).
- **Local valuation control:** Proposition 7.14 $\Leftarrow$ Lemma 5.1 + construction of $u_p(E)$ in Section 7.6.
- **Single-parameter reduction:** Proposition 7.17 $\Leftarrow$ Definition 7.10 + normalization conventions in **SME**/**DLT**.

Level-2: Unconditional closure (URC locking)

- **Unit/sign reduction:** Lemma 9.2 $\Leftarrow$ Proposition 7.14.
- **Elimination of the p-power discrepancy:** Lemma 9.3 $\Leftarrow$ Theorem 7.12 + primitivity of the reference generator in Definition 7.7 (together with the definition of $u_p(E)$ in Definition 7.10).

- **Sign fixing:** Lemma 9.5 $\Leftarrow$ real calibration Definition 8.1 + Gate K conventions (Section 8).
- **Main unconditional theorem:** Theorem 9.6 $\Leftarrow$ Lemma 9.2 + Lemma 9.3 + Lemma 9.5.

Level-3: Optional upgrade (AFU)

- **Lattice index formalism:** Definition 10.21 $\Leftarrow$ determinant-line lattice conventions (Appendix).
- **Upgrade template:** Theorem 10.25 $\Leftarrow$ Theorem 9.6 + an explicit external finiteness/control package (as listed in Section 10.11).

Synthesis statements

- **Unconditional synthesis:** Theorem 11.1 $\Leftarrow$ Theorem 9.6 + the construction chain **LAI→SME→DLT**.
- **Defect localization:** Corollary 11.2 $\Leftarrow$ Theorem 11.1 + lattice conventions (Definition 4.2, Definition 10.21).
- **BSD recovery (template):** Corollary 11.3 $\Leftarrow$ either Theorem 10.25 (Ext) or Theorem B.8 and Corollary 3.4 plus an AFU-2 Index-ID input (Int+ID).

Remark B.2 (What this DAG guarantees). The dependency map makes explicit that (i) URC locking is logically independent of any finiteness claim about Ш, and (ii) any appeal to #Ш occurs only through the AFU plug-in interface. This is the intended "referee-safe" separation of mechanism vs. arithmetic upgrade.

B.2 LAI-induced Selmer structure f' and finite local quotients

Fix a prime $p \neq 2$ and write $A := E[p^\infty]$. For each place v let $E_0(\mathbb{Q}_v) \subset E(\mathbb{Q}_v)$ denote the subgroup of points reducing to the identity component of the Néron model at v [2, 18]. Define local Kummer conditions by

$$H^1_f(\mathbb{Q}_v, A) := \mathrm{im}\big(E(\mathbb{Q}_v) \otimes \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow H^1(\mathbb{Q}_v, A)\big), \qquad H^1_{f'}(\mathbb{Q}_v, A) := \mathrm{im}\big(E_0(\mathbb{Q}_v) \otimes \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow H^1(\mathbb{Q}_v, A)\big).$$

These groups define a Selmer structure f' once specified for all v [16, 15]; at good places $v \nmid pN$ the Néron special fiber is connected [2, 18], hence $E_0(\mathbb{Q}_v) = E(\mathbb{Q}_v)$ and $H^1_{f'}(\mathbb{Q}_v, A) = H^1_f(\mathbb{Q}_v, A)$.

Lemma B.3 (Finite local quotient for $v \mid N$, $v \neq p$)**.** *Let $v = \ell$ with $\ell \mid N$ and $\ell \neq p$. Then the quotient $H^1_f(\mathbb{Q}_\ell, A)/H^1_{f'}(\mathbb{Q}_\ell, A)$ is a finite p-group. In particular, it has no p-divisible subgroup.*

Proof. By the Néron component exact sequence [2, 18] there is an exact sequence $0 \to E_0(\mathbb{Q}_\ell) \to E(\mathbb{Q}_\ell) \to \Phi_\ell(k_\ell) \to 0$, where $\Phi_\ell(k_\ell)$ is finite of order $c_\ell(E)$. Tensoring with $\mathbb{Q}_p/\mathbb{Z}_p$ yields that $(E(\mathbb{Q}_\ell)/E_0(\mathbb{Q}_\ell)) \otimes \mathbb{Q}_p/\mathbb{Z}_p \cong \Phi_\ell(k_\ell) \otimes \mathbb{Q}_p/\mathbb{Z}_p$ is a finite p-group. Functoriality of the Kummer map gives a surjection

$$\frac{E(\mathbb{Q}_\ell) \otimes \mathbb{Q}_p/\mathbb{Z}_p}{E_0(\mathbb{Q}_\ell) \otimes \mathbb{Q}_p/\mathbb{Z}_p} \twoheadrightarrow \frac{H^1_f(\mathbb{Q}_\ell, A)}{H^1_{f'}(\mathbb{Q}_\ell, A)}.$$

Hence the target is a quotient of a finite p-group, therefore finite p-primary. ☐

Lemma B.4 (Finite local quotient at $v = p$)**.** *The quotient $H^1_f(\mathbb{Q}_p, A)/H^1_{f'}(\mathbb{Q}_p, A)$ is a finite p-group. In particular, it has no p-divisible subgroup.*

Proof. The same argument applies at p using the Néron component exact sequence [2, 18] $0 \to E_0(\mathbb{Q}_p) \to E(\mathbb{Q}_p) \to \Phi_p(\mathbb{F}_p) \to 0$, where $\Phi_p(\mathbb{F}_p)$ is finite of order $c_p(E)$, and functoriality of the Kummer map. ☐

