Time and frequency domain analysis of an electronic low-pass filter

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Abstract

The first order RC low pass filter is one of the most important circuits in electronics with the primary use of blocking high frequency waves.

This report will attempt to analyse the time and frequency response of a first order RC low pass filter with RC=0.1s, exploring how the resistance and capacitor tolerances affect the outcome of the results.

The results for the time domain were:

$$RC = 0.096s$$

Representing a 4% error

The results for the frequency domain were:

$$RC = 0.114s, \quad f_c = 1.393Hz$$

Which represents a 14% and 12.5% error, respectively.
All the results found were within the 25% tolerance for RC, hence proving the results were valid.

Nomenclature

C	$Capacitance\ [F]$
R	$Resistance\ [\Omega]$
V	$DC\ Voltage\ [V]$
v	$AC\ Voltage\ [V]$
τ	$Time\ constant\ [s]$
t	$Time\ [s]$
f	$Frequency\ [Hz]$
ω	$Angular\ Frequency\ [Hz]$
ϕ	$Phase\ difference\ [deg]$
ΔL	$Roll\ Off\ [dB/decade]$
G	$Gain\ [dB]$
Z	$Impedance\ [\Omega]$

All Prefixes are SI Standard

Acronyms

LPF	$Low\ pass\ filter$

Subscripts

out	$output$
in	$input$
USB	$USB\ Source$
c	$cutoff$
cap	$capacitor$
R	$resisotr$

Introduction

An electronic Low pass filter (LPF) allows waves with frequencies less than a cut-off frequency, f_c to be passed but attenuates (blocks) waves with frequencies greater than f_c, as illustrated in *Figure 1.*

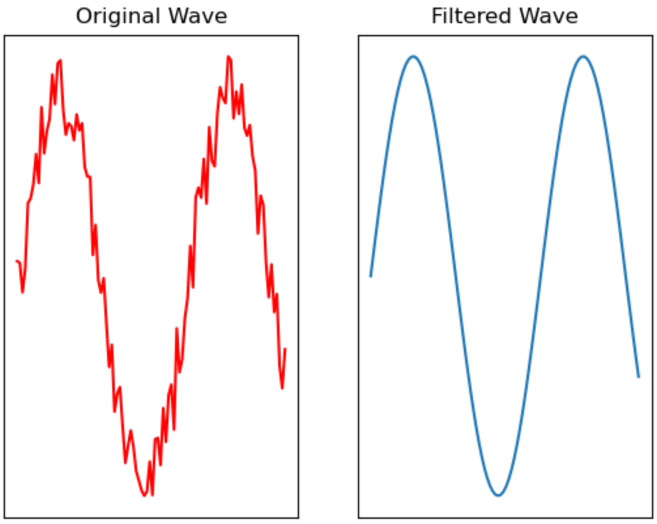

Figure 1-Effect of LPF on noise

Figure 1 clearly demonstrated that the LPF was able to filter out the noise in the input signal. This functionally has many uses, ranging from filtering out unwanted signals from audio to removing high frequency muscle artifact and external interference from electrocardiograms [1].

Objectives

- To calculate the values for time constants and cut off frequency
- To show that our low pass filter behaves according to the theory
- To explore the effects of resistor and capacitor tolerances on our results

Method

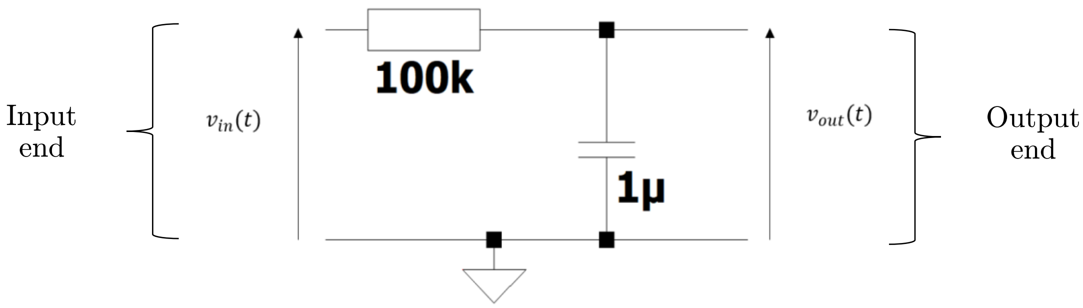

Figure 2- LPF Circuit

Time Domain

The circuit in *Figure 2* was created on a breadboard, with the probe of a oscilloscope connected to the output end. A DC input voltage source (such as a USB) is put across the input ends of the circuit.

