

# The Slice Spectral Sequence for *KR*-Theory

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# Preface

The aim of this project was to understand and explain parts of Hill, Hopkins, and Ravenel's proof of the Kervaire Invariant Conjecture in their paper [HHRa]. This conjecture states that there do not exist framed manifolds with certain surgery theoretic properties, or equivalently, with Kervaire invariant one. Their solution is broken up into three main theorems about a  $C_8$ -spectrum  $\Xi$  that they constructed - the detection, gap, and periodicity theorems.

The detection theorem states that the Kervaire invariant one elements are detected as certain non-zero elements in the homotopy Mackey functors of  $\Xi$  in certain degrees. The gap theorem says that the homotopy Mackey functors of  $\Xi$  have no such elements in degrees  $-1, -2, -3$ . The periodicity theorem says that the homotopy Mackey functors of  $\Xi$  are 256-periodic. Gap and periodicity together imply that these homotopy Mackey functors are zero in the necessary degrees, proving the non-existence of elements of Kervaire invariant one in all dimensions other than 126, where the conjecture remains open.

In this thesis, we look at the theory they developed along the way and apply their techniques, most notably, the slice spectral sequence, to a relatively simpler  $C_2$ -spectrum  $KR$ . We find that this spectrum has similar gap and periodicity theorems and so provides a good toy case to understand the theory.

This thesis is organized as follows. Chapters 1 to 3 give general abstract background required for the later chapters. They are independent from each other and can be read in any order. Chapter 1 is only used in Chapter 8. Chapters 4 and 5 are also independent but have to be read after Chapters 2 and 3. Chapters 1 to 5 mostly consist of the background needed to be able to define everything mentioned above. Chapters 6 to 8 form the meat of the thesis and have to be read in that order. Chapter 6 introduces  $G$ -spectra and homotopy Mackey functors, Chapter 7 defines the  $C_2$ -spectrum  $KR$  and gives its periodicity theorem, and Chapter 8 constructs the slice spectral sequence and uses it to prove the gap theorem for  $KR$ .

# Acknowledgements

I am extremely grateful to my thesis supervisor, Dr. Vigleik Angeltveit, for his guidance and immense patience. He gave me a lot more time than I could have asked for and I always left our meetings with excitement for the subject. His explanations were really helpful because of the additional insight they carried over their textbook accounts. He has made this a very fulfilling project for me and given me a lot of appreciation for this subject.

I am also grateful to my family for their support in my studies and my friends for listening to my ramblings as I tried to refine my understanding of these topics.

# Contents

<b>1</b>	<b>Spectral Sequences</b>	<b>5</b>
1.1	Exact Couples and their Spectral Sequences . . . . .	5
1.2	Convergence of spectral sequences . . . . .	8
1.3	Multiplicative Spectral Sequences . . . . .	14
<b>2</b>	<b>Model Categories</b>	<b>16</b>
2.1	Classical homotopy theory . . . . .	17
2.2	Model Categories . . . . .	19
2.3	Functors of Model Categories . . . . .	23
2.4	Constructing Model Structures . . . . .	25
2.5	Left Bousfield Localization . . . . .	28
<b>3</b>	<b>More Categorical Background</b>	<b>30</b>
3.1	Nerve-Realization Adjunction and Day Convolution . . . . .	30
3.2	Modules in Diagram Categories . . . . .	34
<b>4</b>	<b>Stable Homotopy Theory</b>	<b>35</b>
4.1	Spectra and their Model Structure . . . . .	36
4.2	Stable Homotopy Category . . . . .	38
4.3	Smash Products and Orthogonal Spectra . . . . .	41
<b>5</b>	<b>Equivariant Homotopy Theory</b>	<b>47</b>
5.1	Category of $G$ -spaces . . . . .	47
5.2	Homotopy theory of $G$ -spaces . . . . .	49
<b>6</b>	<b>Equivariant Stable Homotopy Theory</b>	<b>53</b>
6.1	$G$ -spectra . . . . .	53
6.2	Mackey Functors . . . . .	57
6.3	Equivariant Homology and Cohomology . . . . .	64
6.4	Homology of a Point . . . . .	68
6.5	Multiplicative Structure . . . . .	71
<b>7</b>	<b>Vector Bundles and Topological K-Theory</b>	<b>73</b>
7.1	Vector Bundles as Homotopy Invariants . . . . .	73
7.2	Real and Complex $K$ -Theory . . . . .	77
7.3	$KR$ -Theory . . . . .	79
<b>8</b>	<b>The Slice Spectral Sequence</b>	<b>82</b>
8.1	Slice Tower and its Spectral Sequence . . . . .	83
8.2	The Slice Spectral Sequence for $KR$ -Theory . . . . .	91

# 1 Spectral Sequences

Let  $C_*$  be a chain complex with boundary maps  $d_t : C_t \rightarrow C_{t-1}$ . Suppose  $C_*$  has a filtration by subcomplexes

$$C_{0,*} \subset C_{1,*} \subset C_{2,*} \subset \cdots \subset C_*$$

For example,  $C_*$  could be the singular chain complex of a CW complex, and  $C_{s,*}$  that of its  $s$ -skeleton. We are concerned with finding the homology of  $C_*$ . Taking homologies of the entire filtration and its successive quotients, we get

$$\begin{array}{ccccccc} H_*(C_0) & \longrightarrow & H_*(C_1) & \longrightarrow & H_*(C_2) & \longrightarrow & H_*(C_3) \longrightarrow \cdots \\ & \nearrow \text{dotted} & \downarrow & \nearrow \text{dotted} & \downarrow & \nearrow \text{dotted} & \downarrow \\ & & H_*(C_1/C_0) & & H_*(C_2/C_1) & & H_*(C_3/C_2) \end{array}$$

The dotted arrows are the connecting maps of degree  $-1$  that we get from the snake lemma, making each triangle an exact sequence. This is the structure of an unrolled exact couple.

## 1.1 Exact Couples and their Spectral Sequences

**Definition 1.1.** An *unrolled exact couple*  $(A_s, E_s, i_s, j_s, k_s)_{s \in \mathbb{Z}}$  is the data of the following diagram of graded abelian groups

$$\begin{array}{ccccccc} \cdots & \longrightarrow & A_{-1} & \xrightarrow{i_0} & A_0 & \xrightarrow{i_1} & A_1 \xrightarrow{i_2} A_2 \longrightarrow \cdots \\ & & \searrow k_0 & & \downarrow j_0 & \searrow k_1 & \downarrow j_1 \\ & & & & E_0 & & E_1 \end{array}$$

in which each triangle is exact.

Throughout this chapter, a map of graded (or bigraded) abelian groups will be assumed to be a map that shifts degrees by some constant. In the example of the filtered chain complex, the maps  $i_s$  and  $j_s$  preserve degree, while  $k_s$  reduces degree by 1.

We can compactify all this data by defining

$$A = \bigoplus_{s \in \mathbb{Z}} A_s, \quad E = \bigoplus_{s \in \mathbb{Z}} E_s$$

## 1 Spectral Sequences

$$\alpha = \text{diag}(i_s)_{s \in \mathbb{Z}}, \quad \beta = \text{diag}(j_s)_{s \in \mathbb{Z}}, \quad \gamma = \text{diag}(k_s)_{s \in \mathbb{Z}}$$

$A$  and  $E$  are both now bigraded. The first grading is by the *filtration degree*  $s$ , and the second grading is from the *internal grading*  $*$  that was suppressed.

**Definition 1.2.** A *rolled exact couple*  $(A, E, \alpha, \beta, \gamma)$  is an exact triangle of bigraded abelian groups of the form

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & A \\ & \nwarrow \gamma & \downarrow \beta \\ & & E \end{array}$$

In the example of the filtered chain complex,  $\alpha$  has bidegree  $(1, 0)$ ,  $\beta$  has bidegree  $(0, 0)$ , and  $\gamma$  has bidegree  $(-1, -1)$ .

The advantage of an unrolled exact couple is that it makes the filtration explicit. Since our definitions will often depend on the filtration, they would be easier to see in the unrolled diagrams. Rolled exact couples will be easier to manipulate and will declutter notation when the indexing is irrelevant. For example, we have the following key proposition.

**Proposition 1.3.** For the exact couple of Definition 1.2, define  $d : E \rightarrow E$  by  $d = \beta \circ \gamma$ . Then  $d^2 = 0$ , hence we may define  $H(E, d) = \ker d / \text{im } d$ . The derived couple of the exact couple, defined as

$$\begin{array}{ccc} \text{im } \alpha & \xrightarrow{\alpha} & \text{im } \alpha \\ & \nwarrow \gamma & \downarrow \beta \circ \alpha^{-1} \\ & & H(E, d) \end{array}$$

is then a well defined exact couple.

*Proof.* See [Hata, Lemma 5.1]. □

**Remark 1.4.** Deriving the couple we get from an unrolled couple and then unrolling it back, we get the diagram

$$\begin{array}{ccccccc} \cdots & \longrightarrow & i_{-1}(A_{-2}) & \longrightarrow & i_0(A_{-1}) & \longrightarrow & i_1(A_0) & \longrightarrow & i_2(A_1) & \longrightarrow & \cdots \\ & & & & \nwarrow & & \nwarrow & & \nwarrow & & \\ & & & & H_0(E) & & H_1(E) & & & & \end{array} \quad (1.1)$$

which, instead of exact triangles, has a single exact ‘triangular helix’ that spans the entire length of the diagram, and hence this is not an exact couple in the unrolled sense. However, it still rolls into a rolled exact couple and we may derive it again. These further derivations distort the shape of the unrolled version further. This can be seen by computing the bidegrees of the maps in the derived

## 1.1 Exact Couples and their Spectral Sequences

couple: the maps in the place of  $\alpha$  and  $\gamma$  retain their bidegrees, but the map that replaces  $\beta$  has bidegree that of  $\beta$  minus that of  $\alpha$ , and hence reduces by  $(1, 0)$  with every derivation.

For  $r \geq 0$ , denote the  $(r - 1)$ -st derived couple of the above couple by  $(A^r, E^r, \alpha^r, \beta^r, \gamma^r)$ , so that above couple itself is  $(A^1, E^1, \alpha^1, \beta^1, \gamma^1)$ .

For the unrolled exact couple in Definition 1.1, define  $A_\infty = \operatorname{colim}_s A_s$  and  $A_{-\infty} = \operatorname{lim}_s A_s$ . Then the former is filtered by the images of the natural maps from each  $A_s$  and the latter is filtered by the kernels of the natural maps to each  $A_s$ . In the case of a filtered chain complex,  $A_\infty$  is isomorphic to the homology of  $C$ , since the homology functor preserves direct limits. Our goal is to find either  $A_\infty$  or  $A_{-\infty}$  depending on the setting. For example, we might care about the latter in case of cohomology.

The spectral sequence associated to the couple is a structure that lets us compute the successive quotients of these filtrations on either  $A_\infty$  or  $A_{-\infty}$  under some mild conditions. This computation is an infinite process and we can think of each step as a simplification of the exact couple without changing  $A_{\pm\infty}$ . In each step, we replace the exact couple with its derived couple. This is a simplification in the sense that every group in the derived couple is ‘smaller’ and while the limit and colimit of the sequence in the derived couple are unchanged.

The surprising part is that the data of all the  $E^r$  along with the their differentials  $d^r = \beta^r \circ \gamma^r$ , which have bidegrees  $(-r, -1)$ , are all that will be needed to compute the groups we want. Example 1.11 is a simple and familiar example of how this works. The structure of all of this data is that of a spectral sequence.

**Definition 1.5.** A *spectral sequence* is a sequence of bigraded abelian groups  $(E^r)_{r \in \mathbb{N}}$ , a sequence of endomorphisms  $d^r : E^r \rightarrow E^r$  which all square to zero, and a sequence of isomorphisms  $E^{r+1} \cong H(E^r, d^r)$  for all  $r \in \mathbb{N}$ . We say that  $E^r$  is the  $r$ th page of the sequence.

**Definition 1.6.** A spectral sequence  $(E^r, d^r)_{r \in \mathbb{N}}$  has *default grading* if each  $d^r$  has degree  $(-r, -1)$ . It has *slice grading* if each  $d^r$  has bidegree  $(-1, r+1)$ .

There are many ways to grade a spectral sequence coming from applying a linear transformation to the  $(s, t)$ -grid before any computation happens. For example, the *Serre grading* defines  $E_{s,t}^1 = E_{s,t-s}$  to start off the spectral sequence to ensure that it is non-zero in exactly the first quadrant rather than the second octant as in the default grading. The slice grading, which is the grading we will use in Chapter 8, is the result of defining  $E_{s,t}^1 = E_{-s-t,s}$ . This grading is the result of [HHRa] using the *Adams grading* but indexing their spectral sequence to start at the  $E^2$  page. We will have our spectral sequences start in  $E^1$  to keep it consistent with our theory of exact couples, resulting in the slice grading.

Everything in this chapter will apply to any abelian category and we will need that in Section 8.1. We talk about abelian groups here for simplicity.

## 1.2 Convergence of spectral sequences

Our goal is to use the spectral sequence we get from repeated derivations to find  $A_{\pm\infty}$ . Since  $A_{\pm\infty}$  are only defined for an unrolled couple, we will restrict our attention to spectral sequences associated to unrolled exact couples. Once and for all, fix an unrolled exact couple  $(A_s, E_s, i_s, j_s, k_s)_{s \in \mathbb{Z}}$ , denote its  $(r-1)$ -st derivation by  $(A_s^r, E_s^r, i_s^r, j_s^r, k_s^r)$  and define  $E^r = \bigoplus_s E_s^r$ . The internal grading of each  $A$  and  $E$  will not be relevant for a while. Again, as described in Remark 1.4,  $j^r$  reduces degree by  $r-1$  and  $k^r$  reduces degree by 1. Therefore,  $d^r = j^r \circ k^r$  has degree  $-r$ .

In this entire section, we would have to deal with a lot of filtrations on various abelian groups. It will therefore be convenient to use Vakil's diagram calculus [Vak] to keep track of all the components in a visual way. The following proof is a diagrammatic translation of the purely algebraic proof from [Rog, Chapter 2]. Since we have to deal with infinitely many groups, we make one change. We draw the diagram for each group separately and only remember the identifications made by the first isomorphism theorem, instead of drawing their diagrams in an overlapping way as he does.

The idea of the diagram calculus is to draw filtrations on a group in way similar to Venn diagrams. The main difference from Venn diagrams is that the complement of a region that represents a subgroup represents the quotient by that subgroup rather than its complement. In particular, this means that not every region in the Venn diagram is a subgroup, and we need to remember which ones are and which ones are not. We do this by making the diagrams 'directional'; the filtrations in our diagrams will always increase as we go down and to the right.

Suppose we have an abelian group  $A$  with a filtration

$$0 \subset N_1 \subset N_2 \subset N_3 \subset A$$

We then draw  $A$  as



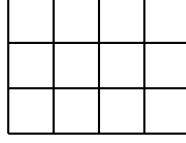
in which the first square represents  $N_1$ , the first two squares together represent  $N_2$ , the first three represent  $N_3$  and the whole diagram represents  $A$ . The middle two squares together will then represent taking the subgroup  $N_3$  and then *removing*  $N_1$  from it, i.e. it represents the subquotient  $N_3/N_1$ . We could also view it as first quotienting by  $N_1$  and then taking a subgroup, which is the same by the isomorphism theorems.

Suppose we have another filtrations on  $A$

$$0 \subset M_1 \subset M_2 \subset A$$



Then we can draw  $A$  as

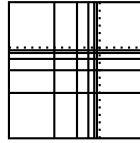


where the top row represents  $M_1$  and the top two rows together represent  $M_2$ . Now the two squares in the center are what we get when we take  $N_3 \cap M_2$  (all but the last row and last column) and remove from it the first row and then the first column. Thus, that region represents

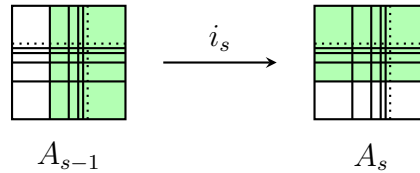
$$\frac{(N_3 \cap M_2)/M_1}{N_1} \cong \frac{N_3 \cap M_2}{M_1 + N_1}$$

The isomorphism theorems will again imply that other ways of getting to these squares will be equivalent.

Now in our case each  $A_s$  has two independent and infinite filtrations. One is the increasing filtration by  $\ker(A_s \rightarrow A_{s+n})$ , and the other is the decreasing filtration by  $\text{im}(A_{s-n} \rightarrow A_s)$ . We follow the convention that the kernel filtration increases as we go to the right, and the image filtration increases as we go down. Then  $A_s$  will be drawn as



where the region to the left of the vertical dotted line is the union of the kernels and the region above the horizontal dotted line is the intersection of the images. Thus, the dotted lines are being used to signify infinite filtrations. The regions that are identified by  $i_s$  via the first isomorphism theorem are the ones below



Also note that the rightmost vertical region in  $A_s$  is the coimage of the natural map  $i_{s,\infty} : A_s \rightarrow A_\infty$ . Since  $A_\infty$  is the union  $\bigcup_s \text{im } i_{s,\infty}$  and the first isomorphism theorem says that  $\text{im } i_{s,\infty} \cong \text{coim } i_{s,\infty}$ , figuring out the group that that region represents would be sufficient for what we want.

We now come to the  $E_s$ . By the exactness of the sequence

$$A_s \rightarrow E_s \rightarrow A_{s-1}$$

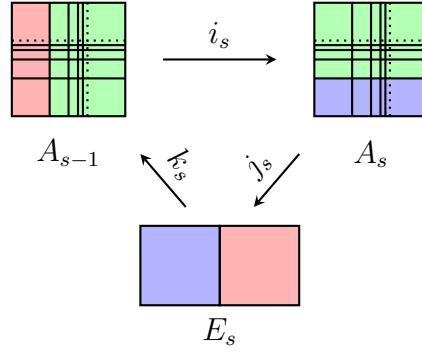
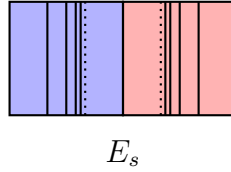


Figure 1.1: Each exact triangle

the kernel and image filtrations (both being filtrations of length 1) induced on  $E_s$  coincide, and thus there is a single filtration. Therefore, the exact triangle could be drawn as in Fig. 1.1, where the like colors are glued. The colored regions in  $E_s$  now inherit filtrations from these isomorphisms, and we get the following diagram for  $E_s$ :



Thus, by definition of this filtration, the rightmost blue region here is the bottom right region in  $A_s$  in Fig. 1.1, and the leftmost red region is the top left region of  $A_{s-1}$ . The former is thus isomorphic to a subquotient of  $A_\infty$ , and the latter to one of  $A_{-\infty}$ . These are the regions that the spectral sequence will allow us to find. Each derivation replaces  $E_s$  with a subquotient of it (the one defined by taking homology with respect to  $d''$ ). It can be checked that this subquotient is precisely the one that throws away the outermost 2 regions of  $E_s$ . Thus at the ‘ $\infty$ -th’ page,  $E_s^\infty$ , not yet formally defined, will be the subquotient represented by the middle two regions.

The couple  $(A_s^2, E_s^2, i_s^2, j_s^2, k_s^2)$  can be drawn as a subquotient of the original couple as in Fig. 1.2, where the gray regions are the ones that were removed.

Again, like colors are being identified by the maps shown. The green color on  $A_{s-1}^2$  is supposed to be glued to one on  $A_s^2$ , but that has been omitted because of overlaps of colors. This diagram can be used to give an alternate proof of Proposition 1.3 for the case of the unrolled couple if one follows the identifications. Each subsequent derivation throws away the outermost remaining regions of each  $E_s^r$  and the bottommost remaining region of each  $A_s^r$ . This gives a clear picture of how we should define the  $\infty$ -th derivation, which may be drawn as in Fig. 1.3 (identifications omitted, gray denotes discarded regions).

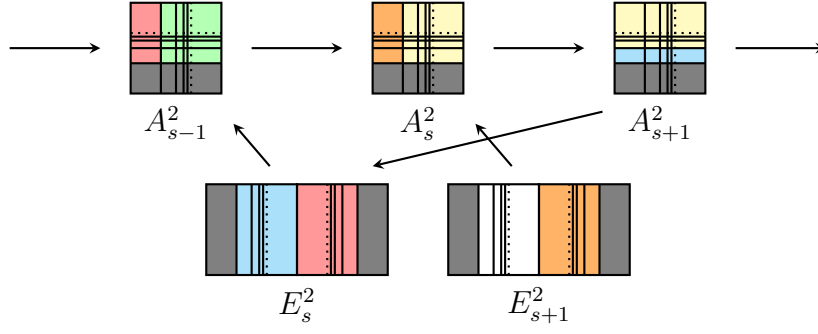
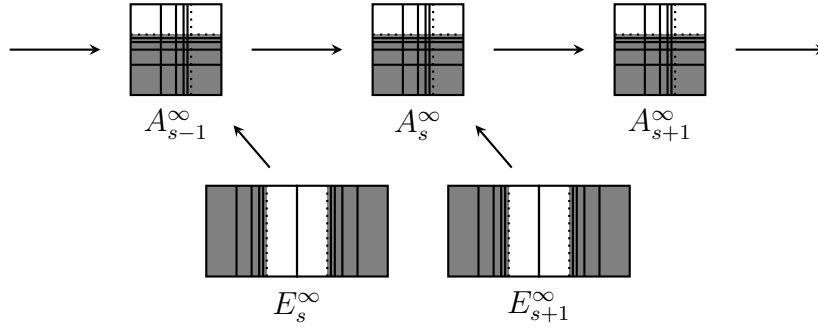


Figure 1.2: The derived couple, unrolled


 Figure 1.3:  $\infty$ -th derivation of the couple

Algebraically, all of this is summarized in the following theorem.

**Theorem 1.7.** Let  $(A_s, E_s, i_s, j_s, k_s)_{s \in \mathbb{Z}}$  be an unrolled exact couple, and  $(A, E, \alpha, \beta, \gamma)$  its corresponding rolled exact couple. Then  $E_s$  has a filtration

$$0 \subset B_s^1 \subset B_s^2 \subset \cdots \subset B_s^\infty \subset \ker k_s = \operatorname{im} j_s \subset Z_s^\infty \subset \cdots \subset Z_s^2 \subset Z_s^1 \subset E_s$$

where

$$B_s^r = j_s \ker(i_{s+r} \circ \cdots \circ i_{s+1}), \quad B_s^\infty = \lim_{\rightarrow} B_s^r = \bigcup_{r=1}^{\infty} B_s^r$$

$$Z_s^r = k_s^{-1} \operatorname{im}(i_{s-1} \circ \cdots \circ i_{s-r}), \quad Z_s^\infty = \lim_{\leftarrow} Z_s^r = \bigcap_{r=1}^{\infty} Z_s^r$$

such that  $E_s^{r+1} \cong \frac{Z_s^r}{B_s^r}$ . The differential  $d_s^r : E_{s+r}^r \rightarrow E_s^r$  is defined as

$$[x] \mapsto [j_s(i_{s+1} \circ \cdots \circ i_{s+r})^{-1} k_s]$$

*Proof.* This can be proved by repeated applications of the first isomorphism theorem, following what the above figures illustrate. See [Rog, Lemma 2.3.1].  $\square$

**Definition 1.8.** The  $\infty$ -th page of a spectral sequence associated to an exact couple is defined as

$$E^\infty = \frac{Z^\infty}{B^\infty}$$

with notation as in Theorem 1.7.

Thus  $E_s^\infty$  is left with 2 regions as shown in Fig. 1.3. We have identifications as shown in Fig. 1.4, which are restrictions of the identifications in Fig. 1.1. The

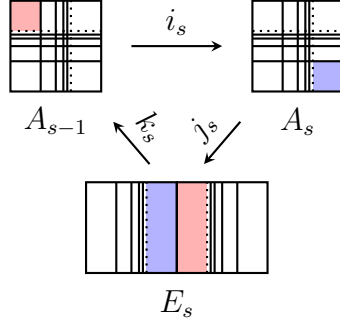


Figure 1.4: Restrictions of the identifications from Fig. 1.1

blue region in  $A_s$  is also identified with

$$\frac{\text{im}(A_s \rightarrow A_\infty)}{\text{im}(A_{s-1} \rightarrow A_s)}$$

This is because the rightmost vertical region in  $A_s$  is identified with the image of the map to  $A_\infty$ , and the image of  $A_{s-1}$  in  $A_\infty$  is exactly the image of the image of  $A_{s-1}$  in  $A_s$ . A similar argument identifies the red region with

$$\frac{\ker(A_\infty \rightarrow A_s)}{\ker(A_\infty \rightarrow A_{s-1})}$$

This information is most useful when either the red or the blue region in  $E_s$  is 0, because then the  $\infty$ -th page of the sequence is giving us the successive quotients of  $A_{\pm\infty}$ . This holds, in particular, when the hypotheses of the next theorem are satisfied.

**Theorem 1.9.** Let  $(A_s, E_s, i_s, j_s, k_s)_{s \in \mathbb{Z}}$  be an unrolled exact couple. If  $A_{-\infty} = 0$ , then

$$E_s^\infty \cong \frac{\text{im}(A_s \rightarrow A_\infty)}{\text{im}(A_{s-1} \rightarrow A_\infty)}$$

$\bigcap_s \text{im}(A_s \rightarrow A_\infty) = 0$  and  $\bigcup_s \text{im}(A_s \rightarrow A_\infty) = A_\infty$ . If  $A_\infty = 0$ , then

$$E_s^\infty \cong \frac{\ker(A_\infty \rightarrow A_s)}{\ker(A_\infty \rightarrow A_{s-1})}$$

$\bigcap_s \ker(A_\infty \rightarrow A_s) = 0$  and  $\bigcup_s \ker(A_\infty \rightarrow A_s) = A_{-\infty}$ . □

This is what we mean when we say that the spectral sequence converges to  $A_{\pm\infty}$ .

**Definition 1.10.** A spectral sequence  $(E^r, d^r)$  associated to an exact couple  $(A_s, E_s, i_s, j_s, k_s)_{s \in \mathbb{Z}}$  is said to converge to a graded abelian group  $X_\bullet$  if there exists a filtration

$$0 \subset \cdots \subset X_{-1,\bullet} \subset X_{0,\bullet} \subset X_{1,\bullet} \subset \cdots \subset X_\bullet$$

with  $\bigcap_s X_s = 0$  and  $\bigcup_s X_s = X$ , along with isomorphisms

$$E_{s,t}^\infty \cong \frac{X_{s,t}}{X_{s-1,t}}$$

In this case, we say

$$E_{s,t}^r \Rightarrow X_t$$

**Example 1.11.** Consider the exact couple of singular homology groups of a CW complex  $X$  filtered by its skeleta. This looks similar to the exact couple in the introduction to this chapter, but with  $H_*(C_s)$  replaced with  $H_*(X_s)$  and  $H_*(C_s/C_{s-1})$  with  $H_*(X_s, X_{s-1})$  a spectral sequence whose  $E^1$  term, under the default grading, is

$$E_{s,t}^1 = H_s(X_t, X_{t-1}) \cong H_s(X_t/X_{t-1})$$

Since this quotient space is a wedge of  $s$ -spheres, this group is non-zero only if  $s = t$ . The map  $d^1 : E_{s,t}^1 \rightarrow E_{s-1,t-1}^1$  is the composite

$$H_*(X_s, X_{s-1}) \xrightarrow{\partial} H_*(X_{s-1}) \rightarrow H_*(X_{s-1}, X_{s-2})$$

which is exactly the definition of the boundary map in cellular homology. Thus the  $E^1$  page is precisely the cellular chain complex of  $X$  sitting on the diagonal. Because of degree reasons, all  $d^r$  for  $r \geq 2$  must be 0, hence  $E^\infty$  is exactly the cellular homology groups  $H_s^{\text{CW}}(X)$  sitting in bidegree  $(s, s)$ . Moreover,  $A_{-\infty} = 0$ . By Theorem 1.9, there must be a filtration

$$0 \subset \cdots \subset M_{-1,*} \subset M_{0,*} \subset M_{1,*} \subset \cdots \subset H_*(X)$$

such that  $E_{s,t}^\infty \cong M_{s,t}$  for each  $s, t \in \mathbb{Z}$ . This can be visualized on the grid as saying that the  $E_{s,t}^\infty$  are the successive quotients of some filtration on  $H_t(X)$ . Since  $E_{s,t}^\infty = 0$  for  $s \neq t$ , this implies

$$H_t(X) = E_{s,s}^\infty = H_t^{\text{CW}}(X)$$

which gives a proof of the agreement of singular and cellular homologies.

## 1.3 Multiplicative Spectral Sequences

The most difficult part of computing with a spectral sequence is to find the differentials  $d^r$ . Trying to do this from the definition of these maps is nearly impossible because of how long winded the definition is and how many identifications are made along the way. Usually, for  $r = 1$ , there would be some geometric interpretation of  $d^r$ . For example, in Example 1.11, we have an interpretation of  $d^1$  in terms of the cellular attaching maps. This often leads to a geometric description of the groups that appear on the  $E^2$  page but not of the differentials. Here, one usually tries to show that certain differentials must be 0 or isomorphisms either because of degree reasons or from partial knowledge of what is being computed by the sequence.

There is one more powerful tool that can be used to determine the differentials - a multiplicative structure on the spectral sequence. This means that each page of spectral sequence comes with a multiplication

$$E_{s_1, t_1}^r \otimes E_{s_2, t_2}^r \rightarrow E_{s_1+s_2, t_1+t_2}^r$$

satisfying the Leibniz rule

$$d(x \cdot y) = x \cdot d(y) + d(x) \cdot y$$

such that the multiplication on each successive page  $E^{r+1}$  is defined by

$$[x] \cdot [y] \rightarrow [x \cdot y]$$

where  $[-]$  denotes equivalence class when  $E^{r+1}$  is identified with  $H(E^r, d^r)$ . This is often used to figure out more differentials from already known ones.

Since the multiplication on each page is completely determined by the one on the previous page, it suffices to figure out the multiplication on  $E^1$ . However, any arbitrary multiplication on  $E^1$  may not descend to the later pages in a well defined manner, so constructing a multiplicative structure on a spectral sequence is harder than just producing a multiplication on the  $E^1$ . It involves using a lot more structure than we get from just the exact couple. Start again with a filtered chain complex  $C$

$$0 = C_{-\infty, *} \subset \cdots \subset C_{-1, *} \subset C_{0, *} \subset C_{1, *} \subset \cdots \subset C_{\infty, *} = C$$

Now instead of taking only the homologies of its successive quotients  $C_{s, *}/C_{s-1, *}$ , we could take homologies of all quotients  $C_{j, *}/C_{i, *}$  where  $i, j \in \mathbb{Z} \cup \{\pm\infty\}$ . If  $i \geq j$ , we define this quotient to be 0. Indexing by  $(*, s)$ , we get a bigraded abelian group  $K_*(i, j) = H_*(C_j/C_i)$  that lives only above the  $i = j$  line.

$K_*$  then has extra structure. Whenever  $i \leq i'$  and  $j \leq j'$ , there is a natural map

$$\eta_{i, j}^{i', j'} : K_*(i, j) \rightarrow K_*(i, j') \rightarrow K_*(i', j')$$

### 1.3 Multiplicative Spectral Sequences

where the first map is induced by the inclusion and the second one by further quotient. For each  $i \leq j \leq k$ , the snake lemma gives a map

$$\partial_{i,j,k} : K_*(j, k) \rightarrow K_{*-1}(i, j)$$

such that the following triangle in this grid is always exact:

$$\cdots \xrightarrow{\partial} K_*(i, j) \xrightarrow{\eta} K_*(i, k) \xrightarrow{\eta} K_*(j, k) \xrightarrow{\partial} K_{*-1}(i, j) \xrightarrow{\eta} \cdots$$

This structure is called a *Cartan-Eilenberg* system and it contains all the data of the exact couple and more. A precise definition can be found in [Rog, Section 3.5].

This structure is helpful here because the groups  $E_{s,t}^r$  that appear on the  $r$ th page of the associated spectral sequence can be identified with quotients of the groups that appear  $r$  units above the diagonal. More specifically,  $E_{s,t}^r$  is isomorphic to a quotient of  $K_t(s, s+r)$  (see [Rog, Proposition 3.5.9]). This lets us construct structure on the later pages of the spectral sequences from structure present on the Cartan-Eilenberg system.

The multiplicative structure on the spectral sequence associated to a Cartan-Eilenberg system is often induced from a pairing on the system. This consists of graded multiplication maps  $r$  units above the diagonal

$$\mu_r : K_*(i, i+r) \otimes K_*(j, j+r) \rightarrow K_*(i+j, i+j+r)$$

These maps are required to commute with  $\eta$  and satisfy one more equation that gives us the Leibniz rule (see [Rog, Definition 6.2.1]). Given such a pairing on a Cartan-Eilenberg system, [Rog, Theorem 6.2.3] proves that we get a multiplicative spectral sequence. In these cases, we also have multiplicative structure on  $A_{\pm\infty}$ , the targets of convergence of the spectral sequence, and the isomorphisms between the successive quotients of these targets and the groups at the  $\infty$ th page are ring isomorphisms.

## 2 Model Categories

Consider the classical homotopy theory in the category  $\mathbf{Top}$ . We have the homotopy equivalence relation  $\sim$ , which allows us to define a quotient functor

$$q : \mathbf{Top} \rightarrow \mathbf{Top}/\sim$$

One key observation is that a functor from  $\mathbf{Top}$  respects the homotopy relation if and only if it takes homotopy equivalences to isomorphisms, and hence  $q$  is universal among both functors that respect the homotopy relation and functors that take homotopy equivalences to isomorphisms. For this reason, we may also write  $\mathbf{Top}/\sim$  as  $\mathbf{Top}[\mathcal{H}^{-1}]$ , where  $\mathcal{H}$  denotes the subcategory of homotopy equivalences. However, every homotopy invariant known to algebraic topology also takes weak homotopy equivalences to isomorphisms, so it is of greater interest to study a functor  $h : \mathbf{Top} \rightarrow \mathbf{Top}[\mathcal{W}^{-1}]$  which is universal among those that take all weak homotopy equivalences to isomorphisms.

Some natural questions arise. Is there an equivalence relation  $\sim_w$  of ‘weak homotopy’ such that the weak homotopy equivalences are exactly the maps that are invertible up to weak homotopy? Can the functor  $h$  described above also be viewed as the functor that quotients by  $\sim_w$ ? These questions are important because the morphism sets  $\mathrm{Hom}_{\mathbf{Top}[\mathcal{W}^{-1}]}(X, Y)$  are difficult to understand, whereas we have a simple description of quotient categories. The answer to these questions is ‘no’, but we can get a simple description of  $\mathbf{Top}[\mathcal{W}^{-1}]$  regardless.

The cellular approximation functor  $\Gamma$  takes weak equivalences to weak equivalences, and is naturally weakly equivalent to the identity functor. This implies that it induces an endofunctor on  $\mathbf{Top}[\mathcal{W}^{-1}]$  that is naturally isomorphic to the identity. Therefore, the full image of this endofunctor, which is  $CW[\mathcal{W}^{-1}]$ , is equivalent to  $\mathbf{Top}[\mathcal{W}^{-1}]$ . Whitehead’s theorem then says that this is equivalent to  $CW[\mathcal{H}^{-1}]$ . Thus,

$$\mathrm{Hom}_{\mathbf{Top}[\mathcal{W}^{-1}]}(X, Y) \cong \mathrm{Hom}_{\mathbf{Top}}(\Gamma X, \Gamma Y)/\sim$$

In homotopy theory, we often want to treat maps as isomorphisms even when they are not, and having results similar to the above in more general settings would be convenient to study the resulting categories. As it was discovered, the notion of a homotopy and the proofs of the cellular approximation and Whitehead theorems can be expressed using Serre cofibrations and Serre fibrations, using there lifting and factorization properties. This led Quillen to define model categories in his book [Qui]. A model category is a category with three distinguished



classes of morphisms called weak equivalences, cofibrations, and fibrations, which satisfy enough properties for an analogous discussion to work. In the following section, we recall what features of these classes allowed us to have this discussion in the first place.

