Chapter 1

Definitions and first examples

Solution 1.1:

Solution 1.2:

Solution 1.3:

Solution 1.4:

Solution 1.5:

Solution 1.6:

Solution 1.7:

Solution 1.8:

Solution 1.9:

Solution 1.10:

Solution 1.11:

Solution 1.12:

Chapter 2

Ideals and homomorphisms

Solution 2.1:

Solution 2.2:

Solution 2.3:

Solution 2.4:

Solution 2.5:

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Solution 2.10:

Solution 2.11:

Solution 2.12:

Chapter 3

Solvable and nilpotent Lie algebras

Solution 3.1:

Solution 3.2:

Solution 3.3 (Lakshay): The relations

$$\begin{aligned} [x, y] &= h \\ [h, x] &= 2x \\ [h, y] &= -2y \end{aligned}$$

hold equally well in characteristic 2, but the last 2 brackets are then 0. Thus $L^1 = [L, L] = \mathbb{C}h$, and $L^2 = [L^1, L] = [\mathbb{C}h, L] = 0$, since $[h, -]$ is uniformly 0.

Solution 3.4:

Solution 3.5:

Solution 3.6:

Solution 3.7 (Lakshay): $K \subset N_L(K)$ follows from K being closed under the bracket operation. Let n be the unique non-negative integer such that $L^n \subsetneq K$ but $L^{n+1} \subset K$, which exists because K is proper and L is nilpotent. Let $z \in L^n \setminus K$. I claim that $z \in N_L(K)$. Indeed,

$$zK \subset [L^n, K] \subset [L^n, L] = L^{n+1} \subset K$$

Solution 3.8:

Solution 3.9 (Lakshay): This is only true if $L \neq 0$, so assume that that is the case.

Since L is nilpotent, it has an ideal K of codimension 1 by Problem 3.7, so there is some $x \in L$ such that $L = K \oplus \mathbb{F}x$ as vector spaces. There is some non zero L^k with $L^{k+1} = 0$, thus $L^k \subset C_L(L) \subset C_L(K) \neq 0$. As the sequence L^i decreases to 0, there is some largest n such that $C_L(K) \subset L^n$. Let $z \in C_L(K) \setminus L^{n+1}$. Define $\delta : L \to L$ by $K \mapsto 0$, $x \mapsto z$, and extending linearly.

Then δ is linear by construction. It is a derivation because for any pair $k_1 + c_1 x$ and $k_2 + c_2 x$ with $k_i \in K$ and $c_i \in \mathbb{C}$, both sides of the Leibniz rule equation become 0. It needs to be shown that it is not an inner derivation. Suppose $\delta = \operatorname{ad} y$ for some $y \in L$. As $\delta(K) = 0$, $y \in C_L(K)$. As $\delta(x) = [y, x] = z$ and $y \in L^n$, $z \in L^{n+1}$, which contradicts the choice of z.

Solution 3.10:

Chapter 4

Theorems of Lie and Cartan

Solution 4.1:

Solution 4.2:

Solution 4.3:

Solution 4.4:

Solution 4.5:

Solution 4.6:

Solution 4.7:

Solution 4.8:

Chapter 5

Killing form

Solution 5.1:

Solution 5.2:

Solution 5.3:

Solution 5.4:

Solution 5.5:

Solution 5.6:

Solution 5.7:

Solution 5.8:

Chapter 6

Complete reducibility of representations

Solution 6.1:

Solution 6.2:

Solution 6.3 (Lakshay): Let $\phi : L \to \mathfrak{gl}(V)$ be an irreducible representation of a solvable Lie algebra L. Let M denote $\phi(L)$, which is a linear Lie algebra isomorphic to a quotient of L and hence also solvable. By Lie's theorem, there is a flag $0 \subset V_0 \subset V_1 \subset \cdots \subset V_n = V$ of V which is stable under M, and if $\dim V > 1$, then V_1 is a non trivial proper subrepresentation of M, so V is not irreducible as a representation of M. However, an M-subrepresentation of V is also an L-subrepresentation, because $l \in L$ acts by $\phi(l)$, and the flag is stable under the action of $\phi(l) \in M$.

Solution 6.4:

Solution 6.5:

Solution 6.6:

Solution 6.7:

Solution 6.8:

Solution 6.9:

Chapter 7

Representations of $\mathfrak{sl}(2, F)$

Solution 7.1:

Solution 7.2:

Solution 7.3:

Solution 7.4:

Solution 7.5:

Solution 7.6:

Solution 7.7:

Chapter 8

Root space decomposition

Solution 8.1:

Solution 8.2:

Solution 8.3:

Solution 8.4:

Solution 8.5:

Solution 8.6:

Solution 8.7:

Solution 8.8:

Solution 8.9:

Solution 8.10:

Solution 8.11: