设二次型 $f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 + 2ax_1x_2 + 2x_1x_3 + 2bx_2x_3$

经过正交变换 $X = CY$ 化成 $f = y_2^2 + 2y_3^2$，其中 $X = (x_1, x_2, x_3)^T$，

$Y = (y_1, y_2, y_3)^T$ 是三维列向量，C 为三阶正交矩阵，求常数 a, b 的值.

解析： 二次型矩阵和标准形矩阵分别为

$$A = \begin{pmatrix} 1 & a & 1 \\ a & 1 & b \\ 1 & b & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

$$\Rightarrow f = X^T A X = Y^T B Y, \quad C^T A C = B.$$

$$|\lambda I - A| = |\lambda I - B|,$$

$$\begin{vmatrix} \lambda-1 & -a & -1 \\ -a & \lambda-1 & -b \\ -1 & -b & \lambda-1 \end{vmatrix} = \begin{vmatrix} \lambda & 0 & 0 \\ 0 & \lambda-1 & 0 \\ 0 & 0 & \lambda-2 \end{vmatrix},$$

$$\lambda^3 - 3\lambda^2 + \left(2 - a^2 - b^2\right)\lambda + \left(a-b\right)^2 = \lambda^3 - 3\lambda^2 + 2\lambda,$$

$$\Rightarrow a = b = 0.$$