# 第五讲　习题课

一.习题1

二.习题2

# 《内容小结》

1. 矩阵的线性运算、乘法、转置、逆矩阵.

2. 线性方程组求解的初等变换法.

3. 矩阵求逆的初等变换法.

# 习题1

**例1** 设 $\alpha=(\frac{1}{2},0,\cdots,0,\frac{1}{2}), A=I-\alpha^{\mathrm{T}}\alpha,\ B=I+2\alpha^{\mathrm{T}}\alpha$,求$AB$.

**解:** 
$$
\begin{aligned}
AB&=(I-\alpha^{\mathrm{T}}\alpha)(I+2\alpha^{\mathrm{T}}\alpha)\\
&=I-\alpha^{\mathrm{T}}\alpha+2\alpha^{\mathrm{T}}\alpha-2\alpha^{\mathrm{T}}\alpha\alpha^{\mathrm{T}}\alpha\\
&=I+\alpha^{\mathrm{T}}\alpha-2\alpha^{\mathrm{T}}(\alpha\alpha^{\mathrm{T}})\alpha
\end{aligned}
$$

注:$\begin{cases}\alpha^{\mathrm{T}}\alpha\text{为矩阵}\\ \alpha\alpha^{\mathrm{T}}\text{为数}\end{cases}$

$$
\alpha\alpha^{\mathrm{T}}=(\frac{1}{2},0,\cdots,0,\frac{1}{2})\begin{pmatrix}\frac{1}{2}\\0\\\vdots\\0\\\frac{1}{2}\end{pmatrix}=\frac{1}{2}\Rightarrow AB=I+\alpha^{\mathrm{T}}\alpha-2\cdot\frac{1}{2}\alpha^{\mathrm{T}}\alpha=I.
$$

$$B=A^{-1}.$$

**例2** 设$A$是实对称矩阵且$A^2=O$，证明：$A=O$.

**分析：** $A^{\mathrm{T}}=A, A^2=O \Rightarrow AA^{\mathrm{T}}=O$

**证：** 设$A=(a_{ij})_{n\times n}$且$A^{\mathrm{T}}=A$，则

$$O=A^2=AA=AA^{\mathrm{T}}=B=(b_{ij})_{n\times n} \Rightarrow b_{ij}=0(\forall i,j)$$

$$\Rightarrow 0=b_{ii}=a_{i1}^2+a_{i2}^2+\cdots+a_{in}^2 \quad (i=1,2,\cdots,n),$$

$A$是实矩阵$\Leftrightarrow a_{ij}\in R$

$$\Rightarrow a_{ij}=0 \quad (i,j=1,2,\cdots,n) \Leftrightarrow A=O.$$

**注：** $A=\begin{pmatrix}0&1\\0&0\end{pmatrix}\neq O \Rightarrow A^2=O \Rightarrow A^2=O \not\Rightarrow A=O$

例3 求$A$的逆矩阵$A^{-1}$.

分析: $(A,I)\to(I,A^{-1})$

$$A=\begin{pmatrix} 0 & a_1 & 0 & \cdots & 0 \\ 0 & 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1} \\ a_n & 0 & 0 & \cdots & 0 \end{pmatrix},\quad \prod_{i=1}^{n} a_i \neq 0.$$

解1:

$$\left(\begin{array}{ccccc|ccccc} 0 & a_1 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & a_2 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 \\ & \vdots & \vdots & & & & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1} & 0 & 0 & 0 & \cdots & 0 \\ a_n & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 \end{array}\right) \xrightarrow[i=n,n-1,\cdots,2]{r_i \leftrightarrow r_{i-1}}$$

$$\rightarrow \left(\begin{array}{ccccc|ccccc} a_n & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & a_1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & a_2 & \cdots & 0 & 0 & 1 & \cdots & 0 & 0 \\ & \cdots & \cdots & \cdots & & & \cdots & \cdots & \cdots & \\ 0 & 0 & 0 & \cdots & a_{n-1} & 0 & 0 & \cdots & 1 & 0 \end{array}\right)$$

$$
\rightarrow \left(\begin{array}{ccccc|ccccc}
1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & \dfrac{1}{a_n} \\
0 & 1 & \cdots & 0 & 0 & \dfrac{1}{a_1} & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \dfrac{1}{a_2} & \cdots & 0 & 0 \\
 & \vdots & & \vdots & & & \vdots & & & \vdots \\
0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & \dfrac{1}{a_{n-1}} & 0
\end{array}\right)
$$

$$\|$$

$$A^{-1}$$

解2: $A=\left(\begin{array}{c|cccc} 0 & a_1 & 0 & \cdots & 0 \\ 0 & 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1} \\ \hline a_n & 0 & 0 & \cdots & 0 \end{array}\right)=\begin{pmatrix} O & A_1 \\ A_2 & O \end{pmatrix}$

