三. 定义计算简单的行列式

例3. 求$\det A$：

$$A=\begin{pmatrix} 1 & -3 & 7 \\ 2 & 4 & -3 \\ -3 & 7 & 2 \end{pmatrix}$$

解

$$\det A = 1(-1)^{1+1}\begin{vmatrix} 4 & -3 \\ 7 & 2 \end{vmatrix} + (-3)(-1)^{1+2}\begin{vmatrix} 2 & -3 \\ -3 & 2 \end{vmatrix}$$

$$+7(-1)^{1+3}\begin{vmatrix} 2 & 4 \\ -3 & 7 \end{vmatrix}$$

$$=(8+21)+3(4-9)+7(14+12)=196$$

例4. 计算下三角行列式 $D_n = \begin{vmatrix} a_{11} & & & \\ a_{21} & a_{22} & & \text{O} \\ \vdots & \vdots & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$

解 $D_n = a_{11} \begin{vmatrix} a_{22} & & & \\ a_{32} & a_{33} & & \text{O} \\ \vdots & \vdots & \ddots & \\ a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix} = a_{11}a_{22} \begin{vmatrix} a_{33} & & & \\ a_{43} & a_{44} & & \text{O} \\ \vdots & \vdots & \ddots & \\ a_{n3} & a_{n4} & \cdots & a_{nn} \end{vmatrix}$

$= \cdots = \boxed{a_{11}a_{22}\cdots a_{nn}}$

同理，$\det(\mathrm{diag}(a_{11}, a_{22}, \cdots, a_{nn})) = a_{11}a_{22}\cdots a_{nn}$

$$\boxed{\det(kI_n) = k^n,\ \det I = 1}$$

例5. 计算斜下三角行列式 $D_n = \begin{vmatrix} & & 0 & & a_n \\ & & & \cdot^{\cdot^{\cdot}} & \\ & & a_2 & & \\ a_1 & & & * & \end{vmatrix}$

解 $D_n = a_n(-1)^{1+n}\begin{vmatrix} & & 0 & & a_{n-1} \\ & & & \cdot^{\cdot^{\cdot}} & \\ & & a_2 & & \\ a_1 & & & * & \end{vmatrix} = (-1)^{n-1}a_n D_{n-1}$

$$= (-1)^{n-1}a_n(-1)^{n-2}a_{n-1}D_{n-2} = \cdots = (-1)^{(n-1)+(n-2)+\cdots+1}a_n a_{n-1}\cdots a_1$$

$$= \boxed{(-1)^{\frac{n(n-1)}{2}}a_1 a_2\cdots a_n}$$

同理， $$D_n=\begin{vmatrix} & 0 & & a_n \\ & & \iddots & \\ & a_2 & & \\ a_1 & & 0 & \end{vmatrix}=(-1)^{\frac{n(n-1)}{2}}a_1a_2\cdots a_n$$

$$D_n=\begin{vmatrix} & 0 & & 1 \\ & & \iddots & \\ & 1 & & \\ 1 & & 0 & \end{vmatrix}=(-1)^{\frac{n(n-1)}{2}}$$

$$D_n = \begin{vmatrix} a_{11} & & & \\ a_{21} & a_{22} & \text{O} & \\ \vdots & \vdots & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = a_{11}a_{22}\cdots a_{nn}$$

$$D_n = \begin{vmatrix} & 0 & & a_n \\ & & \cdot^{\cdot^{\cdot}} & \\ & a_2 & & \\ a_1 & & * & \end{vmatrix} = (-1)^{\frac{n(n-1)}{2}} a_1 a_2 \cdots a_n$$

思考:

$$D_n = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & \cdots & a_{2n} \\ & & \ddots & \vdots \\ & 0 & & a_{nn} \end{vmatrix} = ?$$

$$D_n = \begin{vmatrix} & * & & a_n \\ & & \cdot^{\cdot^{\cdot}} & \\ & a_2 & & \\ a_1 & & 0 & \end{vmatrix} = ?$$

[结束]