The Slice Spectral Sequence for KR-Theory

Lakshay Modi

A thesis presented for the degree of
Master of Science in Mathematics

Chennai Mathematical Institute
India

April 3, 2023

Preface

The aim of this project was to understand and explain parts of Hill, Hopkins, and Ravenel's proof of the Kervaire Invariant Conjecture in their paper [HHRa]. This conjecture states that there do not exist framed manifolds with certain surgery theoretic properties, or equivalently, with Kervaire invariant one. Their solution is broken up into three main theorems about a C_8-spectrum Ξ that they constructed - the detection, gap, and periodicity theorems.

The detection theorem states that the Kervaire invariant one elements are detected as non-zero elements in the homotopy Mackey functors of Ξ in certain degrees. The gap theorem says that the homotopy Mackey functors of Ξ evaluated at the C_8-set C_8/C_8 are 0 in degrees $-1, -2, -3$. The periodicity theorem says that the homotopy Mackey functors of Ξ are 256-periodic. Gap and periodicity together imply that these homotopy Mackey functors evaluate to zero at C_8/C_8 in the necessary degrees, proving the non-existence of elements of Kervaire invariant one in all dimensions other than 126, where the conjecture remains open.

The goal of this thesis is to understand the techniques used in this solution by applying them to a relatively simpler C_2-spectrum KR. This spectrum also has the same gap (evaluation at C_2/C_2) and is 8-periodic, and these theorems are proved by applying the slice spectral sequence introduced in [HHRa] to KR.

This thesis is organized as follows. Chapters 1 to 3 give general abstract background required for the later chapters. They are independent from each other and can be read in any order. Chapter 1 is only used in Chapter 8. Chapters 4 and 5 are also mostly independent, but have to be read after Chapters 2 and 3. Chapters 1 to 5 mostly consist of the background needed to be able to define everything mentioned above. Chapters 6 to 8 form the meat of the thesis have to be read in that order. Chapter 6 introduces G-spectra and homotopy Mackey functors, Chapter 7 defines the C_2-spectrum KR and gives its periodicity theorem, and Chapter 8 defines the slice spectral sequence and uses it to prove the gap theorem for KR.

Contents

1 Spectral Sequences

Let C_* be a chain complex with boundary maps $d_t : C_t \to C_{t-1}$. Suppose C_* has a filtration by subcomplexes

$$C_{0,*} \subset C_{1,*} \subset C_{2,*} \subset \cdots \subset C_*$$

For example, C_* could be the singular chain complex of a CW complex, and $C_{s,*}$ that of its s-skeleton. We are concerned with finding the homology of C_*. Taking homologies of the entire filtration and its successive quotients, we get

$$\begin{tikzcd}
H_*(C_0) \arrow[r] & H_*(C_1) \arrow[r] \arrow[d] & H_*(C_2) \arrow[r] \arrow[d] & H_*(C_3) \arrow[r] \arrow[d] & \cdots \\
 & H_*(C_1/C_0) \arrow[ul, dotted] & H_*(C_2/C_1) \arrow[ul, dotted] & H_*(C_3/C_2) \arrow[ul, dotted] &
\end{tikzcd}$$

The dotted arrows are the connecting maps of degree -1 that we get from the snake lemma, making each triangle an exact sequence. This is the structure of an unrolled exact couple.

1.1 Exact Couples and their Spectral Sequences

Definition 1.1. An *unrolled exact couple* $(A_s, E_s, i_s, j_s, k_s)_{s\in\mathbb{Z}}$ is the data of the following diagram of graded abelian groups

$$\begin{tikzcd}
\cdots \arrow[r] & A_{-1} \arrow[r, "i_0"] & A_0 \arrow[r, "i_1"] \arrow[d, "j_0"'] & A_1 \arrow[r, "i_2"] \arrow[d, "j_1"'] & A_2 \arrow[r] \arrow[d, "j_2"'] & \cdots \\
 & & E_0 \arrow[ul, "k_0"] & E_1 \arrow[ul, "k_1"] & E_2 \arrow[ul, "k_2"] &
\end{tikzcd}$$

in which each triangle is exact.

