第二章 行列式

§2.2 行列式的性质与计算

一. 行列式性质1～性质3

二. 行列式性质4、性质5

三. 行列式的计算

四. 方阵乘积的行列式

五. 几个例题

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一. 行列式性质1～性质3

性质1 行列式按任一行展开，其值相等，即

$$\det A = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}$$

$A_{ij} = (-1)^{i+j} M_{ij}$

M_{ij}：划去A的i行j列后所余下行列式，a_{ij}的余子式

A_{ij}： a_{ij} 的代数余子式

$$D = \begin{vmatrix} 4 & 0 & 0 & 1 \\ 2 & -1 & 3 & 1 \\ 0 & 0 & 0 & 2 \\ 7 & 4 & 3 & 2 \end{vmatrix} = -2\begin{vmatrix} 4 & 0 & 0 \\ 2 & -1 & 3 \\ 7 & 4 & 3 \end{vmatrix} = -2\times 4\begin{vmatrix} -1 & 3 \\ 4 & 3 \end{vmatrix} = -2\times 4\times(-15)$$

例2 计算 $D_n = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & \cdots & a_{2n} \\ & & \ddots & \vdots \\ & 0 & & a_{nn} \end{vmatrix}$

解

$$D_n = a_{nn} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1,n-1} \\ & a_{22} & \cdots & a_{2,n-1} \\ & & \ddots & \vdots \\ & 0 & & a_{n-1,n-1} \end{vmatrix}$$

$$= a_{nn} a_{n-1,n-1} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1,n-2} \\ & a_{22} & \cdots & a_{2,n-2} \\ & & \ddots & \vdots \\ & 0 & & a_{n-2,n-2} \end{vmatrix} = \cdots = a_{11}a_{22}\cdots a_{nn}$$

同理

$$D_n = \begin{vmatrix} & * & & a_n \\ & & \therefore & \\ & a_2 & & \\ a_1 & & 0 & \end{vmatrix}$$

$$= (-1)^{\frac{n(n-1)}{2}} a_1 a_2 \cdots a_n$$

推论 det$A$的某一行全为零 $\Rightarrow \det A = 0$

性质2 det$A$ 的第i行元素与第j行元素对应相等

即 $a_{ik} = a_{jk}$, $i \neq j$, $k=1,\dots,n$ $\Rightarrow \det A = 0$

证 对行列式的阶n用数学归纳法

1°: $n=2$, 显然.

2°: 设结论对n-1阶行列式成立, 对n阶行列式,

按第$k(\neq i, j)$行展开:

$$\det A = a_{k1}A_{k1} + a_{k2}A_{k2} + \cdots + a_{kn}A_{kn}, \quad (k \neq i, j)$$

$M_{kl}(l=1,\dots,n)$: n-1阶行列式, 有两行元对应相等

$$\Rightarrow A_{kl} = 0 \quad (k = 1, \dots, n) \Rightarrow \det A = 0$$

性质3

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{i1}+c_{i1} & b_{i2}+c_{i2} & \cdots & b_{in}+c_{in} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{i1} & b_{i2} & \cdots & b_{in} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ c_{i1} & c_{i2} & \cdots & c_{in} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

证

$$\text{左}\left(\text{按第}i\text{行展开}\right)=(b_{i1}+c_{i1})A_{i1}+\cdots+(b_{in}+c_{in})A_{in}$$

$$=(b_{i1}A_{i1}+\cdots+b_{in}A_{in})+(c_{i1}A_{i1}+\cdots+c_{in}A_{in})$$

$$=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{i1} & b_{i2} & \cdots & b_{in} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}+\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ c_{i1} & c_{i2} & \cdots & c_{in} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

例3

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 5 & 7 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1+4 & 2+5 & 3+6 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 4 & 5 & 6 \end{vmatrix}$$

$$= 0 + 0 = 0$$

[结束]

