五. 几个补充例题

例13 已知 $AA^T = I$ 且 $|A| = -1$，证明：$|-I - A| = 0$.

证：$|-I - A| = |-AA^T - A|$

$$= |A(-A^T - I)|$$

$$= |A||(-A - I)^T|$$

$$= -|-A - I| = -|-I - A|$$

$\therefore \quad |-I - A| = 0$.

例14 设 $\Lambda=\begin{pmatrix}0 & & & & \\ & 1 & & & \\ & & 2 & & \\ & & & \ddots & \\ & & & & n-1\end{pmatrix}$，$P^{-1}BP=\Lambda$，求 $|I+B|$

解：$B=P\Lambda P^{-1}$

$$|I+B|=|I+P\Lambda P^{-1}|=|PIP^{-1}+P\Lambda P^{-1}|$$

$$=|P(I+\Lambda)P^{-1}|=|P||I+\Lambda||P^{-1}|$$

$$=|P||P^{-1}||I+\Lambda|=|I+\Lambda|$$

$$=n!$$

例15 已知 $D_n = \begin{vmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 2 & 0 & \cdots & 0 \\ 1 & 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & n \end{vmatrix}$

求第一行各元素的代数余子式之和:

$$A_{11} + A_{12} + \cdots + A_{1n}$$

解

$$A_{11} + A_{12} + \cdots + A_{1n} = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 0 & \cdots & 0 \\ 1 & 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & n \end{vmatrix} = n!\left(1 - \sum_{j=2}^{n} \frac{1}{j}\right).$$

例16

已知$\alpha,\beta,\gamma_1,\gamma_2$是列向量,并且行列式

$$|A|=|\alpha,\gamma_1,\gamma_2|=4,\quad |B|=|\beta,\gamma_1,\gamma_2|=-1,$$

行列式$|A+B|=?$

解

$$|A+B|=\left|(\alpha,\gamma_1,\gamma_2)+(\beta,\gamma_1,\gamma_2)\right|$$

$$=|\alpha+\beta,2\gamma_1,2\gamma_2|$$

$$=4|\alpha+\beta,\gamma_1,\gamma_2|$$

$$=4\left(|\alpha,\gamma_1,\gamma_2|+|\beta,\gamma_1,\gamma_2|\right)$$

$$=12$$

例17. 计算 $D_n = \begin{vmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 2 & 3 & 4 & \cdots & n & 1 \\ 3 & 4 & 5 & \cdots & 1 & 2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ n-1 & n & 1 & \cdots & n-3 & n-2 \\ n & 1 & 2 & \cdots & n-2 & n-1 \end{vmatrix}$

$$= \begin{vmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 1 & 1 & 1 & \cdots & 1 & 1-n \\ 1 & 1 & 1 & \cdots & 1-n & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & 1-n & \cdots & 1 & 1 \\ 1 & 1-n & 1 & \cdots & 1 & 1 \end{vmatrix} = \begin{vmatrix} \frac{n(n+1)}{2} & 2 & 3 & \cdots & n-1 & n \\ 0 & 1 & 1 & \cdots & 1 & 1-n \\ 0 & 1 & 1 & \cdots & 1-n & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & 1-n & \cdots & 1 & 1 \\ 0 & 1-n & 1 & \cdots & 1 & 1 \end{vmatrix}_n$$

$$=\frac{n(n+1)}{2}\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1-n \\ 1 & 1 & 1 & \cdots & 1-n & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1-n & 1 & \cdots & 1 & 1 \\ 1-n & 1 & 1 & \cdots & 1 & 1 \end{vmatrix}_{n-1} =\frac{n(n+1)}{2}\begin{vmatrix} -1 & 1 & 1 & \cdots & 1 & 1-n \\ -1 & 1 & 1 & \cdots & 1-n & 1 \\ -1 & 1 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ -1 & 1-n & 1 & \cdots & 1 & 1 \\ -1 & 1 & 1 & \cdots & 1 & 1 \end{vmatrix}_{n-1}$$

$$=\frac{n(n+1)}{2}\begin{vmatrix} -1 & 0 & 0 & \cdots & 0 & -n \\ -1 & 0 & 0 & \cdots & -n & 0 \\ -1 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ -1 & -n & 0 & \cdots & 0 & 0 \\ -1 & 0 & 0 & \cdots & 0 & 0 \end{vmatrix}_{n-1}$$

[结束]