Throughout this chapter, a map of graded (or bigraded) abelian groups will be assumed to be a map that shifts degrees by some constant. In the example of the filtered chain complex, the maps i_s and j_s preserve degree, while k_s reduces degree by 1.

We can compactify all this data by defining

$$A = \bigoplus_{s\in\mathbb{Z}} A_s, \qquad E = \bigoplus_{s\in\mathbb{Z}} E_s$$

$$\alpha = \operatorname{diag}(i_s)_{s\in\mathbb{Z}}, \qquad \beta = \operatorname{diag}(j_s)_{s\in\mathbb{Z}}, \qquad \gamma = \operatorname{diag}(k_s)_{s\in\mathbb{Z}}$$

A and E are both now bigraded. The first grading is by the *filtration degree* s, and the second grading is from the *internal grading* $*$ that was suppressed.

Definition 1.2. A *rolled exact couple* $(A, E, \alpha, \beta, \gamma)$ is an exact triangle of bigraded abelian groups of the form

$$\begin{tikzcd} A \arrow[r, "\alpha"] & A \arrow[d, "\beta"] \\ & E \arrow[ul, "\gamma"] \end{tikzcd}$$

In the example of the filtered chain complex, α has bidegree $(1, 0)$, β has bidegree $(0, 0)$, and γ has bidegree $(-1, -1)$.

The advantage of an unrolled exact couple is that it makes the filtration explicit. Since our definitions will often depend on the filtration, they would be easier to see in the unrolled diagrams. Rolled exact couples will be easier to manipulate and will declutter notation when the indexing is irrelevant. For example, we have the following key proposition.

Proposition 1.3. For the exact couple of Definition 1.2, define $d : E \to E$ by $d = \beta \circ \gamma$. Then $d^2 = 0$, hence we may define $H(E, d) = \ker d / \operatorname{im} d$. The derived couple of the exact couple, defined as

$$\begin{tikzcd} \operatorname{im}\alpha \arrow[r, "\alpha"] & \operatorname{im}\alpha \arrow[d, "\beta\circ\alpha^{-1}"] \\ & H(E,d) \arrow[ul, "\gamma"] \end{tikzcd}$$

is then a well defined exact couple.

Proof. Routine, see [Hata, Lemma 5.1]. □

Remark 1.4. Deriving the couple we get from an unrolled couple and then unrolling it back, we get the diagram

$$\begin{tikzcd} \cdots \arrow[r] & i_{-1}(A_{-2}) \arrow[r] & i_0(A_{-1}) \arrow[r] \arrow[dr] & i_1(A_0) \arrow[r] \arrow[dl] & i_2(A_1) \arrow[r] \arrow[dl] & \cdots \\ & & H_0(E) \arrow[ul] & H_1(E) \arrow[ul] & & \end{tikzcd} \tag{1.1}$$

which, instead of exact triangles, has a single exact 'triangular helix' that spans the entire length of the diagram, and hence this is not an exact couple in the unrolled sense. However, it still rolls into a rolled exact couple and we may derive it again. These further derivations distort the shape of the unrolled version further. This can be seen by computing the bidegrees of the maps in the derived couple: the maps in the place of α and γ retain their bidegrees, but the map that replaces β has bidegree that of β minus that of α, and hence reduces by $(1, 0)$ with every derivation.

For $r \geq 0$, denote the $(r-1)$-st derived couple of the above couple by $(A^r, E^r, \alpha^r, \beta^r, \gamma^r)$, so that above couple itself is $(A^1, E^1, \alpha^1, \beta^1, \gamma^1)$.