$\Rightarrow A^{-1}=\begin{pmatrix} O & A_2^{-1} \\ A_1^{-1} & O \end{pmatrix},\quad A_1^{-1}=\begin{pmatrix} \frac{1}{a_1} & & & \\ & \frac{1}{a_2} & & \\ & & \ddots & \\ & & & \frac{1}{a_{n-1}} \end{pmatrix}$

$A_2^{-1}=\frac{1}{a_n}$

$$A^{-1}=\begin{pmatrix} O & A_2^{-1} \\ A_1^{-1} & O \end{pmatrix}=\left(\begin{array}{cccc|c} O & & & & \frac{1}{a_n} \\ \hline \frac{1}{a_1} & & & & \\ & \frac{1}{a_2} & & & \\ & & \ddots & & \\ & & & \frac{1}{a_{n-1}} & O \end{array}\right)$$

**例4** 矩阵$A=\begin{pmatrix}1&0&0\\1&1&0\\1&1&1\end{pmatrix}$，$B=\begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix}$满足：

$$AXA+BXB=AXB+BXA+I$$，求 $X$.

**分析：** 求$X\Leftrightarrow$用$A,B$表示$X$

**解：** $AXA+BXB=AXB+BXA+I \Leftrightarrow$

$$AX(A-B)+BX(B-A)=I \Leftrightarrow AX(A-B)-BX(A-B)=I$$

$$\Leftrightarrow (AX-BX)(A-B)=I \Leftrightarrow (A-B)X(A-B)=I$$

若$A-B$可逆，则 $X=\left[(A-B)^{-1}\right]^2$.

**注：** $(A-B,I)\to(I,(A-B)^{-1})$

$$(A-B,I)=\left(\begin{array}{ccc|ccc}1 & -1 & -1 & 1 & 0 & 0\\ 0 & 1 & -1 & 0 & 1 & 0\\ 0 & 0 & 1 & 0 & 0 & 1\end{array}\right)\to\left(\begin{array}{ccc|ccc}1 & 0 & 0 & 1 & 1 & 2\\ 0 & 1 & 0 & 0 & 1 & 1\\ 0 & 0 & 1 & 0 & 0 & 1\end{array}\right)$$

$$\longrightarrow (A-B)^{-1}=\begin{pmatrix}1 & 1 & 2\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix}$$

$$X=[(A-B)^{-1}]^2=\begin{pmatrix}1 & 1 & 2\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 1 & 2\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix}=\begin{pmatrix}1 & 2 & 5\\ 0 & 1 & 2\\ 0 & 0 & 1\end{pmatrix}.$$

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# 习题2

**例1** 设$A=\begin{pmatrix} 5 & 2 & -4 \\ 2 & 8 & 2 \\ -4 & 2 & 5 \end{pmatrix}$，证明：(1) $A^2-9A=O$;

(2) $A$ 不可逆.

**分析：** $A^2-9A=O \Leftrightarrow A(A-9I)=O$

**证：**

(1) $A^2-9A=A(A-9I)=\begin{pmatrix} 5 & 2 & -4 \\ 2 & 8 & 2 \\ -4 & 2 & 5 \end{pmatrix}\underbrace{\begin{pmatrix} -4 & 2 & -4 \\ 2 & -1 & 2 \\ -4 & 2 & -4 \end{pmatrix}}_{A-9I}=O.$

(2):(I)若$A$ 可逆,则

$$O=A^{-1}O=A^{-1}(A^2-9A)=A^{-1}A(A-9I)=A-9I\neq O$$

$\longrightarrow$ $A$ 不可逆.

