五. 矩阵方幂的计算

例7. 设矩阵 $A=\begin{pmatrix} 4 & 6 & 0 \\ -3 & -5 & 0 \\ -3 & -6 & 1 \end{pmatrix}$，求$A^{10}$.

解:

$$|\lambda I - A| = \begin{vmatrix} \lambda-4 & -6 & 0 \\ 3 & \lambda+5 & 0 \\ 3 & 6 & \lambda-1 \end{vmatrix} = (\lambda+2)(\lambda-1)^2$$

$$\Rightarrow \lambda_1 = -2，\lambda_2 = 1\,(二重).$$

例7. 设矩阵 $A=\begin{pmatrix} 4 & 6 & 0 \\ -3 & -5 & 0 \\ -3 & -6 & 1 \end{pmatrix}$，求$A^{10}$.

$\Rightarrow \lambda_1=-2,\ \lambda_2=1$（二重）.

$$(\lambda_1 I-A)=\begin{pmatrix} -6 & -6 & 0 \\ 3 & 3 & 0 \\ 3 & 6 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} x_1=-x_3, \\ x_2=x_3, \end{cases} \Rightarrow \alpha_1=\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$$\lambda_2 I - A = \begin{pmatrix} -3 & -6 & 0 \\ 3 & 6 & 0 \\ 3 & 6 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow x_1 = -2x_2 + 0x_3 \Rightarrow \alpha_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

令 $P = (\alpha_1 \quad \alpha_2 \quad \alpha_3) = \begin{pmatrix} -1 & -2 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

$$\Rightarrow P^{-1}AP = \begin{pmatrix} -2 & & \\ & 1 & \\ & & 1 \end{pmatrix} = \Lambda$$

$$A = P\Lambda P^{-1}$$

$$A^{10} = P\Lambda^{10}P^{-1}$$

$$= \begin{pmatrix} -1 & -2 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1024 & & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ -1 & -1 & 0 \\ -1 & -2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1022 & -2046 & 0 \\ 1023 & 2047 & 0 \\ 1023 & 2046 & 1 \end{pmatrix}$$

例8. 已知矩阵$A=\begin{pmatrix}-1 & 0 & 2\\ a & 1 & a-2\\ -3 & 0 & 4\end{pmatrix}$有3个线性无关的特征向量, 求$A$的值, 并求$A^n$.

分析: A有3个线性无关的特征向量

$\Rightarrow A$可以相似对角化

$$|\lambda I-A|=\begin{vmatrix}\lambda+1 & 0 & -2\\ -a & \lambda-1 & 2-a\\ 3 & 0 & \lambda-4\end{vmatrix}=(\lambda-1)^2(\lambda-2)$$

$\Rightarrow \lambda_1=1(2重), \lambda_2=2$

对A的2重特征值1: $R(1I-A)=3-2=1$

$$A=\begin{pmatrix}-1 & 0 & 2\\ a & 1 & a-2\\ -3 & 0 & 4\end{pmatrix}$$

$$\Rightarrow R(1I-A)=3-2=1$$

$$I-A=\begin{pmatrix}2 & 0 & -2\\ -a & 0 & 2-a\\ -3 & 0 & -3\end{pmatrix}\to\cdots\to\begin{pmatrix}1 & 0 & -1\\ 0 & 0 & 2-2a\\ 0 & 0 & 0\end{pmatrix}$$

$$\Rightarrow a=1 \quad \Rightarrow A=\begin{pmatrix}-1 & 0 & 2\\ 1 & 1 & -1\\ -3 & 0 & 4\end{pmatrix}$$

$$\Rightarrow \lambda_1 = 1:\ \alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \lambda_2 = 2 : \alpha_3 = \begin{pmatrix} -2 \\ 1 \\ -3 \end{pmatrix}.$$

$$令 P = (\alpha_1, \alpha_2, \alpha_3) = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & -3 \end{pmatrix}$$

$$\Rightarrow A = P \begin{pmatrix} 1 & & \\ & 1 & \\ & & 2 \end{pmatrix} P^{-1} \Rightarrow A^n = \begin{pmatrix} 3 - 2^{n+1} & 0 & 2^{n+1} - 2 \\ 2^n - 1 & 1 & 1 - 2^n \\ 3 - 3 \cdot 2^n & 0 & 3 \cdot 2^n - 2 \end{pmatrix}$$