二.行列式性质4、性质5

性质4（行列式的初等变换）

(1) 将A的某一行乘以数k得到A_1，则

$$\det A_1 = k(\det A)$$

(2) 将A的某一行的$k(\neq 0)$倍加到另一行得到A_2,则

$$\det A_2 = \det A$$

(3) 交换A的两行得到A_3,则 $\det A_3 = -\det A$

证

(1) 将$\det A_1$, $\det A$分别按乘以数k的那一行展开之即得

(2)

$$\det A_2 = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{i1} & \cdots & a_{in} \\ \cdots & \cdots & \cdots \\ a_{j1}+ka_{i1} & \cdots & a_{jn}+ka_{in} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{i1} & \cdots & a_{in} \\ \cdots & \cdots & \cdots \\ a_{j1} & \cdots & a_{jn} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{i1} & \cdots & a_{in} \\ \cdots & \cdots & \cdots \\ ka_{i1} & \cdots & ka_{in} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = \det A + k\cdot 0 = \det A$$

(3)

$$\det A_3 = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{j1} & \cdots & a_{jn} \\ \cdots & \cdots & \cdots \\ a_{i1} & \cdots & a_{in} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} \begin{array}{c} i\text{行} \\ = \\ j\text{行} \end{array} \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{j1} & \cdots & a_{jn} \\ \cdots & \cdots & \cdots \\ a_{j1}+a_{i1} & \cdots & a_{jn}+a_{in} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ -a_{i1} & \cdots & -a_{in} \\ \cdots & \cdots & \cdots \\ a_{j1}+a_{i1} & \cdots & a_{jn}+a_{in} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ -a_{i1} & \cdots & -a_{in} \\ \cdots & \cdots & \cdots \\ a_{j1} & \cdots & a_{jn} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = -\det A$$

推论 $\det A$的某两行元素对应成比例 $\Rightarrow \det A = 0$

应用:

(1) A是n阶矩阵

$$\det(kA) = k^n(\det A)$$

(2) 初等矩阵的行列式:

$$\det(\boldsymbol{E}_{ij}) = \det(\boldsymbol{E}_{ij}\boldsymbol{I}) = -\det \boldsymbol{I} = -1$$

$$\det \boldsymbol{E}_i(\boldsymbol{c}) = \boldsymbol{c} \neq 0;$$

$$\det \boldsymbol{E}_{ij}(\boldsymbol{c}) = 1.$$

(3) 初等矩阵与任一方阵A乘积的行列式:

$$\det(E_{ij}A) = -\det A = (\det E_{ij})(\det A),$$

$$\det(E_i(c)A) = c(\det A) = (\det E_i(c))(\det A),$$

$$\det(E_{ij}(c)A) = \det A = (\det E_{ij}(c))(\det A).$$

设E是初等矩阵,则:

$$\det(EA) = (\det E)(\det A)$$

设$E_1, E_2, \ldots, E_t$是初等矩阵,则:

$$\det(E_1E_2\cdots E_tA) = (\det E_1)\cdots(\det E_t)(\det A)$$

例4 $A=\begin{pmatrix}1&2&3\\2&2&3\\1&1&1\end{pmatrix}$

$$|A|=\begin{vmatrix}1&2&3\\2&2&3\\1&1&1\end{vmatrix}=\begin{vmatrix}1&2&3\\0&-2&-3\\0&-1&-2\end{vmatrix}=\begin{vmatrix}1&2&3\\0&-2&-3\\0&0&-\frac{1}{2}\end{vmatrix}=1$$

$$|2A|=\begin{vmatrix}2&4&6\\4&4&6\\2&2&2\end{vmatrix}=2\cdot2\cdot2\begin{vmatrix}1&2&3\\2&2&3\\1&1&1\end{vmatrix}=8$$

$$2|A|\neq|2A|$$

一般，

$$|kA_{n\times n}|=k^n|A|\neq|kA|.$$

性质5 设A为n阶矩阵，则

$$\det(A^T) = \det A.$$

证

(1) A不可逆时, A可经系列初等行变换化成最后一行全0的阶梯形R, 于是存在初等矩阵 $E_1, E_2, \ldots, E_t$ s.t.

