性质5 设A为n阶矩阵，则

$$\det(A^T) = \det A.$$

证

(1) A不可逆时, A可经系列初等行变换化成最后一行全0的阶梯形R, 于是存在初等矩阵 $E_1, E_2, \ldots, E_t$ s.t.

$$A = E_1 E_2 \cdots E_t R$$

$$\det R = 0 \Rightarrow$$

$$\det A = (\det E_1)\cdots(\det E_t)(\det R) = 0$$

又A不可逆$\Leftrightarrow A^T$不可逆

此时 $\det A^T = 0 = \det A$

(2)当A可逆时： 存在初等矩阵 $E_1, E_2, \dots, E_s$,

$$A = E_1 E_2 \cdots E_s$$

$$\det(A^T) = \det(E_s^T \cdots E_2^T E_1^T)$$

$$= (\det E_s^T) \cdots (\det E_2^T)(\det E_1^T)$$

$$= (\det E_s) \cdots (\det E_2)(\det E_1)$$

$$= (\det E_1 \det E_2 \cdots \det E_s)$$

$$= \det A$$

行列式性质小结：

(1) 按行(列)展开

(2) 三类初等变换

a.换行(列)反号　b.倍乘　c.倍加

(3) 三种为零

a. 有一行(列)全为零,

b. 有两行(列)相同,

c. 有两行(列)成比例.

(4) 一种分解

(5) $D^T = D$.

例5 奇数阶反对称阵的行列式必为零.

证 设$A_{n\times n}$(n为奇数)满足：$A^T=-A$,

于是，$\det A=\det A^T=\det(-A)$

$$=(-1)^n\det A=-\det A$$

$$\det A=0$$

例6 计算 $D=\begin{vmatrix} a^2+\frac{1}{a^2} & a & \frac{1}{a} & 1 \\ b^2+\frac{1}{b^2} & b & \frac{1}{b} & 1 \\ c^2+\frac{1}{c^2} & c & \frac{1}{c} & 1 \\ d^2+\frac{1}{d^2} & d & \frac{1}{d} & 1 \end{vmatrix}$ (已知 $abcd=1$)

解

$$D=\begin{vmatrix} a^2 & a & \frac{1}{a} & 1 \\ b^2 & b & \frac{1}{b} & 1 \\ c^2 & c & \frac{1}{c} & 1 \\ d^2 & d & \frac{1}{d} & 1 \end{vmatrix}+\begin{vmatrix} \frac{1}{a^2} & a & \frac{1}{a} & 1 \\ \frac{1}{b^2} & b & \frac{1}{b} & 1 \\ \frac{1}{c^2} & c & \frac{1}{c} & 1 \\ \frac{1}{d^2} & d & \frac{1}{d} & 1 \end{vmatrix}$$

$$=abcd\begin{vmatrix} a & 1 & \frac{1}{a^2} & \frac{1}{a} \\ b & 1 & \frac{1}{b^2} & \frac{1}{b} \\ c & 1 & \frac{1}{c^2} & \frac{1}{c} \\ d & 1 & \frac{1}{d^2} & \frac{1}{d} \end{vmatrix}+(-1)^3\begin{vmatrix} a & 1 & \frac{1}{a^2} & \frac{1}{a} \\ b & 1 & \frac{1}{b^2} & \frac{1}{b} \\ c & 1 & \frac{1}{c^2} & \frac{1}{c} \\ d & 1 & \frac{1}{d^2} & \frac{1}{d} \end{vmatrix}=0.$$

[结束]