第四章 n维向量空间

4.2 向量组的线性相关性

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一、向量组的线性组合

1. 向量组与矩阵

向量组 同维数的向量所组成的集合.

$A=\left(a_{ij}\right)_{m\times n}$ 有n个m维列向量

$$A=\begin{array}{c}
\begin{array}{ccccccc} \alpha_1 & \alpha_2 & & \alpha_j & & \alpha_n \end{array} \\
\begin{pmatrix}
\boxed{\begin{matrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{matrix}} & \boxed{\begin{matrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{matrix}} & \begin{matrix} \cdots \\ \cdots \\ \vdots \\ \cdots \end{matrix} & \boxed{\begin{matrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{matrix}} & \begin{matrix} \cdots \\ \cdots \\ \vdots \\ \cdots \end{matrix} & \boxed{\begin{matrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{matrix}}
\end{pmatrix}
\end{array}$$

$\alpha_1,\alpha_2,\cdots,\alpha_n$ 称为矩阵A的列向量组

对称地, 矩阵 $A=\left(a_{ij}\right)_{m\times n}$ 有m个n维行向量

$$A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{array}{l} \beta_1 \\ \beta_2 \\ \\ \beta_i \\ \\ \beta_m \end{array}$$

$\beta_1,\beta_2,\cdots,\beta_m$ 称为矩阵A的行向量组.

反之, 给定行(列)向量组, 也可构造矩阵A使得:

A的行(列)向量组恰为给定向量组.

2.线性组合、线性表出的概念

设给定向量β，向量组$\alpha_1, \ldots, \alpha_m$，若存在数 $k_1, \ldots, k_m$使得

$$\beta = k_1\alpha_1 + k_2\alpha_2 + \cdots + k_m\alpha_m,$$

则称向量β为向量组$\alpha_1, \alpha_2, \ldots, \alpha_m$的线性组合，

也称β可由$\alpha_1, \alpha_2, \ldots, \alpha_m$线性表出.

问题: (1) 如何判断向量β可否由某个向量组线性表出?

(2) 如何计算向量β被某向量组线性表出的关系式?

例1.

(1) 零向量是任一向量组的线性组合.

$$0 = 0\alpha_1 + 0\alpha_2 + \cdots + 0\alpha_m.$$

(2) 向量组 $\alpha_1, \alpha_2, \ldots, \alpha_m$ 中任一向量

都可由该向量组自身线性表出.

$$\alpha_i = 0\alpha_1 + \cdots + 0\alpha_{i-1} + 1\alpha_i + 0\alpha_{i+1} + \cdots + 0\alpha_m.$$

(3) 3维几何空间中任一向量可以由 $\vec{i}, \vec{j}, \vec{k}$ 线性表出.

$$(a, b, c) = a\vec{i} + b\vec{j} + c\vec{k}$$

$L(\alpha_1, \ldots, \alpha_m)$: $\alpha_1, \ldots, \alpha_m$ 线性组合的全体.

$$L(\alpha_1, \cdots, \alpha_m) = \{k_1\alpha_1 + \cdots + k_m\alpha_m \mid k_1, \cdots, k_m \in \mathbf{R}\}$$

(1) $\alpha_1, \cdots, \alpha_m \in L(\alpha_1, \cdots, \alpha_m)$;

(2) $L(\alpha_1, \cdots, \alpha_m)$ 是 $\mathbf{R}^n$ 的子空间; $L(\alpha_1, \cdots, \alpha_m) \neq \varnothing$

$$\forall k_1\alpha_1 + \cdots + k_m\alpha_m, l_1\alpha_1 + \cdots + l_m\alpha_m \in L(\alpha_1, \cdots, \alpha_m), c \in \mathbf{R}$$

$$\Rightarrow (k_1 + l_1)\alpha_1 + \cdots + (k_m + l_m)\alpha_m \in L(\alpha_1, \cdots, \alpha_m),$$

$$ck_1\alpha_1 + \cdots + ck_m\alpha_m \in L(\alpha_1, \cdots, \alpha_m).$$

$L(\alpha_1, \cdots, \alpha_m)$ 称为 $\alpha_1, \cdots, \alpha_m$ 生成的子空间.

