Solving convection-diffusion problems

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Abstract

The accurate and efficient numerical solution of convection-diffusion equations plays a pivotal role in various scientific and engineering applications. In this paper, we present a comprehensive study on the performance of three popular numerical methods: Finite Difference Method (FDM), Finite Element Method (FEM), and Finite Volume Method (FVM) - to solving the one-dimensional steady convection-diffusion problem.

Contents

Introduction

The convection-diffusion problem arises in numerous physical phenomena, such as fluid flow with simultaneous diffusion of a chemical concentration or the transport of pollutants in the atmosphere. It involves the interaction between convection which represents the transport of quantities due to volume motion, and diffusion which denotes the spreading or dissipation of quantities due to concentration gradients.

This combination often leads to challenging mathematical equations that are difficult to obtain analytical solutions, so we need numerical methods to solve them. The choice of an appropriate numerical method then becomes crucial to obtain reliable results that satisfy some basic properties. In this paper, we focus on five essential properties to evaluate the numerical methods:

1. transportiveness
2. consistency
3. stability
4. convergence
5. conservativeness.

To satisfy these five properties, we will adopt and evaluate three widely employed numerical methods: the finite Difference Method (FDM), the Finite Element Method (FEM), and the Finite Volume Method (FVM).

We will first start with the traditional version of each method. For FDM, we will implement the Central Difference Scheme a widely used method known for its simplicity and ease of implementation. For FEM, we will employ the Galerkin Approximation Scheme, which allows for flexible meshing and efficient handling of complex geometries. Finally, we will use FVM with Central Difference Scheme that provides a conservative formulation and compatibility with control volume discretization.

After evaluating five properties of these traditional schemes in solving one-dimensional steady convection-diffusion problems, we propose improvements for each method to overcome the limitations of the traditional schemes. For FDM, we introduce the Upwind Difference Scheme and the Artificial Difference Scheme; for FEM, we adopt the Petrov-Galerkin (SUPG) Approximation Scheme; and finally, for FVM, we explore the Upwind Difference Scheme and Hybrid Difference Scheme.

By evaluating the performance of these three methods and their associated schemes, we aim to provide comparisons of numerical methods for solving convection-diffusion problems, helping to select the appropriate numerical method when solving problems under different physical situations.

Convection Diffusion Equations

The general equation of the convection diffusion equation is in form of:

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{\alpha} \cdot \boldsymbol{\nabla} \boldsymbol{u} - \nu \boldsymbol{\nabla}^{\mathbf{2}} \boldsymbol{u} = \boldsymbol{s}$$

- $\boldsymbol{u}(\boldsymbol{x}, t)$ is the variable of interest, which can be the chemical concentration for mass transfer, or the temperature for heat transfer.
- $\boldsymbol{\alpha}(\boldsymbol{x}, t)$ denotes the velocity of convection field,
- ν denotes the diffusivity coefficient (such as the thermal diffusivity or mass diffusivity),
- $\boldsymbol{s}$ represents the source or sinks of the quantity $\boldsymbol{u}(\boldsymbol{x}, t)$.

In the general convection diffusion equation:

- the first term $\frac{\partial \boldsymbol{u}}{\partial t}$ represents the unsteady part, which describes how $\boldsymbol{u}(\boldsymbol{x}, t)$ changes with time t,
- the second term $\boldsymbol{\alpha} \cdot \boldsymbol{\nabla} \boldsymbol{u}$ represents the convective part, which describes the convection of $\boldsymbol{u}(\boldsymbol{x}, t)$,
- the third term $-\nu \boldsymbol{\nabla}^{\mathbf{2}} \boldsymbol{u}$ represents the diffusive part, which describes the diffusion of $\boldsymbol{u}(\boldsymbol{x}, t)$,
- and the last term $\boldsymbol{s}$ represents the source of $\boldsymbol{u}(\boldsymbol{x}, t)$.

Boundary Conditions

The convection diffusion equation is defined on the domain Ω, and the boundary conditions are defined on the boundary of the domain $\partial\Omega$.

The left (inflow side) and the right (outflow side) of the boundary $\partial\Omega$ are respectively denoted as Γ_g and Γ_h. We define $\boldsymbol{u} = \boldsymbol{g}$ on Γ_g and the flux $\dot{n}\boldsymbol{\nabla u} = \boldsymbol{h}$ on Γ_h. The general boundary condition is then written as:

$$\begin{cases} \boldsymbol{u} = \boldsymbol{g} & \text{on } \Gamma_g \\ \dot{n}\boldsymbol{\nabla u} = \boldsymbol{h} & \text{on } \Gamma_h \end{cases}$$

Initial Condition

The initial condition is denoted as $\boldsymbol{u}(\boldsymbol{x}, t = 0) = u_0(\boldsymbol{x})$.

Therefore, the standard general convection diffusion equation is expressed as:

$$\begin{cases} \frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{\alpha} \cdot \boldsymbol{\nabla u} - \nu \boldsymbol{\nabla^2 u} = \boldsymbol{s} \\ \boldsymbol{u} = \boldsymbol{g} & \text{on } \Gamma_g \\ \dot{n}\boldsymbol{\nabla u} = \boldsymbol{h} & \text{on } \Gamma_h \\ \boldsymbol{u}(\boldsymbol{x}, t = 0) = u_0(\boldsymbol{x}) \end{cases}$$

1D Steady-State Convection Diffusion Equations

Consider a steady 1D convection diffusion equation with homogeneous Dirichlet boundary conditions:

$$\begin{cases} \alpha \frac{\partial u}{\partial x} - \nu \frac{\partial^2 u}{\partial x^2} = s & \text{on } [0, 1] \\ u(0) = u(1) = 0 \end{cases}$$

where a, ν, s are all constants.

The Peclet number P_e is a non-dimensional number satisfying $P_e = \frac{a\Delta x}{2\nu}$.

Analytical Solution

When we solve the convective diffusion equation, we can use both analytical and numerical methods.

For our model problem, we can directly inteegrate to solve the exact solution.

$$\begin{aligned} \nu \frac{\partial^2 u}{\partial x^2} &= \alpha \frac{\partial u}{\partial x} - s \\ \Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) &= \frac{\alpha}{\nu}\frac{\partial u}{\partial x} - \frac{s}{\nu} \end{aligned}$$

Assume a function $f(x) = \frac{\partial u}{\partial x}$, then the equation becomes:

$$\begin{aligned} \frac{df}{dx} = \frac{\alpha}{\nu} f - \frac{s}{\nu} &\Rightarrow \frac{1}{\alpha f - s} df = \frac{1}{\nu} dx \\ &\Rightarrow \frac{1}{\alpha} \ln(\alpha f - s) = \frac{x}{\nu} + C \quad \text{, where C is a constant.} \\ &\Rightarrow f = \frac{\partial u}{\partial x} = \exp\left(\frac{\alpha x}{\nu}\right) \cdot C + \frac{s}{\alpha} \\ &\Rightarrow u(x) = \exp\left(\frac{\alpha x}{\nu}\right) \cdot C_1 + \frac{sx}{\alpha} + C_2 \quad \text{, where } C_1, C_2 \text{ are constants.} \end{aligned}$$

Substitute into the boundary conditions that $u(0) = u(1) = 0$, we obtain:

$$\begin{cases} u(0) = C_1 + C_2 = 0 \\ u(1) = \exp\left(\frac{\alpha}{\nu}\right) \cdot C_1 + \frac{s}{\alpha} + C_2 = 0 \end{cases} \qquad \begin{cases} C_1 = \frac{s/\alpha}{1 - \exp\left(\frac{\alpha}{\nu}\right)} \\ C_2 = -\frac{s/\alpha}{1 - \exp\left(\frac{\alpha}{\nu}\right)} \end{cases}$$

Thus, the analytical solution of our model 1D convection diffusion equation is:

$$
\begin{aligned}
u(x) &= \exp(\frac{\alpha x}{\nu}) \cdot \frac{s/\alpha}{1-\exp(\frac{\alpha}{\nu})} + \frac{sx}{\alpha} - \frac{s/\alpha}{1-\exp(\frac{\alpha}{\nu})} \\
&= \frac{sx}{\alpha} - \frac{s}{\alpha} \cdot \frac{1-\exp(\frac{\alpha x}{\nu})}{1-\exp(\frac{\alpha}{\nu})}
\end{aligned}
$$

In reality, it is rarely possible to derive an analytical solution. Therefore, it is very important that numerical simulations are carried out correctly to properly describe convective diffusion problems.

Partition of the Domain

In order to solve the problem numerically, we firstly need to partition the domain of interest $[0, 1]$ into gridpoints to act as the base of the numerical solution.

Now we uniformly mesh the domain of interest $[0, 1]$ into N elements. The domain $[0, 1]$ is then equally partitioned into $N + 1$ gridpoints $x_i \in \{0, 1, ..., N\}$ with the equal space denoted as $\Delta x = \frac{1-0}{N} = \frac{1}{N}$.

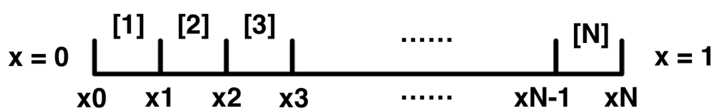


Figure 1: Domain Partition

The numerical solution of $u(x)$ will $u(x)$ by finding the values on the $N + 1$ points of the domain, denoted as $u_i = u(x_i), i \in \{0, 1, ..., N\}$. From the boundary conditions $u(0) = u(1) = 0$, the solution on the starting and the ending gridpoints are respectively $u_0 = 0$ and $u_N = 0$.

With the numerical partition of the domain, we will then implement the numerical methods to solve the one-dimensional steady convection-diffusion problems and discuss the following properties for each method:

1. Transportiveness: it refers that the method should properly handle the propagation and transfer of information without introducing spurious effects. A transportive numerical method will accurately model the movement of quantities, such as fluid flow or wave propagation, without excessive numerical diffusion.

2. Stability / Boundedness: it ensures that the solution remains bounded and does not grow unbounded or oscillate uncontrollably over time. A stable numerical method maintains the desired behavior and prevents numerical instabilities.

3. Consistency: it refers that a consistent numerical method should accurately approximates the underlying mathematical equations being solved.

4. Convergence: it is typically assessed by analyzing the behavior of the numerical errors as the grid or time step size decreases. A convergent numerical method will provide more accurate results as the resolution increases.

5. Conservativeness: it is a property specific to numerical methods used for solving conservation laws, which arise in many areas of physics and engineering. A conservative numerical method ensures that the total amount of a conserved quantity remains constant over time, even after discretization and approximation. This property is important to accurately model systems where mass, energy, momentum, or other conserved quantities must be preserved.

Overall, these are essential considerations to select numerical methods. We will then implement the stated numerical methods and examine them on these properties.

FDM Finite Difference Method

The first method we will implement is the Finite Difference Method (FDM), which is a traditional method to provide discrete solutions on the nodal points by approximating the derivatives.

Then we will firstly use the FDM with Central Difference Scheme to approximate the derivatives to find the numerical solution of our model problem:

$$\begin{cases} \alpha\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2} = s \qquad x \in [0,1] \\ u(0) = u(1) = 0 \end{cases}$$

where a, ν, s are all constants.

FDM with Central Difference Scheme

We expect to obtain the numerical solution on the $N+1$ gridpoints $x_0, x_1, \ldots, x_N$: $\tilde{u} = \sum_{j=0}^{N} u_j$.

The boundary conditions tell us the solution of starting point $x_0 = 0$ and ending point $x_N = 1$ are respectively $u(0) = u_0 = 0$ and $u(1) = u_N = 0$, so we only need to find the solutions on the $N-1$ gridpoints $x_1, \ldots, x_{N-1}$.

By using the Taylor's theorem with Central Difference Scheme, the first order and the second order derivative of $u(x)$ are expressed as:

$$\frac{u_{j+1} - u_{j-1}}{2\Delta x} = u_x + O(\Delta x^2)$$

$$\frac{u_{j+1} - 2u_j + u_{j-1}}{\Delta x^2} = u_{xx} + O(\Delta x^2)$$

Use them to rewrite the convective part and the diffusive part, our equation becomes:

$$\alpha\frac{u_{j+1} - u_{j-1}}{2\Delta x} - \nu\frac{u_{j+1} - 2u_j + u_{j-1}}{\Delta x^2} = s_j \qquad , j \in \{1, ..., N-1\}$$

$$\Rightarrow (-\frac{\alpha}{2\Delta x} - \frac{\nu}{\Delta x^2})u_{j-1} + (\frac{2\nu}{\Delta x^2})u_j + (\frac{\alpha}{2\Delta x} - \frac{\nu}{\Delta x^2})u_{j+1} = s_j \qquad , j \in \{1, ..., N-1\}$$

Thus, the formulation of FDM with Central Difference Scheme is:

$$\alpha \cdot u_{j-1} + b \cdot u_j + c \cdot u_{j+1} = s_j \qquad \text{for } j \in \{1, ..., N-1\}$$

where

$$a = -\frac{\alpha}{2\Delta x} - \frac{\nu}{\Delta x^2}, \quad b = \frac{2\nu}{\Delta x^2}, \quad c = \frac{\alpha}{2\Delta x} - \frac{\nu}{\Delta x^2}$$

Matrix System

We will then convert the formulation into the matrix system:

$$\boldsymbol{Au} = \boldsymbol{s}$$

where $\boldsymbol{s} = \{s_j\}_{j=1}^{N-1} = \{1, ..., 1\}^T$ is the source vector, and $\boldsymbol{u} = \{u_1, \ldots, u_{N-1}\}^T$ is the vector of numerical solutions on the $N-1$ gridpoints.

$$\begin{bmatrix} b & c & 0 & 0 & ... & 0 \\ a & b & c & 0 & ... & 0 \\ 0 & a & b & c & ... & 0 \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & ... & 0 & a & b \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \cdots \\ u_{N-1} \end{pmatrix} = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ ... \\ s_{N-1} \end{pmatrix}$$

Model Example

Then we will try the model example setting to examine the FDM Central Difference method.

Partition the domain $[0, 1]$ into $N = 5$ elements equally with the grid length $\frac{1}{N} = \frac{1}{5}$. The boundary conditions imply that $u_0 = u(0) = 0, u_5 = u(1) = 0$.

Set the unknown constants $\alpha = 5, \nu = 1, s = 1$, then the Peclet number is $P_e = \frac{a\Delta x}{2\nu} = \frac{1}{2} = 0.5 < 1$.

Substitute them into the matrix system of FDM with Central Difference Scheme to solve the $(N - 1) = 4$ numerical solutions $\boldsymbol{u} = \{u_1, u_2, u_3, u_4\}^T$ on gridpoints x_1, x_2, x_3, x_4.

The source vector is expressed as $\boldsymbol{s} = \{s_1, s_2, s_3, s_4\}^T = \{1, 1, 1, 1\}^T$, and a, b, c are calculated as:

$$\begin{cases} a = -\frac{\alpha}{2\Delta x} - \frac{\nu}{\Delta x^2} = -\frac{5}{2/5} - \frac{1}{(1/5)^2} = -37.5 \\ b = \frac{2\nu}{\Delta x^2} = \frac{2}{(1/5)^2} = 50 \\ c = \frac{\alpha}{2\Delta x} - \frac{\nu}{\Delta x^2} = \frac{5}{2/5} - \frac{1}{(1/5)^2} = -12.5 \end{cases}$$

Thus, the matrix system becomes:

$$\begin{bmatrix} 50 & -12.5 & 0 & 0 \\ -37.5 & 50 & -12.5 & 0 \\ 0 & -37.5 & 50 & -12.5 \\ 0 & 0 & -37.5 & 50 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Solve the matrix system to obtain the vector of numerical solutions $\boldsymbol{u}$:

$$\begin{cases} 50u_1 - 12.5u_2 = 1 \\ -37.5u_1 + 50u_2 - 12.5u_3 = 1 \\ -37.5u_2 + 50u_3 - 12.5u_4 = 1 \\ -37.5u_3 + 50u_4 = 1 \end{cases}$$

Combined with $u_0 = u_5 = 0$, the final numerical solution of FDM with Central Difference Scheme is:

$$\boldsymbol{u} = \{0, 0.03834711, 0.07338843, 0.0985124, 0.0938843, 0\}^T$$

We plot them into the diagram and compare with the exact solution:

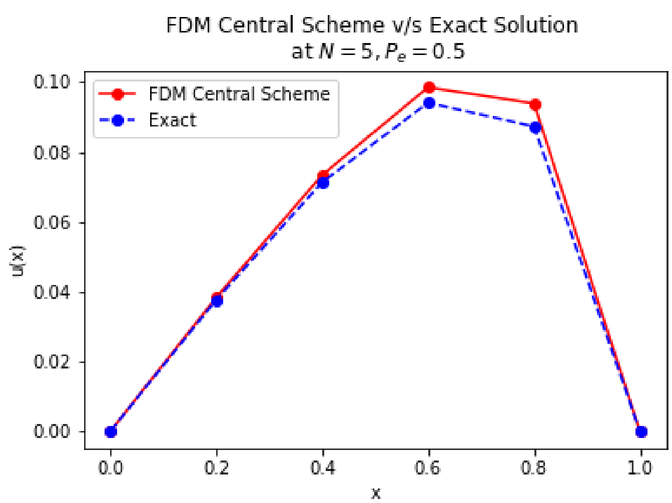


Figure 2: Solution of FDM with Central Difference Scheme at Pe = 0.5, N = 5

The output plot with $P_e = 0.5$ and $N = 5$ shows that the numerical solution of FDM with Central Difference Scheme is in the same trend with the exact solution, but with some errors.

Although finer meshing of the domain will generate more accurate solution, it will also incur higher costs.

Assessment on FDM with Central Difference Scheme

To be more rigorous, we will evaluate the properties of the FDM with Central difference scheme in terms of consistency, stability and convergence.

1. Consistency

The local truncation error T_j at x_j is the difference between numerical solution u_j and exact solution $u(x_j)$.

The Finite Difference Method is k-th order consistent for k > 0, if the local truncation error satisfies:

$$T_j = O(\Delta x^k)$$

As the numerical solution u_j at x_j satisfies:

$$(-\frac{\alpha}{2\Delta x} - \frac{\nu}{\Delta x^2})u_{j-1} + (\frac{2\nu}{\Delta x^2})u_j + (\frac{\alpha}{2\Delta x} - \frac{\nu}{\Delta x^2})u_{j+1} = s_j \qquad j \in \{1, ..., N-1\}$$

with $u_0 = 0, u_N = 1$.

Also, the exact solution $u(x_j)$ at x_j satisfies:

$$\alpha u_x(x_j) - \nu u_{xx}(x_j) = s_j \qquad j = 1, .., N-1$$

The Taylor series gives that:

$$\begin{cases} \frac{u_{j+1}-u_{j-1}}{2\Delta x} = u_x(x_j) + O(\Delta x^2) \\ \frac{u_{j+1}-2u_j+u_{j-1}}{\Delta x^2} = u_{xx}(x_j) + O(\Delta x^2) \end{cases}$$

Thus, the truncation error is expressed as:

$$\begin{aligned} T_j &= [(-\frac{\alpha}{2\Delta x} - \frac{\nu}{\Delta x^2})u_{j-1} + (\frac{2\nu}{\Delta x^2})u_j + (\frac{\alpha}{2\Delta x} - \frac{\nu}{\Delta x^2})u_{j+1} - s_j] - [\alpha u_x(x_j) - \nu u_{xx}(x_j) - s_j] \\ &= O(\Delta x^2) + O(\Delta x^2) \\ &= O(\Delta x^2) \end{aligned}$$

Therefore, the FDM with Central Difference Scheme has the second-order Taylor series truncation error, and this scheme is second-order consistent.

2. Stability

The global error e_i is the absolute value of the difference between numerical solution and exact solution:

$$e_i = |u_i - u(x_i)|$$

where u_i is the i-th numerical solution and $u(x_i)$ is the exact solution at $x = x_i$.

The numerical method is considered to be asymptotically stable if the global error approaches to zero as x increases. This means if the numerical solution is k^{th} order consistent, the global error is required to satisfy:

$$e_i = O(\Delta x^k) \to 0 \text{ as } h \to 0$$

Specifically, for the FDM with Central Difference Scheme with formulation:

$$-a \cdot u_{i-1} + b \cdot u_i - c \cdot u_{i+1} = d_i \qquad \text{for } i \in \{1, ..., N-1\}$$

where:

$$a = -\frac{\alpha}{2\Delta x} - \frac{\nu}{\Delta x^2}, \quad b = \frac{2\nu}{\Delta x^2}, \quad c = \frac{\alpha}{2\Delta x} - \frac{\nu}{\Delta x^2}$$

The solution is stable if the coefficients a, b, c satisfy $a, c \leq 0, b \geq a + c > 0$. Thus, we require:

$$\begin{cases} a = -\frac{\alpha}{2\Delta x} - \frac{\nu}{\Delta x^2} \leq 0, \text{ satisfied.} \\ c = \frac{\alpha}{2\Delta x} - \frac{\nu}{\Delta x^2} \leq 0 \Rightarrow \frac{\alpha \Delta x}{2\nu} = P_e \leq 1 \\ a + c = -\frac{2\nu}{\Delta x^2} < 0, \text{ satisfied.} \\ b = \frac{2\nu}{\Delta x^2} \geq -a - c = \frac{2\nu}{\Delta x^2}, \text{ satisfied.} \end{cases}$$

Therefore, the solution of FDM with Central Difference Scheme does not satisfy the stability if $P_e > 1$.

We will then plot the numerical solutions at different Peclet numbers $P_e = \{0.25, 0.5, 1.0, 5.0\}$ to validate:

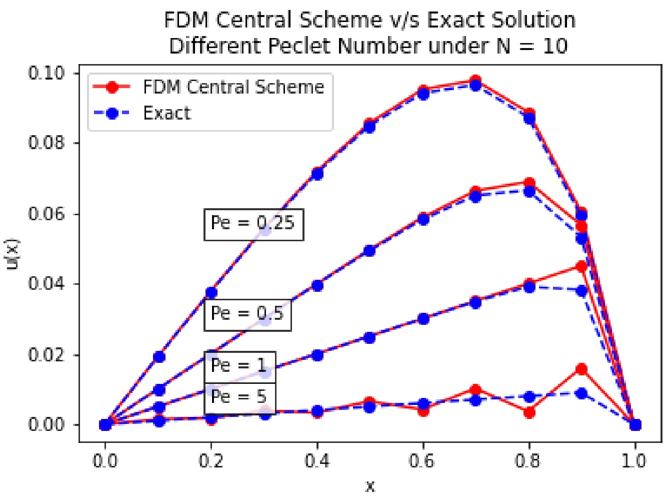


Figure 3: FDM with Central Difference Scheme at N = 10 under different Peclet numbers

From the plot, we observe that:

- At the early part of the domain, the numerical solution is closer to the exact solution compared to the end part of the domain.
- The numerical solution is more accurate when the Peclet number is getting larger.
- Especially when $P_e > 1$, the numerical solution appears node-to-node oscillations around the exact solution. This means that there exists instability problem for $P_e > 1 \Rightarrow \alpha > 2N\nu = 20$, which implies FDM with Central Difference Scheme does not work well for convection-dominated flow.

Although for a fixed convection coefficient α and diffusion coefficient ν, we can refine the mesh to keep $P_e < 1$, it is difficult to seek an extremely fine mesh. Even if we were able to finely mesh the domain, the mesh would constantly reduce convection in the elements, so the elements would still be convection-dominated.

3. Convergence

If the numerical method satisfies both stability and the consistency, then the method is convergent.

Thus, the FDM with Central Difference Scheme is not convergent as it is not stable for $P_e > 1$.

Improvement on the FDM Central Difference Scheme

To address the instability incurred by FDM with Central Difference scheme, we will introduce other finite difference schemes to get rid of the oscillations without expensive costs on meshing.

We suppose the new optimal scheme which satisfies the exact solution as:

$$a_1 u_{j-1} + a_2 u_j + a_3 u_{j+1} = s_j \qquad , j = 1, ..., N-1$$

From the analytical method part, the exact solution is:

$$\frac{1}{\alpha}(x - \frac{1-e^{\gamma x}}{1-e^{\gamma}}), \quad \text{where } \gamma = \frac{\alpha}{\nu}$$

As u_{j-1}, u_j, u_{j+1} satisfies the exact solution, we obtain:

$$\begin{cases} u_{j-1} = \frac{1}{\alpha} \cdot [(x_j - \Delta x) - (\frac{1-e^{\gamma x_j} e^{-2P_e}}{1-e^{\gamma}})] \\ u_j = \frac{1}{\alpha} \cdot [x_j - (\frac{1-e^{\gamma x_j}}{1-e^{\gamma}})] \\ u_{j+1} = \frac{1}{\alpha} \cdot [(x_j + \Delta x) - (\frac{1-e^{\gamma x_j} e^{2P_e}}{1-e^{\gamma}})] \end{cases}$$

The unknowns a_1, a_2, a_3 are calculated as:

$$\Rightarrow \begin{cases} \alpha_1 = -\frac{\alpha}{2\Delta x} - \frac{\alpha \coth(P_e)}{2\Delta x} \\ \alpha_2 = \frac{\alpha \coth(P_e)}{\Delta x} \\ \alpha_3 = +\frac{\alpha}{2\Delta x} - \frac{\alpha \coth(P_e)}{2\Delta x} \end{cases}$$

Thus, the optimal scheme satisfying the exact solution is:

$$(-\frac{\alpha}{2\Delta x} - \frac{\alpha \coth(P_e)}{2\Delta x}) \cdot u_{j-1} + (\frac{\alpha \coth(P_e)}{\Delta x}) \cdot u_j + (\frac{\alpha}{2\Delta x} - \frac{\alpha \coth(P_e)}{2\Delta x}) \cdot u_{j+1} = s_j \qquad , j = 1, ..., N-1$$

As this system is generated from the exact solution, all the transformation methods of it will naturally satisfy the consistency. Therefore, we can rearrange the optimal system to satisfy other properties.