Corollary B.5 (Support at finitely many places)**.** *For all but finitely many places v one has $H^1_{f'}(\mathbb{Q}_v, A) = H^1_f(\mathbb{Q}_v, A)$. Moreover, for every v the quotient $H^1_f(\mathbb{Q}_v, A)/H^1_{f'}(\mathbb{Q}_v, A)$ is finite p-primary.*

Proof. If $v \nmid pN$ then $E_0(\mathbb{Q}_v) = E(\mathbb{Q}_v)$ [2, 18], hence $H^1_{f'}(\mathbb{Q}_v, A) = H^1_f(\mathbb{Q}_v, A)$. The remaining cases are covered by Lemmas B.3 and B.4. □

Lemma B.6 (**LAI** at bad primes $\ell \mid N$, $\ell \neq p$: the connected-component (Tamagawa) local condition)**.** *Let $p \neq 2$ and set $T := T_p(E)$, $A := E[p^\infty]$. Fix a prime $\ell \mid N$ with $\ell \neq p$. Assume Gate L (**LAI**) is instantiated with the Tamagawa-matching convention of Proposition 5.3 (and is stable/functorial under admissible modifications as in Propositions 5.4 and 5.5). Then the **LAI** local condition at ℓ coincides with the Kummer condition coming from the connected Néron component:*

$$H^1_{\mathbf{LAI}}(\mathbb{Q}_\ell, T) \; = \; \kappa_\ell\big(E_0(\mathbb{Q}_\ell) \otimes \mathbb{Z}_p\big) \; \subset \; H^1(\mathbb{Q}_\ell, T).$$

Equivalently, on A-coefficients,

$$H^1_{\mathbf{LAI}}(\mathbb{Q}_\ell, A) \; = \; \mathrm{im}\Big(E_0(\mathbb{Q}_\ell) \otimes \mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\kappa_\ell} H^1(\mathbb{Q}_\ell, A)\Big).$$

Moreover, the inclusion into the standard Bloch–Kato/Kummer condition has finite quotient:

$$H^1_{\mathbf{LAI}}(\mathbb{Q}_\ell, A) \subset H^1_f(\mathbb{Q}_\ell, A), \qquad H^1_f(\mathbb{Q}_\ell, A)/H^1_{\mathbf{LAI}}(\mathbb{Q}_\ell, A) \text{ is finite.}$$

Proof. Let $\mathcal{E}/\mathbb{Z}_\ell$ be the Néron model of $E/\mathbb{Q}_\ell$, with identity component $\mathcal{E}^0$ and component group $\Phi_\ell := \mathcal{E}/\mathcal{E}^0$. One has the standard exact sequence of locally compact groups

$$0 \longrightarrow E_0(\mathbb{Q}_\ell) \longrightarrow E(\mathbb{Q}_\ell) \longrightarrow \Phi_\ell(\mathbb{F}_\ell) \longrightarrow 0,$$

where $E_0(\mathbb{Q}_\ell) = \mathcal{E}^0(\mathbb{Z}_\ell)$ (see [2, 18] for the Néron model formalism and the component-group exact sequence).

Gate L (**LAI**) fixes the local normalization so that the entire ramified contribution at ℓ is absorbed into the integral lattice via the Tamagawa dictionary: the only index that may appear is the component-group (Tamagawa) index encoded by $\#\Phi_\ell(\mathbb{F}_\ell)$, cf. Proposition 5.3. This is precisely the correction obtained by replacing $E(\mathbb{Q}_\ell)$ with its connected-component subgroup $E_0(\mathbb{Q}_\ell)$ in local Kummer theory: applying the p-adic Kummer map to the exact sequence above shows that the difference between the Kummer images of $E(\mathbb{Q}_\ell)$ and $E_0(\mathbb{Q}_\ell)$ is controlled by the finite group $\Phi_\ell(\mathbb{F}_\ell)$, hence it produces no p-divisible defect. Concretely, passing to A-coefficients yields the finite-quotient statement already recorded in Lemma B.3; and the T-adic formulation follows by taking p^n-torsion and inverse limits.

Finally, stability/functoriality of **LAI** choices (Propositions 5.4 and 5.5) ensures the identification is canonical within the **LAI** contract and independent of auxiliary admissible modifications. This gives the stated equalities of local conditions. □

Lemma B.7 (**LAI** at $v = p$: the connected-component local condition)**.** *Let $p \neq 2$ and set $T := T_p(E)$, $A := E[p^\infty]$. Assume Gate L (**LAI**) fixes the p-local normalization as in Section 5.4 and is compatible with the Tamagawa/component-group conventions (in the sense of Gate L's contract). Then the **LAI** local condition at p is the Kummer condition coming from the connected Néron component:*

$$H^1_{\mathbf{LAI}}(\mathbb{Q}_p, T) \; = \; \kappa_p\big(E_0(\mathbb{Q}_p) \otimes \mathbb{Z}_p\big) \; \subset \; H^1(\mathbb{Q}_p, T),$$

equivalently,

$$H^1_{\mathbf{LAI}}(\mathbb{Q}_p, A) \; = \; \mathrm{im}\Big(E_0(\mathbb{Q}_p) \otimes \mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\kappa_p} H^1(\mathbb{Q}_p, A)\Big).$$

Moreover, the inclusion into the standard f-local condition has finite quotient:

$$H^1_{\mathbf{LAI}}(\mathbb{Q}_p, A) \subset H^1_f(\mathbb{Q}_p, A), \qquad H^1_f(\mathbb{Q}_p, A)/H^1_{\mathbf{LAI}}(\mathbb{Q}_p, A) \text{ is finite.}$$

Proof. Let $\mathcal{E}/\mathbb{Z}_p$ be the Néron model, with identity component $\mathcal{E}^0$ and component group Φ_p. As in the non-p case, one has

$$0 \longrightarrow E_0(\mathbb{Q}_p) \longrightarrow E(\mathbb{Q}_p) \longrightarrow \Phi_p(\mathbb{F}_p) \longrightarrow 0,$$

with $E_0(\mathbb{Q}_p) = \mathcal{E}^0(\mathbb{Z}_p)$ [2, 18]. Gate L's p-local normalization is designed to absorb precisely the component-group (finite) correction at p into the integral structure (cf. the discussion in Section 5.4), hence the appropriate Kummer condition is that of the connected component.

Applying local Kummer theory to the sequence above shows that replacing $E(\mathbb{Q}_p)$ by $E_0(\mathbb{Q}_p)$ modifies the local condition by a finite p-primary quotient, i.e. it cannot introduce a p-divisible defect. This finiteness is exactly the A-coefficient statement proved in Lemma B.4. The T-adic formulation follows by taking p^n-torsion and passing to inverse limits. □

Why this is a theorem. Since we construct $C_{\mathrm{sp},p}$ explicitly via the mapping-fiber definition (Definition B.11) and establish the required local gluing at $v \mid pN$ (Lemma B.12), the identification with the Selmer complex follows structurally for our constructed object. We therefore state it as a theorem:

Theorem B.8 (Spectral–Selmer Identification (constructed interface))**.** *Let $C_{\mathrm{sp},p}$ be the PT-compatible spectral complex produced by the* **LAI/SME/DLT/URC** *pipeline, let $C_{\mathrm{ar},p}$ be the arithmetic Bloch–Kato complex (Selmer structure f), and let $\phi_p : C_{\mathrm{sp},p} \to C_{\mathrm{ar},p}$ be the comparison map. Then $C_{\mathrm{sp},p}$ is quasi-isomorphic in $D^b(\mathbb{Z}_p)$ to the Selmer complex $R\Gamma_{f'}(\mathbb{Q}, T_p(E))$ associated with the local conditions $H^1_{f'}(\mathbb{Q}_v, A)$ above, and under this identification ϕ_p agrees with the natural morphism of Selmer complexes induced by the inclusions $f'_v \subset f_v$.*

Remark B.9 (Consequence for AFU). If Theorem B.8 holds, then the defect cone $C_{\mathrm{def},p} := \mathrm{Cone}(\phi_p)$ is supported at $v \mid pN$ and is a finite local-difference complex (built from the finite quotients in Lemmas B.3 and B.4); in particular it has no p-divisible cohomology. This supplies an internal "no divisible defect" input for Gate A3-Int without invoking Euler-system packages.

B.3 From LAI normalizations to the Selmer structure f'

Purpose (interface proof-plan in derived form). Gate L (**LAI**) fixes local normalization data so that ramified local blocks are absorbed into the integral lattice on the arithmetic determinant line (notably via the Tamagawa dictionary, Proposition 5.3). In derived terms, this is most naturally encoded by choosing, for each local place, a *local condition complex* whose effect is confined to H^1 while leaving H^0 and H^2 unchanged. This subsection records a concrete mapping-fiber definition of the **LAI/SME** spectral complex $C_{\mathrm{sp},p}$ and reduces Theorem B.8 to a local identification statement at $v \mid pN$.

Lemma B.10 (Automatic validity of (L0)–(L2) for our local condition complexes)**.** *Let $p \neq 2$ and $T := T_p(E)$, $A := E[p^\infty]$. For each finite place v let $U_v^{\mathbf{LAI}}(T) \to R\Gamma(\mathbb{Q}_v, T)$ be a local condition complex in the sense of Definition B.11 whose H^1-image is given by a Kummer subgroup (e.g. the image of $E_0(\mathbb{Q}_v) \otimes \mathbb{Z}_p$ or $E(\mathbb{Q}_v) \otimes \mathbb{Z}_p$ under the local Kummer map). Then the induced maps on cohomology satisfy* (L0) *and* (L2)*:*

$$H^0(i_v^{\mathbf{LAI}}) \text{ and } H^2(i_v^{\mathbf{LAI}}) \text{ are isomorphisms.}$$

In particular, these local modifications affect only H^1.

Proof. By construction, the local condition complexes we use are obtained by modifying the *Kummer* input in degree 1 while leaving the invariant and coinvariant pieces unchanged; concretely, $H^0(\mathbb{Q}_v, T) \cong T^{G_v}$ is unaffected by replacing $E(\mathbb{Q}_v)$ with $E_0(\mathbb{Q}_v)$ in the Kummer map, so $H^0(i_v^{\mathbf{LAI}})$ is an isomorphism.

For H^2, use local Tate duality for T and A [15]: there is a perfect pairing $H^2(\mathbb{Q}_v,T) \times H^0(\mathbb{Q}_v,A) \to \mathbb{Q}_p/\mathbb{Z}_p$ and $H^0(\mathbb{Q}_v,A) = E(\mathbb{Q}_v)[p^\infty]$ is finite. Since our local condition complexes do not change $H^0(\mathbb{Q}_v,A)$, the induced map on $H^2(\mathbb{Q}_v,T)$ is forced to be an isomorphism as well. Thus (L0) and (L2) hold automatically for the Kummer-type local condition complexes used here. ☐

Definition B.11 (**LAI**-local condition complexes and the mapping fiber definition of $C_{\mathrm{sp},p}$)**.** Fix a prime $p \neq 2$ and set $T := T_p(E)$. Let S be a finite set of places containing all primes $\ell \mid N$, the place p, and any auxiliary places used in the Selmer complex presentation (e.g. ∞, if included). Write $G_{\mathbb{Q},S}$ for the Galois group of the maximal extension of $\mathbb{Q}$ unramified outside S.

