第六讲　习题课

- 一.习题1
- 二.习题2

《内容小结》

1. 行列式的概念、性质及计算.

2. 线性方程组求解的克拉默法则.

3. 矩阵秩的概念、性质及计算.

习题1

例1 设$\alpha_1,\alpha_2,\alpha_3$均为3维向量,矩阵$A=(\alpha_1,\alpha_2,\alpha_3)$,

$B=(\alpha_1+\alpha_2+\alpha_3,\alpha_1+2\alpha_2+4\alpha_3,\alpha_1+3\alpha_2+9\alpha_3)$,

若$|A|=1$,计算$|B|$.

分析: $A\xrightarrow{\text{列初等变换}}B\Rightarrow B\xrightarrow{\text{列初等变换(逆)}}A$

解1: $|B|=|\alpha_1+\alpha_2+\alpha_3,\alpha_1+2\alpha_2+4\alpha_3,\alpha_1+3\alpha_2+9\alpha_3|$

$$\overset{-c_1+c_2}{=}|\alpha_1+\alpha_2+\alpha_3,\alpha_2+3\alpha_3,\alpha_1+3\alpha_2+9\alpha_3|$$

$$\overset{-3c_2+c_3}{=}|\alpha_1+\alpha_2+\alpha_3,\alpha_2+3\alpha_3,\alpha_1|$$

$$\overset{-c_3+c_1}{=}|\alpha_2+\alpha_3,\alpha_2+3\alpha_3,\alpha_1|\overset{-c_1+c_2}{=}|\alpha_2+\alpha_3,2\alpha_3,\alpha_1|$$

$$=2|\alpha_2+\alpha_3,\alpha_3,\alpha_1|=2|\alpha_2,\alpha_3,\alpha_1|=2|A|=2.$$

分析: $A=(\alpha_1,\alpha_2,\alpha_3) \Rightarrow B=AC \Rightarrow |B|=|A|\cdot|C|$

解2: $B=(\alpha_1+\alpha_2+\alpha_3,\alpha_1+2\alpha_2+4\alpha_3,\alpha_1+3\alpha_2+9\alpha_3)$

$$=(\alpha_1,\alpha_2,\alpha_3)\begin{pmatrix}1&1&1\\1&2&3\\1&4&9\end{pmatrix}=AC$$

且 $|C|=\begin{vmatrix}1&1&1\\1&2&3\\1&4&9\end{vmatrix}=\begin{vmatrix}1&1&1\\0&1&2\\0&3&8\end{vmatrix}=2 \Rightarrow |B|=|AC|$

$$=|A|\cdot|C|=1\times2=2$$

例2 设A为3阶矩阵,且$|A|=\dfrac{1}{2}$,计算$\left|\left(\dfrac{1}{3}A\right)^{-1}-10A^{*}\right|$.

$\boxed{\text{分析: } AA^{*}=(\det A)I \Rightarrow A^{*}=(\det A)A^{-1}}$

解:

$$\left(\frac{1}{3}A\right)^{-1}-10A^{*}=3A^{-1}-10|A|A^{-1}$$

$$=3A^{-1}-5A^{-1}$$

$$=(-2)A^{-1}$$

$\boxed{A_{n\times n} \Rightarrow |kA|=k^{n}|A|}$

$$\left|\left(\frac{1}{3}A\right)^{-1}-10A^{*}\right|=|(-2)A^{-1}|=(-2)^{3}|A^{-1}|$$

$$=(-2)^{3}\cdot 2=-16.$$

例3 证明 $$\begin{vmatrix} a_1+b_1x & a_1x+b_1 & c_1 \\ a_2+b_2x & a_2x+b_2 & c_2 \\ a_3+b_3x & a_3x+b_3 & c_3 \end{vmatrix} = (1-x^2)\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

分析：用行列式的初等变换法

证1: $$\begin{vmatrix} a_1+b_1x & a_1x+b_1 & c_1 \\ a_2+b_2x & a_2x+b_2 & c_2 \\ a_3+b_3x & a_3x+b_3 & c_3 \end{vmatrix} \xlongequal{-xc_2+c_1} \begin{vmatrix} a_1(1-x^2) & a_1x+b_1 & c_1 \\ a_2(1-x^2) & a_2x+b_2 & c_2 \\ a_3(1-x^2) & a_3x+b_3 & c_3 \end{vmatrix}$$

$$=(1-x^2)\begin{vmatrix} a_1 & a_1x+b_1 & c_1 \\ a_2 & a_2x+b_2 & c_2 \\ a_3 & a_3x+b_3 & c_3 \end{vmatrix} \xlongequal{-xc_1+c_2} (1-x^2)\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