## 2.1 Classical homotopy theory

We use notation for lifting calculus borrowed from [Rez, Section 15], in which we write  $f \sqsubset g$  for a pair of maps  $f, g$  in a category  $\mathcal{M}$  if for every commutative square

$$\begin{array}{ccc} \bullet & \xrightarrow{\quad} & \bullet \\ f \downarrow & \nearrow & \downarrow g \\ \bullet & \xrightarrow{\quad} & \bullet \end{array}$$

the indicated lift exists and makes the diagram commute. For a class of maps  $\mathcal{S} \subset \mathcal{M}$ , we define its left and right lifting complements as

$$\sqsubset \mathcal{S} := \{f \in \mathcal{M} \mid f \sqsubset s \ \forall s \in \mathcal{S}\}, \quad \mathcal{S}^\sqsubset := \{g \in \mathcal{M} \mid s \sqsubset g \ \forall s \in \mathcal{S}\}$$

respectively.

Recall that in the category **Top** of (compactly generated, weak Hausdorff) topological spaces, we have the class of weak homotopy equivalences  $\mathcal{W}$ , and the class of Serre fibrations is defined as

$$\mathcal{F} := \{i_0 : D^n \hookrightarrow D^n \times I \mid n \geq 0\}^\sqsubset$$

where  $i_t : X \hookrightarrow X \times I$  will always denote the inclusion  $x \mapsto (x, t)$ . Define the class of Serre cofibrations  $\mathcal{C}$  as the class of retracts of relative cell complexes. The motivation for this comes from the following 2 properties, which are analogous to those of Hurewicz (co)fibrations. The proofs of these and their Hurewicz analogues may be found in [MP, Section 17.1 and 17.2].

**Proposition 2.1.** Any map  $f : X \rightarrow Y$  can be factored in two ways:

- (i)  $p \circ \tilde{i}$  where  $p \in \mathcal{F}$  and  $\tilde{i} \in \mathcal{W} \cap \mathcal{C}$ .
- (ii)  $\tilde{p} \circ i$  where  $\tilde{p} \in \mathcal{W} \cap \mathcal{F}$  and  $i \in \mathcal{C}$ .

Moreover, these factorizations can be made functorial, in the sense that they both give functors from the category of maps in **Top** to the category of composable pairs of maps in **Top**.

**Proposition 2.2.** (i)  $\mathcal{C} = \sqsubset(\mathcal{W} \cap \mathcal{F})$  and  $\mathcal{C}^\sqsubset = \mathcal{W} \cap \mathcal{F}$ .

- (ii)  $(\mathcal{W} \cap \mathcal{C})^\sqsubset = \mathcal{F}$  and  $\mathcal{W} \cap \mathcal{C} = \sqsubset \mathcal{F}$ .

## 2 Model Categories

Proposition 2.1 leads immediately to a weaker version cellular approximation theorem. It is weaker in the sense that the cell complexes need not be constructed by attaching cells in increasing order of dimension. We will see that the discussion in the previous section holds because Whitehead's theorem still holds. The standard CW approximation theorem can also be deduced from this using transfinite induction and the fact that there are no homotopically non-trivial maps from a lower dimensional sphere to a higher dimensional one.

**Theorem 2.3** (Cellular approximation). There exists an endofunctor  $\Gamma$  on  $\mathbf{Top}$  which maps each space to a cell complex, and a natural weak equivalence  $\eta : \Gamma \rightarrow 1_{\mathbf{Top}}$ .

*Proof.* We have a functor  $\mathbf{Top} \rightarrow \mathbf{Map}(\mathbf{Top})$  which maps  $X \mapsto (\emptyset \rightarrow X)$ . Compose this with the second functorial factorization of Proposition 2.1, and denote the image of  $X$  by

$$\emptyset \rightarrow \Gamma X \xrightarrow{\sim} X$$

This gives a definition of both  $\Gamma$  and  $\eta$ . By the remark made in that proof, we can ensure that  $\Gamma X$  is a cell complex.  $\square$

**Theorem 2.4** (Whitehead). A map  $f : X \rightarrow Y$ , where  $X$  and  $Y$  are cell complexes, is a weak equivalence if and only if it is a homotopy equivalence.

*Proof.* We know the backward implication to be true. For the forward implication, the idea is to use the Yoneda lemma on  $\mathbf{Top}_c / \sim$ , where  $\mathbf{Top}_c$  is the full subcategory of cell complexes and  $\sim$  is the homotopy relation. So it suffices to show that for any space  $Z$ , the map

$$(\mathbf{Top} / \sim)(f, Z) : (\mathbf{Top} / \sim)(Y, Z) \rightarrow (\mathbf{Top} / \sim)(X, Z)$$

is a bijection. It is enough to prove that the following two lifts exist whenever the square commutes ( $ev_t : U^I \rightarrow U$  denotes evaluation at  $t$ ).

$$\begin{array}{ccc} X & \longrightarrow & Z^I \\ f \downarrow & \nearrow & \downarrow (ev_0, ev_1) \\ Y & \longrightarrow & Z \times Z \end{array} \qquad \begin{array}{ccc} X & \longrightarrow & Z \\ f \downarrow & \nearrow & \\ Y & \longrightarrow & \end{array}$$

These lifts signify injectivity and surjectivity respectively (The first says that if there are two maps  $Y \rightarrow Z$  such that their precomposites with  $f$  are homotopic, then they are homotopic). Alternatively, we could show that  $(\mathbf{Top}_c / \sim)(Z, f)$  is a bijection, for which it suffices to show that the following lifts exist

$$\begin{array}{ccc} Z \sqcup Z & \longrightarrow & X \\ i_0 + i_1 \downarrow & \nearrow & \downarrow f \\ Z \times I & \longrightarrow & Y \end{array} \qquad \begin{array}{ccc} & \nearrow & X \\ & & \downarrow f \\ Z & \longrightarrow & Y \end{array}$$

In the first two diagrams, the maps on the right side are fibrations, and in the latter two diagrams, the maps on the left side are cofibrations. By Proposition 2.2, we need  $f$  to also be a cofibration or a fibration. We can ensure this by applying either factorization of Proposition 2.1 to  $f$ , as they will both factor  $f$  as  $p \circ i$  with  $p \in \mathcal{W} \cap \mathcal{F}$  and  $i \in \mathcal{W} \cap \mathcal{C}$ . The two cases above show that  $p$  and  $i$  are both homotopy equivalences, therefore their composite is a homotopy equivalence.  $\square$

## 2.2 Model Categories

The previous section proves all the claims about the homotopy category in the introduction to this chapter. Thus, similar claims are true in any category in which there is a notion of weak equivalences and (co)fibrations satisfying similar properties. This leads to the notion of a model category.

**Definition 2.5.** A model category is a category  $\mathcal{M}$  along with subcategories  $\mathcal{W}, \mathcal{C}, \mathcal{F}$  called the subcategories of weak equivalences, cofibrations, and fibrations respectively, satisfying the following axioms:

- (A1)  $\mathcal{M}$  is bicomplete, i.e. each small diagram in  $\mathcal{M}$  has a limit and a colimit.
- (A2) If  $f$  and  $g$  are morphisms of  $\mathcal{M}$  such that  $g \circ f$  is defined, and any two of  $f, g, gf$  are in  $\mathcal{W}$ , then so is the third.
- (A3)  $\mathcal{W}$  is closed under retracts in the category of maps in  $\mathcal{M}$ , i.e. for any commutative diagram

$$\begin{array}{ccccc}
 & & 1 & & \\
 & \searrow & & \nearrow & \\
 A & \longrightarrow & X & \longrightarrow & A \\
 \downarrow f & & \downarrow g & & \downarrow f \\
 B & \longrightarrow & Y & \longrightarrow & B \\
 & \searrow & & \nearrow & \\
 & & 1 & & 
 \end{array}$$

if  $g \in \mathcal{W}$ , then  $f \in \mathcal{W}$ .

- (A4) (i)  $\mathcal{C} = \square(\mathcal{W} \cap \mathcal{F})$  and  $\mathcal{C}^\square = \mathcal{W} \cap \mathcal{F}$ .  
 (ii)  $(\mathcal{W} \cap \mathcal{C})^\square = \mathcal{F}$  and  $\mathcal{W} \cap \mathcal{C} = \square\mathcal{F}$ .
- (A5) Any map  $f$  can be functorially factored in two ways:
  - (i)  $p \circ \tilde{i}$  where  $p \in \mathcal{F}$  and  $\tilde{i} \in \mathcal{W} \cap \mathcal{C}$ .
  - (ii)  $\tilde{p} \circ i$  where  $\tilde{p} \in \mathcal{W} \cap \mathcal{F}$  and  $i \in \mathcal{C}$ .

## 2 Model Categories

The original axioms given by Quillen did not include functoriality of the factorizations. For certain subcategories of  $\text{Map } \mathcal{M}$ , these factorizations can be made functorial with some effort, so none of the theory later on in this chapter breaks. However, it is more convenient to assume factorizations to be functorial and does not result in any loss of generality in practice, so it was later added to the axioms. Another such difference is that Quillen only required the existence of finite (co)limits, but we now require that all small (co)limits exist.

These axioms are self dual in the sense that if  $\mathcal{M}$  is a model category, then so is  $\mathcal{M}^{\text{op}}$  with the same  $\mathcal{W}$ , and swapped  $\mathcal{C}$  and  $\mathcal{F}$ . The rest of the section generalizes everything said in Section 2.1, and because of this self duality, all of definitions and theorems come in pairs. All details for this section can be found in [Hov, Chapter 1]. Let  $\mathcal{W} \subset \mathcal{M}$  be categories, and call the maps of  $\mathcal{W}$  weak equivalences.

**Definition 2.6.** The *homotopy category* of  $(\mathcal{M}, \mathcal{W})$  is a category  $\mathcal{M}[\mathcal{W}^{-1}]$  along with a functor  $\ell : \mathcal{M} \rightarrow \mathcal{M}[\mathcal{W}^{-1}]$  that maps weak equivalences to isomorphisms, such that for every category  $\mathcal{D}$  and functor  $\gamma : \mathcal{M} \rightarrow \mathcal{D}$  that maps weak equivalences to isomorphisms, there exists a unique functor  $\hat{\gamma} : \mathcal{M}[\mathcal{W}^{-1}] \rightarrow \mathcal{D}$  such that  $\gamma$  factors as the composite

$$\mathcal{M} \xrightarrow{\ell} \mathcal{M}[\mathcal{W}^{-1}] \xrightarrow{\hat{\gamma}} \mathcal{D}$$

If the subcategory  $\mathcal{W}$  is clear from the context, we may write  $\text{Ho } \mathcal{M}$  for the homotopy category.

There is a standard construction for  $\text{Ho } \mathcal{M}$  but it is not easy to work with, and it can turn a locally small  $\mathcal{M}$  to a non-locally small  $\text{Ho } \mathcal{M}$ . However, when  $(\mathcal{M}, \mathcal{W})$  extends to a model structure, this gets resolved. For the rest of this section, let  $(\mathcal{M}, \mathcal{W}, \mathcal{C}, \mathcal{F})$  be a model category.

**Definition 2.7.** (i) An object  $X \in \mathcal{M}$  is *cofibrant* if the map from the initial object  $\emptyset$  to  $X$  is a cofibration.  $\mathcal{M}_c$  is the full subcategory of cofibrant objects. Dually, an object  $X \in \mathcal{M}$  is *fibrant* if the map from  $X$  to the terminal object  $*$  is a fibration.  $\mathcal{M}_f$  is the full subcategory of fibrant objects.

(ii) An object  $X \in \mathcal{M}$  is *bifibrant* if it is both cofibrant and fibrant.  $\mathcal{M}_{cf}$  is the full subcategory of bifibrant objects.

(iii) A (co)fibration is called *acyclic* if it is also a weak equivalence.

(iv) A *cofibrant replacement* for  $X \in \mathcal{M}$  is a cofibrant object  $X_c$  with a weak equivalence  $X_c \rightarrow X$ . Dually, a *fibrant replacement* of  $X$  is a fibrant object  $X_f$  with a weak equivalence  $X \rightarrow X_f$ .

- (v) A *cylinder object* for  $X \in \mathcal{M}$  is an object  $\text{Cyl}(X)$  along with maps

$$X \sqcup X \xrightarrow{i} \text{Cyl}(X) \xrightarrow{w} X$$

such that  $i \in \mathcal{C}$ ,  $w \in \mathcal{W}$ , and the composite is the fold map. Dually, a *path object* for  $X \in \mathcal{M}$  is an object  $\text{Path}(X)$  along with maps

$$X \xrightarrow{w} \text{Path}(X) \xrightarrow{p} X \times X$$

such that  $i \in \mathcal{C}$ ,  $w \in \mathcal{W}$ , and the composite is the diagonal map.

- (vi) A *left homotopy* between maps  $f, g : X \rightarrow Y$  is a choice of object  $\text{Cyl}(X)$  along with a map  $\text{Cyl}(X) \rightarrow Y$  such that the composite

$$X \sqcup X \xrightarrow{i} \text{Cyl}(X) \rightarrow Y$$

is  $f + g$ . Dually, a *right homotopy* between maps  $f, g : X \rightarrow Y$  is a choice of object  $\text{Path}(Y)$  along with a map  $X \rightarrow \text{Path}(Y)$  such that the composite

$$X \rightarrow \text{Path}(Y) \xrightarrow{p} Y \times Y$$

is  $(f, g)$ .

- (vii)  $f, g : X \rightarrow Y$  are *homotopic*, denoted  $f \sim g$ , if there exists a left homotopy as well as a right homotopy between them.

Note that functorial (co)fibrant replacements can be constructed by applying (A5) to the maps  $\emptyset \rightarrow X \rightarrow *$ . These will be called the *canonical (co)fibrant replacements*. These have the special feature that the weak equivalence between the object and its replacement is also a fibration or a cofibration. This can be used to compare two different replacements directly through a weak equivalence rather than through a zigzag of them.

**Lemma 2.8.** Let  $Q, R$  denote the canonical cofibrant and fibrant replacements functors respectively. Suppose we have cofibrant and fibrant replacements  $X_c$  and  $X_f$  respectively of  $X \in \mathcal{M}$ . Then there exist weak equivalences  $X_c \rightarrow QX \rightarrow RQX$  and  $QRX \rightarrow RX \rightarrow X_f$ .

*Proof.* We prove this only in the case of  $X_c$ . The weak equivalence  $QX \rightarrow RQX$  is clear. For  $X_c \rightarrow QX$ , take the lift

$$\begin{array}{ccc} \emptyset & \longrightarrow & QX \\ \downarrow & \nearrow \text{dashed} & \downarrow \\ X_c & \longrightarrow & X \end{array}$$

and apply (A2) to prove that the lift is a weak equivalence. □

## 2 Model Categories

Similarly,  $\text{Cyl}(X)$  and  $\text{Path}(X)$  always exist using (A5), but are only defined up to weak equivalence. This notion of homotopy does not behave as expected in all cases, but the following proposition shows that it is well behaved whenever the domains of all involved maps are cofibrant, and all the codomains are fibrant.

**Proposition 2.9.** (i) Let  $X \in \mathcal{M}$  be cofibrant and  $Y \in \mathcal{M}$  be fibrant.

If there exists either a left homotopy or a right homotopy between  $f, g : X \rightarrow Y$ , then  $f \sim g$ . Moreover, any choice of cylinder (path) object for  $X$  admits a left (right) homotopy through it.

(ii) For  $X$  cofibrant and  $Y$  fibrant,  $\sim$  is an equivalence relation on  $\mathcal{M}(X, Y)$ .

(iii) For  $X, Y$  cofibrant,  $Y, Z$  fibrant, and homotopic pairs of maps

$$f_1, g_1 : X \rightarrow Y, \quad f_2, g_2 : Y \rightarrow Z$$

we have  $f_2 f_1 \cong g_2 g_1$ .

(iv) A functor  $\mathcal{M}_{cf} \rightarrow \mathcal{D}$  for any category  $\mathcal{D}$  maps homotopy equivalences to isomorphisms if and only if it respects  $\sim$ .

*Proof.* For (i)-(iii), see [Hov, Proposition 1.2.5]. For (iv), see [Hov, Corollary 1.2.9].  $\square$

**Theorem 2.10.** There is a functor  $\Gamma : \mathcal{M} \rightarrow \mathcal{M}_{cf}$  connected to  $1_{\mathcal{M}}$  via some zigzag of weak equivalences. In particular, there is a functor  $Q : \mathcal{M} \rightarrow \mathcal{M}$  with natural weak equivalences

$$\Gamma \xleftarrow{\alpha} Q \xrightarrow{\eta} 1_{\mathcal{M}}$$

The inclusion  $\mathcal{M} \rightarrow \mathcal{M}_{cf}$  is omitted from the notation.

*Proof.* Let  $Q$  denote the cofibrant replacement functor obtained by factoring  $\emptyset \rightarrow X$  and let  $R$  denote the fibrant replacement functor obtained by factoring  $X \rightarrow *$ . Then there is a natural acyclic cofibration  $\eta : 1_{\mathcal{M}} \rightarrow R$ . Take  $\Gamma = RQ$  and  $\alpha = \varepsilon Q$ .  $\square$

Dually, we could have taken  $\Gamma = QR$ , in which case both weak equivalences would need to be flipped. In either case, we have

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\Gamma} & \mathcal{M}_{cf} \\ \downarrow & & \downarrow \\ \mathcal{M}[\mathcal{W}^{-1}] & \dashrightarrow & \mathcal{M}_{cf}[(\mathcal{W} \cap \mathcal{M}_{cf})^{-1}] \end{array}$$

where the dotted arrow is induced by the  $\Gamma$  using the universal property of  $\mathcal{M}[\mathcal{W}^{-1}]$ . The universal property applies because  $\Gamma$  maps weak equivalences to weak equivalences, which may be checked using (A2) and the natural weak equivalences of Theorem 2.10. The inclusion of  $\mathcal{M}_{cf}$  into  $\mathcal{M}$  induces a functor the other way. These two functors form an equivalence of categories, where the relevant natural isomorphisms come from the natural weak equivalences of Theorem 2.10. We have proved

**Corollary 2.11.** The bifibrant replacement  $\Gamma$  induces an equivalence of categories

$$\mathrm{Ho} \mathcal{M} \simeq \mathrm{Ho} \mathcal{M}_{cf}$$

so for any  $X, Y \in \mathcal{M}$ , we have an isomorphism

$$(\mathrm{Ho} \mathcal{M})(X, Y) \cong (\mathrm{Ho} \mathcal{M})(\Gamma X, \Gamma Y)$$

A generalization of Whitehead's theorem describes  $\mathcal{M}(\Gamma X, \Gamma Y)$ .

**Theorem 2.12.** Let  $X, Y \in \mathcal{M}$  be bifibrant, and  $f : X \rightarrow Y$ . Then  $f$  is a weak equivalence if and only if it is a homotopy equivalence.

*Proof.* The forward implication is just as in Theorem 2.4. This time, the backward implication requires some work. This is where (A3) is used. See [Hov, Proposition 1.2.8].  $\square$

Thus a functor on  $\mathcal{M}_{cf}$  maps weak equivalences to isomorphisms if and only if it maps homotopy equivalences to isomorphisms if and only if it respects the homotopy relation, proving that  $\mathrm{Ho} \mathcal{M} \simeq \mathrm{Ho} \mathcal{M}_{cf} \simeq \mathcal{M}_{cf} / \sim$ .

**Theorem 2.13** (Fundamental theorem). Let  $\mathcal{M}$  be a model category with a bifibrant replacement functor  $\Gamma$ . Then  $\mathrm{Ho} \mathcal{M}$  has the same objects as  $\mathcal{M}$  with morphism sets

$$\mathrm{Ho} \mathcal{M}(X, Y) \cong \mathcal{M}(\Gamma X, \Gamma Y) / \sim \cong \mathcal{M}(X_c, Y_f) / \sim$$

where  $X_c$  is any cofibrant replacement of  $X$  and  $Y_f$  is any fibrant replacement of  $Y$ .  $\square$

*Proof.* Only the last isomorphism is new. It follows from [Hov, Proposition 1.2.5(iv)] applied to the weak equivalences of Lemma 2.8.  $\square$

## 2.3 Functors of Model Categories

The purpose of a model category is to aid in the study of its homotopy category, so a functor of model categories should be defined such that it also induces a

## 2 Model Categories

functor of the homotopy categories. Let  $F : \mathcal{M} \rightarrow \mathcal{N}$  be an ordinary functor of model categories. To induce a functor  $\mathrm{Ho} F : \mathrm{Ho} \mathcal{M} \rightarrow \mathrm{Ho} \mathcal{N}$ , it needs to at least map homotopic maps of bifibrant objects to homotopic maps. This leads to two kinds of functors, ones that preserve left homotopies and ones that preserve right homotopies. It is important to note that most functors in practice do not preserve weak equivalences, which is why we need a more general notion of a functor.

Let  $f, g : X \rightarrow Y$  be maps between bifibrant objects in  $\mathcal{M}$ , and  $H : \mathrm{Cyl}(X) \rightarrow Y$  a left homotopy between them. By a functor  $F : \mathcal{M} \rightarrow \mathcal{N}$  preserving the left homotopy  $H$ , we mean that  $FH$  is a left homotopy between  $Ff, Fg : FX \rightarrow FY$ . For this,  $F \mathrm{Cyl}(X)$  needs to be a cylinder object for  $FX$ . Thus  $F$  should preserve the coproduct  $X \sqcup X$ , the cofibration  $X \sqcup X \rightarrow \mathrm{Cyl}(X)$ , and the weak equivalence  $\mathrm{Cyl}(X) \rightarrow X$ . A dual discussion could be carried out for right homotopies. Since preserving (co)products is involved, it is more convenient, and usually sufficient, to deal with adjoint pairs. Recall that  $Q$  and  $R$  denote the cofibrant and fibrant replacements respectively.

**Definition 2.14.** An adjunction  $F : \mathcal{M} \rightleftarrows \mathcal{N} : U$  is a *Quillen adjunction* if the left adjoint  $F$  preserves cofibrations and the right adjoint  $U$  preserves fibrations.

This ensures that the other conditions for  $F$  and  $U$  to preserve left and right homotopies respectively are satisfied (see [Rie, Lemma 11.3.11 and 11.3.14]; note that  $\mathrm{Cyl}(X)$  is cofibrant as it receives a cofibration from a cofibrant object). In fact, this gives the following stronger result. Ken Brown's lemma (see [Rie, Lemma 11.3.14]) implies that  $F$  preserves weak equivalences between cofibrant objects, so  $F \circ Q$ , where  $Q$  is a functorial cofibrant replacement, induces a functor  $\mathrm{Ho}(FQ) : \mathcal{M} \rightarrow \mathcal{N}$ . Similarly,  $U \circ R$  induces  $\mathrm{Ho}(UR) : \mathcal{N} \rightarrow \mathcal{M}$  where  $R$  is a functorial fibrant replacement. These are called the *total left and right derived functors* of  $F$  and  $U$ , and denoted  $\mathbb{L}F$  and  $\mathbb{R}U$ , respectively. This is a natural definition because of [Rie, Theorem 2.2.8], which says that  $\mathbb{L}F = \mathrm{Ran}_\ell F$  and  $\mathbb{R}U = \mathrm{Lan}_\ell U$ , where  $\ell$  is the functor to the homotopy category (see Definition 2.6). In this sense, these are the universal homotopical approximations of  $F$  and  $U$ .

**Theorem 2.15.** The total derived functors of the Quillen adjunction of Definition 2.14 form an adjunction  $\mathbb{L}F \dashv \mathbb{R}G$ .

*Proof.* See [Hov, Lemma 1.3.10]. □

We call this the derived adjunction of  $F \dashv U$ . Now we consider equivalences of model categories. They should induce equivalences on the homotopy categories. Since any categorical equivalence can be upgraded to an adjoint equivalence ([Lan, Theorem IV.4.1]), we will not lose much generality by defining them to be Quillen adjunctions satisfying the necessary conditions.



**Definition 2.16.** A *Quillen equivalence* is a Quillen adjunction  $F : \mathcal{M} \rightleftarrows \mathcal{N} : U$ , such that for every cofibrant  $X \in \mathcal{M}$  and fibrant  $Y \in \mathcal{N}$ , the bijection

$$\mathcal{M}(FX, Y) \cong \mathcal{N}(X, UY)$$

restricts to a bijection of weak equivalences.

**Theorem 2.17.** The derived adjunction of a Quillen adjunction is an adjoint equivalence if and only if it is a Quillen equivalence.

*Proof.* See [Hov, Proposition 1.3.13]. □

## 2.4 Constructing Model Structures

Out of all the axioms, (A5) is the most difficult to verify. For it, we need the small object argument, which is summarized below. A detailed exposition can be found in [Hov, Section 2.1]. First, we have the following generalization of a relative cell complex.

**Definition 2.18.** Let  $\mathcal{I}$  be a class of maps in a category  $\mathcal{M}$ . A *relative  $\mathcal{I}$ -cell complex* is a map that can be constructed as a transfinite composite of pushouts of coproducts of maps in  $\mathcal{I}$ . The class of relative  $\mathcal{I}$ -cell complexes is denoted by  $\overline{\mathcal{I}}$ .  $X \in \mathcal{M}$  is called a *cellular object* if  $(\emptyset \rightarrow X) \in \overline{\mathcal{I}}$ .

**Definition 2.19.** Let  $\mathcal{I}$  be a class of maps in a category  $\mathcal{M}$ . An object  $A \in \mathcal{M}$  is *small relative to  $\mathcal{I}$*  if there exists an ordinal  $\kappa$  such that for all regular cardinals  $\gamma > \kappa$  and directed systems  $(B_\beta)_{\beta < \gamma}$  indexed by  $\gamma$  and containing only the maps from  $\mathcal{I}$ , the natural map below is a bijection

$$\operatorname{colim}_\beta \operatorname{Hom}(A, B_\beta) \rightarrow \operatorname{Hom}(A, \operatorname{colim}_\beta B_\beta)$$

Then  $A$  is small in the sense that a map to colimit of the directed system must be factor through some stage in the system itself.

**Theorem 2.20** (Small object argument). Let  $\mathcal{I}$  be a set of maps in a category  $\mathcal{M}$  such that each of its domains is small relative to  $\overline{\mathcal{I}}$ . Then any map  $f \in \mathcal{M}$  factors functorially as a map in  $\overline{\mathcal{I}}$  followed by a map in  $\mathcal{I}^\square$ .

*Proof.* The idea is to factor  $f$  as  $p \circ i$  through some  $i \in \overline{\mathcal{I}}$  in a way that ‘formally adjoints solutions’ to all the lifting problems involving  $f$  to the factor  $p$ . This factor now may have new possible lifting problems of its own so we might need to continue this process (transfinitely) inductively. The smallness hypothesis on the domains of  $\mathcal{I}$  then ensures that there is a large enough ordinal  $\kappa$  such that any lifting problems against  $\mathcal{I}$  involving the second factor at the  $\kappa$ -th stage factor

## 2 Model Categories

through some earlier stage and hence have a solution in the next stage. See [Hov, Theorem 2.1.14] for details.  $\square$

Since  $\overline{\mathcal{I}}$  is contained in  ${}^{\square}(\mathcal{I}^{\square})$  (use (A4)), applying it to a set  $\mathcal{I}$  of ‘generating cofibrations’ and a set  $\mathcal{J}$  of ‘generating acyclic cofibrations’ would give us the desired factorizations.

**Theorem 2.21.** Let  $\mathcal{M}$  be a category,  $\mathcal{W}$  a subcategory, and  $\mathcal{I}$  and  $\mathcal{J}$  sets of maps. Let  $\mathcal{F} = \mathcal{J}^{\square}$  and  $\mathcal{C} = {}^{\square}(\mathcal{I}^{\square})$ . If these definitions satisfy the model category axioms (A1) to (A4), and the domains of  $\mathcal{I}, \mathcal{J}$  are small relative to  $\overline{\mathcal{I}}, \overline{\mathcal{J}}$  respectively, then  $(\mathcal{M}, \mathcal{W}, \mathcal{C}, \mathcal{F})$  is a model category.

*Proof.* This is direct from the the small object argument.  $\square$

Such a model category is called a *cofibrantly generated model category*. The small object argument says that not only do we have a cofibrant replacement, but we can in fact have a functorial cellular approximation. In fact, every cofibration is a retract of a relative  $\mathcal{I}$ -cell.

**Proposition 2.22.** Let  $(\mathcal{M}, \mathcal{W}, \mathcal{C}, \mathcal{F})$  be a cofibrantly generated model category with generating sets  $\mathcal{I}$  and  $\mathcal{J}$ . Then  $\mathcal{C}$  is exactly the subcategory of retracts (in the category of maps in  $\mathcal{M}$ ) of the maps in  $\overline{\mathcal{I}}$  and  $\mathcal{C} \cap \mathcal{W}$  is the subcategory of retracts of the maps in  $\overline{\mathcal{J}}$ .

*Proof.* For any  $f \in \mathcal{C}$ , factor it as a map in  $\overline{\mathcal{I}}$  followed by one in  $\mathcal{I}^{\square}$ . Show that  $f$  is then a retract of the former. See [Hov, Lemma 1.1.9] for the last step.  $\square$

This justifies the definition of Serre cofibrations given before Proposition 2.1.

Given sets of maps  $\mathcal{W}, \mathcal{I}, \mathcal{J}$  in  $\mathcal{M}$ , the hardest part of checking the hypotheses of Theorem 2.21 is (A4) and that the small object argument can be applied. Since  $\mathcal{C}$  and  $\mathcal{F}$  are already defined using some lifting properties, the number of checks to be made for (A4) can be significantly reduced. See [Hov, Theorem 2.1.19] for a precise statement.

**Example 2.23.** As discussed in the introduction to this chapter, **Top** has two model structures of interest. In the *Strøm model structure*,  $\mathcal{W}$  is the subcategory of homotopy equivalences,  $\mathcal{C}$  the Hurewicz cofibrations, and  $\mathcal{F}$  the Hurewicz fibrations. Then every object is bifibrant, a cylinder object for  $X$  is  $X \times I$ , and a path object for  $Y$  is  $Y^I$ . This model structure is not cofibrantly generated, and the verification of the axioms is done using point-set level characterizations of the cofibrations. On the other hand, the *Quillen model structure* with  $\mathcal{W}$  weak equivalences, and  $\mathcal{C}$  and  $\mathcal{F}$  the Serre (co)fibrations, has generating sets

$$\mathcal{I} = \{S^n \hookrightarrow D^{n+1} \mid n \geq 0\}, \quad \mathcal{J} = \{i_0 : D^n \hookrightarrow D^n \times I \mid n \geq 0\}$$

All objects are fibrant, but only the retracts of cell complexes are cofibrant. The same choices of cylinder and path objects work.

**Example 2.24.** The suspension-loop adjunction on  $\mathbf{Top}_*$  is a Quillen adjunction and hence it descends to an adjunction on  $\mathbf{Ho Top}_*$ . The total left derived functor  $\mathbb{L}\Sigma$  of the suspension functor is computed by cofibrant replacement and then ordinary suspension - hence it coincides with the suspension on cofibrant objects, and in particular  $(\mathbb{L}\Sigma)S^n = S^{n+1}$ .

We will use two other ways to produce model structures from existing ones. The first is to transfer a model structure along an adjunction. The other, Bousfield localization, is treated in the next section.

**Theorem 2.25.** (Transferred model structure) Let  $(\mathcal{M})$  be a cofibrantly generated model category with generating sets  $\mathcal{I}$  and  $\mathcal{J}$ ,  $\mathcal{N}$  any bicomplete category, and suppose there is an adjunction  $F : \mathcal{M} \rightleftarrows \mathcal{N} : U$ . If

- (i)  $F\mathcal{I}$  and  $F\mathcal{J}$  permit the small object argument, and
- (ii)  $U$  takes relative  $F\mathcal{J}$ -cell complexes to weak equivalences in  $\mathcal{M}$ ,

then there exists a cofibrantly generated model structure on  $\mathcal{N}$  with generating sets  $F\mathcal{I}$  and  $F\mathcal{J}$ , weak equivalences  $U^{-1}\mathcal{W}$  and fibrations  $U^{-1}\mathcal{F}$ .

*Proof.* Check that  $F\mathcal{I}$ ,  $F\mathcal{J}$  and  $U^{-1}\mathcal{W}$  satisfy the conditions for Theorem 2.21 (the refined statement cited in the paragraph after the theorem is useful here). Then, via the adjunction,  $(F\mathcal{J})^\square = U^{-1}\mathcal{F}$  is equivalent to  $\mathcal{J}^\square = \mathcal{F}$  so the fibrations are as stated.  $\square$

**Remark 2.26.** With the transferred model structure on  $\mathcal{N}$  in the above scenario, the adjunction  $F \dashv U$  becomes a Quillen adjunction, but not necessarily a Quillen equivalence. Also note that there may exist other model structures on  $\mathcal{N}$  for which this adjunction is a Quillen adjunction, so the transferred model structure is not the unique one with this property.

**Corollary 2.27** (Projective model structure). Let  $\mathcal{M}$  be a cofibrantly generated model category with generating sets  $\mathcal{I}$  and  $\mathcal{J}$  and let  $\mathcal{D}$  be a small category. Then the functor category  $\mathcal{M}^{\mathcal{D}}$  has a cofibrantly generated model structure in which the weak equivalences and fibrations are level-wise. The generating sets are given by the images of the generating sets under the left adjoints to the evaluation-at- $d$  functors  $\mathcal{M}^{\mathcal{D}} \rightarrow \mathcal{M}$  for each  $d \in \mathcal{D}$  (which exist because  $\mathcal{D}$  is small and  $\mathcal{M}$  is cocomplete).

*Proof.* Let  $|\mathcal{D}|$  be the discrete category on the objects of  $\mathcal{D}$  with the inclusion  $i : |\mathcal{D}| \rightarrow \mathcal{D}$ . Then the product category  $\mathcal{M}^{|\mathcal{D}|}$  has a product model structure

in which a map is in  $\mathcal{W}$ ,  $\mathcal{C}$  or  $\mathcal{F}$  if and only if each of its coordinates is. This structure is cofibrantly generated: a generating (acyclic) cofibration is a map which is identity on the initial object in all but one coordinate, and is in  $\mathcal{I}$  (resp.  $\mathcal{J}$ ) in this coordinate. The precomposition functor  $i^* : \mathcal{M}^{\mathcal{D}} \rightarrow \mathcal{M}^{|\mathcal{D}|}$  has a left adjoint given by  $\text{Lan}_i$ . This adjunction satisfies the hypotheses of Theorem 2.25 and then the transferred model structure on  $\mathcal{M}^{\mathcal{D}}$  is as described.  $\square$

**Remark 2.28.** In our applications, we will need the above theorem in the enriched setting. This means that  $\mathcal{D}$  will be a  $\mathcal{V}$ -category and  $\mathcal{M}$  a model  $\mathcal{V}$ -category for some closed symmetric monoidal model category  $\mathcal{V}$ . This requires defining enriched model and monoidal categories which we avoid. The idea in both cases is that we want this additional structure on a model category in a way that it is compatible with its homotopy theoretic structure and descends to a similar structure on its model category. For the precise statement of the above theorem as we will be applying it, see [nLa, Theorem 3.1].