(4) $\mathbf{R}^n = L(\varepsilon_1, \varepsilon_2, ..., \varepsilon_n)$, 其中

$$\varepsilon_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \varepsilon_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \cdots, \varepsilon_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

即, 任一n维向量均可由$\varepsilon_1, \varepsilon_2, ..., \varepsilon_n$ 线性表出:

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1\varepsilon_1 + x_2\varepsilon_2 + \cdots + x_n\varepsilon_n.$$

3. 线性表出的充要条件和计算

定理1. 设 $A=(\alpha_1,\dots,\alpha_n)$，则如下条件等价：

(1) $b\in L(\alpha_1,\dots,\alpha_n)$ (2) $AX=b$ 有解； (3) $R(A)=R(\overline{A})$.

证： $(1)\Leftrightarrow(2)$ $b\in L(\alpha_1,\dots,\alpha_n)$

$$\Updownarrow$$

有数 $x_1,\dots,x_n$ 使得 $\boxed{x_1\alpha_1+\cdots+x_n\alpha_n=b}$

$$\Updownarrow$$

有数 $x_1,x_2,\dots,x_n$ 使得 $\boxed{(\alpha_1,\cdots,\alpha_n)\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}=b,}$

$$\Updownarrow$$

$AX=b$ 有解

定理1. 设 $A=(\alpha_1,\dots,\alpha_n)$, 则如下条件等价:

(1) $b\in L(a_1,\dots,a_n)$ (2) $AX=b$ 有解; (3) $R(A)=R(\overline{A})$.

(2)⇔(3)

设 $\mathrm{R}(A)=r$,

$$\overline{A}=\left(A\,\middle|\,b\right)\xrightarrow{\text{行初等变换}}\left(\begin{array}{ccccc|c} c_{11} & \cdots & c_{1s} & \cdots & c_{1n} & d_1 \\ & \ddots & \vdots & & \vdots & \vdots \\ & & c_{rs} & \cdots & c_{rn} & d_r \\ & & & & 0 & d_{r+1} \\ & & O & & & \\ & & & & & \end{array}\right)=\left(B\,\middle|\,d\right).$$

$AX=b$ 与 $BX=d$ 同解, 所以

$AX=b$ 有解 $\Leftrightarrow d_{r+1}=0 \Leftrightarrow R\left(B\,\middle|\,d\right)=R(B)=r$

$\Leftrightarrow R\left(\overline{A}\right)=R(A)$

例2. 将$b=\begin{pmatrix}1\\0\\-4\end{pmatrix}$用$\alpha_1=\begin{pmatrix}0\\1\\1\end{pmatrix},\alpha_2=\begin{pmatrix}1\\0\\1\end{pmatrix},\alpha_3=\begin{pmatrix}1\\1\\0\end{pmatrix}$线性表出.

解:

$$(A|b)=(\alpha_1,\alpha_2,\alpha_3|b)=\left(\begin{array}{ccc|c}0&1&1&1\\1&0&1&0\\1&1&0&-4\end{array}\right)\to\left(\begin{array}{ccc|c}1&0&1&0\\0&1&1&1\\1&1&0&-4\end{array}\right)$$

$$\to\left(\begin{array}{ccc|c}1&0&1&0\\0&1&1&1\\0&1&-1&-4\end{array}\right)\to\left(\begin{array}{ccc|c}1&0&1&0\\0&1&1&1\\0&0&-2&-5\end{array}\right)\to\left(\begin{array}{ccc|c}1&0&0&-\frac{5}{2}\\0&1&0&-\frac{3}{2}\\0&0&1&\frac{5}{2}\end{array}\right)$$

$$\Rightarrow b=-\frac{5}{2}\alpha_1-\frac{3}{2}\alpha_2+\frac{5}{2}\alpha_3$$