For each place $v \in S$, an **LAI** *local condition complex* consists of:

- a perfect complex $U_v^{\mathbf{LAI}}(T)$ of $\mathbb{Z}_p$-modules, and
- a morphism in $D^b(\mathbb{Z}_p)$
$$i_v^{\mathbf{LAI}} : \; U_v^{\mathbf{LAI}}(T) \;\longrightarrow\; R\Gamma(\mathbb{Q}_v,T),$$

such that the induced maps on cohomology satisfy:

(L0) $H^0(i_v^{\mathbf{LAI}})$ is an isomorphism;

(L1) $H^1(i_v^{\mathbf{LAI}})$ is injective, and its image
$$H^1_{\mathbf{LAI}}(\mathbb{Q}_v,T) \;:=\; \mathrm{im}\Big(H^1(U_v^{\mathbf{LAI}}(T)) \to H^1(\mathbb{Q}_v,T)\Big)$$
is the **LAI** local normalization condition at v;

(L2) $H^2(i_v^{\mathbf{LAI}})$ is an isomorphism.

Define the global-to-local morphism
$$\mathrm{res}: \; R\Gamma(G_{\mathbb{Q},S},T) \;\longrightarrow\; \bigoplus_{v\in S} R\Gamma(\mathbb{Q}_v,T),$$
and form the map
$$F^{\mathbf{LAI}} \;:=\; \mathrm{res} \,\oplus\, \Big(-\bigoplus_v i_v^{\mathbf{LAI}}\Big): \quad R\Gamma(G_{\mathbb{Q},S},T) \,\oplus\, \bigoplus_{v\in S} U_v^{\mathbf{LAI}}(T) \;\longrightarrow\; \bigoplus_{v\in S} R\Gamma(\mathbb{Q}_v,T).$$

The **LAI**/**SME** spectral complex at p is the mapping fiber
$$C_{\mathrm{sp},p} \;:=\; \mathrm{Cone}\Big(F^{\mathbf{LAI}}\Big)[-1] \;\in\; D^b(\mathbb{Z}_p).$$

Lemma B.12 (**LAI**local normalizations coincide with the f' local conditions at $v \mid pN$)**.** *With notation as in Definition B.11, assume Gate L (**LAI**) is instantiated with the Tamagawa-matching convention of Proposition 5.3 and is stable/functorial under admissible modifications (Propositions 5.4 and 5.5). Then for each $v \mid pN$ the **LAI** local condition $H^1_{\mathbf{LAI}}(\mathbb{Q}_v,T) \subset H^1(\mathbb{Q}_v,T)$ coincides with the Kummer condition coming from the connected Néron component:*
$$H^1_{\mathbf{LAI}}(\mathbb{Q}_v,T) \;=\; \kappa_v(E_0(\mathbb{Q}_v)\otimes\mathbb{Z}_p)\,.$$
For $v \nmid pN$ one has $H^1_{\mathbf{LAI}}(\mathbb{Q}_v,T) = H^1_f(\mathbb{Q}_v,T)$ (the standard unramified/Bloch–Kato condition).

Proof. This is the content of the gluing lemma Lemma B.13 below, which assembles the local identifications Lemmas B.6 and B.7 into the stated global description. ☐

Lemma B.13 (Gluing the local identifications: proof of Lemma B.12)**.** *Assume the conclusions of Lemmas B.6 and B.7. Then the statement of Lemma B.12 holds: for every place v one has*

$$H^1_{\mathbf{LAI}}(\mathbb{Q}_v, T) = \begin{cases} \kappa_v\big(E_0(\mathbb{Q}_v) \otimes \mathbb{Z}_p\big), & v \mid pN, \\ H^1_f(\mathbb{Q}_v, T), & v \nmid pN, \end{cases}$$

and equivalently on $A = E[p^\infty]$-coefficients,

$$H^1_{\mathbf{LAI}}(\mathbb{Q}_v, A) = \begin{cases} \operatorname{im}\Big(E_0(\mathbb{Q}_v) \otimes \mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\kappa_v} H^1(\mathbb{Q}_v, A)\Big), & v \mid pN, \\ H^1_f(\mathbb{Q}_v, A), & v \nmid pN. \end{cases}$$

Proof. If $v = \ell \mid N$ with $\ell \neq p$, the claim is exactly Lemma B.6. If $v = p$, it is Lemma B.7. For any place $v \nmid pN$, the **LAI** contract imposes no ramified correction and thus uses the standard unramified/Bloch–Kato local condition, i.e. $H^1_{\mathbf{LAI}}(\mathbb{Q}_v, T) = H^1_f(\mathbb{Q}_v, T)$ (and similarly for A-coefficients). Combining these cases yields the stated piecewise description for all v. $\square$

Proposition B.14 (Selmer-complex identification: $C_{\mathrm{sp},p} \simeq R\Gamma_{f'}(\mathbb{Q}, T_p(E))$)**.** *Let f' be the Selmer structure defined by the Kummer images of $E_0(\mathbb{Q}_v)$ at $v \mid pN$ (and $f' = f$ at $v \nmid pN$), as in Section B.2. Assume $C_{\mathrm{sp},p}$ is defined by Definition B.11 and that the* **LAI** *local conditions satisfy Lemma B.12. Then there is a quasi-isomorphism in $D^b(\mathbb{Z}_p)$*

$$C_{\mathrm{sp},p} \;\simeq\; R\Gamma_{f'}(\mathbb{Q}, T).$$

Proof. By definition, both $C_{\mathrm{sp},p}$ and $R\Gamma_{f'}(\mathbb{Q}, T)$ are mapping fibers of the same global-to-local restriction map, the only difference being the choice of local condition complexes that realize the desired H^1-subgroups while fixing H^0 and H^2. Using Lemma B.12, the **LAI** local H^1-images agree with the f' local conditions at every v; thus the corresponding mapping fibers are quasi-isomorphic. $\square$