## 2.5 Left Bousfield Localization

Sometimes we would have a model structure on some category  $\mathcal{M}$  but its weak equivalences would not be the ones we want. Often, we will just want to add more weak equivalences to the already existing ones. This is called localization because it is defined by a universal property similar to that of localization of rings. Since functors of model categories are of two kinds (left and right Quillen), we get two kinds of localizations. All localizations that show up for us would be left localizations so we restrict our attention to those.

**Definition 2.29.** Let  $\mathcal{S}$  be class of maps in a model category  $\mathcal{M}$ . A *left localization of  $\mathcal{M}$  at  $\mathcal{S}$*  is a model category  $L_{\mathcal{S}}\mathcal{M}$  and a left Quillen functor  $\ell_{\mathcal{S}} : \mathcal{M} \rightarrow L_{\mathcal{S}}\mathcal{M}$  taking maps in  $\mathcal{S}$  to weak equivalences such that for any model category  $\mathcal{N}$  and left Quillen functor  $F : \mathcal{M} \rightarrow \mathcal{N}$  taking maps in  $\mathcal{S}$  to weak equivalences, there exists a unique left Quillen functor making the following commute

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\ell_{\mathcal{S}}} & L_{\mathcal{S}}\mathcal{M} \\ & \searrow F & \downarrow \exists! \\ & & \mathcal{N} \end{array}$$

Now suppose we have model categories  $(\mathcal{M}, \mathcal{W}, \mathcal{C}, \mathcal{F})$  and  $(\mathcal{M}', \mathcal{W}', \mathcal{C}', \mathcal{F}')$  with the same underlying category  $\mathcal{M} = \mathcal{M}'$ , the same cofibrations  $\mathcal{C} = \mathcal{C}'$ , and with  $\mathcal{W} \subseteq \mathcal{W}'$ . We claim that then  $\mathcal{M}'$  along with the identity functor  $\mathcal{M} \rightarrow \mathcal{M}'$  is a left localization of  $\mathcal{M}$  at  $\mathcal{W}'$ . Clearly it is a left adjoint and preserves cofibrations and acyclic cofibrations so it is left Quillen. It also takes maps in  $\mathcal{W}'$  to  $\mathcal{W}$ . Then given any functor  $F : \mathcal{M} \rightarrow \mathcal{N}$  with the same properties,  $F$  itself works as the unique witness for the universal property above. This special

case of left localization is called a *left Bousfield localization*. From here on,  $L_{\mathcal{S}}\mathcal{M}$  will only refer to a left Bousfield localization.

If the left Bousfield localization of  $(\mathcal{M}, \mathcal{W}, \mathcal{C}, \mathcal{F})$  at  $\mathcal{S}$  exists, then its subcategory of weak equivalences will contain  $\mathcal{S} \cup \mathcal{W}$  but could be much larger. The weak equivalences in this localized model structure are called the  $\mathcal{S}$ -local equivalences. We want to know not only whether the left Bousfield localization exists, but also what the  $\mathcal{S}$ -local equivalences are (it will not be good to end up with a model structure with too many weak equivalences).

The characterization of the  $\mathcal{S}$ -local equivalences in our cases of concern requires talking about (co)simplicial resolutions and homotopy function complexes in model categories. This is out of the scope of this thesis, but details can be found in [Hir, Chapter 3]. We summarize the general idea here along with the assumptions that we will be making.

The left Bousfield localization of  $\mathcal{M}$  at  $\mathcal{S}$  exists whenever  $\mathcal{S}$  is a (small) set in a *left proper* and *cellular* model category. The idea is to find the cofibrant generating sets for  $L_{\mathcal{S}}\mathcal{M}$  given such sets  $\mathcal{I}$  and  $\mathcal{J}$  for  $\mathcal{M}$  (cellular model categories are cofibrantly generated). Since the cofibrations do not change,  $\mathcal{I}$  need not change.  $\mathcal{J}$  needs to be replaced with a larger set. A naive choice for this would be all cofibrations that are  $\mathcal{S}$ -local equivalences, but this would usually be a proper class.  $\mathcal{S}$  being a set and cellularity are combinatorial conditions that enable cutting down this into a generating set. Left properness is then needed to show that this is indeed a model structure.

This is all applicable for us because the Quillen model structure on  $\mathbf{Top}$  is left proper and and cellular. Both of these properties are preserved under the model categorical operations of turning a category into a pointed one, so  $\mathbf{Top}_*$  also has these properties. The projective model structure on the category of functors into a left proper and cellular model category also has these properties. Lastly, left Bousfield localization itself also preserves these properties. All model structures in this thesis will be constructed from the Quillen model structure using these constructions, so all left Bousfield localizations will exist.

However, in our applications,  $\mathcal{S}$  will not be a set. This is fixed by finding a ‘generating set’ within the class  $\mathcal{S}$  and then applying the above theorem. We will simply assume that the left Bousfield localizations that we want exist and that their  $\mathcal{S}$ -local equivalences are exactly the ones that we need.

## 3 More Categorical Background

The goal of this chapter is to introduce the nerve-realization adjunction and Day convolution from enriched category theory and the results of [MMSS] on modules over diagrams. The nerve-realization adjunction will be used to describe the weak equivalences in Theorems 4.12, 5.7 and 6.7. Day convolution will be used to define the smash product of spectra and the box product of Mackey functors. The results on modules in diagram categories motivate the definitions of orthogonal and symmetric spectra in Section 4.3. A reader willing to accept these results may skip this chapter or come back to it later.

We assume the definitions of enriched categories, functors, natural transformations and (co)end calculus. For an introduction to these, see [Rie, Chapter 1, 3]. The enriching category will always be denoted  $(\mathcal{V}, \odot)$  and it will always be assumed to be closed symmetric monoidal and cocomplete.

### 3.1 Nerve-Realization Adjunction and Day Convolution

The starting point of this discussion is the enriched Yoneda lemma.  $\text{Nat}(F, G)$  denotes the object of natural transformations  $F \rightarrow G$ .

**Theorem 3.1** (Yoneda Lemma). Let  $\mathcal{D}$  be a  $\mathcal{V}$ -category. Let  $F : \mathcal{D}^{\text{op}} \rightarrow \mathcal{V}$  be a  $\mathcal{V}$ -functor and  $m \in \mathcal{D}$ . Then there is a natural isomorphism in  $\mathcal{V}$

$$\text{Nat}(\mathcal{D}(-, m), F) \cong F(m) \quad \square$$

This tells us that  $\mathcal{D}(-, m)$  is *freely generated by the identity element in  $\mathcal{D}(m, m)$* , in the sense that defining a map from this functor to  $F$  is equivalent to describing just the image of this generator. Our algebraic intuition says then that any functor should have a description using generators and relations. The most straightforward way to construct such a description would be to take every ‘element of the functor’ as a generator. Thus for each  $m \in \mathcal{D}$ , we want ‘ $F(m)$  many’ copies of  $\mathcal{D}(-, m)$ . This gives a description of  $F$  as a colimit of representable functors indexed by the comma category  $\mathcal{D} \downarrow F$  where  $\mathcal{D}$  is treated as a subcategory of the presheaf category  $\widehat{\mathcal{D}}$  via the Yoneda embedding (see [Lan, Theorem III.7.1] for a proof for ordinary categories). This can be packaged nicely with a coend.

**Theorem 3.2** (Coyoneda Lemma). Let  $\mathcal{D}$  be a  $\mathcal{V}$ -category and  $X : \mathcal{D}^{\text{op}} \rightarrow \mathcal{V}$  a  $\mathcal{V}$ -functor. Then there is a canonical isomorphism

$$X \cong \int^{m \in \mathcal{D}} X(m) \odot \mathcal{D}(-, m) \quad \square$$

This discussion says that every presheaf is canonically a colimit of representable presheaves or that the representable presheaves are *dense* in the presheaf category. This allows us to extend functors on  $\mathcal{D}$  to ones that preserve colimits on  $\mathcal{D}$ , and gives us a universal property of the presheaf category.

**Theorem 3.3.** Let  $\mathcal{D}, \mathcal{N}$  be  $\mathcal{V}$ -categories and  $F : \mathcal{D} \rightarrow \mathcal{N}$  a  $\mathcal{V}$ -functor. If  $\mathcal{D}$  is small and  $\mathcal{N}$  is cocomplete, then up to natural isomorphism, there exists a unique  $\mathcal{V}$ -functor  $\overline{F} : \widehat{\mathcal{D}} \rightarrow \mathcal{N}$  which has a right adjoint and makes the following diagram commute, where  $y$  is the Yoneda embedding into the presheaf category  $\widehat{\mathcal{D}}$ .

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{y} & \widehat{\mathcal{D}} \\ & \searrow F & \downarrow \overline{F} \\ & & \mathcal{N} \end{array}$$

*Proof.* Define  $\overline{F}(\mathcal{D}(-, m)) = Fm$  and extend using Theorem 3.2. That is, for  $X \in \widehat{\mathcal{D}}$ , define

$$\overline{F}X = \int^{m \in \mathcal{D}} X(m) \odot F(m)$$

This can also be described as the left Kan extension of  $F$  along the Yoneda embedding. The right adjoint to this is the functor

$$\begin{aligned} \mathbb{N}F : \mathcal{N} &\rightarrow \widehat{\mathcal{D}} \\ n &\mapsto \mathcal{N}(F-, n) \end{aligned}$$

The adjunction isomorphism for a pair  $X \in \widehat{\mathcal{D}}$  and  $n \in \mathcal{N}$  can be defined easily when  $X$  is representable. Other cases can be reduced to the representable one by writing  $X$  as a colimit of representables and using the fact that  $\text{Hom}$  turns coends in the first coordinate into ends.  $\square$

This universal property says that the presheaf category of  $\mathcal{D}$  is its *free cocompletion*. It cocompletes  $\mathcal{D}$  in the most general way possible: i.e. it is cocomplete and  $\mathcal{D}$  embeds into it as a dense subcategory. ‘Free’ here refers to the fact that any existing colimits may be destroyed, which is necessary because  $F$  is not required to be cocontinuous while its extension  $\overline{F}$  is.

**Definition 3.4.** The functor  $\mathbb{N}F$  as above is called the *nerve* induced by  $F$  and  $\overline{F}$  is called the *realization* induced by  $F$ . The adjunction above is called the *nerve-realization adjunction* induced by  $F$ .



### 3 More Categorical Background

**Example 3.5.** The nerve-realization adjunction induced by the unenriched inclusion of the discrete category  $\mathbf{1}$  on one object into  $\mathbf{Top}$  as the singleton space is the free-forgetful adjunction between  $\mathbf{Top}$  and  $[\mathbf{1}, \mathbf{Set}] \cong \mathbf{Set}$ .

**Example 3.6.** The nerve-realization adjunction induced by the unenriched inclusion of the simplex category  $\Delta$  into  $\mathbf{Top}$  is the familiar adjunction between  $\mathbf{Top}$  and  $\mathbf{SSet}$ . The nerve takes a space to its singular simplicial set. The realization is geometric realization. This is a Quillen equivalence.

The nerve may be thought of as the data one gets from probing an arbitrary object of  $\mathcal{N}$  by objects in  $\mathcal{D}$ , which is often a subcategory of ‘nice’ objects and morphisms sitting in  $\mathcal{N}$  via an inclusion  $F$ . The realization is then an attempt to best reconstruct an object given a presheaf, which is the form that such probing data usually takes. The above examples show that probing by the singleton space is not enough to capture all homotopical data, whereas probing by the simplices is. However, adding enrichments changes things.

**Example 3.7.** Treating  $\mathbf{1}$  as a  $\mathbf{Top}$ -category, the same inclusion of  $\mathbf{1}$  into  $\mathbf{Top}$  is a  $\mathbf{Top}$ -functor. The enriched nerve-realization functor is then an isomorphism of categories. More generally, the  $\mathcal{V}$ -enriched nerve-realization adjunction induced by the inclusion of the unit into  $\mathcal{V}$  is always an equivalence.

Another important application of the coyoneda lemma is Day convolution. Suppose  $(\mathcal{D}, +)$  is a symmetric monoidal  $\mathcal{V}$ -category which is not necessarily closed. We can use this to define a *closed* symmetric monoidal product  $\square$  on  $\widehat{\mathcal{D}}$ . Being closed would imply that  $\square$  preserves colimits in both coordinates. This gives us a candidate definition, where we define  $\mathcal{D}(-, m) \square \mathcal{D}(-, n) = \mathcal{D}(-, m + n)$  and extend as

$$\begin{aligned} X \square Y &= \left( \int^m X(m) \odot \mathcal{D}(-, n) \right) \square \left( \int^n Y(n) \odot \mathcal{D}(-, n) \right) \\ &= \int^{m, n} X(m) \odot Y(n) \odot (\mathcal{D}(-, m) \square \mathcal{D}(-, n)) \\ &= \int^{m, n} X(m) \odot Y(n) \odot (\mathcal{D}(-, m + n)) \end{aligned}$$

**Definition 3.8.** Let  $\mathcal{D}$  be a small symmetric monoidal  $\mathcal{V}$ -category. The *Day convolution product*  $X \square Y$  of  $X, Y \in \widehat{\mathcal{D}}$  is defined as the coend

$$X \square Y = \int^{m, n \in \mathcal{D}} X(m) \odot Y(n) \odot \mathcal{D}(-, m + n)$$

We could also have arrived at the same definition by applying the nerve-realization adjunction to one coordinate of  $+$  and then to the other since we want to create a right adjoint to it in each coordinate. From the coend formula for Kan extensions (see [Lan, Theorem X.4.1]), we can infer the following important property of  $\square$ .



**Proposition 3.9.** For  $X, Y : \mathcal{D}^{\text{op}} \rightarrow \mathcal{V}$ , their Day convolution product  $X \square Y$  is precisely the left Kan extension

$$\begin{array}{ccc} \mathcal{M}^{\text{op}} \otimes \mathcal{M}^{\text{op}} & \xrightarrow{X \otimes Y} & \mathcal{V} \otimes \mathcal{V} \xrightarrow{\odot} \mathcal{V} \\ & \searrow + & \nearrow X \square Y \\ & & \mathcal{M}^{\text{op}} \end{array}$$

That is, we have an isomorphism

$$\text{Nat}((X \square Y) \circ +, -) \cong \text{Nat}(\odot \circ (X \otimes Y), -) \quad \square$$

This means that defining a natural transformation  $X \square Y \rightarrow Z$  is equivalent to defining maps  $X(m) \otimes Y(n) \rightarrow Z(m+n)$  natural in  $m, n \in \mathcal{D}$ .

**Theorem 3.10** (Day Convolution Theorem). Let  $\mathcal{D}$  be a small symmetric monoidal  $\mathcal{V}$ -category, and  $\square$  the Day convolution product on  $\widehat{\mathcal{D}}$ . Then  $(\widehat{\mathcal{D}}, \square)$  is a closed symmetric monoidal category.

*Proof.* To check symmetry and associativity, we can use the universal property above. For example, we can show that

$$\text{Nat}(X \square (Y \square Z), -) \cong \text{Nat}((X \square Y) \square Z, -)$$

For the unit, first check that  $X \square \mathcal{D}(-, m) \cong X(- + m)$  by checking it for representable  $X$  and then extending by coends. Then the unit for  $\square$  is  $\mathcal{D}(-, 0)$  where 0 is the unit of  $+$ . For  $\square$  to be closed, we must have internal hom  $[-, -]$  with

$$\begin{aligned} [X, Y](m) &\cong \text{Nat}(\mathcal{D}(-, m), [X, Y]) \\ &\cong \text{Nat}(\mathcal{D}(-, m) \square X, Y) \\ &\cong \text{Nat}(X(- + m), Y) \end{aligned}$$

Taking this as the definition of the internal hom, the adjunctions are easily checked.  $\square$

We also get a copresheaf version of the Day convolution theorem. Let  $\overline{\mathcal{D}}$  denote the category of copresheaves on  $\mathcal{D}$ , i.e.  $\mathcal{V}$ -functors  $\mathcal{D} \rightarrow \mathcal{V}$ . Note that  $\mathcal{D}^{\text{op}}$  is also symmetric monoidal under  $+$ . For copresheaves  $X, Y : \mathcal{D} \rightarrow \mathcal{V}$ , or equivalently presheaves  $X, Y : (\mathcal{D}^{\text{op}})^{\text{op}} \rightarrow \mathcal{V}$ , define their Day convolution product as

$$\begin{aligned} X \square Y &= \int^{m, n \in \mathcal{D}^{\text{op}}} X(m) \odot Y(n) \odot \mathcal{D}^{\text{op}}(-, m+n) \\ &= \int^{m, n \in \mathcal{D}} X(m) \odot Y(n) \odot \mathcal{D}(m+n, -) \end{aligned}$$

Then  $(\overline{\mathcal{D}}, \square)$  is a closed symmetric monoidal category. We will need to use both versions of this theorem in the later chapters.

## 3.2 Modules in Diagram Categories

Let  $(\mathcal{D}, +)$  be a small symmetric monoidal  $\mathcal{V}$ -category. Then  $(\overline{\mathcal{D}}, \square)$  is closed symmetric monoidal. Let  $\mathbb{S}$  be a monoid object in  $\overline{\mathcal{D}}$  and consider the category  $\mathbf{Mod}_{\mathbb{S}}$  of right  $\mathbb{S}$ -modules. This situation will be of interest in Section 4.3.

The following theorem is a generalization of the case  $\mathcal{V} = \mathbf{Ab}$  and  $\mathcal{D}$  the terminal category. Then  $\overline{\mathcal{D}} \cong \mathbf{Ab}$  with the same tensor product, so  $\mathbb{S}$  is simply a ring. The category  $\mathbf{Mod}_{\mathbb{S}}$  is then equivalent to  $\overline{\mathcal{J}}$ , where  $\mathcal{J}$  is  $\mathbb{S}$  viewed as a one object  $\mathbf{Ab}$ -category in the usual way. This happens because  $\mathcal{J}$  is equivalent to the full subcategory of  $\mathbf{Mod}_{\mathbb{S}}$  spanned by the free cyclic  $\mathbb{S}$ -modules and the maps between these modules capture all possible ways in which elements of  $\mathbb{S}$  could act on an element of an arbitrary module.

**Theorem 3.11.** For any monoid  $\mathbb{S}$  as above, there exists a small monoidal  $\mathcal{V}$ -category  $\mathcal{J}$  such that  $\mathbf{Mod}_{\mathbb{S}} \cong \overline{\mathcal{J}}$ . Furthermore, if  $\mathbb{S}$  is a commutative monoid, then  $\mathcal{J}$  has a symmetric monoidal product such that the induced Day convolution product on  $\overline{\mathcal{J}}$  coincides with the usual monoidal product of  $\mathbb{S}$ -modules over  $\mathbb{S}$ .

*Proof.* Construct  $\mathcal{J}$  as the full subcategory of  $\mathbf{Mod}_{\mathbb{S}}$  spanned by the free modules generated that can be generated by a single element, i.e. the  $\mathbb{S}$ -modules  $\mathcal{D}(n, -) \square \mathbb{S}$  for each  $n \in \mathcal{D}$ . See [MMSS, Theorem 2.2, Section 23] for details.  $\square$

## 4 Stable Homotopy Theory

Homotopy groups are notoriously difficult to compute. On the other hand, we have some techniques available which allow the computation of (co)homology theories. For example, they are abelian groups in all dimensions, allowing the use of spectral sequences without annoying edge cases. Homology also has the suspension isomorphism  $\tilde{H}_\bullet(Y) \cong \tilde{H}_{\bullet+1}(\Sigma Y)$  for each  $Y \in \mathbf{Top}_*$  which is not present in homotopy. For example,

$$\pi_2(S^1) \cong 0 \quad \pi_3(S^2) \cong \mathbb{Z}$$

where the former comes from covering theory, and the latter from the long exact sequence of the Hopf fibration  $S^3 \rightarrow S^2$ .

However, there is still a natural comparison map  $\pi_n(Y) \rightarrow \pi_{n+1}(\Sigma Y)$  by Example 2.24. We are led to the definition of the *stable homotopy groups* of  $Y$

$$\pi_n^s(Y) = \operatorname{colim}_k \pi_{n+k}(\Sigma^k X)$$

These stable homotopy groups then have a suspension isomorphism almost by construction. In fact, they form a generalized homology theory. More generally, for any pair  $X, Y \in \mathbf{Top}_*$ , one could define the set of *stable homotopy classes of maps*

$$[X, Y]^s = \operatorname{colim}_k [\Sigma^k X, \Sigma^k Y]$$

One could now try to construct a model structure on  $\mathbf{Top}_*$  such that

$$\operatorname{Ho} \mathbf{Top}_*(X, Y) = [X, Y]^s$$

with respect to this model structure, but it turns out that that is not possible - this category does not have arbitrary coproducts while the homotopy category of a model category does (this is the Spanier-Whitehead category, [BR, Chapter 1]). Instead, we define a larger category  $\mathbf{Sp}$  where we can still talk about stable homotopy classes of maps with a fully faithful embedding  $\mathbf{Top}_* \rightarrow \mathbf{Sp}$ . For more detailed motivations for stable homotopy groups and spectra, see [Rog, Section 9.1] which describes some of their history, [Mar, Chapter 1] for an axiomatic introduction, or [Greb, Section 1.2] for an introduction through  $\infty$ -categorical stabilization. For us, suspension will mean smashing with  $S^1$  on the right, i.e.  $\Sigma := - \wedge S^1$ .

## 4.1 Spectra and their Model Structure

**Definition 4.1.** (i) A *(sequential) spectrum*  $X$  is a sequence of spaces  $(X_n)_{n \in \mathbb{N}}$  along with maps

$$\sigma_n : \Sigma X_n \rightarrow X_{n+1}$$

(ii) A map  $f : X \rightarrow Y$  of sequential spectra is a sequence of maps  $f_n : X_n \rightarrow Y_n$  such that for each  $n \in \mathbb{N}$ , the following square commutes.

$$\begin{array}{ccc} \Sigma X_n & \xrightarrow{\sigma_n} & X_{n+1} \\ \Sigma f_n \downarrow & & \downarrow f_{n+1} \\ \Sigma Y_n & \xrightarrow{\sigma_n} & Y_{n+1} \end{array}$$

Thus we have a category of spectra, which will be denoted  $\mathbf{Sp}$ .

(iii) The  $n$ th *(stable) homotopy group* of a spectrum  $X$  is

$$\pi_n(X) := \operatorname{colim}_k \pi_{n+k} X_k$$

where the colimit is over the diagram in which the map  $\pi_{n+k} X_k \rightarrow \pi_{n+k+1} X_{k+1}$  is the composite

$$[S^{n+k}, X_k] \xrightarrow{\mathbb{L}\Sigma} [S^{n+k+1}, \Sigma X_k] \xrightarrow{(\sigma_k)_*} [S^{n+k+1}, X_{k+1}]$$

This definition also makes sense for  $n < 0$  by starting the diagram from  $k \geq |n|$ . Thus we have functors  $\pi_n : \mathbf{Sp} \rightarrow \mathbf{Ab}$  for every  $n \in \mathbb{Z}$ .

(iv) A map of spectra  $f : X \rightarrow Y$  is a *stable equivalence* if  $\pi_n(f)$  is an isomorphism for all  $n \in \mathbb{Z}$ .

(v) For  $X$  a based space, the suspension spectrum of  $X$ , denoted  $\Sigma^\infty X$ , is defined by

$$(\Sigma^\infty X)_n = X \wedge S^n$$

with structure maps the isomorphisms  $\sigma_n : X \wedge S^n \wedge S^1 \rightarrow X \wedge S^{n+1}$ . This is left adjoint to the functor  $\Omega^\infty : \mathbf{Sp} \rightarrow \mathbf{Top}_*$  that takes the zeroth space of a given spectrum.

Then, for any pointed space  $X$ , its stable homotopy groups coincide with the homotopy groups of its suspension spectrum. Our goal is to study spectra up to homotopy, which would subsume the study of spaces up to stable homotopy since  $\Sigma^\infty$  is fully faithful. Thus, we want a model structure on spectra in which the weak equivalences are the stable equivalences. In this section, we construct this model structure on spectra.

## 4.1 Spectra and their Model Structure

The definition of a spectrum is very similar to that of a graded module over a graded ring. By composing structure maps as

$$\Sigma^k X_n \xrightarrow{\Sigma^{k-1}\sigma_n} \Sigma^{k-1} X_{n+1} \xrightarrow{\Sigma^{k-2}\sigma_{n+1}} \dots \xrightarrow{\sigma_{n+k-1}} X_{n+k}$$

gives us a  $k$ -fold structure map

$$\sigma_{n,k} : X_n \wedge S^k \rightarrow X_{n+k}$$

so we could have equally defined a spectrum as a sequence of spaces with structure maps  $X_n \wedge S^k \rightarrow X_{n+k}$  for every  $n, k \in \mathbb{N}$  satisfying the composition condition described above. Moreover, the sphere spectrum  $\mathbb{S} := \Sigma^\infty S^0$  is analogous to a graded ring as its  $k$ -fold structure maps are

$$S^k \wedge S^n \xrightarrow{\sim} S^{k+n}$$

which makes it look like a graded ring.

To make this precise, let  $\mathcal{D}_{\mathbb{N}}$  denote the discrete category with objects the natural numbers. This can be viewed as a category enriched over  $\mathbf{Top}_*$ . It has a symmetric monoidal product given by natural number addition.

**Definition 4.2.** A *sequential space*  $X$  is a sequence  $(X_n)_{n \in \mathbb{N}}$  of pointed spaces. Equivalently, it is a copresheaf  $X : \mathcal{D}_{\mathbb{N}} \rightarrow \mathbf{Top}_*$ .

Theorem 3.10 then gives us a closed symmetric monoidal product on the category of sequential spaces defined as

$$(X \wedge Y)_n = \int^{i,j} X_i \wedge Y_j \wedge \mathcal{D}_{\mathbb{N}}(i+j, n) = \bigvee_{i+j=n} X_i \wedge Y_j$$

The unit of this product is  $\mathbb{1}$ , with  $\mathbb{1}_0 = S^0$  and  $\mathbb{1}_n = *$  for  $n \in \mathbb{N}_{>0}$ . The sphere sequential spectrum  $\mathbb{S}$  is a monoid in this category and a spectrum is exactly a right module over this monoid. Thus  $\mathbf{Sp}$  is equivalent to  $\mathbf{Mod}_{\mathbb{S}}$ .

We can now apply Theorem 3.11 to find that there exists a  $\mathbf{Top}_*$ -category  $\mathcal{J}_{\mathbb{N}}$  such that  $\mathbf{Mod}_{\mathbb{S}} \cong \overline{\mathcal{J}_{\mathbb{N}}}$ . Then Corollary 2.27 gives us a cofibrantly generated model structure called the *projective model structure* on this category in which the weak equivalences are the level-wise weak equivalences of  $\mathbf{Top}_*$ . The corollary also describes the generating cofibrations  $\mathcal{I}$  as the maps  $S(m-1, n) \hookrightarrow D(m, n)$  where

$$(S(m, n))_k = \begin{cases} * & \text{if } k < n \\ S^{m-n+k} & \text{if } k \geq n \end{cases}$$

$$(D(m, n))_k = \begin{cases} * & \text{if } k < n \\ D^{m-n+k} & \text{if } k \geq n \end{cases}$$

## 4 Stable Homotopy Theory

and the inclusion is the obvious level-wise inclusion.  $S(m-1, n) \hookrightarrow D(m, n)$  is the result of applying the left adjoint of the evaluation-at- $n$  functor to the inclusion of pointed spaces  $S^{m-1} \hookrightarrow D^m$ .

This lets us define CW-spectra. Say that  $D(m, n)$  has dimension  $m - n$  and define a CW spectrum to be a spectrum obtained by attaching these disks in increasing order of dimension. This also admits a point-set level description: A CW spectrum is a spectrum in which every space is a pointed CW complex and the structure maps are inclusions of subcomplexes (see [Adaa, Section III.2]). Note that  $S(n, 0) \cong \Sigma^\infty S^n$ .

However, this is definitely not the model structure we want on spectra because there are more stable equivalences than there are level-wise weak equivalences of spectra. For example, the level-wise inclusion  $S(0, 1) \hookrightarrow S(0, 0) = \mathbb{S}$  is a stable equivalence since the colimit defining the stable homotopy groups is not affected by a finite number of changes in the spectra. Therefore, we want to change the model structure by expanding its subcategory of weak equivalences. We achieve this using left Bousfield localization at the class of stable equivalences.

**Theorem 4.3** (Stable model structure). There exists a cofibrantly generated model structure on  $\mathbf{Sp}$  in which the weak equivalences are the stable equivalences and the cofibrations are the cofibrations in the projective model structure. The fibrant objects in this model structure are the  $\Omega$ -spectra, which are spectra  $X$  in which the adjoints of the structure maps  $X_n \rightarrow \Omega X_{n+1}$  are weak equivalences.

*Proof.* See [HHRb, Section 7.3]. □

Since we have the same generating cofibrations and more weak equivalences, CW-spectra are cellular objects and  $I_+ \wedge X$ , defined by  $(I_+ \wedge X)_n = I_+ \wedge X_n$ , is a cylinder object for any spectrum  $X$ , so left homotopies are exactly the compatible level-wise homotopies.

## 4.2 Stable Homotopy Category

The first thing to check is whether we achieved our goal of constructing a model category in which homotopy classes of maps are the stable homotopy classes.

**Proposition 4.4.** Let  $A, B \in \mathbf{Top}_*$  with  $A$  a CW complex with finitely many cells. Then  $\mathcal{SH}(\Sigma^\infty A, \Sigma^\infty B) \cong [A, B]^s$  with the right side as defined in the introduction of this chapter.

*Proof.* See [BR, Lemma 5.1.2]. □

Recall the spectra  $S(m, n)$  from the discussion preceding Theorem 4.3. We say that  $S(m, n)$  has dimension  $m$ . It can be checked using similar methods that there are no non-trivial maps in  $\mathcal{SH}$  from a lower dimensional sphere to a higher dimensional one. This implies that we can turn our cellular approximation theorem into a CW approximation theorem where the CW spectra are defined as relative  $\mathcal{I}$ -cells in which the cells are attached in a non-decreasing order of dimension. These CW spectra can be defined as spectra in which each space is a pointed CW complex and the structure maps are inclusions of subcomplexes.

Define suspension and loop functors  $\Sigma, \Omega : \mathbf{Sp} \rightarrow \mathbf{Sp}$  by applying the ordinary suspension or loop functors level-wise.

**Theorem 4.5.** Suspension and loop form a Quillen equivalence  $\Sigma \dashv \Omega$  on  $\mathbf{Sp}$ .

*Proof.* See [BR, Theorem 2.3.14]. □

We can also define the smash product of a space with a spectrum and a spectrum of maps from a space to a spectrum level-wise. That is, if  $X$  is a space and  $E$  a spectrum, we define  $(X \wedge E)_n = X \wedge E_n$  and  $(\text{Map}(X, E))_n = \text{Map}(X, E_n)$  with the obvious structure maps. Then  $X \wedge -$  is left adjoint to  $\text{Map}(X, -)$ .

Being a colimit of maps between suspensions,  $[\Sigma^\infty A, \Sigma^\infty B]$  is an abelian group for any  $A, B \in \mathbf{Top}_*$ . In fact,  $\mathcal{SH}$  is an  $\mathbf{Ab}$ -enriched category. This follows from the fact that  $\Sigma^2$  is both an auto-equivalence and an abelian cogroup functor up to stable homotopy - there is a natural multiplication  $\Sigma^2 X \rightarrow \Sigma^2 X \vee \Sigma^2 X$  (suspension of the stable pinch map or equivalently the stable pinch map for  $\Sigma X$ ) that makes  $\Sigma^2 X$  into an abelian cogroup object in  $\mathcal{SH}$ . Thus every object  $X \in \mathcal{SH}$  is naturally an abelian cogroup object making  $\mathcal{SH}(X, -)$  an abelian group object and hence  $\mathbf{Ab}$ -valued.

In particular, any finite coproducts that exist in  $\mathcal{SH}$  are also products and vice versa. While homotopy categories rarely contain (co)limits, [Ram] proves that products and coproducts in the homotopy category of a model category always exist and can be computed at the point-set level. The  $\mathbf{Ab}$ -enrichment of  $\mathcal{SH}$  then proves that finite products and coproducts of spectra coincide up to homotopy.

We can define homotopy (co)fiber of a map of spectra level-wise. Since  $\Sigma$  and  $\Omega$  are equivalences, we are able to continue the homotopy (co)fiber sequence of a map in the ‘wrong’ direction too. We get the expected property that the homotopy groups of a fiber sequence give a long exact sequence, but we also get the same property for cofiber sequences (see [Rog, Proposition 9.3.7]). A consequence of this and the five lemma is that a sequence is a fiber sequence if and only if it is a cofiber sequence. Another important property of cofiber sequences is that they are preserved by the smash product with a fixed spectrum. This is because homotopy cofibers can be described as pushouts and the smash product has a right adjoint. These properties of cofiber sequences allow us to build generalized homology and cohomology theories given any spectrum.

**Theorem 4.6.** Let  $E$  be a spectrum. For each  $n \in \mathbb{Z}$ , define

$$\begin{aligned} E^n(X) &= \mathcal{SH}(\Sigma^\infty X, S^n \wedge E) \\ E_n(X) &= \mathcal{SH}(S^n, X \wedge E) \cong \pi_n(X \wedge E) \end{aligned}$$

Then  $E_\bullet$  is a generalized homology theory and  $E^\bullet$  is a generalized cohomology theory.

*Proof.* The homotopy invariance of  $E^\bullet$  and  $E_\bullet$  follows from the fact that  $\Sigma^\infty$  takes weak equivalences to stable equivalences. The remaining axioms (additivity, exactness for any cofiber sequence, suspension isomorphism) follow from the preceding discussion and the properties of the functors used in these definitions.  $\square$

Perhaps a more surprising result is that all generalized cohomology theories are built this way.

**Lemma 4.7** (Brown representability). A contravariant functor  $K : (\text{Ho CW}_*^c)^{\text{op}} \rightarrow \text{Set}$  from the homotopy category of pointed connected CW complexes is representable if and only if it converts wedge sums to direct products and satisfies the Mayer-Vietoris axiom: For any CW complex  $X$ , subcomplexes  $A_1, A_2$  that cover  $X$ , and elements  $a_i \in KA_i$  with the same restriction to  $K(A_1 \cap A_2)$ , there exists some  $x \in KX$  such whose restriction to  $KA_i$  is  $a_i$ .

*Proof.* The harder part is to prove that a functor satisfying these conditions is representable. This is prove by constructing the representing CW complex  $Z$  cell by cell. The conditions imply that it is enough to ensure that  $\text{Ho CW}_*^c(X, Z) \cong KX$  when  $X$  is a sphere, thus we just need  $\pi_n Z \cong KS^n$ . See [Put] for details.  $\square$

**Corollary 4.8.** For any generalized cohomology theory  $E^\bullet$ , there exists an  $\Omega$ -spectrum  $Z$  such that  $Z^\bullet \cong E^\bullet$ .