$$A = E_1 E_2 \cdots E_t R$$

$$\det R = 0 \Rightarrow$$

$$\det A = (\det E_1)\cdots(\det E_t)(\det R) = 0$$

又A不可逆$\Leftrightarrow A^T$不可逆

此时 $\det A^T = 0 = \det A$

(2)当A可逆时： 存在初等矩阵 $\boldsymbol{E}_1, \boldsymbol{E}_2, \dots, \boldsymbol{E}_s$,

$$\boldsymbol{A} = \boldsymbol{E}_1\boldsymbol{E}_2\cdots\boldsymbol{E}_s$$

$$\det(A^T) = \det(E_s^T \cdots E_2^T E_1^T)$$

$$= (\det E_s^T)\cdots(\det E_2^T)(\det E_1^T)$$

$$= (\det E_s)\cdots(\det E_2)(\det E_1)$$

$$= (\det \boldsymbol{E}_1 \det \boldsymbol{E}_2 \cdots \det \boldsymbol{E}_s)$$

$$= \det \boldsymbol{A}$$

行列式性质小结：

(1) 按行(列)展开

(2) 三类初等变换

a.换行(列)反号　*b*.倍乘　*c*.倍加

(3) 三种为零

a. 有一行(列)全为零,

b. 有两行(列)相同,

c. 有两行(列)成比例.

(4) 一种分解

(5) $D^T = D$.

例5 奇数阶反对称阵的行列式必为零.

证 设$A_{n\times n}$(n为奇数)满足：$A^T=-A$,

于是，$\det A=\det A^T=\det(-A)$

$$=(-1)^n\det A=-\det A$$

$$\det A=0$$

例6 计算 $D=\begin{vmatrix} a^2+\frac{1}{a^2} & a & \frac{1}{a} & 1 \\ b^2+\frac{1}{b^2} & b & \frac{1}{b} & 1 \\ c^2+\frac{1}{c^2} & c & \frac{1}{c} & 1 \\ d^2+\frac{1}{d^2} & d & \frac{1}{d} & 1 \end{vmatrix}$ (已知 $abcd=1$)

解

$$D=\begin{vmatrix} a^2 & a & \frac{1}{a} & 1 \\ b^2 & b & \frac{1}{b} & 1 \\ c^2 & c & \frac{1}{c} & 1 \\ d^2 & d & \frac{1}{d} & 1 \end{vmatrix}+\begin{vmatrix} \frac{1}{a^2} & a & \frac{1}{a} & 1 \\ \frac{1}{b^2} & b & \frac{1}{b} & 1 \\ \frac{1}{c^2} & c & \frac{1}{c} & 1 \\ \frac{1}{d^2} & d & \frac{1}{d} & 1 \end{vmatrix}$$

$$=abcd\begin{vmatrix} a & 1 & \frac{1}{a^2} & \frac{1}{a} \\ b & 1 & \frac{1}{b^2} & \frac{1}{b} \\ c & 1 & \frac{1}{c^2} & \frac{1}{c} \\ d & 1 & \frac{1}{d^2} & \frac{1}{d} \end{vmatrix}+(-1)^3\begin{vmatrix} a & 1 & \frac{1}{a^2} & \frac{1}{a} \\ b & 1 & \frac{1}{b^2} & \frac{1}{b} \\ c & 1 & \frac{1}{c^2} & \frac{1}{c} \\ d & 1 & \frac{1}{d^2} & \frac{1}{d} \end{vmatrix}=0.$$

[结束]

性质5 设A为n阶矩阵，则

$$\det(A^T) = \det A.$$

证

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$$A = E_1 E_2 \cdots E_t R$$