For the unrolled exact couple in Definition 1.1, define $A_\infty = \operatorname{colim}_s A_s$ and $A_{-\infty} = \lim_s A_s$. Then the former is filtered by the images of the natural maps from each A_s and the latter is filtered by the kernels of the natural maps to each A_s. In the case of a filtered chain complex, A_∞ is isomorphic to the homology of C, since the homology functor preserves direct limits. Our goal is to find either A_∞ or $A_{-\infty}$ depending on the setting. For example, we might care about the latter in case of cohomology.

The spectral sequence associated to the couple is a structure that will let us compute the successive quotients of these filtrations on either A_∞ or $A_{-\infty}$ under some mild conditions. This computation is an infinite process and we can think of each step as a simplification of the exact couple without changing $A_{\pm\infty}$. More precisely, in each step, we replace the exact couple with its derived couple as in Remark 1.4. This is a simplification in the sense that every group in the derived couple is smaller and the limit and colimit of the sequence in the derived couple remain the same.

The surprising part is that the data of all the E^r along with the their differentials $d^r = \beta^r \circ \gamma^r$, which have bidegrees $(-r, -1)$, are all that will be needed to compute the groups we want. Example 1.11 is a simple and familiar example of how this works. The structure of all of this data is that of a spectral sequence.

Definition 1.5. A *spectral sequence* is a sequence of bigraded abelian groups $(E^r)_{r\in\mathbb{N}}$, a sequence of endomorphisms $d^r : E^r \to E^r$ which all square to zero, and a sequence of isomorphisms $E^{r+1} \cong H(E^r, d^r)$ for all $r \in \mathbb{N}$. We say that E^r is the r*th page* of the sequence.

Definition 1.6. A spectral sequence $(E^r, d^r)_{r\in\mathbb{N}}$ has *default grading* if each d^r has degree $(-r, -1)$. It has *slice grading* if each d^r has bidegree $(-1, r+1)$.

There are many ways to grade a spectral sequence coming from applying a linear transformation to the (s, t)-grid. For example, the *Serre grading* puts the group $E^r_{s,t-s}$ in bidegree (s, t) to ensure that the Serre spectral sequence is non-zero exactly in the first quadrant rather than the second octant. The slice grading, which is the grading we will use in Chapter 8, is the result of putting $E^r_{t-s,t}$ in bidegree (s, t). This grading is the result of [HHRa] using the *Adams grading* but indexing their spectral sequence to start at the E^2 page. We will have our spectral sequences start in E^1, resulting in the slice grading.

Everything in this chapter will apply equally well to any abelian category and we will need that in Section 8.1. We talk about abelian groups here just for simplicity.

1.2 Convergence of spectral sequences

Our goal is to use the spectral sequence we get from repeated derivations to find $A_{\pm\infty}$. Since $A_{\pm\infty}$ are only defined for an unrolled couple, we will restrict our attention to spectral sequences associated to unrolled exact couples. Once and for all, fix an unrolled exact couple $(A_s, E_s, i_s, j_s, k_s)_{s\in\mathbb{Z}}$, denote its $(r-1)$-st derivation by $(A_s^r, E_s^r, i_s^r, j_s^r, k_s^r)$ and define $E^r = \bigoplus_s E_s^r$. The internal grading of each A and E will not be relevant for a while. Again, as described in Remark 1.4, j^r reduces degree by $r-1$ and k^r reduces degree by 1. Therefore, $d^r = j^r \circ k^r$ has degree $-r$.

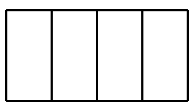
In this entire section, we would have to deal with a lot of filtrations on various abelian groups. It will therefore be convenient to use Vakil's diagram calculus [Vak] to keep track of all the components in a visual way. The following proof is a diagrammatic translation of the purely algebraic proof from [Rog, Chapter 2]. Since we have to deal with infinitely many groups, we make one change. We draw the diagram for each group separately and only remember the identifications made by the first isomorphism theorem, instead of drawing their diagrams in an overlapping way as he does.