Proposition 1

We retain the structure of the convective part in the Central Difference scheme and add a new term $\hat{\nu}$ to the diffusive part. The new difference system after adjusting the diffusive part becomes:

$$\alpha\frac{u_{j+1} - u_{j-1}}{2\Delta x} - (\nu + \hat{\nu})\frac{u_{j+1} - 2u_j + u_{j-1}}{\Delta x^2} = s_j \qquad , j = 1, ..., N-1$$

Equivalent it to the optimal system, we obtain:

$$\begin{aligned} \Rightarrow s_j &= (-\frac{\alpha}{2\Delta x} - \frac{\nu + \hat{\nu}}{\Delta x^2}) \cdot u_{j-1} + \frac{2(\nu + \hat{\nu})}{\Delta x^2} \cdot u_j + (\frac{\alpha}{2\Delta x} - \frac{\nu + \hat{\nu}}{\Delta x^2}) \cdot u_{j+1} \\ &= (-\frac{\alpha}{2\Delta x} - \frac{\alpha \coth(P_e)}{2\Delta x}) \cdot u_{j-1} + (\frac{\alpha \coth(P_e)}{\Delta x}) \cdot u_j + (\frac{\alpha}{2\Delta x} - \frac{\alpha \coth(P_e)}{2\Delta x}) \cdot u_{j+1} \end{aligned}$$

The equivalence gives:

$$\nu + \hat{\nu} = \frac{\alpha \coth(P_e)\Delta x}{2}$$

As $P_e = \frac{\alpha \Delta x}{2\nu}$, we then have:

$$\Rightarrow \hat{\nu} = (\frac{\alpha \coth(P_e)\Delta x}{2\nu P_e} - \frac{1}{P_e}) \cdot \nu P_e = (\coth(P_e) - \frac{1}{P_e}) \cdot \nu P_e = \beta \nu P_e$$

where $\beta = \coth(P_e) - \frac{1}{P_e}$.

This method is called the FDM with Artificial Difference scheme, where $\hat{\nu}$ is the artificial diffusion which is the negative diffusion of the central difference scheme.

Note that when $\hat{\nu} = 0$, the new scheme comes back to the Central Difference Scheme.

Therefore, the formulation of FDM with Artificial Difference scheme is:

$$\alpha\frac{u_{j+1} - u_{j-1}}{2\Delta x} - (\nu + \hat{\nu})\frac{u_{j+1} - 2u_j + u_{j-1}}{\Delta x^2} = s_j \qquad , j = 1, ..., N-1$$

where $\hat{\nu} = \beta\nu P_e$ and $\beta = \coth(P_e) - \frac{1}{P_e}$.

Proposition 2

We can also retain the structure of the diffusion part in the FDM Central Difference scheme and alter the convective part.

$$(-\frac{\alpha}{2\Delta x} - \frac{\alpha\coth(P_e)}{2\Delta x}) \cdot u_{j-1} + (\frac{\alpha\coth(P_e)}{\Delta x}) \cdot u_j + (\frac{\alpha}{2\Delta x} - \frac{\alpha\coth(P_e)}{2\Delta x}) \cdot u_{j+1} = s_j \qquad , j = 1, ..., N-1$$

Substitute into $\beta = \coth(P_e) - \frac{1}{P_e}$ and $P_e = \frac{\alpha\Delta x}{2\nu}$, we obtain:

$$\begin{aligned} s_j &= (-\frac{\alpha}{2\Delta x} - \frac{\alpha(\beta + \frac{1}{P_e})}{2\Delta x}) \cdot u_{j-1} + (\frac{\alpha(\beta + \frac{1}{P_e})}{\Delta x}) \cdot u_j + (\frac{\alpha}{2\Delta x} - \frac{\alpha(\beta + \frac{1}{P_e})}{2\Delta x}) \cdot u_{j+1} \\ &= (-\frac{\alpha(1+\beta)}{2\Delta x} - \frac{\nu}{\Delta x^2}) \cdot u_{j-1} + (\frac{\alpha\beta}{\Delta x} + \frac{\alpha^2}{2\nu}) \cdot u_j + (\frac{\alpha(1-\beta)}{2\Delta x} - \frac{\nu}{\Delta x^2}) \cdot u_{j+1} \\ &= (\frac{1-\beta}{2}) \cdot \alpha\frac{u_{j+1} - u_j}{\Delta x} + (\frac{1+\beta}{2}) \cdot \alpha\frac{u_j - u_{j-1}}{\Delta x} - \nu\frac{u_{j+1} - 2u_j + u_{j-1}}{\Delta x^2} \end{aligned}$$

This method is called the FDM with Upwind Difference scheme, which adjusted the convective part in the Central Difference Scheme.

Therefore, the formulation of FDM with Upwind Difference scheme is expressed as:

$$(\frac{1-\beta}{2}) \cdot \alpha\frac{u_{j+1} - u_j}{\Delta x} + (\frac{1+\beta}{2}) \cdot \alpha\frac{u_j - u_{j-1}}{\Delta x} - \nu\frac{u_{j+1} - 2u_j + u_{j-1}}{\Delta x^2} = s_j \qquad j = 1, .., N-1$$

where $\beta = \coth(P_e) - \frac{1}{P_e}$.

Therefore, the methods to improve the Central Difference Scheme are respectively:

1. Artificial Difference Scheme: alter the diffusive term by adding terms to diffusion: $\nu \Rightarrow \nu = \nu + \hat{\nu}$.
2. Upwind Difference Scheme: alter the convective term by making it upwind.

Both methods will generate the same answer. We will then implement the FDM with Artificial Difference scheme and FDM with Upwind Difference scheme to solve the model problem and assess their performances.

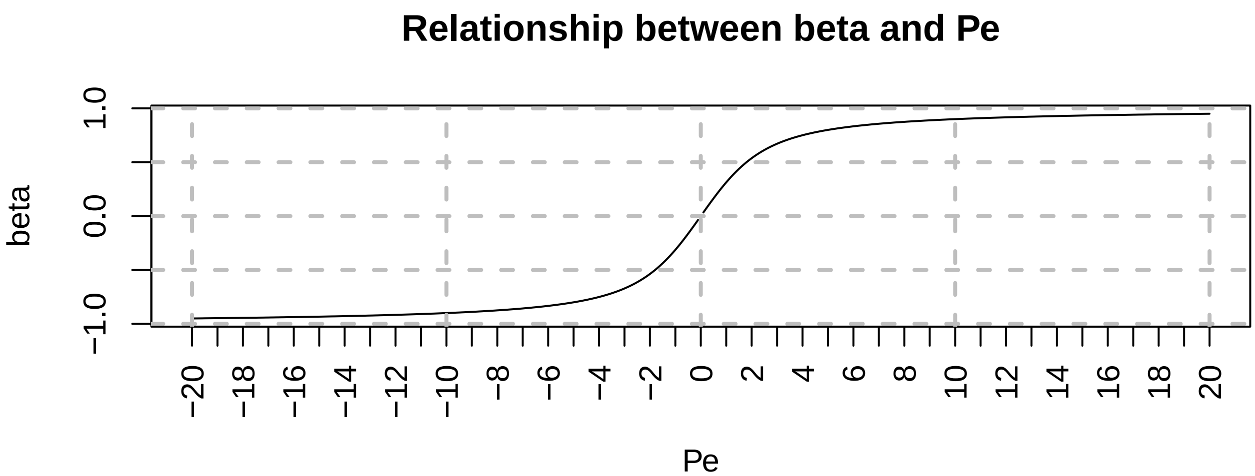
FDM with Artificial Difference Scheme

The system of FDM with Artificial Difference Scheme is formulated by adding terms to the diffusion coefficient to get rid of the instabilities and keep the convective term invariant:

$$\alpha \frac{u_{j+1} - u_{j-1}}{2\Delta x} - (\nu + \hat{\nu}) \frac{u_{j+1} - 2u_j + u_{j-1}}{\Delta x^2} = s_j \qquad , j = 1, ..., N-1$$

where $\hat{\nu} = \beta \frac{a \Delta x}{2} = \beta \nu P_e$, and $\beta = \coth(P_e) - \frac{1}{P_e}$.

The relationship between β and P_e is plotted as below:



The parameter β follows the piecewise linear function:

$$\Rightarrow \beta = \begin{cases} -1 & \text{for } P_e < -3 \\ \frac{P_e}{3} & \text{for } -3 < P_e < 3 \\ 1 & \text{for } P_e > 3 \end{cases}$$

Note that when $\beta = 0$, $P_e \to 0 \Rightarrow \hat{\nu} = 0$, there is no need for artificial diffusion, the difference scheme comes back to the Central Difference Scheme.

Matrix System

With the formulation of FDM with Artificial Difference Scheme:

$$a u_{j-1} + b u_j + c u_{j+1} = s_j \qquad j = 1, ..., N-1$$

We convert it into the matrix system $\boldsymbol{Au = s}$, where $\boldsymbol{s} = \{s_j\}_{j=1}^{N-1} = \{1, ..., 1\}^T$ is the source vector and $\boldsymbol{u} = \{u_1, \ldots, u_{N-1}\}^T$ is the vector of numerical solutions on the $N-1$ grid points.

$$\Rightarrow \begin{bmatrix} b & c & 0 & 0 & ... & 0 \\ a & b & c & 0 & ... & 0 \\ 0 & a & b & c & ... & 0 \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & ... & 0 & a & b \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \cdots \\ u_{N-1} \end{pmatrix} = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ ... \\ s_{N-1} \end{pmatrix}$$

with:

$$a = -\frac{\alpha}{2\Delta x} - \frac{\nu + \hat{\nu}}{\Delta x^2}, b = \frac{2(\nu + \hat{\nu})}{\Delta x^2}, c = \frac{\alpha}{2\Delta x} - \frac{\nu + \hat{\nu}}{\Delta x^2}$$

where $\hat{\nu} = \beta \frac{a \Delta x}{2} = \beta \nu P_e$ and $\beta = \coth(P_e) - \frac{1}{P_e}$.

Model Example

We will then solve the model example by using the FDM with Artificial Difference Scheme:

$$N = 5, \alpha = 5, \nu = 1, s_j = 1 \Rightarrow P_e = 0.5, \beta = \coth(0.5) - 2, \hat{\nu} = \frac{\coth(0.5)}{2} - 1$$

The values of a, b, c are:

$$\begin{cases} a = -\frac{\alpha}{2\Delta x} - \frac{\nu+\hat{\nu}}{\Delta x^2} = -\frac{5}{2/5} - \frac{\coth(0.5)}{2/25} \approx -39.54942 \\ b = \frac{2(\nu+\hat{\nu})}{\Delta x^2} = \frac{\coth(0.5)}{1/25} = 25\coth(0.5) = 54.09884 \\ c = \frac{\alpha}{2\Delta x} - \frac{\nu+\hat{\nu}}{\Delta x^2} \approx -14.54942 \end{cases}$$

Thus, the matrix system becomes:

$$\begin{bmatrix} 54.09884 & -14.54942 & 0 & 0 \\ -39.54942 & 54.09884 & -14.54942 & 0 \\ 0 & -39.54942 & 54.09884 & -14.54942 \\ 0 & 0 & -39.54942 & 54.09884 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Solve the matrix system to obtain the vector of numerical solutions $\boldsymbol{u}$:

$$\begin{cases} 54.09884u_1 - 14.54942u_2 = 1 \\ -39.54942u_1 + 54.09884u_2 - 14.54942u_3 = 1 \\ -39.54942u_2 + 54.09884u_3 - 14.54942u_4 = 1 \\ -39.54942u_3 + 54.09884u_4 = 1 \end{cases}$$

Combined with $u_0 = u_5 = 0$, the final numerical solution of FDM with Artificial Difference Scheme is:

$$\boldsymbol{u} = \{0, 0.03766875, 0.07133177, 0.09410606, 0.08728173, 0\}^T$$

We plot them in the following diagram and compare it with the exact solution:

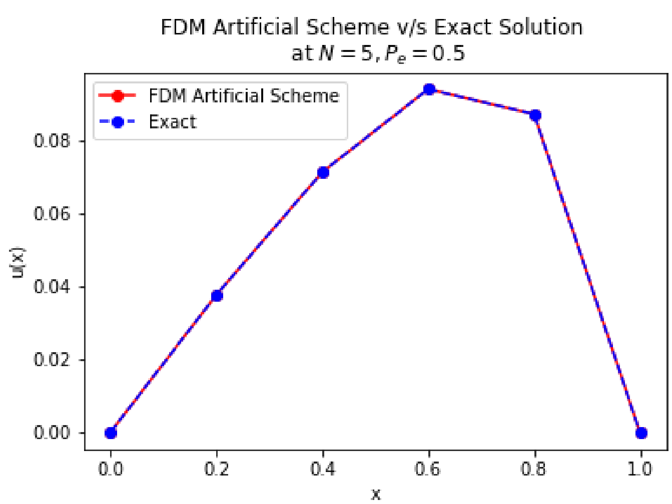


Figure 4: Solution of FDM Artificial Difference Scheme at Pe = 0.5 and N = 5

The output of the numerical solution of FDM with Artificial Difference scheme under $P_e = 0.5$ is very close to the exact solution, and there is not oscillations appearing in the solution. The stability and the accuracy is much improved compared to the same solution from FDM with Central Difference Scheme.

Assessment on FDM with Artificial Difference Scheme

1. Consistency

The numerical solution u_j satisfies the formulation of FDM with Artificial Difference Scheme:

$$au_{j-1} + bu_j + cu_{j+1} = s_j \qquad j = 1, ..., N-1$$

with:

$$a = -\frac{\alpha}{2\Delta x} - \frac{\nu + \hat{\nu}}{\Delta x^2}, b = \frac{2(\nu + \hat{\nu})}{\Delta x^2}, c = \frac{\alpha}{2\Delta x} - \frac{\nu + \hat{\nu}}{\Delta x^2}$$

where $\hat{\nu} = \beta \frac{a\Delta x}{2} = \beta\nu P_e$ and $\beta = \coth(P_e) - \frac{1}{P_e}$.

Also, the exact solution $u(x_j)$ at x_j satisfies:

$$\alpha u_x(x_j) - \nu u_{xx}(x_j) = s_j \qquad j = 1, .., N-1$$

With Taylor series gives that:

$$\begin{cases} \frac{u_{j+1} - u_{j-1}}{2\Delta x} = u_x(x_j) + O(\Delta x^2) \\ \frac{u_{j+1} - 2u_j + u_{j-1}}{\Delta x^2} = u_{xx}(x_j) + O(\Delta x^2) \end{cases}$$

The truncation error is expressed as:

$$\begin{aligned} T_j &= [(-\frac{\alpha}{2\Delta x} - \frac{\nu + \hat{\nu}}{\Delta x^2})u_{j-1} - u(x_{j-1})] + [(\frac{2(\nu + \hat{\nu})}{\Delta x^2})u_j - u(x_j)] + [(\frac{\alpha}{2\Delta x} - \frac{\nu + \hat{\nu}}{\Delta x^2})u_{j+1} - u(x_{j+1})] - (s_j - s_j) \\ &= O(\Delta x^2) + O(\Delta x^2) \\ &= O(\Delta x^2) \end{aligned}$$

Therefore, the FDM with Artificial Difference Scheme has the second-order Taylor series truncation error, and this scheme is second-order consistent.

2. Stability

FDM with Artificial Difference Scheme is stable if the coefficients a, b, and c satisfy $a, c \geq 0, b \geq a + c > 0$:

$$\Rightarrow \begin{cases} a = -\frac{\alpha}{2\Delta x} - \frac{\nu+\hat{\nu}}{\Delta x^2} \leq 0, \text{ satisfied.} \\ c = \frac{\alpha}{2\Delta x} - \frac{\nu+\hat{\nu}}{\Delta x^2} \leq 0 \\ a + c = -\frac{2(\nu+\hat{\nu})}{\Delta x^2} < 0, \text{ satisfied.} \\ b = \frac{2(\nu+\hat{\nu})}{\Delta x^2} \geq -a - c = -\frac{2(\nu+\hat{\nu})}{\Delta x^2}, \text{ satisfied.} \end{cases}$$

As $P_e = \frac{\alpha\Delta x}{2\nu}, \beta = \coth(P_e) - \frac{1}{P_e}, \hat{\nu} = \beta\frac{a\Delta x}{2} = \beta\nu P_e$, we have:

$$c = \frac{\alpha}{2\Delta x} - \frac{\nu + \hat{\nu}}{\Delta x^2} = \frac{P_e}{\Delta x^2 \nu}(1 - \coth(P_e))$$

Since $\coth(P_e) > 1 \Rightarrow 1 - \coth(P_e) < 0$ and $P_e, \Delta x, \nu$ are non-negative, we have:

$$c = \frac{P_e}{\Delta x^2 \nu}(1 - \coth(P_e)) \leq 0 \qquad \text{satisfied}$$

Therefore, FDM with Artificial Difference Scheme satisfies the requirement of stability, this scheme is stable.

We will then validate the stability by plotting the solution of FDM with Artificial difference scheme under different Peclet numbers $P_e = \{0.25, 0.5, 1.0, 5.0\}$ at the same level of partition $N = 10$.

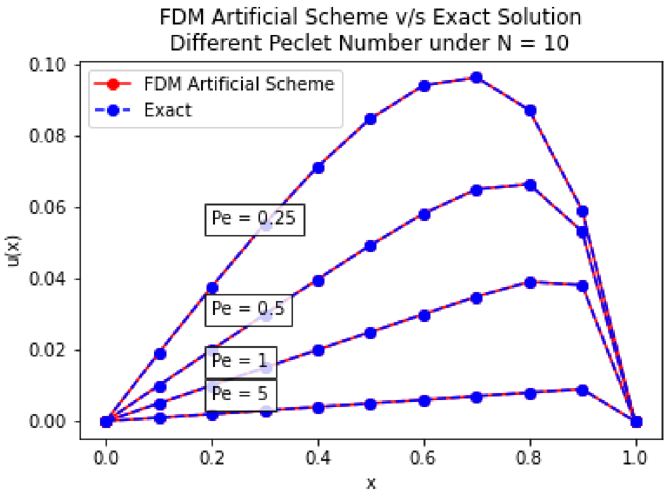


Figure 5: FDM Artificial Difference Scheme at N = 10 under different Peclet numbers

As shown, the solution of FDM with Artificial Difference Scheme do not oscillate around the exact solution, in line with our mathematical proposition that this difference scheme satisfies stability and consistency.

3. Convergence

Since FDM with Artificial Difference Scheme is first-order consistent and satisfies stability, this scheme is then convergent.

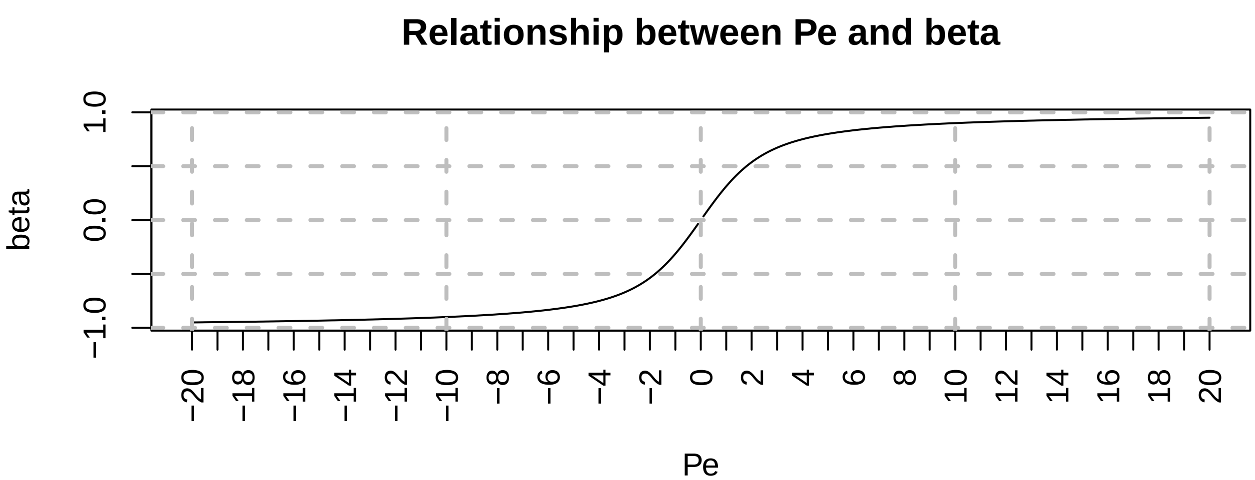
FDM with Upwind Difference Scheme

Then, we will try the second proposition FDM with Upwind Difference Scheme that altering the convective part and keeping the diffusive part invariant. The formulation is this scheme is expressed as:

$$(\frac{1-\beta}{2})\cdot\alpha\frac{u_{j+1}-u_j}{\Delta x}+(\frac{1+\beta}{2})\cdot\alpha\frac{u_j-u_{j-1}}{\Delta x}-\nu\frac{u_{j+1}-2u_{j-1}+u_{j-1}}{\Delta x^2}=1$$

where $P_e=\frac{\alpha\Delta x}{2\nu}, \beta=\coth(P_e)-\frac{1}{P_e}$.

The relationship between β and P_e is plotted as:



From the plot, we can observe that the value of P_e determines the upwind direction.

- When $P_e>3\Rightarrow\beta\to1$, the convective part becomes $\alpha\frac{u_j-u_{j-1}}{\Delta x}$, which is the positive direction.

It implies that we will only reply on j and $j-1$ rather than information from $j+1$. This conforms to the physical realization that the positive upstream flow is independent of the future velocity u_{j+1}.

- When $P_e < -3 \Rightarrow \beta \to -1$, the convective part becomes: $\alpha \frac{u_{j+1}-u_j}{\Delta x}$, which is the negative direction.

  The convective flow goes to the opposite direction, we only rely on j and $j-1$ in this case.
- Note that when $P_e \to 0$, $\beta = 0$, the convection gets back to $\alpha \frac{u_{j+1}-u_{j-1}}{2\Delta x}$ the Central Difference Scheme.