$\det R = 0 \Rightarrow$

$$\det A = (\det E_1)\cdots(\det E_t)(\det R) = 0$$

又A不可逆$\Leftrightarrow A^T$不可逆

此时 $\det A^T = 0 = \det A$

(2)当*A*可逆时： 存在初等矩阵 $\boldsymbol{E}_1, \boldsymbol{E}_2, \ldots, \boldsymbol{E}_s$,

$$\boldsymbol{A} = \boldsymbol{E}_1\boldsymbol{E}_2\cdots\boldsymbol{E}_s$$

$$\det(A^T) = \det(E_s^T \cdots E_2^T E_1^T)$$

$$= (\det E_s^T)\cdots(\det E_2^T)(\det E_1^T)$$

$$= (\det E_s)\cdots(\det E_2)(\det E_1)$$

$$= (\det \boldsymbol{E}_1 \det \boldsymbol{E}_2 \cdots \det \boldsymbol{E}_s)$$

$$= \det \boldsymbol{A}$$

行列式性质小结：

(1) 按行(列)展开

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a.换行(列)反号　　b.倍乘　　c.倍加

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$$=(-1)^n\det A=-\det A$$

$$\det A=0$$

例6 计算 $D=\begin{vmatrix} a^2+\frac{1}{a^2} & a & \frac{1}{a} & 1 \\ b^2+\frac{1}{b^2} & b & \frac{1}{b} & 1 \\ c^2+\frac{1}{c^2} & c & \frac{1}{c} & 1 \\ d^2+\frac{1}{d^2} & d & \frac{1}{d} & 1 \end{vmatrix}$ (已知 $abcd=1$)

解

$$D=\begin{vmatrix} a^2 & a & \frac{1}{a} & 1 \\ b^2 & b & \frac{1}{b} & 1 \\ c^2 & c & \frac{1}{c} & 1 \\ d^2 & d & \frac{1}{d} & 1 \end{vmatrix}+\begin{vmatrix} \frac{1}{a^2} & a & \frac{1}{a} & 1 \\ \frac{1}{b^2} & b & \frac{1}{b} & 1 \\ \frac{1}{c^2} & c & \frac{1}{c} & 1 \\ \frac{1}{d^2} & d & \frac{1}{d} & 1 \end{vmatrix}$$

$$=abcd\begin{vmatrix} a & 1 & \frac{1}{a^2} & \frac{1}{a} \\ b & 1 & \frac{1}{b^2} & \frac{1}{b} \\ c & 1 & \frac{1}{c^2} & \frac{1}{c} \\ d & 1 & \frac{1}{d^2} & \frac{1}{d} \end{vmatrix}+(-1)^3\begin{vmatrix} a & 1 & \frac{1}{a^2} & \frac{1}{a} \\ b & 1 & \frac{1}{b^2} & \frac{1}{b} \\ c & 1 & \frac{1}{c^2} & \frac{1}{c} \\ d & 1 & \frac{1}{d^2} & \frac{1}{d} \end{vmatrix}=0.$$

[结束]

三. 行列式的计算

例7. 设 $A=\begin{pmatrix} 1 & -3 & 7 \\ 2 & 4 & -3 \\ -3 & 7 & 2 \end{pmatrix}$，求 $\det A$.

解.

$$\det A=\begin{vmatrix} 1 & -3 & 7 \\ 0 & 10 & -17 \\ 0 & -2 & 23 \end{vmatrix}=\begin{vmatrix} 1 & -3 & 7 \\ 0 & 10 & -17 \\ 0 & 0 & \frac{196}{10} \end{vmatrix}=196$$

例8. 计算 $D=\begin{vmatrix}1 & 4 & -1 & 4\\ 2 & 1 & 4 & 3\\ 4 & 2 & 3 & 11\\ 3 & 0 & 9 & 2\end{vmatrix}$

解.

$$D=\begin{vmatrix}-7 & 0 & -17 & -8\\ 2 & 1 & 4 & 3\\ 0 & 0 & -5 & 5\\ 3 & 0 & 9 & 2\end{vmatrix}=(-1)^{2+2}\begin{vmatrix}-7 & -17 & -8\\ 0 & -5 & 5\\ 3 & 9 & 2\end{vmatrix}$$

$$=\begin{vmatrix}-7 & -25 & -8\\ 0 & 0 & 5\\ 3 & 11 & 2\end{vmatrix}=-5\begin{vmatrix}-7 & -25\\ 3 & 11\end{vmatrix}=10$$