分析: $\det(AB) = (\det A)(\det B)$

证2: $$\begin{pmatrix} a_1+b_1x & a_1x+b_1 & c_1 \\ a_2+b_2x & a_2x+b_2 & c_2 \\ a_3+b_3x & a_3x+b_3 & c_3 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}\begin{pmatrix} 1 & x & 0 \\ x & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{vmatrix} a_1+b_1x & a_1x+b_1 & c_1 \\ a_2+b_2x & a_2x+b_2 & c_2 \\ a_3+b_3x & a_3x+b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}\begin{vmatrix} 1 & x & 0 \\ x & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= (1-x^2)\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

例4 设$A=\begin{pmatrix}3&1&1&4\\ \lambda&4&10&1\\ 1&7&17&3\\ 2&2&4&3\end{pmatrix}$，讨论 $R(A)$ 与 λ的关系.

分析：$A\rightarrow B$(行阶梯形) $\Rightarrow R(A)=r_B$(B的非零行行数)

解：$A=\begin{pmatrix}3&1&1&4\\ \lambda&4&10&1\\ 1&7&17&3\\ 2&2&4&3\end{pmatrix}\rightarrow\begin{pmatrix}1&7&17&3\\ 3&1&1&4\\ 2&2&4&3\\ \lambda&4&10&1\end{pmatrix}\xrightarrow[-\lambda r_1+r_4]{\substack{-3r_1+r_2\\ -2r_1+r_3}}$

$$\begin{pmatrix}1&7&17&3\\ 0&-20&-50&-5\\ 0&-12&-30&-3\\ 0&4-7\lambda&10-17\lambda&1-3\lambda\end{pmatrix}\xrightarrow[-\frac{1}{3}r_3]{-\frac{1}{5}r_2}\begin{pmatrix}1&7&17&3\\ 0&4&10&1\\ 0&4&10&1\\ 0&4-7\lambda&10-17\lambda&1-3\lambda\end{pmatrix}$$

$$\xrightarrow[-r_2+r_4]{-r_2+r_3} \begin{pmatrix} 1 & 7 & 17 & 3 \\ 0 & 4 & 10 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -7\lambda & -17\lambda & -3\lambda \end{pmatrix} \xrightarrow{-r_4} \begin{pmatrix} 1 & 7 & 17 & 3 \\ 0 & 4 & 10 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 7\lambda & 17\lambda & 3\lambda \end{pmatrix}$$

(1) $\lambda = 0 \Rightarrow r = 2 \Rightarrow R(A) = 2;$

(2) $\lambda \neq 0 \Rightarrow r = ?$

$$\begin{pmatrix} 1 & 7 & 17 & 3 \\ 0 & 4 & 10 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 7\lambda & 17\lambda & 3\lambda \end{pmatrix} \xrightarrow[\frac{1}{\lambda}r_4]{r_3 \leftrightarrow r_4} \begin{pmatrix} 1 & 7 & 17 & 3 \\ 0 & 4 & 10 & 1 \\ 0 & 7 & 17 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{-\frac{7}{4}r_2+r_3}$$

$$\begin{pmatrix} 1 & 7 & 17 & 3 \\ 0 & 4 & 10 & 1 \\ 0 & 0 & \dfrac{-1}{2} & \dfrac{5}{4} \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow r = 3 \Rightarrow R(A) = 3.$$

故 $\begin{cases} (1)\ \lambda = 0 \Rightarrow R(A) = 2 \\ (2)\ \lambda \neq 0 \Rightarrow R(A) = 3. \end{cases}$

例5 计算行列式 $D=\begin{vmatrix} 7 & 6 & 5 & 4 & 3 & 2 \\ 9 & 7 & 8 & 9 & 4 & 3 \\ 7 & 4 & 9 & 7 & 0 & 0 \\ 5 & 3 & 6 & 1 & 0 & 0 \\ 0 & 0 & 5 & 6 & 0 & 0 \\ 0 & 0 & 6 & 8 & 0 & 0 \end{vmatrix}$.

解： $D=\begin{vmatrix} 3 & 2 \\ 4 & 3 \end{vmatrix}\cdot(-1)^{(1+2)+(5+6)}\begin{vmatrix} 7 & 4 & 9 & 7 \\ 5 & 3 & 6 & 1 \\ 0 & 0 & 5 & 6 \\ 0 & 0 & 6 & 8 \end{vmatrix}=\begin{vmatrix} 7 & 4 \\ 5 & 3 \end{vmatrix}\cdot\begin{vmatrix} 5 & 6 \\ 6 & 8 \end{vmatrix}$

$=4.$