第二章 行列式

§2.1 n阶行列式的定义

一. 一阶、二阶和三阶行列式

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一. 一阶、二阶和三阶行列式

$$A_{n\times n}X=b$$

$n=1$　$a_{11}x=b_1(a_{11}\neq 0)$　　$x=\dfrac{b_1}{a_{11}}$

$n=2$　$\begin{cases}a_{11}x_1+a_{12}x_2=b_1\\ a_{21}x_1+a_{22}x_2=b_2\end{cases}$　$(a_{11}a_{22}-a_{12}a_{21}\neq 0)$　$\begin{cases}x_1=\dfrac{b_1a_{22}-a_{12}b_2}{a_{11}a_{22}-a_{12}a_{21}}\\ x_2=\dfrac{a_{11}b_2-b_1a_{21}}{a_{11}a_{22}-a_{12}a_{21}}\end{cases}$

$n=3$　$\begin{cases}a_{11}x_1+a_{12}x_2+a_{13}x_3=b_1\\ a_{21}x_1+a_{22}x_2+a_{23}x_3=b_2\\ a_{31}x_1+a_{32}x_2+a_{33}x_3=b_3\end{cases}$

$$x_1=\frac{b_1a_{22}a_{33}+a_{12}a_{23}b_3+a_{13}b_2a_{32}-b_1a_{23}a_{32}-a_{12}b_2a_{33}-a_{13}a_{22}b_3}{a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{11}a_{23}a_{32}-a_{12}a_{21}a_{33}-a_{13}a_{22}a_{31}}$$

$$A_{n\times n}X=b$$

$n=1$ $\quad x=\dfrac{b_1}{a_{11}}$

一阶行列式 $|A|=|a_{11}|=a_{11}$

如:行列式 $|-5|=-5,\ |3|=3$

$n=2$ $\quad \begin{cases} x_1=\dfrac{b_1a_{22}-a_{12}b_2}{a_{11}a_{22}-a_{12}a_{21}} \\ x_2=\dfrac{a_{11}b_2-b_1a_{21}}{a_{11}a_{22}-a_{12}a_{21}} \end{cases}$

二阶行列式 $|A|=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}=a_{11}a_{22}-a_{12}a_{21}$

如: $\begin{vmatrix} 2 & -1 \\ 3 & -3 \end{vmatrix}=-6+3=-3$

$n=3$ $\quad x_1=\dfrac{b_1a_{22}a_{33}+a_{12}a_{23}b_3+a_{13}b_2a_{32}-b_1a_{23}a_{32}-a_{12}b_2a_{33}-a_{13}a_{22}b_3}{a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{11}a_{23}a_{32}-a_{12}a_{21}a_{33}-a_{13}a_{22}a_{31}}$

……

三阶行列式 $|A|=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}=\begin{matrix} a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32} \\ -a_{11}a_{23}a_{32}-a_{12}a_{21}a_{33}-a_{13}a_{22}a_{31} \end{matrix}$

三阶行列式的记忆法

(对角线法)

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

例1. 计算三阶行列式 $D = \begin{vmatrix} 1 & 2 & -4 \\ -2 & 2 & 1 \\ -3 & 4 & -2 \end{vmatrix}$

解 $D = 1\times2\times(-2) \quad +2\times1\times(-3) \quad +(-4)\times(-2)\times4$

$-(-4)\times2\times(-3) \;\; -2\times(-2)\times(-2) \;\; -1\times1\times4 \;\; = -14$

例2. 求解方程 $\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & x \\ 4 & 9 & x^2 \end{vmatrix} = 0.$

解 方程左端

$$D = 3x^2 + 4x + 18 - 9x - 2x^2 - 12 = x^2 - 5x + 6,$$

由 $x^2 - 5x + 6 = 0$ 解得 $x = 2$ 或 $x = 3$.

[结束]

二. n阶行列式的定义

1. 二、三阶行列式的规律观察

$$\begin{vmatrix} a_{11} & a_{12} \\ \boxed{a_{21}} & \boxed{a_{22}} \end{vmatrix} = a_{11}a_{22} + a_{12}(-1)a_{21} = a_{11}A_{11} + a_{12}A_{12}$$

$$A_{11} = (-1)^{1+1}|a_{22}|, \quad A_{12} = (-1)^{1+2}|a_{21}|$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \underline{a_{11}a_{22}a_{33}} + \underline{a_{12}a_{23}a_{31}} + \underline{a_{13}a_{21}a_{32}}$$

$$-\underline{a_{11}a_{23}a_{32}} - \underline{a_{12}a_{21}a_{33}} - \underline{a_{13}a_{22}a_{31}}$$

$$= a_{11}(\underline{a_{22}a_{33} - a_{23}a_{32}}) - a_{12}(\underline{a_{21}a_{33} - a_{23}a_{31}}) + a_{13}(\underline{a_{21}a_{32} - a_{22}a_{31}})$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}(-1)\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

A_{11}, A_{12}, A_{13} 分别称为 a_{11}, a_{12}, a_{13} 的**代数余子式**.

