

第五讲 习题课

► 一.习题1

二.习题2

《内容小结》

1. 矩阵的线性运算、乘法、转置、逆矩阵.
2. 线性方程组求解的初等变换法.
3. 矩阵求逆的初等变换法.

习题1

例1 设 $\alpha = (\frac{1}{2}, 0, \dots, 0, \frac{1}{2})$, $A = I - \alpha^T \alpha$, $B = I + 2\alpha^T \alpha$, 求 AB .

解:
$$\begin{aligned} AB &= (I - \alpha^T \alpha)(I + 2\alpha^T \alpha) \\ &= I - \alpha^T \alpha + 2\alpha^T \alpha - 2\alpha^T \alpha \alpha^T \alpha \\ &= I + \alpha^T \alpha - 2\alpha^T (\alpha \alpha^T) \alpha \end{aligned}$$

注: $\begin{cases} \alpha^T \alpha \text{ 为矩阵} \\ \alpha \alpha^T \text{ 为数} \end{cases}$

$$\alpha \alpha^T = (\frac{1}{2}, 0, \dots, 0, \frac{1}{2}) \begin{pmatrix} \frac{1}{2} \\ 2 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 2 \end{pmatrix} = \frac{1}{2} \Rightarrow AB = I + \alpha^T \alpha - 2 \cdot \frac{1}{2} \alpha^T \alpha = I.$$
$$B = A^{-1}.$$

例2 设 A 是实对称矩阵且 $A^2 = O$, 证明: $A = O$.

分析: $A^T = A, A^2 = O \Rightarrow AA^T = O$

证: 设 $A = (a_{ij})_{n \times n}$ 且 $A^T = A$, 则

$$O = A^2 = AA = AA^T = B = (b_{ij})_{n \times n} \Rightarrow b_{ij} = 0 (\forall i, j)$$

$$\Rightarrow 0 = b_{ii} = a_{i1}^2 + a_{i2}^2 + \cdots + a_{in}^2 \quad (i = 1, 2, \cdots, n),$$

$$A \text{ 是实矩阵} \Leftrightarrow a_{ij} \in R$$

$$\Rightarrow a_{ij} = 0 \quad (i, j = 1, 2, \cdots, n) \Leftrightarrow A = O.$$

注: $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq O \Rightarrow A^2 = O \Rightarrow A^2 = O \not\Rightarrow A = O$

例3 求 A 的逆矩阵 A^{-1} .

分析: $(A, I) \rightarrow (I, A^{-1})$

$$A = \begin{pmatrix} 0 & a_1 & 0 & \cdots & 0 \\ 0 & 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1} \\ a_n & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \prod_{i=1}^n a_i \neq 0.$$

解1:

$$\left(\begin{array}{ccccc|ccccc} 0 & a_1 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & a_2 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & & \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1} & 0 & 0 & 0 & \cdots & 0 \\ a_n & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 \end{array} \right) \xrightarrow[i=n, n-1, \dots, 2]{r_i \leftrightarrow r_{i-1}} \rightarrow$$

$$\rightarrow \left(\begin{array}{ccccc|ccccc} a_n & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & a_1 & 0 & \dots & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & a_2 & \dots & 0 & 0 & 1 & \dots & 0 & 0 \\ & \dots & \dots & \dots & & & \dots & \dots & \dots & \\ 0 & 0 & 0 & \dots & a_{n-1} & 0 & 0 & \dots & 1 & 0 \end{array} \right)$$

$$\rightarrow \left(\begin{array}{ccccc|ccccc} \mathbf{1} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \frac{\mathbf{1}}{a_n} \\ \mathbf{0} & \mathbf{1} & \cdots & \mathbf{0} & \mathbf{0} & \frac{\mathbf{1}}{a_1} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{\mathbf{1}}{a_2} & \cdots & \mathbf{0} & \mathbf{0} \\ & \vdots & & \vdots & & \vdots & & & & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \cdots & \frac{\mathbf{1}}{a_{n-1}} & \mathbf{0} \end{array} \right)$$

\parallel
 A^{-1}

解2: $A = \left(\begin{array}{c|cccc} 0 & a_1 & 0 & \cdots & 0 \\ 0 & 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1} \\ \hline a_n & 0 & 0 & \cdots & 0 \end{array} \right) = \begin{pmatrix} O & A_1 \\ A_2 & O \end{pmatrix}$

$$\Rightarrow A^{-1} = \begin{pmatrix} O & A_2^{-1} \\ A_1^{-1} & O \end{pmatrix}, \quad A_1^{-1} = \begin{pmatrix} \frac{1}{a_1} & & & \\ & \frac{1}{a_2} & & \\ & & \ddots & \\ & & & \frac{1}{a_{n-1}} \end{pmatrix}$$

$$A_2^{-1} = \frac{1}{a_n}$$

$$A^{-1} = \begin{pmatrix} O & A_2^{-1} \\ A_1^{-1} & O \end{pmatrix} = \left(\begin{array}{ccc|c} O & & & \frac{1}{a_n} \\ \hline \frac{1}{a_1} & & & \\ & \frac{1}{a_2} & & \\ & & \ddots & \\ & & & \frac{1}{a_{n-1}} \\ & & & O \end{array} \right)$$

例4 矩阵 $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ 满足:

$$AXA + BXB = AXB + BXA + I, \text{ 求 } X.$$

分析: 求 $X \Leftrightarrow$ 用 A, B 表示 X

解: $AXA + BXB = AXB + BXA + I \Leftrightarrow$

$$AX(A - B) + BX(B - A) = I \Leftrightarrow AX(A - B) - BX(A - B) = I$$

$$\Leftrightarrow (AX - BX)(A - B) = I \Leftrightarrow (A - B)X(A - B) = I$$

若 $A - B$ 可逆, 则 $X = [(A - B)^{-1}]^2$.

注: $(A - B, I) \rightarrow (I, (A - B)^{-1})$

$$(A - B, I) = \left(\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right) \rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right)$$

$$\longrightarrow (A - B)^{-1} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$X = [(A - B)^{-1}]^2 = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$