例9. 计算 $D_n=\begin{vmatrix} x & y & \cdots & y \\ y & x & \cdots & y \\ \cdots & \cdots & \cdots & \cdots \\ y & y & \cdots & x \end{vmatrix}$

解（逐列相加）

$$D_n=\begin{vmatrix} x+(n-1)y & y & \cdots & y \\ x+(n-1)y & x & \cdots & y \\ \cdots & \cdots & \cdots & \cdots \\ x+(n-1)y & y & \cdots & x \end{vmatrix}=(x+(n-1)y)\begin{vmatrix} 1 & y & \cdots & y \\ 1 & x & \cdots & y \\ \cdots & \cdots & \cdots & \cdots \\ 1 & y & \cdots & x \end{vmatrix}$$

$$=(x+(n-1)y)\begin{vmatrix} 1 & y & \cdots & y \\ 0 & x-y & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & x-y \end{vmatrix}=\left[x+\left(n-1\right)y\right]\left(x-y\right)^{n-1}$$

例10. 证明范德蒙行列式($n\geq 2$)

$$V_n=\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_n \\ x_1^2 & x_2^2 & x_3^2 & \cdots & x_n^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_1^{n-1} & x_2^{n-1} & x_3^{n-1} & \cdots & x_n^{n-1} \end{vmatrix}=\prod_{1\leq j<i\leq n}(x_i-x_j),$$

证 $n=2$: $\begin{vmatrix} 1 & 1 \\ x_1 & x_2 \end{vmatrix}=x_2-x_1$, 结论成立

设对于n-1阶结论成立，对于n阶：

$$V_n=\begin{vmatrix}1&1&1&\cdots&1\\0&x_2-x_1&x_3-x_1&\cdots&x_n-x_1\\0&x_2(x_2-x_1)&x_3(x_3-x_1)&\cdots&x_n(x_n-x_1)\\\vdots&\vdots&\vdots&\vdots&\vdots\\0&x_2^{n-2}(x_2-x_1)&x_3^{n-2}(x_3-x_1)&\cdots&x_n^{n-2}(x_n-x_1)\end{vmatrix}$$

$$=\begin{vmatrix}x_2-x_1&x_3-x_1&\cdots&x_n-x_1\\x_2(x_2-x_1)&x_3(x_3-x_1)&\cdots&x_n(x_n-x_1)\\\cdots&\cdots&\cdots&\cdots\\x_2^{n-2}(x_2-x_1)&x_3^{n-2}(x_3-x_1)&\cdots&x_n^{n-2}(x_n-x_1)\end{vmatrix}$$

$$=(x_2-x_1)(x_3-x_1)\cdots(x_n-x_1)\begin{vmatrix}1&1&\cdots&1\\x_2&x_3&\cdots&x_n\\\cdots&\cdots&\cdots&\cdots\\x_2^{n-2}&x_3^{n-2}&\cdots&x_n^{n-2}\end{vmatrix}$$

n-1阶范德蒙行列式

$$V_n=(x_2-x_1)(x_3-x_1)\cdots(x_n-x_1)\prod_{2\le j<i\le n}(x_i-x_j)=\prod_{1\le j<i\le n}(x_i-x_j)$$

例11

$$D=\begin{vmatrix} a & a^2 & a^3 & a^4 \\ b & b^2 & b^3 & b^4 \\ c & c^2 & c^3 & c^4 \\ d & d^2 & d^3 & d^4 \end{vmatrix}=abcd\begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix}$$

$$=abcd\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{vmatrix}$$

$$=abcd\ (d-c)(d-b)(d-a)(c-b)(c-a)(b-a)$$

例12. 计算 $D_n = \begin{vmatrix} 1+a_1 & a_2 & \cdots & a_n \\ a_1 & 1+a_2 & \cdots & a_n \\ \cdots & \cdots & \cdots & \cdots \\ a_1 & a_2 & \cdots & 1+a_n \end{vmatrix}$

