二. 分块矩阵的运算

加法: 同型矩阵 $A = \begin{pmatrix} A_{11} & \cdots & A_{1s} \\ \vdots & & \vdots \\ A_{r1} & \cdots & A_{rs} \end{pmatrix}, B = \begin{pmatrix} B_{11} & \cdots & B_{1s} \\ \vdots & & \vdots \\ B_{r1} & \cdots & B_{rs} \end{pmatrix}$

$$A + B = \begin{pmatrix} A_{11} + B_{11} & \cdots & A_{1s} + B_{1s} \\ \vdots & & \vdots \\ A_{r1} + B_{r1} & \cdots & A_{rs} + B_{rs} \end{pmatrix}.$$

要求: 两矩阵的行、列分块方法一致

数乘: 分块矩阵 $A = (A_{ij})_{s \times t}$, $\quad kA = (kA_{ij})_{s \times t}$.

乘法： $A = (a_{ij})_{m \times n}, \ B = (b_{ij})_{n \times p},$

$$AB = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1s} \\ \vdots & \vdots & & \vdots \\ A_{r1} & A_{r2} & \cdots & A_{rs} \end{pmatrix} \begin{pmatrix} B_{11} & \cdots & B_{1t} \\ B_{21} & \cdots & B_{2t} \\ \cdots & \cdots & \cdots \\ B_{s1} & \cdots & B_{st} \end{pmatrix} = C = (C_{kl})$$

其中C是 $\boldsymbol{r \times t}$ 矩阵

$$C_{kl} = \sum_{i=1}^{s} A_{ki} B_{il} \quad (k = 1,...,r; \ l = 1,...,t)$$

要求： 1) 矩阵A的列数=矩阵B的行数
　　　　2) 矩阵A列的分法=矩阵B行的分法

例1. 求 AB: $A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 3 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 3 & -1 \\ 0 & 2 & 1 & 4 \end{pmatrix}$

解:

$$AB = \left(\begin{array}{cc|cc} 1 & -1 & 0 & 0 \\ 3 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 2 & -1 \end{array}\right) \left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ \hline 0 & 1 & 3 & -1 \\ 0 & 2 & 1 & 4 \end{array}\right) = \begin{pmatrix} A_1 & O \\ O & A_2 \end{pmatrix}\begin{pmatrix} B_1 & O \\ O & B_2 \end{pmatrix}$$

$$= \begin{pmatrix} A_1 B_1 & O \\ O & A_2 B_2 \end{pmatrix} = \left(\begin{array}{c|ccc} 2 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 5 & -6 \end{array}\right)$$

设 A, B 均为 n 阶矩阵，则

$$A = \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_m \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & & \\ & \ddots & \\ & & B_m \end{pmatrix},$$

$$AB = \begin{pmatrix} A_1 B_1 & & \\ & \ddots & \\ & & A_m B_m \end{pmatrix},$$

$$A^k = ?$$

注意： 将矩阵分块作乘法其分法不是唯一的.如,例1中

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 3 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 3 & -1 \\ 0 & 2 & 1 & 4 \end{pmatrix} = \begin{pmatrix} A_1 & O \\ O & A_2 \end{pmatrix} \begin{pmatrix} B_1 & O \\ O & B_2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 3 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 3 & -1 \\ 0 & 2 & 1 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 3 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 3 & -1 \\ 0 & 2 & 1 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} A_{11} & O \\ O & A_{22} \end{pmatrix} \begin{pmatrix} B_{11} & O & O \\ O & B_{22} & B_{23} \end{pmatrix}$$

$$= \begin{pmatrix} A_{11}B_{11} & O & O \\ O & A_{22}B_{22} & A_{22}B_{23} \end{pmatrix}$$

例2. 如何分块求AB:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 2 & 0 & 1 & 0 \\ 1 & 3 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{pmatrix}$$

解

$$A = \begin{pmatrix} I_2 & O_{2\times 3} \\ A_1 & I_3 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & I_2 \\ -I_3 & O_{3\times 2} \end{pmatrix}$$

$$AB = \begin{pmatrix} I_2 & O_{2\times 3} \\ A_1 & I_3 \end{pmatrix}\begin{pmatrix} B_1 & I_2 \\ -I_3 & O_{3\times 2} \end{pmatrix} = \begin{pmatrix} B_1 & I_2 \\ A_1 B_1 - I_3 & A_1 \end{pmatrix}$$

转置:

$$A = (A_{kl})_{s \times t} \Rightarrow A^T = (B_{lk})_{t \times s}$$

$$其中 B_{lk} = A_{kl}^T, \quad l = 1, \ldots, t; \quad k = 1, \ldots, s$$

例3.

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{pmatrix}$$

$$\Rightarrow A^T = \begin{pmatrix} A_{11}^T & A_{21}^T \\ A_{12}^T & A_{22}^T \\ A_{13}^T & A_{23}^T \end{pmatrix}$$

逆:

$$D = \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix}, (d_1, ..., d_n \neq 0) \Rightarrow D^{-1} = \begin{pmatrix} \frac{1}{d_1} & & \\ & \ddots & \\ & & \frac{1}{d_n} \end{pmatrix}$$

$$A = \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_m \end{pmatrix}, A_1, ..., A_m 均可逆$$

$$\Rightarrow A^{-1} = \begin{pmatrix} A_1^{-1} & & \\ & \ddots & \\ & & A_m^{-1} \end{pmatrix}$$

$$A = \begin{pmatrix} & & A_1 \\ & \ddots & \\ A_m & & \end{pmatrix}, A_1, ..., A_m \text{均可逆}$$

$$\Rightarrow A^{-1} = \begin{pmatrix} & & A_m^{-1} \\ & \ddots & \\ A_1^{-1} & & \end{pmatrix}.$$

特殊： $A = \begin{pmatrix} & & d_1 \\ & \ddots & \\ d_m & & \end{pmatrix}, d_1 d_2 ... d_m \neq 0$

$$\Rightarrow A^{-1} = \begin{pmatrix} & & d_m^{-1} \\ & \ddots & \\ d_1^{-1} & & \end{pmatrix}$$

例**4.** 设矩阵 $A = \begin{pmatrix} 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 & 0 \end{pmatrix}$，求 A 的逆.

解

$$A = \left(\begin{array}{cc|ccc} 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ \hline 1 & 2 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 & 0 \end{array}\right) = \begin{pmatrix} O & A_1 \\ A_2 & O \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} O & A_2^{-1} \\ A_1^{-1} & O \end{pmatrix} = \cdots\cdots$$

[结束]