The idea of the diagram calculus is to draw filtrations on a group in way similar to Venn diagrams. The main difference from Venn diagrams is that the complement of a region that represents a subgroup represents the quotient by that subgroup rather than its complement. In particular, this means that not every region in the Venn diagram is a subgroup, and we need to remember which ones are and which ones are not. We do this by making the diagrams 'directional'; the filtrations in our diagrams will always increase as we go down and to the right.

Suppose we have an abelian group A with a filtration

$$0 \subset N_1 \subset N_2 \subset N_3 \subset A$$

We then draw A as

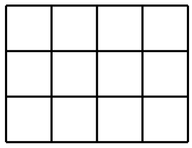


in which the first square represents N_1, the first two squares together represent N_2, the first three represent N_3 and the whole diagram represents A. The middle two squares together will then represent taking the subgroup N_3 and then *removing* N_1 from it, i.e. it represents the subquotient N_3/N_1. We could also view it as first quotienting by N_1 and then taking a subgroup, which is the same by the isomorphism theorems.

Suppose we have another filtrations on A

$$0 \subset M_1 \subset M_2 \subset A$$

Then we can draw A as

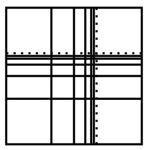


where the top row represents M_1 and the top two rows together represent M_2. Now the two squares in the center are what we get when we take $N_3 \cap M_2$ (all but the last row and last column) and remove from it the first row and then the first column. Thus, that region represents

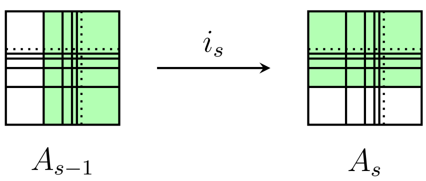
$$\frac{(N_3 \cap M_2)/M_1}{N_1} \cong \frac{N_3 \cap M_2}{M_1 + N_1}$$

The isomorphism theorems will again imply that other ways of getting to these squares will be equivalent.

Now in our case each A_s has two independent and infinite filtrations. One is the increasing filtration by $\ker(A_s \to A_{s+n})$, and the other is the decreasing filtration by $\operatorname{im}(A_{s-n} \to A_s)$. We follow the convention that the kernel filtration increases as we go to the right, and the image filtration increases as we go down. Then A_s will be drawn as



where the region to the left of the vertical dotted line is the union of the kernels and the region above the horizontal dotted line is the intersection of the images. Thus, the dotted lines are being used to signify infinite filtrations. The regions that are identified by i_s via the first isomorphism theorem are the ones below



Also note that the rightmost vertical region in A_s is the coimage of the natural map $i_{s,\infty} : A_s \to A_\infty$. Since A_∞ is the union $\bigcup_s \operatorname{im} i_{s,\infty}$ and the first isomorphism theorem says that $\operatorname{im} i_{s,\infty} \cong \operatorname{coim} i_{s,\infty}$, figuring out the group that that region represents would be sufficient for what we want.

We now come to the E_s. By the exactness of the sequence

$$A_s \to E_s \to A_{s-1}$$

the kernel and image filtrations (both being filtrations of length 1) induced on E_s coincide, and thus there is a single filtration. Therefore, the exact triangle could be drawn as in Fig. 1.1, where the like colors are glued. The colored regions

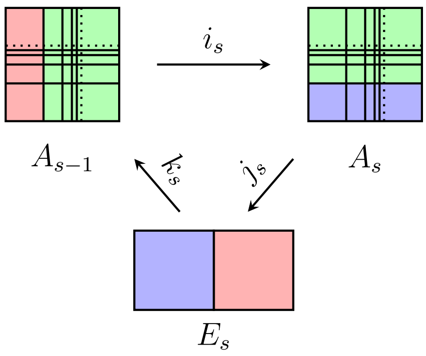
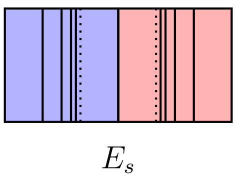