2. n阶行列式的定义

$$A=(a_{ij})_{n\times n} \longrightarrow \det A=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

(1) 当$n=1$时，$\det A=\det(a_{11})=a_{11}$;

(2) 当$n\geq 2$时，

$$\det A=a_{11}A_{11}+a_{12}A_{12}+\cdots+a_{1n}A_{1n},$$

其中 $A_{1j}=(-1)^{1+j}M_{1j}$, M_{1j}为划去A的第1行第j列后所得的$n-1$阶行列式,称为a_{1j}的余子式,A_{1j}称为a_{1j}的代数余子式. 记号 $\det A$, $|A|$

	行列式	矩阵
(1)	数	数表
(2)	D_n	$A_{m\times n}$
(3)	$\vert\ \ \vert$	$(\ \ ),[\ \ ]$
(4)	$\vert A\vert=\det A \leftarrow$ ------	------ $A_{n\times n}$

[结束]

三. 定义计算简单的行列式

例3. 求$\det A$:
$$A=\begin{pmatrix}1 & -3 & 7\\ 2 & 4 & -3\\ -3 & 7 & 2\end{pmatrix}$$

解
$$\det A = 1(-1)^{1+1}\begin{vmatrix}4 & -3\\ 7 & 2\end{vmatrix} + (-3)(-1)^{1+2}\begin{vmatrix}2 & -3\\ -3 & 2\end{vmatrix}$$
$$+7(-1)^{1+3}\begin{vmatrix}2 & 4\\ -3 & 7\end{vmatrix}$$
$$=(8+21)+3(4-9)+7(14+12)=196$$

例4. 计算下三角行列式 $D_n = \begin{vmatrix} a_{11} & & & \\ a_{21} & a_{22} & & O \\ \vdots & \vdots & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$

解 $D_n = a_{11}\begin{vmatrix} a_{22} & & & \\ a_{32} & a_{33} & & O \\ \vdots & \vdots & \ddots & \\ a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix} = a_{11}a_{22}\begin{vmatrix} a_{33} & & & \\ a_{43} & a_{44} & & O \\ \vdots & \vdots & \ddots & \\ a_{n3} & a_{n4} & \cdots & a_{nn} \end{vmatrix}$

$$= \cdots = \boxed{a_{11}a_{22}\cdots a_{nn}}$$

同理，$\det(\mathrm{diag}(a_{11}, a_{22}, \cdots, a_{nn})) = a_{11}a_{22}\cdots a_{nn}$

$$\boxed{\det(kI_n) = k^n, \ \det I = 1}$$

例5. 计算斜下三角行列式 $D_n = \begin{vmatrix} & & 0 & & a_n \\ & & & \cdot^{\cdot^{\cdot}} & \\ & & a_2 & & \\ a_1 & & & * & \end{vmatrix}$

解 $D_n = a_n(-1)^{1+n} \begin{vmatrix} & & 0 & & a_{n-1} \\ & & & \cdot^{\cdot^{\cdot}} & \\ & & a_2 & & \\ a_1 & & & * & \end{vmatrix} = (-1)^{n-1} a_n D_{n-1}$

$$= (-1)^{n-1} a_n (-1)^{n-2} a_{n-1} D_{n-2} = \cdots = (-1)^{(n-1)+(n-2)+\cdots+1} a_n a_{n-1} \cdots a_1$$

$$= \boxed{(-1)^{\frac{n(n-1)}{2}} a_1 a_2 \cdots a_n}$$

同理，$D_n = \begin{vmatrix} & 0 & & a_n \\ & & \iddots & \\ & a_2 & & \\ a_1 & & 0 & \end{vmatrix} = (-1)^{\frac{n(n-1)}{2}} a_1 a_2 \cdots a_n$

$$D_n = \begin{vmatrix} & 0 & & 1 \\ & & \iddots & \\ & 1 & & \\ 1 & & 0 & \end{vmatrix} = (-1)^{\frac{n(n-1)}{2}}$$

$$D_n=\begin{vmatrix} a_{11} & & & \\ a_{21} & a_{22} & \mathrm{O} & \\ \vdots & \vdots & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}=a_{11}a_{22}\cdots a_{nn}$$

$$D_n=\begin{vmatrix} & 0 & & a_n \\ & & ⋰ & \\ & a_2 & & \\ a_1 & & * & \end{vmatrix}=(-1)^{\frac{n(n-1)}{2}}a_1a_2\cdots a_n$$

思考:

$$D_n=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & \cdots & a_{2n} \\ & & \ddots & \vdots \\ & 0 & & a_{nn} \end{vmatrix}=?$$

$$D_n=\begin{vmatrix} & * & & a_n \\ & & ⋰ & \\ & a_2 & & \\ a_1 & & 0 & \end{vmatrix}=?$$

[结束]