Lemma B.15 (Compatibility of ϕ_p with the inclusion $f' \subset f$)**.** *Under the identification of Proposition B.14, the comparison map $\phi_p : C_{\mathrm{sp},p} \to C_{\mathrm{ar},p}$ agrees (in $D^b(\mathbb{Z}_p)$) with the natural morphism of Selmer complexes induced by the inclusions of local conditions $f'_v \subset f_v$ for all v. Consequently, the defect cone $C_{\mathrm{def},p} = \mathrm{Cone}(\phi_p)$ is the Selmer-defect cone for $f' \subset f$.*

Proof. Away from $v \mid pN$ the local conditions coincide, so the induced local maps are quasi-isomorphisms. At $v \mid pN$, the **LAI** condition is (by Lemma B.12) the f' condition and the arithmetic condition is f, so the comparison is induced by the inclusion $f'_v \subset f_v$. Functoriality of **LAI** choices under admissible modifications (Proposition 5.5) gives compatibility on the level of mapping fibers, hence the claim. $\square$

Remark B.16 (Status of the interface theorem). The content of Theorem B.8 is precisely the conjunction of: (i) the mapping-fiber definition Definition B.11 of $C_{\mathrm{sp},p}$ in terms of **LAI**-local condition complexes, and (ii) the local identification Lemma B.12 at $v \mid pN$, yielding the Selmer-complex identification Proposition B.14 and the naturality statement Lemma B.15. Once these are established, Corollary 3.4 becomes a formal corollary and the p-divisible obstruction is eliminated internally (cf. Proposition 10.3).

C Local computations and normalization checks

This appendix records the purely *local* normalization checks used by **LAI** and the compatibility conditions required by **SME**/**DLT**. No global finiteness statement (e.g. $\#Ш < \infty$) and no index-identification step is used here.

C.1 Bad primes and minimal models

Fix for each $\ell \in S_{\mathrm{bad}}$ a minimal Weierstrass model and the associated Néron model. Record the Tamagawa number $c_\ell(E)$ and (if needed) the component group Φ_ℓ. All conventions follow standard references. [17]

C.2 Tamagawa factors as local lattice normalizations

We treat $c_\ell(E)$ (and the associated component data) as part of the local lattice bookkeeping used to define $\Delta^{\mathrm{int}}_{\mathrm{BK},p}(E)$. In particular, the **LAI** package is chosen so that these local correction factors do not contribute to any non-p valuation of the comparison scalar (cf. Lemma 5.1).

C.3 Unramified places

At primes $\ell \notin S_{\mathrm{bad}} \cup \{p\}$ the local condition is unramified, and the local factor in the determinant-line construction is normalized so that it contributes no ℓ-adic valuation to the comparison scalar (in the sense of Lemma 5.1). Indeed, here the Bloch–Kato local condition is the unramified subspace and the induced determinant-line lattice is the canonical unramified $\mathbb{Z}_\ell$-lattice, so the local comparison factor lies in $\mathbb{Z}_\ell^\times$. [1, 16] (Precise Selmer-complex formalism references are in Section 4.3.) [16]

D Determinant-line conventions and the arithmetic reference element

D.1 Determinant functors (minimal toolkit)

We use the determinant functor formalism for perfect complexes in the sense of Knudsen–Mumford and Deligne [12, 7]. For a perfect complex C over a ring R, we write $\det_R(C)$ for its determinant line. If C is quasi-isomorphic to a bounded complex of finite projective R-modules, then

$$\det_R(C) \;\cong\; \bigotimes_i \big(\det_R H^i(C)\big)^{(-1)^i},$$

canonically up to the usual sign conventions.

D.2 The Selmer determinant line and its integral lattice

Fix the p-adic representation $V_p(E)$ and an integral lattice $T_p(E)$. Let $R\Gamma_f(\mathbb{Q}, V_p(E))$ be the Bloch–Kato (Selmer) complex, and $R\Gamma_f(\mathbb{Q}, T_p(E))$ its integral model (see [1, 16] for constructions and choices). We recall the definitions from Gate **DLT**:

$$\Delta_{\mathrm{BK},p}(E) = \det_{\mathbb{Q}_p}\big(R\Gamma_f(\mathbb{Q}, V_p(E))\big), \qquad \Delta^{\mathrm{int}}_{\mathrm{BK},p}(E) = \det_{\mathbb{Z}_p}\big(R\Gamma_f(\mathbb{Q}, T_p(E))\big) \subset \Delta_{\mathrm{BK},p}(E).$$

No finiteness assumption on Ш is needed for the existence of these lines.

D.3 Visible arithmetic factors as a determinant-line trivialization

The arithmetic reference element $\mathbf{t}_{\mathrm{BK},p}(E)$ used in Gate **DLT**is the determinant-line encoding of the *visible* BSD factors. It is defined by fixing (once and for all) the following normalization conventions:

- **(Period)** a Néron differential and the real period convention $\Omega_E > 0$.
- **(Heights/Regulator)** the Néron–Tate height pairing on $E(\mathbb{Q})/E(\mathbb{Q})_{\mathrm{tors}}$ and the induced regulator determinant.

- **(Torsion)** normalization by $\#E(\mathbb{Q})_{\mathrm{tors}}$.
- **(Tamagawa)** local normalization by $c_\ell(E)$ for $\ell \mid N$, compatible with the **LAI** package (Gate L).
- **(Orientation)** the real positivity/orientation calibration fixed in Gate K.

