五. 正交矩阵

将例3中的$\gamma_1,\gamma_2,\gamma_3$作为列向量组构造矩阵$A$:

$$A=(\gamma_1\quad\gamma_2\quad\gamma_3)=\begin{pmatrix}\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0\\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}}\end{pmatrix}$$

$$AA^T=\begin{pmatrix}\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0\\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\\ -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}}\\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\end{pmatrix}=I$$

若实矩阵 A 满足 $AA^T=A^TA=I$，则称 A 为<u>正交矩阵</u>。

性质:

(1) $A^{-1}=A^T$,

(2) $|A|=\pm 1$,

(3) 正交矩阵的乘积也是正交矩阵.

设 $A^TA=AA^T=I \quad B^TB=BB^T=I$,则:

$$(AB)^T(AB)=B^TA^TAB=B^TB=I .$$

(4) A 为正交矩阵 $\Leftrightarrow$ A 的行(列)向量组都是规范正交向量组.

思考: $A^*, A^{-1}, A^T, A+B, A-B$ 是正交矩阵吗?

(4) A为正交矩阵$\Leftrightarrow$ A 的行(列)向量组都是规范正交向量组.

证明: 设 $A=\begin{pmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_n\end{pmatrix}, A^T=\left(\alpha_1^T,\alpha_2^T,\cdots,\alpha_n^T\right)$, 则

$$AA^T=\begin{pmatrix}\alpha_1\alpha_1^T & \alpha_1\alpha_2^T & \cdots & \alpha_1\alpha_n^T\\ \alpha_2\alpha_1^T & \alpha_2\alpha_2^T & \cdots & \alpha_2\alpha_n^T\\ \cdots & \cdots & \cdots & \cdots\\ \alpha_n\alpha_1^T & \alpha_n\alpha_2^T & \cdots & \alpha_n\alpha_n^T\end{pmatrix}=I$$

$$\Leftrightarrow \alpha_i\alpha_i^T=1,\quad \alpha_i\alpha_j^T=0\,(i\neq j).$$

$$\Leftrightarrow (\alpha_i,\alpha_i)=1,\quad (\alpha_i,\alpha_j)=0\,(i\neq j).$$

例4. 设$A=(\alpha_1,\alpha_2,\alpha_3)$为正交矩阵，

$$\beta_1=\frac{1}{3}(2\alpha_1+2\alpha_2-\alpha_3),\quad \beta_2=\frac{1}{3}(2\alpha_1-\alpha_2+2\alpha_3),$$

$$\beta_3=\frac{1}{3}(\alpha_1-2\alpha_2-2\alpha_3),$$

证明：$B=(\beta_1,\beta_2,\beta_3)$是正交矩阵.

方法1: 证明 $(\beta_i,\beta_j)=0\,(i\neq j)$，$\|\beta_i\|=1,(i=1,2,3)$.

$$(\beta_1,\beta_1)=\frac{1}{9}(2\alpha_1+2\alpha_2-\alpha_3,2\alpha_1+2\alpha_2-\alpha_3)=\frac{1}{9}(4+4+1)=1$$

$$(\beta_1,\beta_2)=\frac{1}{9}(2\alpha_1+2\alpha_2-\alpha_3,2\alpha_1-\alpha_2+2\alpha_3)=\frac{1}{9}(4-2-2)=0$$

$$(\beta_1,\beta_3)=\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

例4. 设$A=(\alpha_1,\alpha_2,\alpha_3)$为正交矩阵,

$$\beta_1=\frac{1}{3}(2\alpha_1+2\alpha_2-\alpha_3),\quad \beta_2=\frac{1}{3}(2\alpha_1-\alpha_2+2\alpha_3),$$

$$\beta_3=\frac{1}{3}(\alpha_1-2\alpha_2-2\alpha_3),$$

证明:$B=(\beta_1,\beta_2,\beta_3)$是正交矩阵.

方法2: 证明$B^TB=I$即可:

$$B=(\beta_1,\beta_2,\beta_3)=\frac{1}{3}\underbrace{(\alpha_1,\alpha_2,\alpha_3)}_{A}\underbrace{\begin{pmatrix}2&2&1\\2&-1&-2\\-1&2&-2\end{pmatrix}}_{C}=\frac{1}{3}AC$$

$$B^TB=\frac{1}{9}(AC)^T(AC)=\frac{1}{9}C^TA^TAC=\frac{1}{9}C^TC=I$$

例5. 设$A=\left(a_{ij}\right)_{3\times 3}$是3阶正交矩阵，$a_{11}=1, b=(1,0,0)^T$，

求线性方程组 $AX=b$ 的解.

证明:

$$\left.\begin{array}{r} a_{11}=1 \\ a_{11}^2+a_{12}^2+a_{13}^2=1 \\ a_{11}^2+a_{21}^2+a_{31}^2=1 \end{array}\right\} \Rightarrow a_{12}=a_{13}=a_{21}=a_{31}=0$$

$$\Rightarrow AX=b: \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{有解} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

A正交 $\Rightarrow$ A可逆 $\Rightarrow$ $AX=b$有唯一解 $\Rightarrow X=\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

例6. 设 A 是奇数阶正交矩阵且 $\det A=1$.

证明： 1 是 A 的特征值.

分析: A正交 $\Rightarrow A^TA=I$ $\quad |I-A|=0?$

$$|I-A| = |A^TA-A| = |(A^T-I)A|$$

$$= |A^T-I| \cdot |A| = |A^T-I^T|$$

$$= |(A-I)^T| = |A-I|$$

$$= (-1)^n |I-A| = -|I-A|$$

$$\Rightarrow |I-A|=0$$

所以 1 是 A 的特征值.