At time, $t = 0$, the DC supply is switched on, such that $v_{in}(t) = V_{USB}$,(in this experiment $V_{USB} \approx 5.077V$)then the signal on the oscilloscope was recorded until there was no change in the $v_{out}(t)$ measured by the oscilloscope. The recorded signal was saved as a CSV file for later analysis.

Frequency Domain

The circuit in *Figure 2* was created on a breadboard, with the probes of a oscilloscope connected to the input and output ends. Then, the input end was connected to an AC voltage source (such as an AWG). The input voltage will be an AC sine wave with a DC offset, in the form:

$$v_{in}(t) = V_{peak}\sin(\omega t) + V_{offset}$$

$$\Rightarrow v_{in}(t) = V_{peak}\sin(2\pi f t) + V \qquad \because \omega = 2\pi f$$

For the experiment conducted, $V_{peak} = 1V,\ \ V_{offset} = 1V$

The input is set to a frequency f, this will be our variable. The values of frequencies used were:

$$f\ in\ \{0.1, 0.2, 0.5, 1, 2, 5, 10, 20, 50, 100\}\ Hz$$

For each value of frequency, the root-mean square (RMS) voltage of the input and output, and the phase difference between the input and output were measured using the oscilloscope.

Theory

Time Domain

For charging a capacitor. Kirchhoff's Voltage Law:

$$V_{in} = V_{resistor} + V_{cap}$$
$$V_{in} = IR + V_{cap}$$
$$V_{in} = \frac{dQ}{dt}R + V_{cap}$$
$$V_{in} - V_{cap} = RC\frac{dV}{dt} \qquad \because Q = CV$$

This gives us a differential equation, solving:

$$\int_0^t \frac{dt}{RC} = \int_0^{V_{cap}} \frac{dV}{V_{in} - V_{cap}}$$

Gives us the expression for the charging of a capacitor:

$$V_{cap} = \begin{cases} 0 & t < 0 \\ V_{in}\left(1 - e^{-\frac{t}{\tau}}\right) & t \geq 0 \end{cases}$$

$$where\ \tau = RC$$

In our experiments, $V_{in} = 4.99V \approx V_{USB} \quad \tau = 100k\Omega \times 1\mu F = 0.1s$

So, the graph of charging, theoretically should look like *Figure 3.*

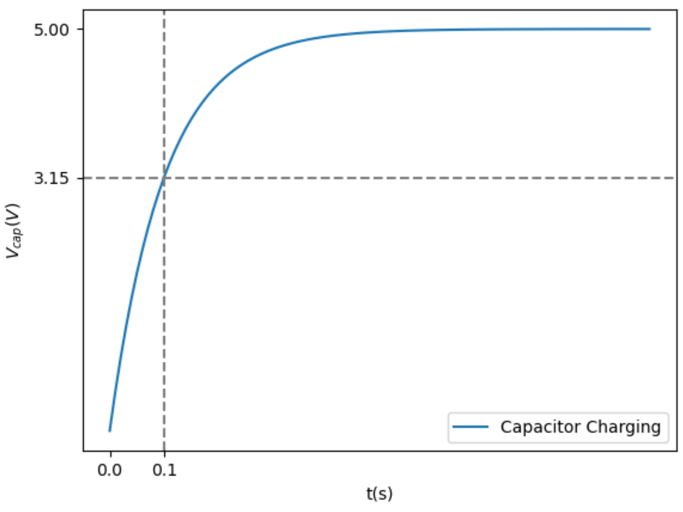

Figure 3- Capacitor Charging Graph

An important property to remember is that when:

$$t = \tau \qquad V_{cap} = (1 - \frac{1}{e})V_{in}$$

This is useful because it will enable you to get the value of time constant from just reading the graph.

Frequency Domain

As we are working with AC voltages, it is trivial that we use the complex numbers to represent voltages and impedances, since it allows us to avoid solving complicated differential equations.

