三．相似对角化的判定(1)

定理3. n阶矩阵A与对角矩阵相似$\Leftrightarrow$

A有n个线性无关的特征向量

证：充分性 设A有n个线性无关的特征向量：$P_1,\cdots,P_n$.

$$AP_1=\lambda_1 P_1,\cdots,AP_n=\lambda_n P_n$$

$$A(P_1,\cdots,P_n)=(AP_1,\cdots,AP_n)=(\lambda_1 P_1,\cdots,\lambda_n P_n)=\underbrace{(P_1,\cdots,P_n)}_{P}\underbrace{\begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix}}_{\Lambda}$$

$P_1,\cdots,P_n$线性无关$\Rightarrow P$可逆

$$\Rightarrow AP=P\Lambda,\quad P^{-1}AP=\Lambda \quad \Rightarrow A\sim\Lambda=\begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix}$$

必要性: 设$P^{-1}AP=\Lambda=\begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix}$

$\Rightarrow AP=P\Lambda.$ 设 $P=(P_1,\cdots,P_n)$,

$$\Rightarrow A(P_1,\cdots,P_n)=(P_1,\cdots,P_n)\begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix}$$

$$=(\lambda_1P_1,\cdots,\lambda_nP_n)$$

$$\left.\begin{array}{l}\Rightarrow AP_1=\lambda_1P_1,\cdots,AP_n=\lambda_nP_n\\ P\text{可逆}\Rightarrow P_1,\cdots,P_n\text{线性无关}\end{array}\right\}\Rightarrow$$

$\Rightarrow P_1,\cdots,P_n$ 是A的n个线性无关的特征向量.

定理4. $A\alpha_1=\lambda_1\alpha_1,\cdots,A\alpha_m=\lambda_m\alpha_m,\ \alpha_i\neq 0(i=1,\cdots,m),$

$\lambda_1,\cdots,\lambda_m$ 互异 $\Rightarrow \alpha_1,\cdots,\alpha_m$线性无关.

证： 设$k_1\alpha_1+\cdots+k_m\alpha_m=0,$ 左乘$A^i(i=1,\cdots,m-1)$

$$A\alpha_j=\lambda_j\alpha_j\Rightarrow A^i\alpha_j=\lambda_j^i\alpha_j\,(j=1,\cdots,m)\Rightarrow$$

$$\Rightarrow 0=k_1A^i\alpha_1+\cdots+k_mA^i\alpha_m=k_1\lambda_1^i\alpha_1+\cdots+k_m\lambda_m^i\alpha_m$$

$$\Rightarrow\begin{cases}k_1\alpha_1+\cdots+k_m\alpha_m=0\\ k_1\lambda_1\alpha_1+\cdots+k_m\lambda_m\alpha_m=0\\ \cdots\cdots\cdots\\ k_1\lambda_1^{m-1}\alpha_1+\cdots+k_m\lambda_m^{m-1}\alpha_m=0\end{cases}$$

$$\Rightarrow \begin{cases} k_1\alpha_1 + \cdots + k_m\alpha_m = 0 \\ k_1\lambda_1\alpha_1 + \cdots + k_m\lambda_m\alpha_m = 0 \\ \cdots\cdots\cdots \\ k_1\lambda_1^{m-1}\alpha_1 + \cdots + k_m\lambda_m^{m-1}\alpha_m = 0 \end{cases}$$

$$\left.\begin{array}{r} \Rightarrow (k_1\alpha_1,\cdots,k_m\alpha_m)\underbrace{\begin{pmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{m-1} \\ \vdots & \vdots & & \vdots \\ 1 & \lambda_m & \cdots & \lambda_m^{m-1} \end{pmatrix}_{m\times m}}_{T} = (0,\cdots,0) \\ \lambda_1,\cdots,\lambda_m\text{互异} \Rightarrow T\text{可逆} \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r} \Rightarrow (k_1\alpha_1,\cdots,k_m\alpha_m) = (0,\cdots,0) \\ \alpha_i \neq 0 (i = 1,\cdots,m) \end{array}\right\} \Rightarrow k_1 = \cdots = k_m = 0$$

推论1. A 的特征值互异, 则 A 与对角矩阵相似.

证： 设 $\lambda_1,\cdots,\lambda_n$ 是A的互异特征值，

$\alpha_1,\cdots,\alpha_n$ 是它们对应的特征向量

则 $\alpha_1,\cdots,\alpha_n$ 线性无关

$\Rightarrow A$与对角矩阵相似.

可以证明**推论2.**

设 $\lambda_1,\cdots,\lambda_k$ 是矩阵A的不同特征值.

$\alpha_{i1},\cdots,\alpha_{ir_i}$ 是λ_i的线性无关的特征向量.

$\Rightarrow \alpha_{11},\cdots,\alpha_{1r_1},\cdots,\alpha_{k1},\cdots,\alpha_{kr_k}$ 线性无关

例2. 设 A 是 3 阶矩阵且 $I+A$, $3I-A$, $I-3A$ 均不可逆.

证明: (1) A 可逆; (2) A 与对角矩阵相似.

证: (1) $I+A$ 不可逆 $\Rightarrow |I+A|=0$

$$\Rightarrow |-I-A|=(-1)^3|I+A|=0 \Rightarrow \lambda_1=-1 \text{ 是 } A \text{ 特征值}.$$

$3I-A$ 不可逆 $\Rightarrow |3I-A|=0 \Rightarrow \lambda_2=3$ 是 A 的特征值.

$I-3A$ 不可逆 $\Rightarrow |I-3A|=3^3\left|\frac{1}{3}I-A\right|=0$

$$\Rightarrow \left|\frac{1}{3}I-A\right|=0 \Rightarrow \lambda_3=\frac{1}{3} \text{ 是 } A \text{ 的特征值}.$$

A 的特征值均非零, 故 A 可逆.

例2. 设 A 是 3 阶矩阵且 $I+A$, $3I-A$, $I-3A$ 均不可逆 .

证明：(1) A 可逆； (2) A 与对角矩阵相似.

(2) 3阶方阵A有3个互异特征值,

故A与对角阵相似，且

$$A \sim \Lambda = \begin{pmatrix} -1 & & \\ & 3 & \\ & & 1/3 \end{pmatrix}.$$