We will then use the Upwind Difference Scheme to solve our model problem. Rearrange the formulation to:

$$(-\frac{\alpha(1+\beta)}{2\Delta x} - \frac{\nu}{\Delta x^2}) \cdot u_{j-1} + (-\frac{\alpha(1-\beta)}{2\Delta x} + \frac{\alpha(1+\beta)}{2\Delta x} + \frac{2\nu}{\Delta x^2}) \cdot u_j + (\frac{\alpha(1-\beta)}{2\Delta x} - \frac{\nu}{\Delta x^2}) \cdot u_{j+1} = s_j$$

$$\Rightarrow (-\frac{\alpha(1+\beta)}{2\Delta x} - \frac{\nu}{\Delta x^2}) \cdot u_{j-1} + (\frac{\alpha\beta}{\Delta x} + \frac{2\nu}{\Delta x^2}) \cdot u_j + (\frac{\alpha(1-\beta)}{2\Delta x} - \frac{\nu}{\Delta x^2}) \cdot u_{j+1} = s_j$$

Thus, the formulation of FDM with Upwind Difference scheme is expressed as:

$$\alpha \cdot u_{j-1} + b \cdot u_j + c \cdot u_{j+1} = s_j \qquad \text{for } j \in \{1, ..., N-1\}$$

where

$$a = -\frac{\alpha(1+\beta)}{2\Delta x} - \frac{\nu}{\Delta x^2}, \quad b = \frac{\alpha\beta}{\Delta x} + \frac{2\nu}{\Delta x^2}, \quad c = \frac{\alpha(1-\beta)}{2\Delta x} - \frac{\nu}{\Delta x^2}$$

Matrix System

Convert the formulation into the matrix system: $\boldsymbol{Au} = \boldsymbol{s}$, where $\boldsymbol{s} = \{s_j\}_{j=1}^{N-1} = \{1, ..., 1\}^T$ is the source vector, and $\boldsymbol{u} = \{u_1, \ldots, u_{N-1}\}^T$ is the vector of numerical solutions on the $N-1$ grid points.

$$\begin{bmatrix} b & c & 0 & 0 & ... & 0 \\ a & b & c & 0 & ... & 0 \\ 0 & a & b & c & ... & 0 \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & ... & 0 & a & b \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \cdots \\ u_{N-1} \end{pmatrix} = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ ... \\ s_{N-1} \end{pmatrix}$$

Model Example

With the settings of model example: $N = 5, \alpha = 5, \nu = 1, s_j = 1$, we have

$$\Rightarrow P_e = 0.5, \quad \beta = \coth(0.5) - 2 = 0.163953, \quad \hat{\nu} = \frac{\coth(0.5)}{2} - 1 = 0.0819765$$

The values of a, b, c are respectively calculated as:

$$\begin{cases} a = -\frac{\alpha(1+\beta)}{2\Delta x} - \frac{\nu}{\Delta x^2} = -\frac{5(1+0.163953)}{2/5} - \frac{1}{(1/5)^2} = -39.54942 \\ b = \frac{\alpha\beta}{\Delta x} + \frac{2\nu}{\Delta x^2} = \frac{5\times0.163953}{1/5} + \frac{2\times1}{1/25} = 54.09882 \\ c = \frac{\alpha(1-0.163953)}{2\Delta x} - \frac{\nu}{\Delta x^2} = \frac{5(1-0.163953)}{2/5} - \frac{1}{(1/5)^2} = -14.54942 \end{cases}$$

The matrix system becomes:

$$\begin{bmatrix} 54.09882 & -14.54942 & 0 & 0 \\ -39.54942 & 54.09882 & -14.54942 & 0 \\ 0 & -39.54942 & 54.09882 & -14.54942 \\ 0 & 0 & -39.54942 & 54.09882 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

We will now solve the matrix system to obtain the vector of numerical solutions $\boldsymbol{u}$:

$$\begin{cases} 54.09882u_1 - 25u_2 = 1 \\ -39.54942u_1 + 54.09882u_2 - 14.54942u_3 = 1 \\ -39.54942u_2 + 54.09882u_3 - 14.54942u_4 = 1 \\ -39.54942u_3 + 54.09882u_4 = 1 \end{cases}$$

Combined with $u_0 = u_5 = 0$, the final numerical solution of FDM with Upwind Difference scheme is:

$$\boldsymbol{u} = \{0, 0.03766875, 0.07133177, 0.09410606, 0.08728173, 0\}^T$$

We plot them into the diagram and compare with the exact solution:

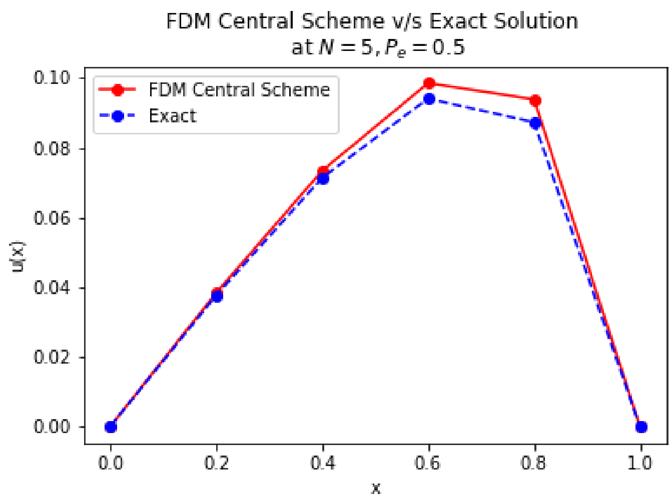


Figure 6: Solution of FDM with Upwind Difference Scheme at Pe = 0.5, N = 5

The numerical solution of FDM with Upwind Difference Scheme at $P_e = 0.5$ is much improved, and has no or at least not visible oscillations. Thus, Upwind Difference Scheme achieves better stability and accuracy.

Assessment on FDM with Upwind Difference Scheme

1. Consistency

The numerical solution u_j satisfies:

$$(-\frac{\alpha(1+\beta)}{2\Delta x} - \frac{\nu}{\Delta x^2}) \cdot u_{j-1} + (\frac{\alpha\beta}{\Delta x} + \frac{2\nu}{\Delta x^2}) \cdot u_j + (\frac{\alpha(1-\beta)}{2\Delta x} - \frac{\nu}{\Delta x^2}) \cdot u_{j+1} = s_j \qquad j = 1, .., N-1$$

Also, the exact solution $u(x_j)$ at x_j satisfies:

$$\alpha u_x(x_j) - \nu u_{xx}(x_j) = s_j \qquad j = 1, .., N-1$$

With the Taylor series:

$$\frac{u_{j+1} - u_{j-1}}{2\Delta x} = u_x(x_j) + O(\Delta x^2) \quad , \quad \frac{u_{j+1} - 2u_j + u_{j-1}}{\Delta x^2} = u_{xx}(x_j) + O(\Delta x^2)$$

The truncation error is calculated as:

$$T_j = [(-\frac{\alpha(1+\beta)}{2\Delta x} - \frac{\nu}{\Delta x^2}) \cdot u_{j-1} + (\frac{\alpha\beta}{\Delta x} + \frac{2\nu}{\Delta x^2}) \cdot u_j + (\frac{\alpha(1-\beta)}{2\Delta x} - \frac{\nu}{\Delta x^2}) \cdot u_{j+1} - s_j] - [\alpha u_x(x_j) - \nu u_{xx}(x_j) - s_j]$$
$$= O(\Delta x^2) + O(\Delta x^2) = O(\Delta x^2)$$

Therefore, the FDM with Upwind Difference Scheme has the second-order Taylor series truncation error, and this scheme is second-order consistent.

2. Stability

As stated before, the solution of FDM with Upwind Difference Scheme is stable if the coefficients a, b, and c satisfy $a, c \geq 0, b \geq a + c > 0$. Thus, for the formulation of FDM with Upwind Difference Scheme:

$$\alpha \cdot u_{j-1} + b \cdot u_j + c \cdot u_{j+1} = s_j \qquad \text{for } j \in \{1, ..., N-1\}$$

where

$$a = -\frac{\alpha(1+\beta)}{2\Delta x} - \frac{\nu}{\Delta x^2}, \quad b = \frac{\alpha\beta}{\Delta x} + \frac{2\nu}{\Delta x^2}, \quad c = \frac{\alpha(1-\beta)}{2\Delta x} - \frac{\nu}{\Delta x^2}$$

We require:

$$\begin{cases} a = -\frac{\alpha(1+\beta)}{2\Delta x} - \frac{\nu}{\Delta x^2} \leq 0, \text{ satisfied.} \\ c = \frac{\alpha(1-\beta)}{2\Delta x} - \frac{\nu}{\Delta x^2} \leq 0 \\ a + c = -\frac{\alpha\beta}{\Delta x} - \frac{2\nu}{\Delta x^2} < 0, \text{ satisfied.} \\ b = \frac{\alpha\beta}{\Delta x} + \frac{2\nu}{\Delta x^2} \geq -a - c = \frac{\alpha\beta}{\Delta x} + \frac{2\nu}{\Delta x^2}, \text{ satisfied.} \end{cases}$$

As $P_e = \frac{\alpha\Delta x}{2\nu}, \beta = \coth(P_e) - \frac{1}{P_e}$, we have:

$$c = \frac{\alpha(1-\beta)}{2\Delta x} - \frac{\nu}{\Delta x^2} = \frac{P_e}{\Delta x^2 \nu}(1 - \coth(P_e))$$

Since $\coth(P_e) > 1 \Rightarrow 1 - \coth(P_e) < 0$ and $P_e, \Delta x, \nu$ are non-negative, we have:

$$c = \frac{P_e}{\Delta x^2 \nu}(1 - \coth(P_e)) \leq 0 \qquad \text{satisfied}$$

Therefore, FDM with Upwind Difference Scheme satisfies the requirement of stability, this scheme is stable.

We will then display the solutions of FDM with Upwind Difference scheme under different Peclet number $P_e = \{0.25, 0.5, 1.0, 5.0\}$ with the same level of partition $N = 10$:

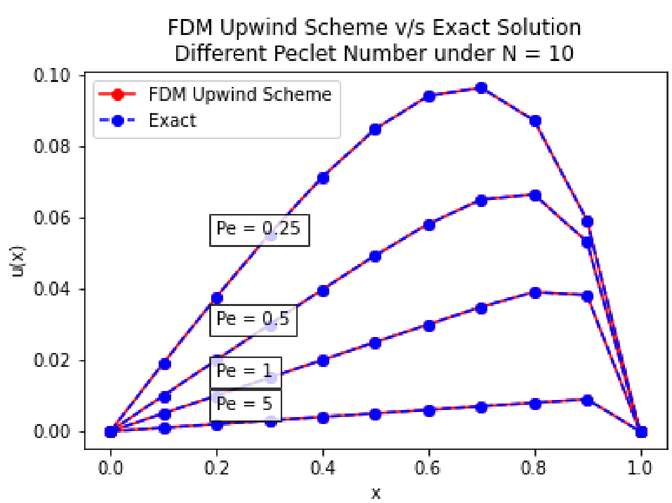


Figure 7: FDM Upwind Difference Scheme at N = 10 under different Peclet numbers

Thus, FDM with Upwind Difference Scheme gives numerical solutions that do not oscillate around the exact solution, in line with our claim that FDM with Upwind Difference Scheme brings stability and consistency.

3. Convergence

FDM with Upwind Difference Scheme is 1st-order consistent and satisfies stability, this scheme is convergent.

Comparison of Accuracy of FDM with different difference scheme

We will now compare these three difference schemes under the setting that the Peclet number is at the critical value $P_e = 1$ with the number of partitions $N = 10$.

The numerical solutions of FDM Central, Artificial, and Upwind difference schemes are plotted as below:

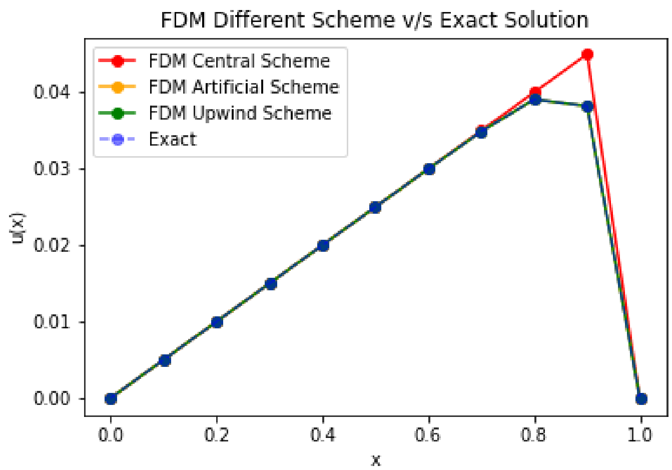


Figure 8: Comparison of FDM Central, Artificial, Upwind Difference Scheme at Pe = 1.0 N = 10

We can observe that the FDM central difference scheme is particularly far from the exact solution in the latter part of the domain, while the FDM Artificial difference scheme and the FDM Upwind difference scheme overlap with the exact solution throughout the domain.

Thus, we will specifically compare the global errors of the FDM Artificial difference scheme and the FDM Upwind difference scheme to see which causes less errors.

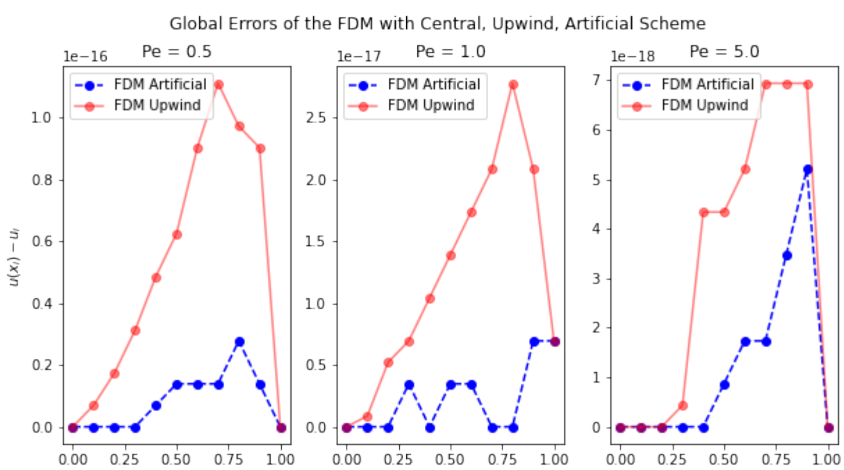


Figure 9: Global Errors of FDM Artificial and Upwind Difference Scheme

Comparison of Stability of FDM with different difference scheme

We will then compare the stability of three FDM difference schemes: the central difference scheme, the upwind difference scheme, and the artificial difference scheme for Peclet numbers greater than 1, e.g. $P_e = [5, 10, 20, 50]$.

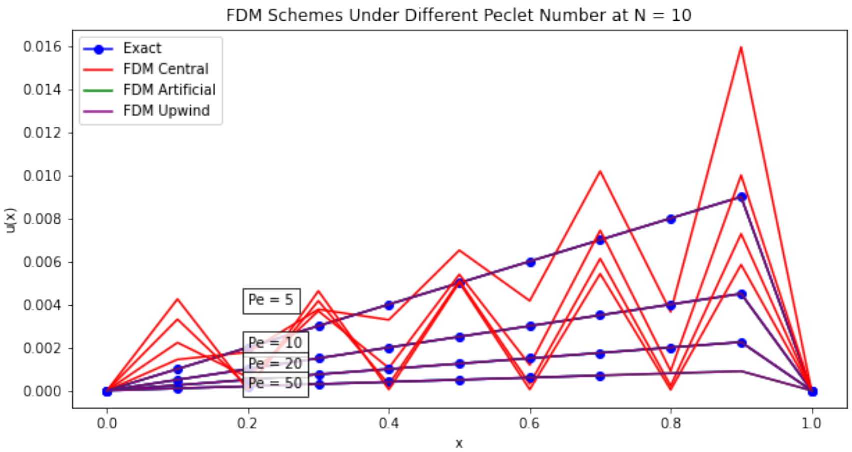


Figure 10: Different FDM Schemes under different Pe at N = 10

FEM Finite Element Method

We implement the Finite Element Method (FEM) to solve our 1D steady-state convection diffusion model equation:

$$\begin{cases} \alpha\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2} = s & \text{on } [0,1] \\ u(0) = u(1) = 0 \end{cases}$$

where a, ν, s are all constants.

Compared to FEM, FDM is more traditional and requires less computational power but also has less accuracy in some cases. FEM permits to get a higher order of accuracy, but requires more on the quality of the mesh.

FEM with Galerkin Approximation Scheme

On an equidistant mesh $x_j = \frac{j}{N}$ for $j = 0, 1, .., N$, we will implement the Galerkin approximation scheme with piecewise polynomials to obtain the numerical solution, denoted as $u^h \in S^h$ coming from the S^h space:

$$u^h := \sum_{j=0}^{N} u_j \varphi_j(x) \in S^N$$

Assume a residual R that describes the difference between the numerical solution and the exact solution:

$$R = \alpha u_x^h - \nu u_{xx}^h - s$$

When the numerical solution u^h is equal to the exact solution, the residual R becomes zero.

Variation Formulation of Galerkin Approximation Scheme

For the problem with non-homogeneous boundary conditions, the numerical solution u^h consists of the homogeneous solution $v^h \in V^h$ and the particular solution g^h:

$$u^h = v^h + g^h$$

While for our model problem with homogeneous boundary conditions, the numerical solution we pursue is exactly the homogeneous solution: $u^h = v^h$.

For any weighted function w^h coming from the V^h space, the trial function $v^h \in S^h$ satisfies the equivalence of the inner products on both sides of the equation:

$$< R, v^h >=< s, v^h >$$

$$\Rightarrow \int_0^1 w^h \alpha \frac{\partial v^h}{\partial x} dx + \int_0^1 \nu \frac{\partial w^h}{\partial x} \frac{\partial v^h}{\partial x} dx - [\nu w^h \frac{\partial v^h}{\partial x}]_0^1 = \int_0^1 s \cdot w^h dx$$

As v^h satisfies the homogeneous Dirichlet boundary conditions:

$$v^h(0) = v^h(1) = 0$$

The equation becomes:

$$\int_0^1 w^h \alpha \frac{\partial v^h}{\partial x} dx + \int_0^1 \nu \frac{\partial w^h}{\partial x} \frac{\partial v^h}{\partial x} dx = \int_0^1 s \cdot w^h dx$$

Written in the short form as:

$$B_G(w^h, v^h) = F_G(w^h, s)$$

$B_G(w^h, v^h)$ denotes the Galerkin approximation, written as:

$$B_G(w^h, v^h) = \int_0^1 \alpha w^h \frac{\partial v^h}{\partial x} dx + \int_0^1 \nu \frac{\partial w^h}{\partial x} \frac{\partial v^h}{\partial x} dx$$

$F_G(w^h, s)$ denotes the source integral, written as

$$F_G(w^h, s) = \int_0^1 s \cdot w^h dx$$

In the Galerkin Approximation scheme, we take the special case that the weighted function is the same with the trial solution: $w^h = v^h$. Then the system becomes:

$$B_G(v^h, v^h) = F_G(v^h, s)$$

where:

$$\begin{cases} B_G(v^h, v^h) = \int_0^1 \alpha v^h \frac{\partial v^h}{\partial x} dx + \int_0^1 \nu \frac{\partial v^h}{\partial x} \frac{\partial v^h}{\partial x} dx \\ F_G(v^h, s) = \int_0^1 s \cdot v^h dx \end{cases}$$

Basis Function

We will use the linear basis function $\varphi(x)$ as the trial solution v^h in the FEM Galerkin formulation:

$$\begin{cases} \varphi_{k-1}(x) = \frac{x_k - x}{\Delta x} = k - \frac{x}{\Delta x} \\ \varphi_k(x) = \frac{x - x_{k-1}}{\Delta x} = \frac{x}{\Delta x} - (k-1) \end{cases} \qquad \begin{cases} \frac{d\varphi_{k-1}(x)}{dx} = -\frac{1}{\Delta x} \\ \frac{d\varphi_k(x)}{dx} = \frac{1}{\Delta x} \end{cases}$$

Substitute into the trial solution $u^N(x) := \sum_{i=0}^N u_i \varphi_i(x)$, the formulation of FEM Galerkin Scheme becomes:

$$\Rightarrow \begin{cases} B_G(v^h, v^h) = \sum_{i=0}^N u_i \sum_{k=1}^N \int_{e^k} \frac{d\varphi_i(x)}{dx} \cdot (\alpha \varphi_j(x) + \nu \frac{d\varphi_j(x)}{dx}) \, dx \\ F_G(v^h, s) = \sum_{k=1}^N \int_{e^k} s_j \varphi_j(x) \, dx \end{cases}$$

Then, we will transform $B_G(v^h, v^h)$ and $F_G(v^h, s)$ into the matrix system $\boldsymbol{A}\boldsymbol{u} = \boldsymbol{d}$:

$$\boldsymbol{A} = \{B_{ij}^k\} \qquad \boldsymbol{d} = \{F_j^k\}$$

We will then evaluate them on the element $e^k = [x_{k-1}, x_k],$ where $k = 1, ..., N.$ Note that $x_k = k\Delta x$.

For B_{ij}, we have:

$$B_{ij} = \sum_{k=1}^{N} B_{ij}^k = \int_0^1 \frac{d\varphi_i(x)}{dx} \cdot (\alpha\varphi_j(x) + \nu \frac{d\varphi_j(x)}{dx})\ dx \qquad , a_{ij}^k \neq 0 \text{ only for } i, j \in \{k-1, k\}$$

where:

$$B_{ij}^k = \int_{e^k} \frac{d\varphi_i(x)}{dx} \cdot (\alpha\varphi_j(x) + \nu \frac{d\varphi_j(x)}{dx})\ dx$$

For F_j, we have:

$$F_j = \sum_{k=1}^{N} d_j^k = \int_0^1 s_j\varphi_j(x)\ dx \qquad d_j^k \neq 0 \text{ only for } j \in \{k-1, k\}$$

where:

$$F_j^k = \int_{e^k} s_j\varphi_j(x)\ dx$$

Thus, the matrix $\boldsymbol{A}$ on k-th element becomes:

$$A^k = \begin{bmatrix} a_{k-1,k-1} & a_{k,k-1} \\ a_{k-1,k} & a_{k,k} \end{bmatrix}$$

where:

$$\begin{aligned} a_{k-1,k-1} &= \int_{e^k} \frac{d\varphi_{k-1}(x)}{dx} \cdot (\alpha\varphi_{k-1}(x) + \nu \frac{d\varphi_{k-1}(x)}{dx})\ dx \\ &= \int_{x_{k-1}}^{x_k} -\frac{1}{\Delta x} \cdot (\alpha(k - \frac{x}{\Delta x}) - \nu \frac{1}{\Delta x})\ dx \\ &= \frac{1}{\Delta x^2} \int_{x_{k-1}}^{x_k} (-\alpha k \Delta x + \alpha x + \nu) dx \\ &= -\alpha k + \alpha(k - \frac{1}{2}) + \frac{\nu}{\Delta x} \\ &= -\frac{1}{2}\alpha + \frac{\nu}{\Delta x} \end{aligned} \quad , \quad \begin{aligned} a_{k,k-1} &= \int_{x_{k-1}}^{x_k} [\alpha \frac{d\varphi_k(x)}{dx} \varphi_{k-1}(x) + \nu \frac{d\varphi_k(x)}{dx} \frac{d\varphi_{k-1}(x)}{dx}]\ dx \\ &= \int_{x_{k-1}}^{x_k} [\alpha \frac{1}{\Delta x}(k - \frac{x}{\Delta x}) + \nu \frac{1}{\Delta x} \frac{-1}{\Delta x}]\ dx \\ &= \frac{1}{\Delta x^2} \int_{(k-1)\Delta x}^{k\Delta x} [\alpha k \Delta x - \alpha x - \nu]\ dx \\ &= \alpha k - \alpha(k - \frac{1}{2}) - \frac{\nu}{\Delta x} \\ &= \frac{1}{2}\alpha - \frac{\nu}{\Delta x} \end{aligned}$$

$$\begin{aligned} a_{k-1,k} &= \int_{x_{k-1}}^{x_k} [\alpha \frac{d\varphi_{k-1}(x)}{dx} \varphi_k(x) + \nu \frac{d\varphi_{k-1}(x)}{dx} \frac{d\varphi_k(x)}{dx}]\ dx \\ &= \int_{x_{k-1}}^{x_k} [\alpha \frac{-1}{\Delta x}(\frac{x}{\Delta x} - (k-1)) + \nu \frac{-1}{\Delta x} \frac{1}{\Delta x}]\ dx \\ &= \frac{1}{\Delta x^2} \int_{(k-1)\Delta x}^{k\Delta x} [-\alpha x + \alpha(k-1) - \nu]\ dx \\ &= -\alpha(k - \frac{1}{2}) + \alpha(k-1) - \frac{\nu}{\Delta x} \\ &= -\frac{1}{2}\alpha - \frac{\nu}{\Delta x} \end{aligned} \quad , \quad \begin{aligned} a_{k,k} &= \int_{x_k}^{x_k} [\alpha \frac{d\varphi_k(x)}{dx} \varphi_k(x) + \nu \frac{d\varphi_k(x)}{dx} \frac{d\varphi_k(x)}{dx}]\ dx \\ &= \int_{x_{k-1}}^{x_k} [\alpha \frac{1}{\Delta x}(\frac{x}{\Delta x} - (k-1)) + \nu \frac{1}{\Delta x} \frac{1}{\Delta x}]\ dx \\ &= \frac{1}{\Delta x^2} \int_{(k-1)\Delta x}^{k\Delta x} [\alpha x - \alpha(k-1) + \nu]\ dx \\ &= \alpha(k - \frac{1}{2}) - \alpha(k-1) + \frac{\nu}{\Delta x} \\ &= \frac{1}{2}\alpha + \frac{\nu}{\Delta x} \end{aligned}$$

The global stiffness matrix A is then computed as:

$$\boldsymbol{A} = \begin{bmatrix} \cdots & \cdots & \cdots & \cdots \\ \cdots & -\frac{1}{2}\alpha + \frac{\nu}{\Delta x} & \frac{1}{2}\alpha - \frac{\nu}{\Delta x} & \cdots \\ \cdots & -\frac{1}{2}\alpha - \frac{\nu}{\Delta x} & \frac{1}{2}\alpha + \frac{\nu}{\Delta x} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}\alpha + \frac{\nu}{\Delta x} & \frac{1}{2}\alpha - \frac{\nu}{\Delta x} & 0 & 0 \\ -\frac{1}{2}\alpha - \frac{\nu}{\Delta x} & \frac{1}{2}\alpha + \frac{\nu}{\Delta x} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \ldots + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2}\alpha + \frac{\nu}{\Delta x} & \frac{1}{2}\alpha - \frac{\nu}{\Delta x} \\ 0 & 0 & -\frac{1}{2}\alpha - \frac{\nu}{\Delta x} & \frac{1}{2}\alpha + \frac{\nu}{\Delta x} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2\nu}{\Delta x} & \frac{1}{2}\alpha - \frac{\nu}{\Delta x} & 0 & 0 \\ -\frac{1}{2}\alpha - \frac{\nu}{\Delta x} & \frac{2\nu}{\Delta x} & \frac{1}{2}\alpha - \frac{\nu}{\Delta x} & 0 \\ 0 & -\frac{1}{2}\alpha - \frac{\nu}{\Delta x} & \frac{2\nu}{\Delta x} & \frac{1}{2}\alpha - \frac{\nu}{\Delta x} \\ 0 & 0 & -\frac{1}{2}\alpha - \frac{\nu}{\Delta x} & \frac{2\nu}{\Delta x} \end{bmatrix}$$

The vector $\boldsymbol{d}$ on k-th element becomes $d^k = \begin{bmatrix} d_{k-1} \\ d_k \end{bmatrix}$, where:

$$
\begin{aligned}
d^{k-1} &= \int_{x_{k-1}}^{x_k} s_{k-1}\varphi_{k-1}(x)dx & d^k &= \int_{x_{k-1}}^{x_k} s_{k-1}\varphi_k(x)dx \\
&= \int_{(k-1)\Delta x}^{k\Delta x} s_{k-1}(k - \frac{x}{\Delta x})dx \quad , & &= \int_{(k-1)\Delta x}^{k\Delta x} s_{k-1}(\frac{x}{\Delta x} - (k-1))dx \\
&= \frac{1}{2}s_{k-1}\Delta x & &= \frac{1}{2}s_{k-1}\Delta x
\end{aligned}
$$

As the source s_k are constants, the element d^k becomes $d^k = s_k\Delta x \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$. Thus, the total vector $\boldsymbol{d}$ is:

$$
\boldsymbol{d} = \begin{bmatrix} d^1, & d^2, & d^3, & d^4 \end{bmatrix}^T = s\Delta x \begin{bmatrix} 1, & 1, & 1, & 1 \end{bmatrix}^T
$$

Matrix System

Insert them into the global matrix system $\boldsymbol{Au} = \boldsymbol{d}$:

$$
\begin{bmatrix} b & c & 0 & 0 \\ a & b & c & 0 \\ 0 & a & b & c \\ 0 & 0 & a & b \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = s\Delta x \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}
$$

where:

$$
a = -\frac{1}{2}\alpha - \frac{\nu}{\Delta x}, \quad b = \frac{2\nu}{\Delta x}, \quad c = \frac{1}{2}\alpha - \frac{\nu}{\Delta x}
$$

Model Example

We will then try the model example setting to examine the FEM Galerkin Approximation Scheme:

$$
N = 5, \alpha = 5, \nu = 1, s_j = 1 \Rightarrow P_e = 0.5
$$

Then, the values of a, b, c are calculated as:

$$
\begin{cases}
a = -\frac{1}{2}\alpha - \frac{\nu}{\Delta x} = -\frac{5}{2} - \frac{1}{1/5} = -7.5 \\
b = \frac{2\nu}{\Delta x} = \frac{2}{1/5} = 10 \\
c = \frac{1}{2}\alpha - \frac{\nu}{\Delta x} = \frac{5}{2} - \frac{1}{1/5} = -2.5
\end{cases}
$$

The matrix system becomes:

$$
\begin{bmatrix} 10 & -2.5 & 0 & 0 \\ -7.5 & 10 & -2.5 & 0 \\ 0 & -7.5 & 10 & -2.5 \\ 0 & 0 & -7.5 & 10 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0.2 \\ 0.2 \\ 0.2 \\ 0.2 \end{bmatrix}
$$

We find that this matrix system is the same with the matrix system of FDM with Central Difference Scheme.