Via the potential divider equation

$$v_{out}(t) = v_{in}(t) \times \frac{Z_c}{Z_R + Z_R}$$

Define Complex Gain, g_c as $g_c = \frac{v_{out}(t)}{v_{in}(t)}$

$$g_c = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{1}{1 + j\omega RC}$$

$$g_c = \frac{1 - j\omega RC}{1 + (\omega RC)^2} = \frac{1}{\sqrt{1 + (\omega RC)^2}} e^{-\mathrm{j}\arctan \omega RC}$$

Which can be rewritten as:

$$g_c = g e^{\phi}$$

$$g = \frac{1}{\sqrt{1 + (\omega RC)^2}}, \phi = -\arctan \omega RC$$

Where; $g\ is\ gain, \phi$ is the phase difference between the output and input

Usually gain, is measured in decibels (dB), so let us define gain in decibels as G, where:

$$G = 20 \log_{10} g = 20 \log_{10} \frac{1}{\sqrt{1 + (\omega RC)^2}}$$

We want to know at what frequency does the signal get attenuated. A useful way to determine this cut-off frequency, is to find the frequency where

$$g = \frac{1}{\sqrt{2}}\ or\ G = -3dB$$

This occurs when:

$$\omega_c RC = 1$$

$$\Rightarrow \omega_c = \frac{1}{RC}$$

$$\Rightarrow f_c = \frac{1}{2\pi RC}$$

So theoretically: $f_c = 1.591 Hz$

Then the expected graphs should look like *Figure 4*, the graphs are usually called a Bode Plots:

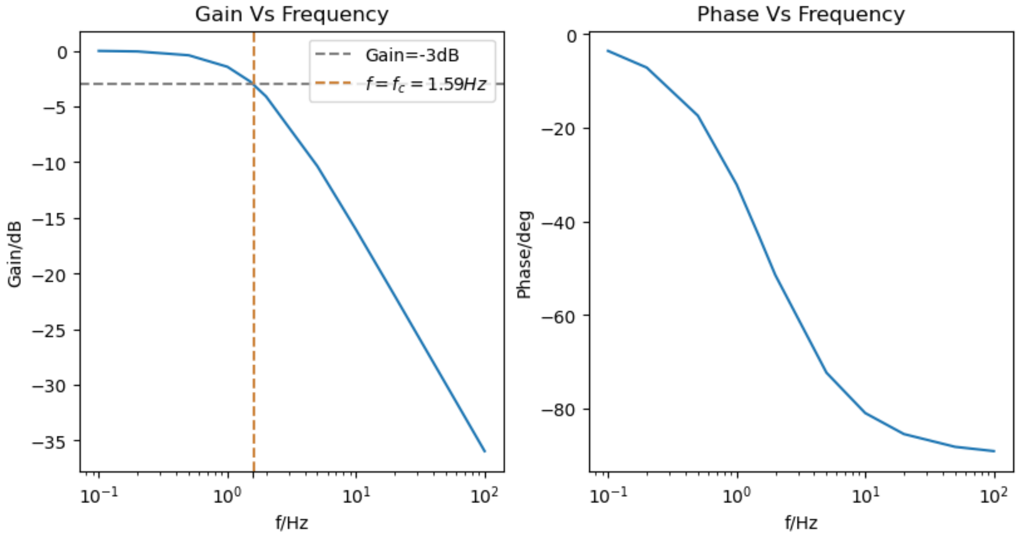

Figure 4- Gain and phase vs frequency

Roll off is a measure of the how much the signal gets attenuated past the cut-off frequency.

Remember that:

$$G = 20\log_{10}\frac{1}{\sqrt{1+(\omega RC)^2}}$$

$$= -10\log_{10}(1+\omega^2/\omega_c^2) \quad \because \ \omega_c = \frac{1}{RC}$$

When past the cut-off frequency $\frac{\omega}{\omega_c} > 1$, hence we can define Roll-off, L, as

$$L \approx -20\log_{10}(\omega/\omega_c)$$

An increase in one decade means, $\omega = 10\omega_c$, So

$$\Delta L = -20\log_{10} 10 \ dB/decade$$

$$= -20 \ dB/decade$$

This shows us past the cut-off frequency, increasing the frequency by 10x, results in a decrease in Gain by a factor of 20.