解. 加边法

$$D_n = \begin{vmatrix} 1 & a_1 & a_2 & \cdots & a_n \\ 0 & 1+a_1 & a_2 & \cdots & a_n \\ 0 & a_1 & 1+a_2 & \cdots & a_n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & a_1 & a_2 & \cdots & 1+a_n \end{vmatrix} = \begin{vmatrix} 1 & a_1 & a_2 & \cdots & a_n \\ -1 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & 0 & 0 & \cdots & 1 \end{vmatrix}$$

$$
= \begin{vmatrix} 1+\sum_{i=1}^{n} a_i & a_1 & a_2 & \cdots & a_n \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{vmatrix} = 1+\sum_{i=1}^{n} a_i
$$

其它方法

拆边法

逐行（列）相加法

先猜测，后归纳

$$
\begin{vmatrix} 1+a_1 & a_2 & \cdots & a_n \\ a_1 & 1+a_2 & \cdots & a_n \\ \cdots & \cdots & \cdots & \cdots \\ a_1 & a_2 & \cdots & 1+a_n \end{vmatrix}
$$

[结束]

四. 方阵乘积的行列式

问题： 1. 可逆矩阵与行列式的关系；
2. 矩阵乘积的行列式.

定理1 方阵A可逆的充要条件为$\det A\neq 0$.

证 设 $A \xrightarrow{\text{行初等变换}} R$（简化行阶梯形）

即存在初等矩阵 $E_1, \ldots, E_t$ 使得 $A = E_1 \cdots E_t R$

$\Leftarrow$:已知$\det A \neq 0$ 若A不可逆,

则R的最后一行的元全为零, 所以$\det R = 0$

$\det A = (\det E_1)\cdots(\det E_t)(\det R) = 0$, 矛盾.

$\Rightarrow$: 若A可逆, 则$R=I$,

$\det A = (\det E_1)\cdots(\det E_t)(\det I) \neq 0$.

定理2 设A, B为n阶方阵，则

$$\det(AB) = (\det A)(\det B).$$

证 设$A \xrightarrow{\text{行初等变换}} R$（简化行阶梯形）

即存在初等矩阵 $E_1, \ldots, E_t$使得 $A = E_1 \cdots E_t R$

$$\det(AB) = \det(E_1 \cdots E_t RB)$$

$$= (\det E_1) \cdots (\det E_t)(\det(RB)).$$

若A可逆，则$R=I$,

$$\det(AB) = (\det E_1) \cdots (\det E_t)(\det(IB)) = (\det A)(\det B).$$

若A不可逆，则R的最后一行全为零，RB的最后一行全为零.

$$\det(AB) = 0$$

$$(\det A)(\det B) = 0(\det B) = 0.$$

推论1 设A_i(i=1, …, t)为n阶矩阵，则

$$\det(A_1A_2\cdots A_t)=(\det A_1)\cdots(\det A_t).$$

推论2 设A, B为n阶矩阵，且$AB=I$ (或$BA=I$), 则

$$B=A^{-1}$$

证

$$\det(AB)=(\det A)(\det B)=\det I=1.$$

所以 $\det A\neq 0$, 于是A可逆

$$A^{-1}AB=A^{-1}I=A^{-1}$$

$$B=A^{-1}$$

应用 $\det(A^{-1})=\dfrac{1}{\det A}$

[结束]

五. 几个补充例题

例13 已知$AA^T=I$ 且 $|A|=-1$，证明：$|-I-A|=0$.