Figure 1.1: Each exact triangle

in E_s now inherit filtrations from these isomorphisms, and we get the following diagram for E_s:



Thus, by definition of this filtration, the rightmost blue region here is the bottom right region in A_s in Fig. 1.1, and the leftmost red region is the top left region of A_{s-1}. The former is thus isomorphic to a subquotient of A_∞, and the latter to one of $A_{-\infty}$. These are the regions that the spectral sequence will allow us to find. Each derivation replaces E_s with a subquotient of it (the one defined by taking homology with respect to d^r). It can be checked that this subquotient is precisely the one that throws away the outermost 2 regions of E_s. Thus at the '∞-th' page, E_s^∞, not yet formally defined, will be the subquotient represented by the middle two regions.

The couple $(A_s^2, E_s^2, i_s^2, j_s^2, k_s^2)$ can be drawn as a subquotient of the original couple as in Fig. 1.2, where the gray regions are the ones that were removed.

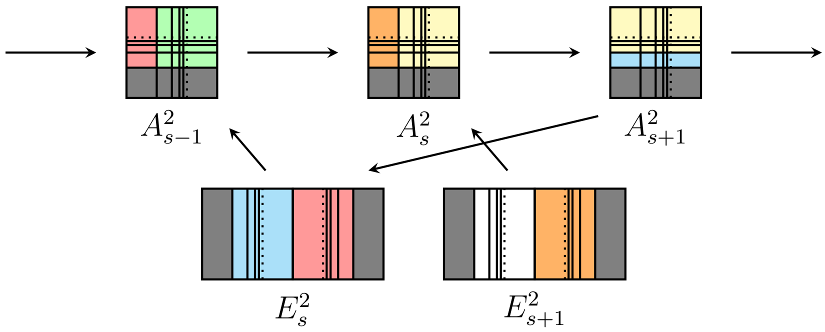


Figure 1.2: The derived couple, unrolled

Again, like colors are being identified by the maps shown. The green color on A^2_{s-1} is supposed to be glued to one on A^2_s, but that has been omitted because of overlaps of colors. This diagram can be used to give an alternate proof of Proposition 1.3 for the case of the unrolled couple if one follows the identifications. Each subsequent derivation throws away the outermost remaining regions of each E^r_s and the bottommost remaining region of each A^r_s. This gives a clear picture of how we should define the ∞-th derivation, which may be drawn as in Fig. 1.3 (identifications omitted, gray denotes discarded regions).

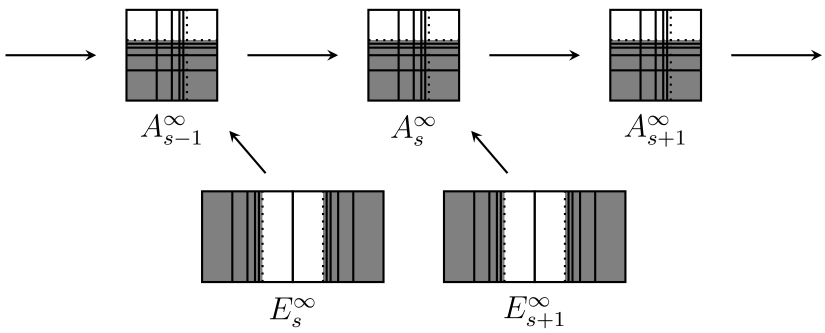


Figure 1.3: ∞-th derivation of the couple

Algebraically, all of this is summarized in the following theorem.