We will now solve the matrix system to obtain the vector of numerical solutions $\boldsymbol{u}$:

$$
\begin{cases}
10u_1 - 2.5u_2 = 0.2 \\
-7.5u_1 + 10u_2 - 2.5u_3 = 0.2 \\
-7.5u_2 + 10u_3 - 2.5u_4 = 0.2 \\
-7.5u_3 + 10u_4 = 0.2
\end{cases}
$$

Combined with $u_0 = u_5 = 0$, the nodal coefficients of FEM with Galerkin Approximation Scheme is:

$$\boldsymbol{u} = \{0, 0.03834711, 0.07338843, 0.0985124, 0.0938843, 0\}^T$$

Combined with the basis functions $\varphi(x) = \begin{cases} \varphi_{k-1}(x) = \frac{x_k - x}{\Delta x} = k - \frac{x}{\Delta x} \\ \varphi_k(x) = \frac{x - x_{k-1}}{\Delta x} = \frac{x}{\Delta x} - (k-1) \end{cases}$

Insert the basis functions and the nodal coefficients into the trial solution $u^N(x) := \sum_{i=0}^{N} u_i \varphi_i(x)$, we obtain the final solutions of FEM with Galerkin Approximation scheme:

$$u^N(x_i) = u_i \cdot [\varphi_{i-1}(x_i) + \varphi_i(x_i)]$$

$$\begin{cases} u^N(x_0) = u_0 \varphi_k(x_0) = 0.0 \\ u^N(x_1) = u_1 \times (\varphi_{k-1}(x_1) + \varphi_k(x_k)) = 0.03834711 \times (k - \frac{1 \cdot \Delta x}{\Delta x} + \frac{1 \cdot \Delta x}{\Delta x} - (k-1)) = 0.03834710743801653 \\ u^N(x_2) = u_2 \times (\varphi_{k-1}(x_2) + \varphi_k(x_2)) = 0.07338842975206612 \\ u^N(x_3) = u_3 \times (\varphi_{k-1}(x_3) + \varphi_k(x_3)) = 0.09851239669421488 \\ u^N(x_4) = u_4 \times (\varphi_{k-1}(x_4) + \varphi_k(x_4)) = 0.09388429752066117 \\ u^N(x_5) = u_5 \times \varphi_{k-1}(x_5) = 0.0 \end{cases}$$

The numerical solution the FEM with Galerkin Approximation Scheme is shown as below:

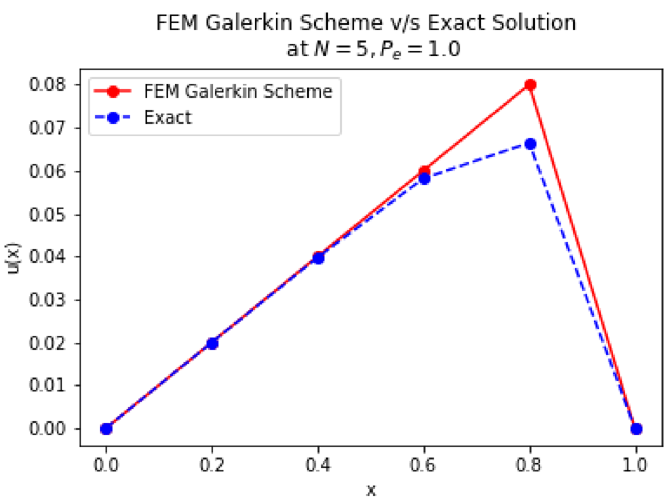


Figure 11: Solution of FEM with Galerkin Approximation Scheme at Pe = 0.5, N = 5

The plot shows that the numerical solutions of FEM with Galerkin Approximation Schemee conforms to the exact solution but with errors mainly in the later part of the domain.

Assessment on FEM with Galerkin Approximation Scheme

1. Consistency

FEM with Galerkin Approximation Scheme is originally consistent, since this scheme is residual based, which means that the exact solution will also satisfy the Galerkin formulation to become $R = 0$.

2. Boundedness

Denote the term $I(v^h, v^h)$ by $I(v^h, v^h) = \int_0^1 a v^h \frac{\partial v^h}{\partial x} dx$, it becomes:

$$\begin{aligned} I(v^h, v^h) &= [a(v^h)^2]_0^1 - \int_0^1 a v^h \frac{\partial v^h}{\partial x} dx \\ &= (0 - 0) - I(v^h, v^h) \end{aligned} \qquad \Rightarrow I(v^h, v^h) = 0$$

Thus, $B_G(w^h, v^h)$ and $F_G(v^h, s)$ respectivly satisfies the requirement of boundednss:

$$B_G(w^h, v^h) = I(v^h, v^h) + \int_0^1 v\frac{\partial v^h}{\partial x}\frac{\partial v^h}{\partial x}dx = \int_0^1 v\frac{\partial v^h}{\partial x}\frac{\partial v^h}{\partial x}dx \leq ||\nu(\frac{\partial v^h}{\partial x})^2||$$

$$F_G(v^h, s) = B_G(v^h, v^h) \leq ||\nu(\frac{\partial v^h}{\partial x})^2||$$

3. Stability

In the limit of the diffusivity term $\nu \to 0$, the term $F_G(v^h, s) = 0$. Thus, the flow becomes the convection dominated flow, and the Galerkin method is unstable under this situation.

We will evaluate the stability of FEM with Galerkin Approximation Scheme at different Peclet numbers $P_e = \{0.25, 0.5, 1, 5\}$ with the number of partitions $N = 10$:

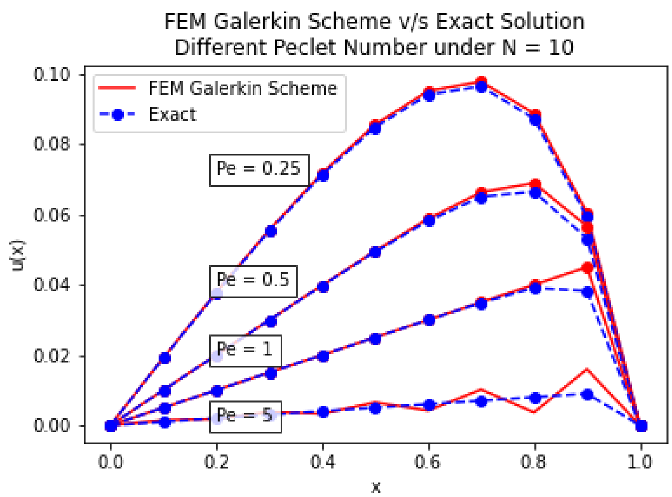


Figure 12: FEM with Galerkin Approximation Scheme at N = 10 under different Peclet numbers

When the Peclet number is larger than 1, there exists node-to-node oscillations around the exact solution. The scheme becomes unstable, the same as FDM with Central Difference Scheme.

Improvements on FEM with Galerkin Approximation Scheme

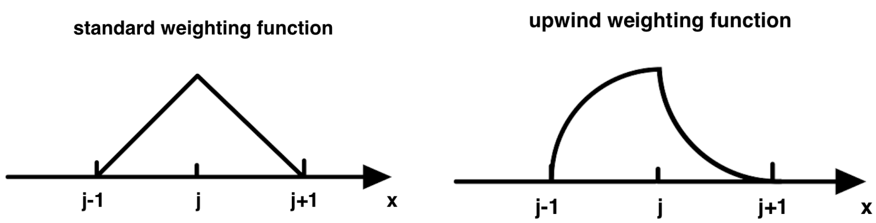
To tackle the instability, we consider two improvements: the Artificial scheme and the Upwind scheme.

Proposition 1. Artificial scheme

If we use the artificial scheme by adding the numerical diffusion term $\hat{\nu}$ to the diffusivity ν, the residual R will be inconsistent. The original equation will then be changed, the artificial scheme solves the stability problem but loses consistency, so we cannot use the Artificial scheme to improve.

Proposition 2. Upwind scheme

If we use the Upwind Scheme by adjusting the convective term, the weighting function w will change as:



The upstream is a little heavier than the downstream. The functional spaces for u and v remain unchanged, and the weighting functions remain invariant from the Galerkin to the Upwind scheme.

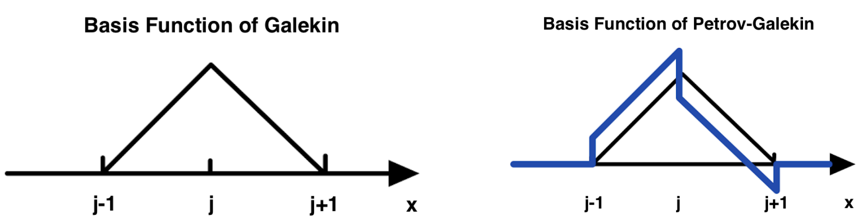
Therefore, the FEM with Upwind Difference scheme solves the instability and maintains the consistency.

This method is also called the Streamline Upwind Petrov-Galerkin method (SUPG), proposed by Hughes and Brooks in 1982 [1]. We will then implement this method to solve our model problem.

FEM with Petrov-Galerkin Approximation Scheme (SUPG)

To adjust the convection term, we replace the weighting function w with $w + \hat{w}$, where $\hat{w} = \tau\alpha w_x$, $\tau = \beta\frac{\Delta x}{2|\alpha|}$, $\beta = \coth(P_e) - \frac{1}{P_e}$. The weighting function $w \in N_\alpha$ is then changed to $\hat{w} \in \hat{N}_\alpha$.

The basis functions of Petrov-Galerkin Scheme will be altered to:



Variational Formulation of FEM with Petrov-Galerkin Approximation Scheme

We write the formulation of FEM with Petrov-Galerkin Approximation Scheme as:

$$(w, R)_\Omega + \sum_e (\hat{w}, R)_{\Omega_e} = 0$$

where $(w, R)_\Omega$ term represents the Galerkin Scheme term which is unstable and the $\sum_e (\hat{w}, R)_{\Omega_e}$ term is added as the Petrov-Galerkin scheme to satisfy the stability requirement. Note that since $\sum_e (\hat{w}, R)_{\Omega_e}$ is no longer continuous function, we cannot integrate by parts, so we denote the field as Ω_e instead of Ω.

The formulation system of SUPG is then expressed as:

$$B_{SUPG}(w^h, F^h) = F_{SUPG}(w^h, s)$$

where:

$$\begin{cases} B_{SUPG}(w^h, F^h) = B_G(w^h, v^h) + \sum_{e=1}^{N} (\tau\alpha w_x^h, \alpha v_x^h - \nu v_{xx}^h)_e \\ F_{SUPG}(w^h, s) = F_G(w^h, s) + \sum_{e=1}^{N} (\tau\alpha w_x^h, s)_e \end{cases}$$

We will still use the trial function to act as the weighting function, which gives:

$$\begin{cases} B_{SUPG}(v^h, F^h) = B_G(v^h, v^h) + B_{extra} \\ F_{SUPG}(v^h, s) = F_G(v^h, s) + F_{extra} \end{cases}$$

The Galerkin scheme part is written as:

$$\begin{cases} B_G(v^h, v^h) = \int_0^1 \alpha v^h \frac{\partial v^h}{\partial x} dx + \int_0^1 \nu \frac{\partial v^h}{\partial x} \frac{\partial v^h}{\partial x} dx \\ F_G(v^h, s) = \int_0^1 s \cdot v^h dx \end{cases}$$

And the extra part is calculated as:

$$\begin{cases} B_{extra} = \sum_{e=1}^N (\tau \alpha v_x^h, \alpha v_x^h - \nu v_{xx}^h)_e = \sum_{k=1}^N \int_{e^k} (\tau \alpha^2 v_x^h v_x^h - \tau \alpha \nu v_x^h v_{xx}^h) \ \ dx \\ F_{extra} = \sum_{e=1}^N (\tau \alpha w_x^h, s)_e = \sum_{k=1}^N \int_{e^k} \tau \alpha v_x^h s \ \ dx \end{cases}$$

Basis Function

We will use the linear basis function $\varphi(x)$ to act as the trial solution v^h in the FEM Galerkin formulation

$$\begin{cases} \varphi_{k-1}(x) = \frac{x_k - x}{\Delta x} = k - \frac{x}{\Delta x} \\ \varphi_k(x) = \frac{x - x_{k-1}}{\Delta x} = \frac{x}{\Delta x} - (k-1) \end{cases} \quad ; \quad \begin{cases} \frac{d\varphi_{k-1}(x)}{dx} = -\frac{1}{\Delta x} \\ \frac{d\varphi_k(x)}{dx} = \frac{1}{\Delta x} \end{cases} \quad ; \quad \frac{d^2\varphi(x)}{dx^2} = 0$$

Thus, the extra terms are calculated as:

$$\begin{cases} B_{extra} = \sum_{k=1}^N \int_{e^k} (\tau \alpha^2 v_x^h v_x^h - \tau \alpha \nu v_x^h v_{xx}^h) \ \ dx = \tau \alpha^2 \sum_{j=1}^N \sum_{k=1}^N \int_{e^k} \frac{d\varphi_i}{dx} \frac{d\varphi_j}{dx} \ \ dx \\ F_{extra} = \sum_{k=1}^N \int_{e^k} \tau \alpha v_x^h s \ \ dx = \tau \alpha \sum_{j=1}^N \sum_{k=1}^N \int_{e^k} s \frac{d\varphi_j}{dx} \ \ dx \end{cases}$$

We will use the formulation of the extra term of Petrov-Galerkin Scheme to generate the extra system:

$$\boldsymbol{A_{extra} u = d_{extra}}$$

The matrix $\boldsymbol{A_{extra}}$ on k-th element is calculated as:

$$A_{extra}^k = \begin{bmatrix} a_{k-1,k-1} & a_{k,k-1} \\ a_{k-1,k} & a_{k,k} \end{bmatrix}$$

where:

$$a_{k-1,k-1} = \tau \alpha^2 \int_{x_{k-1}}^{x_k} \frac{d\varphi_{k-1}}{dx} \frac{d\varphi_{k-1}}{dx} dx = \tau \alpha^2 \int_{x_{k-1}}^{x_k} (-\frac{1}{\Delta x})^2 dx = \frac{\tau \alpha^2}{\Delta x}$$

$$a_{k-1,k} = \tau \alpha^2 \int_{x_{k-1}}^{x_k} \frac{d\varphi_{k-1}}{dx} \frac{d\varphi_k}{dx} dx = \tau \alpha^2 \int_{x_{k-1}}^{x_k} (-\frac{1}{\Delta x} \cdot \frac{1}{\Delta x}) dx = -\frac{\tau \alpha^2}{\Delta x}$$

$$a_{k,k-1} = \tau \alpha^2 \int_{x_{k-1}}^{x_k} \frac{d\varphi_k}{dx} \frac{d\varphi_{k-1}}{dx} dx = \tau \alpha^2 \int_{x_{k-1}}^{x_k} (-\frac{1}{\Delta x} \cdot \frac{1}{\Delta x})^2 dx = -\frac{\tau \alpha^2}{\Delta x}$$

$$a_{k,k} = \tau \alpha^2 \int_{x_{k-1}}^{x_k} \frac{d\varphi_k}{dx} \frac{d\varphi_k}{dx} dx = \tau \alpha^2 \int_{x_{k-1}}^{x_k} (\frac{1}{\Delta x})^2 dx = \frac{\tau \alpha^2}{\Delta x}$$

The matrix of $\boldsymbol{A_{extra}}$ on k-th element becomes:

$$\Rightarrow A_{extra}^k = \begin{bmatrix} a_{k-1,k-1} & a_{k,k-1} \\ a_{k-1,k} & a_{k,k} \end{bmatrix} = \begin{bmatrix} \frac{\tau \alpha^2}{\Delta x} & -\frac{\tau \alpha^2}{\Delta x} \\ -\frac{\tau \alpha^2}{\Delta x} & \frac{\tau \alpha^2}{\Delta x} \end{bmatrix}$$

Then, the global stiffness matrix of the extra term $\boldsymbol{A_{extra}}$ becomes:

$$\boldsymbol{A_{extra}} = \begin{bmatrix} \cdots & \cdots & \cdots & \cdots \\ \cdots & \frac{\tau\alpha^2}{\Delta x} & -\frac{\tau\alpha^2}{\Delta x} & \cdots \\ \cdots & -\frac{\tau\alpha^2}{\Delta x} & \frac{\tau\alpha^2}{\Delta x} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix} = \begin{bmatrix} \frac{2\tau\alpha^2}{\Delta x} & -\frac{\tau\alpha^2}{\Delta x} & 0 & 0 \\ -\frac{\tau\alpha^2}{\Delta x} & \frac{2\tau\alpha^2}{\Delta x} & -\frac{\tau\alpha^2}{\Delta x} & 0 \\ 0 & -\frac{\tau\alpha^2}{\Delta x} & \frac{2\tau\alpha^2}{\Delta x} & -\frac{\tau\alpha^2}{\Delta x} \\ 0 & 0 & -\frac{\tau\alpha^2}{\Delta x} & \frac{2\tau\alpha^2}{\Delta x} \end{bmatrix}$$

The source vector of the extra term generated by F_{extra} is in form of $d_{extra} = \begin{bmatrix} d_{k-1} \\ d_k \end{bmatrix}$, where:

$$d^{k-1} = \tau \int_{x_{k-1}}^{x_k} s_{k-1} \frac{d\varphi_{k-1}}{dx} dx = \tau s_{k-1} \int_{x_{k-1}}^{x_k} (-\frac{1}{\Delta x}) dx = -\tau s_{k-1}$$

$$d^k = \tau \int_{x_{k-1}}^{x_k} s \frac{d\varphi_k}{dx} dx = \tau s_k \int_{x_{k-1}}^{x_k} (\frac{1}{\Delta x}) dx = \tau s_k$$

As the source s_k are constants, the element d^k is in form of $d^k = \tau s_k \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. The vector $\boldsymbol{d_{extra}}$ becomes:

$$\boldsymbol{d_{extra}} = \begin{bmatrix} d^1, & d^2, & d^3, & d^4 \end{bmatrix}^T = \tau s_k \begin{bmatrix} 1-1, & 1-1, & 1-1, & 1-1 \end{bmatrix}^T = \begin{bmatrix} 0, & 0, & 0, & 0 \end{bmatrix}^T$$

Matrix System

The formulation of the Galerkin Approximation Scheme has been calculated as:

$$B_G(v^h, v^h) = \sum_{i=0}^{N} u_i \sum_{k=1}^{N} \int_{e^k} \frac{d\varphi_i(x)}{dx} \cdot (\alpha\varphi_j(x) + \nu \frac{d\varphi_j(x)}{dx})\ dx$$

$$F_G(v^h, s) = \sum_{k=1}^{N} \int_{e^k} s_j \varphi_j(x)\ dx$$

Write the matrix system generated by the Galerkin Approximation Scheme as:

$$\boldsymbol{A_G u = d_G}$$

$$\begin{bmatrix} b & c & 0 & 0 \\ a & b & c & 0 \\ 0 & a & b & c \\ 0 & 0 & a & b \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = s\Delta x \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

with:

$$a = -\frac{1}{2}\alpha - \frac{\nu}{\Delta x}, \quad b = \frac{2\nu}{\Delta x}, \quad c = \frac{1}{2}\alpha - \frac{\nu}{\Delta x}$$

The matrix system generated by the extra term from the Petrov-Galerkin Scheme is expressed as:

$$\boldsymbol{A_{extra} u = d_{extra}}$$

$$\begin{bmatrix} \frac{2\tau\alpha^2}{\Delta x} & -\frac{\tau\alpha^2}{\Delta x} & 0 & 0 \\ -\frac{\tau\alpha^2}{\Delta x} & \frac{2\tau\alpha^2}{\Delta x} & -\frac{\tau\alpha^2}{\Delta x} & 0 \\ 0 & -\frac{\tau\alpha^2}{\Delta x} & \frac{2\tau\alpha^2}{\Delta x} & -\frac{\tau\alpha^2}{\Delta x} \\ 0 & 0 & -\frac{\tau\alpha^2}{\Delta x} & \frac{2\tau\alpha^2}{\Delta x} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus, we add up two matrix systems to obtain the total matrix system of FEM with Petrov-Galerkin Scheme:

$$\boldsymbol{A_{SUPG} u = A_G + A_{extra} u = d_G + d_{extra} = d_{SUPG}}$$

$$\Rightarrow \begin{bmatrix} b & c & 0 & 0 \\ a & b & c & 0 \\ 0 & a & b & c \\ 0 & 0 & a & b \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = s\Delta x \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

with:

$$a = -\frac{1}{2}\alpha - \frac{\nu}{\Delta x} - \frac{\tau\alpha^2}{\Delta x}, \quad b = \frac{2\nu}{\Delta x} + \frac{2\tau\alpha^2}{\Delta x}, \quad c = \frac{1}{2}\alpha - \frac{\nu}{\Delta x} - \frac{\tau\alpha^2}{\Delta x}$$

Model Example

We will then try the model example to examine the FEM Petrov-Galerkin Approximation method:

$$N = 5, \alpha = 5, \nu = 1, s_j = 1 \Rightarrow P_e = 0.5, \beta = \coth(0.5) - 2 = 0.163953, \tau = \beta\frac{\Delta x}{2|\alpha|} = 0.00327906$$

The values of a, b, c are then calculated as:

$$\begin{cases} a = -\frac{1}{2}\alpha - \frac{\nu}{\Delta x} - \frac{\tau\alpha^2}{\Delta x} = -\frac{5}{2} - \frac{1}{1/5} - \frac{0.00327906\times 25}{1/5} = -7.909883 \\ b = \frac{2\nu}{\Delta x} + \frac{2\tau\alpha^2}{\Delta x} = \frac{2}{1/5} + \frac{2\times 0.00327906\times 25}{1/5} = 10.81977 \\ c = \frac{1}{2}\alpha - \frac{\nu}{\Delta x} - \frac{\tau\alpha^2}{\Delta x} = \frac{5}{2} - \frac{1}{1/5} - \frac{0.00327906\times 25}{1/5} = -2.909883 \end{cases}$$