*Proof.* Apply Lemma 4.7 to each functor  $E^n$  for  $n \geq 0$  to get a representing CW complex  $Z_n$ . The suspension axiom gives a natural isomorphism  $[X, Z_n] \cong [\Sigma X, Z_{n+1}] \cong [X, \Omega Z_{n+1}]$ , which, by the Yoneda lemma, must correspond to a homotopy equivalence  $Z_n \rightarrow \Omega Z_{n+1}$ . These spaces define an  $\Omega$ -spectrum  $Z$ , and to check that this has the correct value on each space, it is enough to check it on CW complexes. Since suspension spectra of CW complexes are cofibrant and  $\Omega$ -spectra are fibrant, this can be checked by computing the level-wise homotopy classes and using the fact that  $\Sigma^\infty$  is left adjoint to taking the 0th space. The abelian group structure on both must be the same because the suspension isomorphism being a group homomorphism implies that the two group structures satisfy the interchange identity, allowing use of the Eckmann-Hilton argument.  $\square$



A similar result holds for generalized homology theories satisfying a mild condition (see [Adaa, Remark III.6.5]).

We can also show the existence of Eilenberg-MacLane spectra. An Eilenberg-MacLane spectrum  $HA$  for an abelian group  $A$  is a spectrum such that  $\pi_0(HA) \cong A$  and  $\pi_n(HA) = 0$  otherwise. One construction of these is as  $(HA)_n = K(A, n)$  with structure maps the adjoints of the weak equivalences  $K(A, n) \rightarrow \Omega K(A, n+1)$  and it is an easy check that the homotopy groups are as claimed. The Eilenberg-MacLane spectra for other indices can be constructed as suspensions or loops of  $HA$ .

**Proposition 4.9.** For any  $A \in \mathbf{Ab}$ ,  $HA_\bullet \cong H_\bullet(-; A)$  and  $HA^\bullet \cong H^\bullet(-; A)$ .

*Proof.* Check that these theories agree on all spheres and hence on CW complexes.  $\square$

## 4.3 Smash Products and Orthogonal Spectra

Lemma 4.7 told us that spectra represent cohomology theories as contravariant functors into the category of graded abelian groups, but many of our favorite cohomology theories have extra multiplicative structure. For example, ordinary cohomology with coefficients in a ring is valued in graded rings which are monoid objects in the category of graded abelian groups. We would like such cohomology theories to be represented by ring spectra - spectra that are monoid objects in  $\mathbf{Sp}$  under some symmetric monoidal product on  $\mathbf{Sp}$ . In [Adaa, Section III.4], Adams constructs a symmetric smash product on  $\mathcal{SH}$  with much effort which achieves this but this is not ideal because then we cannot exploit it at the point-set level of  $\mathbf{Sp}$ .

Recognizing spectra as right  $\mathbb{S}$ -modules tells us that the reason it is so difficult to construct a good monoidal product on  $\mathbf{Sp}$  is that  $\mathbb{S}$  is not a commutative monoid, so the general machinery for smashing two right modules over it (see [Lan, Exercise VII.4.6]) fails. This can be fixed by adding more symmetries of (the spaces that appear in)  $\mathbb{S}$  to the domain  $\mathcal{D}_{\mathbb{N}}$ . The category of  $\mathbb{S}$ -modules that we get then will not be equivalent to  $\mathbf{Sp}$ , but with an appropriate model structure, it will be Quillen equivalent. Here is a sketch of how this works for the interested reader, concluding in Definition 4.11.

Let  $\Sigma(n)$  denote the symmetric group on  $n$  letters and  $O(n)$  the  $n$ th orthogonal group. Let  $\mathcal{D}_{\Sigma}$  be the  $\mathbf{Top}_*$ -category with objects the natural numbers and Hom spaces  $\mathcal{D}_{\Sigma}(n, n) := (\Sigma(n))_+$ . Let  $\mathcal{D}_{\mathbf{O}}$  be the  $\mathbf{Top}_*$ -category with objects the natural numbers and Hom spaces  $\mathcal{D}_{\mathbf{O}}(n, n) := (O(n))_+$ . All other Hom spaces are defined to be trivial.

**Definition 4.10.** A *symmetric space* is a copresheaf on  $\mathcal{D}_\Sigma$  and an *orthogonal space* is a copresheaf on  $\mathcal{D}_\mathbf{O}$ . Equivalently, a symmetric/orthogonal space is a sequence  $(X_n)_{n \in \mathbb{N}}$  of pointed spaces with a left action of the  $n$ th symmetric/orthogonal group acting on  $X_n$ .

The embeddings  $\Sigma(m) \times \Sigma(n) \rightarrow \Sigma(m+n)$  and  $O(m) \times O(n) \rightarrow O(m+n)$  allow us to extend the monoidal structure on  $\mathcal{D}_\mathbb{N}$  to those on  $\mathcal{D}_\Sigma$  and  $\mathcal{D}_\mathbf{O}$ . Day convolution then makes the categories of symmetric and orthogonal spaces into closed symmetric monoidal categories.  $\mathbb{S}$  then extends to a symmetric space  $\mathbb{S}_\Sigma$  and an orthogonal space  $\mathbb{S}_\mathbf{O}$  with the actions being the one-point compactifications of the usual actions of  $\Sigma(n)$  and  $O(n)$  on  $\mathbb{R}^n$ . Both of these are still monoids. We sketch this for the symmetric case.  $S \wedge S$  is described as the coend

$$(S \wedge S)_n = \int_{i,j} S^i \wedge S^j \wedge \mathcal{D}_\Sigma(i+j, n)$$

The multiplication  $\mathbb{S} \wedge \mathbb{S} \rightarrow \mathbb{S}$  is the one induced by the maps

$$S^i \wedge S^j \wedge \mathcal{D}_\Sigma(i+j, n) \rightarrow S^n$$

The domain is trivial when  $n \neq i+j$ . If  $n = i+j$ , this is the composite

$$S^i \wedge S^j \wedge (\Sigma(n))_+ \cong S^n \wedge (\Sigma(n))_+ \rightarrow S^n$$

where the second map is the action map (following a twist). This is a commutative monoid because in the diagram

$$\begin{array}{ccc} S^i \wedge S^j \wedge (\Sigma(i+j))_+ & \xrightarrow{\tau} & S^j \wedge S^i \wedge (\Sigma(j+i))_+ \\ & \searrow & \swarrow \\ & S^{i+j} & \end{array}$$

the twist map  $\tau$  exchanges the two spheres and also precomposes the permutation with the permutation

$$1, 2, \dots, n \mapsto i+1, \dots, n, 1, \dots, i$$

and both of these changes cancel out. Therefore,  $\mathbb{S}_\Sigma$  is a commutative monoid in the copresheaf category on  $\mathcal{D}_\Sigma$  which gives us a closed symmetric monoidal category of modules over it. Similar arguments can be made for  $\mathbb{S}_\mathbf{O}$ . We define a symmetric spectrum as a right module over  $\mathbb{S}_\Sigma$  and an orthogonal spectrum as a right module over  $\mathbb{S}_\mathbf{O}$ . From this point on, we talk only about orthogonal spectra because of Remark 4.13.

**Definition 4.11.** An orthogonal spectrum is a sequence of spaces  $(X_n)_{n \in \mathbb{N}}$  with a left action of  $O(n)$  on  $X_n$  and  $O(n) \times O(k)$ -equivariant structure

maps

$$\sigma_{n,k} : X_n \wedge S^k \rightarrow X_{n+k}$$

such that  $\sigma_{n,0} = \text{id}_{X_n}$  and the composite

$$X_n \wedge S^k \xrightarrow{\sigma_{n,k} \wedge S^l} X_{n+k} \wedge S^l \xrightarrow{\sigma_{n+k,l}} X_{n+k+l}$$

is the map  $\sigma_{n,k+l}$  for each  $n, k, l \in \mathbb{N}$ .

A map of orthogonal spectra is a map between them as  $\mathbb{S}_{\mathbf{O}}$ -modules, which amounts to a map of the underlying spectra such that the level-wise maps are  $O(n)$ -equivariant. The category of orthogonal spectra is denoted  $\mathbf{Sp}_{\mathbf{O}}$ .

Theorem 3.11 now gives us a  $\mathbf{Top}_*$ -category  $\mathcal{J}_{\mathbf{O}}$  such that  $\mathbf{Sp}_{\mathbf{O}}$  is equivalent to the category of copresheaves on  $\mathcal{J}_{\mathbf{O}}$ . There is a natural inclusion  $\mathcal{J}_{\mathbb{N}} \rightarrow \mathcal{J}_{\mathbf{O}}$  which induces, by precomposition, a forgetful functor  $\mathbf{Sp}_{\mathbf{O}} \rightarrow \mathbf{Sp}_{\mathbb{N}}$ . This has a left adjoint given by left Kan extension since  $\mathcal{J}_{\mathbb{N}}$  is small and  $\mathbf{Top}_*$  is cocomplete. By composing left adjoints, this proves that once again we have a left adjoint to the functor

$$\begin{aligned} \Omega^\infty : \mathbf{Sp}_{\mathbf{O}} &\rightarrow \mathbf{Top}_* \\ X &\mapsto X_0 \end{aligned}$$

which we call the *suspension spectrum functor* and denote by  $\Sigma^\infty$ .

We finally have a closed symmetric monoidal category of spectra. The smash product of spectra is defined by Day convolution (Theorem 3.10) as the coend

$$(X \wedge Y)_k = \int^{m,n} X_m \wedge Y_n \wedge \mathcal{J}_{\mathbf{O}}(m+n, k)$$

The closed structure is given by internal homs

$$\text{Map}(X, Y)_n = \mathbf{Sp}_{\mathbf{O}}(X(-+n), Y)$$

For  $n = 0$  in this, we get the mapping space of diagram maps  $X \rightarrow Y$ , and we denote this by  $\text{Map}_0(X, Y)$ . Then the functor  $- \wedge E : \mathbf{Top}_* \rightarrow \mathbf{Sp}_{\mathbf{O}}$  is left adjoint to  $\text{Map}_0(E, -)$ . A consequence of these adjunctions is the natural isomorphism

$$\Sigma^\infty X \wedge E \cong X \wedge E$$

which can be verified by checking the maps out of both. As it does not change meaning in this or any other case, we omit  $\Sigma^\infty$  from our notation. We will often say “the spectrum  $X$ ”, which is to be understood as “the spectrum  $\Sigma^\infty X$ ”.

All the proofs about sequential spectra from Section 4.1 now work here and we are able to define a stable model structure on orthogonal spectra. A map of orthogonal spectra is classed a stable equivalence if it induces an isomorphism on all stable homotopy groups, which are defined just as in Definition 4.1.

**Theorem 4.12.** There exists a model structure called the *stable model structure* on  $\mathbf{Sp}_{\mathbf{O}}$  in which the weak equivalences are the stable equivalences of orthogonal spectra. The free-forgetful adjunction between  $\mathbf{Sp}$  and  $\mathbf{Sp}_{\mathbf{O}}$  described above is then a Quillen equivalence.

*Proof.* See [HHRb, Section 7.3] for the first statement and [MMSS, Section 10] for the second one. The Quillen equivalence can be briefly described as follows. The category  $\mathcal{D}_{\mathbb{N}}$  embeds into  $\mathcal{D}_{\mathbf{O}}$ , which extends to an embedding of  $\mathcal{J}_{\mathbb{N}}$  into  $\mathcal{J}_{\mathbf{O}}$ . From an orthogonal spectrum, we get a sequential one by restriction to this subcategory (equivalently, by forgetting the orthogonal group actions), and we go the other way using left Kan extension along this embedding. Checking that this is a Quillen equivalence reduces to the fact that the map

$$O(n)_+ \wedge_{O(k-1)} S^n \rightarrow O(n)_+ \wedge_{O(k)} S^n$$

is a  $(2n + k - 1)$ -equivalence (and hence a stable homotopy equivalence) for all  $k \leq n \in \mathbb{N}$  (see Definition 5.1).  $\square$

**Remark 4.13.** Symmetric spectra are conceptually simpler than orthogonal spectra because adding the symmetric group actions is the least invasive way to fix the non-commutativity of  $\mathbb{S}$  as a monoid. However, their homotopy theory is not as well behaved. If we want  $\mathbf{Sp}_{\Sigma}$  to be Quillen equivalent to  $\mathbf{Sp}$ , then the definition of stable equivalences will not be the expected one and will instead have to rely on cohomology. The reason we prefer orthogonal spectra is that their stable equivalences are exactly those of their underlying sequential spectra. In many cases, orthogonal spectra also add some features that are not present in sequential spectra, like the following results about shifts and suspensions. They are also the only kind of spectra out of these that generalize easily to the equivariant case (see the introduction in Section 6.1). From this point on, ‘spectrum’ always refers to an orthogonal spectrum.

We can describe the generating cofibrations, desuspensions, and the negative dimensional spheres using the shift functors. The generating cofibrations for  $\mathbf{Sp}_{\mathbf{O}}$  under the stable model structure are the same as those under the projective model structure, and Corollary 2.27 says that to describe those, we need to find the left adjoints to the evaluation-at- $n$  functors  $\mathbf{Sp}_{\mathbf{O}} \rightarrow \mathbf{Top}_*$ . Define

$$\mathrm{sh}^n X = X_{n+-}$$

i.e. the composite of  $X : \mathcal{J}_{\mathbf{O}} \rightarrow \mathbf{Top}_*$  with  $n + - : \mathcal{J}_{\mathbf{O}} \rightarrow \mathcal{J}_{\mathbf{O}}$ . This shifts the spaces that appear in  $X$  down by  $n$ , deleting the bottom  $n$  of them and restricting the orthogonal group actions appropriately. This is precisely the nerve induced by the functor  $\mathcal{J}_{\mathbf{O}} \rightarrow \overline{\mathcal{J}}_{\mathbf{O}}$  given by  $m \mapsto \mathcal{J}_{\mathbf{O}}(n + m, -)$  and hence has the realization  $\mathrm{sh}^{-n}$  as a left adjoint. It can be checked that  $\mathrm{sh}^n$  preserves fibrations and acyclic fibrations. The right adjoint preserving acyclic fibrations

is equivalent to the left adjoint  $\mathrm{sh}^{-n}$  preserving cofibrations, and hence this is a Quillen adjunction.

Since the evaluation-at- $n$  functor is isomorphic to  $\Omega^\infty \mathrm{sh}^n$ , its left adjoint can be written as the composite of the left adjoints  $\mathrm{sh}^{-n} \Sigma^\infty$ . This will give us an explicit description of these left adjoints once we know what the realization  $\mathrm{sh}^{-n}$  looks like. This realization functor shifts the spaces that appear in a given spectrum up by  $n$ , induces up the actions of the orthogonal groups (see Proposition 5.2), and inserts the zero space  $*$  in the lower degrees. This gives an explicit description of the generating cofibrations for  $\mathbf{Sp}_\mathbf{O}$  that looks similar to that for  $\mathbf{Sp}$ .

**Proposition 4.14.** There is a natural weak equivalence  $\Sigma \rightarrow \mathrm{sh}$ .

*Proof.* See [Sch, Proposition 2.15]. □

**Corollary 4.15.** The left derived functors  $\mathbb{L}\Sigma$  and  $\mathbb{L}(\mathrm{sh}^{-1})$  on  $\mathcal{SH}$  are essential inverses.

*Proof.* The above proves that  $\mathbb{L}\Sigma \cong \mathbb{R}\mathrm{sh}$ . Since  $\mathbb{L}\Sigma \dashv \mathbb{R}\Omega$  is an adjoint equivalence,  $\mathbb{R}\Omega$  is also the left adjoint of  $\mathbb{L}\Sigma$ . The result follows from the uniqueness of left adjoints and essential inverses up to natural isomorphism. □

The shift operator allows us to say some things about duals and inverses, which we briefly discuss here. A *weak dual* of an object is the internal hom from it to the unit, generalizing the dual of a vector space. The *dual* of an object is required to satisfy some more properties but it need not exist for every object. Briefly, viewing the symmetric monoidal category as a 2-category with one object, a dual for an object is exactly a two-sided adjoint to it as a 1-morphism. See [Wic, Lecture 8] for a more concrete definition.

In  $\mathcal{SH}$ , we can often construct duals of spaces (more precisely, of the suspension spectra of spaces) in the following way. Suppose we embed a space  $X$  into some  $S^n$ . Then the complement  $S^n \setminus X$  is stably independent of the chosen embedding; the spectrum  $\Sigma^{-(n-1)}(S^n \setminus X)$ , up to homotopy, does not depend on the embedding, and is the dual of  $X$ . If  $X$  is a manifold (in which case, Whitney's theorem implies the existence of an embedding  $X \rightarrow S^n$ ), this can be identified with the  $n$ -fold desuspension of the Thom space of the normal bundle of the embedding  $X \hookrightarrow S^n$ . Applying the Thom isomorphism theorem then gives us a proof of Poincaré duality. See [Wic, Lecture 10]. This is relevant for us because it highlights the connection between embeddability, stability, and Poincaré duality, and hence serves as a guide when we try to define the stabilization of equivariant homotopy theory in Chapter 6.

Dualization is a contravariant functor on the subcategory of dualizable objects. This gives another construction of duals for finite CW spectra. For this, we need to know that a finite coproduct of dualizable objects is dualizable (with the

#### 4 Stable Homotopy Theory

dual being the coproduct of duals), and that the homotopy cofiber of a map of dualizable spectra is dualizable (with its dual being the desuspension of the cofiber of the dual map). Since the  $k$ -skeleton  $\mathrm{sk}_k X$  of a CW spectrum  $X$  is obtained as the homotopy cofiber of a map

$$\bigvee S^{k+1} \rightarrow \mathrm{sk}_{k-1} X$$

this allows construction of the duals. The advantage of this construction is that we can inductively prove that the dual has a finite CW structure too because the homotopy cofiber of a map between CW spectra is a CW spectrum. In fact, we find that the dual has exactly the same number of cells but the dimension of each cell gets negated.

An *invertible object* is an object  $X$  such that there is some  $Y$  with  $X \wedge Y$  isomorphic to the unit of the symmetric monoidal category. For example, the sphere  $S^n$  for  $n \geq 0$  is invertible with inverse

$$S^{-n} := \mathrm{sh}^{-n} S^0$$

because

$$S^n \wedge S^{-n} = S^n \wedge \mathrm{sh}^{-n} S^0 \cong \mathrm{sh}^{-n} S^n$$

where the isomorphism follows from the fact that  $\mathrm{sh}^{-n} S^0 \cong \mathcal{J}_{\mathbf{O}}(n, -)$  (because they are both free  $\mathbf{Top}_*$ -functors generated by one point at level  $n$ ) and how the Day convolution product behaves with representable functors (see the proof of Theorem 3.10). Corollary 4.15 tells us that this is isomorphic (in  $\mathcal{SH}$ ) to  $S^0$ , and recall that  $S^0$ , or more precisely,  $\Sigma^\infty S^0 \cong \mathbb{S}$ , is the monoidal unit in  $\mathcal{SH}$ .

An invertible object is always dualizable with the dual being an inverse, which is a consequence of the fact that every equivalence in a 2-category can be improved to an adjoint equivalence. Therefore, the dual of  $S^n$  is  $S^{-n}$ . Since  $S^n$  has a finite CW structure, so does  $S^{-n}$ . This is not very interesting in this case as we already had a CW structure on  $S^{-n}$ , but will be interesting in the equivariant case.

# 5 Equivariant Homotopy Theory

Given a group  $G$ , a  $G$ -space is a topological space with an action of  $G$  on it. We will mostly be concerned with the case of  $G = C_2$ , the cyclic group with 2 elements, but the theory will be developed in the general setting. The point of studying these spaces is highlighted in the introduction of [Grea]. It is also summarized in Section 7.3.

Throughout this thesis,  $G$  will be a finite group with identity  $e \in G$ .

## 5.1 Category of $G$ -spaces

A (left)  $G$ -space is a topological space along with a continuous map

$$G \times X \rightarrow X$$

usually written with juxtaposition  $(g, x) \mapsto gx$ , satisfying the usual action axioms

$$g_1(g_2x) = (g_1g_2)x, \quad ex = x$$

We then have the category  $\mathbf{Top}^G$  of  $G$ -spaces in which the morphisms are the  $G$ -maps, i.e. maps  $f : X \rightarrow Y$  such that

$$f(gx) = g(f(x)) \quad \forall g \in G, x \in X, y \in Y$$

A right  $G$ -space is defined similarly. A  $G$ -space will be a left  $G$ -space unless otherwise specified.

The category of pointed  $G$ -spaces  $\mathbf{Top}_*^G$  is defined similarly, where the action of  $G$  is required to fix the basepoint. For the rest of this chapter, a map will be a  $G$ -map unless otherwise specified.

Both  $\mathbf{Top}^G$  and  $\mathbf{Top}_*^G$  are bicomplete. They can be viewed as the categories of functors from a category  $\mathcal{BG}$  to  $\mathbf{Top}$  and  $\mathbf{Top}_*$  respectively, where  $\mathcal{BG}$  is  $G$  viewed as a category with one object. As such, a limit or colimit of a diagram in one of these categories is computed as it is in the non-equivariant case and then endowed with the unique  $G$ -action compatible with the actions of all the  $G$ -spaces in the diagram.

In particular,  $G$  acts diagonally on a product of (pointed)  $G$ -spaces, and hence also on its quotient, the smash product. This makes these categories symmetric monoidal. We also have an internal hom. For  $G$ -spaces  $X, Y$ , this is defined

as the set  $\mathbf{Top}(X, Y)$  of all (not necessarily equivariant) maps with its usual compact-open topology and  $G$ -action defined by

$$(gf)(x) = g(f(g^{-1}x))$$

for all  $g \in G$ ,  $f \in \mathbf{Top}(X, Y)$  and  $x \in X$ . In the pointed case, we take the pointed space of pointed maps with the same action. We denote these internal homs by  $\mathbf{Map}(X, Y)$  and  $\mathbf{Map}_*(X, Y)$  respectively.

Given any space  $X$ , we can view it as a  $G$ -space by endowing it with the trivial  $G$ -action. This gives a functor

$$\mathrm{Triv}_G : \mathbf{Top} \rightarrow \mathbf{Top}^G$$

We also have restriction functors for every subgroup  $H \leq G$

$$\mathrm{Res}_H^G : \mathbf{Top}^G \rightarrow \mathbf{Top}^H$$

that forget the actions of all elements not in  $H$ . Both of these are part of important adjunctions as follows.

**Definition 5.1.** The *product over  $H$*  of a right  $H$ -space  $X$  with a left  $H$ -space  $Y$  is

$$X \times_H Y := (X \times Y) / \sim$$

where  $\sim$  is the equivalence relation generated by  $(gh, y) \sim (x, hy)$  for all  $x \in X, y \in Y, h \in H$ .

A smash product over  $H$  when  $X$  and  $Y$  are pointed is defined as a similar quotient of the smash product. Note that the (smash) product over  $H$  is a non-equivariant space. However, if  $X$  also has a compatible left action of a group  $G$ , then  $G$  has a left action on  $X \times_H Y$  by acting on  $X$ . If  $H \leq G$ , then  $G$  has a left  $G$ -action and a right action of (in particular)  $H$ . This is analogous to tensor products of right  $R$ -modules with left  $R$ -modules to get abelian groups.

**Proposition 5.2.** (i)  $\mathrm{Triv}_G$  has a right adjoint called the *fixed point functor*,  $(-)^G : \mathbf{Top}^G \rightarrow \mathbf{Top}$ . This functor takes a  $G$ -space  $X$  to its subspace of fixed points  $X^G$ .

(ii) For each  $H \leq G$ ,  $\mathrm{Res}_H^G$  has a left adjoint called *induction*

$$\mathrm{Ind}_H^G := G \times_H - : \mathbf{Top}^H \rightarrow \mathbf{Top}^G$$

When  $X$  has a trivial  $H$ -action, then  $G \times_H X \cong G/H \times X$ . All of these definitions and results can again be adapted to pointed  $G$ -spaces. In this case,  $\mathrm{Ind}_H^G X = G_+ \wedge_H X$ .



## 5.2 Homotopy theory of $G$ -spaces

A  $G$ -homotopy between  $f, g : X \rightarrow Y$  is a  $G$ -map  $H : X \times I \rightarrow Y$  where  $G$  acts trivially on  $I = [0, 1] \in \mathbb{R}$  with

$$H(-, 0) = f, \quad H(-, 1) = g$$

$G$ -homotopy equivalences can then be defined in the usual way. An analogous definition can be made for the pointed case. Our objective is to study the homotopy theory of  $\mathbf{Top}^G$  and  $\mathbf{Top}_*^G$ , which we do by putting model structures on both of these categories.

Viewing  $\mathbf{Top}^G$  as the category of functors  $\mathcal{B}G \rightarrow \mathbf{Top}$  gives us a projective model structure on  $\mathbf{Top}^G$  via Corollary 2.27, but a weak equivalence in this model structure is a weak homotopy equivalence of the underlying non-equivariant spaces, so this is not the model structure we want. To get at the right model structure, we begin by identifying the  $G$ -CW complexes and then proving a version of Whitehead's theorem that tells us what  $G$ -weak equivalences should be.

Our notion of  $G$ -CW complexes should be broad enough that it captures every  $G$ -space that we would care about, such as  $G$ -manifolds or spaces constructed from orthogonal representations of  $G$ . There are many equivalent versions which differ in their choices of cells, but the one below is the most commonly used for its simplicity. It is to be thought of as a CW-complex with a  $G$ -action that only permutes cells. In the following,  $S^n$  and  $D^n$  are regarded as  $G$ -spaces with trivial actions (i.e. we are suppressing  $\mathrm{Triv}_G$  from our notation).

**Definition 5.3.** A  $G$ -CW complex is a  $G$ -space  $X$  with a filtration  $X_0 \subset X_1 \subset X_2 \subset \dots$  where  $X_{-1}$  is empty and  $X_n$  is obtained from  $X_{n-1}$  by attaching the induced up  $n$ -disks  $G/H \times D^n$  along maps on their boundaries  $G/H \times S^{n-1} \rightarrow X_{n-1}$ . We call this an  $n$ -dimensional  $G$ -cell of type  $G/H$ .

We also have, for any such  $G$ -cell, the adjoint  $S^{n-1} \rightarrow X_{n-1}^H$  of the attaching map obtained by applying both the adjunctions of Proposition 5.2. Indeed, a  $G$ -map  $G/H \times S^{n-1} \cong G \times_H S^{n-1} \rightarrow X_{n-1}$  is equivalent to an  $H$ -map  $S^{n-1} \rightarrow X$ , which is then equivalent to a non-equivariant map  $S^{n-1} \rightarrow X^H$ . This is usually how we will describe the attaching map. We could also take the union of all the  $G$ -orbits used to go from  $X_{n-1}$  to  $X_n$  to get some  $G$ -set  $K$ , and then  $X_n$  is the pushout

$$\begin{array}{ccc} K \times S^{n-1} & \hookrightarrow & K \times D^n \\ \downarrow & & \downarrow \\ X_{n-1} & \xrightarrow{\quad \quad} & X_n \end{array}$$

Every  $G$ -manifold has a  $G$ -CW complex structure, justifying this definition to some extent (see [III]). As with any CW construction, May's HELP Lemma [Mayb, Section 10.3] gives us the following version of Whitehead's theorem for

these  $G$ -CW complexes. This is spelled out in greater detail in [Blu, Theorem 1.2.11]. The idea, similar to that of the non-equivariant case, is that  $G$ -CW complexes are built from attaching cones along  $G/H \times S^n$ , so probing  $X$  by these suffices to infer its homotopy type. But again, maps of this kind are equivalent to maps  $S^n \rightarrow X^H$  by Proposition 5.2. Of course, the model theoretic proof of Whitehead's theorem cannot be applied since we have not yet constructed the correct model structure on  $G$ -spaces.

**Theorem 5.4.** Let  $f : X \rightarrow Y$  be a map of  $G$ -CW complexes. Then  $f$  is a homotopy equivalence if and only if  $f^H : X^H \rightarrow Y^H$  is a weak equivalence for every  $H \leq G$ .  $\square$

**Definition 5.5.** A map  $f : X \rightarrow Y$  of  $G$ -spaces is a weak equivalence if  $f^H$  is a weak equivalence for every  $H \leq G$ .

This leads to the following observation. Let  $\mathcal{O}_G$  be the full  $\mathbf{Top}$ -subcategory of  $\mathbf{Top}^G$  spanned by the  $G$ -orbits  $G/H$  with the discrete topology (the topological enrichment here will not matter as each morphism set in  $\mathcal{O}_G$  has the discrete topology). Then a map  $X \rightarrow Y$  is a weak equivalence if and only if the nerve induced by the inclusion  $\mathcal{O}_G \rightarrow \mathbf{Top}^G$  maps it to a weak equivalence in the presheaf category  $\widehat{\mathcal{O}_G}$  with the projective model structure. It can be checked that this nerve-realization adjunction satisfies the criteria of Theorem 2.25, so it may be used to induce a model structure on  $\mathbf{Top}^G$  with the correct weak equivalences. We have thus the following description of the model structure on  $\mathbf{Top}^G$ .

**Theorem 5.6.** There is a cofibrantly generated model structure on  $\mathbf{Top}^G$  with generating sets

$$\begin{aligned} \mathcal{I} &= \{G/H \times S^n \hookrightarrow G/H \times D^{n+1} \mid n \in \mathbb{N}, H \leq G\} \\ \mathcal{J} &= \{G/H \times D^n \hookrightarrow G/H \times D^n \times I \mid n \in \mathbb{N}, H \leq G\} \end{aligned}$$

where each inclusion is the standard one.  $f : X \rightarrow Y$  is a weak equivalence (resp. fibration) if and only if  $f^H$  is a weak equivalence (resp. fibration) for each  $H \leq G$ .  $\square$

This also justifies our definition of  $G$ -CW complexes as it gives us a  $G$ -CW approximation theorem (see the paragraph preceding Theorem 2.3). We will write

$$[X, Y]^G := \mathrm{Ho} \mathbf{Top}^G(X, Y), \quad [X, Y]_*^G := \mathrm{Ho} \mathbf{Top}_*^G(X, Y)$$

for (pointed)  $G$ -spaces  $X, Y$ .

Generally, a transferred model structure need not be Quillen equivalent to the model structure it was transferred from, but we do have this result in this case.

**Theorem 5.7** (Elmendorf). The topological nerve-realization adjunction

$$\begin{array}{ccc} \mathcal{O}_G & \xrightarrow{y} & \widehat{\mathcal{O}}_G \\ & \searrow & \uparrow (-) \\ & & \mathbf{Top}^G \end{array}$$

is a Quillen equivalence.

*Proof.* See [Blu, Section 1.3]. □

**Example 5.8.** Let  $G = C_6$ . We think of this as  $\mathbb{Z}/6\mathbb{Z}$  under addition. Then  $\mathcal{O}_G$  is the category

$$\begin{array}{ccccc} & & C_6/C_6 & & \\ & \nearrow & & \nwarrow & \\ C_6/3C_6 & & & & C_6/2C_6 \\ & \nwarrow \scriptstyle 1 & \uparrow & \nearrow \scriptstyle 1 & \\ & C_6/e & & & \end{array}$$

where the identity maps and the endomorphisms have been omitted. The arrow labelled  $i$  is the one that maps the equivalence class 0 to the equivalence class of  $i$ . The unlabelled arrows are to be interpreted as being labelled by 0. The nerve  $\mathbf{Top}^G \rightarrow \widehat{\mathcal{O}}_G$  takes a space  $X$  to the presheaf

$$\begin{array}{ccccc} & & X^{C_6} & & \\ & \nwarrow & & \nearrow & \\ X^{3C_6} & & & & X^{2C_6} \\ & \nwarrow \scriptstyle 1 & \downarrow & \nearrow \scriptstyle 1 & \\ & X & & & \end{array}$$

where each map is the inclusion followed by the action of the appropriate element.

Everything done till now can also be adapted to pointed  $G$ -spaces. In this case, we change the enrichment of  $\mathcal{O}_G$  to be over  $\mathbf{Top}_*$  by adding a disjoint basepoint to each Hom space and use the nerve-realization adjunction induced by the composite

$$\mathcal{O}_G \hookrightarrow \mathbf{Top}^G \xrightarrow{(-)_+} \mathbf{Top}_*^G$$

with the understanding that the newly added basepoints are mapped to the basepoints of the Hom spaces in  $\mathbf{Top}^G$ , i.e. the constant maps. It can be checked that the nerve of a pointed  $G$ -space has a description similar to the one in Example 5.8, except that now all the spaces appearing in the diagram must be

pointed. More succinctly, the above is a case of free change of enrichments, as  $(-)_+$  is the free functor  $\mathbf{Top} \rightarrow \mathbf{Top}_*$ , and hence the form that the diagrams take does not change. The pointed version of Elmendorf's theorem then holds and tells us the structure of the homotopy invariants.

**Definition 5.9.** A  $G$ -coefficient system is a functor  $\mathcal{O}_G^{\text{op}} \rightarrow \mathbf{Ab}$ . More generally, for a category  $\mathcal{C}$ , a  $\mathcal{C}$ -valued  $G$ -coefficient system is a functor  $\mathcal{O}_G^{\text{op}} \rightarrow \mathcal{C}$ .

**Definition 5.10.** Let  $X$  be a pointed  $G$ -space, and let  $\tilde{X} : \mathcal{O}_G \rightarrow \mathbf{Top}_*$  be its nerve as described above. Define the  $n$ th homotopy coefficient system of  $X$  to be the composite

$$\mathcal{O}_G \xrightarrow{\tilde{X}} \mathbf{Top}_* \xrightarrow{\pi_n} \mathbf{Ab}$$

Of course, this is only a  $G$ -coefficient system for  $n \geq 2$  since  $\pi_0$  and  $\pi_1$  are  $\mathbf{Set}$  and  $\mathbf{Grp}$  valued respectively. In those cases, we instead have  $\mathbf{Set}$ - or  $\mathbf{Grp}$ -valued  $G$ -coefficient systems. In the stable case, we will not have to worry about these edge cases.

Unpacking the definition, this says that the coefficient system  $\pi_n(X)$  has values

$$\pi_n(X)(G/H) \cong \pi_n(X^H) \cong [S^n, X^H]_*^e \cong [(G/H)_+ \wedge S^n, X]_*^G$$

for each  $H \leq G$ . It is clear that these homotopy coefficient systems together form a complete set of invariants on the category of presheaves on  $\mathcal{O}_G$ , i.e. a map  $f$  of such presheaves is a weak equivalence if and only if  $\pi_n(f)$  is a weak equivalence for every  $n \geq 0$  and every compatible choice of basepoints in the domain of  $f$ . Since  $G$ -CW complexes are realizations of cofibrant presheaves on  $\mathcal{O}_G$  (because the  $G$ -cell inclusions are) and every  $G$ -space is fibrant, a map of  $G$ -CW complexes is a weak equivalence if and only if its image under each  $\pi_n$  is an isomorphism. This justifies our definition of the homotopy coefficient systems.

One could also then define the  $n$ th equivariant (co)homology of a  $G$ -space  $X$  as the composite

$$\mathcal{O}_G \xrightarrow{\tilde{X}} \mathbf{Top} \xrightarrow{H_n \text{ or } H^n} \mathbf{Ab}$$

where  $\tilde{X}$  is the nerve of  $X$ . This is not the right definition of these functors because it lacks one of the most important features of non-equivariant (co)homology - Poincaré duality. We defer equivariant (co)homology and Eilenberg-MacLane spaces to the next chapter.