证：$|-I-A|=|-AA^T-A|$

$$=|A(-A^T-I)|$$

$$=|A||(-A-I)^T|$$

$$=-|\ -A-I|=-|\ -I-A|$$

$$\therefore\quad |-I-A|=0.$$

例14 设 $\Lambda = \begin{pmatrix} 0 & & & & \\ & 1 & & & \\ & & 2 & & \\ & & & \ddots & \\ & & & & n-1 \end{pmatrix}$，$P^{-1}BP = \Lambda$，求 $|I+B|$

解：$B = P\Lambda P^{-1}$

$$|I+B| = |I+P\Lambda P^{-1}| = |PIP^{-1} + P\Lambda P^{-1}|$$

$$= |P(I+\Lambda)P^{-1}| = |P||I+\Lambda||P^{-1}|$$

$$= |P||P^{-1}||I+\Lambda| = |I+\Lambda|$$

$$= n!$$

例15 已知 $D_n = \begin{vmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 2 & 0 & \cdots & 0 \\ 1 & 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & n \end{vmatrix}$

求第一行各元素的代数余子式之和:

$$A_{11} + A_{12} + \cdots + A_{1n}$$

解

$$A_{11} + A_{12} + \cdots + A_{1n} = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 0 & \cdots & 0 \\ 1 & 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & n \end{vmatrix} = n!\left(1 - \sum_{j=2}^{n} \frac{1}{j}\right).$$

例16 已知$\alpha,\beta,\gamma_1,\gamma_2$是列向量,并且行列式

$$|A|=|\alpha,\gamma_1,\gamma_2|=4,\quad |B|=|\beta,\gamma_1,\gamma_2|=-1,$$

行列式$|A+B|=?$

解

$$|A+B|=|(\alpha,\gamma_1,\gamma_2)+(\beta,\gamma_1,\gamma_2)|$$

$$=|\alpha+\beta,2\gamma_1,2\gamma_2|$$

$$=4|\alpha+\beta,\gamma_1,\gamma_2|$$

$$=4(|\alpha,\gamma_1,\gamma_2|+|\beta,\gamma_1,\gamma_2|)$$

$$=12$$

例17. 计算 $D_n=\begin{vmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 2 & 3 & 4 & \cdots & n & 1 \\ 3 & 4 & 5 & \cdots & 1 & 2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ n-1 & n & 1 & \cdots & n-3 & n-2 \\ n & 1 & 2 & \cdots & n-2 & n-1 \end{vmatrix}$

$$=\begin{vmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 1 & 1 & 1 & \cdots & 1 & 1-n \\ 1 & 1 & 1 & \cdots & 1-n & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & 1-n & \cdots & 1 & 1 \\ 1 & 1-n & 1 & \cdots & 1 & 1 \end{vmatrix}=\begin{vmatrix} \frac{n(n+1)}{2} & 2 & 3 & \cdots & n-1 & n \\ 0 & 1 & 1 & \cdots & 1 & 1-n \\ 0 & 1 & 1 & \cdots & 1-n & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & 1-n & \cdots & 1 & 1 \\ 0 & 1-n & 1 & \cdots & 1 & 1 \end{vmatrix}_n$$

$$=\frac{n(n+1)}{2}\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1-n \\ 1 & 1 & 1 & \cdots & 1-n & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1-n & 1 & \cdots & 1 & 1 \\ 1-n & 1 & 1 & \cdots & 1 & 1 \end{vmatrix}_{n-1} = \frac{n(n+1)}{2}\begin{vmatrix} -1 & 1 & 1 & \cdots & 1 & 1-n \\ -1 & 1 & 1 & \cdots & 1-n & 1 \\ -1 & 1 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ -1 & 1-n & 1 & \cdots & 1 & 1 \\ -1 & 1 & 1 & \cdots & 1 & 1 \end{vmatrix}_{n-1}$$

$$=\frac{n(n+1)}{2}\begin{vmatrix} -1 & 0 & 0 & \cdots & 0 & -n \\ -1 & 0 & 0 & \cdots & -n & 0 \\ -1 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ -1 & -n & 0 & \cdots & 0 & 0 \\ -1 & 0 & 0 & \cdots & 0 & 0 \end{vmatrix}_{n-1}$$

[结束]