The matrix system becomes

$$\begin{bmatrix} 10.81977 & -2.909883 & 0 & 0 \\ -7.909883 & 10.81977 & -2.909883 & 0 \\ 0 & -7.909883 & 10.81977 & -2.909883 \\ 0 & 0 & -7.909883 & 10.81977 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0.2 \\ 0.2 \\ 0.2 \\ 0.2 \end{bmatrix}$$

Combined with $u_0 = u_5 = 0$, the nodal coefficients vector of FEM with Petrov-Galerkin Scheme is:

$$\boldsymbol{u} = \{0, 0.03766875, 0.07133177, 0.09410606, 0.08728173, 0\}^T$$

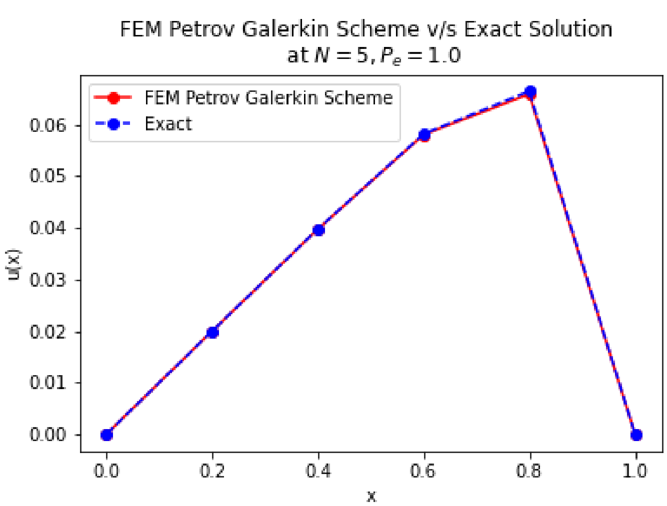
Combined with the basis functions $\varphi_{k-1}(x) = \frac{x_k - x}{\Delta x} = k - \frac{x}{\Delta x} \quad , \quad \varphi_k(x) = \frac{x - x_{k-1}}{\Delta x} = \frac{x}{\Delta x} - (k-1)$,

And insert nodal coefficients into the trial solution $u^N(x) := \sum_{i=0}^{N} u_i \varphi_i(x)$, we obtain the nodal values:

$$u^N(x_i) = u_i \cdot [\varphi_{i-1}(x_i) + \varphi_i(x_i)]$$

$$\begin{cases} u^N(x_0) = u_0 \varphi_k(x_0) = 0.0 \\ u^N(x_1) = u_1 \times (\varphi_{k-1}(x_1) + \varphi_k(x_k)) = 0.03766875 \times (k - \frac{1\cdot\Delta x}{\Delta x} + \frac{1\cdot\Delta x}{\Delta x} - (k-1)) = 0.03766875380879208 \\ u^N(x_2) = u_2 \times (\varphi_{k-1}(x_2) + \varphi_k(x_2)) = 0.07133176964956722 \\ u^N(x_3) = u_3 \times (\varphi_{k-1}(x_3) + \varphi_k(x_3)) = 0.0941060607623135 \\ u^N(x_4) = u_4 \times (\varphi_{k-1}(x_4) + \varphi_k(x_4)) = 0.08728172931176616 \\ u^N(x_5) = u_5 \times \varphi_{k-1}(x_5) = 0.0 \end{cases}$$

The numerical solution the FEM with Petrov-Galerkin Approximation Scheme is shown as below:



Assessment on FEM with Petrov-Galerkin Approximation Scheme

1. Consistency

FEM with Petrov-Galerkin Approximation Scheme is also consistent, since this scheme is residual based, which means that the exact solution will also satisfy the variational formulation to become $R = 0$.

2. Stability

FEM with Petrov-Galerkin Approximation Scheme is improved to achieve stability as it adds the stablized term τ, which is the stabilization parameter.

Note that when $P_e \to 0$, $\tau = 0$, the Petrov-Galerkin Scheme comes back to the Galerkin Scheme.

When $P_e \to \infty$, the flow is a convection-dominated and the Petrov-Galerkin scheme provides the stablity.

We will then evaluate the stability of FEM with Petrov-Galerkin Approximation Scheme at different Peclet numbers $P_e = \{0.25, 0.5, 1, 5\}$ with the number of partitions $N = 10$:

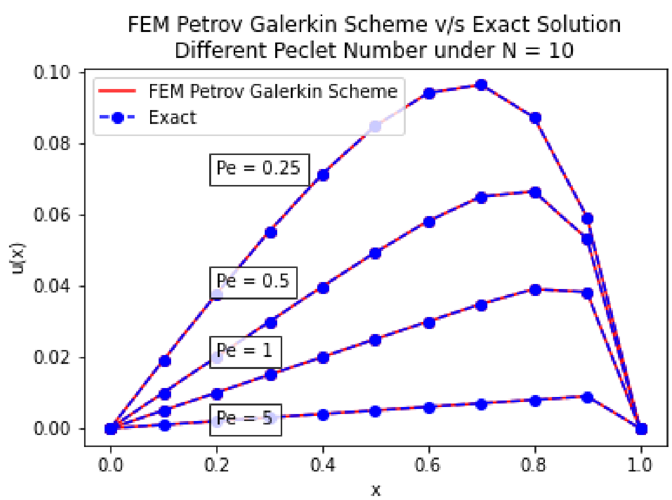


Figure 13: FEM with Petrov-Galerkin Approximation Scheme at N = 10 under different Peclet numbers

As we can observed, the numerical solution of FEM with Petrov-Galerkin Approxination Scheme is closer to the exact solution, and there exists no or at least not visible oscillations around the exact solution.

Therefore, FEM with Petrov-Galerkin Approxination Scheme achieves both consistency and stability.

3. Continuity

As $N_a + \hat{N}_a$ is not zero anymore, it loses the elementary boundaries. Thus, the Petrov-Galerkin scheme achieves consistency and stability but incurs the discontinuity. So we cannot integrate by parts due to the discontinuity for the terms including u. However, we can resolve it by using the elementary integration.

Comparison of Accuracy for FEM Galerkin and Petrov-Galerkin Scheme

We will now compare the FEM with Galerkin Approximation Scheme with FEM with Petrov-Galerkin Approximation Scheme at the critical value of Peclet number $P_e = 1$ with the number of partitions $N = 10$.

As shown, the numerical solution of Petrov-Galerkin scheme is closer even overlaps with the exact solution.

Specifically, we investigate the global errors generated by FEM Galerkin and Petrov-Galerkin schemes for different values of Peclet number under the same number of partitons $N = 10$.

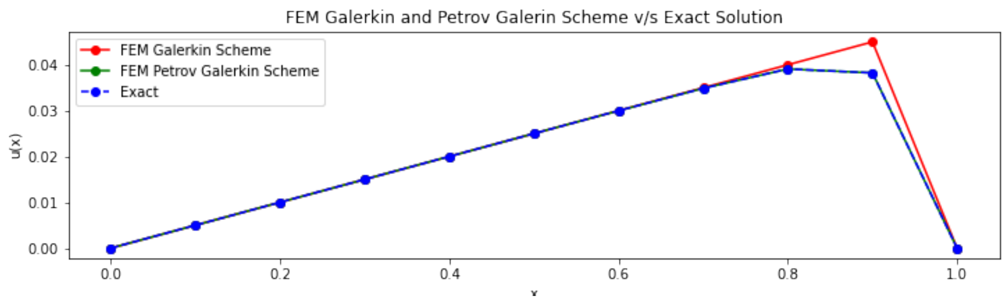


Figure 14: Comparison of FEM Galerkin and Petrov Galerin Scheme at Pe = 1.0 N = 10

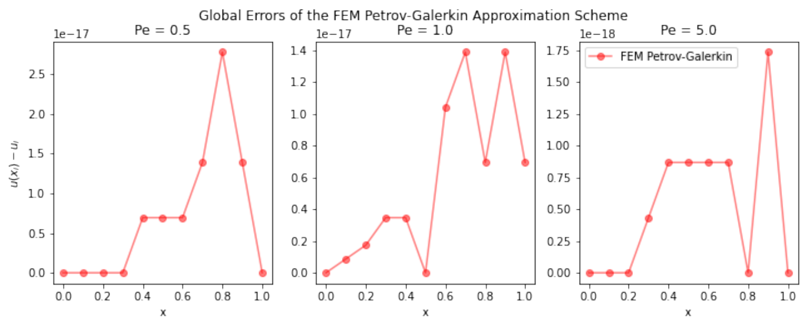


Figure 15: Global Errors of FEM Petrov-Galerkin Approximation Scheme

Comparison of Stability for FEM Galerkin and Petrov-Galerkin Scheme

Then, we will compare the stability of FEM with Galerkin scheme and FEM with Petrov-Galerkin scheme for large Peclet numbers $P_e > 1$:

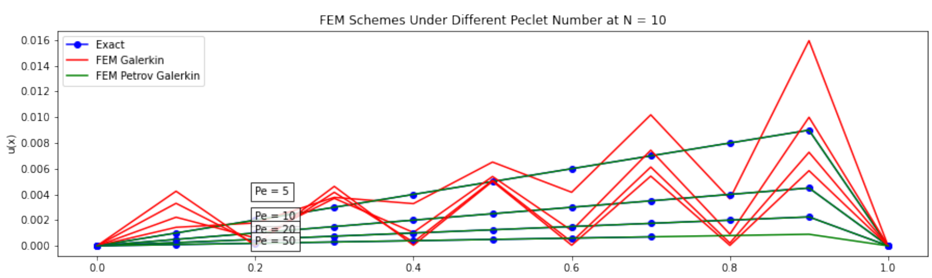


Figure 16: Different FEM Schemes under different Peclet numbers at N = 10

As we observed, there does not exist oscillations around the exact solution for FEM with Petrov-Galekrin Approximation Scheme. Also, it is much more accurate at high Peclet nubers $P_e > 1$ compared to FEM with Galerkin Approximation Scheme.

FVM Finite Volume Method

Finite Volume Method (FVM) evaluates the flux at surfaces for each finite volume, so it will automatically satisfy the law of balance of fluxes. This means that the flow-in fluxes equals the flow-out fluxes. Therefore, FVM will automatically satisfy the discretized conservativeness.

We will then implement FVM to solve our 1D steady convection-diffusion problem:

$$\begin{cases} \alpha \frac{\partial u}{\partial x} - \nu \frac{\partial^2 u}{\partial x^2} = s & \text{on } [0,1] \\ u(0) = u(1) = 0 & \alpha, \nu, s \text{ are constants.} \end{cases}$$

Partition of the domain

The parition for FVM is a little different from previous partition. On the domain $[0,1]$ partitioned $N+1$ nodes $x_0, x_1, ..., x_N$, we firstly take a small example of two adjacent elements bounded by x_W, x_P, x_E:

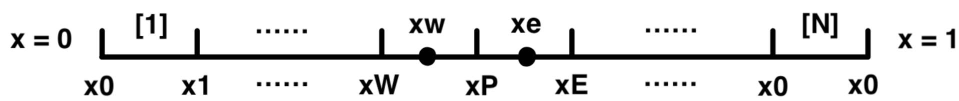


Figure 17: Partition of the domain for FVM

Denote the middle node between x_W and x_P by x_w, and the middle node between x_P and x_E by x_e. The length between x_W and x_P is the same as which between x_P and x_E, denoted by Δx.

We will now integrate the model equation on the part between two middle points x_w and x_e,

$$\int_{x_w}^{x_e} \alpha \frac{\partial u}{\partial x} - \int_{x_w}^{x_e} \nu \frac{\partial^2 u}{\partial x^2} = \int_{x_w}^{x_e} s$$

Since the source term s is constant, $s_e - s_w = 0$. The equation becomes:

$$(\alpha A u)_{x_e} - (\alpha A u)_{x_w} = (\nu A \frac{du}{dx})_{x_e} - (\nu A \frac{du}{dx})_{x_w}$$

A is the area, the same for left and right hand sides $A_w = A_e = A$ as the elements are equally divided.

Therefore, the integrated convection-diffusion equation is in form of:

$$\alpha u_e - \alpha u_w = (\nu \frac{du}{dx})_e - (\nu \frac{du}{dx})_w$$

We will then transform the integrated equation according to different difference schemes to solve the model problem.

FVM with Central Difference Scheme

Formulation of FVM with Central Difference Scheme

The first difference scheme we will use is the most traditional one: FVM with Central Difference Scheme.

Rewrite the u_e, u_w terms as

$$u_w = \frac{(u_W + u_P)}{2}, u_e = \frac{(u_P + u_E)}{2}$$

And the first derivative $\frac{du}{dx}$ is expressed as:

$$(\nu \frac{du}{dx})_w = \nu \frac{u_P - u_W}{\Delta x} \quad ; \quad (\nu \frac{du}{dx})_e = \nu \frac{u_E - u_P}{\Delta x}$$

Substitute them into the equation, we obtain:

$$[\alpha\frac{(u_P+u_E)}{2}-\alpha\frac{(u_W+u_P)}{2}]-[\nu\frac{u_E-u_P}{\Delta x}-\nu\frac{u_P-u_W}{\Delta x}]=s\cdot\Delta x$$
$$\Rightarrow -u_W\cdot(\frac{\alpha}{2}+\frac{\nu}{\Delta x})+u_P\cdot(\frac{2\nu}{\Delta x})-u_E\cdot(-\frac{\alpha}{2}+\frac{\nu}{\Delta x})=s\cdot\Delta x$$

Extend the equation to the whole domain $x_0, x_1, ..., x_N$, the formulation of FVM with Central scheme is:

$$-a\cdot u_{j-1}+b\cdot u_j-c\cdot u_{j+1}=s_j\cdot\Delta x \qquad j=1,...,N$$

where:

$$a=\frac{\alpha}{2}+\frac{\nu}{\Delta x},\quad b=\frac{2\nu}{\Delta x},\quad c=-\frac{\alpha}{2}+\frac{\nu}{\Delta x}$$

Matrix System

We convert the formulation into the matrix system $\boldsymbol{Au}=\boldsymbol{s}$, where $\boldsymbol{s}=\{s_j\}_{j=1}^{N-1}=\{1,...,1\}^T$ is the source vector, and $\boldsymbol{u}=\{u_1,\dots,u_{N-1}\}^T$ is the vector of numerical solutions on the $N-1$ gridpoints.

$$\Rightarrow\begin{bmatrix} b & -c & 0 & 0 & ... & 0\\ -a & b & -c & 0 & ... & 0\\ 0 & -a & b & -c & ... & 0\\ ... & ... & ... & ... & ... & ...\\ 0 & 0 & ... & 0 & -a & b\end{bmatrix}\begin{pmatrix}u_1\\u_2\\u_3\\ \dots\\u_{N-1}\end{pmatrix}=\Delta x\begin{pmatrix}s_1\\s_2\\s_3\\ ...\\s_{N-1}\end{pmatrix}$$

Model Example

We will then try the model example setting to examine the FVM with Central Difference Scheme:

$$N=5,\alpha=5,\nu=1,s_j=1$$

The values of a,b,c are calculated as:

$$\begin{cases} a=\frac{\alpha}{2}+\frac{\nu}{\Delta x}=\frac{5}{2}+\frac{1}{1/5}=7.5\\ b=\frac{2\nu}{\Delta x}=\frac{2}{1/5}=10\\ c=-\frac{\alpha}{2}+\frac{\nu}{\Delta x}=-\frac{5}{2}+\frac{1}{1/5}=2.5\end{cases}$$

Thus, the matrix system becomes:

$$\begin{bmatrix} 10 & -2.5 & 0 & 0\\ -7.5 & 10 & -2.5 & 0\\ 0 & -7.5 & 10 & -2.5\\ 0 & 0 & -7.5 & 10\end{bmatrix}\begin{pmatrix}u_1\\u_2\\u_3\\u_4\end{pmatrix}=0.2\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$$

We will now solve the matrix system to obtain the vector of numerical solutions $\boldsymbol{u}$:

$$\begin{cases} 10u_1-2.5u_2=0.2\\ -7.5u_1+10u_2-2.5u_3=0.2\\ -7.5u_2+10u_3-2.5u_4=0.2\\ -7.5u_3+10u_4=0.2\end{cases}$$

Combined with $u_0=u_5=0$, the final numerical solution of FVM with Central Difference scheme is:

$$\boldsymbol{u}=\{0,0.03766875,0.07133177,0.09410606,0.08728173,0\}^T$$

We plot them into the diagram and compare with the exact solution:

As we can observe, the solution FVM with Central Difference Scheme is in the same trend with the exact solution at Peclet number $P_e=0.5$, although with some errors.

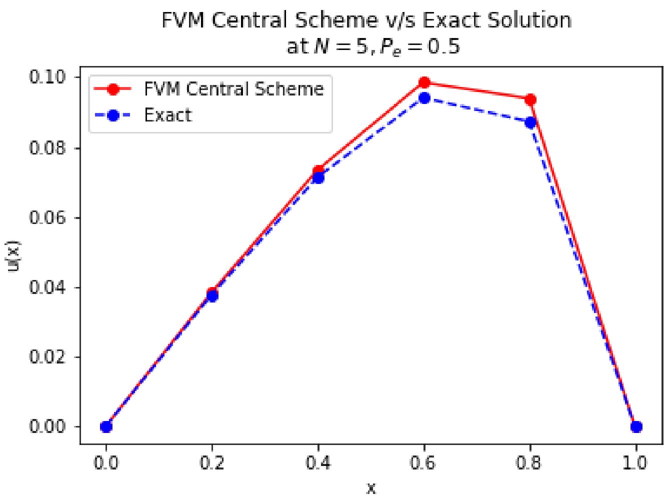


Figure 18: Solution of FVM with Central Difference Scheme at Pe = 0.5, N = 5

Assessment on FVM with Central Difference Scheme

As we stated before, FVM originally satisfies the conservativeness at the discrete level. Then, we will examine FVM with Central Difference Scheme in terms of transportiveness, stability, consistency, and convergence.

1. Transportiveness

The coefficient of the middle point u_P is influenced by both the west point u_w and the east point u_e.

Suppose there is a upwind convective flowing from west to east, the impact from west side will be much larger than the impact from east side. However, the Central Difference Scheme considers the impact from the west and the east equally, it is then not able to identify the flow direction.

Therefore, FVM with Central Difference Scheme does not satisfiy the transportiveness for problems at high Peclet number, which also proves that it does not work well for the convection-dominated flow.

2. Stability

The essential requirement for stability / boundedness is that all coefficients of the discretized equations should have the same sign, usually all positive. This means that the increase of u at one node will result in the increase of u at neighboring nodes, which implies convergence and prevents the oscillations.

For the east coefficient $c = -\frac{\alpha}{2} + \frac{\nu}{\Delta x}$, α is the convective coefficient and ν is the diffusive coefficient. If the equation is convection dominated, α will be much larger than ν, then the coefficient of the east node may become negative. If the coefficient c is negative, we have:

$$c < 0 \quad \Rightarrow -\frac{\alpha}{2} + \frac{\nu}{\Delta x} < 0 \quad \Rightarrow \frac{\alpha \Delta x}{2\nu} > 1 \quad \Rightarrow P_e > 1$$

Thus, if $P_e > 1$, the coefficient of east point u_e is negative, which violates the boundedness requirement.

We will then display the solutions at different Peclet numbers $P_e = \{0.25, 0.5, 1, 5\}$ to validate the stability in the figure 19.

As we observed, there exists node-to-node oscillations around the exact solution for solutions at $P_e > 1$. Thus, FVM with Central Difference Scheme is unstable at Peclet numbers $P_e > 1$.

The global errors of FVM with Central Difference Scheme at different Peclet numbers $P_e = \{0.1, 0.5, 1, 5, 10, 20\}$ also validates that the global errors at $P_e < 1$ is much lower than which at $P_e > 1$, shown in the figure 20.

Therefore, FVM with Central Difference scheme is not suitable for convection-dominated equations at $P_e > 1$.

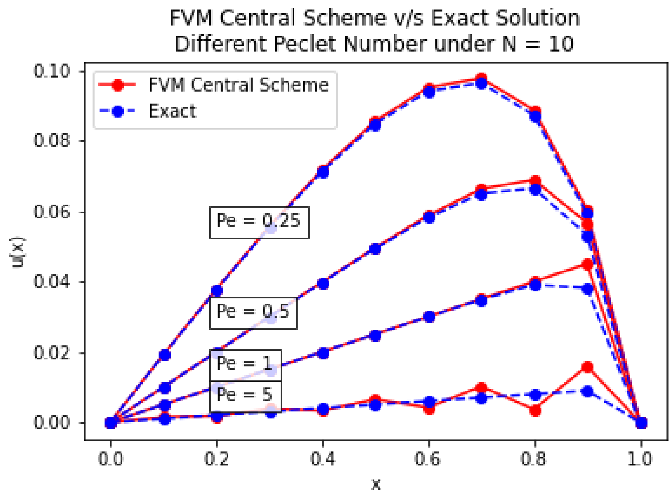


Figure 19: Solution of FVM with Central Difference Scheme at N = 10 under different Peclet numbers

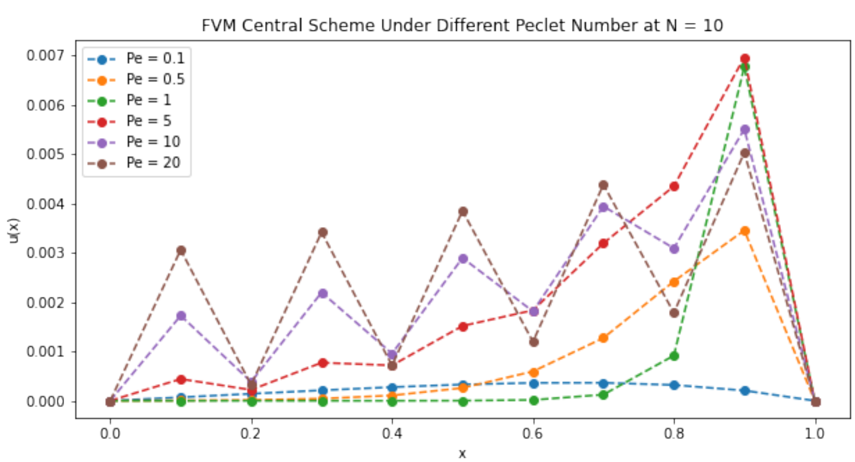


Figure 20: FVM Central Schemes under different Pe at N = 10

3. Consistency

The consistency of FVM is decided by the Taylor series truncation error T_j , which is the difference between numerical solution u_j and exact solution $u(x_j)$.

FVM is k-th order consistent for k > 0, if the local truncation error satisfies:

$$T_j = O(\Delta x^k)$$

The numerical solution u_j of FVM with Central Difference Scheme at gridpoint x_j satisfies:

$$(-\frac{\alpha}{2} - \frac{\nu}{\Delta x}) \cdot u_{j-1} + (\frac{2\nu}{\Delta x} \cdot)u_j + (\frac{\alpha}{2} - \frac{\nu}{\Delta x}) \cdot u_{j+1} = s_j \cdot \Delta x \qquad j = 1, ..., N$$

The exact solution $u(x_j)$ at gridpoint x_j satisfies:

$$\alpha u_x - \nu u_{xx} = s$$

With Taylor expansion, the equation of exact solution becomes:

$$\frac{u_{j+1} - u_{j-1}}{2\Delta x} = u_x(x_j) + O(\Delta x^2)$$

$$\frac{u_{j+1} - 2u_j + u_{j-1}}{\Delta x^2} = u_{xx}(x_j) + O(\Delta x^2)$$

Thus, the Taylor series truncation error is the subtraction of the numerical and the exact solution:

$$\begin{aligned}
T_j &= (-\frac{\alpha}{2\Delta x} - \frac{\nu}{\Delta x^2} + \frac{\alpha}{2\Delta x} + \frac{\nu}{\Delta x^2}) \cdot u_{j-1} + O(\Delta x^2) + (\frac{2\nu}{\Delta x^2} + \frac{2\nu}{\Delta x^2}) \cdot u_j + O(\Delta x^2) \\
&+ (\frac{\alpha}{2\Delta x} - \frac{\nu}{\Delta x} - \frac{\alpha}{2\Delta x} + \frac{\nu}{\Delta x}) \cdot u_{j+1} + O(\Delta x^2) - (s_j - s_j) \\
&= O(\Delta x^2) + O(\Delta x^2) \\
&= O(\Delta x^2)
\end{aligned}$$

FVM with Central Difference Scheme has the second-order Taylor series truncation error. Therefore, this scheme is second-order consistent.

4. Convergence

Since FVM with Central Difference Scheme is second-order consistent but does not satisfy the stability at $P_e > 1$, it is therefore not a convergent scheme.

