六、$f(A), A^{-1}, A^*$的特征值与特征向量

例6. 设 A 是 n 阶方阵, α是特征值λ的特征向量:

$$A\alpha=\lambda\alpha \Rightarrow A^2\alpha = A(A\alpha)=A(\lambda\alpha)=\lambda A\alpha=\lambda^2\alpha$$

$\Rightarrow \alpha$是A^2的特征值λ^2的特征向量.

一般的, 设 $A^k\alpha=\lambda^k\alpha$

$$\Rightarrow A^{k+1}\alpha = A(A^k\alpha)=A(\lambda^k\alpha)=\lambda^k A\alpha=\lambda^{k+1}\alpha$$

$\Rightarrow \alpha$是A^{k+1}的特征值λ^{k+1}的特征向量.

$f(x)$是一元多项式, 则

$\Rightarrow \alpha$是$f(A)$的特征值$f(\lambda)$的特征向量.

$f(x)$是一元多项式，α是特征值λ的特征向量.

$\Rightarrow \alpha$是$f(A)$的特征值$f(\lambda)$的特征向量.

设$f(x)=a_k x^k + a_{k-1}x^{k-1}+\cdots+a_1 x + a_0$

$$\Rightarrow f(A)=a_k A^k + a_{k-1}A^{k-1}+\cdots+a_1 A + a_0 I$$

$$\Rightarrow f(A)\alpha = a_k A^k\alpha + a_{k-1}A^{k-1}\alpha+\cdots+a_1 A\alpha + a_0 I\alpha$$

$$= a_k \lambda^k\alpha + a_{k-1}\lambda^{k-1}\alpha+\cdots+a_1 \lambda\alpha + a_0 \alpha$$

$$= \left(a_k \lambda^k + a_{k-1}\lambda^{k-1}+\cdots+a_1 \lambda + a_0\right)\alpha$$

$$= f(\lambda)\alpha$$

$f(x)$是一元多项式，α是特征值λ的特征向量.

$$\Rightarrow f(A)\alpha = f(\lambda)\alpha$$

结论: 设 n 阶方阵 A 满足 $f(A)=O$, 则A的特征值λ满足:

$$f(\lambda)=0.$$

证: 设α是特征值λ的特征向量, 则:

$$\left.\begin{array}{r} A\alpha=\lambda\alpha \Rightarrow 0=f(A)\alpha=f(\lambda)\alpha \\ \alpha\neq 0 \end{array}\right\} \Rightarrow f(\lambda)=0.$$

例7. 设方阵 A 满足 $A^2=A$，则A的特征值为0或者1.

例8. 设 A 是 n 阶可逆方阵, α是特征值λ的特征向量:

$$A\alpha = \lambda\alpha$$

$$\left.\begin{array}{r} \Rightarrow \alpha = A^{-1}(\lambda\alpha) = \lambda\left(A^{-1}\alpha\right) \\ A\text{可逆} \quad \Rightarrow \lambda \neq 0 \end{array}\right\} \Rightarrow A^{-1}\alpha = \frac{1}{\lambda}\alpha$$

$\Rightarrow \alpha$是A^{-1}的特征值λ^{-1}的特征向量.

$$A\alpha = \lambda\alpha$$

$$\left.\begin{array}{r} \Rightarrow A^{*}A\alpha = \lambda A^{*}\alpha \Rightarrow \ |A|\alpha = \lambda A^{*}\alpha \\ A\text{可逆} \quad \Rightarrow \lambda \neq 0 \end{array}\right\} \Rightarrow A^{*}\alpha = \frac{|A|}{\lambda}\alpha$$

$\Rightarrow \alpha$是A^{*}的特征值$\dfrac{|A|}{\lambda}$的特征向量.