第六讲　习题课

一.习题1

二.习题2

《内容小结》

1. 行列式的概念、性质及计算.

2. 线性方程组求解的克拉默法则.

3. 矩阵秩的概念、性质及计算.

习题1

例1 设$\alpha_1,\alpha_2,\alpha_3$均为3维向量,矩阵$A=(\alpha_1,\alpha_2,\alpha_3)$,

$$B=(\alpha_1+\alpha_2+\alpha_3,\alpha_1+2\alpha_2+4\alpha_3,\alpha_1+3\alpha_2+9\alpha_3),$$

若$|A|=1$,计算$|B|$.

$$\boxed{\text{分析: } A \xrightarrow{\text{列初等变换}} B \Rightarrow B \xrightarrow{\text{列初等变换(逆)}} A}$$

解1: $$|B|=|\alpha_1+\alpha_2+\alpha_3,\alpha_1+2\alpha_2+4\alpha_3,\alpha_1+3\alpha_2+9\alpha_3|$$

$$\overset{-c_1+c_2}{=}|\alpha_1+\alpha_2+\alpha_3,\alpha_2+3\alpha_3,\alpha_1+3\alpha_2+9\alpha_3|$$

$$\overset{-3c_2+c_3}{=}|\alpha_1+\alpha_2+\alpha_3,\alpha_2+3\alpha_3,\alpha_1|$$

$$\overset{-c_3+c_1}{=}|\alpha_2+\alpha_3,\alpha_2+3\alpha_3,\alpha_1|\overset{-c_1+c_2}{=}|\alpha_2+\alpha_3,2\alpha_3,\alpha_1|$$

$$=2|\alpha_2+\alpha_3,\alpha_3,\alpha_1|=2|\alpha_2,\alpha_3,\alpha_1|=2|A|=2.$$

分析: $A=(\alpha_1,\alpha_2,\alpha_3) \Rightarrow B=AC \Rightarrow |B|=|A|\cdot|C|$

解2: $B=(\alpha_1+\alpha_2+\alpha_3,\alpha_1+2\alpha_2+4\alpha_3,\alpha_1+3\alpha_2+9\alpha_3)$

$$=(\alpha_1,\alpha_2,\alpha_3)\begin{pmatrix}1&1&1\\1&2&3\\1&4&9\end{pmatrix}=AC$$

且 $|C|=\begin{vmatrix}1&1&1\\1&2&3\\1&4&9\end{vmatrix}=\begin{vmatrix}1&1&1\\0&1&2\\0&3&8\end{vmatrix}=2 \Rightarrow |B|=|AC|$

$$=|A|\cdot|C|=1\times 2=2$$

例2 设A为3阶矩阵,且 $|A|=\frac{1}{2}$, 计算$|(\frac{1}{3}A)^{-1}-10A^*|$.

$$\boxed{\text{分析: } AA^*=(\det A)I \Rightarrow A^*=(\det A)A^{-1}}$$

解: $$(\frac{1}{3}A)^{-1}-10A^*=3A^{-1}-10|A|A^{-1}$$

$$=3A^{-1}-5A^{-1}$$

$$=(-2)A^{-1}$$

$$\boxed{A_{n\times n} \Rightarrow |kA|=k^n|A|}$$

$$|(\frac{1}{3}A)^{-1}-10A^*|=|(-2)A^{-1}|=(-2)^3|A^{-1}|$$

$$=(-2)^3\bullet 2 \quad =-16.$$

例3 证明 $\begin{vmatrix} a_1+b_1x & a_1x+b_1 & c_1 \\ a_2+b_2x & a_2x+b_2 & c_2 \\ a_3+b_3x & a_3x+b_3 & c_3 \end{vmatrix} = (1-x^2)\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