FVM with Upwind Difference Scheme

As we mentioned in the previous transportiveness part, the Central Difference Scheme does not imply the direction of convection. In order to tackle this issue, we will use the same idea with the proposition 1 of FDM part. We introduce FVM with Upwind Difference Scheme, which only considers the impact from the west point u_w in the positive direction and the impact from the east point u_e in the negative direction.

Now we use FVM with Upwind Difference Scheme to solve model problem and examine its performance.

Formulation of FVM with Upwind Difference Scheme

Starting from the initial integrated convection-diffusion equation:

$$\alpha u_e - \alpha u_w = \nu(\frac{du}{dx})_e - \nu(\frac{du}{dx})_w$$

When is the flow is in the positive direction (from west u_w to east u_e), the convective term α is positive. The middle point u_w is decided by u_W and u_P as usual, but the middle point u_e is decided by u_P.

Write the relationship as:

$$\begin{cases} u_w = u_W, & u_w = \frac{(u_W+u_P)}{2} \\ u_e = u_P, & u_e = \frac{(u_P+u_E)}{2} \end{cases}$$

The formulation of FVM with Upwind Difference Scheme at the positive convective direction becomes:

$$\alpha u_P - \alpha u_W = \frac{\nu}{\Delta x}(u_E - u_P) - \frac{\nu}{\Delta x}(u_P - u_W)$$

$$\Rightarrow -u_W \cdot (\alpha + \frac{\nu}{\Delta x}) + u_P \cdot (\alpha + \frac{2\nu}{\Delta x}) - u_E \cdot (\frac{\nu}{\Delta x}) = 0$$

When is the flow is in the negative direction (from east u_e to west u_w), the convective term α is negative. The middle point u_e is decided by u_E and u_P as original, but the middle point u_w is decided by u_P.

Write the relationship as:

$$\begin{cases} u_w = u_P, & u_w = \frac{(u_W+u_P)}{2} \\ u_e = u_E, & u_e = \frac{(u_P+u_E)}{2} \end{cases}$$

The formulation of FVM with Upwind Difference Scheme at the negative convective direction becomes:

$$\alpha u_E - \alpha u_P = \frac{\nu}{\Delta x}(u_E - u_P) - \frac{\nu}{\Delta x}(u_P - u_W)$$

$$\Rightarrow -u_W \cdot (\frac{\nu}{\Delta x}) + u_P \cdot (-\alpha + \frac{2\nu}{\Delta x}) - u_E \cdot (-\alpha + \frac{\nu}{\Delta x}) = 0$$

Therefore, we combine the two direction cases into one equation system:

$$-a \cdot u_W + b \cdot u_P - c \cdot u_E = 0$$

where:

$$a = \begin{cases} \alpha + \frac{\nu}{\Delta x} & \text{, if } \alpha > 0 \\ \frac{\nu}{\Delta x} & \text{, if } \alpha < 0 \end{cases} \qquad \Rightarrow a = \max(\alpha, 0) + \frac{\nu}{\Delta x}$$

$$b = \begin{cases} \alpha + \frac{2\nu}{\Delta x} & \text{, if } \alpha > 0 \\ -\alpha + \frac{2\nu}{\Delta x} & \text{, if } \alpha < 0 \end{cases} \qquad \Rightarrow b = |\alpha| + \frac{2\nu}{\Delta x}$$

$$c = \begin{cases} \frac{\nu}{\Delta x} & \text{, if } \alpha > 0 \\ -\alpha + \frac{\nu}{\Delta x} & \text{, if } \alpha < 0 \end{cases} \qquad \Rightarrow c = \max(-\alpha, 0) + \frac{\nu}{\Delta x}$$

Extend the equation to the whole domain $x_0, x_1, ..., x_N$, we have:

$$-a \cdot u_{j-1} + b \cdot u_j - c \cdot u_{j+1} = 0 \qquad j = 1, ..., N$$

Matrix System

We will then convert the formulation of FVM with Upwind Scheme into the matrix system: $\boldsymbol{Au} = \boldsymbol{s}$, where $\boldsymbol{s} = \{s_j\}_{j=1}^{N-1} = \{1, ..., 1\}^T$ is the source vector and $\boldsymbol{u} = \{u_1, \ldots, u_{N-1}\}^T$ is the numerical solutions.

$$\Rightarrow \begin{bmatrix} b & -c & 0 & 0 & ... & 0 \\ -a & b & -c & 0 & ... & 0 \\ 0 & -a & b & -c & ... & 0 \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & ... & 0 & -a & b \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \ldots \\ u_{N-1} \end{pmatrix} = \Delta x \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ ... \\ s_{N-1} \end{pmatrix}$$

Model Example

Try the model example setting to examine the FVM with Upwind Difference Scheme:

$$N = 5, \alpha = 5, \nu = 1, s_j = 1 \quad \Rightarrow P_e = \frac{\alpha}{2\nu N} = \frac{5}{2 \times 5} = 0.5$$

The values of a, b, c are calculated as:

$$\begin{cases} a = \max(\alpha, 0) + \frac{\nu}{\Delta x} = 5 + \frac{1}{1/5} = 10 \\ b = |\alpha| + \frac{2\nu}{\Delta x} = 5 + \frac{2}{1/5} = 15 \\ c = \max(-\alpha, 0) + \frac{\nu}{\Delta x} = 0 + \frac{1}{1/5} = 5 \end{cases}$$

Thus, we obtain and solve the matrix system as:

$$\begin{bmatrix} 15 & -5 & 0 & 0 \\ -10 & 15 & -5 & 0 \\ 0 & -10 & 15 & -5 \\ 0 & 0 & -10 & 15 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = 0.2 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad \Rightarrow \quad \begin{cases} 15u_1 - 5u_2 = 0.2 \\ -10u_1 + 15u_2 - 5u_3 = 0.2 \\ -10u_2 + 15u_3 - 5u_4 = 0.2 \\ -10u_3 + 15u_4 = 0.2 \end{cases}$$

Combined with $u_0 = u_5 = 0$, the final numerical solution of FVM with Upwind Difference Scheme is:

$$\boldsymbol{u} = \{0, 0.03354839, 0.06064516, 0.07483871, 0.06322581, 0\}^T$$

We plot the numerical solutions as below and compare with the exact solution:

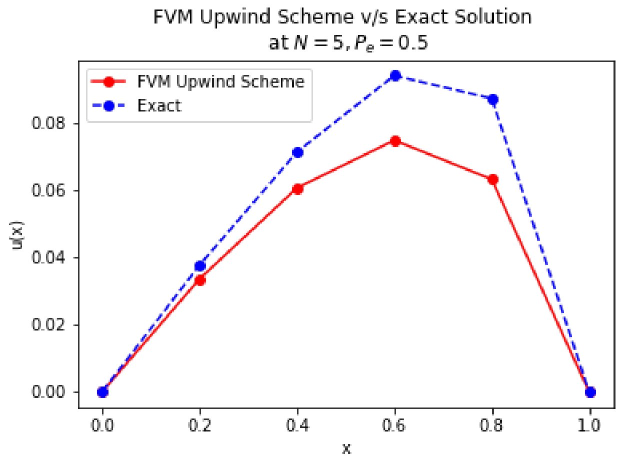


Figure 21: Solution of FVM with Upwind Difference Scheme at Pe = 0.5 N = 5

We can observe that the Upwind Difference Scheme does not perform well in our model example setting, as the equation is not convection-dominated at the Peclet number $P_e = 0.5$.

However, we will try a convection-dominated equation to examine the scheme. Suppose the convective coefficient is $\alpha = 50$ and keep other parameters fixed, the Peclet number becomes $P_e = \frac{\alpha}{2\nu N} = \frac{50}{2\times 5} = 5$.

The solutions of Central Difference scheme and Upwind Difference Scheme are plotted below:

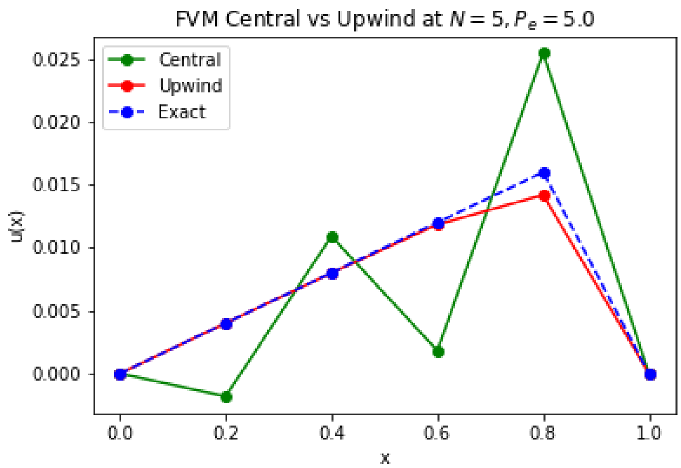


Figure 22: Comparison of FVM Central and Upwind Schemes at Pe = 5.0 N = 5 Plot

Compare the numerical solutions given by two different schemes, we find that the Upwind Difference Scheme performs much better than the Central Difference Scheme at high Peclet number $P_e > 1$. There is also not wiggles around the exact solution for Upwind Difference Scheme, which may implies stability.

Assessment on FVM with Upwind Difference Scheme

Except for satisfying the discretized conservativeness, we will then evaluate the FVM with Upwind Difference Scheme in terms of transportiveness, stability, consistency, and convergence.

1. Transportiveness

The FVM with Upwind Difference Scheme particularly solves the problem of transportiveness, since it considers both the positive and negative convection direction.

2. Stability

The coefficients of the west, the middle, and the east nodes are all secured to be positive in Upwind Difference scheme. Thus, there will not be oscillations appearing in the solution.

We plot the solution of FVM with Upwind Difference Scheme at different Peclet numbers $P_e = \{0.25, 0.5, 1, 5\}$ under the same number of partitions $N = 10$. As we can observed, the solutions are always smooth without wiggles. Therefore, FVM with Upwind Difference Scheme is a stable scheme.

3. Consistency

The solutions of the model example under Central Difference Scheme and Upwind Difference Scheme have clearly displayed that the Upwind scheme is less accurate than Central scheme at low Peclet number.

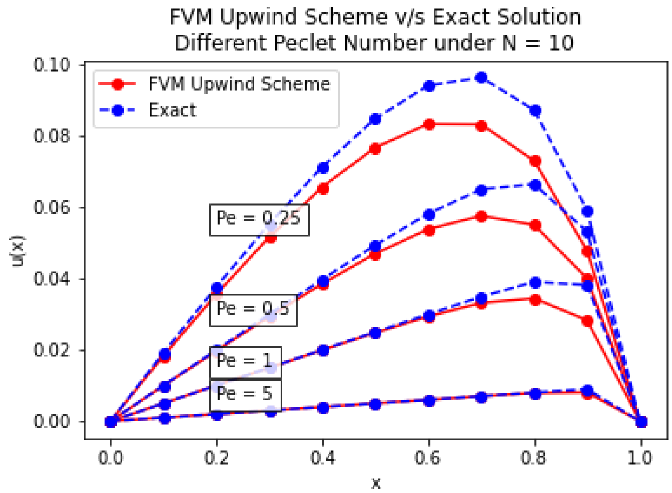


Figure 23: Solutions of FVM with Upwind Difference Scheme at N = 10 under different Peclet numbers

The global errors of Upwind Difference Scheme at Peclet numbers $P_e = \{0.1, 0.5, 1, 5, 10, 50\}$ is plotted as:

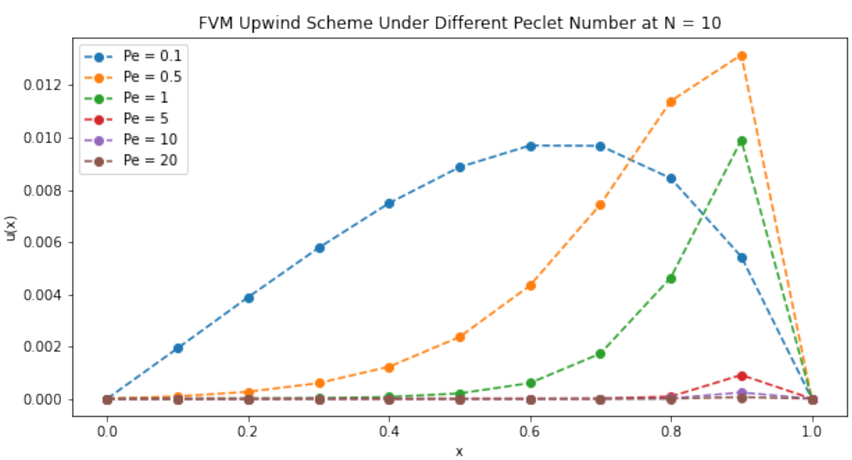


Figure 24: Solution of FVM Upwind Difference Scheme at different Peclet numbers under N = 10

As we can observe, the global errors at Peclet number $P_e > 1$ is much lower than which at $P_e < 1$.

The reason is that the Upwind Difference Scheme essentially implements the backward difference scheme, thus the accuracy is only first-order on the basis of the Taylor series truncation error.

Therefore, FVM with Upwind Difference scheme is first-order consistent. It is not suitable for diffusion-dominated equations ($P_e < 1$), instead only accurate for equations with high convective coefficients ($P_e > 1$).

4. Convergence

As FVM with Upwind Difference scheme is first-order consistent and satisfies stability, it is then convergent.

FVM with Hybrid Difference Scheme

In order to tackle both the transportiveness issue of Central Difference scheme and the accuracy of Upwind Difference scheme at low Peclet number, we will implement a method combining the Central and Upwind Difference scheme to keep the advantages and offset the drawbacks of each other.

Formulation

Since Central Difference Scheme works well for diffusion-dominated problem, and Upwind Difference Scheme is appropriate for convection-dominated problem, the Hybrid Difference Scheme will implements the Central Scheme at $P_e < 1$ and the Upwind Scheme at $P_e > 1$. Specifically, the formulation consists of:

$$\text{at } P_e < 1, \begin{cases} \text{Convection: Central Difference Scheme} \\ \text{Diffusion : Central Difference Scheme} \end{cases}$$

$$\Rightarrow [\alpha\frac{(u_P + u_E)}{2} - \alpha\frac{(u_W + u_P)}{2}] - [\nu\frac{u_E - u_P}{\Delta x} - \nu\frac{u_P - u_W}{\Delta x}] = s \cdot \Delta x \qquad \text{at } P_e < 1$$

$$\text{at } P_e > 1, \begin{cases} \text{Convection: Upwind Difference Scheme} \\ \text{Diffusion : Set to zero} \end{cases}$$

$$\Rightarrow \begin{cases} \alpha u_P - \alpha u_W = 0 \text{ , for the positive convective direction} \\ \alpha u_E - \alpha u_P = 0 \text{ , for the negative convective direction} \end{cases} \qquad \text{at } P_e > 1$$

Thus, the formulation of FVM with Hybrid Difference Scheme is expressed as:

$$-a \cdot u_W + b \cdot u_P - c \cdot u_E = s \cdot \Delta x$$

$$a = \max(\frac{\alpha}{2} + \frac{\nu}{\Delta x}, \alpha, 0) \ \ , \ \ c = \max(-\frac{\alpha}{2} + \frac{\nu}{\Delta x}, 0, -\alpha) \ \ , \ \ b = a + c$$

Model Example

We will then try the model example setting to examine the FVM with Hybrid Difference Scheme:

$$N = 5, \alpha = 5, \nu = 1, s_j = 1 \quad \Rightarrow \quad P_e = \frac{\alpha}{2\nu N} = \frac{5}{2 \times 5} = 0.5$$

The values of a, b, c are calculated as:

$$\begin{cases} a = \max(\frac{\alpha}{2} + \frac{\nu}{\Delta x}, \alpha, 0) = \max(\frac{5}{2} + \frac{1}{1/5}, 5, 0) = 7.5 \\ c = \max(-\frac{5}{2} + \frac{1}{1/5}, 0, -5) = 2.5 \\ b = a + c = 10 \end{cases}$$

Thus, the matrix system becomes:

$$\begin{bmatrix} 10 & -2.5 & 0 & 0 \\ -7.5 & 10 & -2.5 & 0 \\ 0 & -7.5 & 10 & -2.5 \\ 0 & 0 & -7.5 & 10 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = 0.2 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

We will now solve the matrix system to obtain the vector of numerical solutions $\boldsymbol{u}$:

$$\begin{cases} 10u_1 - 2.5u_2 = 0.2 \\ -7.5u_1 + 15u_2 - 2.5u_3 = 0.2 \\ -7.5u_2 + 15u_3 - 2.5u_4 = 0.2 \\ -7.5u_3 + 15u_4 = 0.2 \end{cases}$$

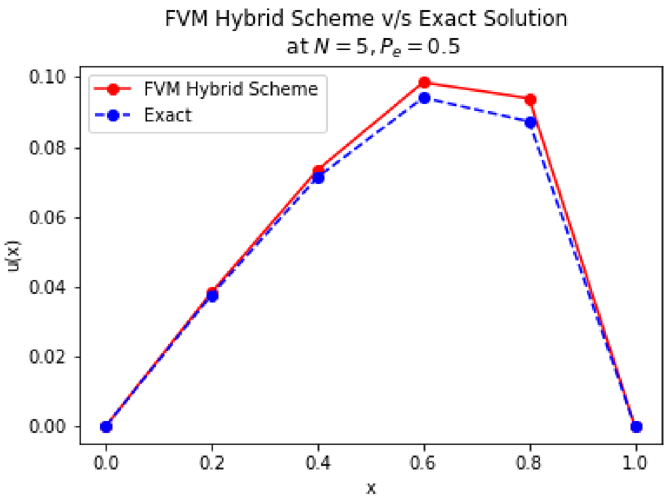


Figure 25: Solution of FVM with Hybrid Difference Scheme at Pe = 0.5 N = 5

Combined with $u_0 = u_5 = 0$, the final numerical solution of FDM with Artificial Difference scheme is:

$$\boldsymbol{u} = \{0, 0.03834711, 0.07338843, 0.0985124, 0.0938843, 0\}^T$$

We plot them into the diagram and compare with the exact solution:

As the model example is diffusion-dominated at $P_e = 0.5$, the solution of FVM with Hybrid Difference Scheme is then identical to the solution of FVM with Central Difference Scheme.

We will then compare the outputs of FVM with Central, Upwind, Hybrid Difference Scheme at Peclet numbers $P_e = \{0.25, 0.5, 1, 5, 10, 50\}$:

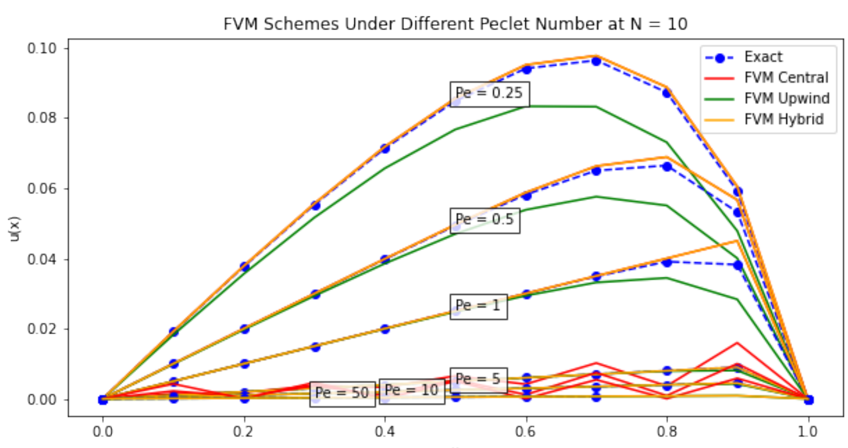


Figure 26: Different FVM Schemes under different Peclet numbers at N = 10

The Hybrid Difference Scheme always overlaps with the better solution between Central Scheme and Upwind Scheme, no matter for high Peclet number or low Peclet number. Thus, the Hybrid Difference Scheme always auto-select the best scheme between FVM Central and FVM Hybrid methods.

Assessment on FVM with Hybrid Difference Scheme

1. Transportiveness

FVM with Hybrid Difference Scheme satisfies the transportiveness for convection-dominated problem, as it is automatically switched to FVM with Upwind Difference Scheme at high Peclet number $P_e > 1$.

2. Stability

The Hybrid Difference Scheme inherits the stability of the Upwind Difference scheme. As we can observe below, there is no oscillations appearing in the solutions of Hybrid Scheme at $P_e = \{0.25, 0.5, 1, 5\}$.

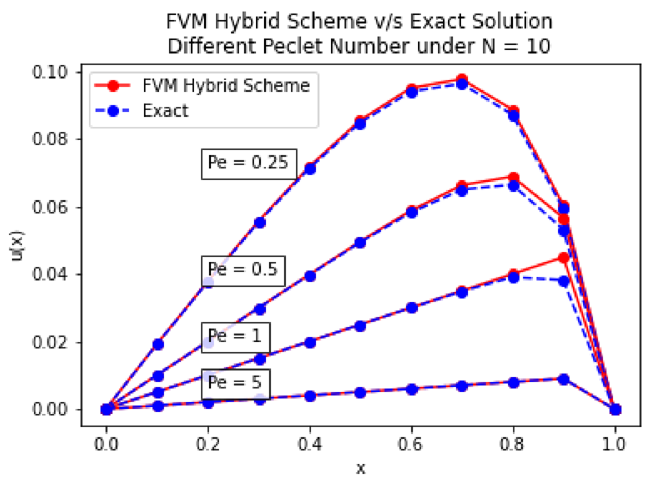


Figure 27: FVM with Hybrid Difference Scheme at N = 10 under different Peclet numbers

3. Consistency

We will firstly compare the global errors of FVM Central, Upwind, and Hybrid schemes at $P_e = \{0.5, 1, 5\}$.

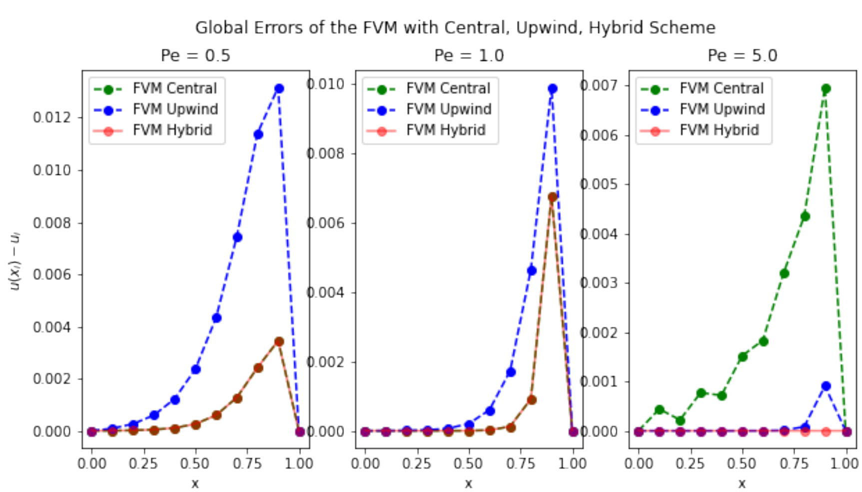


Figure 28: Global Errors of FVM Central, Upwind and Hybrid Difference Scheme

No matter how large the error is, FVM with Hybrid Difference Scheme always has the least global errors. Specifically, we observe the global errors of the Hybrid scheme for different Peclet numbers from below:

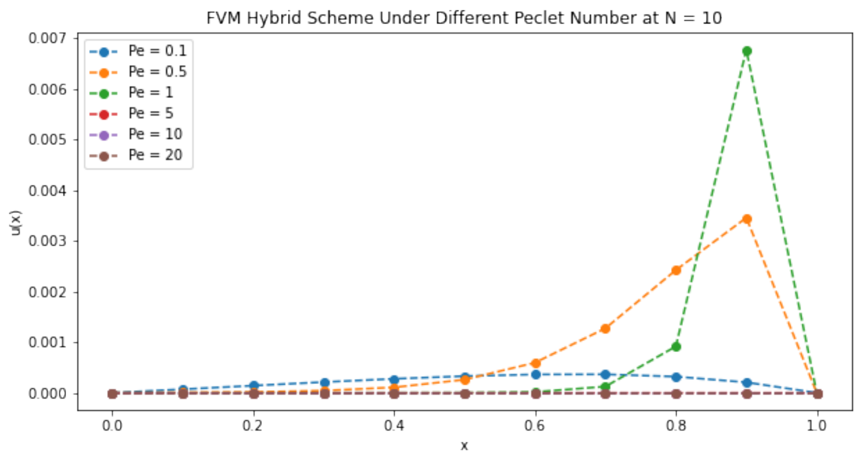


Figure 29: FVM with Hybrid Difference Scheme under different Peclet numbers at N = 10

We find that except for the particular high global error at the critical value of the Peclet number $P_e = 1$, other global errors are kept at the low level less than 5×10^{-3}.

However, the Taylor series truncation error of FVM with Hybrid Difference Scheme is only first-order.

4. Convergence

Since FVM with Hybrid Difference Scheme is first-order consistent and stable, this scheme is convergent.