Tolerances

R has a tolerance of $\pm 5\%$ and C has a tolerance of $\pm 20\%$,
So, it is trivial that,

$$RC \ has \ a \ tolerance \ of \pm 25\%$$

Using the theoretical values for $RC = 0.1s, V_{in} = 5V$, we can plot a graph showing the area of tolerance range in *Figure 5 & 6.*

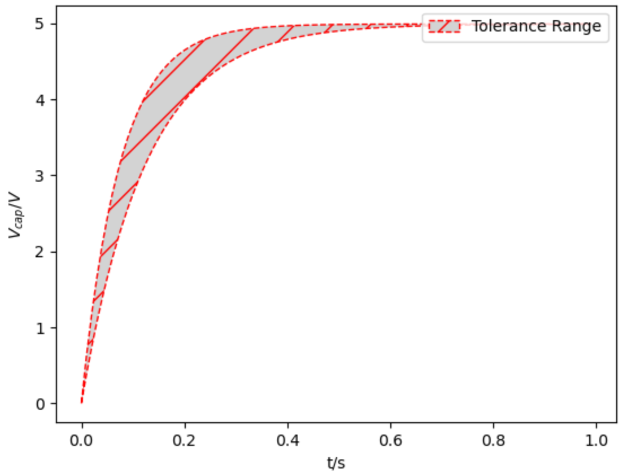

Figure 5- Tolerance effect on charging

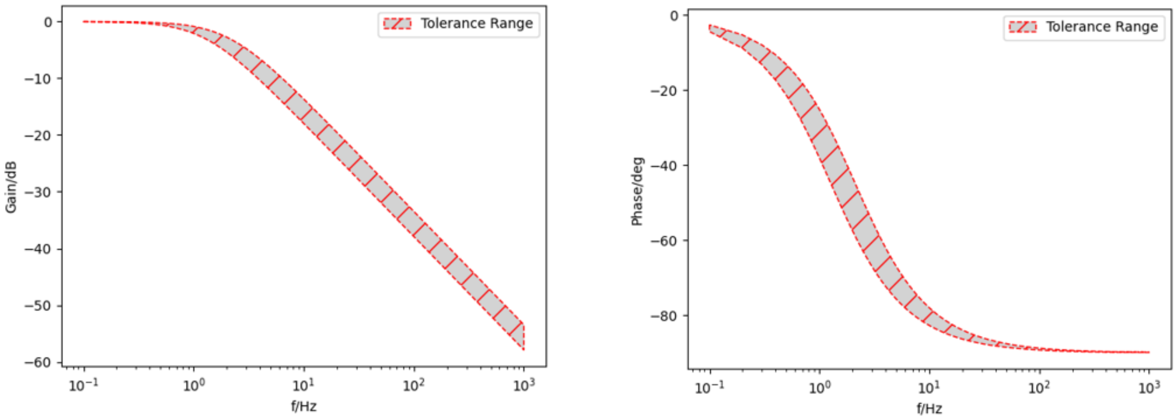

Figure 6- Tolerance effect on gain and phase

These graphs will be useful to later determine if our results can be considered accurate and within the given tolerances of the resistor and capacitor.

Results

Time Domain

The data collected contained over 5000 data points, which would make it difficult to analyse, so a small sample of data (which are evenly spread out from each other) was chosen as shown in *Table 1.*

When the capacitor started charging, the time t was not exactly zero, this makes it quite annoying to analyse, so to fix it, we define:

$$t_d = \tau \ln(1 - \frac{V_{cap}}{V_{in}})$$

So, in actuality,

$$V_{cap} = \begin{cases} 0 & t < t_d \\ V_{in}\left(1 - e^{-\frac{t_{measured}+t_d}{\tau}}\right) & t \geq t_d \end{cases}$$

Then we add t_d to all values of t

$$t = t_{measured} + t_d$$

Thus leaving us with the intended equation,

$$V_{cap} = \begin{cases} 0 & t < 0 \\ V_{in}\left(1 - e^{-\frac{t}{\tau}}\right) & t \geq 0 \end{cases}$$

Time, t/s	Voltage, V_{cap}/V
0.000	0.896
0.100	3.299
0.200	4.278
0.300	4.723
0.400	4.901
0.500	4.990
0.600	4.990
0.700	4.990
0.800	4.990
0.900	4.990
1.000	4.990

Table 1- Small sample of data collected

Using this table, and the Python Library, SciPy to do a curve fit with the data and use the previously calculated tolerance in RC to plot two extra lines representing a $\pm$ error in RC, as shown in *Figure 7.*