分析：用行列式的初等变换法

证1：

$$\begin{vmatrix} a_1+b_1x & a_1x+b_1 & c_1 \\ a_2+b_2x & a_2x+b_2 & c_2 \\ a_3+b_3x & a_3x+b_3 & c_3 \end{vmatrix} \overset{-xc_2+c_1}{=\!=\!=} \begin{vmatrix} a_1(1-x^2) & a_1x+b_1 & c_1 \\ a_2(1-x^2) & a_2x+b_2 & c_2 \\ a_3(1-x^2) & a_3x+b_3 & c_3 \end{vmatrix}$$

$$=(1-x^2)\begin{vmatrix} a_1 & a_1x+b_1 & c_1 \\ a_2 & a_2x+b_2 & c_2 \\ a_3 & a_3x+b_3 & c_3 \end{vmatrix} \overset{-xc_1+c_2}{=\!=\!=\!=} (1-x^2)\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

分析: $\det(AB)=(\det A)(\det B)$

证2: $$\begin{pmatrix} a_1+b_1x & a_1x+b_1 & c_1 \\ a_2+b_2x & a_2x+b_2 & c_2 \\ a_3+b_3x & a_3x+b_3 & c_3 \end{pmatrix}=\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}\begin{pmatrix} 1 & x & 0 \\ x & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{vmatrix} a_1+b_1x & a_1x+b_1 & c_1 \\ a_2+b_2x & a_2x+b_2 & c_2 \\ a_3+b_3x & a_3x+b_3 & c_3 \end{vmatrix}=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}\begin{vmatrix} 1 & x & 0 \\ x & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$=(1-x^2)\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

例4 设$A=\begin{pmatrix} 3 & 1 & 1 & 4 \\ \lambda & 4 & 10 & 1 \\ 1 & 7 & 17 & 3 \\ 2 & 2 & 4 & 3 \end{pmatrix}$，讨论 $R(A)$ 与 λ的关系.

分析: $A \to B$(行阶梯形) $\Rightarrow R(A)=r_B$(B的非零行行数)

解: $$A=\begin{pmatrix} 3 & 1 & 1 & 4 \\ \lambda & 4 & 10 & 1 \\ 1 & 7 & 17 & 3 \\ 2 & 2 & 4 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 7 & 17 & 3 \\ 3 & 1 & 1 & 4 \\ 2 & 2 & 4 & 3 \\ \lambda & 4 & 10 & 1 \end{pmatrix} \xrightarrow[-\lambda r_1+r_4]{\substack{-3r_1+r_2 \\ -2r_1+r_3}}$$

$$\begin{pmatrix} 1 & 7 & 17 & 3 \\ 0 & -20 & -50 & -5 \\ 0 & -12 & -30 & -3 \\ 0 & 4-7\lambda & 10-17\lambda & 1-3\lambda \end{pmatrix} \xrightarrow[-\frac{1}{3}r_3]{-\frac{1}{5}r_2} \begin{pmatrix} 1 & 7 & 17 & 3 \\ 0 & 4 & 10 & 1 \\ 0 & 4 & 10 & 1 \\ 0 & 4-7\lambda & 10-17\lambda & 1-3\lambda \end{pmatrix}$$

$$\xrightarrow[-r_2+r_4]{-r_2+r_3}\begin{pmatrix}1&7&17&3\\0&4&10&1\\0&0&0&0\\0&-7\lambda&-17\lambda&-3\lambda\end{pmatrix}\xrightarrow{-r_4}\begin{pmatrix}1&7&17&3\\0&4&10&1\\0&0&0&0\\0&7\lambda&17\lambda&3\lambda\end{pmatrix}$$

(1) $\lambda=0\Rightarrow r=2\Rightarrow R(A)=2;$

(2) $\lambda\neq 0\Rightarrow r=?$

$$\begin{pmatrix}1&7&17&3\\0&4&10&1\\0&0&0&0\\0&7\lambda&17\lambda&3\lambda\end{pmatrix}\xrightarrow[\frac{1}{\lambda}r_4]{r_3\leftrightarrow r_4}\begin{pmatrix}1&7&17&3\\0&4&10&1\\0&7&17&3\\0&0&0&0\end{pmatrix}\xrightarrow{-\frac{7}{4}r_2+r_3}$$

$$\begin{pmatrix} 1 & 7 & 17 & 3 \\ 0 & 4 & 10 & 1 \\ 0 & 0 & \frac{-1}{2} & \frac{5}{4} \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow r=3 \Rightarrow R(A)=3.$$