## 6 Equivariant Stable Homotopy Theory

We now wish to stabilize the equivariant homotopy theory. One could again invert the suspension functor  $- \wedge S^1 : \mathbf{Top}_*^G \rightarrow \mathbf{Top}_*^G$  by taking spectrum objects in  $\mathbf{Top}_*^G$ , i.e. sequences  $(X_n)_{n \in \mathbb{N}}$  in  $\mathbf{Top}_*^G$  with structure maps  $X_n \wedge S^1 \rightarrow X_{n+1}$ . Such spectra are called the naive  $G$ -spectra. This is not the stability we want.

If we want some of our most important theorems, such as Poincaré duality, to generalize to the equivariant setting, then we need to be able to embed  $G$ -manifolds equivariantly into spheres. However, since the spheres  $S^n$  have trivial  $G$ -action, this will be impossible for any manifolds with non-trivial action. An equivariant version of Whitney's theorem says that every  $G$ -manifold embeds into some finite dimensional orthogonal  $G$ -representation  $V$  and hence into the one-point compactification  $S^V$  of  $V$ , where  $G$  acts trivially on the point at infinity. These spaces  $S^V$  are called the  $G$ -representation spheres. Therefore, the functors that we want to invert are  $\Sigma^V := - \wedge S^V$  rather than the 'trivial' suspensions  $\Sigma^n$ . Since every finite dimensional  $G$ -representation is a summand of a multiple of the regular  $G$ -representation  $\rho$  and  $S^V \wedge S^W \cong S^{V \oplus W}$ , inverting all  $\Sigma^V$  is equivalent to inverting just  $\Sigma^\rho$ . For us,  $G$ -representations will always be finite dimensional and orthogonal. We write  $|V|$  for the dimension of  $V$ .

### 6.1 $G$ -spectra

One attempt at a definition of  $G$ -spectra could be as follows. A  $G$ -spectrum would be a  $G$ -space  $X_V$  for each  $G$ -representation  $V$  with compatible structure maps  $\sigma_{V,W} : \Sigma^W X_V \rightarrow X_{V \oplus W}$ . However, this compatibility is hard to state because the direct sum is only associative up to coherent natural isomorphisms and not on-the-nose. Therefore, for every  $G$ -isomorphism  $V \rightarrow W$ , we should also have an isomorphism  $X_V \rightarrow X_W$  satisfying some compatibility conditions with the structure maps. Once again, we would realize these objects as right modules over a diagram  $\mathbb{S}$  defined by  $\mathbb{S}_V = S^V$ .  $\mathbb{S}$  would be a functor from the  $\mathbf{Top}_*^G$ -category  $\mathcal{D}^G$  with objects the  $G$ -representations and the morphism object  $\mathcal{D}^G(V, W)$  the space of the  $G$ -isomorphisms  $V \rightarrow W$  with a disjoint basepoint and a trivial  $G$ -action. For  $G$  the trivial group, this gives a category equivalent to that of orthogonal spectra.

However,  $\mathbb{S}$  would not be a commutative monoid. If we try to follow the

argument for commutativity in the case of symmetric spectra (preceding Definition 4.11), we see the problem. The  $G$ -isomorphisms  $V \oplus W \rightarrow W \oplus V$  do not include the linear isomorphisms which ‘interchange  $V$  and  $W$ ’, i.e. the isomorphisms given by block matrices of the form

$$\begin{bmatrix} 0 & I_{|V|} \\ I_{|W|} & 0 \end{bmatrix}$$

upon a choice of bases of  $V$  and  $W$ . In order to make  $\mathbb{S}$  a commutative monoid, we need the morphism objects to contain all of these isomorphisms.

Define  $\mathcal{D}_O^G$  as the  $\mathbf{Top}_*^G$ -category with objects the  $G$ -representations and morphism objects  $\mathcal{D}_O^G(V, W) = O(V, W)_+$  where  $O(V, W)$  denotes the  $G$ -space of linear isomorphisms  $V \rightarrow W$  that preserve the inner product. The action of  $G$  on  $O(V, W)$  is given by conjugation (as in the internal hom of  $G$ -spaces). We will write  $O(V) := O(V, V)$  and  $n \in \mathbb{N}$  for  $\mathbb{R}^n$  with the trivial action when clear from context.  $\mathcal{D}_O^G$  has a symmetric monoidal product defined by direct sum and hence its copresheaf category is closed symmetric monoidal by Theorem 3.10 ( $\mathcal{D}_O^G$  is not small, but it has a small skeleton, which is enough for all relevant coends to exist).  $\mathbb{S}_V := S^V$  defines a commutative monoid in this category. We define an orthogonal  $G$ -spectrum to be a right module over  $\mathbb{S}$ , or equivalently a copresheaf on a small category  $\mathcal{J}_O^G$  as in Theorem 3.11. Unravelling this, we get the following definition.

**Definition 6.1.** (i) An *(orthogonal, genuine)  $G$ -spectrum* consists of a pointed  $G$ -space  $X_V$  for every  $G$ -representation  $V$  along with  $G$ -equivariant action maps

$$\alpha_{V,W} : O(V, W)_+ \wedge X_V \rightarrow X_W$$

for  $G$ -representations  $V, W$  of the same dimension and  $G$ -equivariant structure maps

$$\sigma_{V,W} : \Sigma^W X_V \rightarrow X_{V \oplus W}$$

for  $G$ -representations  $V, W$  such that:

- For  $G$ -representations  $U, V, W$ ,

$$\begin{array}{ccc} \Sigma^{V \oplus W} X_U & \xleftarrow{\sim} & \Sigma^W \Sigma^V X_U \xrightarrow{\Sigma^W \sigma_{U,V}} \Sigma^W X_{U \oplus V} \\ \sigma_{U, V \oplus W} \downarrow & & \downarrow \sigma_{U \oplus V, W} \\ X_{U \oplus (V \oplus W)} & \xrightarrow{\sim} & X_{(U \oplus V) \oplus W} \end{array}$$

commutes, where the bottom map is the action of the canonical associator isomorphism  $\alpha : U \oplus (V \oplus W) \rightarrow (U \oplus V) \oplus W$ .

- The actions are associative and unital: the two maps  $O(V, W)_+ \wedge O(U, V)_+ \wedge X_U \rightarrow X_W$  built from the action maps coincide, and the identity map in  $O(V, V)$  acts by identity on  $X_V$ .

- The action maps on  $X$  and on  $\mathbb{S}$  are compatible, i.e. the following diagram commutes for any  $G$ -representations  $V, V', W, W'$ .

$$\begin{array}{ccc}
O(W, W')_+ \wedge O(V, V')_+ \wedge X_V \wedge S^W & \longrightarrow & X_{V'} \wedge S^{W'} \\
\downarrow & & \downarrow \\
O(V \oplus W, V' \oplus W')_+ \wedge X_{V \oplus W} & \longrightarrow & X_{V' \oplus W'}
\end{array}$$

- The composite,

$$X_V \wedge S^0 \xrightarrow{\sim} X_V \xrightarrow{\sim} X_{V \oplus 0}$$

where both maps are (induced by) the canonical isomorphisms, coincides with the structure map  $\sigma_{V,0} : \Sigma^0 X_V \rightarrow X_{V \oplus 0}$ .

- (ii) A map  $f : X \rightarrow Y$  of  $G$ -spectra is a collection of pointed  $G$ -maps  $f_V : X_V \rightarrow Y_V$  for each  $G$ -representation  $V$  which commute with the structure and action maps. Thus we have a category of orthogonal  $G$ -spectra, which will be denoted  $\mathbf{Sp}_O^G$ .

**Proposition 6.2.** For any  $G$ -representations  $V, W$  of the same dimension, there is a natural  $G$ -homeomorphism

$$X_W \cong O(V, W)_+ \wedge_{O(V)} X_V$$

*Proof.* The map from the right side to the left is given by the action map, which factors through this smash over  $O(V)$  because of the associativity condition on the action. The inverse can be described by choosing any isomorphism  $h : V \rightarrow W$ ; it maps  $x \in X_W$  to the class of  $(h, \alpha_{V,W}(h^{-1}, x))$ .  $\square$

**Remark 6.3.** From this result and the coherence conditions on the structure and action maps, we can infer that a  $G$ -spectrum is completely determined by its restriction to the integer degrees, i.e. define  $\mathcal{J}_{tr}^G \subset \mathcal{J}_O^G$  to be the full subcategory spanned by the positive integers, and then  $X : \mathcal{J}_O^G \rightarrow \mathbf{Top}_*^G$  is determined by its restriction to  $\mathcal{J}_{tr}^G$ . This gives an equivalence of categories between  $\mathbf{Sp}_O^G$  and the so-called *naïve orthogonal  $G$ -spectra*, which are the copresheaves on  $\mathcal{J}_{tr}^G$ . If we unravel the definition of this structure, we find that this is exactly an orthogonal spectrum with an action of  $G$ , or equivalently an ‘orthogonal spectrum object’ in  $\mathbf{Top}_*^G$ . Direct sum restricts to a symmetric monoidal product on  $\mathcal{J}_{tr}^G$  which induces a product on naïve  $G$ -spectra. This agrees with the Day convolution product on genuine  $G$ -spectra, because direct sum of representations adds dimensions. The stable model structure is easier to define on genuine  $G$ -spectra while point-set level constructions are easier on the naïve  $G$ -spectra because we need to give less data.

**Definition 6.4.** Let  $H \leq G$  be groups. Define the *restriction functor*  $\text{Res}_H^G : \mathbf{Sp}_O^G \rightarrow \mathbf{Sp}_O^H$  by restricting the action to  $H$ :

$$(\text{Res}_H^G X)_n := \text{Res}_{H \times O(n)}^{G \times O(n)} X_n$$

This has a left adjoint called *induction*

$$(\text{Ind}_H^G Y)_n := \text{Ind}_{H \times O(n)}^{G \times O(n)} Y_n$$

We also have  $\text{Triv}_G : \mathbf{Sp}_O^G \rightarrow \mathbf{Sp}_O^G$  that takes a spectrum to itself with the trivial  $G$ -action with the orthogonal groups acting the same way

$$(\text{Triv}_G X)_n := \text{Triv}_G X_n$$

This has a right adjoint  $(-)^G$  which takes fixed points at each level under the restricted action of the orthogonal groups

$$(X^G)_n := (X_n)^G$$

For  $X$  a based  $G$ -space, the *suspension  $G$ -spectrum* of  $X$  is the image of  $X$  under the left adjoint  $\Sigma^\infty$  of the functor  $\Omega^\infty : \mathbf{Sp}_O^G \rightarrow \mathbf{Top}_*^G$  that evaluates a  $G$ -spectrum at the zero representation.

We can also define shift operators on  $G$ -spectra. For any  $G$ -representation  $V$ ,

$$\text{sh}^V X := X_{V \oplus -}$$

Since this is the nerve functor induced by  $W \mapsto \mathcal{J}_O^G(-, V \oplus W)$ , it has a left adjoint  $\text{sh}^{-V}$ .

We get a projective model structure on genuine  $G$ -spectra by viewing them as copresheaves on  $\mathcal{J}_O^G$ . The generating cofibrations in this model structure can be described as in Chapter 4; they will be the inclusions  $S(H, m-1, V) \hookrightarrow D(H, m, V)$  where

$$S(H, m, V) = \text{sh}^{-V} \Sigma^\infty((G/H)_+ \wedge S^m)$$

$$D(H, m, V) = \text{sh}^{-V} \Sigma^\infty((G/H)_+ \wedge D^m)$$

We can now define  $G$ -CW spectra. We will require that the cells of a  $G$ -CW spectrum be the  $D(H, m, V)$  where  $V$  is an integer  $n$ . Define the dimension of the sphere  $S(H, m, n)$  to be  $m - n$ , and then every map from a lower dimensional sphere to higher dimensional one is trivial up to level-wise homotopies (this is not true if we let  $V$  be any representation). Let  $\mathcal{S}$  denote the set of these inclusions  $S(H, m, n) \hookrightarrow D(H, m, n)$ . We define  $G$ -CW spectra as the  $\mathcal{S}$ -cellular objects in which the cells are attached in dimension order.

We need to define the stable equivalences. Since we are replacing the ordinary spheres with representation spheres, our homotopy groups must also be indexed by  $G$ -representations rather than just the integers and the colimit should



be taken over all representation spheres. Since we are doing these things stably, we get negative degree homotopy groups too. Thus, the homotopy groups will be indexed over the class of formal differences  $V - W$  where  $V, W$  are  $G$ -representations. Similar to Definition 5.10, the homotopy groups should actually be richer - they should allow evaluation at every  $G$ -orbit.

**Definition 6.5.** (i) For  $G$ -representations  $V_1, V_2$ , the  $(V_1 - V_2)$ th homotopy group of  $X \in \mathbf{Sp}_O^G$  at a  $G$ -orbit  $G/H$  is

$$\pi_{V_1 - V_2}(X)(G/H) := \operatorname{colim}_{G\text{-rep } W \geq V_2} [(G/H)_+ \wedge S^{(W - V_2) \oplus V_1}, X_W]_*^G$$

where the colimit is over  $G$ -representations that contain  $V_2$ , which form a directed set under inclusion.  $W - V_2$  denotes the orthogonal complement of  $V_2$  in  $W$ .

(ii) A map of  $G$ -spectra  $f : X \rightarrow Y$  is a *stable equivalence* if  $\pi_{V - W}(f)(G/H)$  is an isomorphism for all  $G$ -representations  $V, W$  and  $H \leq G$ .

Everything said about orthogonal spectra in Chapter 4 now applies. We take the left Bousfield localization of the model structure at the stable equivalences to get the stable model structure. The set of generating cofibrations is again  $\mathcal{S}$  as described above. Define the  $G$ -equivariant stable homotopy category as  $\mathcal{SH}^G := \operatorname{Ho} \mathbf{Sp}^G$ .

Day convolution gives us a closed symmetric monoidal product, which we again call the smash product  $\wedge$ . We have an internal hom, which is a  $G$ -spectrum  $\operatorname{Map}_G(X, Y)$ . The  $V$ th pointed  $G$ -space in this  $G$ -spectrum is the space of (non-equivariant) spectrum maps  $\operatorname{sh}^V X = X(- \oplus V) \rightarrow Y$  with  $G$  acting on it by conjugation (see Remark 6.3 to understand ‘spectrum maps’). Taking the fixed point spectrum of this, we get the spectrum  $\operatorname{Map}^G(X, Y)$  of maps from  $X \rightarrow Y$ , giving a spectral enrichment on  $\mathbf{Sp}_O^G$ . More explicitly, the  $n$ th space of  $\operatorname{Map}^G(X, Y)$  is the space of equivariant maps  $X(- \oplus n) \rightarrow Y$ , which is the subspace of compatible maps in the product of mapping spaces  $X(k \oplus n) \rightarrow Y(k)$ . As stated before,  $n$  here refers to the canonical  $n$ -dimension trivial  $G$ -representation  $\mathbb{R}^n$ .

As in the non-equivariant case,  $S^V$  becomes an invertible object in  $\mathcal{SH}^G$ . The discussion at the end of Chapter 4 says that it must be dualizable and with dual and inverse a finite  $G$ -CW spectrum  $S^{-V}$ .

## 6.2 Mackey Functors

Similar to the unstable case, we have more structure on the homotopy groups of  $G$ -spectra and the following stable analogue of Elmendorf’s theorem tells us

what this structure should be. As in Theorem 5.7, we probe each spectrum by the  $G$ -orbits. This probing will be spectrally enriched. However, there is a simplification here. A spectrally enriched functor preserves all finite coproducts so instead of probing with just the  $G$ -orbits, we may probe with all finite  $G$ -sets without making our presheaf category any larger. This is a simplification because the subcategory of the finite  $G$ -sets will then have finite colimits, making its homotopy category additive rather than just pre-additive.

**Definition 6.6.** The *spectral Burnside category*  $\mathcal{D}$  is the full spectrally enriched subcategory of  $\mathbf{Sp}_{\mathbf{O}}^G$  spanned by the fibrant replacements of  $\Sigma^\infty T_+$ , the suspension spectra of finite  $G$ -sets  $T$ .

**Theorem 6.7.** The spectral nerve-realization adjunction

$$\mathbb{R}\mathcal{D} : \widehat{\mathcal{D}} \rightarrow \mathbf{Sp}_{\mathbf{O}}^G : \mathbb{N}\mathcal{D}$$

is a Quillen equivalence.

*Proof.* This is a special case of [GM, Theorem 1.36].  $\square$

As in Section 5.2, the 0th homotopy functor evaluated at  $X \in \mathbf{Sp}_{\mathbf{O}}^G$  should be defined as the composite

$$\mathcal{D} \xrightarrow{\tilde{X}} \mathbf{Sp}_{\mathbf{O}} \xrightarrow{\pi_0} \mathbf{Ab}$$

where  $\tilde{X} \in \widehat{\mathcal{D}}$  is the nerve of  $X$ . The evaluation of this composite at a finite  $G$ -set  $T$  is

$$\begin{aligned} \pi_0(X)(T) &= \pi_0(\tilde{X}(T)) \\ &= \mathcal{SH}(\Sigma^\infty S^0, \mathrm{Map}^G(\Sigma^\infty T_+, X)) \\ &= \mathcal{SH}^G(\Sigma^\infty S^0, \mathrm{Map}_G(\Sigma^\infty T_+, X)) \\ &\cong \mathcal{SH}^G(T_+ \wedge \Sigma^\infty S^0, X) \end{aligned}$$

The fibrant replacement in Definition 6.6 will not matter since that does not affect the set of maps in  $\mathcal{SH}^G$ . Similarly, we want to define the  $(V_1 - V_2)$ th homotopy functor evaluated at  $X \in \mathbf{Sp}^G$  to be a functor  $\mathcal{D} \rightarrow \mathbf{Ab}$  given by

$$\pi_{V_1 - V_2}(X) = \mathcal{SH}^G(-_+ \wedge S^{V_1}, S^{V_2} \wedge X)$$

where  $-$  takes values in finite  $G$ -sets. This functor is additive as  $\wedge$  distributes over  $\vee$  and  $\mathcal{SH}^G$  is additive. It also factors through  $\mathrm{Ho} \mathcal{D}$  because it takes hom sets at the homotopy level. Therefore, the homotopy functors on  $\mathbf{Sp}_{\mathbf{O}}^G$  should be valued in additive functors  $\mathrm{Ho} \mathcal{D} \rightarrow \mathbf{Ab}$ , so we need to understand the additive category  $\mathrm{Ho}(\mathcal{D})$ . It can be described algebraically as follows.

**Definition 6.8.** A *span*  $X \rightarrow Y$  in a category  $\mathcal{C}$  is a diagram

$$X \leftarrow Z \rightarrow Y$$

in  $\mathcal{C}$ . We call  $Z$  the *spanning object*. An isomorphism of spans from  $X \rightarrow Y$  is an isomorphism of spanning objects making the following diagram commute.

$$\begin{array}{ccc} & Z_1 & \\ \swarrow & \downarrow \cong & \searrow \\ X & & Y \\ \nwarrow & \downarrow & \nearrow \\ & Z_2 & \end{array}$$

If  $\mathcal{C}$  has finite coproducts, the isomorphism classes of spans  $X \rightarrow Y$  form an abelian monoid where the addition is defined by taking coproducts of the spanning objects.

**Definition 6.9.** The *Grothendieck group*  $GM$  of a commutative monoid  $M$  is the abelian group of formal differences  $m_1 - m_2$  of elements of  $M$  where we identify  $m_1 - m_2 \sim n_1 - n_2$  if there exists  $p \in M$  such that  $m_1 + n_2 + p = n_1 + m_2 + p$ .  $G$  is then functorial and is the left adjoint of the inclusion of the category of abelian groups into that of commutative monoids.

**Definition 6.10.** The *(algebraic) Burnside category*  $\mathcal{B}_G$  is the additive category in which the objects are the finite  $G$ -sets and the abelian group  $\mathcal{B}_G(X, Y)$  is the Grothendieck group of the abelian monoid of isomorphism classes of spans of finite  $G$ -sets. The composition of spans is defined by pullbacks of spans and then extended to the negatives.

$$Y \leftarrow U_2 \rightarrow Z \quad \circ \quad X \leftarrow U_1 \rightarrow Y \quad = \quad X \leftarrow U_1 \times_Y U_2 \rightarrow Z$$

This is an additive category because it has finite biproducts which are given by disjoint union.

**Proposition 6.11.** There is an equivalence of additive categories  $\mathcal{B}_G \rightarrow \text{Ho}(\mathcal{D})$ .

*Proof.* On an object  $X$ , this equivalence is defined as

$$X \mapsto \Sigma^\infty X_+$$

On a basis element  $X \leftarrow U \rightarrow Y$  of  $\mathcal{B}_G$ , this is defined to be the composite

$$\Sigma^\infty X_+ \rightarrow \Sigma^\infty U_+ \rightarrow \Sigma^\infty Y_+$$

where the second map is the obvious one, and the first one is the transfer map associated to the map  $X \leftarrow U$  constructed via the Pontryagin-Thom map. For details, see [Maya, Section IX.3].  $\square$

**Definition 6.12.** A  $G$ -Mackey functor is an additive functor  $\mathcal{B}_G^{\text{op}} \rightarrow \mathbf{Ab}$ .

A Mackey functor is typically denoted by a symbol with an underline. The following definition extends (i) of Definition 6.5.

**Definition 6.13.** For  $X \in \mathbf{Sp}_O^G$  and  $G$ -representations  $V_1, V_2$ , the  $(V_1 - V_2)$ th homotopy Mackey functor is defined on a  $G$ -set  $T \in \mathcal{B}_G$  as

$$(\underline{\pi}_{V_1 - V_2} X)(T) := [T_+ \wedge S^{V_1}, S^{V_2} \wedge X]$$

The integer-graded homotopy Mackey functors can be defined as

$$\underline{\pi}_n(X) := \begin{cases} \underline{\pi}_{\mathbb{R}^n}(X) & \text{if } n \geq 0 \\ \underline{\pi}_{-\mathbb{R}^{|n|}}(X) & \text{otherwise} \end{cases}$$

Since the generating cofibrations can be taken to be just the inclusions  $S(H, m-1, n) \hookrightarrow D(H, m, n)$  rather than the general case where  $n$  is replaced with  $V$ , a map of  $G$ -CW spectra is a weak equivalence if and only if it induces isomorphisms on the integer-graded homotopy Mackey functors.

**Remark 6.14.** Mackey functors are the result of us inverting the functor  $\Sigma^V$  for every  $G$ -representation  $V$ . If, instead, we only inverted the ordinary suspensions, we would get coefficient systems. Indeed, it can be checked that in that case, the homotopy category of the full subcategory of the naive suspension  $G$ -spectra of the orbits would be exactly the free  $\mathbf{Ab}$ -enrichment of  $\mathcal{O}_G$ , so contravariant additive functors from it to  $\mathbf{Ab}$  would be equivalent to coefficient systems.

The rest of this section is devoted to the structure and examples of Mackey functors. Any map  $X \rightarrow Y$  in  $\mathcal{B}_G$  can be factored as

$$X \leftarrow U \rightarrow Y = X \leftarrow U = U = U \rightarrow Y$$

i.e. as a composite of a purely ‘contravariant part’ and a purely ‘covariant part’. Let  $\mathcal{F}_G$  denote the category of finite  $G$ -sets.

**Lemma 6.15.** The data of a Mackey functor  $\underline{M} : \mathcal{B}_G^{\text{op}} \rightarrow \mathbf{Ab}$  is equivalent to a pair of functors  $M_* : \mathcal{F}_G \rightarrow \mathbf{Ab}$  and  $M^* : \mathcal{F}_G^{\text{op}} \rightarrow \mathbf{Ab}$  that agree on objects and convert disjoint unions to direct sums such that for every pullback square as on the left,

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ g \downarrow & \lrcorner & \downarrow i \\ C & \xrightarrow{h} & D \end{array} \qquad \begin{array}{ccc} MA & \xleftarrow{M^*f} & MB \\ M_*g \downarrow & & \downarrow M_*i \\ MC & \xleftarrow{M^*h} & MD \end{array}$$

the square on the right commutes.

*Proof.* Use the decomposition described above. For example, given such a pair of functors, define  $\underline{M}(X \leftarrow U \rightarrow Y) = M_*(U \rightarrow Y) \circ M^*(X \leftarrow U)$ . Functoriality is equivalent to the condition on pullbacks and additivity is equivalent to the condition on disjoint unions.  $\square$

For a map  $f \in \mathcal{F}_G$ ,  $M_*f$  is called the *transfer map* induced by  $f$ , and  $M^*f$  is called the *restriction map* induced by  $f$ . It can be checked, directly through the construction of  $\mathcal{B}_G$  or using the universal property described above, that  $\mathcal{B}_G^{\text{op}} \cong \mathcal{B}_G$ . Any Mackey functor  $\underline{M}$  is completely determined by its restriction to the orbits because of the condition on disjoint unions. This means that we can represent Mackey functors by their *Lewis diagrams*, which are diagrams that show the image of each  $G$ -orbit and all the transfer and restriction maps. For example, a  $C_2$ -Mackey functor  $\underline{M}$  is completely determined by the data

$$\begin{array}{c} \underline{M}(C_2/C_2) \\ \text{Res} \downarrow \quad \uparrow \text{Tr} \\ \underline{M}(C_2/e) \\ \uparrow \end{array}$$

We write  $C_2/e$  here to emphasize that we are viewing  $C_2$  as merely a  $C_2$ -set. The transfers and restrictions induced by the identity maps are omitted as they will be identity. The indicated restriction and transfer are the ones induced by the unique map  $C_2/e \rightarrow C_2/C_2$ . The looped arrow is both the restriction and transfer induced by the unique non-identity endomorphism on  $C_2/e$ . These coincide because for any  $G$ -orbit  $X$  and map  $X \rightarrow X$  (which must always be an isomorphism), there is an isomorphism of spans

$$\begin{array}{ccccc} & & X & & \\ & \swarrow = & \downarrow f & \searrow f & \\ X & & & & X \\ & \nwarrow f^{-1} & \downarrow & \nearrow = & \\ & & X & & \end{array}$$

In our case,  $X = C_2/e$  and  $f = f^{-1}$  is the swap map, so that the transfer induced by it is  $\underline{M}$  applied to the span at the top and the restriction is  $\underline{M}$  applied to the span at the bottom. For other groups, the transfer induced by  $f$  is the restriction induced by  $f^{-1}$ , but the information of one is found in the other. This gives an action of the *Weyl group* of  $H$

$$WH := \text{Aut}(G/H) \cong NH/H$$

on  $\underline{M}(G/H)$  for every  $H \leq G$ , where  $NH$  denotes the normalizer of  $H$  in  $G$ . Thus, the arrows in a Lewis diagram are of three forms: the transfers and restrictions along non-endomorphisms and Weyl group actions on each object.

**Example 6.16.** For any Mackey functor  $\underline{M}$  and finite  $G$ -set  $X$ , we can produce a *precomposite Mackey functor*  $\underline{M}_X$  as  $\underline{M}_X(-) = \underline{M}(X \times -)$ . This is a Mackey functor because  $X \times -$  preserves pullbacks.

**Example 6.17.** For each  $X \in \mathcal{B}_G$ , we have the Mackey functor  $\underline{A}_X = \mathcal{B}_G(-, X)$  represented by  $X$ . For  $*$  the terminal  $G$ -set, we write  $\underline{A}_* = \underline{A}$  and call this the *Burnside Mackey functor*. The notation here is consistent with that of the last example because  $\mathcal{B}_G(X \times -, Y) \cong \mathcal{B}_G(-, X \times Y)$  and  $X \times * \cong X$ .

**Example 6.18.** Let  $N$  be a  $\mathbb{Z}[G]$ -module. Its *fixed point Mackey functor*  $\underline{N}$  is defined by

$$\begin{aligned} \underline{N}(X) &= \text{Hom}_{\mathbb{Z}[G]}(X, N) \\ N^*(f : X \rightarrow Y) &= (q \mapsto q \circ f) \\ N_*(f : X \rightarrow Y) &= \left( p \mapsto \left( y \mapsto \sum_{x \in f^{-1}(y)} p(x) \right) \right) \end{aligned}$$

i.e.  $N^* = \text{Hom}_{\mathbb{Z}[G]}(-, N)$  and transfers are defined by ‘summing over fibers’. The terminology comes from the fact that

$$\underline{N}(G/H) \cong N^H$$

**Example 6.19.** Let  $N$  be an abelian group. The *constant  $N$  Mackey functor*  $\underline{N}$  is the fixed point Mackey functor of  $N$  regarded as a  $\mathbb{Z}[G]$ -module where  $G$  acts trivially. The restriction of the contravariant part  $N^*$  to the  $G$ -orbits is then the constant  $N$  functor, hence the name. The transfer along a map  $G/H \rightarrow G/K$  (which is necessarily surjective), is multiplication by the cardinality of each fiber. In particular, the transfer along the projection  $G \rightarrow G/H$  is multiplication by the index  $[G : H]$ .

**Example 6.20.** Let  $N$  be a  $\mathbb{Z}[G]$ -module. Its *fixed quotient Mackey functor*  $\underline{N}^{\text{op}}$  is defined by

$$\begin{aligned} N_* &= \mathbb{Z}[-] \otimes_{\mathbb{Z}[G]} N \\ N^*(f : X \rightarrow Y) &= \left( y \otimes n \mapsto \sum_{x \in f^{-1}(y)} x \otimes n \right) \end{aligned}$$

where, for a  $G$ -set  $X$ ,  $\mathbb{Z}[X]$  denotes the  $\mathbb{Z}[G]$ -module with basis  $X$ . The terminology comes from the fact that

$$\underline{N}(G/H) \cong N / \sim$$

where  $\sim$  is the equivalence relation generated by  $n \sim hn$  for all  $h \in H$ ,  $n \in N$ .

**Example 6.21.** Let  $N$  be an abelian group. The *coconstant  $N$  Mackey functor*  $\underline{N}^{\text{op}}$  is the fixed quotient Mackey functor of  $N$  regarded as a  $\mathbb{Z}[G]$ -module where  $G$  acts trivially. The restriction of the covariant part  $N_*$  to the  $G$ -orbits is then the constant  $N$  functor, hence the name. The restriction map induced by  $G \rightarrow G/H$  is multiplication by  $[G : H]$ .

The Lewis diagrams of some of these can be drawn for  $G = C_2$  as follows. We assign special symbols to some of them so that labelling figures with them is easier later on.  $\underline{\mathbb{Z}}_-$  denotes the  $\mathbb{Z}[C_2]$ -module  $\mathbb{Z}$  with the sign action of  $C_2$ ,

$$X = \begin{bmatrix} 1 & 2 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \Delta = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \nabla = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

and  $s$  denotes the map on  $\mathbb{Z}^2$  that swaps the two copies of  $\mathbb{Z}$ .

Description	$\underline{A}$	$\underline{A}_{C_2/e}$	$\underline{\mathbb{Z}}$	$\underline{\mathbb{Z}}^{\text{op}}$	$\underline{\mathbb{Z}}_-$	$\underline{\mathbb{Z}}_-^{\text{op}}$
Symbol			$\square$	$\overline{\square}$	$\odot$	$\overline{\odot}$
Lewis Diagram	$\begin{array}{c} \mathbb{Z}^2 \\ \begin{array}{c} \downarrow \quad \uparrow \\ X \quad Y \end{array} \\ \mathbb{Z} \\ \uparrow \\ 1 \end{array}$	$\begin{array}{c} \mathbb{Z} \\ \begin{array}{c} \downarrow \quad \uparrow \\ \Delta \quad \nabla \end{array} \\ \mathbb{Z}^2 \\ \uparrow \\ s \end{array}$	$\begin{array}{c} \mathbb{Z} \\ \begin{array}{c} \downarrow \quad \uparrow \\ 1 \quad 2 \end{array} \\ \mathbb{Z} \\ \uparrow \\ 1 \end{array}$	$\begin{array}{c} \mathbb{Z} \\ \begin{array}{c} \downarrow \quad \uparrow \\ 2 \quad 1 \end{array} \\ \mathbb{Z} \\ \uparrow \\ 1 \end{array}$	$\begin{array}{c} 0 \\ \begin{array}{c} \downarrow \quad \uparrow \\ 0 \quad 0 \end{array} \\ \mathbb{Z} \\ \uparrow \\ -1 \end{array}$	$\begin{array}{c} \mathbb{Z}/2\mathbb{Z} \\ \begin{array}{c} \downarrow \quad \uparrow \\ 0 \quad 1 \end{array} \\ \mathbb{Z} \\ \uparrow \\ -1 \end{array}$

Figure 6.1: Examples of  $C_2$ -Mackey functors. Note that  $\underline{A}_{C_2/e} \cong \underline{\mathbb{Z}}_{C_2/e}$ .

Maps of Mackey functors are natural transformations between them. The category of Mackey functors, denoted  $\mathcal{M}_G$ , is then a category of **Ab**-valued additive functors and hence an abelian category in which direct sums, kernels and cokernels are computed level-wise. Moreover, Cartesian product induces a symmetric monoidal product on  $\mathcal{B}_G$ , which gives the category of Mackey functors a closed symmetric monoidal structure via Theorem 3.10.

**Definition 6.22.** The *box product*  $\square$  is the Day convolution product induced on the category of  $G$ -Mackey functors by the Cartesian product on  $\mathcal{B}_G$ . The internal hom of Mackey functors is denoted  $\underline{\text{Hom}}(-, -)$ .

**Proposition 6.23.** Let  $X, Y$  be finite  $G$ -sets,  $N, P$  modules over  $\mathbb{Z}[G]$ , and  $\underline{M}, \underline{M}_1, \underline{M}_2 \in \mathcal{M}_G$ . There are natural isomorphisms

- (i)  $\underline{A}_X \oplus \underline{A}_Y \cong \underline{A}_{X \amalg Y}$ .
- (ii)  $\underline{A}_X \square \underline{M} \cong \underline{M}_X$ .
- (iii)  $\underline{\text{Hom}}(\underline{A}_X, \underline{M}) \cong \underline{M}_X$

$$(iv) \quad (\underline{M}_1 \square \underline{M}_2)(G/e) \cong \underline{M}_1(G/e) \otimes \underline{M}_2(G/e).$$

*Proof.* (i) From the Yoneda lemma,

$$\text{Nat}(A_{X \amalg Y}, -) \cong -(X \amalg Y) \cong -(X) \oplus -(Y) \cong \text{Nat}(A_X, -) \oplus \text{Nat}(A_Y, -)$$

(ii) As mentioned in the proof of Theorem 3.10, this is true when  $\underline{M}$  is representable and follows in the general case by applying Theorem 3.2.

(iii)  $\underline{\text{Hom}}(\underline{A}_X, \underline{M})(Y) \cong \text{Hom}(\underline{A}_{X \times Y}, \underline{M}) \cong \underline{M}(X \times Y) \cong \underline{M}_X(Y)$ .

(iv) Check that this is true for the functors  $\underline{A}_X$  using their description as representables and then extend the result using Theorem 3.2.  $\square$

### 6.3 Equivariant Homology and Cohomology

Equivariant homology theory was deferred to this chapter because it is stable and hence should be valued in Mackey functors rather than coefficient systems even for  $G$ -spaces. We want our homology functor to be a generalized equivariant homology theory. That is, it should be a collection of functors  $\tilde{E}_{V-W} : \text{Ho Top}_*^G$  for each pair  $V, W$  of  $G$ -representations which take wedge sums to direct sums, take cofiber sequences to exact sequences, have a suspension isomorphism

$$E_\bullet(-) \cong E_{\bullet+V}(\Sigma^V -)$$

for every representation  $V$ , and have functorial isomorphisms  $E_V \cong E_W$  for  $G$ -isomorphisms  $V \cong W$ . We first define an integer-graded cellular homology theory, and later extend it to one graded by the formal differences of representations.