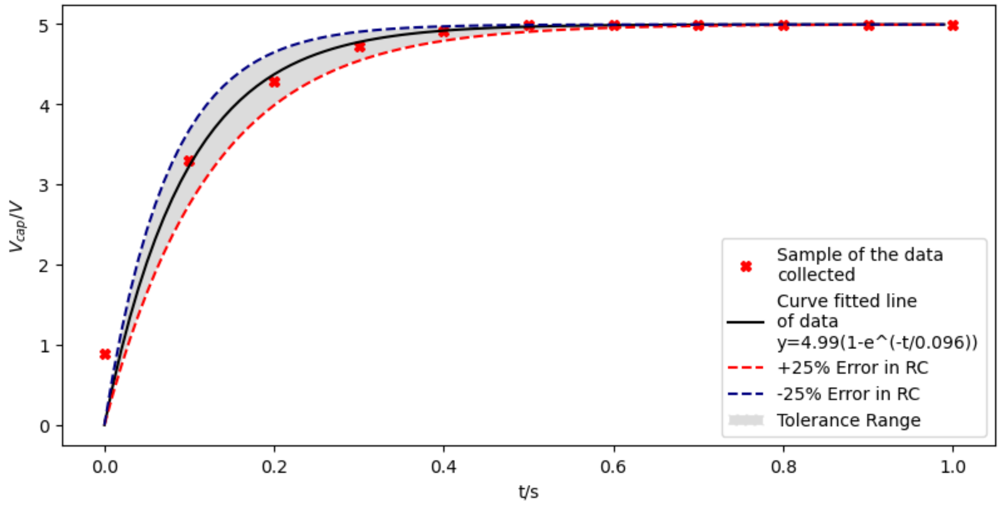

Figure 7 - Time Domain results

Frequency Domain

The measured values of v_{in} and v_{out} were used to calculate the gain and the phase using the formulas mentioned in the Frequency Domain Theory section.

The results are recorded in *Table 2*

Frequency (Hz)	vin RMS (mV) Probe A	vout RMS (mV) Probe B	Measured gain (dB)	Measured phase (degrees)
0.1	733.5	729	-0.05	0.00
0.2	720.7	708.5	-0.15	-10.40
0.5	698	649.8	-0.62	-21.31
1	704.1	563.3	-1.94	-37.62
2	702	393.5	-5.03	-54.45
5	699.1	184.8	-11.56	-75.58
10	702.3	96.77	-17.22	-80.32
20	701.7	49.32	-23.06	-83.30
50	700	19.04	-31.31	-91.35
100	702.5	10.39	-36.60	-90.04

Table 2- Frequency domain data

Just like in the time domain analysis, SciPy was used to do a curve fit with the data and use the previously calculated tolerance in RC to plot two extra lines representing a $\pm$ error in RC, as shown in *Figure 8 and Figure 9 .*

