五. 线性相关基本定理

$$k_1\alpha_1+\cdots+k_m\alpha_m=0 \Leftrightarrow k_1\alpha_1^T+\cdots+k_m\alpha_m^T=0$$

行向量组 $\alpha_1, \dots, \alpha_m$ 与列向量组 $\alpha_1^T, \dots, \alpha_m^T$

具有相同的线性相关性

结论: 只需讨论列向量组的线性相关性.

问题:

(1) 线性相关性与线性表出有何关系?

(2) 线性相关性的具体意义何在?

定理3. 若$\alpha_1, \ldots, \alpha_m$线性相关，

则$\alpha_1, \ldots, \alpha_m, \alpha_{m+1}, \ldots, \alpha_n$线性相关.

证： $\alpha_1, \ldots, \alpha_m$线性相关，

存在不全为零的数$k_1, \ldots, k_m$使得

$$k_1\alpha_1 + \ldots + k_m\alpha_m = 0.$$

$$\Rightarrow k_1\alpha_1 + \ldots + k_m\alpha_m + 0\alpha_{m+1} + \ldots + 0\alpha_n = 0.$$

$k_1, \ldots, k_m, 0, \ldots, 0$不全为零，故$\alpha_1, \ldots, \alpha_n$线性相关.

部分相关，则整体相关
整体无关，则部分无关

定理4. $\alpha_1, \alpha_2, \ldots, \alpha_m\,(m\geq 2)$ 线性相关

$\Leftrightarrow$ 某个向量可由其余 $m-1$ 个向量线性表出.

证: " $\Leftarrow$ " 不妨设 α_1 可由 $\alpha_2, \ldots, \alpha_m$ 线性表出

$$\Rightarrow \exists\, k_2, \cdots, k_m,\ s.t.$$

$$\alpha_1 = k_2\alpha_2 + \cdots + k_m\alpha_m$$

$$\Rightarrow -1\alpha_1 + k_2\alpha_2 + \cdots + k_m\alpha_m = 0$$

$-1, k_2, \cdots, k_m$ 不全为零

$\Rightarrow \alpha_1, \alpha_2, \ldots, \alpha_m$ 线性相关.

定理4. $\alpha_1, \alpha_2, \ldots, \alpha_m\,(m\geq 2)$ 线性相关

$\Leftrightarrow$ 某个向量可由其余 $m-1$ 个向量线性表出.

"$\Rightarrow$" 设 $\alpha_1, \alpha_2, \ldots, \alpha_m\,(m\geq 2)$ 线性相关,

则有不全为零的数 $k_1, k_2, \ldots, k_m$ 使得:

$$k_1\alpha_1 + k_2\alpha_2 + \ldots + k_m\alpha_m = 0.$$

不妨设 $k_1 \neq 0$, 则:

$$\alpha_1 = -\frac{k_2}{k_1}\alpha_2 - \cdots - \frac{k_m}{k_1}\alpha_m,$$

即 α_1 可由 $\alpha_2, \ldots, \alpha_m$ 线性表出.

定理4. $\alpha_1, \alpha_2, \ldots, \alpha_m\ (m\geq 2)$ 线性相关

$\Leftrightarrow$ 某个向量可由其余 $m-1$ 个向量线性表出.

逆否命题:

$\alpha_1, \alpha_2, \ldots, \alpha_m$ 线性无关 $\Leftrightarrow$

任一向量都不能由其余向量线性表出.

线性相关的意义:

$\alpha_1, \ldots, \alpha_m\ (m\geq 2)$ 线性相关 $\Leftrightarrow$ 某个向量多余.

$\alpha_1, \ldots, \alpha_m\ (m\geq 2)$ 线性无关 $\Leftrightarrow$ 任一向量都不多余.

定理5. 若 $\alpha_1, \alpha_2, \ldots, \alpha_m$ 线性无关，

$\alpha_1, \alpha_2, \ldots, \alpha_m, \beta$ 线性相关，

则 β 可由 $\alpha_1, \alpha_2, \ldots, \alpha_m$ 惟一线性表出.

证：可表出性： 因 $\alpha_1, \alpha_2, \ldots, \alpha_m, \beta$ 线性相关，

有不全为零的数 $k_1, k_2, \ldots, k_m, k$ 使

$$k_1\alpha_1 + k_2\alpha_2 + \ldots + k_m\alpha_m + k\beta = 0.$$

若 $k = 0$，则

$$\left.\begin{array}{r} k_1\alpha_1 + k_2\alpha_2 + \ldots + k_m\alpha_m = 0 \\ \alpha_1, \alpha_2, \ldots, \alpha_m \text{ 线性无关} \end{array}\right\} \Rightarrow$$

$$k = k_1 = k_2 = \ldots = k_m = 0, \text{ 矛盾！}$$

所以 $k \neq 0$，

$$\Rightarrow \beta = -\frac{k_1}{k}\alpha_1 - \frac{k_2}{k}\alpha_2 - \cdots - \frac{k_m}{k}\alpha_m.$$

定理5. 若 $\boxed{\alpha_1, \alpha_2, \dots, \alpha_m \text{ 线性无关,}}$

$\alpha_1, \alpha_2, \dots, \alpha_m, \beta$ 线性相关，

则 β 可由 $\alpha_1, \alpha_2, \dots, \alpha_m$ 惟一线性表出.

惟一性: 设 $\beta = k_1\alpha_1 + k_2\alpha_2 + \cdots + k_m\alpha_m$,

$\boxed{k_i = l_i ?}$

$$\beta = l_1\alpha_1 + l_2\alpha_2 + \cdots + l_m\alpha_m,$$

$$\Rightarrow 0 = (k_1 - l_1)\alpha_1 + (k_2 - l_2)\alpha_2 + \cdots + (k_m - l_m)\alpha_m,$$

因 $\alpha_1, \alpha_2, \dots, \alpha_m$ 线性无关，所以

$$k_1 - l_1 = k_2 - l_2 = \cdots = k_m - l_m = 0,$$

$\Rightarrow k_1 = l_1, \cdots, k_m = l_m$. 故表式惟一.

例6. 设 $\boxed{\alpha_1, \alpha_2, \alpha_3 \text{线性相关}}$, $\boxed{\alpha_2, \alpha_3, \alpha_4 \text{线性无关}}$, 证明:

(1) α_1可由α_2, α_3线性表出;

(2) α_4不可由$\alpha_1, \alpha_2, \alpha_3$线性表出.

证: (1) $\left.\begin{array}{r} \alpha_2, \alpha_3, \alpha_4\text{线性无关} \Rightarrow \alpha_2, \alpha_3\text{线性无关} \\ \alpha_1, \alpha_2, \alpha_3\text{线性相关} \end{array}\right\} \Rightarrow$

$\Rightarrow \boxed{\alpha_1\text{可由}\alpha_2, \alpha_3\text{线性表出.}}$

(2) 反证: $\left.\begin{array}{r} \text{设}\alpha_4\text{可由}\alpha_1, \alpha_2, \alpha_3\text{线性表出} \\ \alpha_1\text{可由}\alpha_2, \alpha_3\text{线性表出} \end{array}\right\} \Rightarrow$

$\left.\begin{array}{l} \Rightarrow \alpha_4\text{可由}\alpha_2, \alpha_3\text{线性表出} \\ \qquad \alpha_2, \alpha_3, \alpha_4\text{线性无关} \end{array}\right\} \Rightarrow$ 矛盾!

$\boxed{\text{故}\alpha_4\text{不可由}\alpha_1, \alpha_2, \alpha_3\text{线性表出!}}$