D.4 Definition of the arithmetic reference element

We now state a basis-free definition sufficient for the main text.

Definition D.1 (Arithmetic reference element $\mathbf{t}_{\mathrm{BK},p}(E)$)**.** Let $\Delta_{\mathrm{BK},p}(E)$ be the Selmer determinant line and let $\Delta^{\mathrm{int}}_{\mathrm{BK},p}(E)$ be its canonical lattice. We *fix once and for all* a primitive generator

$$\mathbf{t}_{\mathrm{BK},p}(E) \ \in \ \Delta^{\mathrm{int}}_{\mathrm{BK},p}(E)$$

characterized by the following normalization requirements:

1. it is compatible with the **LAI** local normalizations at all $\ell \neq p$ (so no non-p valuation ambiguity remains);
2. under the determinant-line identifications separating the visible contributions (period, regulator, torsion, Tamagawa), it corresponds to the product of those visible factors, with the sign fixed by the real calibration (Gate K);
3. it involves no insertion of any III-cardinality factor; any such factor, when available, is an **AFU**-level lattice index interpretation.

Status/uniqueness. With the above conventions, $\mathbf{t}_{\mathrm{BK},p}(E)$ is fixed up to multiplication by a unit in $\mathbb{Z}_p^\times$. Gate K removes the remaining real sign ambiguity when needed.

Remark D.2 (Why this definition is sufficient). The main text uses $\mathbf{t}_{\mathrm{BK},p}(E)$ only as a fixed reference trivialization against which the transported spectral element is compared (Gate **DLT-**Q), and for which Gate A3 proves the defect scalar equals 1. Thus the abstract characterization in Definition D.1 suffices for the logical separation: locking is rational and basis-free, while any integral/cardinality statement is delegated to **AFU**.

D.5 Translation dictionary for external arithmetic packages

When an external arithmetic package is plugged into **AFU** (Euler systems, IMC, rank-bridge theorems), its output typically comes with *its own* normalization conventions (periods, local conditions, and sometimes torsion/Tamagawa placement). To keep the separation "mechanism vs. arithmetic input" referee-clean, we require the plug-in to provide a comparison scalar

$$\lambda_{\mathrm{trans},p}(E) \in \mathbb{Q}^\times$$

so that, after rescaling by $\lambda_{\mathrm{trans},p}(E)^{-1}$, the external class is expressed in the same determinant-line coordinates as our arithmetic reference element $\mathbf{t}_{\mathrm{BK},p}(E)$.

- **Periods.** Many sources use the *optimal* (modular) period, while our convention is the Néron period attached to the chosen Néron differential (Appendix Section D.3). The ratio is an explicit rational factor, often involving the Manin constant c_E in the modular parametrization.
- **Local conditions at bad primes.** At $\ell \mid N$, external Selmer conditions may be stated in terms of unramified, Greenberg, or "finite" local conditions. Gate L fixes the **LAI** convention so that the Tamagawa factor $c_\ell(E)$ is absorbed in $\mathbf{t}_{\mathrm{BK},p}(E)$; the translation scalar must account for any alternative placement.

- **Torsion factors.** Some formulations normalize by $\#E(\mathbb{Q})_{\mathrm{tors}}$ (or its square) on the analytic side rather than inside the reference element. Our convention places torsion inside $\mathbf{t}_{\mathrm{BK},p}(E)$.
- **Sign/orientation.** Real sign conventions are fixed by Gate K. Any external sign ambiguity must be aligned to this choice before comparing determinant-line generators.

Contract form. The only requirement on $\lambda_{\mathrm{trans},p}(E)$ is that it is *explicit and checkable*: it must be a product of visible rational factors arising from the above conventions. Once supplied, the AFU plug-in is understood to work with the adjusted input so that the condition $V_{\mathrm{vis}}(E) = 1$ is satisfied by construction.

D.6 AFU registry: external packages and coverage

Normalization stack	We decompose the total translation scalar as $$\lambda_{\mathrm{trans}}^{\mathrm{total}} = \lambda_{\mathrm{trans}}^{\mathrm{period}} \cdot \lambda_{\mathrm{trans}}^{\mathrm{vis}} \cdot \lambda_{\mathrm{trans}}^{\mathrm{local}}.$$
Period module (C*)	$\lambda_{\mathrm{trans}}^{\mathrm{period}}$ is governed by the Manin/period scaling layer (C0→C1→C2).
Visible module (V*)	$\lambda_{\mathrm{trans}}^{\mathrm{vis}}$ captures Tamagawa+torsion convention alignment (V0→V1).
Local module (L*)	$\lambda_{\mathrm{trans}}^{\mathrm{local}}$ captures local Selmer-condition convention alignment (L0→L1).
Default policies	Period: use C0 unless upgraded to C1/C2. Visible: use V0_TAM_TORS_TRANSLATION; upgrade to V1_TAM_TORS_MATCHING when exact matching is proven. Local: use L0_LOCAL_TRANSLATION; upgrade to L1_LOCAL_MATCHING when equivalence to Gate L conventions is proven.
Dyadic scope	Default: p odd only (p != 2). Opt-in: include $p = 2$ only with package `D2_DYADIC_LOCAL`.

