四. Gram-Schmidt 正交化方法

已知 $\alpha_1,\cdots,\alpha_n$ 线性无关, 试求正交向量组 $\beta_1,\cdots,\beta_n$ 使得 $\alpha_1,\cdots,\alpha_i$ 与 $\beta_1,\cdots,\beta_i$ 等价?

思路: 归纳法 令 $\beta_1=\alpha_1$

令 $\beta_2=\alpha_2+k\beta_1$, 选取适当的 k 使得 $(\beta_2,\beta_1)=0$,

$$(\alpha_2+k\beta_1,\beta_1)=(\alpha_2,\beta_1)+k(\beta_1,\beta_1)=0,$$

$$\Rightarrow k=-\frac{(\alpha_2,\beta_1)}{(\beta_1,\beta_1)},\quad \beta_2=\alpha_2-\frac{(\alpha_2,\beta_1)}{(\beta_1,\beta_1)}\beta_1 .$$

令 $\beta_3 = \alpha_3 + k_1\beta_1 + k_2\beta_2$,

求k_1, k_2 使得 $(\beta_1, \beta_3) = (\beta_2, \beta_3) = 0$

$$0 = (\beta_1, \beta_3) = (\beta_1, \alpha_3) + k_1(\beta_1, \beta_1) \Rightarrow k_1 = -\frac{(\alpha_3, \beta_1)}{(\beta_1, \beta_1)},$$

$$0 = (\beta_2, \beta_3) = (\beta_2, \alpha_3) + k_2(\beta_2, \beta_2) \Rightarrow k_2 = -\frac{(\alpha_3, \beta_2)}{(\beta_2, \beta_2)}$$

$$\Rightarrow \beta_3 = \alpha_3 - \frac{(\alpha_3, \beta_1)}{(\beta_1, \beta_1)}\beta_1 - \frac{(\alpha_3, \beta_2)}{(\beta_2, \beta_2)}\beta_2$$

一般的, 类似可得

$$\beta_s=\alpha_s-\frac{(\alpha_s,\beta_1)}{(\beta_1,\beta_1)}\beta_1-\frac{(\alpha_s,\beta_2)}{(\beta_2,\beta_2)}\beta_2-\cdots-\frac{(\alpha_s,\beta_{s-1})}{(\beta_{s-1},\beta_{s-1})}\beta_{s-1}.$$

$$s=2,\cdots,n$$

进而, 再令 $\gamma_i=\frac{1}{\|\beta_i\|}\beta_i \quad (i=1,2,\cdots,n)$,

则 $\gamma_1,\gamma_2,\cdots,\gamma_s$ 是规范正交组, 并且

$\alpha_1,\cdots,\alpha_i$ 与 $\gamma_1,\cdots,\gamma_i$ 等价.

$$\beta_1 = \alpha_1$$

$$\beta_2 = \alpha_2 - \frac{(\alpha_2, \beta_1)}{(\beta_1, \beta_1)}\beta_1$$

$$\beta_3 = \alpha_3 - \frac{(\alpha_3, \beta_1)}{(\beta_1, \beta_1)}\beta_1 - \frac{(\alpha_3, \beta_2)}{(\beta_2, \beta_2)}\beta_2$$

$$\cdots\cdots\cdots\cdots$$

$$\beta_s = \alpha_s - \frac{(\alpha_s, \beta_1)}{(\beta_1, \beta_1)}\beta_1 - \frac{(\alpha_s, \beta_2)}{(\beta_2, \beta_2)}\beta_2 - \cdots - \frac{(\alpha_s, \beta_{s-1})}{(\beta_{s-1}, \beta_{s-1})}\beta_{s-1} .$$

$$\cdots\cdots\cdots\cdots$$

$$\gamma_i = \frac{1}{\|\beta_i\|}\beta_i \quad (i = 1, 2, \cdots, n),$$

例3. 将 $\alpha_1=(1,1,1),\alpha_2=(1,2,1),\alpha_3=(0,-1,1)$ 规范正交化.

解: **(1) 正交化**

$$\beta_1=\alpha_1=(1,1,1),$$

$$\beta_2=\alpha_2-\frac{(\alpha_2,\beta_1)}{(\beta_1,\beta_1)}\beta_1=(1,2,1)-\frac{4}{3}(1,1,1)=\frac{1}{3}(-1,2,-1),$$

$$\beta_3=\alpha_3-\frac{(\alpha_3,\beta_1)}{(\beta_1,\beta_1)}\beta_1-\frac{(\alpha_3,\beta_2)}{(\beta_2,\beta_2)}\beta_2=\cdots=\frac{1}{2}(-1,0,1),$$

$$\beta_1=(1,1,1),\ \beta_2=\frac{1}{3}(-1,2,-1),\ \beta_3=\frac{1}{2}(-1,0,1).$$

(2) 单位化

$$\gamma_1=\frac{1}{\|\beta_1\|}\beta_1=\frac{1}{\sqrt{3}}(1,1,1)$$

$$\gamma_2=\frac{1}{\|\beta_2\|}\beta_2=\frac{1}{\sqrt{6}}(-1,2,-1)$$

$$\gamma_3=\frac{1}{\|\beta_3\|}\beta_3=\frac{1}{\sqrt{2}}(-1,0,1).$$

注意：将 $\beta=\frac{1}{k}\alpha$ 单位化，只需将 α 单位化即可. 为什么？