二. 实对称矩阵的特征值与特征向量

定理1. 实对称矩阵的特征值都是实数.

证明: 设 $A\in\mathbb{R}^{n\times n}, A^T=A$, $\alpha=(z_1,z_2,\cdots,z_n)^T\neq 0$,

$A\alpha=\lambda\alpha$. 求证: $\lambda=\bar{\lambda}$.

$$A\alpha=\lambda\alpha \Rightarrow \overline{A\alpha}=\overline{\lambda\alpha} \Rightarrow \bar{A}\bar{\alpha}=\bar{\lambda}\bar{\alpha}$$

$$\Rightarrow \bar{\alpha}^T\bar{A}^T=\bar{\lambda}\bar{\alpha}^T \Rightarrow \bar{\alpha}^T A=\bar{\lambda}\bar{\alpha}^T$$

$$\Rightarrow \bar{\alpha}^T A\alpha=\bar{\lambda}\bar{\alpha}^T\alpha \Rightarrow \lambda\bar{\alpha}^T\alpha=\bar{\lambda}\bar{\alpha}^T\alpha,$$

$$\Rightarrow (\lambda-\bar{\lambda})\bar{\alpha}^T\alpha=0,$$

$$\bar{\alpha}^T\alpha=\bar{z}_1 z_1+\bar{z}_2 z_2+\cdots+\bar{z}_n z_n>0 \qquad \Rightarrow \lambda=\bar{\lambda}.$$

推论: 实对称矩阵A的任一特征值都有一个实特征向量.

Why ?

定理2: 实对称矩阵不同特征值的实特征向量相互正交.

证. 设 $A\alpha_1 = \lambda_1\alpha_1, A\alpha_2 = \lambda_2\alpha_2, \lambda_1 \neq \lambda_2,$

α_1, α_2是非零实向量. 求证: $\alpha_1^T\alpha_2 = 0.$

$$A\alpha_1 = \lambda_1\alpha_1 \Rightarrow \alpha_1^T A^T = \lambda_1\alpha_1^T \Rightarrow \alpha_1^T A = \lambda_1\alpha_1^T$$

$$\Rightarrow \alpha_1^T A\alpha_2 = \lambda_1\alpha_1^T\alpha_2 \Rightarrow \lambda_2\alpha_1^T\alpha_2 = \lambda_1\alpha_1^T\alpha_2$$

$$\Rightarrow (\lambda_1 - \lambda_2)\alpha_1^T\alpha_2 = 0 \Rightarrow (\alpha_1, \alpha_2) = \alpha_1^T\alpha_2 = 0.$$

$$(\lambda_1 - \lambda_2 \neq 0)$$