计算行列式$D_n = \begin{vmatrix} a & a+b & a+b & \cdots & a+b \\ a-b & a & a+b & \cdots & a+b \\ a-b & a-b & a & \cdots & a+b \\ \vdots & \vdots & \vdots & & \vdots \\ a-b & a-b & a-b & \cdots & a \end{vmatrix}$.

[解析]

方法一：根据行列式的结构,可将第1行的-1倍加到第$i(i=2,\ldots,n)$行.

$$D_n = \begin{vmatrix} a & a+b & a+b & \cdots & a+b \\ a-b & a & a+b & \cdots & a+b \\ a-b & a-b & a & \cdots & a+b \\ \vdots & \vdots & \vdots & & \vdots \\ a-b & a-b & a-b & \cdots & a \end{vmatrix} \overset{-r_1+r_i}{\underset{i=2,\cdots,n}{=}} \begin{vmatrix} a & a+b & a+b & \cdots & a+b \\ -b & -b & 0 & \cdots & 0 \\ -b & -2b & -b & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -b & -2b & -2b & \cdots & -b \end{vmatrix}$$

【解析】

$$D_n=\begin{vmatrix} a & a+b & a+b & \cdots & a+b \\ a-b & a & a+b & \cdots & a+b \\ a-b & a-b & a & \cdots & a+b \\ \vdots & \vdots & \vdots & & \vdots \\ a-b & a+b & a-b & \cdots & a \end{vmatrix} \overset{-r_1+r_i}{\underset{i=2,\cdots,n}{=}} \begin{vmatrix} a & a+b & a+b & \cdots & a+b \\ -b & -b & 0 & \cdots & 0 \\ -b & -2b & -b & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -b & -2b & -2b & \cdots & -b \end{vmatrix}$$

方法一：然后，依次将第i-1行的-1倍加到第i (i=n,...,3)行.

$$D_n \overset{-r_{i-1}+r_i,\ i=n,\cdots,3}{\underset{-c_i+c_{i-1},\ i=n,\cdots,2}{=\!=\!=}} \begin{vmatrix} a & a+b & a+b & \cdots & a+b & a+b \\ -b & -b & 0 & \cdots & 0 & 0 \\ 0 & -b & -b & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -b & -b \end{vmatrix}$$

思考：列数是不是偶数？

[解析]

$$D_n \xlongequal[-c_i + c_{i-1},\, i = n,\cdots,2]{-r_{i-1} + r_i,\, i = n,\cdots,3} \begin{vmatrix} a & a+b & a+b & \cdots & a+b & a+b \\ -b & -b & 0 & \cdots & 0 & 0 \\ 0 & -b & -b & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -b & -b \end{vmatrix}$$

当 n 为偶数时

(1) 当n为偶数时，依次第i列的-1倍加到第i-1 (i=n,...,2)列.

$$D_n = \begin{vmatrix} -b & a+b & 0 & \cdots & 0 & a+b \\ 0 & -b & 0 & \cdots & 0 & 0 \\ 0 & 0 & -b & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -b \end{vmatrix} = (-b)^n = \boxed{b^n}$$

[解析]

$$D_n \xlongequal[-c_i+c_{i-1},\, i=n,\cdots,2]{-r_{i-1}+r_i,\, i=n,\cdots,3} \begin{vmatrix} a & a+b & a+b & \cdots & a+b & a+b \\ -b & -b & 0 & \cdots & 0 & 0 \\ 0 & \boxed{-b \quad -b} & & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \boxed{-b \quad -b} & \end{vmatrix}$$

当n为奇数时，还剩下第一列

(2) 当n为奇数时，依次将第i列的-1倍加到第i-1 ($i=n,...,3$)列，再按第一列展开.

$$D_n = \begin{vmatrix} a & 0 & a+b & \cdots & 0 & a+b \\ -b & -b & 0 & \cdots & 0 & 0 \\ 0 & \boxed{0 \quad -b} & & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \boxed{0 \quad -b} & \end{vmatrix} = a(-b)^{n-1} = \boxed{ab^{n-1}}$$

计算行列式$D_n=\begin{vmatrix} a & a+b & a+b & \cdots & a+b \\ a-b & a & a+b & \cdots & a+b \\ a-b & a-b & a & \cdots & a+b \\ \vdots & \vdots & \vdots & & \vdots \\ a-b & a-b & a-b & \cdots & a \end{vmatrix}$.

[解析]

方法二：注意到该行列式具有很好的“对称性”，因此先通过初等变换进行“降阶”，再利用递推关系求解.

$$D_n=\begin{vmatrix} a & a+b & a+b & \cdots & a+b \\ a-b & a & a+b & \cdots & a+b \\ a-b & a-b & a & \cdots & a+b \\ \vdots & \vdots & \vdots & & \vdots \\ a-b & a-b & a-b & \cdots & a \end{vmatrix} \overset{-r_2+r_1}{=} \begin{vmatrix} b & b & 0 & \cdots & 0 \\ a-b & a & a+b & \cdots & a+b \\ a-b & a-b & a & \cdots & a+b \\ \vdots & \vdots & \vdots & & \vdots \\ a-b & a-b & a-b & \cdots & a \end{vmatrix}$$

[解析]

$$D_n=\begin{vmatrix} a & a+b & a+b & \cdots & a+b \\ a-b & a & a+b & \cdots & a+b \\ a-b & a-b & a & \cdots & a+b \\ \vdots & \vdots & \vdots & & \vdots \\ a-b & a-b & a-b & \cdots & a \end{vmatrix} \overset{-r_2+r_1}{=} \begin{vmatrix} b & b & 0 & \cdots & 0 \\ a-b & a & a+b & \cdots & a+b \\ a-b & a-b & a & \cdots & a+b \\ \vdots & \vdots & \vdots & & \vdots \\ a-b & a-b & a-b & \cdots & a \end{vmatrix}$$

方法二：再将上述变换后的行列式，实行初等列变换，以保持“对称性”，获得递推关系如下：

$$D_n \overset{-c_2+c_1}{=} \begin{vmatrix} 0 & b & 0 & \cdots & 0 \\ -b & a & a+b & \cdots & a+b \\ 0 & a-b & & & \\ \vdots & \vdots & & D_{n-2} & \\ 0 & a-b & & & \end{vmatrix} = b^2 D_{n-2}$$

$$\text{计算行列式}D_n=\begin{vmatrix} a & a+b & a+b & \cdots & a+b \\ a-b & a & a+b & \cdots & a+b \\ a-b & a-b & a & \cdots & a+b \\ \vdots & \vdots & \vdots & & \vdots \\ a-b & a-b & a-b & \cdots & a \end{vmatrix}.$$

【解析】方法二：利用上述递推关系.

注意到：$D_1=a, D_2=b^2$.

(1) 当n为偶数时

$$D_n=b^2D_{n-2}=b^4D_{n-4}=\cdots=b^{n-2}D_2=\boxed{b^n}$$

(2) 当n为奇数时

$$D_n=b^2D_{n-2}=b^4D_{n-4}=\cdots=b^{n-1}D_1=\boxed{ab^{n-1}}$$