Let  $X$  be a  $G$ -CW complex (or a  $G$ -CW spectrum). Define its  $n$ th chain Mackey functor with coefficients to be the free Mackey functor generated by its  $n$ -cells

$$\underline{C}_n(X) = \bigoplus_{\substack{\sigma \text{ an } n\text{-cell} \\ \text{of type } G/H}} \underline{A}_{G/H}\{\sigma\}$$

where  $\underline{A}_{G/H}\{\sigma\}$  is a copy of  $\underline{A}_{G/H}$  freely generated by an element  $\langle \sigma \rangle \in \underline{A}_{G/H}(G/H)$ . The boundary map

$$d_n : \underline{C}_n(X) \rightarrow \underline{C}_{n-1}(X)$$

on the summand corresponding to  $\sigma$  is described as follows, generalizing the definition of the boundary operators in non-equivariant cellular homology. Let

$$\phi : G/H \times S^{n-1} \rightarrow X_{n-1}$$

denote the attaching map of  $\sigma$ . Composing this with the quotient  $X_{n-1} \rightarrow X_{n-1}/X_{n-2}$  we get a map from  $G/H \times S^{n-1}$  to a wedge of  $(G/K)_+ \wedge S^{n-1}$  for



various orbits  $G/K$ . Each wedge summand here corresponds to an  $(n-1)$ -cell  $\tau$  of type  $G/K$ . Consider the composite

$$G/H \times S^{n-1} \xrightarrow{\phi} X^{n-1} \rightarrow X^{n-1}/X^{n-2} \rightarrow (G/K)_+ \wedge S^{n-1}$$

where the last map is the one that collapses all summands other than the one corresponding to  $\tau$ . This map is  $G$ -homotopic to a basepoint-preserving map, which makes it factor (up to homotopy) through a map  $\phi_\tau : (G/H)_+ \wedge S^{n-1} \rightarrow (G/K)_+ \wedge S^{n-1}$ . We call  $\phi_\tau$  the *collapsed adjoint attaching map* of  $\tau$ .  $\phi_\tau$  represents a stable homotopy class of maps  $G/H \rightarrow G/K$  and hence is a morphism in  $\text{Ho } \mathcal{D} \simeq \mathcal{B}_G$ . We define the boundary operator

$$\underline{A}_{G/H}\{\sigma\} \rightarrow \bigoplus_{\substack{\tau \text{ an } (n-1)\text{-cell} \\ \text{of type } G/K}} \underline{A}_{G/K}\{\tau\}$$

to be the sum of the maps  $\underline{A}_{G/H}\{\sigma\} \rightarrow \underline{A}_{G/K}\{\tau\}$  induced by  $\phi_\tau$ . This is well defined because all but finitely many of these  $\phi_\tau$  must be nullhomotopic by the compactness of  $S^{n-1}$ . Here, the ‘map induced by  $G/H \rightarrow G/K$ ’ refers to the natural transformation  $\underline{A}_{G/H} \rightarrow \underline{A}_{G/K}$  defined level-wise as post-composition by the map of  $\text{Ho } \mathcal{D}$  represented by  $\phi_\tau$ .

We can also define homology and cohomology with arbitrary coefficients.

**Definition 6.24.** Let  $X$  be a  $G$ -CW spectrum,  $\underline{M} \in \mathcal{M}_G$ , and  $\underline{C}_\bullet(X)$  the chain complex of Mackey functors described above. The *cellular chain complex for  $X$  with coefficients in  $\underline{M}$*  is

$$\underline{C}_\bullet(X; \underline{M}) := \underline{C}_\bullet(X) \square \underline{M}$$

The  $n$ th homology Mackey functor for  $X$  with coefficients in  $\underline{M}$ , denoted by  $\underline{H}_n(X; \underline{M})$ , is the  $n$ th homology of this chain complex.

The *cellular cochain complex for  $X$  with coefficients in  $\underline{M}$*  is

$$\underline{C}^\bullet(X; \underline{M}) := \underline{\text{Hom}}(\underline{C}_\bullet(X), \underline{M})$$

The  $n$ th cohomology Mackey functor for  $X$  with coefficients in  $\underline{M}$ , denoted by  $\underline{H}^n(X; \underline{M})$ , is the  $n$ th cohomology of this cochain complex.

Since  $\underline{C}_\bullet(X)$  is a direct sum of precomposites of the Burnside Mackey functor, Proposition 6.23 simplifies the above definition to give the usual definition of the cellular (co)chains (as seen in [HHRb, Definition 8.5.14]).

The hard part of any computation here would be to figure out the boundary or coboundary maps. For this, we need to figure out the map of Mackey functors induced by  $\phi_\tau$  as a representative of a map in  $\text{Ho } \mathcal{D}$ . Since we have a more algebraic description  $\mathcal{B}_G$  of  $\text{Ho } \mathcal{D}$ , the first step is to figure out what map  $\phi_\tau$  represents in  $\mathcal{B}_G$ . Because of Remark 6.14 and the fact that  $\phi_\tau$  are maps between

trivial suspensions of  $G$ -orbits (trivial in the sense that we are smashing by the trivial  $G$ -representation), we expect the map in  $\mathcal{B}_G$  corresponding to  $\phi_\tau$  to be a linear combination of spans of the type  $G/H \xrightarrow{=} G/H \rightarrow G/K$ . This is indeed what we get.

A  $G$ -map  $(G/H)_+ \wedge S^n \rightarrow (G/K)_+ \wedge S^n$  is equivalent to an  $H$ -map  $S^n \rightarrow (G/K)_+ \wedge S^n$ , which is equivalent to a non-equivariant map  $S^n \rightarrow ((G/K)_+ \wedge S^n)^H$ . The codomain here is always a wedge sum of spheres. From non-equivariant theory, the (stable or unstable) homotopy class of such a map is completely determined by the degrees of its composites with the maps that collapse all but one wedge summand. The addition of stable homotopy classes of such maps can be carried out by simply adding these degrees. This allows us to decompose the stable homotopy class of  $\phi_\tau$  above into a sum of maps for which all but one of these degrees are zero and the non-zero degree is  $\pm 1$ .

Now take one of these summands  $f : S^n \rightarrow (G/K)_+ \wedge S^n$  as above and suppose that it has degree 1 onto the wedge summand corresponding to the coset  $gK$ . Hence this is homotopic to the map which maps homeomorphically onto that wedge summand. Viewing  $f$  again as a map  $(G/H)_+ \wedge S^n \rightarrow (G/K)_+ \wedge S^n$ , we see that it is now an  $n$ -fold suspension of a map  $(G/H)_+ \rightarrow (G/K)_+$ . Under the equivalence  $\text{Ho } \mathcal{D} \simeq \mathcal{B}_G$ , this corresponds to the span  $G/H \xrightarrow{=} G/H \rightarrow G/K$ , where the map on the right is the one above. Thus, we have decomposed the stable homotopy class represented by  $\phi_\tau$  into a sum of spans.

Now we need to figure out what map of Mackey functors is induced by said spans. In the case of homology with coefficients in  $\underline{A}$ , this span is supposed to induce a map from  $\underline{A}_{G/H}(X) \cong \underline{A}(G/H \times X)$  to  $\underline{A}_{G/K}(X) \cong \underline{A}(G/K \times X)$  for each  $X$ . Using this and the representable functor description of  $\underline{A}_{G/H}$ , we can view the induced map either as post-composition by the given span or the functor  $\underline{A}$  applied to it. In the second case, this is exactly the description of the transfer map induced by  $G/H \rightarrow G/K$ .

For the cellular chain complex with coefficients in some  $\underline{M} \in \mathcal{M}_G$ , we need to take the box product with  $\underline{M}$ . Using the natural isomorphisms of Proposition 6.23, we get a commutative square

$$\begin{array}{ccc} (\underline{A}_{G/H} \square \underline{M})(X) & \longrightarrow & (\underline{A}_{G/K} \square \underline{M})(X) \\ \parallel & & \parallel \\ \underline{M}(G/H \times X) & \xrightarrow{\quad tr \quad} & \underline{M}(G/K \times X) \end{array}$$

which again identifies the map that we are concerned with as the transfer map induced by the given map of orbits. Therefore, the boundary maps will always be linear combinations of transfers. For the cellular cochain complex, we have

natural isomorphisms

$$\begin{array}{ccc}
 \underline{\mathrm{Hom}}(\underline{A}_{G/H}, \underline{M})(X) & \longleftarrow & \underline{\mathrm{Hom}}(\underline{A}_{G/K}, \underline{M})(X) \\
 \parallel & & \parallel \\
 \underline{M}(G/H \times X) & \xleftarrow{\quad res \quad} & \underline{M}(G/K \times X)
 \end{array}$$

identifying the map with the map induced by the span  $G/K \leftarrow G/H = G/H$ , which is exactly the restriction induced by  $G/H \rightarrow G/K$ . Therefore, the coboundary maps will always be linear combinations of restrictions.

So far, we have been grading everything by formal differences of representations. There is a lot of redundancy in this indexing as isomorphic pairs of representations index isomorphic homotopies/(co)homologies, and, for example,  $V - 0$  and  $(V \oplus W) - W$  will always index isomorphic Mackey functors. Therefore, the indexing really is by the *real representation ring* of  $G$ , denoted  $RO(G)$ , which is the Grothendieck group (Definition 6.9) of the abelian monoid of finite-dimensional orthogonal representations of  $G$  under  $\oplus$ . The multiplication on this ring is induced by  $\otimes$  but is not relevant to us. It can be checked that whenever two formal differences are equal in  $RO(G)$ , they must index isomorphic Mackey functors in all the cases described above.

**Remark 6.25.** Formalizing this  $RO(G)$ -grading requires some effort. We cannot carelessly treat these isomorphisms of Mackey functors as equalities as there may not be canonical isomorphisms, which could lead to coherence issues when we want to describe further structure (such as the graded multiplication in Section 6.5; see [Adab, Section 6] for an explanation of where this goes wrong). One approach is to fix beforehand a representative  $V - W$  for each element of  $RO(G)$  and an isomorphism between every pair of isomorphic representations, which avoids the coherence issue by doing everything on-the-nose and explicitly. In this thesis, whenever we have to work with theoretical details, we assume that the grading is by formal differences, but when we do any computations, we only do them once for each representative, effectively computing the  $RO(C_2)$ -graded version while avoiding the issues that come with it.

An equivariant version of the Brown representability theorem also holds and implies a version of Corollary 4.8 for any  $RO(G)$ -graded cohomology theory. In this case, it is more complicated to infer the structure maps from the suspension isomorphism than in Corollary 4.8 because the suspension isomorphisms only give us weak equivalences of the representing objects uniquely up to homotopy, whereas the definition of a  $G$ -spectrum requires these to be on-the-nose. See [Maya, Theorem XIII.3.1] for the proof.

We then have, corresponding to reduced cellular cohomology theory with coefficients in  $\underline{M}$ , a representing  $G$ -spectrum  $H\underline{M}$ . We call this the *Eilenberg-MacLane  $G$ -spectrum* for the Mackey functor  $\underline{M}$ . Computing its  $n$ th homotopy Mackey functor would then give us the 0th reduced cohomology Mackey functor

of  $S^n$ , or equivalently the  $-n$ th reduced cohomology Mackey functor of  $S^0$ , with coefficients in  $\underline{M}$ . Therefore,

$$\pi_n(H\underline{M}) = \begin{cases} \underline{M} & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

Note that the  $RO(G)$ -graded homotopy Mackey functors of  $H\underline{M}$  may be more complicated, but these integer-graded ones determine  $H\underline{M}$  up to stable equivalence. It can also be checked that  $\underline{H}_\bullet(-; \underline{M}) \cong \mathcal{SH}^G(S^\bullet, - \wedge H\underline{M})$  by checking this on the spheres.

## 6.4 Homology of a Point

We compute the  $RO(C_2)$ -graded homology  $C_2$ -Mackey functors for a point treated as a space with trivial  $C_2$ -action with coefficients in the constant  $\mathbb{Z}$  Mackey functor  $\underline{\mathbb{Z}}$ . This computation will be used in the final computation of this thesis in Section 8.2. Every orthogonal  $C_2$ -representation can be written as a direct sum  $p + q\sigma$  of  $p$  copies of the trivial representation and  $q$  copies of the sign representation. Therefore  $RO(C_2)$  is the abelian group

$$\{p + q\sigma \mid p, q \in \mathbb{Z}\} \cong \mathbb{Z}^2$$

We have the following isomorphisms, where  $*$  denotes the singleton space.

$$\begin{aligned} \underline{H}_{p+q\sigma}(*; \underline{\mathbb{Z}}) &\cong \tilde{\underline{H}}_{p+q\sigma}(S^0; \underline{\mathbb{Z}}) \\ &\cong [S^{p+q\sigma}, S^0 \wedge H\underline{\mathbb{Z}}] \\ &\cong [S^p, S^{-q\sigma} \wedge H\underline{\mathbb{Z}}] \\ &\cong [S^{q\sigma}, S^{-p} \wedge H\underline{\mathbb{Z}}] \end{aligned}$$

Therefore, we can convert the computation of one  $RO(G)$ -graded homology into many integer-graded computations of homologies and cohomologies depending on the sign of  $q$ .

**Proposition 6.26.** Let  $*$  denote the terminal  $C_2$ -space. Then

$$\underline{H}_{p+q\sigma}(*; \underline{\mathbb{Z}}) \cong \pi_{p+q\sigma}(H\underline{\mathbb{Z}}) \cong \tilde{\underline{H}}_p(S^{-q\sigma}; \underline{\mathbb{Z}}) \cong \tilde{\underline{H}}^{-p}(S^{q\sigma}; \underline{\mathbb{Z}})$$

To compute these using cellular chains, we need  $C_2$ -CW complex structures on  $S^{n\sigma}$  for every  $n \geq 0$ . Therefore, for now, we can eliminate the need to figure out the  $G$ -CW structure on the  $S^{-n\sigma}$ .

$S^{n\sigma}$  will have two 0-cells, one representing 0 and one the point at infinity, both of type  $C_2/C_2$ . This gives us  $S^{0\rho} = S^0$ . Then we attach a 1-cell of type  $C_2/e$  along the attaching map  $C_2/e \times S^0 \rightarrow S^0$  given by projection onto the second coordinate. These represent the positive and negative halves of the same axis.

Continuing this way, we will end up with a structure in which there are two 0-cells of type  $C_2/C_2$  and one  $i$ -cell of type  $C_2/e$  for each  $i \in \{1, \dots, n\}$ . Therefore, the cellular chain complex with coefficients in  $\mathbb{Z}$  would be

$$\begin{array}{ccccccc} 0 & & 1 & & 2 & & \cdots & & n-1 & & n \\ \mathbb{Z} \oplus \mathbb{Z} & \xleftarrow{d_1} & \mathbb{Z}_{C_2/e} & \xleftarrow{d_2} & \mathbb{Z}_{C_2/e} & \xleftarrow{\quad} & \cdots & \xleftarrow{\quad} & \mathbb{Z}_{C_2/e} & \xleftarrow{d_n} & \mathbb{Z}_{C_2/e} \end{array}$$

To compute the boundary maps, we will need to compute degrees of the attaching maps. The 1-cell has the adjoint attaching map  $S^0 \rightarrow S^0$ , where the second copy of  $S^0$  is really to be thought of as two 0-cells  $D^0 \sqcup D^0$ . Quotienting by the  $-1$ -skeleton (which is empty and hence adds a disjoint point), we get  $S^0 \vee S^0$ . The composites with the two collapse maps then have degrees 1 and  $-1$ . Therefore the boundary map will be the transfer induced by  $C_2/e \rightarrow C_2/C_2$  to the first summand and the negative of the same to the second summand.

For  $n > 1$ , we have the collapsed adjoint attaching map  $S^{n-1} \rightarrow (C_2/e)_+ \wedge S^{n-1}$  for the  $n$ -cell. This has the degree assignment

$$\begin{aligned} \{e\} &\mapsto 1 \\ \{t\} &\mapsto (-1)^n \end{aligned}$$

Since the map onto the summand corresponding to  $e$  is homotopic to the identity and the map onto the other summand is homotopic to the antipodal map. Therefore the boundary map will be the transfer induced by the identity map on  $C_2/e$  plus or minus the transfer induced by the switch map on the same.

The chain complex for coefficients in  $\mathbb{Z}$ , expanded out as Lewis diagrams, looks like

$$\begin{array}{ccccccc} 0 & & 1 & & 2 & & \cdots & & n-1 & & n \\ \mathbb{Z} \oplus \mathbb{Z} & \xleftarrow{(2,-2)} & \mathbb{Z} & \xleftarrow{0} & \mathbb{Z} & \xleftarrow{2} & \cdots & \xleftarrow{\quad} & \mathbb{Z} & \xleftarrow{1+(-1)^n} & \mathbb{Z} \\ \downarrow \scriptstyle 1 \uparrow \scriptstyle 2 & & \Delta \downarrow \scriptstyle \uparrow \nabla & & \Delta \downarrow \scriptstyle \uparrow \nabla & & & & \Delta \downarrow \scriptstyle \uparrow \nabla & & \Delta \downarrow \scriptstyle \uparrow \nabla \\ \mathbb{Z} \oplus \mathbb{Z} & \xleftarrow{A} & \mathbb{Z} \oplus \mathbb{Z} & \xleftarrow{B_{-1}} & \mathbb{Z} \oplus \mathbb{Z} & \xleftarrow{B_1} & \cdots & \xleftarrow{\quad} & \mathbb{Z} \oplus \mathbb{Z} & \xleftarrow{B_{(-1)^n}} & \mathbb{Z} \oplus \mathbb{Z} \\ \uparrow \scriptstyle 1 & & \uparrow \scriptstyle s & & \uparrow \scriptstyle s & & & & \uparrow \scriptstyle s & & \uparrow \scriptstyle s \end{array}$$

where  $\Delta$  denotes the diagonal map  $\mathbb{Z} \rightarrow \mathbb{Z} \oplus \mathbb{Z}$ ,  $\nabla$  denotes the fold map,  $s$  denotes map that swaps the two copies, and

$$\begin{aligned} A &= \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \\ B_\varepsilon &= \begin{bmatrix} 1 & \varepsilon \\ \varepsilon & 1 \end{bmatrix} \end{aligned}$$

## 6 Equivariant Stable Homotopy Theory

To figure these out, one has to use an explicit isomorphism  $C_2/e \times C_2/e \cong C_2/e \sqcup C_2/e$  and apply additivity of the Mackey functor  $\underline{\mathbb{Z}}$  to find the bottom row of groups along with all the maps in and out of all of them.

Similarly, we find that the cochain complex of  $S^{n\rho}$  with coefficients in  $\underline{\mathbb{Z}}$  is

$$\begin{array}{ccccccc}
 0 & 1 & 2 & \cdots & n-1 & n \\
 \\
 \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{[1 \ -1]} \mathbb{Z} & \xrightarrow{0} \mathbb{Z} & \xrightarrow{2} \cdots & \longrightarrow \mathbb{Z} & \xrightarrow{1+(-1)^n} \mathbb{Z} \\
 \downarrow \scriptstyle 1 \uparrow \scriptstyle 2 & \Delta \downarrow \scriptstyle \uparrow \nabla & \Delta \downarrow \scriptstyle \uparrow \nabla & & \Delta \downarrow \scriptstyle \uparrow \nabla & \Delta \downarrow \scriptstyle \uparrow \nabla \\
 \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{A^T} \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{B_{-1}} \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{B_1} \cdots & \longrightarrow \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{B_{(-1)^n}} \mathbb{Z} \oplus \mathbb{Z} \\
 \uparrow \scriptstyle 1 & \uparrow \scriptstyle s & \uparrow \scriptstyle s & & \uparrow \scriptstyle s & \uparrow \scriptstyle s
 \end{array}$$

where  $A^T$  denotes the transpose of  $A$ .

An important feature to note is that the bottom row of these diagrams are exactly the chain and cochain complexes for  $S^{n\sigma}$  treated as a non-equivariant space. This is a consequence of the fact that a  $C_2$ -map out of  $C_2/e \times S^n$  is equivalent to a non-equivariant map out of  $S^n$ .

Now recall that we need to compute the reduced homology and cohomology for these, since our aim was to compute the unreduced versions for a point. The reduced theories, as for any generalized cohomology theories, are defined on pointed  $G$ -spaces by taking the quotient by the split inclusion

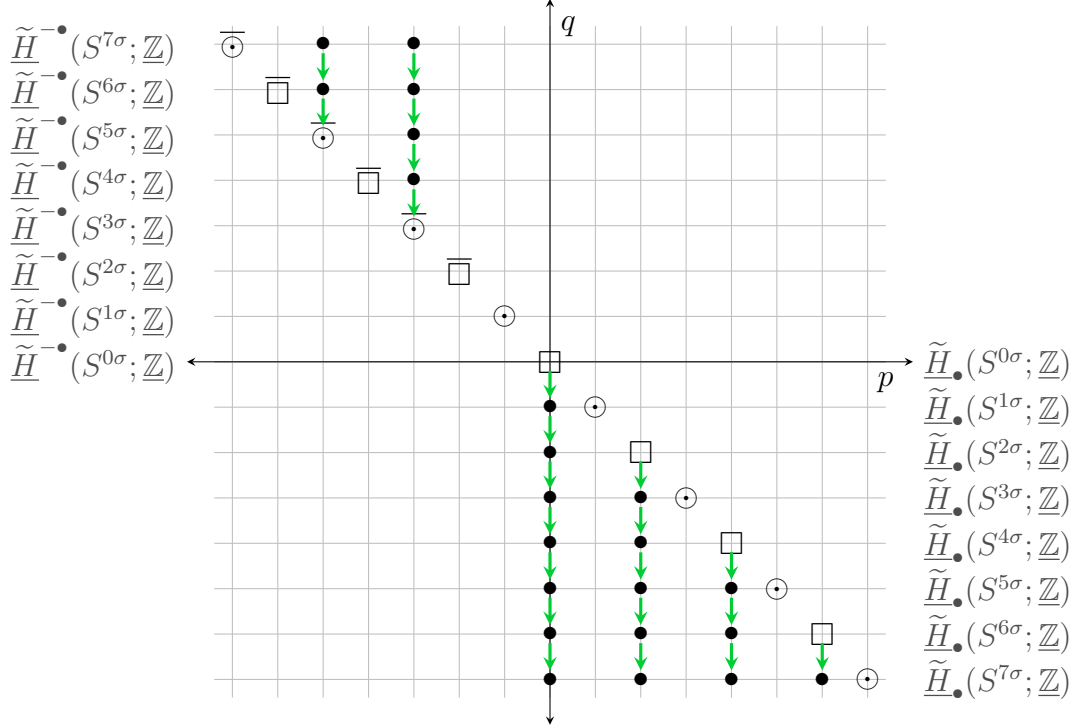
$$\underline{H}_\bullet(*) \rightarrow \underline{H}_\bullet(X)$$

induced by the inclusion of the basepoint  $* \rightarrow X$ . Let this be the inclusion as either 0 or the point at  $\infty$ , making this a cellular inclusion. Then the effect on chain and cochain complexes is of killing the  $\underline{\mathbb{Z}}$  summand corresponding to that point. Alternatively, we could define the augmented chain complex  $\tilde{\underline{C}}_\bullet(X)$ , which is  $\underline{C}_\bullet(X)$  but with  $\tilde{\underline{C}}_{-1}(X) = \underline{\mathbb{Z}}$  with augmentation maps as usually defined.

Finally, computing the homologies and cohomologies of the above complexes and plotting them according to Proposition 6.26, we get the following theorem. To understand which Mackey functors the symbols in the diagram refer to, see Fig. 6.1. The symbol  $\bullet$  refers to a Mackey functor  $\underline{F}$  which has Lewis diagram

$$\begin{array}{c}
 \mathbb{Z}/2\mathbb{Z} \\
 0 \downarrow \uparrow 0 \\
 0 \\
 \uparrow \\
 0
 \end{array}$$

**Theorem 6.27.** Let  $*$  be the terminal  $C_2$ -space. Then its homology  $\underline{H}_{p+q\sigma}(*; \mathbb{Z})$  with coefficients in  $\mathbb{Z}$  is as shown in the following grid.



The green arrows describe part of the multiplicative structure and are explained in Section 6.5.

Note that on evaluating all the Mackey functors at  $C_2/e$ , we get a  $\mathbb{Z}$  on each anti-diagonal lattice point as we would expect from the non-equivariant case.

## 6.5 Multiplicative Structure

Similar to the cross and cup products in non-equivariant cohomology theory with coefficients in a ring, we get products on equivariant cohomology with coefficients in a monoid object of  $\mathcal{M}_G$ , which is a Mackey functor  $\underline{M}$  with maps  $\underline{A} \rightarrow \underline{M}$  and  $\underline{M} \square \underline{M} \rightarrow \underline{M}$  satisfying the appropriate associativity and identity conditions. Such a Mackey functor is called a *Green functor*.

For  $C_2$ , these are described as follows. We first describe a cross product

$$\underline{\tilde{H}}^{p_1+q_1\sigma}(X; \underline{M}) \square \underline{\tilde{H}}^{p_2+q_2\sigma}(X; \underline{M}) \rightarrow \underline{\tilde{H}}^{p_1+p_2+(q_1+q_2)\sigma}(X \wedge X; \underline{M})$$

This map comes from one at the level of chain complexes. Let

$$\begin{aligned} x &\in \underline{C}_{p_1}(S^{-q_1\sigma} \wedge X)(G/H) \\ y &\in \underline{C}_{p_2}(S^{-q_2\sigma} \wedge X)(G/K) \end{aligned}$$

be  $p_1$ - and  $p_2$ -cells of types  $G/H$  and  $G/K$  respectively. The  $G$ -CW structures on  $S^{-q_1\sigma} \wedge X$  and  $S^{-q_2\sigma} \wedge X$  give us a product  $G$ -CW structure on  $S^{-(q_1+q_2)\sigma} \wedge X \wedge X$ , which contains a cell  $x \wedge y$  of type  $G/H \times G/K$ . This product may not be an orbit; regardless,  $x \wedge y$  is a union of cells and by the additivity of Mackey functors, the sum of those cells is of type  $G/H \times G/K$ . We extend this product to the case where  $x$  and  $y$  are not cells using the fact that the cells freely generate the chain Mackey functors to get a product map

$$\underline{C}_{p_1+q_1\sigma}(X) \square \underline{C}_{p_2+q_2\sigma}(X) \rightarrow \underline{C}_{p_1+p_2+(q_1+q_2)\sigma}(X \wedge X)$$

Taking the box product of this with the map  $\underline{M} \square \underline{M} \rightarrow \underline{M}$  gives us the cross product on homology. This induces a map the other way on cochain complexes, giving a cross product on cohomology. The cup product of two elements can be defined by taking their cross product and pulling the result back along the diagonal map  $X \rightarrow X \wedge X$  (which, at the level of cochain complexes, has to be done along a cellular approximation).

Because of the isomorphisms

$$\begin{aligned} \tilde{H}^{V-W}(S^0; \mathbb{Z}) &\cong \mathcal{SH}^{C_2}(S^0, S^{V-W} \wedge H\mathbb{Z}) \\ &\cong \mathcal{SH}^{C_2}(S^{W-V}, S^0 \wedge H\mathbb{Z}) \cong \tilde{H}_{W-V}(S^0; \mathbb{Z}) \end{aligned}$$

the homology of a point can be viewed as a re-indexing of the cohomology of a point, and hence has a cup product structure on it. Part of this multiplicative structure is indicated using the green arrows in Theorem 6.27. They indicate all the cases in which multiplication by the generator of  $\tilde{H}_{-\sigma}(S^0; \mathbb{Z})$  takes the generator of the source of the arrow to the unique non-zero element at the top level of the target.

To elaborate, if Mackey functors  $\underline{M}$  and  $\underline{N}$  are generated by elements  $(m_i)_{i \in I}$  and  $(n_j)_{j \in J}$ , then  $\underline{M} \square \underline{N}$  is generated by the elements  $(m_i \square n_j)_{i \in I, j \in J}$ , and hence a map from  $\underline{M} \square \underline{N}$  is completely determined by the images of these generators. In this case,  $\underline{\mathbb{Z}}$  and  $\underline{F}$  are generated by one element each in  $\underline{\mathbb{Z}}(C_2/C_2)$  and  $\underline{F}(C_2/C_2)$  respectively, so the box product of any pair of these functors is generated by a single element in the group at  $C_2/C_2 \times C_2/C_2 \cong C_2/C_2$ .

In order to figure out the multiplicative structure on the homology of a point, we would need to find explicit  $G$ -CW structures on each  $S^{-n\sigma}$  so that we can express everything in terms of cochains. We can use the following fact about the dual of a finite  $G$ -CW spectrum  $X$ : The dual of  $X$  has, for each  $n$ -cell of  $X$ , a  $-n$ -cell, whose collapsed adjoint attaching maps are, up to stable homotopy, the duals of those of the former. Here, the dual of a span refers to the span itself, but viewed as a morphism going the other way. We can see that this does indeed match up with the chain and cochain complexes used in the calculations. We omit the calculation of the product structure.



# 7 Vector Bundles and Topological K-Theory

A vector bundle is a structure on a space that generalizes the tangent and normal bundles on manifolds. To say that  $B$  has a vector bundle structure over it is to say that each point of  $B$  has a vector space over it, which ‘varies continuously’ over  $B$ . For us, vector bundles will be used as homotopy invariants of spaces.

## 7.1 Vector Bundles as Homotopy Invariants

**Definition 7.1.** A *real vector bundle* over a space  $B$  is a map  $p : E \rightarrow B$  of spaces such there exists an open cover  $(U_\alpha)_{\alpha \in \Lambda}$  of  $B$  with homeomorphisms  $\phi_\alpha : p^{-1}(U_\alpha) \cong U_\alpha \times \mathbb{R}^{n_\alpha}$  for some  $n_\alpha \in \mathbb{N}$  satisfying the following conditions

(i) The triangle

$$\begin{array}{ccc} p^{-1}(U_\alpha) & \xrightarrow{\phi} & U_\alpha \times \mathbb{R}^{n_\alpha} \\ & \searrow p & \swarrow \pi_1 \\ & U_\alpha & \end{array}$$

commutes for each  $\alpha \in \Lambda$ , where  $\pi_1$  denotes the first projection.

(ii) For each pair  $\alpha, \beta \in \Lambda$ , the *transition map*

$$(\phi_\beta \circ \phi_\alpha^{-1})|_{(U_\alpha \cap U_\beta) \times \mathbb{R}^{n_\alpha}} : (U_\alpha \cap U_\beta) \times \mathbb{R}^{n_\alpha} \rightarrow (U_\alpha \cap U_\beta) \times \mathbb{R}^{n_\beta}$$

is a linear isomorphism on each fiber  $\{b\} \times \mathbb{R}^{n_\alpha}$  for  $b \in U_\alpha \cap U_\beta$ .

We call  $B$  the *base space*,  $E$  the *total space*, and  $p$  the *projection map* of the vector bundle. (i) is called the *local triviality* condition and each  $U_\alpha$  is called a *trivializing neighbourhood*.

A *complex vector bundle* is defined similarly, except each occurrence of  $\mathbb{R}^n$  is replaced with  $\mathbb{C}^n$ . Vector bundles locally look like projection maps from products, so we think of them as ‘twisted products’ with  $\mathbb{R}^n$  (for example, the Möbius strip, if thought of as a strip of infinite width, is a real vector bundle over  $S^1$ ). The condition on the transition maps implies that the  $n_\alpha$  are constant over each

## 7 Vector Bundles and Topological K-Theory

connected component. If they are all equal to some  $n \in \mathbb{N}$ , we say that the bundle is  $n$ -dimensional. In the following, unless otherwise specified, every statement is about both real and complex vector bundles and  $k$  denotes either  $\mathbb{R}$  or  $\mathbb{C}$ .

Each fiber  $p^{-1}(b)$  in a bundle  $p : E \rightarrow B$  is canonically a vector space. This comes from choosing a trivializing neighbourhood  $U_\alpha$  containing  $b$  and transferring the vector space structure along the homeomorphism  $\phi_\alpha : p^{-1}(U_\alpha) \rightarrow U_\alpha \times k^{n_\alpha}$ . The condition on the transition maps ensures that this structure does not depend on the  $U_\alpha$  chosen.

A morphism of vector bundles  $p_1 \rightarrow p_2$  over  $B$  with total spaces  $E_1, E_2$  respectively is a map  $f : E_1 \rightarrow E_2$  which respects the projection maps in the sense that

$$p_2 f = p_1$$

and restricts to a linear map on each fiber. This is called an injection/surjection/isomorphism if  $f$  is one.

The following examples describe some important constructions of vector bundles. For the detailed constructions and proofs, see [MS, Section 3].

**Example 7.2.** The projection  $\pi_1 : B \times k^n \rightarrow B$  is a vector bundle over  $B$  with trivializing cover  $\{B\}$ . This is called the *trivial vector bundle* over  $B$  of dimension  $n$ .

**Example 7.3.** Let  $p : E \rightarrow B$  be a vector bundle and  $f : B' \rightarrow B$  any map. Then the *pullback bundle* of  $p$  along  $f$  is the map  $f^*p : E' \rightarrow B'$  in the pullback diagram

$$\begin{array}{ccc} E' & \longrightarrow & E \\ f^*p \downarrow & \lrcorner & \downarrow p \\ B' & \xrightarrow{f} & B \end{array}$$

It can be checked that this is always a vector bundle. In particular, if  $f$  is an inclusion of a subspace, then  $E' = p^{-1}B'$  and  $f^*p$  is called the *restriction* of  $p$  to  $B'$ , denoted  $p|_{B'}$ .

The direct sum and tensor product of vector spaces extend to give us these operations on vector bundles over  $B$  by applying these constructions fiber-wise. In the following, let  $p_i : E_i \rightarrow B$  be vector bundles for  $i = 1, 2$ .

**Example 7.4.** There exists a vector bundle  $p_1 \oplus p_2$  over  $B$  with inclusions  $p_i \rightarrow p_1 \oplus p_2$  such that each fiber  $(p_1 \oplus p_2)^{-1}(b)$  is a direct sum of the images of the fibers  $p_i^{-1}(b)$  under these inclusions. This is both the product and the coproduct in the category of vector bundles over  $B$  and is called the *Whitney sum* of  $p_1$  and  $p_2$ .

**Example 7.5.** There exists a vector bundle  $p_1 \otimes p_2$ , called the *tensor product* of  $p_1$  and  $p_2$ , with a map  $f : p_1 \oplus p_2 \rightarrow p_1 \otimes p_2$  commuting with the projection maps and bilinear on each fiber (and hence not a map of vector bundles) such that for any vector bundle  $q$  over  $B$  and any map  $g : p_1 \oplus p_2 \rightarrow q$  commuting with the projections and bilinear on each fiber, there is a unique factoring of  $g$  through  $f$  via a map of vector bundles. The fibers of  $p_1 \otimes p_2$  are the tensor products of the fibers of  $p_i$ .

Let  $\text{Vect}_k X$  denote the set of isomorphism classes of vector bundles over  $X$ . Example 7.3 says that any map  $f : X \rightarrow Y$  induces a map of sets  $f^* : \text{Vect}_k Y \rightarrow \text{Vect}_k X$ . Since a pullback of a pullback is the same as a pullback along the composite,  $\text{Vect}_k$  is a functor  $\text{Top}^{\text{op}} \rightarrow \text{Set}$ . Moreover,  $f^*$  preserves the Whitney sum of vector bundles, which can be proved by checking that the Whitney sum satisfies the universal property of the appropriate pullback. The Whitney sum makes  $\text{Vect}_k X$  into a commutative monoid and hence  $\text{Vect}_k$  is a functor  $\text{Top}^{\text{op}} \rightarrow \text{CMon}$ , the category of commutative monoids.