Comparison of FDM, FEM, FVM

After evaluating three numerical methods with different schemes, we will then compare the best scheme of each numerical method to solve our 1D steady convection-diffusion problem: FDM with Artificial Difference Scheme, FEM with Petrov-Galerkin Approximation Scheme, and FVM with Hybrid Difference Scheme.

We will evaluate their performance in solving convection-diffusion problems with different Peclet numbers in terms of accuracy and stability.

First, the global errors of the three methods are compared as follows:

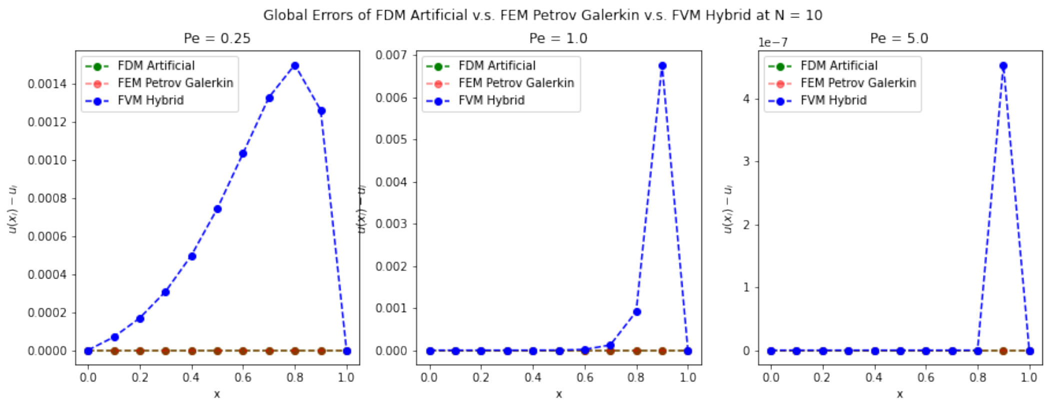


Figure 30: Global Errors of FDM Artificial v.s. FEM Petrov Galerkin v.s. FVM Hybrid at N = 10

As we can observe, FVM with Hybrid Difference Scheme is less accurate, compared to another two methods. The accuracy comparison between FDM with Artificial Difference Scheme and FEM with Petrov-Galerkin Approximation Scheme is not obvious and cannot be distinguished.

Therefore, we will specifically compare the global errors of FDM and FEM for different number of partitions N at the same critical value of Peclet number $P_e = 1$, as plotted below:

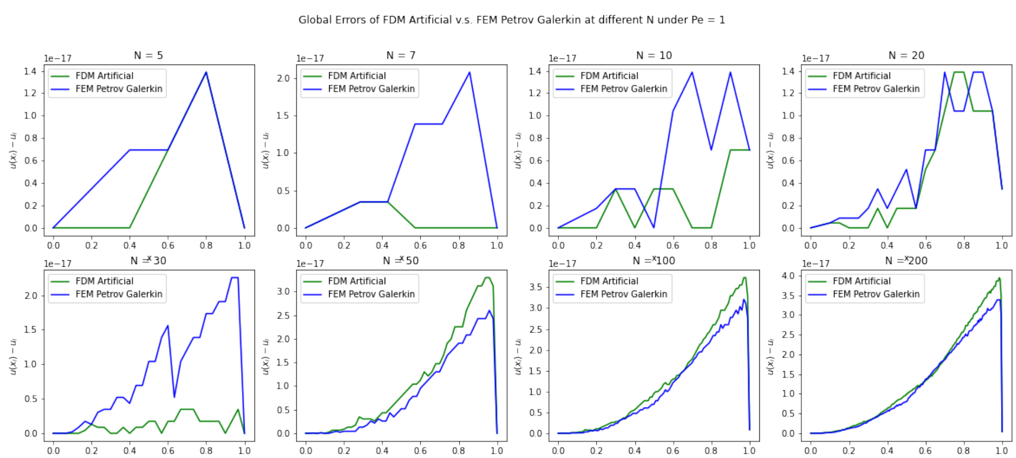


Figure 31: Global Errors of FDM Artificial v.s. FEM Petrov Galerkin at different N at Pe = 1

From the global error plot, it is not obvious to determine the best method, at least in this case.

However, as we analyzed previously, FDM with Artificial Difference Scheme introduces additional numerical diffusion, which may have an impact on accuracy in other cases.

Furthermore, FDM is originally only suitable for simple geometries and regular meshes. Therefore, in real physical scenarios, FDM may not be a good choice for problems that are difficult to discretize with structured meshes. However, FEM with the Petrov-Galerkin approximation scheme is highly versatile and can handle irregular geometries and complex domains.

In addition, for the high-dimensional case, FEM initially offers higher accuracy than FDM, so FEM will be more suitable for accurate simulations in engineering and physics.

Extension to 2D Problems

After the implementation and discussion of numerical methods used for one-dimensional convection-diffusion problems, we will then briefly implement the same methodology to extend the convection-diffusion problem to two-dimension about (x, y).

The model equation of the two-dimensional convection-diffusion problem is written in form of:

$$\begin{aligned}&\alpha\frac{\partial u}{\partial x} + \alpha\frac{\partial u}{\partial y} - \nu\frac{\partial^2 u}{\partial x^2} - \nu\frac{\partial^2 u}{\partial y^2} = s \qquad (x, y) \in \Omega = [0,1]\times[0,1]\\&u|_{\partial\Omega} = 0\end{aligned}$$

Partition of the domain

We partition the domain of interest $[0,1]\times[0,1]$ to obtain the basis of the numerical solution.

Suppose we have a square grid consisting of $N+1$ grid points on x axis and $M+1$ grid points on y axis. They are exactly $(N+1)\times(M+1)$ nodes supporting $N\times M$ elements, each element is a small block with the same area $\frac{1}{\Delta x}\times\frac{1}{\Delta y} = \frac{1}{N}\times\frac{1}{M}$.

Thus, we will be solving the numerical solution $\tilde{u}(x, y) = \sum_{i=1}^{N}\sum_{j=1}^{M} u_{i,j}$, $x \in \{x_0, ..., x_N\}, y \in \{y_0, .., y_M\}$.

2D FDM with Central Difference Scheme

2D Formulation of FDM with Central Difference Scheme

The Taylor Expansion of Central Difference Scheme gives that:

$$\begin{cases}u_x = \frac{u_{i+1,j}-u_{i-1,j}}{2\Delta x}\\ u_y = \frac{u_{i,j+1}-u_{i,j-1}}{2\Delta y}\end{cases} \qquad \begin{cases}u_{xx} = \frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{\Delta x^2}\\ u_{yy} = \frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{\Delta y^2}\end{cases}$$

Substitute into our model equation, we obtain the formulation:

$$\alpha u_x + \alpha u_y - \nu u_{xx} - \nu u_{yy} = s$$

$$\Rightarrow \alpha\frac{u_{i+1,j}-u_{i-1,j}}{2\Delta x} + \alpha\frac{u_{i,j+1}-u_{i,j-1}}{2\Delta y} - \nu\frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{\Delta x^2} - \nu\frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{\Delta y^2} = s_j$$

where $i = 1, .., N-1, j = 1, .., M-1$

$$\Rightarrow \alpha\frac{u_{i+1,j}-u_{i-1,j}}{2\Delta x} + \alpha\frac{u_{i,j+1}-u_{i,j-1}}{2\Delta y} - \nu\frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{\Delta x^2} - \nu\frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{\Delta y^2} = s_j$$

$$u_{i-1,j}\times(-\frac{\alpha}{2\Delta x}-\frac{\nu}{\Delta x^2})+u_{i+1,j}\times(\frac{\alpha}{2\Delta x}-\frac{\nu}{\Delta x^2})+u_{i,j}\times(\frac{2\nu}{\Delta x^2})+u_{i,j-1}\times(-\frac{\alpha}{2\Delta y}-\frac{\nu}{\Delta y^2})+u_{i,j+1}\times(\frac{\alpha}{2\Delta y}-\frac{\nu}{\Delta y^2}) = s_j$$

Matrix System

We will transform the formulation of FDM with Central Difference Scheme into the matrix system:

$$\boldsymbol{Au} = \boldsymbol{s}$$

$$\Rightarrow \begin{bmatrix} \boldsymbol{A_{diag}} & \boldsymbol{A_{right}} & \mathbf{0} & \mathbf{0} \\ \boldsymbol{A_{left}} & \boldsymbol{A_{diag}} & \boldsymbol{A_{right}} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{A_{left}} & \boldsymbol{A_{diag}} & \boldsymbol{A_{right}} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{A_{left}} & \boldsymbol{A_{diag}} \end{bmatrix} \begin{bmatrix} u_{1,1} & u_{1,2} & u_{1,3} & u_{1,4} \\ u_{2,1} & u_{2,2} & u_{2,3} & u_{2,4} \\ u_{3,1} & u_{3,2} & u_{3,3} & u_{3,4} \\ u_{4,1} & u_{4,2} & u_{4,3} & u_{4,4} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

where

$$\boldsymbol{A_{diag}} = \begin{bmatrix} b_y & c_y & 0 & 0 \\ a_y & b_y & c_y & 0 \\ 0 & a_y & b_y & c_y \\ 0 & 0 & a_y & b_y \end{bmatrix} \quad , \quad \boldsymbol{A_{left}} = \begin{bmatrix} a_x & 0 & 0 & 0 \\ 0 & a_x & 0 & 0 \\ 0 & 0 & a_x & 0 \\ 0 & 0 & 0 & a_x \end{bmatrix} \quad , \quad \boldsymbol{A_{right}} = \begin{bmatrix} c_x & 0 & 0 & 0 \\ 0 & c_x & 0 & 0 \\ 0 & 0 & c_x & 0 \\ 0 & 0 & 0 & c_x \end{bmatrix}$$

with:

$$\begin{aligned} a_x &= (-\frac{\alpha}{2\Delta x} - \frac{\nu}{\Delta x^2}) \quad , \quad c_x = (\frac{\alpha}{2\Delta x} - \frac{\nu}{\Delta x^2}) \\ b_x &= b_y = (\frac{2\nu}{\Delta x^2}) + (\frac{2\nu}{\Delta x^2}) = \frac{4\nu}{\Delta x^2} \\ a_y &= (-\frac{\alpha}{2\Delta y} - \frac{\nu}{\Delta y^2}) \quad , \quad c_y = (\frac{\alpha}{2\Delta y} - \frac{\nu}{\Delta y^2}) \end{aligned}$$

Model Example

We will now examine the method using our model example setting:

$$N = M = 5, \alpha = 5, \nu = 1, s = 1 \quad \Rightarrow \quad P_e = \frac{\alpha \Delta x}{2\nu} = 0.5$$

Thus, the values of $a_x, b_x, c_x, a_y, b_y, c_y$ are respectively:

$$\begin{aligned} a_x &= (-\frac{\alpha}{2\Delta x} - \frac{\nu}{\Delta x^2}) = (-\frac{5}{2/5} - \frac{1}{(1/5)^2}) = -37.5 \\ c_x &= (\frac{\alpha}{2\Delta x} - \frac{\nu}{\Delta x^2}) = (\frac{5}{2/5} - \frac{1}{(1/5)^2}) = -12.5 \\ b_x &= b_y = \frac{4\nu}{\Delta x^2} = \frac{4}{(1/5)^2} = 100 \\ a_y &= (-\frac{\alpha}{2\Delta y} - \frac{\nu}{\Delta y^2}) = (-\frac{5}{2/5} - \frac{1}{(1/5)^2}) = -37.5 \\ c_y &= (\frac{\alpha}{2\Delta y} - \frac{\nu}{\Delta y^2}) = (\frac{5}{2/5} - \frac{1}{(1/5)^2}) = -12.5 \end{aligned}$$

The matrix system becomes:

$$\begin{bmatrix} \boldsymbol{A_{diag}} & \boldsymbol{A_{right}} & \mathbf{0} & \mathbf{0} \\ \boldsymbol{A_{left}} & \boldsymbol{A_{diag}} & \boldsymbol{A_{right}} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{A_{left}} & \boldsymbol{A_{diag}} & \boldsymbol{A_{right}} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{A_{left}} & \boldsymbol{A_{diag}} \end{bmatrix} \begin{bmatrix} u_{1,1} & u_{1,2} & u_{1,3} & u_{1,4} \\ u_{2,1} & u_{2,2} & u_{2,3} & u_{2,4} \\ u_{3,1} & u_{3,2} & u_{3,3} & u_{3,4} \\ u_{4,1} & u_{4,2} & u_{4,3} & u_{4,4} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

where

$$\boldsymbol{A_{diag}} = \begin{bmatrix} 100 & -12.5 & 0 & 0 \\ -37.5 & 100 & -12.5 & 0 \\ 0 & -37.5 & 100 & -12.5 \\ 0 & 0 & -37.5 & 100 \end{bmatrix}$$

$$\boldsymbol{A_{right}} = \begin{bmatrix} -12.5 & 0 & 0 & 0 \\ 0 & -12.5 & 0 & 0 \\ 0 & 0 & -12.5 & 0 \\ 0 & 0 & 0 & -12.5 \end{bmatrix} \quad , \quad \boldsymbol{A_{left}} = \begin{bmatrix} -37.5 & 0 & 0 & 0 \\ 0 & -37.5 & 0 & 0 \\ 0 & 0 & -37.5 & 0 \\ 0 & 0 & 0 & -37.5 \end{bmatrix}$$

Combined with the homogeneous Dirichlet boundary conditions that $\boldsymbol{u}(0,y) = \boldsymbol{u}(1,y) = \boldsymbol{u}(x,0) = \boldsymbol{u}(x,1) = \boldsymbol{0}$, the numerical solutions are solved as:

$$\begin{bmatrix} 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0.01619948 & 0.02479792 & 0.02880335 & 0.02650047 & 0. \\ 0. & 0.02479792 & 0.04098159 & 0.04953258 & 0.04559373 & 0. \\ 0. & 0.02880335 & 0.04953258 & 0.06131211 & 0.05665069 & 0. \\ 0. & 0.02650047 & 0.04559373 & 0.05665069 & 0.05248802 & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. \end{bmatrix}$$

We will then plot the numerical solution as below:

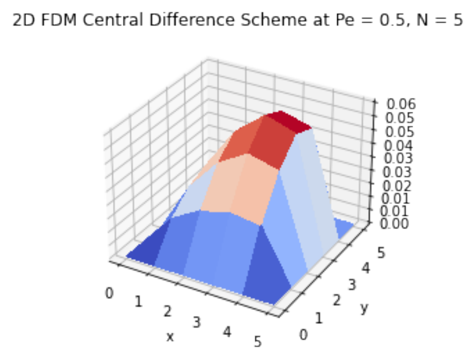


Figure 32: 2D FDM Central Difference Scheme at Pe = 0.5, N = 5

As we can observe, the numerical solution of FDM with Central Difference Scheme is well-performed. However, as we know the Central Difference Scheme is unstable at $P_e > 1$ in 1D problem, we solve the numerical solutions at $P_e = \{1, 5, 10\}$ and plot as below:

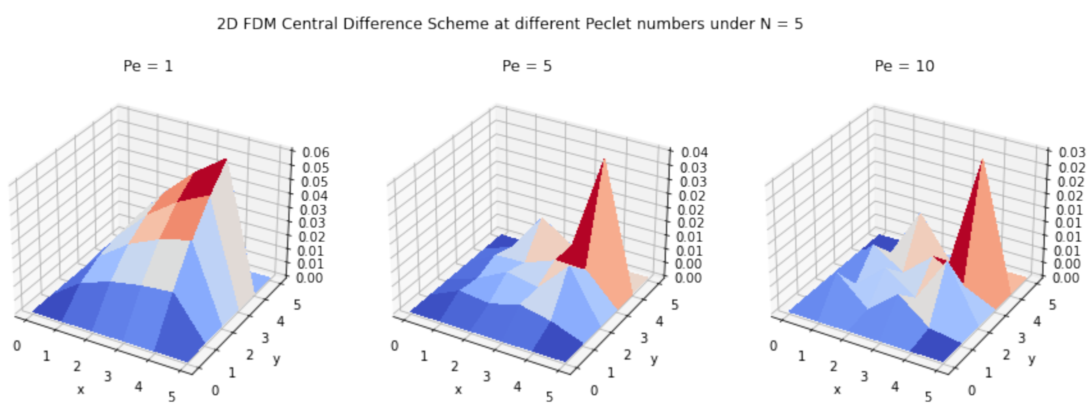


Figure 33: 2D FDM Central Difference Scheme at different Peclet numbers

The solution of FDM with Central Difference Scheme appears oscillations at high Peclet numbers, which is the same as our conclusion in one-dimensional problem.

2D FDM with Artificial Difference Scheme

We will then apply the improvements on FDM with Central Difference Scheme: FDM with Artificial Difference Scheme and Upwind Difference Scheme.

Firstly, for the Artificial Difference Scheme, we will add an artificial diffusion term $\hat{\nu}$ to the diffusivity.

2D Formulation of FDM with Artificial Difference Scheme

The formulation of FDM with Artificial Difference Scheme is in form of:

$$\alpha\frac{u_{i+1,j}-u_{i-1,j}}{2\Delta x}+\alpha\frac{u_{i,j+1}-u_{i,j-1}}{2\Delta y}-(\nu+\hat{\nu})\frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{\Delta x^2}-(\nu+\hat{\nu})\frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{\Delta y^2}=s_j$$

where $\hat{\nu}=\beta\nu P_e, \beta=\coth(P_e)-\frac{1}{P_e}$.

$$\Rightarrow a_x\cdot u_{i-1,j}+b_x\cdot u_{i,j}+c_x\cdot u_{i+1,j}+a_y\cdot u_{i,j-1}++c_y\cdot u_{i,j+1}=s_{ij} \quad ,i=1,..,N-1,j=1,...,M-1$$

$$\begin{aligned}
a_x &= (-\frac{\alpha}{2\Delta x}-\frac{(\nu+\hat{\nu})}{\Delta x^2}) \quad , \quad c_x=(\frac{\alpha}{2\Delta x}-\frac{(\nu+\hat{\nu})}{\Delta x^2}) \\
b_x &= b_y=\frac{4(\nu+\hat{\nu})}{\Delta x^2} \\
a_y &= (-\frac{\alpha}{2\Delta y}-\frac{(\nu+\hat{\nu})}{\Delta y^2}) \quad , \quad c_y=(\frac{\alpha}{2\Delta y}-\frac{(\nu+\hat{\nu})}{\Delta y^2})
\end{aligned}$$

Matrix System

We will then convert the formulation of FDM with Artificial Difference Scheme into the matrix system:

$$\boldsymbol{Au}=\boldsymbol{s}$$

$$\begin{bmatrix} \boldsymbol{A_{diag}} & \boldsymbol{A_{right}} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{A_{left}} & \boldsymbol{A_{diag}} & \boldsymbol{A_{right}} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A_{left}} & \boldsymbol{A_{diag}} & \boldsymbol{A_{right}} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{A_{left}} & \boldsymbol{A_{diag}} \end{bmatrix} \begin{bmatrix} u_{1,1} & u_{1,2} & u_{1,3} & u_{1,4} \\ u_{2,1} & u_{2,2} & u_{2,3} & u_{2,4} \\ u_{3,1} & u_{3,2} & u_{3,3} & u_{3,4} \\ u_{4,1} & u_{4,2} & u_{4,3} & u_{4,4} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Model Example

Apply the matrix system at the model example setting $N=M=5,\alpha=5,\nu=1,s=1\Rightarrow P_e=\frac{\alpha\Delta x}{2\nu}=0.5,\hat{\nu}=\coth(P_e)P_e\nu-\nu=\coth(0.5)\cdot 0.5-1$, the values of a_x,b_x,c_x,a_y,c_y become:

$$\begin{aligned}
a_x &= (-\frac{\alpha}{2\Delta x}-\frac{(\nu+\hat{\nu})}{\Delta x^2})=(-\frac{5}{2/5}-\frac{(\coth(0.5)\cdot 0.5)}{(1/5)^2})\approx -39.54941767 \\
c_x &= (\frac{\alpha}{2\Delta x}-\frac{(\nu+\hat{\nu})}{\Delta x^2})=(\frac{5}{2/5}-\frac{(\coth(0.5)\cdot 0.5)}{(1/5)^2})\approx -14.54941767 \\
b_x &= b_y=\frac{4(\nu+\hat{\nu})}{\Delta x^2}=\frac{4(\coth(0.5)\cdot 0.5)}{(1/5)^2}\approx 108.1976707 \\
a_y &= (-\frac{\alpha}{2\Delta y}-\frac{(1+\hat{\nu})}{\Delta y^2})=(-\frac{5}{2/5}-\frac{(\coth(0.5)\cdot 0.5)}{(1/5)^2})\approx -39.54941767 \\
c_y &= (\frac{\alpha}{2\Delta y}-\frac{(1+\hat{\nu})}{\Delta y^2})=(\frac{5}{2/5}-\frac{(\coth(0.5)\cdot 0.5)}{(1/5)^2})\approx -14.54941767
\end{aligned}$$

Thus, $\boldsymbol{A_{diag}}, \boldsymbol{A_{left}}, \boldsymbol{A_{right}}$ become:

$$\boldsymbol{A_{diag}}=\begin{bmatrix} 108.1976707 & -14.54941767 & 0 & 0 \\ -39.54941767 & 108.1976707 & -14.54941767 & 0 \\ 0 & -39.54941767 & 108.1976707 & -14.54941767 \\ 0 & 0 & -39.54941767 & 108.1976707 \end{bmatrix}$$

$$
\boldsymbol{A_{right}} = \begin{bmatrix} -14.54941767 & 0 & 0 & 0 \\ 0 & -14.54941767 & 0 & 0 \\ 0 & 0 & -14.54941767 & 0 \\ 0 & 0 & 0 & -14.54941767 \end{bmatrix}
$$

$$
\boldsymbol{A_{left}} = \begin{bmatrix} -39.54941767 & 0 & 0 & 0 \\ 0 & -39.54941767 & 0 & 0 \\ 0 & 0 & -39.54941767 & 0 \\ 0 & 0 & 0 & -39.54941767 \end{bmatrix}
$$

Combined with the boundary conditions, the numerical solution is therefore calculated as:

$$
\boldsymbol{u} = \begin{bmatrix} 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0.01570866 & 0.0240436 & 0.02779316 & 0.02516255 & 0. \\ 0. & 0.0240436 & 0.03957673 & 0.04743453 & 0.04284197 & 0. \\ 0. & 0.02779316 & 0.04743453 & 0.0580463 & 0.05252645 & 0. \\ 0. & 0.02516255 & 0.04284197 & 0.05252645 & 0.04764225 & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. \end{bmatrix}
$$

We plot the numerical solution of FDM with Artificial Scheme for our 2D problem in the below diagram:

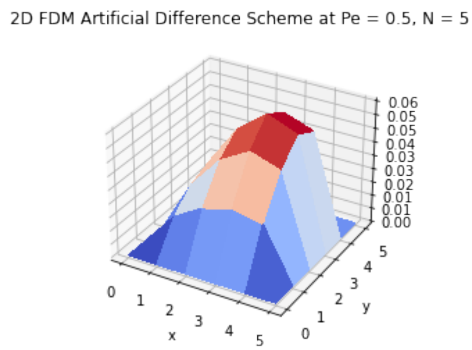


Figure 34: 2D FDM Artificial Difference Scheme at Pe = 0.5, N = 5

Compare numerical solutions at high Peclet number to examine the stability for convection–dominated flow:

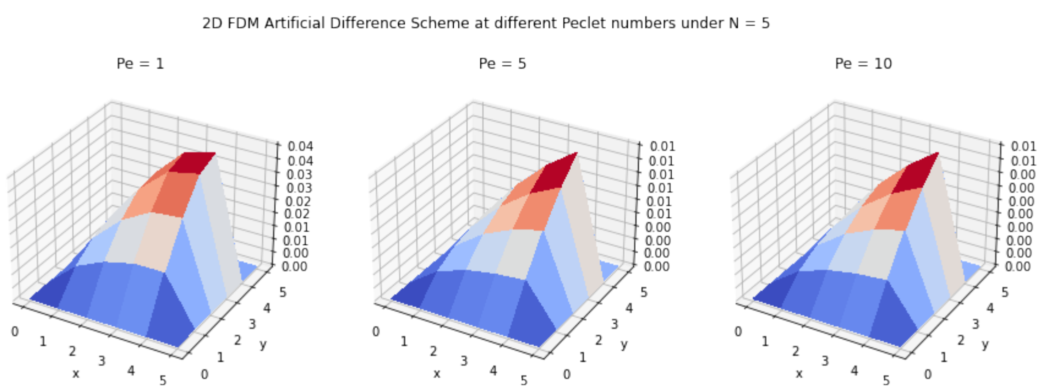


Figure 35: 2D FDM Artificial Difference Scheme at different Peclet numbers

As we observed, there is no oscillations in the numerical solution of FDM with Upwind Difference Scheme even for high Peclet numbers $P_e > 1$.

2D FDM with Upwind Difference Scheme

Then, we implement another improvement that altering the convective term of the formulation.