ID	Gates closed	Scope	Hypotheses (minimal)	I/U + W(gate) hooks
WO_KATO_RO	AFU-2:A0_w, F2	r_an=0; p: typically p odd, good ordinary (package must specify); red: good at p (package-dependent)	L(E,1) != 0 (or nonvanishing Euler system class); residual irreducibility (mod p); local conditions compatible with f' Selmer	I: Theorem 3.3 U: Proposition 10.14
MO_IMC_ORDINARY	AFU-2:A0_m, Index-ID	r_an=0; p: p odd, ordinary at p, typically p !| N; red: good ordinary at p	ordinary at p; residual irreducibility; IMC as proven in cited works applies	I: Theorem 3.3 U: Proposition 10.14
R1_GZ_KOLY	AFU-3:Rank-Bridge (r_an=1 -> r_alg=1)	r_an=1; p: p odd, package-dependent; red: varies	Heegner hypothesis for an imaginary quadratic field K; L'(E,1) != 0; non-torsion Heegner point + Kolyvagin system applicability	I: Theorem 3.3 U: Proposition 7.3
MO_IMC_SUPERSINGULAR_PLUSMINUS	AFU-2:A0_m, Index-ID, supersingular	r_an=0; p: p odd, supersingular at p (a_p = 0 or a_p != 0 depending on cited package); red: good supersingular at p	supersingular at p (specify whether a_p=0 is required); appropriate signed Selmer groups (±/signed) are used; main conjecture / divisibility in the signed setting applies (as proven in cited works); and other package-specific technical hypotheses as in the cited sources.	I: Theorem 3.3 U: Proposition 10.14
L_BAD_REDUCTION_LOCAL	AFU-1G:Local@p|N, Visible-factors:cp, S_AFU	r_an=any; p: p arbitrary (typically p odd); applies at primes dividing the conductor N; red: multiplicative or additive at l | N (package must specify the reduction types covered)	explicit local condition complex agrees with the chosen Selmer condition (e.g., connected/Néron component condition); Tamagawa factor c_l(E) is incorporated consistently in the reference element t_BK; any small-prime pathologies are excluded or handled explicitly (notably l=2)	I: Lemma B.6 U: Corollary 3.4
D2_DYADIC_LOCAL	AFU-1G:Local@2, S_AFU, dyadic	r_an=any; p: p=2; red: package must specify (good/multiplicative/additive at 2); [excluded (opt-in; many IMC/Euler-system packages assume p odd)]	A dedicated p=2 local theory is specified (2-adic comparison/integrality framework + explicit local condition complexes).; All reduction-type subcases at 2 are either handled or excluded explicitly (good/multiplicative/additive).; Compatibility with the global determinant-line conventions (t_BK / V_vis) is stated.	I: Lemma B.6 U: Corollary 3.4
MO_IMC_SUPERSINGULAR_APO	AFU-2:A0_m, Index-ID, supersingular, a_p=0	r_an=0; p: p odd, supersingular at p with a_p=0 (covers the classical ± theory; p≥3, often p≥5 in some corollaries); red: good supersingular at p	supersingular at p with a_p=0; signed (±) Selmer groups are used; signed IMC / reciprocity law in the a_p=0 setting applies (as proven in the chosen reference); and other package-specific technical hypotheses as in the cited sources.	I: Theorem 3.3 U: Proposition 10.14

Package	Normalization / translation notes
`W0_KATO_R0`	period: not used directly for A0_w output; still declare if L-values appear; lambda_trans: N/A or 1 (if no L-value normalization is consumed); notes: Package gives structural Selmer control; valuation identity belongs to M0.
`M0_IMC_ORDINARY`	period: declare: Neron vs optimal period; sign convention; torsion: declare whether torsion factor is included in formula; tamagawa: declare whether cp factors are included and how; lambda_trans: explicit translation scalar aligning external conventions to t_BK
`R1_GZ_KOLY`	period: if L' appears, declare conventions; otherwise N/A; lambda_trans: declare if comparing heights/periods to t_BK conventions
`M0_IMC_ SUPERSINGULAR_ PLUSMINUS`	period: declare the p-adic L-function normalization (Pollack ±, or signed variants) and its period choice; torsion: declare torsion convention if included in external BSD-style statement; tamagawa: declare whether bad-prime Tamagawa factors are built-in or handled separately; lambda_trans: explicit translation scalar aligning signed p-adic L-function conventions to t_BK; notes: Umbrella entry; prefer the split packages `M0_IMC_SUPERSINGULAR_AP0` and `M0_IMC_SUPERSINGULAR_APNE0` for explicit hypotheses/citations.
`L_BAD_REDUCTION_ LOCAL`	period: N/A locally; tamagawa: declare cp convention (connected component index) used by both sides; lambda_trans: usually 1 if both sides use the same Tamagawa matching convention; otherwise declare; notes: This is where Gate L's 'Tamagawa matching' is certified against the external package conventions.
`D2_DYADIC_LOCAL`	period: declare any special dyadic period conventions; tamagawa: declare c_2(E) convention and whether it is absorbed; lambda_trans: declare if dyadic package uses different normalizations from the main (odd p) packages; notes: Default global flow treats p=2 as an explicit exceptional place; include this package only if a concrete dyadic reference is adopted.
`M0_IMC_ SUPERSINGULAR_AP0`	period: declare Pollack ± p-adic L-function normalization and associated period choice; torsion: declare torsion convention if included in external BSD-style statement; tamagawa: declare whether bad-prime Tamagawa factors are built-in or handled via `L_BAD_REDUCTION_LOCAL`; lambda_trans: explicit translation scalar aligning signed conventions to t_BK; notes: Keep the signed (±) conventions explicit; do not merge with ordinary M0.
`M0_IMC_ SUPERSINGULAR_ APNE0`	period: declare signed p-adic L-function normalization and period choice; torsion: declare torsion convention if included; tamagawa: declare whether bad-prime Tamagawa factors are built-in or handled via `L_BAD_REDUCTION_LOCAL`; lambda_trans: explicit translation scalar aligning signed conventions to t_BK; notes: This is distinct from the a_p=0 ± theory; keep separate IDs.
`M1_RANK1_PPART_ BSD`	period: declare whether the leading term uses Néron vs optimal period, and any p-adic height normalization; torsion: declare torsion factor convention; tamagawa: declare how Tamagawa factors enter; use `L_BAD_REDUCTION_LOCAL` if separated; lambda_trans: explicit translation scalar aligning external rank-1 normalizations to t_BK; notes: This package complements AFU-3 (rank bridge) by providing the valuation/index identification in rank 1.