Theorem 1.7. Let $(A_s, E_s, i_s, j_s, k_s)_{s\in\mathbb{Z}}$ be an unrolled exact couple, and $(A, E, \alpha, \beta, \gamma)$ its corresponding rolled exact couple. Then E_s has a filtration

$$0 \subset B_s^1 \subset B_s^2 \subset \cdots \subset B_s^\infty \subset \ker k_s = \operatorname{im} j_s \subset Z_s^\infty \subset \cdots \subset Z_s^2 \subset Z_s^1 \subset E_s$$

where

$$B_s^r = j_s \ker(i_{s+r} \circ \cdots \circ i_{s+1}), \qquad B_s^\infty = \lim_{\rightarrow} B^r = \bigcup_{r=1}^{\infty} B^r$$

$$Z_s^r = k_s^{-1} \operatorname{im}(i_{s-1} \circ \cdots \circ i_{s-r}), \qquad Z_s^\infty = \lim_{\leftarrow} Z^r = \bigcap_{r=1}^{\infty} Z^r$$

such that $E_s^{r+1} \cong \frac{Z_s^r}{B_s^r}$. The differential $d_s^r : E_{s+r}^r \to E_s^r$ is defined as

$$[x] \mapsto [j_s(i_{s+1} \circ \cdots \circ i_{s+r})^{-1} k_s]$$

Proof. This can be proved by repeated applications of the first isomorphism theorem, following what the above figures illustrate. See [Rog, Lemma 2.3.1]. □

Definition 1.8. The ∞-th page of a spectral sequence associated to an exact couple is defined as

$$E^\infty = \frac{Z^\infty}{B^\infty}$$

with notation as in Theorem 1.7.

Thus E_s^∞ is left with 2 regions as shown in Fig. 1.3. We have identifications as shown in Fig. 1.4, which are restrictions of the identifications in Fig. 1.1. The

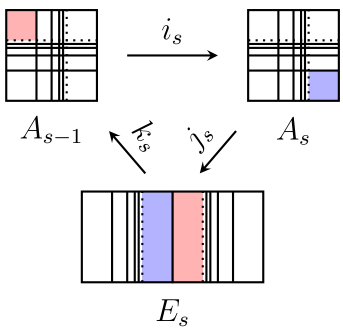


Figure 1.4: Restrictions of the identifications from Fig. 1.1

blue region in A_s is also identified with

$$\frac{\operatorname{im}(A_s \to A_\infty)}{\operatorname{im}(A_{s-1} \to A_s)}$$

This is because the rightmost vertical region in A_s is identified with the image of the map to A_∞, and the image of A_{s-1} in A_∞ is exactly the image of the image of A_{s-1} in A_s. A similar argument identifies the red region with

$$\frac{\ker(A_{-\infty} \to A_s)}{\ker(A_{-\infty} \to A_{s-1})}$$

This information is most useful when either the red or the blue region in E_s is 0, because then the ∞-th page of the sequence is giving us the successive quotients of $A_{\pm\infty}$. This holds, in particular, when the hypotheses of the next theorem are satisfied.

Theorem 1.9. Let $(A_s, E_s, i_s, j_s, k_s)_{s\in\mathbb{Z}}$ be an unrolled exact couple. If $A_{-\infty} = 0$, then

$$E_s^\infty \cong \frac{\operatorname{im}(A_s \to A_\infty)}{\operatorname{im}(A_{s-1} \to A_\infty)}$$

$\bigcap_s \operatorname{im}(A_s \to A_\infty) = 0$ and $\bigcup_s \operatorname{im}(A_s \to A_\infty) = A_\infty$. If $A_\infty = 0$, then

$$E_s^\infty \cong \frac{\ker(A_{-\infty} \to A_s)}{\ker(A_{-\infty} \to A_{s-1})}$$

$\bigcap_s \ker(A_{-\infty} \to A_s) = 0$ and $\bigcup_s \ker(A_{-\infty} \to A_s) = A_{-\infty}$. □

This is what we mean when we say that the spectral sequence converges to $A_{\pm\infty}$.