**Proposition 7.6.** Let  $f, g : X \rightarrow Y$  be homotopic maps of paracompact topological spaces. Then  $f^* = g^* : \text{Vect}_k Y \rightarrow \text{Vect}_k X$ .

*Proof.* Pull back a vector bundle along a homotopy  $H : X \times I \rightarrow Y$  to get a bundle over  $X \times I$ . From functoriality, the restrictions of this bundle to the top and bottom faces of  $X \times I$  are the same as the pullbacks of the original bundle along  $f$  and  $g$  respectively. The paracompactness of  $X$  then implies that these two restrictions are isomorphic. See [Hatb, Theorem 1.6] for details.  $\square$

**Example 7.7.** A way to construct vector bundles is using the *clutching construction*. Let  $X$  be a CW complex and  $A_1, A_2$  subcomplexes that cover it. Let  $p_i : E_i \rightarrow A_i$  be vector bundles for  $i = 1, 2$  whose restrictions  $p'_i$  to the intersection  $A_1 \cap A_2$  are isomorphic via a map  $\phi : p_1^{-1}(A_1 \cap A_2) \rightarrow p_2^{-1}(A_1 \cap A_2)$ . We can then glue these bundles along  $\phi$  to get a bundle on  $X$ . That is, we define  $E$  as the pushout

$$\begin{array}{ccccc} p_1^{-1}(A_1 \cap A_2) & \xrightarrow{\phi} & p_2^{-1}(A_1 \cap A_2) & \longrightarrow & E_2 \\ \downarrow & & & & \downarrow \\ E_1 & \longrightarrow & & \xrightarrow{\quad r \quad} & E \end{array}$$

and the glued vector bundle  $p : E \rightarrow X$  is described by the natural map from the pushout. We need  $A_i$  to be subcomplexes rather than just arbitrary subspaces to ensure that local triviality is satisfied. We could instead have asked that the  $A_i$  be open subspaces, in which case the proof of local triviality would be even simpler. The tuple  $(p_1, p_2, \phi)$  is called the *clutching data* of a vector bundle over  $X$ . See the argument in [AB, Page 234-235].

This clutching data is often given in the following form. Suppose first that the restrictions of the bundles to  $A_1 \cap A_2$  are trivial bundles of dimension  $n$ . Fix

bundle identifications

$$f_i : p_i^{-1}(A_1 \cap A_2) \cong (A_1 \cap A_2) \times k^n$$

$\phi$  is then viewed as a bundle automorphism of  $(A_1 \cap A_2) \times k^n$ . Currying this map, we have a map  $A_1 \cap A_2 \rightarrow \text{Map}(k^n, (A_1 \cap A_2) \times k^n)$ . This is equivalent to two maps  $A_1 \cap A_2 \rightarrow \text{Map}(k^n, A_1 \cap A_2)$  and  $A_1 \cap A_2 \rightarrow \text{Map}(k^n, k^n)$ . Since bundle isomorphisms are fiber-wise linear, the former is redundant and the latter is a map  $A_1 \cap A_2 \rightarrow \text{GL}_n(k)$ . Thus the isomorphism  $\phi$  is completely specified by a continuous map  $A_1 \cap A_2 \rightarrow \text{GL}_n(k)$ , which is called the *clutching function* for gluing  $p_1$  and  $p_2$ . In the case where the restrictions to the intersection are non-trivial, we cover the intersection by trivializing neighbourhoods and give clutching functions on each trivialization separately in a compatible way (satisfying an appropriate cocycle condition).

Over paracompact spaces, homotopic clutching functions produce isomorphic vector bundles: the homotopy between them can be used to produce a clutching function for a bundle over  $X \times I$ , after which we can use the argument in Proposition 7.6.

The clutching construction shows that as a **Set**-valued functor on pointed connected CW complexes,  $\text{Vect}_k : (\text{Ho } CW_*^c)^{\text{op}} \rightarrow \mathbf{Set}$  satisfies the Mayer-Vietoris axiom of Lemma 4.7. However,  $\text{Vect}_k$  is not the correct functor for pointed spaces as it does not take wedge sums to direct products. For example,  $* \vee * \cong *$ , the terminal space, but  $\text{Vect}_k(* \vee *) \cong \mathbb{N} \cong \text{Vect}_k(*)$ .

To fix this, we define a ‘reduced’ functor  $\widetilde{\text{Vect}}_k$  by defining  $\widetilde{\text{Vect}}_k(X)$  as the kernel of the map  $\text{Vect}_k(X) \rightarrow \text{Vect}_k(*)$  induced by the inclusion of the basepoint into  $X$ . Equivalently, we get this by quotienting  $\text{Vect}_k(X)$  by the submonoid of trivial bundles. This functor then satisfies the conditions for Lemma 4.7 and is hence represented by a CW complex.

**Definition 7.8.** The *classifying space for real vector bundles*  $BO$  is the CW complex representing  $\widetilde{\text{Vect}}_{\mathbb{R}}$ . The *classifying space for complex vector bundles*  $BU$  is the CW complex representing  $\widetilde{\text{Vect}}_{\mathbb{C}}$ .

The  $O$  and  $U$  stand for ‘orthogonal’ and ‘unitary’ respectively. This is because the transition maps of an  $n$ -dimensional vector bundle come from the groups  $\text{GL}_n(k)$ , but by Gram-Schmidt orthonormalization we may assume that they are elements of  $O(n)$  or  $U(n)$ .  $BO(n)$  and  $BU(n)$  are defined as spaces that represent the functors taking a CW complex to the set of isomorphism classes of principal  $O(n)$ - or  $U(n)$ -bundles on it. By the theory of associated bundles, these functors are naturally isomorphic to the ones that take a CW complex to the set of isomorphism classes of  $n$ -dimensional real or complex bundles.

## 7.2 Real and Complex $K$ -Theory

Since  $\widetilde{\text{Vect}}_k$  is representable, it will take cofiber sequences to exact sequences of pointed sets (and hence of commutative monoids). This suggests that we should be able to construct a generalized cohomology theory out of these. The first issue is that  $\widetilde{\text{Vect}}_k$  only satisfies the axioms that we need on paracompact spaces and CW complexes. This can be fixed by applying the CW approximation functor  $\Gamma$  before applying  $\widetilde{\text{Vect}}_k$  as CW complexes are paracompact. The second issue is that  $\widetilde{\text{Vect}}_k$  takes values in commutative monoids instead of abelian groups. This can be fixed by then applying the Grothendieck group functor  $G$  of Definition 6.9 to the resulting monoid. Therefore we define

$$\widetilde{KO}^0 = G \circ \widetilde{\text{Vect}}_{\mathbb{R}} \circ \Gamma : \text{Top}^{op} \rightarrow \text{Ab}$$

in the real case and

$$\widetilde{K}^0 = G \circ \widetilde{\text{Vect}}_{\mathbb{C}} \circ \Gamma : \text{Top}^{op} \rightarrow \text{Ab}$$

in the complex case. We can infer that these take weak equivalences to isomorphisms, wedge sums to direct products, and cofiber sequences to exact sequences (see [Hatb, Chapter 2] for direct proofs). We could also first define unreduced versions of these functors using  $\text{Vect}_k$  instead of  $\widetilde{\text{Vect}}_k$  and then define the reduced versions by taking the kernels of the maps induced by the inclusion of the basepoint. These definitions are equivalent because the short exact sequence involving the reduced and unreduced functors is always split by the map induced by the projection onto the singleton.

For these to be cohomology theories, we need to define functors  $\widetilde{KO}^n, \widetilde{K}^n : \text{Top} \rightarrow \text{Ab}$ , all satisfying these axioms along with the suspension axiom. We define these functors in the negative degrees using the suspension axiom itself.

$$\widetilde{KO}^{-n}(X) := \widetilde{KO}^0(\Sigma^n X), \quad \widetilde{K}^{-n}(X) := \widetilde{K}^0(\Sigma^n X)$$

This is almost a cohomology theory. The following major result allows us to extend these definitions to positive degrees.

**Theorem 7.9** (Bott periodicity). There are natural isomorphisms

$$\widetilde{KO}^0(\Sigma^8 X) \rightarrow \widetilde{KO}^0(X), \quad \widetilde{K}^0(\Sigma^2 X) \rightarrow \widetilde{K}^0(X)$$

*Proof.* See [AB] or [Hatb, Section 2.2] for proofs of the  $K$  statement. The statement about  $KO$  is harder and is best proved using  $KR$ -theory (Section 7.3).  $\square$

We can therefore define  $\widetilde{K}^n = \widetilde{K}^{-n}$  to get a cohomology theory called *reduced complex  $K$ -theory* or simply  $K$ -theory. Similarly extending  $KO$  to positive degrees by setting  $\widetilde{KO}^n = \widetilde{KO}^{n-8}$ , we get a cohomology theory called *reduced real  $K$ -theory* or  $KO$ -theory.

## 7 Vector Bundles and Topological K-Theory

By Lemma 4.7, there exist spectra  $KU$  and  $KO$  representing complex and real  $K$ -theory respectively. In particular, this means that

$$\pi_n(KU) \cong [S^n, KU] \cong \tilde{K}^{-n}(S^0) \cong \begin{cases} \tilde{K}^0(S^0) & \text{if } n \text{ is even} \\ \tilde{K}^0(S^1) & \text{if } n \text{ is odd} \end{cases}$$

A complex vector bundle over  $S^0$  is exactly a pair of complex vector bundles, whose monoid under addition is  $\mathbb{N}^2$ . Reducing and taking the Grothendieck group, we get  $\mathbb{Z}$ . Using clutching functions, one can show that there are no non-trivial complex vector bundles over  $S^1$ . Indeed, covering  $S^1$  by two intervals  $A, B$  with intersection  $S^0$ , any complex vector bundle  $p$  over  $S^1$  is glued together from the clutching data  $(p|_A, p|_B, \text{id}_{p^{-1}(A \cap B)})$ . Since  $A, B$  are contractible,  $p_A$  and  $p_B$  are trivial of some dimension  $n$ . Then  $p$  is completely determined by a clutching function  $A \cap B \simeq S^0 \rightarrow \text{GL}_n(\mathbb{C})$ .  $\text{GL}_n(\mathbb{C})$  is path connected for  $n \geq 1$ , so any such clutching function is nullhomotopic, and hence  $p$  must be trivial.

Therefore, the homotopy groups of  $KU$  are  $\mathbb{Z}$  in even degrees and 0 in odd degrees. In this way, we can use the periodicity theorem to determine all complex vector bundles over all spheres (up to addition of trivial bundles - a non-trivial bundle and a trivial bundle can sum to a trivial bundle). Similar calculations can be done for  $KO$ . We will need the following result in Section 8.2.

**Proposition 7.10.**  $\pi_3(KO) \cong \widetilde{KO}^0(S^3) \cong 0$ .

*Proof sketch.* Cover  $S^3$  by two copies  $A, B$  of  $D^3$  intersecting in  $S^2$ . As above, since  $A, B$  are contractible, any real vector bundle  $p$  over  $S^3$  is completely determined by a clutching function  $S^2 \rightarrow \text{GL}_n(\mathbb{R})$  for some  $n \in \mathbb{N}$ . The bundle depends only on the homotopy class of this function and  $\text{GL}_n(\mathbb{R})$  deformation retracts to  $O(n)$  so we can equivalently use a clutching function  $S^2 \rightarrow O(n)$ . We can assume that its image lies in the positive determinant component because if not, we can apply a fiber-wise reflection to the trivialization of  $p|_A$  and compose the clutching function with multiplication by the same reflection. Therefore, a non-trivial bundle  $p$  of dimension  $n$  implies the existence of a non-zero element of  $\pi_2(SO(n))$ . The covering map

$$S^3 \cong SU(2) \rightarrow SO(3)$$

shows that  $\pi_2(SO(3)) \cong 0$  and the fiber sequence

$$SO(n-1) \rightarrow SO(n) \rightarrow S^{n-1}$$

shows the same for  $SO(n)$  for  $n \geq 3$  by induction. Thus, adding a trivial bundle of dimension 3 to any bundle gives the trivial bundle, which completes the proof.  $\square$

## 7.3 $KR$ -Theory

There is a certain feature called a *complex orientation* that  $K$ -theory has but  $KO$ -theory does not. This makes working with  $K$  a lot easier than those with  $KO$ . For example, there exists a universal complex oriented cohomology theory  $MU$  which allows us to study some aspects of all such theories at once. There is also a correspondence between certain complex oriented cohomology theories and algebraic structures called *formal groups*. There is no good real analogue of this theory.

$KR$ -theory provides, in particular, a way to study real  $K$ -theory using complex  $K$ -theory. It is a  $C_2$ -equivariant cohomology theory constructed from  $KU$  using the conjugation action of  $C_2$  on  $\mathbb{C}$ , and so contains the information of  $KO$  in its fixed points. More precisely, it is defined as follows. The action of the non-identity element of  $C_2$  will always be denoted by an overline. Since the action of this element on  $\mathbb{C}$  is complex conjugation, this will also be used to denote complex conjugation.

**Definition 7.11.** Let  $X$  be a  $C_2$ -space. A *vector bundle with conjugation* over  $X$  is a  $C_2$ -map  $p : E \rightarrow X$  which is non-equivariantly a complex vector bundle such that the  $C_2$ -action restricted to the fiber of a point  $b$

$$\bar{\cdot} : p^{-1}(b) \rightarrow p^{-1}(\bar{b})$$

coincides with complex conjugation. That is, for any  $\alpha \in \mathbb{C}$ ,  $v \in p^{-1}(b)$ , we have

$$\overline{\alpha v} = \bar{\alpha} \bar{v}$$

Atiyah introduced these vector bundles and their associated cohomology theory in his paper [Ati]. There, he called  $C_2$ -spaces *real spaces* and such bundles *real vector bundles over a real space*. This is why the associated cohomology theory on pointed  $C_2$ -spaces, which we construct in the following, is now often called reduced Real  $K$ -theory (with a capital R) or  $KR$ -theory. We avoid using the word ‘real’ to avoid confusion.

We can define Whitney sums and tensor products of such vector bundles as we did before. We are able to mimic much of the theory from the previous sections, except now our cohomology theory should be valued in  $C_2$ -Mackey functors.

Now suppose  $X$  is a pointed space, which we also treat as a  $C_2$ -space with the trivial action. A vector bundle with conjugation over  $X$  would be a complex vector bundle in which each fiber has a complex conjugation on it. There is an isomorphism  $\text{Vect}_{C_2}(X) \cong \text{Vect}_{\mathbb{R}}(X)$  given by taking the fixed points under conjugation, with inverse given by tensoring a given bundle with the trivial 1-dimensional complex vector bundle over  $\mathbb{R}$ .

We can also study complex vector bundles over  $X$  using vector bundles with conjugation using the space  $\text{Ind}_e^{C_2} X = (C_2/e)_+ \wedge X$ . There is an isomorphism

## 7 Vector Bundles and Topological K-Theory

$\widetilde{\text{Vect}}_{C_2}(\text{Ind}_e^{C_2} X) \cong \widetilde{\text{Vect}}_{\mathbb{C}}(X)$ . This isomorphism is given by restricting a given bundle on  $\text{Ind}_e^{C_2} X$  to the subspace  $\{e\} \times X$ , with inverse given by inducing up a bundle  $E \rightarrow X$  by taking  $(C_2/e)_+ \wedge E \rightarrow (C_2/e)_+ \wedge X$ . In this way, the study of these bundles will subsume that of both complex and real vector bundles.

The  $C_2$ -cohomology theory  $\widetilde{KR}^\bullet$  is constructed as follows. For a  $C_2$ -space  $X$ , define  $\text{Vect}_{C_2}(X)$  as the commutative monoid of vector bundles with conjugation over  $X$  under Whitney sum. If  $X$  is pointed, define  $\widetilde{\text{Vect}}_{C_2}(X)$  as the kernel of the map  $\text{Vect}_{C_2}(X) \rightarrow \text{Vect}_{C_2}(*)$ . For a finite  $G$ -set  $T$ , define a  $C_2$ -Mackey functor

$$\widetilde{KR}^0(X)(T) = G(\widetilde{\text{Vect}}_{C_2}(\Gamma(T_+ \wedge X)))$$

where  $\Gamma$  is the  $G$ -CW approximation functor and  $G$  is the Grothendieck group functor. For a map  $f : T_1 \rightarrow T_2$  of  $C_2$ -sets, the restriction along  $f$  is given by pullback along  $f_+ \wedge X$ . For the transfer along  $f$ , suppose we have a bundle over  $T_{1+} \wedge X$ . The transferred bundle on  $T_{2+} \wedge X$  is then defined component by component: if  $t \in T_2$ , then the bundle over the subspace  $\{t\} \times X$  is defined as the Whitney sum of the bundles over  $\{s\} \times X$  for all  $s \in f^{-1}(t)$  (i.e. summing over the fibers, similar to Example 6.19; in fact, the proof that this is a Mackey functor will be essentially the same).

We define  $\widetilde{KR}^{-V}(X) = \widetilde{KR}^0(\Sigma^V X)$  for each  $V \in \mathcal{R}_{C_2}$ . To extend this to a  $RO(C_2)$ -graded cohomology theory, we need a version of the Bott periodicity theorem.

**Theorem 7.12** (Bott periodicity). There is a natural isomorphism  $\widetilde{KR}^{-V} \rightarrow \widetilde{KR}^{-V-\rho}(X)$  where  $\rho$  denotes the regular representation of  $C_2$ .

*Proof.* See [Ati, Theorem 2.3] □

This allows extending  $\widetilde{KR}^\bullet$  to a cohomology theory since every  $V \in RO(C_2)$  can be written as  $n\rho - W$  for some  $n \in \mathbb{Z}$  and  $W \in \mathcal{R}_{C_2}$ . This is called (reduced)  $KR$ -theory, and its representing  $C_2$ -spectrum is denoted  $KR$ . The underlying non-equivariant spectrum of this is  $\text{Res}_e^{C_2} KR \simeq KU$ . Indeed, the stable homotopy class of a non-equivariant map  $S^n \rightarrow KR$  is equivalent to one of a  $C_2$ -map  $(C_2/e)_+ \wedge S^n \rightarrow KR$ . This is equivalent to a vector bundle with conjugation over  $(C_2/e)_+ \wedge S^n$ , which is equivalent to complex vector bundle over  $S^n$ , which is equivalent to the stable homotopy class of a map  $S^n \rightarrow KU$ , proving that  $\text{Res}_e^{C_2} KR$  and  $KU$  are stably equivalent. Thus, the periodicity theorem for  $KU$  can be deduced from the above. Atiyah was also able to give a simple proof of the periodicity theorem for  $KO$  by applying the following to spaces with trivial  $C_2$ -actions.

**Theorem 7.13.** There is a natural isomorphism  $\underline{KR}^V(X) \rightarrow \underline{KR}^{V-8}(X)$ .

*Proof.* See [Ati, Theorem 3.10]. □



This is the periodicity theorem for  $KR$  mentioned in the preface.

Lastly,  $K$ ,  $KO$ , and  $KR$  are all multiplicative cohomology theories. We sketch this for  $KR$ . We begin with a multiplication on the negative degrees. Let  $a \in \widetilde{KR}^{-V}(X)(S)$  and  $b \in \widetilde{KR}^{-W}(X)(T)$  for  $X$  a pointed  $C_2$ -space,  $V, W \in \mathcal{R}_{C_2}$ , and  $C_2$ -sets  $S, T$ . These then correspond to formal differences of vector bundles with conjugation. Suppose first that they correspond to actual bundles over the  $C_2$ -spaces

$$S_+ \wedge S^V \wedge X \quad \text{and} \quad T_+ \wedge S^W \wedge X$$

respectively. The smash product of these bundles is then a vector bundle with conjugation over

$$S_+ \wedge T_+ \wedge S^{V+W} \wedge X \wedge X \cong (S \times T)_+ \wedge S^{V+W} \wedge X \wedge X$$

We can pull this back along the map induced by the diagonal map  $X \rightarrow X \wedge X$  to get a bundle over  $(S \times T)_+ \wedge S^{V+W} \wedge X$ , or an element

$$a \star b \in \widetilde{KR}^{-V-W}(X)(S \times T)$$

By extending this to the negatives of the bundles, this describes a natural multiplication

$$\widetilde{KR}^{-V}(X)(S) \otimes \widetilde{KR}^{-W}(X)(T) \rightarrow \widetilde{KR}^{-V-W}(X)(S \times T)$$

which, by the universal property of Day convolution (Proposition 3.9), corresponds to a map

$$\widetilde{KR}^{-V}(X) \square \widetilde{KR}^{-W}(X) \rightarrow \widetilde{KR}^{-V-W}(X)$$

We also similarly get a multiplication map

$$\widetilde{KR}^{-V}(S^0) \square \widetilde{KR}^{-W}(X) \rightarrow \widetilde{KR}^{-V-W}(X)$$

by following the above outline, except that in the last step we pull back the bundle along the isomorphism  $X \rightarrow S^0 \wedge X$  instead of the diagonal map. Now [Ati] shows that the Bott periodicity isomorphism in Theorem 7.12 is described by multiplication by an element in  $\widetilde{KR}^{-\rho}(S^0)(C_2/C_2)$ . This gives us the required compatibility of periodicity and multiplication to be able to extend the multiplication to all of  $KR$ .

In fact, this multiplication on the cohomology theory corresponds to a point-set level multiplicative structure (that of a monoid object) on  $KR$  described by maps  $KR \wedge KR \rightarrow KR$  and  $S^0 \rightarrow KR$ . Note that this depends on the point-set model we choose for  $KR$ ; for an arbitrary model, these maps may only exist/have the desired properties up to homotopy. This will give us the multiplicative structure on the slice spectral sequence of  $KR$  in the next chapter.

A statement of Bott periodicity in fact gives a weak equivalence  $KR_0 \rightarrow \Omega^\rho KR_0$ , which implies the existence of a stable equivalence  $\Sigma^\rho KR \rightarrow KR$ . This map is often denoted by  $\beta$ . This requires that we have an explicit model of  $KR$  as a spectrum. We will use the bifibrant model used by [Duga] (bifibrancy dependent on the model of  $BU$ ).

## 8 The Slice Spectral Sequence

The goal of this chapter is to compute the homotopy Mackey functors of  $KR$  using the slice spectral sequence and prove the gap theorem (Corollary 8.20). This computation will be done by breaking  $KR$  apart into simpler pieces using an equivariant analogue of the Postnikov tower.

For a non-equivariant spectrum  $X$ , its Postnikov tower is a sequence of spectra

$$\cdots \rightarrow \mathbb{P}^1 X \rightarrow \mathbb{P}^0 X \rightarrow \mathbb{P}^{-1} X \rightarrow \cdots$$

with maps  $X \rightarrow \mathbb{P}^n X$  commuting with the maps in the sequence such that  $\pi_i(\mathbb{P}^n X) \cong 0$  for  $i > n$ , each map  $X \rightarrow \mathbb{P}^n X$  induces an isomorphism on  $\pi_i$  for  $i \leq n$ , and each map  $\mathbb{P}^n X \rightarrow \mathbb{P}^{n-1} X$  induces an isomorphism on  $\pi_i$  for  $i < n$ . The spectrum  $\mathbb{P}^n X$  can be constructed by repeatedly killing all maps from each  $S^i$  to  $X$  for  $i > n$  by attaching cells along them. For each  $n$ , the homotopy fiber  $\mathbb{P}_n^n$  of  $\mathbb{P}^n \rightarrow \mathbb{P}^{n-1}$ , which we will call the *n*th Postnikov component of  $X$ , is  $\Sigma^n H(\pi_n X)$  where  $HA$  denotes the Eilenberg-MacLane spectrum with  $\pi_0(HA) \cong A$  (and the suspension above moves the non-zero homotopy group up to the *n*th degree).

$\mathbb{P}^n X$  is called the *n*-truncation of  $X$ . Dually, we can construct the *n*-connected covers  $\mathbb{P}_{n+1} X$  of  $X$ .  $\mathbb{P}_{n+1} X$  has trivial homotopy groups at and below degree  $n$  and has a natural map to  $X$  that induces isomorphisms on all homotopy groups above degree  $n$ . This can be constructed as the homotopy fiber of the map  $X \rightarrow \mathbb{P}^n X$ . These fit together into the *Whitehead tower* of  $X$

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \mathbb{P}_1 X & \longrightarrow & \mathbb{P}_0 X & \longrightarrow & \mathbb{P}_{-1} X \longrightarrow \cdots \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \Sigma H(\pi_1 X) & & H(\pi_0 X) & & \Sigma^{-1} H(\pi_{-1} X) \end{array}$$

with the indicated homotopy cofibers.

Applying the homotopy functor  $\pi_*$ , we get an unrolled exact couple of graded abelian groups in which  $A_s = \mathbb{P}_{-s} X$  in the notation from Chapter 1. The inverse limit  $A_{-\infty}$  for this exact couple is 0 and the direct limit  $A_{\infty}$  is  $\pi_*(X)$ . Therefore we can apply Theorem 1.9 to get a spectral sequence converging to  $\pi_* X$ . We could have done this with the Postnikov tower too to get the same spectral sequence, but the multiplicative structure is easier to see in the Whitehead tower, as we will see later.

However, this spectral sequence will give us no useful information. The  $E^1$

page, under the slice grading of Definition 1.6, is given by

$$E_{s,t}^1 = E_{-s-t,s} = \pi_s(\Sigma^{s+t} H(\pi_{s+t} X)) \cong \begin{cases} \pi_s(X) & \text{if } t = 0 \\ 0 & \text{otherwise} \end{cases}$$

Since the  $r$ th differential  $d^r$  has bidegree  $(-1, r+1)$  under this grading, this spectral sequence cannot support any differentials on any page and hence there is no simplification before the  $E^\infty$  page. The filtration on  $\pi_*(X)$  whose successive quotients are being computed by this spectral sequence is trivial in the sense that it merely separates out the various degrees without easing the computation of any of them. However, there is an equivariant analogue of the Postnikov tower which does not suffer from this issue.

## 8.1 Slice Tower and its Spectral Sequence

One obvious equivariant analogue of the Postnikov tower for a  $G$ -spectrum  $X$  could be constructed by killing maps into  $X$  from  $(G/H)_+ \wedge S^i \cong \text{Ind}_H^G S^i$  for all  $i > n$  and  $H \leq G$  to get  $\mathbb{P}^n X$ , because the  $i$ th homotopy Mackey functor of  $X$  is zero exactly when there are no non-trivial maps in  $\mathcal{SH}^G$  from each  $(G/H)_+ \wedge S^i$  to  $X$ . This would then also be preserved by restriction of action in the sense that  $\mathbb{P}^n(\text{Res}_H^G X) \simeq \text{Res}_H^G(\mathbb{P}^n X)$  for any  $n \in \mathbb{Z}$ ,  $H \leq G$  (the proof of the analogous property of the slice tower is in Proposition 8.13). We will call this tower the Postnikov tower.

However, the resulting spectral sequence would once again suffer from the same problem as the non-equivariant one. This is a consequence of the fact that there are no homotopically non-trivial maps  $\text{Ind}_H^G S^m \rightarrow \text{Ind}_K^G S^n$  for any  $H, K \leq G$  and  $n > m$ , so that killing the higher homotopy groups does not interfere with the lower ones. This is no longer true if we include representation spheres; there may be homotopically non-trivial maps  $S^V \rightarrow S^W$  even if  $V < W$ .

Moreover, the construction proposed above would not have another good property of the non-equivariant Postnikov tower. Suspending a spectrum  $X$  moves its homotopy groups up one degree, so the Postnikov tower simply gets suspended and shifted. We do get this property for  $G$ -spectra, but only when we suspend by smashing with  $S^1$ . It is instead more natural to have this stability property be with respect to smashing with  $S^{\rho_G}$ , the representation sphere for the regular  $G$ -representation  $\rho_G$ , especially since we wish to use this spectral sequence on  $KR$  which is  $\rho_{C_2}$ -periodic. Note, however, that stability under smashing with  $S^{\rho_G}$  will not ensure stability under smashing with every  $G$ -representation.

To ensure both stability under smashing with the regular representation and preservation by restriction of action, we replace the induced up spheres with the induced up regular representation spheres:

$$\widehat{S}(m, H) := \text{Ind}_H^G S^{m\rho_H} = G_+ \wedge_H S^{m\rho_H}$$

## 8 The Slice Spectral Sequence

for  $m \in \mathbb{Z}$  and  $H \leq G$ .  $\widehat{S}(m, H)$  is called a slice sphere of type  $(m, H)$ . Define the dimension of a slice sphere to be that of its underlying spectrum (which would be a wedge of spheres)

$$\dim \widehat{S}(m, H) := m|H|$$

We construct  $P^n X$  by killing all maps into  $X$  from slice spheres of dimension greater than  $n$ . We will say that  $P^n X$  is the  $n$ th slice section of  $X$ . In the case of the Postnikov sections  $\mathbb{P}^n X$ , we can show that this construction is possible by attaching cells in the increasing order of dimensions and using the fact that attaching a higher dimensional sphere does not affect the lower degree homotopy groups. This argument does not work here as it is. However, we can use the small object argument for this.

Suppose we have some set  $\mathcal{A}$  of  $G$ -spectra and a  $G$ -spectrum  $X \in \mathbf{Sp}_O^G$ . Assume that  $\mathcal{A}$  consists of only cofibrant objects and  $X$  is bifibrant so that we may use left homotopies. We wish to construct a  $G$ -spectrum  $P^{\mathcal{A}} X$  such that for any  $A \in \mathcal{A}$ , all maps  $A \rightarrow P^{\mathcal{A}} X$  in  $\mathcal{SH}^G$  are trivial. If we insist also that  $P^{\mathcal{A}} X$  be bifibrant, then this means, from the definition of a left homotopy, that every map  $A \rightarrow P^{\mathcal{A}} X$  extends to a map  $CA \rightarrow P^{\mathcal{A}} X$  where  $CA$  denotes the cone on  $A$ . This is equivalent to saying that the map  $P^{\mathcal{A}} X \rightarrow *$  is in  $\mathcal{S}^{\square}$  where  $\mathcal{S}$  is the set of the canonical inclusions

$$\mathcal{S} = \{A \hookrightarrow CA \mid A \in \mathcal{A}\}$$

The fibrancy of  $P^{\mathcal{A}} X$  is equivalent to this map being in  $\mathcal{J}^{\square}$ , where  $\mathcal{J}$  is the set of generating acyclic cofibrations of  $\mathbf{Sp}_O^G$ , so altogether we want  $(P^{\mathcal{A}} X \rightarrow *) \in (\mathcal{S} \cup \mathcal{J})^{\square}$ .

We also want to ensure that  $P^{\mathcal{A}} X$  is obtained from  $X$  in the least intrusive way possible; we want it to be obtained by just attaching cells along maps from  $\mathcal{A}$ . This means that there should be a relative  $\mathcal{S}$ -cell  $X \rightarrow P^{\mathcal{A}} X$ , but we will instead only ask that this map be a relative  $(\mathcal{S} \cup \mathcal{J})$ -cell, which doesn't pose any issues as relative  $\mathcal{J}$ -cells are weak equivalences. Therefore, we want to factor the map  $X \rightarrow *$  into a map in  $\overline{\mathcal{S} \cup \mathcal{J}}$  followed by a map in  $(\mathcal{S} \cup \mathcal{J})^{\square}$ . If every  $A \in \mathcal{A}$  and every disk (domains of  $\mathcal{J}$ ) is small relative to  $\overline{\mathcal{S} \cup \mathcal{J}}$ , then Theorem 2.20 gives us such a factorization, and therefore a construction of such a  $G$ -spectrum  $P^{\mathcal{A}} X$ .

However, the  $P^{\mathcal{A}} X$  constructed here will not necessarily be unique with these properties, even up to homotopy. For example, if  $G$  is the trivial group,  $\mathcal{A}$  contains only  $S^n$ , and  $X$  has trivial  $n$ th homotopy group, then the standard proof of the small object argument would construct  $P^{\mathcal{A}} X$  by attaching more copies of  $D^{n+1}$  to  $X$  even though it doesn't need to. This will not affect the  $n$ th homotopy group, but the higher homotopy groups could change. Therefore, we need more to ensure the uniqueness of these spectra even up to homotopy.

Suppose we had two different extensions of  $A \rightarrow X$  to maps  $CA \rightarrow X$ . They would then glue into a single map  $\Sigma A \rightarrow X$ . If we ensure that this map then

## 8.1 Slice Tower and its Spectral Sequence

extends to a map  $C\Sigma A \rightarrow X$ , then the two extensions would be left homotopic. Therefore, the set  $\mathcal{A}$  should be closed under suspensions. To ensure that  $P^{\mathcal{A}}X$  is cofibrant, it is enough to require that all maps in  $\mathcal{S}$  be cofibrations, as then the map  $X \rightarrow P^{\mathcal{A}}X$  would be a cofibration with a cofibrant domain, making its codomain cofibrant. For maps in  $\mathcal{S}$  to be cofibrations, it is enough that all objects in  $\mathcal{A}$  be cofibrant.

We want to construct the  $n$ th slice section by killing maps from the set  $\left\{ \widehat{S}(m, H) \mid m|H| > n \right\}$ . We take the closure of this set under suspensions.

$$\mathcal{A}_n := \left\{ \Sigma^k \widehat{S}(m, H) \mid m|H| > n, k \in \mathbb{N} \right\}$$

Let  $\mathcal{S}_n$  denote the set of canonical inclusions  $A \rightarrow CA$  for  $A \in \mathcal{A}_n$ . The above construction applies because the suspensions of the slice spheres are small relative to  $\overline{\mathcal{S}_n \cup \mathcal{J}}$  (a consequence of compactness of the spheres; see [Hov, Proposition 2.4.2]). Denote the  $G$ -spectrum we get from the above construction by  $P^n X$ . This has the following universal property.

**Definition 8.1.** A fibrant  $G$ -spectrum  $Y$  is called *slice  $n$ -truncated* if every map  $A \rightarrow Y$  for  $A \in \mathcal{A}_n$  extends to a map  $CA \rightarrow Y$ . Equivalently,  $Y$  is slice  $n$ -truncated if  $Y \rightarrow *$  is in  $(\mathcal{S}_n \cup \mathcal{J})^{\square}$ .

**Lemma 8.2.** Let  $X \rightarrow P^n X$  be as above. Any map  $X \rightarrow Y$  where  $Y$  is slice  $n$ -truncated factors through a map  $P^n X \rightarrow Y$ . This factoring is unique up to left homotopy.

*Proof.* Such a factoring exists because in the following square, the map on the left is in  $\overline{\mathcal{S}_n \cup \mathcal{J}}$  and the map on the right is in  $(\mathcal{S}_n \cup \mathcal{J})^{\square}$ .

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & \nearrow \text{dotted} & \downarrow \\ P^n X & \longrightarrow & * \end{array}$$

Given any two such factorizations, a left homotopy between them can be built by extending the identity homotopy on the map  $X \rightarrow Y$  cell by cell to a homotopy between their extensions. The induction step uses the fact that for a  $G$ -spectrum  $T$ , gluing together a cone, a cylinder, and then a cone on  $T$  gives the suspension of  $T$ , which is why we closed  $\mathcal{A}_n$  under suspension.  $\square$

**Corollary 8.3.** For any bifibrant  $G$ -spectrum  $X$  and maps  $X \rightarrow P^n X$ ,  $X \rightarrow \tilde{P}^n X$  in  $\overline{\mathcal{S}_n \cup \mathcal{J}}$  to slice  $n$ -truncated  $G$ -spectra,  $P^n X$  and  $\tilde{P}^n X$  are weakly equivalent.

