

Computing Exact Power and Sample Size for Hotelling's T^2 -test and Related Multivariate Procedures

Jimmy Thomas Efird, Ph.D., M.Sc.^{1,2}

Kavitha Alimineti, B.Sc.¹

¹Biostatistics and Data Management Facility, John A. Burns School of Medicine, Honolulu, HI; ²Department of Biostatistics and Epidemiology, UCSF School of Medicine, San Francisco, CA

ABSTRACT

Multiple comparisons often arise in medical and epidemiologic research and several methods exist for adjusting statistical significance levels. In this paper, we discuss the multivariate method attributed to Harold Hotelling for handling multiplicity in the p-variate two-sample case. A generalized method is derived for computing the exact power for this test. The technique may be easily adapted to other multivariate procedures that utilize the T^2 -statistic.

INTRODUCTION

Given the case of p-variate observations from two multivariate normally-distributed populations with common covariance matrix, Hotelling's T^2 -statistic may be used to test the equality of the vector of means associated with the above samples. If the null hypothesis of equal means is rejected, then simultaneous confidence intervals may be constructed to determine which components of the mean vector differ between the two groups. Below, we derive the exact power for Hotelling's T^2 -test.

METHODOLOGY

Let \mathbf{d} denote the vector of mean differences and \mathbf{S} the common sample covariance matrix based on random samples of sizes n_1 and n_2 from two p-variate normally-distributed populations. A direct extension of the univariate pooled t-test to multivariate space (Kelsey, et al., 1996; Kramer, 1972) gives the exact power for Hotelling's T-test as

$$T_{\text{power}} = \sqrt{(n_1 n_2) / (n_1 + n_2) \mathbf{d}' \mathbf{S}^{-1} \mathbf{d}} - \sqrt{\frac{(n_1 + n_2 - 2)p}{(n_1 + n_2 - p - 1)}} F_{p, n_1 + n_2 - p - 1}, \quad (1)$$

where $F_{p, n_1 + n_2 - p - 1}$ is the 100(1 - α) centile of the F-distribution with p and $n_1 + n_2 - p - 1$ degrees of freedom (Johnson and Wichern, 1982). When p=1, we see that equation number (1) gives the exact power for the univariate pooled t-test.

EXAMPLE

We wish to compare two drugs used in the treatment of HPV infection with respect to mean expression levels of cytokines IL-2 and IL-10. If we are given that the vector of mean differences equals

$$\mathbf{d} = \begin{bmatrix} -2.60 \\ 2.17 \end{bmatrix}' \quad (2)$$

and that the sample covariance matrix equals

$$\mathbf{S} = \begin{bmatrix} 7.22 & 0.320 \\ 0.320 & 6.992 \end{bmatrix}, \quad (3)$$

then from Figure 1, we see that a sample of size $n_1=n_2=15$ will have power ≥ 0.82 at the $\alpha = 0.05$ level of significance to reject the null hypothesis (e.g., mean expression levels for the cytokines examined are equal for each drug) given that it is false.

SAS® CODE

The SAS® code used to compute the data points plotted in Figure 1 is shown in Figure 2.

Figure 2

```
proc iml workspace=500;
  start main;
    alpha=0.05;
    p=2;
    d={-2.60,2.17}`;
    s={7.220 0.320,0.320 6.992};
    do n1=5 to 30;
      do n2=5 to 30 by 5;
        power=probt(sqrt((n1*n2/(n1+n2))*d*inv(s)*d`)
          -sqrt(((p*(n1+n2-2))/(n1+n2-1-p))*
            finv(1-alpha,p,n1+n2-1-p)),n1+n2-2);
        print n1 n2 power;
      end;
    end;
  finish;
run main;
```

CONCLUSION

By taking advantage of the covariance structure among variables, Hotelling's T^2 -test presents a simple way to control for multiplicity when comparing the equality of multiple means in two samples. In this paper, we have derived a method for computing the exact power for Hotelling's T^2 -test. The method is a direct extension to multivariate space of the formula used to compute power in the univariate pooled t-test and may be easily adapted to other multivariate procedures that utilize the T^2 -statistic.

REFERENCES

Johnson R, Wichern D. Applied Multivariate Statistical Analysis. Prentice-Hall, Inc.: Englewood Cliffs, NJ., 1982.

Kelsey J, Whittemore A, Evans A, Thompson W. Methods in Observational Epidemiology. Oxford: New York, 1996.

Kramer C. A First Course in Methods of Multivariate Analysis. Virginia Polytechnic Institute and State University: Blacksburg, VI, 1972.

ACKNOWLEDGMENTS

The authors thank Paige Bracci (Department of Epidemiology and Biostatistics, UCSF) for reviewing this manuscript and providing useful comments. This manuscript was supported by NIH Grant Number G12RR003061 from the National Center for Research Resources.

CONTACT INFORMATION

Comments and questions may be directed to the authors at:

Jimmy Thomas Efird, Ph.D., M.Sc.
Director, Biostatistics and Data Management Facility
Associate Professor
John A. Burns School of Medicine
1960 East-West Road
D-103
Honolulu, HI 96822-2319
Email: Jimmy.efird@stanfordalumni.org

SAS and all other SAS Institute Inc. product or service names are registered trademarks or trademarks of SAS Institute Inc. in the USA and other countries. ® indicates registration.

Figure 1: Exact Power Analysis for Hotelling's T-Square Test

Alpha=0.05, $d = [-2.60, 2.17]$, $S = \begin{bmatrix} 7.220 & 0.320 \\ 0.320 & 6.992 \end{bmatrix}$