$A$可逆 $\Leftrightarrow AX=0$ 只有零解

**(II)** 记 $A-9I=\begin{pmatrix} -4 & 2 & -4 \\ 2 & -1 & 2 \\ -4 & 2 & -4 \end{pmatrix}=(\beta_1,\beta_2,\beta_3)$,则

$$A^2-9A=O \Leftrightarrow A(A-9I)=O \Leftrightarrow A(\beta_1,\beta_2,\beta_3)=O$$

$$\Rightarrow \begin{cases} A\beta_i=0 \\ \beta_i\neq 0 \end{cases} (i=1,2,3) \longrightarrow A \text{ 不可逆.}$$

**例2** 设$A_{n\times n}$，证明:存在$B_{n\times n}$,$C_{n\times n}$,使得 $A=B+C$,

(其中$B^{\mathrm{T}}=B, C^{\mathrm{T}}=-C$)

分析:$\begin{cases} A=B+C \\ A^{\mathrm{T}}=B^{\mathrm{T}}+C^{\mathrm{T}}=B-C \end{cases} \Rightarrow B=?\ C=?$

证: $A=B+C \quad (1)$

$$\Rightarrow A^{\mathrm{T}}=(B+C)^{\mathrm{T}}=B^{\mathrm{T}}+C^{\mathrm{T}}=B-C \quad (2)$$

$$(1)+(2) \Rightarrow A+A^{\mathrm{T}}=2B \Rightarrow B=\frac{1}{2}(A+A^{\mathrm{T}})$$

$$(1)-(2) \Rightarrow A-A^{\mathrm{T}}=2C \Rightarrow C=\frac{1}{2}(A-A^{\mathrm{T}})$$

例3 设 $A=\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$, (1)证明：$(I-A)^{-1}=I-\frac{1}{2}A$;

(2) 写出 $(I-A)^{-1}$.

分析: $A\Rightarrow(I-A)\bullet B=I\Rightarrow B=(I-A)^{-1}$

证1:

(1) $A=\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}\Rightarrow A=\begin{pmatrix}1\\1\\1\end{pmatrix}\begin{pmatrix}1&1&1\end{pmatrix}\Rightarrow$

$$A^2=\begin{pmatrix}1\\1\\1\end{pmatrix}\underbrace{\begin{pmatrix}1&1&1\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix}}_{=3}\begin{pmatrix}1&1&1\end{pmatrix}=3\begin{pmatrix}1\\1\\1\end{pmatrix}\begin{pmatrix}1&1&1\end{pmatrix}=3A$$

即 $A^2-3A=0 \Leftrightarrow A^2-3A+2I=2I$

$$\Leftrightarrow (I-A)(2I-A)=2I$$

$$\Leftrightarrow (I-A)(I-\frac{1}{2}A)=I$$

$$\Rightarrow (I-A)^{-1}=I-\frac{1}{2}A$$

(2) $(I-A)^{-1}=I-\frac{1}{2}A=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}-\frac{1}{2}\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$

$$=\frac{1}{2}\begin{pmatrix}1&-1&-1\\-1&1&-1\\-1&-1&1\end{pmatrix}$$

**分析:** $A \Rightarrow I-A \Rightarrow (I-A, I) \rightarrow (I, (I-A)^{-1})$

**证2:** $A=\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix} \Rightarrow I-A=\begin{pmatrix}0&-1&-1\\-1&0&-1\\-1&-1&0\end{pmatrix}$

$$\Rightarrow (I-A\,|\,I)=\left(\begin{array}{ccc|ccc}0&-1&-1&1&0&0\\-1&0&-1&0&1&0\\-1&-1&0&0&0&1\end{array}\right)\longrightarrow$$

$$\left(\begin{array}{ccc|ccc}0&1&1&-1&0&0\\1&0&1&0&-1&0\\1&1&0&0&0&-1\end{array}\right)\longrightarrow\left(\begin{array}{ccc|ccc}2&2&2&-1&-1&-1\\1&0&1&0&-1&0\\1&1&0&0&0&-1\end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 2 & 2 & 2 & -1 & -1 & -1 \\ 1 & 0 & 1 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 & 0 & -1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & \frac{-1}{2} & \frac{-1}{2} & \frac{-1}{2} \\ 1 & 0 & 1 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 & 0 & -1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & \frac{-1}{2} & \frac{-1}{2} & \frac{-1}{2} \\ 0 & -1 & 0 & \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ 0 & 0 & -1 & \frac{1}{2} & \frac{1}{2} & \frac{-1}{2} \end{array}\right) \rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{-1}{2} & \frac{-1}{2} \\ 0 & 1 & 0 & \frac{-1}{2} & \frac{1}{2} & \frac{-1}{2} \\ 0 & 0 & 1 & \frac{-1}{2} & \frac{-1}{2} & \frac{1}{2} \end{array}\right)$$

$$\Rightarrow \quad (I-A)^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} = I - \frac{1}{2}A.$$