三. 正交向量组与标准正交基

正交向量组:

α与β正交:$(\alpha, \beta) = 0$.

正交向量组:$\alpha_1, \alpha_2, \cdots, \alpha_s$

两两正交且不含零向量.

如:$\alpha_1 = (1, 1, 1), \alpha_2 = (-1, 2, -1), \alpha_3 = (-1, 0, 1)$

$$(\alpha_1, \alpha_2) = (\alpha_1, \alpha_3) = (\alpha_2, \alpha_3) = 0$$

$\alpha_1, \alpha_2, \alpha_3$:正交向量组

例1. 设 A 是 n 阶反对称矩阵, x 是 n 维列向量,

且 $Ax=y$. 证明: x 与 y 正交.

分析: A反对称 $\Rightarrow A^T=-A$

x 与 y 正交? $\Leftrightarrow (x,y)=0$?

证: $(x,y)=x^Ty=x^TAx$

两端同取转置

$$(x,y)=x^TA^Tx=-x^TAx=-x^Ty=-(x,y)$$

$$\Rightarrow (x,y)=0$$

定理1. 正交向量组必然线性无关.

证: 设 $\alpha_1, \alpha_2, \cdots, \alpha_s$ 是正交向量组, 且

$$k_1\alpha_1 + k_2\alpha_2 + \cdots + k_s\alpha_s = 0$$

$$\Rightarrow (\alpha_1, k_1\alpha_1 + k_2\alpha_2 + \cdots + k_s\alpha_s)$$

$$= k_1(\alpha_1, \alpha_1) + k_2(\alpha_1, \alpha_2) + \cdots + k_s(\alpha_1, \alpha_s)$$

$$= k_1(\alpha_1, \alpha_1) = 0,$$

$\because \ (\alpha_1, \alpha_1) > 0, \quad \therefore \ k_1 = 0,$

同理: $k_2 = k_3 = \cdots = k_s = 0,$

$\therefore \ \alpha_1, \alpha_2, \cdots, \alpha_s$ 线性无关

线性无关向量组未必是正交向量组.

如：$\alpha_1=(1,0,0),\alpha_2=(1,1,0),\alpha_3=(1,1,1)$

例2. 设 $\alpha_1=(1,1,1),\alpha_2=(1,-2,1)$,

求 α_3, 使 $\alpha_1,\alpha_2,\alpha_3$ 为正交向量组.

解: 设 $\alpha_3=(x_1,x_2,x_3)$, 则:

$$(\alpha_1,\alpha_3)=x_1+x_2+x_3=0$$

$$(\alpha_2,\alpha_3)=x_1-2x_2+x_3=0$$

$$\alpha_3=(1,0,-1).$$

标准正交向量组

$\alpha_1,\alpha_2,\cdots,\alpha_s$ 满足：

(1) $(\alpha_i,\alpha_j)=0,(i\neq j,\alpha_i\neq 0,\alpha_j\neq 0)$

(2) $\|\alpha_i\|=1,(i=1,2,\cdots,s)$

则称 $\alpha_1,\alpha_2,\cdots,\alpha_s$ 是标准(规范)正交向量组.

如 $\varepsilon_1=(1,0,\cdots,0),\varepsilon_2=(0,1,\cdots,0),\cdots,\varepsilon_n=(0,0,\cdots,1)$

是$\mathbb{R}^n$的标准正交基.

$$\alpha_1=\left(\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}\right),\alpha_2=\left(-\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}\right),\alpha_3=(0,1,0)$$

是$\mathbb{R}^3$的标准正交基.