*Proof.* From the discussion above,  $P^n X$  and  $\tilde{P}^n X$  are bifibrant. The universal property described in Lemma 8.2 implies that they are homotopy equivalent and hence weak equivalent.  $\square$

Therefore,  $P^n X$  is the unique slice  $n$ -truncated  $G$ -spectrum up to weak equivalence that is obtained from  $X$  by attaching cones along maps from the slice spheres. We call  $P^n X$  the  $n$ th slice section of  $X$ . It is universal in  $\mathcal{SH}^G$  among slice  $n$ -truncated spectra receiving maps from  $X$ .

**Theorem 8.4.** For each bifibrant  $G$ -spectrum  $X$ , there exists a tower under  $X$

$$\dots P^1 X \rightarrow P^0 X \rightarrow P^{-1} X \rightarrow \dots$$

which is unique up to natural weak equivalence. This is called the *slice tower of  $X$* .

*Proof.* For each  $n \in \mathbb{Z}$ , apply Lemma 8.2 to get a factoring of  $X \rightarrow P^{n-1} X$  through  $X \rightarrow P^n X$ . The lemma applies because  $P^{n-1} X$  is slice  $(n-1)$ -truncated and hence also slice  $n$ -truncated.  $\square$

We can also define a slice analogue of the Whitehead tower for a bifibrant  $X$ .

**Definition 8.5.** The  $(n+1)$ st coslice section of  $X$ , denoted  $P_{n+1} X$ , is the homotopy fiber of the map  $X \rightarrow P^n X$ .

**Definition 8.6.** The *coslice tower* of  $X$  is the tower

$$\dots \rightarrow P_1 X \rightarrow P_0 X \rightarrow P_{-1} X \rightarrow \dots$$

where the maps are induced by the universal property of homotopy fibers applied to the diagram

$$\begin{array}{ccccc} & P_n X & & & \\ & \uparrow & \searrow & & \\ P_{n+1} X & \longrightarrow & X & \longrightarrow & P^n X \\ & & \searrow & \downarrow & \\ & & & P^{n-1} X & \end{array}$$

The existence and uniqueness (up to homotopy) of the coslice tower for a bifibrant  $G$ -spectrum is implied by the same for the slice tower.

**Definition 8.7.** A  $G$ -spectrum  $X$  is *slice  $n$ -connected* if  $\mathcal{SH}^G(X, Y) = 0$  for every slice  $n$ -truncated  $Y$ .

The  $(n+1)$ st coslice section  $P_{n+1} X$  of any bifibrant  $G$ -spectrum  $X$  is slice  $n$ -connected. We can say more. The first map in the cofiber sequence

$$X \rightarrow P^{n-1} X \rightarrow (P^{n-1} X)/X \simeq \Sigma P_n X$$

## 8.1 Slice Tower and its Spectral Sequence

is a relative  $(\mathcal{S}_{n-1} \cup \mathcal{J})$ -cell, so the map  $* \rightarrow \Sigma P_n X$  to the quotient is too. As  $\Sigma P_n X$  has no fibrancy requirement, we may assume without loss of generality, by replacing it with a weakly equivalent  $G$ -spectrum, that this map is a relative  $\mathcal{S}_{n-1}$ -cell. Therefore, there is a filtration on  $\Sigma P_n X$  such that the *associated graded* of this filtration (the wedge sum of its successive quotients) is a wedge sum of  $G$ -spectra of the form  $\Sigma^{k+1} \widehat{S}(m, H)$  with  $k \geq 0$ ,  $m|H| \geq n$ . Desuspending the entire filtration, we get a construction of  $P_n X$  whose associated graded is a desuspension of the associated graded above, i.e. a wedge sum of  $\Sigma^k \widehat{S}(m, H)$  with  $k \geq 0$ ,  $m|H| \geq n$ , or a wedge sum of elements of  $\mathcal{A}_{n-1}$ .

**Proposition 8.8.** There is a model of the  $n$ th coslice section of any bifibrant  $G$ -spectrum which admits a finite filtration with associated graded a wedge sum of elements of  $\mathcal{A}_{n-1}$ . Any  $G$ -spectrum  $X$  filtered with cofibrations

$$* = X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots \rightarrow X_m = X$$

whose associated graded is such a wedge sum is slice  $n$ -connected.

*Proof.* The above discussion proves the first statement. For the second statement, induct on the filtration. Any wedge of elements of  $\mathcal{A}_{n-1}$  is slice  $n$ -connected since wedge sum is the coproduct in  $\mathcal{SH}^G$ . For each  $k = 1, \dots, n$ , there is a cofiber sequence

$$X_{k-1} \rightarrow X_k \rightarrow C_k$$

where  $C_k$  is a wedge of elements of  $\mathcal{A}_{n-1}$  (because it is a summand of the associated graded) and hence slice  $n$ -connected. For any slice  $n$ -truncated  $G$ -spectrum  $Z$ , applying  $\mathcal{SH}^G(-, Z)$  to the above gives an exact sequence in which the outer two abelian groups are trivial, and hence so is the middle one.  $\square$

**Proposition 8.9.** For any bifibrant  $G$ -spectrum  $X$  and integers  $i, j$ , we have weak equivalences

- (i)  $P_i P_j X \simeq P_{\max(i,j)} X$ .
- (ii)  $P^i P^j X \simeq P^{\min(i,j)} X$ .
- (iii)  $P^j P_i X \simeq P_i P^j X$ .

*Proof.* (i) and (ii) are direct consequences of the universal properties. For (iii), assume that  $j \geq i$  because otherwise, both sides are trivial. Then consider the

## 8 The Slice Spectral Sequence

commutative solid square

$$\begin{array}{ccccc}
 ? & \dashrightarrow & P_{j+1}X & \dashrightarrow & * \\
 \downarrow & & \downarrow & & \downarrow \\
 P_i X & \dashrightarrow & X & \longrightarrow & P^{i-1}X \\
 \downarrow & & \downarrow & & \downarrow \\
 P_i P^j X & \dashrightarrow & P^j X & \longrightarrow & P^{i-1}X
 \end{array}$$

We construct the left column by taking homotopy fibers of the horizontal maps in the square, and the top row by taking homotopy fibers of the square the other way. We get  $?$  as the homotopy fiber of the top row, or equivalently (since limits commute with limits) as the homotopy fiber of the left column. The former says that  $? \simeq P_{j+1}X \simeq P_{j+1}P_i X$  using (i), so the latter implies the result.  $\square$

**Definition 8.10.** Define  $P_i^j X := P_i P^j X$ . We call  $P_n^n$  the  $n$ th slice of  $X$ .

Applying the integer-graded homotopy Mackey functor functor  $\pi_*$  to the slice tower (and its successive homotopy fibers) and to the coslice tower (and its successive homotopy cofibers), we get two unrolled exact couples. The stable equivalence in Proposition 8.9(iii) implies that the  $E_s$  terms (notation as in Chapter 1) of both are the same and that they give the same spectral sequence. The exact couple in the coslice case is

$$\begin{array}{ccccccc}
 \cdots & \longrightarrow & \pi_*(P_2 X) & \longrightarrow & \pi_*(P_1 X) & \longrightarrow & \pi_*(P_0 X) & \longrightarrow & \pi_*(P_{-1} X) & \longrightarrow & \cdots \\
 & & \swarrow \text{dashed} & & \downarrow & \swarrow \text{dashed} & \downarrow & \swarrow \text{dashed} & \downarrow & & \\
 & & & & \pi_*(P_1^1 X) & & \pi_*(P_0^0 X) & & \pi_*(P_{-1}^{-1} X) & & 
 \end{array}$$

In the notation of Chapter 1, we have

$$\begin{aligned}
 \underline{A}_{s,t} &= \pi_t(P_{-s} X) \\
 \underline{E}_{s,t} &= \pi_t(P_{-s}^{-s} X)
 \end{aligned}$$

**Remark 8.11.** When an exact couple has decreasing indexing, we call it a *cohomological exact couple*, as opposed to the exact couples we saw in Chapter 1 which are called *homological exact couples*. If we do not make the change in indexing by putting negative signs as above, we get *cohomological spectral sequences*, with the only difference being in grading. To keep indexing related confusion to the minimum, we work only with homological indexings.

**Definition 8.12.** The *slice spectral sequence* of a bifibrant  $G$ -spectrum  $X$  is the spectral sequence of Mackey functors  $(\underline{E}^r, d^r)$  associated to the exact



couple defined above under the slice grading. Its  $\underline{E}^1$  page is described as

$$\underline{E}_{s,t}^1 = \underline{E}_{-s-t,s} = \pi_s(P_{s+t}^{s+t} X)$$

and its differentials  $d^r$  have bidegree  $(-1, r+1)$ .

We would like this spectral sequence to converge to the integer-graded homotopy groups of  $X$ . We will prove this using Theorem 1.9 by showing that  $\underline{A}_{-\infty} = 0$  and  $\underline{A}_{\infty} = \pi_*(X)$ .

**Proposition 8.13.** Let  $H \leq G$ .

- (i) Let  $X$  be a slice  $n$ -truncated  $G$ -spectrum.  $\text{Res}_H^G X$  is a slice  $n$ -truncated  $H$ -spectrum.
- (ii) For  $n \geq 0$ ,  $(G/H)_+ \wedge S^n$  is slice  $(n-1)$ -connected.
- (iii) For  $n < 0$ ,  $(G/H)_+ \wedge S^n$  is slice  $(n|G| - 1)$ -connected.

*Proof.* (i) We need to show that  $\text{Res}_H^G X \rightarrow *$  is in the right complement of a set of maps, each of which has the form  $\text{Ind}_K^H f$  for a  $K$ -map  $f$  where  $K \leq H$ . Using adjointness, it suffices to show that  $X \rightarrow *$  is in the right complement of the set of maps of the form  $\text{Ind}_H^G \text{Ind}_K^H f = \text{Ind}_K^G f$  where  $f$  is as above, but these maps are contained in  $\mathcal{S}_n \cup \mathcal{J}$  and hence lift against  $X \rightarrow *$ .

- (ii) See [HHRb, Proposition 11.2.6](i).  $\tau_n^G$  there denotes the full subcategory of slice  $(n-1)$ -connected  $G$ -spectra. The proofs rely on the fact that in a cofiber sequence  $X \rightarrow Y \rightarrow Z$ , if  $X$  is slice  $n$ -connected, then  $Y$  is slice  $n$ -connected if and only if  $Z$  is. This is a consequence of the fact that applying the functor  $\mathcal{SH}^G(-, T)$  to a cofiber sequence gives a long exact sequence of abelian groups.

- (iii) See [HHRb, Proposition 11.2.6](ii). □

**Corollary 8.14.** The slice spectral sequence for a bifibrant  $G$ -spectrum  $X$  converges to its homotopy Mackey functors  $\pi_* X$ , that is, for each  $s$ , there is a filtration

$$\cdots \subset \underline{F}_1 \subset \underline{F}_0 \subset \underline{F}_{-1} \subset \cdots \subset \pi_s(X)$$

such that

$$\underline{E}_{s,t}^{\infty} \cong \frac{F_t}{F_{t-1}}$$

*Proof.* Proposition 8.13(ii) implies that the exact couple associated to the coslice has  $\underline{A}_{-\infty} = 0$ , while (iii) implies that  $\underline{A}_{\infty} \cong \pi_* X$  (those are statements about the vanishing of certain homotopy Mackey functors of the slice sections, but we can infer them for the coslice sections using the long exact sequence a fiber sequence).

## 8 The Slice Spectral Sequence

Theorem 1.9 gives us the required result. Note that our grading here is different from the one in the theorem.  $\square$

Now suppose a bifibrant  $G$ -spectrum  $X$  has an (on-the-nose) associative multiplication map  $X \wedge X \rightarrow X$ . We show that the slice spectral sequence of  $X$  has multiplicative structure in the sense of Section 1.3.

**Proposition 8.15.** Let  $X$  be a bifibrant  $G$ -spectrum. Then  $P_m X \wedge P_n X$  is slice  $(m + n - 1)$ -connected.

*Proof.* Take  $P_m X$  and  $P_n X$  to be filtered such that their associated gradeds are wedge sums of elements of  $\mathcal{A}_{m-1}$  and  $\mathcal{A}_{n-1}$  respectively. Then  $P_m X \wedge P_n X$  has the smashed filtration whose associated graded is a wedge sum of smash products of elements of  $\mathcal{A}_{m-1}$  with those of  $\mathcal{A}_{n-1}$ . All such smash products are wedge sums of elements of  $\mathcal{A}_{m+n-2}$ . This can be proved using the fact that for any  $H$ -space or spectrum  $A$  and any  $G$ -space or spectrum  $B$ , there is a  $G$ -homeomorphism

$$(\mathrm{Ind}_H^G A) \wedge B \cong \mathrm{Ind}_H^G (A \wedge \mathrm{Res}_H^G B) \quad \square$$

This allows us to describe ‘graded multiplication’ maps on the coslice sections as in the following diagram, where we can use the universal property of  $P_{m+n} X$  as the homotopy fiber of the map  $X \rightarrow P^{m+n-1} X$ .

$$\begin{array}{ccccc} & & & & P_{m+n} X \\ & & & \nearrow & \downarrow \\ P_m X \wedge P_n X & \longrightarrow & X \wedge X & \longrightarrow & X \\ & & & & \downarrow \\ & & & & P^{m+n-1} X \end{array}$$

Recall that for a multiplicative spectral sequence, we need graded multiplication maps on an entire Cartan-Eilenberg system. We get a Cartan-Eilenberg system here by defining

$$\underline{K}_*(i, j) := \pi_*(P_{-i}^{-j-1} X)$$

with notation as in Section 1.3. We need pairings

$$\mu_r : \underline{K}_*(i, i+r) \otimes \underline{K}_*(j, j+r) \rightarrow \underline{K}_{i+j, i+j+r}$$

which come from maps

$$P_{-i}^{-i-r-1} \wedge P_{-j}^{-j-r-1} \rightarrow P_{-i-j}^{-i-j-r-1}$$

We get these by explicitly modelling  $P_{-k}^{-k-r-1}$  as the mapping cone of  $P_{-k-r} \rightarrow P_{-k}$  for each  $k$ . See the discussion preceding [Dugb, Theorem 6.2] for a description of this product in the case of spaces which also works for  $G$ -spectra.

The hardest and most important part of working with the slice spectral sequence is finding the slices. This is enabled, in part, by the following.

**Proposition 8.16.** Let  $X$  be a bifibrant  $G$ -spectrum.

- (i)  $X$  is slice 0-truncated if and only if it is 0-truncated.
- (ii)  $P_0^0 X \simeq \mathbb{P}_0^0 X$ , i.e. the 0th slice and 0th Postnikov component of  $X$  are weakly equivalent.
- (iii)  $\Sigma^\rho(P^n X) \simeq P^{n+|G|}(\Sigma^\rho X)$ . The same can be said of the coslice sections and the slices of  $X$ .

*Proof.* (i) Proposition 8.13 proves that slice 0-truncated implies 0-truncated. For the converse, we need to show that any map from an  $n$ -fold suspension of a slice sphere of positive dimension to  $X$  is trivial up to homotopy, where  $n \geq 0$ . Any suspension of a positive dimensional slice sphere admits a finite filtration whose associated graded is a wedge sum of  $(G/H)_+ \wedge S^n$  with  $n > 0$  (given by the skeletal filtration for the  $G$ -CW structure on the representation spheres that has no non-basepoint 0-cells). We can now reason as in Proposition 8.8.

- (ii) The map  $X \rightarrow \mathbb{P}^0 X$  also serves as the 0th slice section, so its homotopy fiber  $\mathbb{P}_0^0 X$  is also the 0th slice of  $X$ .
- (iii) Smashing a slice sphere of dimension  $m$  with  $S^{\rho_G}$  gives a slice sphere of dimension  $m + |G|$  because of the homeomorphisms

$$\begin{aligned}
 (\mathrm{Ind}_H^G S^{k\rho_H}) \wedge S^{\rho_G} &\cong \mathrm{Ind}_H^G (S^{k\rho_H} \wedge \mathrm{Res}_H^G S^{\rho_G}) \\
 &\cong \mathrm{Ind}_H^G (S^{k\rho_H} \wedge S^{[G:H]\rho_H}) \\
 &\cong \mathrm{Ind}_H^G S^{(k+[G:H])\rho_H}
 \end{aligned}$$

This also shows that every slice sphere of dimension can be obtained this way. Therefore, elements of  $\mathcal{A}_{n+|G|}$  are precisely those of  $\mathcal{A}_n$  smashed with  $S^{n\rho}$ . The first statement follows from the universal property of the slice sections, and then the second one from the fact that  $\Sigma^{\rho_G}$  preserves cofiber sequences.  $\square$

Thus, the slice tower and the Postnikov tower agree in degree 0 and the main feature differentiating them is that the Postnikov tower is stable with respect to  $\Sigma$  while the slice tower is stable with respect to  $\Sigma^\rho$ .

## 8.2 The Slice Spectral Sequence for $KR$ -Theory

Our goal is to compute the homotopy groups of  $KR$ , the spectrum representing  $KR$ -theory, and proving its gap and periodicity theorems as mentioned in the preface. This will be done by finding the coslice tower for  $KR$  and applying the slice spectral sequence. The main tool that will help here is the following cofiber sequence.

**Theorem 8.17.** There exists a cofiber sequence

$$\Sigma^\rho kr \xrightarrow{\beta} kr \rightarrow H\mathbb{Z}$$

where  $kr := \mathbb{P}_0 KR$  and  $\beta$  is the map induced by the Bott map  $\Sigma^\rho KR \rightarrow KR$ ; it is the unique map that makes

$$\begin{array}{ccc} \Sigma^\rho kr & \xrightarrow{\beta} & kr \\ \Sigma^\rho a_0 \downarrow & & \downarrow a_0 \\ \Sigma^\rho KR & \xrightarrow{\beta} & KR \end{array}$$

commute, where  $a_0$  is the canonical map  $P_0 KR \rightarrow KR$ .

*Proof.* See [Duga, Section 6]. □

By Proposition 8.16(iii),  $P_0(KR) \simeq \mathbb{P}_0 KR = kr$ . By Proposition 8.16, the  $2n$ th coslice section of  $KR$  is  $P_{2n} KR = \Sigma^{n\rho} kr$  for each  $n \in \mathbb{Z}$ .

**Lemma 8.18.** For each  $n \in \mathbb{Z}$ , the natural map  $P_{2n} KR \rightarrow P_{2n-1} KR$  is a weak equivalence.

*Proof.* We check that  $P_{2n} KR$  also satisfies the universal property of  $P_{2n-1} KR$ . Let  $X \rightarrow KR$  be a map from a slice  $(2n-2)$ -connected  $G$ -spectrum  $X$ . Since  $X$  is also then slice  $(2n-3)$ -connected, this factors through  $P_{2n-2} KR \cong \Sigma^{(n-1)\rho} kr$  as the dashed map

$$\begin{array}{ccccc} & & X & & \\ & \swarrow \text{dotted} & \downarrow \text{dashed} & \searrow & \\ \Sigma^{n\rho} kr & \longrightarrow & \Sigma^{(n-1)\rho} kr & \longrightarrow & KR \\ & & \downarrow & & \\ & & \Sigma^{(n-1)\rho} H\mathbb{Z} & & \end{array}$$

$H\mathbb{Z}$  is 0-truncated and hence slice 0-truncated, so  $\Sigma^{(n-1)\rho} H\mathbb{Z}$  is slice  $(2n-2)$ -truncated. Since  $X$  is slice  $(2n-2)$ -connected, the vertical composite in the above diagram is nullhomotopic and hence there is a factoring  $X \rightarrow \Sigma^{n\rho} kr$  through the homotopy fiber. This dotted map is unique up to homotopy because any other choice for the dotted arrow would lead to a different choice for the dashed arrow that makes the diagram commute, but the dashed arrow was homotopically unique because of the universal property of  $P_{2n-2} KR$ . □

**Proposition 8.19.** The coslice tower of  $KR$  is

$$\begin{array}{ccccccc}
 & P_{-2}KR & & P_{-1}KR & & P_0KR & & P_1KR \\
 & \wr & & \wr & & \wr & & \wr \\
 \cdots & \xrightarrow{\Sigma^\rho \beta} & \Sigma^\rho kr & = & \Sigma^\rho kr & \xrightarrow{\beta} & kr & = & kr & \xrightarrow{\Sigma^{-\rho} \beta} & \cdots \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & \Sigma^\rho H\mathbb{Z} & & * & & H\mathbb{Z} & & * & & 
 \end{array}$$

where the bottom row show the slices.

We can now use the slice spectral sequence to compute the homotopy Mackey functors of  $KR$ . We have

$$\underline{E}_{s,t}^1 = \pi_s(P_{s+t}^{s+t} X) \Rightarrow \pi_s(X)$$

Since  $P_{s+t}^{s+t} = *$  whenever  $s+t$  is odd, the odd anti-diagonals of the grid are zero. On the  $2n$ th anti-diagonal ( $s+t = 2n$ ), we have the homotopy Mackey functors of  $\Sigma^{n\rho} H\mathbb{Z}$ . There are isomorphisms

$$\begin{aligned}
 \pi_s(\Sigma^{n\rho} H\mathbb{Z}) &\cong [S^s, S^{n\rho} \wedge H\mathbb{Z}] \\
 &\cong \widetilde{H}_s(S^{n\rho}) \\
 &\cong \underline{H}_{s-n-n\sigma}(*).
 \end{aligned}$$

Therefore, whenever  $s+t = 2n$ ,

$$\underline{E}_{s,t}^1 \cong \underline{H}_{\frac{s-t}{2} - \frac{s+t}{2}\sigma}(*).$$

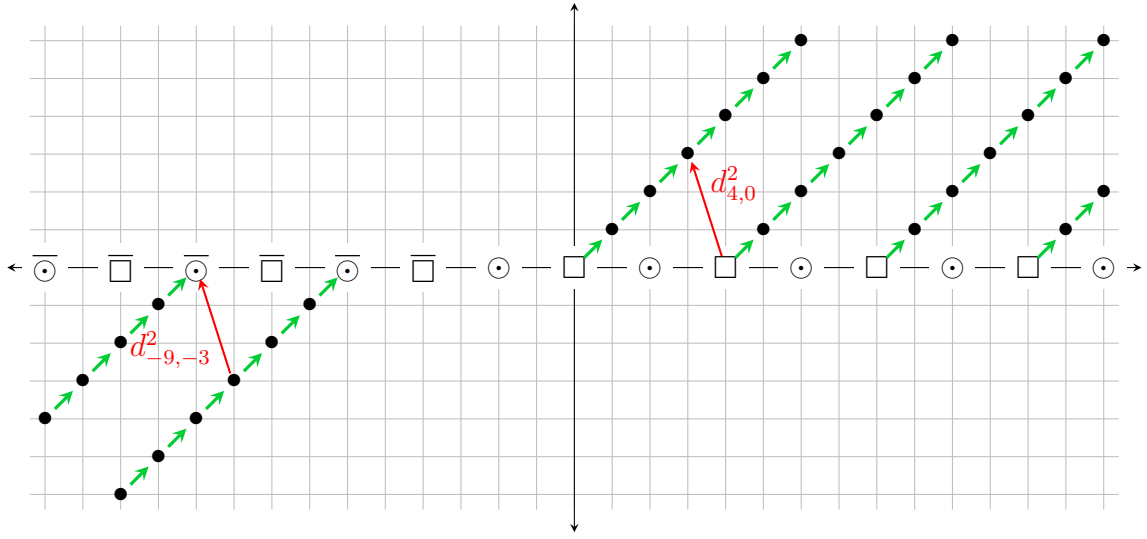
We have already computed the  $RO(G)$ -graded homology of a point in Theorem 6.27. Using notation from there, we draw this page (see Fig. 8.1). The arrows are explained later.

Our next task is to figure out the differentials. The  $d^1$  differentials have bidegree  $(-1, 2)$  and there is no non-zero pair of Mackey functors on the  $\underline{E}^1$  page differing by that bidegree, so  $d^1 = 0$ . Therefore,  $\underline{E}^1 \cong \underline{E}^2$ .

Figuring out the  $d^2$  differentials is trickier. First, we prove that

$$d_{4,0}^2 : \underline{E}_{4,0}^2 \rightarrow \underline{E}_{3,3}^2$$

(labelled in Fig. 8.1) is non-zero. Suppose this were not the case. Then  $\underline{E}_{3,3}^2$  must survive to  $\underline{E}^\infty$ . This is because, for  $r > 2$ , every  $d^r$  originating in  $\underline{E}_{3,3}^r$  would land on a zero Mackey functor and every  $d^r$  that lands on  $\underline{E}_{3,3}^r$  must originate in a zero Mackey functor, so this group will not be killed by taking homologies on the later pages. This would imply that  $\pi_3(KR) \neq 0$ , and in particular the evaluation of this homotopy Mackey functor at the top level of its Lewis diagram, i.e. at  $C_2/C_2$ , would be non-zero. But since  $S^3$  has a trivial  $C_2$ -action, a vector bundle


 Figure 8.1: The  $E^1 \cong E^2$  page of the slice spectral sequence of  $KR$ 

with conjugation over  $C_2/C_2 \wedge S^3$  is equivalent to a real vector bundle over  $S^3$ , and hence

$$\pi_3(KR)(C_2/C_2) \cong \pi_3(KO)$$

Proposition 7.10 tells us that this is zero. Therefore,  $d_{4,0}^2$  must be surjective at the top level, which determines it as the map

$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow & \mathbb{Z}/2\mathbb{Z} \\ \downarrow \uparrow & & \downarrow \uparrow \\ \mathbb{Z} & \longrightarrow & 0 \end{array}$$

To deduce the rest of the  $d^2$  differentials, we need the multiplicative structure of this spectral sequence coming from the multiplication  $KR \wedge KR \rightarrow KR$ . Figuring out the exact multiplicative structure on the  $\underline{E}^1$  page requires working out the multiplication maps on each slice. Since the  $\underline{E}^1$  page is the homology of a point (after some linear transformation), we expect it to have the same multiplicative structure which is indeed the case. This is because the product induced on  $H\underline{\mathbb{Z}}$  by the product on the coslice tower induces the integer-multiplication-product on  $\pi_0(H\underline{\mathbb{Z}}) = \underline{\mathbb{Z}}$  (because tensor product of vector bundles multiplies their dimensions), and an obstruction theory argument shows that products on Eilenberg-MacLane spectra are completely determined by the product they induce on  $\pi_0$ , so this must be the usual product on  $H\underline{\mathbb{Z}}$ .

We can now figure out the differential

$$d_{5,1}^2 : \underline{E}_{5,1}^2 \rightarrow \underline{E}_{4,4}^2$$

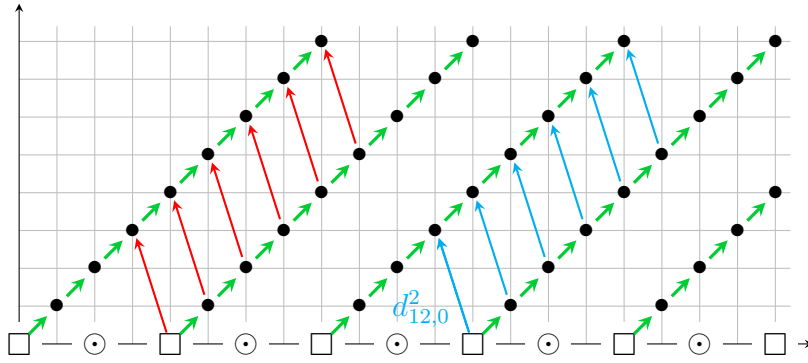
or more generally,

$$d_{4+k,k}^2 : \underline{E}_{4+k,k}^2 \rightarrow \underline{E}_{3+k,3+k}^2$$

inductively for  $k \geq 1$  using the Leibniz formula. Write the generator of  $\underline{E}_{4+k,k}^2$  as  $xy$  where  $x$  is the generator of  $\underline{E}_{4+k-1,k-1}^2$  and  $y$  is the generator of  $\underline{E}_{1,1}^2$ . Then we have

$$\begin{aligned} d_{4+k,k}^2(xy) &= (d_{4+k-1,k-1}^2 x)y + x(d_{1,1}^2 y) \\ &= (d_{4+k-1,k-1}^2 x)y \end{aligned}$$

which is the product of the generators of  $\underline{E}_{3+k-1,3+k-1}^2$  and of  $\underline{E}_{1,1}^2$ . The multiplicative structure tells us that this is the generator of  $\underline{E}_{3+k,3+k}^2$ . The first quadrant then looks like the following, where we have shown that all the red arrows are non-zero differentials.



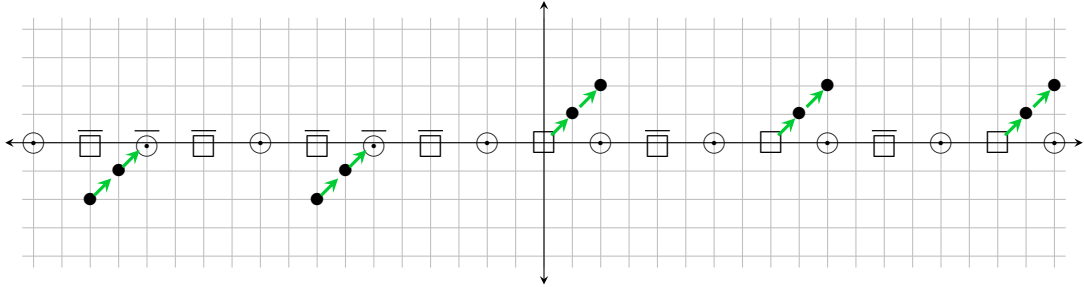
and the differentials are supposed to compose to 0. Therefore, the only way this Mackey functor dies is if  $d_{4+8n,0}^2$  surjects onto it.

This proves that the non-zero  $d^2$  differentials in the first quadrant are arranged as above. These must be all of the non-zero differentials because there is no room left for any others if they are to satisfy the composing to 0 property. Now we can do something similar for the third quadrant. We first prove that the differential

$$d_{-9,-3}^2 : \underline{E}_{-9,-3}^2 \rightarrow \underline{E}_{-10,0}^2$$

must be non-zero (labelled in Fig. 8.1). If it is not, then the Mackey functor  $\underline{E}_{-9,-3}^2$  will survive to the  $\infty$ th page, implying  $\pi_{-9}(KR) \neq 0$ . This contradicts the analysis of the first quadrant as both the Mackey functors in the column  $s = 7$  got killed, giving  $\pi_7(KR) = 0$ , Theorem 7.13, and Theorem 7.12 implies that  $\pi_7(KR) \cong \pi_{-9}(KR)$ .

Using the Leibniz formula and periodicity, we can show that the  $d^2$  differentials in the second quadrant are arranged with a pattern similar to that of the first quadrant. Here, the multiplication maps go the other way, so the argument will go backwards; assuming that the differential diagonally below and to the left of a non-zero differential is zero, we would get a contradiction by deducing that said non-zero differential must be zero too. Having found all the differentials, we can draw  $\underline{E}^3$ .



From this page onwards, all differentials will have to be zero because there is no pair of non-zero Mackey functors in this grid that differs by bidegree  $(-1, r+1)$  for  $r \geq 3$ . Therefore, this is isomorphic to the  $\underline{E}^\infty$  page of this spectral sequence.

Now we can compute the homotopy Mackey functors of  $KR$ . Because of the 8-periodicity, we only need to do this in degrees 0 through 7. For degree 0, the  $\underline{E}^\infty$  page tells us that there is a filtration on  $\pi_0(KR)$  whose intersection is 0 and union is  $\pi_0(KR)$  such that its successive quotients are exactly the Mackey functors  $\underline{E}_{0,t}^\infty$  as  $t$  varies. Since this 0th column of the  $\underline{E}^\infty$  page has only one non-zero Mackey functor  $\mathbb{Z}$ , we find that this must be the value of  $\pi_0(KR)$ . Similarly, we find the homotopy Mackey functors of  $KR$  in all degrees other than 2, since in the column  $s = 2$  of the  $\underline{E}^\infty$  page, there are two non-zero Mackey functors.

This tells us that there is a 2-step filtration on  $\pi_2(KR)$

$$0 = \underline{M}_0 \subset \underline{M}_1 \subset \underline{M}_2 = \pi_2(KR)$$



with successive quotients

$$\frac{M_1}{M_0} \cong \underline{F} \quad \frac{M_2}{M_1} \cong \underline{\mathbb{Z}}_-$$

where we are writing  $\underline{F}$  for the Mackey functor being drawn as  $\bullet$ . This does not uniquely determine  $\pi_2(KR)$ . Indeed, all this says is that there is a short exact sequence

$$0 \rightarrow \underline{F} \rightarrow \pi_2(KR) \rightarrow \underline{\mathbb{Z}}_- \rightarrow 0$$

Looking at the Lewis diagrams tells us that there are exactly two Mackey functors that can fit in the middle of this sequence, one being  $\underline{F} \oplus \underline{\mathbb{Z}}_-$  and the other being  $\underline{\mathbb{Z}}_-^{\text{op}}$ .

$$\begin{array}{ccc} \mathbb{Z}/2\mathbb{Z} & & \mathbb{Z}/2\mathbb{Z} \\ 0 \downarrow \uparrow 0 & & 0 \downarrow \uparrow 1 \\ \mathbb{Z} & & \mathbb{Z} \\ \uparrow & & \uparrow \\ -1 & & -1 \end{array}$$

Once again, the 8-periodicity of  $KR$  comes to the rescue, because we have  $\pi_{-6}(KR) = \underline{\mathbb{Z}}_-^{\text{op}}$  directly by looking at the  $E^\infty$  page. In particular, this implies that the above short exact sequence of Mackey functors is not split. Similar reasoning shows that the extension at  $t = -8$  must be a non-split extension of  $\underline{F}$  by  $\underline{\mathbb{Z}}_-^{\text{op}}$  since  $\pi_{-8}KR \cong \pi_0KR \cong \underline{\mathbb{Z}} \not\cong \underline{F} \oplus \underline{\mathbb{Z}}_-^{\text{op}}$ . We have found all the homotopy Mackey functors of  $KR$ .

$$\begin{aligned} \pi_0(KR) &\cong \underline{\mathbb{Z}} \\ \pi_1(KR) &\cong \underline{F} \\ \pi_2(KR) &\cong \underline{\mathbb{Z}}_-^{\text{op}} \\ \pi_3(KR) &\cong 0 \\ \pi_4(KR) &\cong \underline{\mathbb{Z}}_-^{\text{op}} \\ \pi_5(KR) &\cong 0 \\ \pi_6(KR) &\cong \underline{\mathbb{Z}}_- \\ \pi_7(KR) &\cong 0 \end{aligned}$$

These can be extended to the  $RO(G)$ -graded homotopy Mackey functors of  $KR$  using periodicity (Theorems 7.12 and 7.13). In particular, we have proved the  $KR$  analogue of [HHRa]'s gap theorem of  $\Xi$  as mentioned in the preface.

**Corollary 8.20** (Gap theorem). The homotopy  $C_2$ -Mackey functors of  $KR$  in degrees  $-1, -2, -3$  evaluated at the top level of their Lewis diagrams,  $C_2/C_2$ , are 0.  $\square$

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