故 $\begin{cases} (1)\ \lambda=0 \Rightarrow R(A)=2 \\ (2)\ \lambda\neq 0 \Rightarrow R(A)=3. \end{cases}$

例5 计算行列式 $D=\begin{vmatrix} 7 & 6 & 5 & 4 & 3 & 2 \\ 9 & 7 & 8 & 9 & 4 & 3 \\ 7 & 4 & 9 & 7 & 0 & 0 \\ 5 & 3 & 6 & 1 & 0 & 0 \\ 0 & 0 & 5 & 6 & 0 & 0 \\ 0 & 0 & 6 & 8 & 0 & 0 \end{vmatrix}$.

解：$D=\begin{vmatrix} 3 & 2 \\ 4 & 3 \end{vmatrix}\cdot(-1)^{(1+2)+(5+6)}\begin{vmatrix} 7 & 4 & 9 & 7 \\ 5 & 3 & 6 & 1 \\ 0 & 0 & 5 & 6 \\ 0 & 0 & 6 & 8 \end{vmatrix}=\begin{vmatrix} 7 & 4 \\ 5 & 3 \end{vmatrix}\cdot\begin{vmatrix} 5 & 6 \\ 6 & 8 \end{vmatrix}$

$=4.$

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习题2

例1 设矩阵$A=\begin{pmatrix} k & 1 & 1 & 1 \\ 1 & k & 1 & 1 \\ 1 & 1 & k & 1 \\ 1 & 1 & 1 & k \end{pmatrix}$且 $R(A)=3$, 求 k.

$R(A)=3<4$
$\Rightarrow |A|=0.$

$|A|=0$
$\Leftrightarrow R(A)<4$

解: $|A| \xlongequal[i=2,3,4]{r_i+r_1} (k+3)\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & k & 1 & 1 \\ 1 & 1 & k & 1 \\ 1 & 1 & 1 & k \end{vmatrix}$

$\xlongequal[i=2,3,4]{-c_1+c_i} (k+3)\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & k-1 & 0 & 0 \\ 1 & 0 & k-1 & 0 \\ 1 & 0 & 0 & k-1 \end{vmatrix} = (k+3)(k-1)^3$

(1) $|A|=0 \Rightarrow k=-3$或$k=1.$

(2) $k=1 \Rightarrow A=\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \Rightarrow R(A)=1.$

这与秩$R(A)=3$相矛盾！ 故$k=-3.$

例2 λ,μ取何值时,线性方程组$\begin{cases} \lambda x_1 + x_2 + x_3 = 0 \\ x_1 + \mu x_2 + x_3 = 0 \\ x_1 + 2\mu x_2 + x_3 = 0 \end{cases}$

有非零解?当$\mu=1$时,求其全部非零解.

$Ax = 0(x \neq 0)$

$\Leftrightarrow |A| = 0.$

解: (1) $|A| = \begin{vmatrix} \lambda & 1 & 1 \\ 1 & \mu & 1 \\ 1 & 2\mu & 1 \end{vmatrix} = \mu - \mu\lambda = \mu(1-\lambda) = 0$

即$\mu=0$或$\lambda=1$时齐次线性方程组有非零解.

