设 $\begin{cases} x_n = 4x_{n-1} - 5y_{n-1}, \\ y_n = 2x_{n-1} - 3y_{n-1}, \end{cases}$ 且 $x_0 = 2,\ y_0 = 1$，求 x_{100}.

[解析] 将递推关系写成矩阵形式: $\begin{cases} x_n = 4x_{n-1} - 5y_{n-1}, \\ y_n = 2x_{n-1} - 3y_{n-1}, \end{cases}$

$$\Rightarrow \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} 4 & -5 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix} \quad \text{令 } A = \begin{pmatrix} 4 & -5 \\ 2 & -3 \end{pmatrix} \Rightarrow \begin{pmatrix} x_n \\ y_n \end{pmatrix} = A^n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

先计算特征值: $|\lambda I - A| = \begin{vmatrix} \lambda - 4 & 5 \\ -2 & \lambda + 3 \end{vmatrix} = (\lambda - 2)(\lambda + 1) \Rightarrow \lambda_1 = 2, \lambda_2 = -1$

再计算特征向量: $\lambda_1 = 2$: $\alpha_1 = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$ $\quad \lambda_2 = -1$: $\alpha_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

令 $P=(\alpha_1,\alpha_2)=\begin{pmatrix}5&1\\2&1\end{pmatrix}\Rightarrow A=P\begin{pmatrix}2&\\&-1\end{pmatrix}P^{-1}$ 计算得 $P^{-1}=\frac{1}{3}\begin{pmatrix}1&-1\\-2&5\end{pmatrix}$,

$$\Rightarrow A^{100}=P\begin{pmatrix}2&\\&-1\end{pmatrix}^{100}P^{-1}=\frac{1}{3}\begin{pmatrix}5&1\\2&1\end{pmatrix}\begin{pmatrix}2^{100}&\\&1\end{pmatrix}\begin{pmatrix}1&-1\\-2&5\end{pmatrix}$$

$$=\frac{1}{3}\begin{pmatrix}5\cdot2^{100}&1\\2\cdot2^{100}&1\end{pmatrix}\begin{pmatrix}1&-1\\-2&5\end{pmatrix}=\frac{1}{3}\begin{pmatrix}5\cdot2^{100}-2&-5\cdot2^{100}+5\\2\cdot2^{100}-2&-2\cdot2^{100}+5\end{pmatrix}$$

$$\Rightarrow\begin{pmatrix}x_{100}\\y_{100}\end{pmatrix}=A^{100}\begin{pmatrix}x_0\\y_0\end{pmatrix}=\frac{1}{3}\begin{pmatrix}5\cdot2^{100}-2&-5\cdot2^{100}+5\\2\cdot2^{100}-2&-2\cdot2^{100}+5\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix}=\frac{1}{3}\begin{pmatrix}5\cdot2^{100}+1\\2\cdot2^{100}-1\end{pmatrix}$$

$$\Rightarrow \boxed{x_{100}=\frac{5\cdot2^{100}+1}{3}}$$