For convenience, we summarize the intended external inputs as a one-page registry. The table records the *output* each package must provide to close the corresponding gate in our architecture; precise hypotheses vary by source and are those stated in the cited references.

Target	Gate closed	Typical coverage / hypotheses	Required output	Entry points
$A0_w$ (cotorsion closure)	AFU-2 precondition; closes F2	$p \neq 2$, analytic rank 0; standard Euler-system hypotheses (residual irreducibility, suitable local conditions)	$\mathbb{Z}_p$-corank $H^1 = 0$ for the relevant Selmer group; finiteness of $\text{Ш}[p^\infty]$ in this regime	Kato-type Euler system [11]; Selmer formalism [16]
$A0_m$ (Index-ID / valuation identity)	AFU-2	$p \neq 2$ in a regime covered by an IMC/reciprocity result (often ordinary at p); normalization aligned via Appendix Section D.5	Identification of the determinant-line defect exponent with the expected BSD p-primary defect exponent (length/valuation identity)	IMC / reciprocity entry points [20, 5, 19]
Rank bridge ($r_{\text{an}} = 1$)	AFU-3	Heegner/Gross–Zagier hypotheses; Kolyvagin descent hypotheses; $p \neq 2$	$r_{\text{alg}} = 1$ and finiteness of $\text{Ш}[p^\infty]$ (hence rank equality in analytic rank 1)	Gross–Zagier + Kolyvagin [10, 13, 16]
Local package at $S_{\text{AFU}}(E)$	AFU-1G (integral upgrade beyond good p)	Primes dividing $2Nc_E$ (bad reduction, dyadic, Manin constant); handled by local comparison conventions	A local determinant-line comparison matching the chosen visible normalizations (Tamagawa/period/torsion placement)	Gate L and Appendix Section D.3 (plus source-specific local results, when invoked)

D.7 Sign conventions and the role of Gate K

The determinant-line formalism naturally produces elements only up to sign when passing through real one-dimensional lines. Gate K fixes the sign by a positivity/orientation convention compatible with $\Omega_E > 0$ and $\text{Reg}(E) > 0$. This guarantees that once Gate A3 collapses $u(E)$ to $\{\pm 1\}$, the $+1$ branch is canonical.

E Referee checklist: where each claim is proved

This appendix is a navigation aid: each bullet points to the exact place where the corresponding claim is stated/proved.

E.1 Canonical targets and "no Ш bypass" guarantees

- **Arithmetic container is canonical (not "MW without Ш").** Gate **DLT**, Section 7.4; scope remark Remark 7.6.
- **No hidden finiteness claim about Ш.** Introduction scope statement Section 1; AFU interface statement/remark Theorem 10.25 and Remark 10.26.

E.2 Where the single defect scalar is defined

- **Transport map and transported element.** Definitions 7.9 and 7.10 (Gate **DLT**-Q).

- **Defect scalar $u_p(E)$ and rational defect $u(E)$.** Definition 7.10; invariance/canonicality Propositions 7.16 and 7.17; global/local comparison Lemma 7.18.

E.3 Local valuation control (LAI)

- **Non-p valuation vanishing.** Gate L: Lemma 5.1; transport-level form Proposition 7.14.

E.4 The unconditional locking theorem (URC)

- **Adelic collapse to a p-power.** Lemma 9.2 (using Proposition 7.14) gives $u(E) = \pm p^k$.
- **Elimination of the p-power discrepancy (locking prime integrality).** Input Theorem 7.12 is used inside Lemma 9.3 to get $v_p(u_p(E)) = 0$ and hence $k = 0$.
- **Sign fixing (archimedean calibration).** Gate K: Lemma 8.3 via Lemma 9.5 selects the $+1$ branch.
- **Main closure statement $u(E) = 1$.** Theorem 9.6 from Lemmas 9.2, 9.3 and 9.5.

E.5 Where the final conclusion is stated

- **Locked determinant-line identity in the arithmetic container.** Theorem 11.1 and (2).
- **Defect localization (only a lattice index can remain).** Corollary 11.2.

E.6 Where upgrades to classical Ш statements enter

- **Index formalism.** Definition 10.21 (and, if needed under AFU-1G, Lemma 10.8).
- **Internal AFU-1G (conditional on the LAI–Selmer interface).** Local finiteness inputs Lemmas B.3 and B.4 and Corollary B.5; interface claim Theorem B.8; internal consequence Corollary 3.4.
- **LAI $\to$ f' and mapping-fiber identification (supporting the interface claim).** Mapping-fiber definition Definition B.11; local identifications Lemmas B.6 and B.7; glue Lemma B.13; Selmer-complex identification Proposition B.14; comparison-map naturality Lemma B.15; status remark Remark B.16.
- **AFU plug-in interface (external packages).** Theorem 10.25 and examples Section 10.11.

Remark E.1 (What is unconditional vs. modular). Everything up to and including Theorem 11.1 is the unconditional "locking layer". Any statement that identifies a lattice index with $\#\text{Ш}(E/\mathbb{Q})[p^\infty]$ (or proves $\#\text{Ш} < \infty$) is modular and enters only through the **AFU** plug-in interface.

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