(2) $\mu=1$时,要有非零解,只能$\lambda=1$. 此时

$$A=\begin{pmatrix}1&1&1\\1&1&1\\1&2&1\end{pmatrix}\to\begin{pmatrix}1&1&1\\0&0&0\\0&1&0\end{pmatrix}\to\begin{pmatrix}1&1&1\\0&1&0\\0&0&0\end{pmatrix}\to\begin{pmatrix}1&0&1\\0&1&0\\0&0&0\end{pmatrix},$$

同解方程组为:

$$\begin{cases}x_1+x_3=0\\ x_2=0\end{cases}\Leftrightarrow\begin{cases}x_1=-x_3\\ x_2=0\end{cases}\Leftrightarrow\begin{cases}x_1=-k\\ x_2=0\\ x_3=k\end{cases},(k\neq 0).$$

例3 若一元n次方程

$$p(x)=a_0+a_1x+a_2x^2+\cdots+a_nx^n=0$$

有$n+1$个不同的根，证明：

$$p(x)=a_0+a_1x+a_2x^2+\cdots+a_nx^n\equiv 0$$

$$\boxed{\text{分析: } p(x)\equiv 0 \Leftrightarrow a_0=a_1=a_2=\cdots=a_n=0}$$

证: 设 $x_1,x_2,\cdots,x_n,x_{n+1}$ 为其$n+1$个不同的根,即

$$\begin{cases} a_0+a_1x_1+a_2x_1^2+\cdots+a_nx_1^n=0 \\ a_0+a_1x_2+a_2x_2^2+\cdots+a_nx_2^n=0 \\ \qquad\vdots \\ a_0+a_1x_{n+1}+a_2x_{n+1}^2+\cdots+a_nx_{n+1}^n=0 \end{cases}$$

$$\Leftrightarrow \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^n \\ 1 & x_2 & x_2^2 & \cdots & x_2^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \\ 1 & x_{n+1} & x_{n+1}^2 & \cdots & x_{n+1}^n \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$V = \prod_{1 \le j < i \le n+1} (x_i - x_j) \ne 0 \quad \Rightarrow a_0 = a_1 = \cdots = a_n = 0$$

$$\Rightarrow \ a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \equiv 0$$

例4 设n阶行列式

$$D_n=\begin{vmatrix}1 & 2 & 3 & \cdots & n\\ 1 & 2 & 0 & \cdots & 0\\ 1 & 0 & 3 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & 0 & 0 & \cdots & n\end{vmatrix}$$

求第一行各元素的代数余子式之和:

$$A_{11}+A_{12}+\cdots+A_{1n}.$$

$$\det A = a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n}$$

$$A_{11} + A_{12} + \cdots + A_{1n} = 1 \cdot A_{11} + 1 \cdot A_{12} + \cdots + 1 \cdot A_{1n}$$

解: 第一行各元素的代数余子式之和可以表示成

$$A_{11} + A_{12} + \cdots + A_{1n} = 1 \cdot A_{11} + 1 \cdot A_{12} + \cdots + 1 \cdot A_{1n}$$

$$= \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 0 & \cdots & 0 \\ 1 & 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & n \end{vmatrix} = \begin{vmatrix} 1 - \sum_{j=2}^{n} \frac{1}{j} & 1 & 1 & \cdots & 1 \\ 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & n \end{vmatrix} = n!\left(1 - \sum_{j=2}^{n} \frac{1}{j}\right).$$

例5 设A,B为n阶可逆矩阵，证明：$\begin{cases}(1)\ (AB)^*=B^*A^* \\ (2)\ (A^*)^{-1}=(A^{-1})^*\end{cases}$

$$\boxed{(AB)^{-1}=B^{-1}A^{-1}}$$

证：$AA^*=(\det A)I \Rightarrow A^*=(\det A)A^{-1}$

$$\Rightarrow (AB)^*=(\det(AB))(AB)^{-1}=(\det A)(\det B)B^{-1}A^{-1}$$

$$=(\det B)B^{-1}(\det A)A^{-1} =B^*A^*$$

$$\Rightarrow I= I^*= (AA^{-1})^*=(A^{-1})^*A^* \Rightarrow (A^*)^{-1}=(A^{-1})^*$$

$$\boxed{I^*=(\det I)I^{-1}=I}$$