Definition 1.10. A spectral sequence (E^r, d^r) associated to an exact couple $(A_s, E_s, i_s, j_s, k_s)_{s\in\mathbb{Z}}$ is said to converge to a graded abelian group $X_\bullet$ if there exists a filtration

$$0 \subset \cdots \subset X_{-1,\bullet} \subset X_{0,\bullet} \subset X_{1,\bullet} \subset \cdots \subset X_\bullet$$

with $\bigcap_s X_s = 0$ and $\bigcup_s X_s = X$, along with isomorphisms

$$E_{s,t}^\infty \cong \frac{X_{s,t}}{X_{s-1,t}}$$

In this case, we say

$$E_{s,t}^r \Rightarrow X_t$$

Example 1.11. Consider the exact couple of singular homology groups of a CW complex X filtered by its skeleta. This looks similar to the exact couple in the introduction to this chapter, but with $H_*(C_s)$ replaced with $H_*(X_s)$ and $H_*(C_s/C_{s-1})$ with $H_*(X_s, X_{s-1})$ a spectral sequence whose E^1 term, under the default grading, is

$$E_{s,t}^1 = H_s(X_t, X_{t-1}) \cong H_s(X_t/X_{t-1})$$

Since this quotient space is a wedge of s-spheres, this group is non-zero only if $s = t$. The map $d^1 : E^1_{s,t} \to E^1_{s-1,t-1}$ is the composite

$$H_*(X_s, X_{s-1}) \xrightarrow{\partial} H_*(X_{s-1}) \to H_*(X_{s-1}, X_{s-2})$$

which is exactly the definition of the boundary map in cellular homology. Thus the E^1 page is precisely the cellular chain complex of X sitting on the diagonal. Because of degree reasons, all d^r for $r \geq 2$ must be 0, hence E^∞ is exactly the cellular homology groups $H^{\mathrm{CW}}_s(X)$ sitting in bidegree (s, s). Moreover, $A_{-\infty} = 0$. By Theorem 1.9, there must be a filtration

$$0 \subset \cdots \subset M_{-1,*} \subset M_{0,*} \subset M_{1,*} \subset \cdots \subset H_*(X)$$

such that $E^\infty_{s,t} \cong M_{s,t}$ for each $s, t \in \mathbb{Z}$. This can be visualized on the grid as saying that the as s varies, $E^\infty_{s,t}$ are the successive quotients of some filtration on $H_t(X)$. Since $E^\infty_{s,t} = 0$ for $s \neq t$, this implies

$$H_t(X) = E^\infty_{s,s} = H^{\mathrm{CW}}_t(X)$$

which gives a proof of the agreement of singular and cellular homologies.

1.3 Multiplicative Spectral Sequences

The most difficult part of computing with a spectral sequence is to find the differentials d^r. Trying to do this from the definition of these maps is nearly impossible because of how long winded the definition is and how many identifications are made along the way. Usually, for $r = 1$, there would be some geometric interpretation of d^r. For example, in Example 1.11, we have an interpretation of d^1 in terms of the cellular attaching maps. This often leads to a geometric description of the groups that appear on the E^2 page but not of the differentials. Here, one usually tries to show that certain differentials must be 0 or isomorphisms either because of degree reasons or from partial knowledge of what is being computed by the sequence.

There is one more powerful tool that can be used to determine the differentials - a multiplicative structure on the spectral sequence. This means that each page of spectral sequence comes with a multiplication

$$E^r_{s_1,t_1} \otimes E^r_{s_2,t_2} \to E^r_{s_1+s_2,t_1+t_2}$$

satisfying the Leibniz rule

$$d(x \cdot y) = x \cdot d(y) + d(x) \cdot x$$

such that the multiplication on each successive page E^{r+1} is defined by

$$[x] \cdot [y] \to [x \cdot y]$$

where $[-]$ denotes equivalence class when E^{r+1} is identified with $H(E^r, d^r)$. This is often used to figure out more differentials from already known ones.