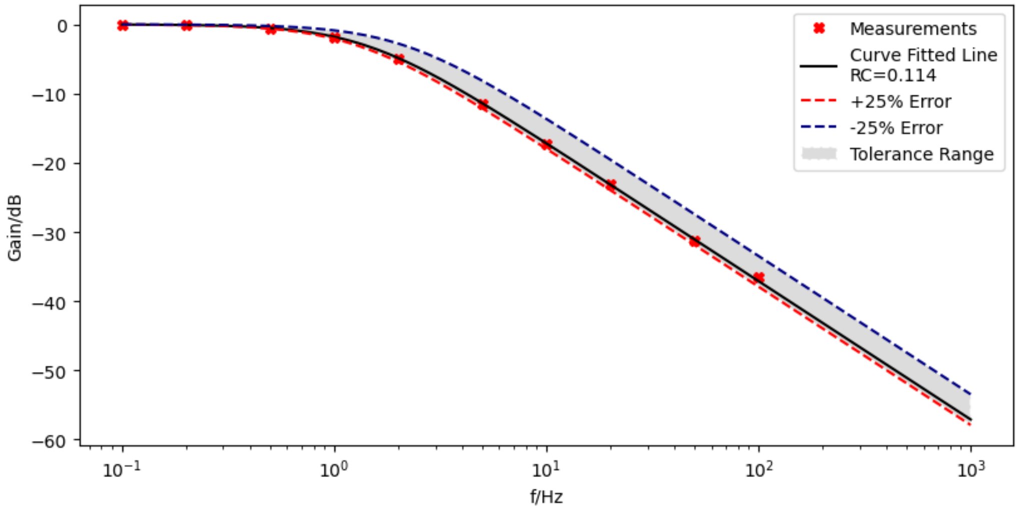

Figure 8 - Gain vs frequency

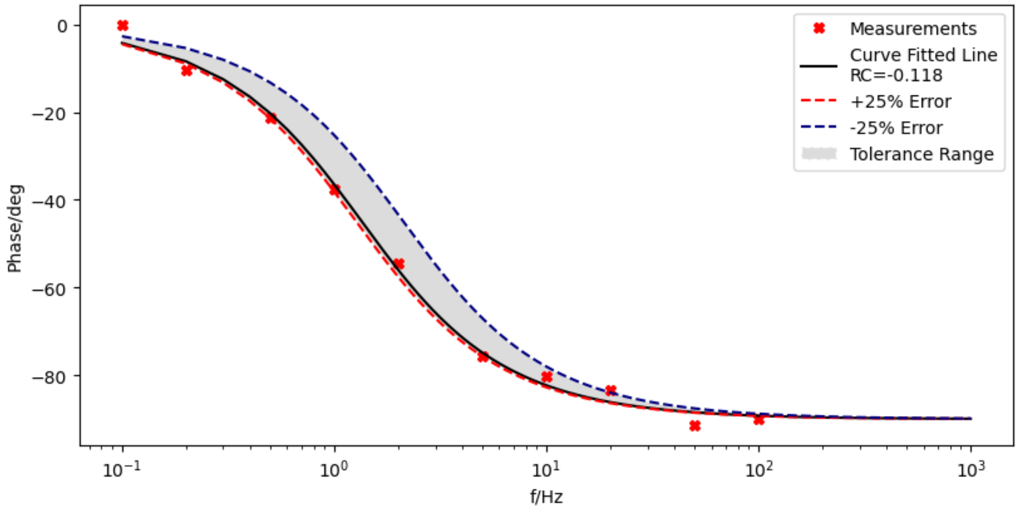

Figure 9- Phase vs frequency

Discussion

Time Domain

Figure 7 accurately portrays the behaviour of a capacitor charging where there is a large rate of increase in V initially which slowly reduces over time resulting in a constant voltage, when it is fully charged.

Using the `curve_fit()` function of SciPy to calculate the value for time constant, gave:

$$\tau = 0.096s$$

Which represents a 4% error.

Figure 7 also demonstrated that the measured values were within the error lines, hence the error in RC did not affect my results that much since it was so low

Substituting this value into the Excel [2] worksheet provided by the CUED, it confirms that the value of τ obtained fits my data most accurately.

Frequency Domain

Figure 8 shows the characteristic behaviour of a low pass filter, where before the cut-off frequency the gain is unity, then past it, the signal gets attenuated with a roll-off of 20dB/decade.

Using the `curve_fit()` function of SciPy to calculate the value for time constant,

$$RC = 0.114s$$

Which represents an error of 14%.

Figure 8 and *Figure 9* shows that the measured values were very close to the lower boundary for the error in RC.

Substituting this value into the Excel [2] worksheet provided by the CUED, it confirms that the value of RC obtained fits my data most accurately.

From theory, we found that

$$G = 20\log_{10}\frac{1}{\sqrt{1+(\omega RC)^2}}$$

So:

$$f = \frac{\sqrt{10^{-0.1G}-1}}{2\pi RC}$$

At the cut-off point, $f = f_c \quad and\ G = -3dB$

Hence $f_c = 1.393Hz$

Which represents an error of 12.5% from the theoretical values.

Conclusion

We were able to conclude that:

- Results were with the tolerances of the resistor and capacitor
- Results accurately represent the equations that were derived from theory
- We were able to show the behaviours of a low pass filter in a time and frequency domain

References

[1] C. Watford, "Understanding ECG Filtering," [Online]. Available: http://ems12lead.com/2014/03/10/understanding-ecg-filtering/#:~:text=Low%2Dpass%20filters%20on%20the,do%20not%20alter%20repolarization%20signals.. [Accessed 17 November 2020].

[2] CUED, "1P3: Physical Principles of Electronics and Electromagnetics," [Online]. Available: https://www.vle.cam.ac.uk/course/view.php?id=69811. [Accessed 19 11 2020].

To access the Python code I made, to curve fit and plot, go to:
https://github.com/LakeeSiv/1AExpositon