2D Formulation of FDM with Upwind Difference Scheme

The formulation of FDM with Upwind Difference Scheme is in form of:

$$\alpha(\frac{1-\beta}{2})\frac{u_{i+1,j}-u_{i,j}}{\Delta x}+\alpha(\frac{1+\beta}{2})\frac{u_{i,j}-u_{i-1,j}}{\Delta x}+\alpha(\frac{1-\beta}{2})\frac{u_{i,j+1}-u_{i,j}}{\Delta y}+\alpha(\frac{1+\beta}{2})\frac{u_{i,j}-u_{i,j-1}}{\Delta y}$$
$$-\nu\frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{\Delta x^2}-\nu\frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{\Delta y^2}=s_{ij} \qquad i=1,...,N-1, j=1,...,M-1$$

where $\beta=\coth(P_e)-\frac{1}{P_e}$.

$$\Rightarrow a_x\cdot u_{i-1,j}+b_x\cdot u_{i,j}+c_x\cdot u_{i+1,j}+a_y\cdot u_{i,j-1}+c_y\cdot u_{i,j+1}=s_{ij} \qquad ,i=1,..,N-1, j=1,...,M-1$$

$$\begin{aligned}
&a_x=(-\frac{1+\beta}{2}\frac{\alpha}{\Delta x}-\frac{\nu}{\Delta x^2}) \quad , \quad c_x=(\frac{1+\beta}{2}\frac{\alpha}{\Delta x}-\frac{\nu}{\Delta x^2})\\
&b_x=b_y=\frac{4\nu}{\Delta x^2}\\
&a_y=(-\frac{1+\beta}{2}\frac{\alpha}{\Delta y}-\frac{\nu}{\Delta x^2}) \quad , \quad c_y=(\frac{1+\beta}{2}\frac{\alpha}{\Delta y}-\frac{\nu}{\Delta y^2})
\end{aligned}$$

Matrix System

We will then convert the formulation of FDM with Artificial Difference Scheme into the matrix system:

$$\boldsymbol{Au}=\boldsymbol{s}$$

$$\begin{bmatrix}\boldsymbol{A_{diag}} & \boldsymbol{A_{right}} & \boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{A_{left}} & \boldsymbol{A_{diag}} & \boldsymbol{A_{right}} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{A_{left}} & \boldsymbol{A_{diag}} & \boldsymbol{A_{right}}\\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{A_{left}} & \boldsymbol{A_{diag}}\end{bmatrix}\begin{bmatrix}u_{1,1} & u_{1,2} & u_{1,3} & u_{1,4}\\ u_{2,1} & u_{2,2} & u_{2,3} & u_{2,4}\\ u_{3,1} & u_{3,2} & u_{3,3} & u_{3,4}\\ u_{4,1} & u_{4,2} & u_{4,3} & u_{4,4}\end{bmatrix}=\begin{bmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{bmatrix}$$

Model Example

Apply the matrix system at the model example setting $N=M=5,\alpha=5,\nu=1,s=1\Rightarrow P_e=\frac{\alpha\Delta x}{2\nu}=0.5,\beta=\coth(P_e)-\frac{1}{P_e}\approx 0.1639534137$, the values of a_x,b_x,c_x,a_y,c_y become:

$$\begin{cases}
a_x=(-\frac{1+0.163953414}{2}\frac{5}{1/5}-\frac{1}{(1/5)^2})=-39.54941767\\
c_x=(\frac{1+0.163953414}{2}\frac{5}{1/5}-\frac{1}{(1/5)^2})=-10.45058233\\
b_x=b_y=\frac{4}{(1/5)^2}=100\\
a_y=(-\frac{1+0.163953414}{2}\frac{5}{1/5}-\frac{1}{(1/5)^2})=-39.54941767\\
c_y=(\frac{1+0.163953414}{2}\frac{5}{1/5}-\frac{1}{(1/5)^2})=-10.45058233
\end{cases}$$

Thus, $\boldsymbol{A_{diag}},\boldsymbol{A_{left}},\boldsymbol{A_{right}}$ become:

$$\boldsymbol{A_{diag}}=\begin{bmatrix}100 & -10.45058233 & 0 & 0\\ -39.54941767 & 108.1976707 & -10.45058233 & 0\\ 0 & -39.54941767 & 108.1976707 & -10.45058233\\ 0 & 0 & -39.54941767 & 108.1976707\end{bmatrix}$$

$$\boldsymbol{A_{right}}=\begin{bmatrix}-10.45058233 & 0 & 0 & 0\\ 0 & -10.45058233 & 0 & 0\\ 0 & 0 & -10.45058233 & 0\\ 0 & 0 & 0 & -10.45058233\end{bmatrix}$$

$$
\boldsymbol{A_{left}} = \begin{bmatrix} -39.54941767 & 0 & 0 & 0 \\ 0 & -39.54941767 & 0 & 0 \\ 0 & 0 & -39.54941767 & 0 \\ 0 & 0 & 0 & -39.54941767 \end{bmatrix}
$$

Combined with the boundary conditions, the numerical solution is therefore calculated as:

$$
\boldsymbol{u} = \begin{bmatrix} 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0.0146943 & 0.02245952 & 0.02627781 & 0.02497305 & 0. \\ 0. & 0.02245952 & 0.03733596 & 0.04579046 & 0.04382847 & 0. \\ 0. & 0.02627781 & 0.04579046 & 0.05790355 & 0.05590037 & 0. \\ 0. & 0.02497305 & 0.04382847 & 0.05590037 & 0.05421654 & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. \end{bmatrix}
$$

We plot the numerical solutions as below:

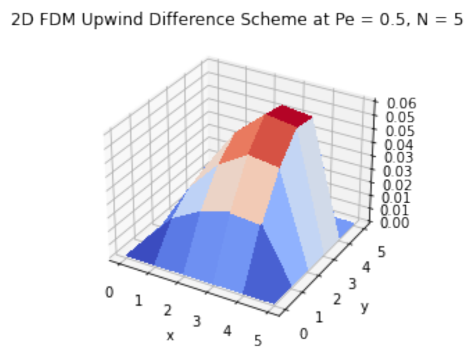


Figure 36: 2D FDM Upwind Difference Scheme at Pe = 0.5, N = 5

Then, we evaluate the stability of FDM with Upwind Difference Schemes by plotting the numerical solutions at different Peclet numbers $P_e \in \{0.25, 1, 5, 10\}$.

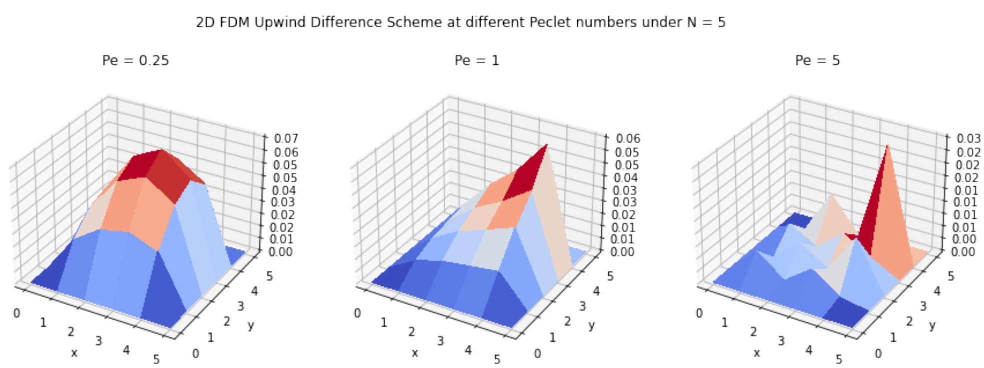
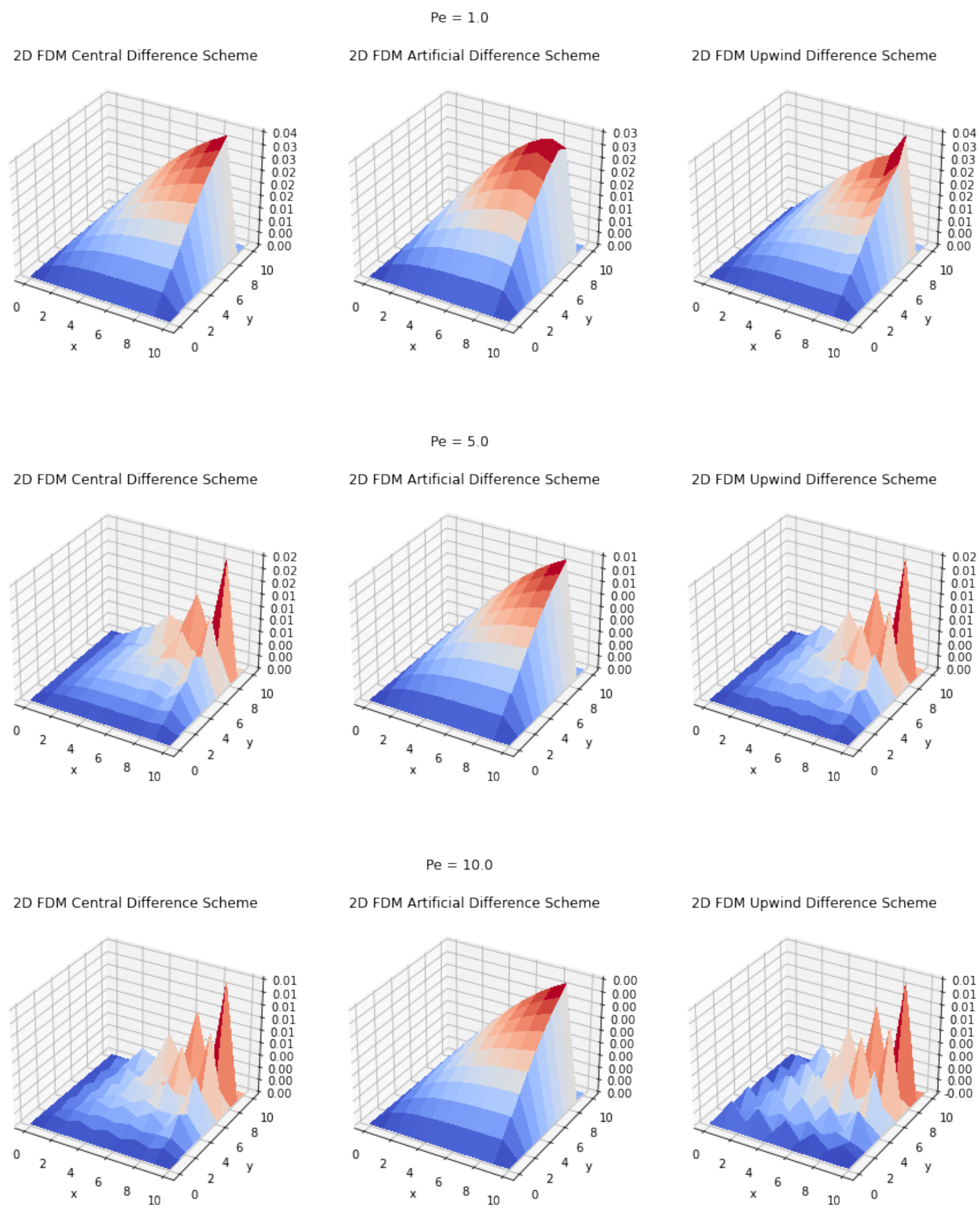


Figure 37: 2D FDM Upwind Difference Scheme at different Peclet numbers

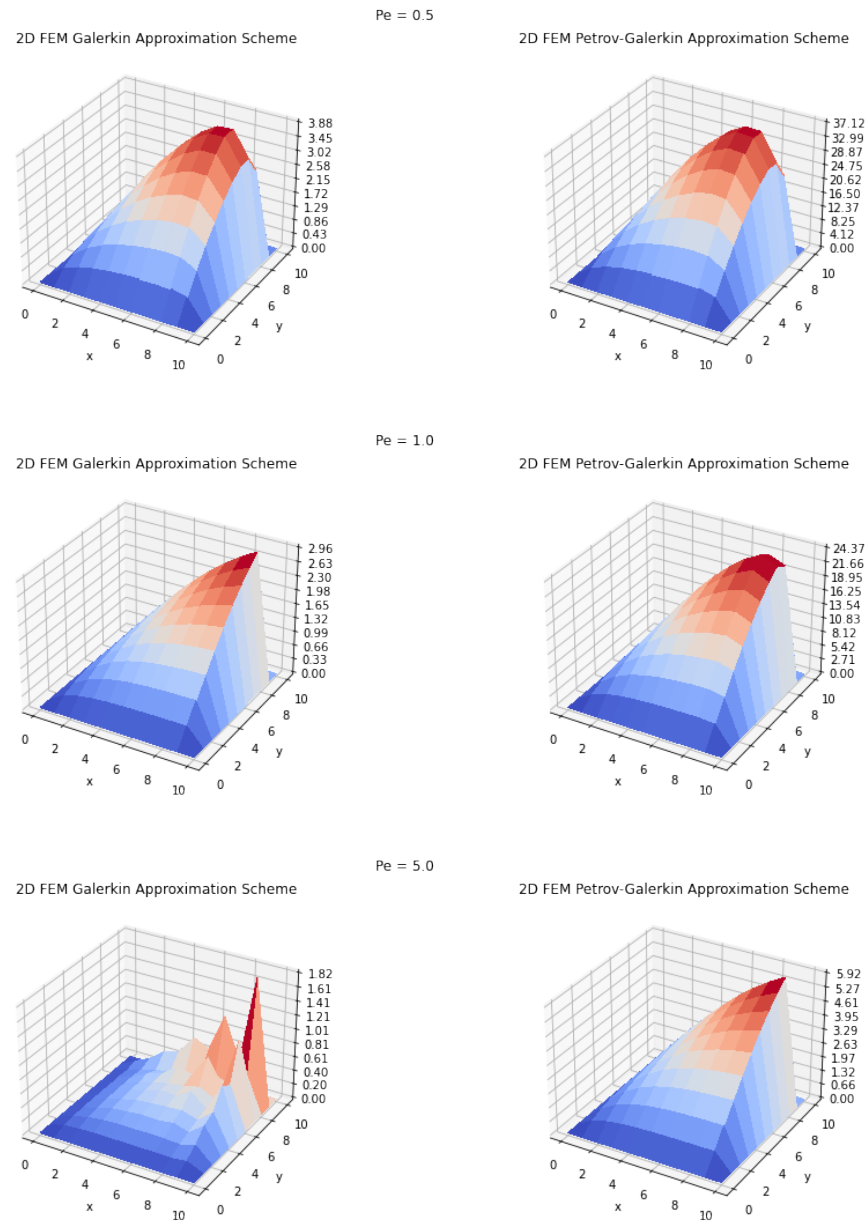
Finally, we compare the outputs of FDM with different schemes at different Peclet numbers $P_e = \{1, 5, 10\}$ as below diagram shows. The FDM with Artificial Difference Scheme performs particularly well, it satisfies both stability and consistency in all three cases.



2D FEM with Galerkin and Petrov-Galekrin Approximation Scheme

The FEM with Galerkin Approximation Scheme and FEM with Petrov-Galekrin Approximation Scheme in 2D problems are the same as which in 1D problems.

The outputs of our model 2D steady convection-diffusion problem at different Peclet numbers $P_e = \{0.5, 1, 5, 10\}$ are plotted as below:



Therefore, FEM with Petrov-Galekrin Approximation Scheme is stable and consistent, the same as our conclusion in 1D FEM part.

Selection of Numerical Methods

FDM is often suitable for problems with regular grid structures and simple geometries, such as problems defined on rectangular or Cartesian grids. It is particularly effective for problems with smooth solutions and isotropic properties. FDM is relatively straightforward to implement and computationally efficient for structured grids. FDM is a good choice when accuracy requirements are moderate and the solution behavior is well-understood.

FEM is well-suited for problems with complex geometries, irregular domains, and problems that involve heterogeneous or anisotropic materials. It offers flexibility in handling arbitrary geometries by using unstructured or adaptive meshes. FEM is particularly effective for problems with localized phenomena, such as stress concentrations or boundary layers. It provides accurate solutions and allows for the inclusion of various boundary conditions and material properties. FEM is widely used in structural mechanics, heat transfer, and fluid dynamics.

FVM is often chosen for problems with conservation laws, such as fluid flow, heat transfer, or mass transport problems. It is suitable for problems involving control volumes or cells, where the solution is discretized at the center of each cell. FVM naturally ensures the conservation of quantities across the cell interfaces and is robust for capturing shocks, discontinuities, or strong gradients. FVM is commonly used in computational fluid dynamics (CFD) and has applications in various engineering disciplines. It is particularly suitable for problems with complex flow behaviors, such as turbulent flows or multiphase flows.

It's important to note that the choice between FDM, FEM, and FVM is not always mutually exclusive, and there can be overlaps depending on the specific problem and the available computational resources. Researchers often select the method that best matches the problem requirements in terms of accuracy, efficiency, and the ability to handle the desired geometries and physics.

Other Numerical Methods

There are many other numerical methods available to solve the convection-diffusion problems, such as the Method of Lines, the Spectral Methods, and the Boundary Element Method (BEM).

Worth mentioning, with the propagation of the modern technology, there are also some numerical methods arised. Particularly, Physics-Informed Neural Networks (PINN) is very popular to solve the complex partial differential equations (PDEs), which is a data-driven approach based on the neural networks by applying the physical principles as the constraints, to model the behavior of physical systems without explicitly relying on traditional numerical discretization methods.

The underlying frustructure of PINN is sketched as below. The network consists of an input, middle layers (called hidden layers), and an output. There are many nodes with weights and biases on each hidden layer.

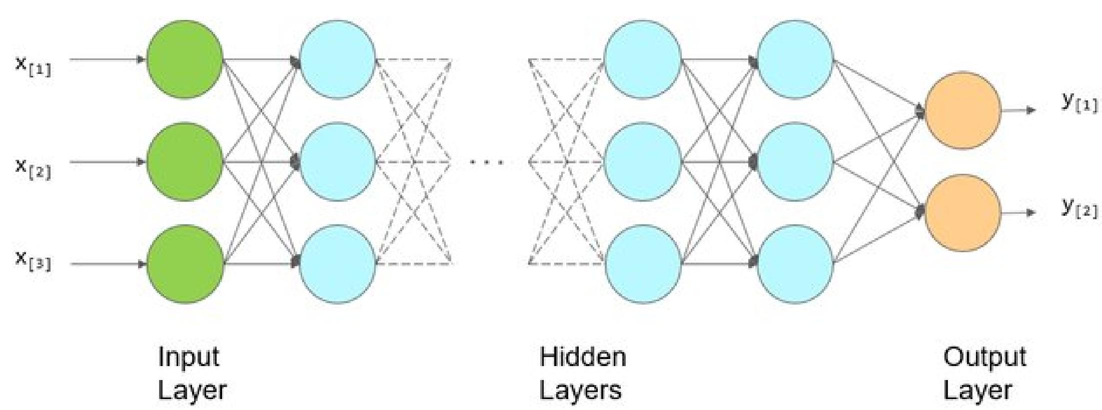


Figure 38: Guo, Y. (2020, August). [3] *PINN Basic Framework*

We use the residuals from the Galerkin Variational Formulation as the loss function. The overall central idea is simply to minimise the loss and use this to refine the parameters of the network (weights and biases) to obtain a least square error. In the end, the linear combination of the trained parameters and inputs is the final solution to the equation we want.

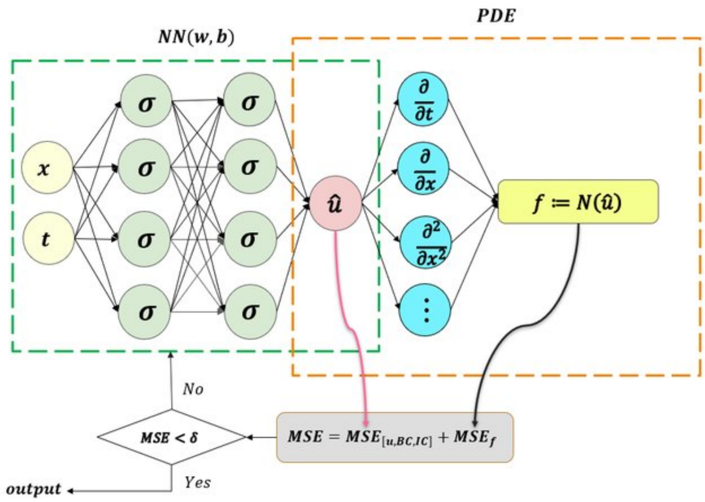


Figure 39: Guo, Y. (2020, August). [3] *PINN Optimization Structure.*

To solve our model 1D convective diffusion problem using PINN, we can use a popular Python library: PyDEns [2]. PyDEns helps to solve PDEs using deep neural networks with a very user-friendly interface. For example, for our one-dimensional stable convective diffusion problem, we plot the solution as follows:

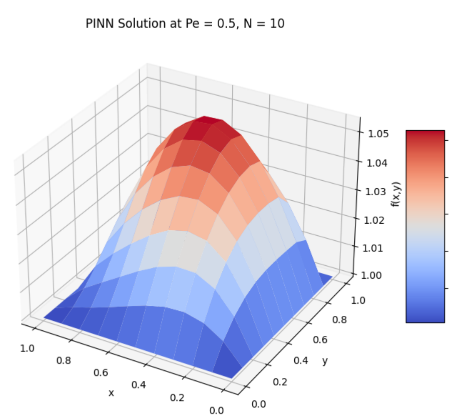


Figure 40: PINN Solution at Pe = 0.5, N = 10

As we can observe, the solution is well-performed at $P_e = 0.5$ case. Then we try another setting that $P_e = 10.0$ under the same number of partitions $N_x = N_y = 10$ to evaluate the performance:

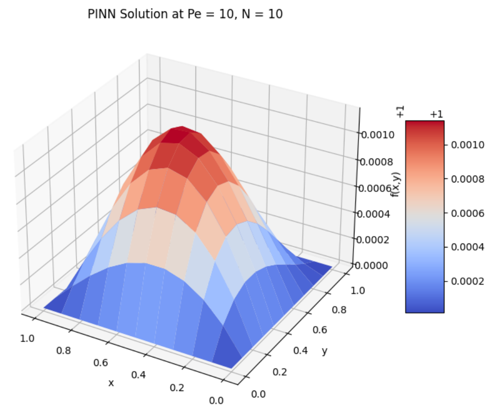


Figure 41: PINN Solution at Pe = 10, N = 10

There is no wiggles appearing in the solution, so the method satisfies both stability and consistency.

This is only a brief example for PINN method. More remarkably, PINN can handle complex geometries and irregular domains as it does not rely on a structured grid. It can also handle noisy or incomplete data, as it learns from existing observations and incorporates physics to fill in the missing information. In addition, PINN can reduce computational costs by learning an implicit representation of the solution compared to traditional numerical methods.

However, PINN is relatively new and is still in the research progress stage. Its practical implementation and performance depend on factors such as the complexity of the problem, the quality of available data, the architecture of the neural network, and the optimization algorithms used for training.

Codes

Except for the reference diagram, all diagrams in this project are generated by code written by the author. The code supporting this project is Python based and is attached to the following github repository:

https://github.com/LakeYang0818/Solving-Convection-Diffusion-Problems.

Summary

In this paper, we implemented three numerical methods—the Finite Difference Method (FDM), the Finite Element Method (FEM), and the Finite Volume Method (FVM) to solve the one-dimensional steady convection-diffusion problem.

Initially, we implemented and evaluated the traditional versions of each method, including the Central Difference Scheme for FDM, the Galerkin Approximation Scheme for FEM, and the FVM with Central Difference Scheme. By identifying the limitations of these traditional schemes, we proposed the improvements that Upwind Difference Scheme and Artificial Difference Scheme for FDM, Petrov-Galerkin (SUPG) Approximation Scheme for FEM, and Upwind Difference Scheme and Hybrid Difference Scheme for FVM.

Finally, by comparing the performances of these methods with associated schemes based on five essential properties: transportiveness, consistency, stability, convergence, and conservativeness, we aim to provide insights for selecting the appropriate numerical method when solving convection-diffusion problems under different physical scenarios.

References

[1] Alexander N. Brooks *et al.* (2003) *Streamline Upwind/petrov-galerkin formulations for convection dominated flows with particular emphasis on the incompressible navier-stokes equations*, *Computer Methods in Applied Mechanics and Engineering.* Available at: https://www.sciencedirect.com/science/article/abs/pii/0045782582900718 (Accessed: 04 May 2023).

[2] Koryagin, A., Khudorozkov, R. and Tsimfer, S. (2019). PyDEns: a Python Framework for Solving Differential Equations with Neural Networks. arXiv:1909.11544 [cs, stat]. [online] Available at: https://arxiv.org/abs/1909.11544 [Accessed 05 May 2023].

[3] Yanan, G. Solving partial differential equations using deep learning and physical … Available at: https://www.researchgate.net/publication/343897221_Solving_Partial_Differential_Equations_Using_Deep_Learning_and_Physical_Constraints (Accessed: 05 May 2023).