Since the multiplication on each page is completely determined by the one on the previous page, it suffices to figure out the multiplication on E^1. However, any arbitrary multiplication on E^1 may not descend to the later pages in a well defined manner, so constructing a multiplicative structure on a spectral sequence is harder than just producting a multiplication on the E^1. It involves using a lot more structure than we get from just the exact couple. Start again with a filtered chain complex C

$$0 = C_{-\infty,*} \subset \cdots \subset C_{-1,*} \subset C_{0,*} \subset C_{1,*} \subset \cdots \subset C_{\infty,*} = C$$

Now instead of taking only the homologies of its successive quotients $C_{s,*}/C_{s-1,*}$, we could take homologies of all quotients $C_{j,*}/C_{i,*}$ where $i, j \in \mathbb{Z} \cup \{\pm\infty\}$. If $i \geq j$, we define this quotient to be 0. Indexing by (u, s), we get a bigraded abelian group $K_*(i, j) = H_*(C_j/C_i)$ that lives only above the $i = j$ line.

K_* then has extra structure. Whenever $i \leq i'$ and $j \leq j'$, there is a natural map

$$\eta_{i,j}^{i',j'} : K_*(i, j) \to K_*(i, j') \to K_*(i', j')$$

where the first map is induced by the inclusion and the second one by further quotient. For each $i \leq j \leq k$, the snake lemma gives a map

$$\partial_{i,j,k} : K_*(j, k) \to K_{*-1}(i, j)$$

such that the following triangle in this grid is always exact:

$$\cdots \xrightarrow{\partial} K_*(i, j) \xrightarrow{\eta} K_*(i, k) \xrightarrow{\eta} K_*(j, k) \xrightarrow{\partial} K_{*-1}(i, j) \xrightarrow{\eta} \cdots$$

This structure is called a *Cartan-Eilenberg* system and it contains all the data of the exact couple and more. A precise definition can be found in [Rog, Section 3.5].

This structure is helpful here because the groups $E^r_{s,t}$ that appear on the rth page of the associated spectral sequence can be identified with subgroups of the groups that appear r units above the diagonal. More specifically, $E^r_{s,t}$ is isomorphic to a quotient of $K_t(s, s + r)$ (see [Rog, Proposition 3.5.9]). This lets us construct structure on the later pages of the spectral sequences from structure present on the Cartan-Eilenberg system.

The multiplicative structure on the spectral sequence associated to a Cartan-Eilenberg system is often induced from a pairing on the system. This consists of graded multiplication maps r units above the diagonal

$$\mu_r : K_*(i, i + r) \otimes K_*(j, j + r) \to K_*(i + j, i + j + r)$$

These maps are required to commute with η and satisfy one more equation that gives us the Leibniz rule (see [Rog, Definition 6.2.1]). Given such a pairing on a

Cartan-Eilenberg system, [Rog, Theorem 6.2.3] proves that we get a multiplicative spectral sequence. In these cases, we also have multiplicative structure on $A_{\pm\infty}$, the targets of convergence of the spectral sequence, and the isomorphisms between the successive quotients of these targets and the groups at the ∞th page are ring isomorphisms.

2 Model Categories

Consider the classical homotopy theory in the category Top. We have the homotopy equivalence relation $\sim$, which allows us to define a quotient functor

$$q : \mathsf{Top} \to \mathsf{Top}/\sim$$

One key observation is that a functor from Top respects the homotopy relation if and only if it takes homotopy equivalences to isomorphisms, and hence q is universal among both functors that respect the homotopy relation and functors that take homotopy equivalences to isomorphisms. For this reason, we may also write $\mathsf{Top}/\sim$ as $\mathsf{Top}[\mathcal{H}^{-1}]$, where $\mathcal{H}$ denotes the subcategory of homotopy equivalences. However, every homotopy invariant known to algebraic topology also takes weak homotopy equivalences to isomorphisms, so it is of greater interest to study a functor $h : \mathsf{Top} \to \mathsf{Top}[\mathcal{W}^{-1}]$ which is universal among those that take all weak homotopy equivalences to isomorphisms.

Some natural questions arise. Is there an equivalence relation $\sim_w$ of 'weak homotopy' such that the weak homotopy equivalences are exactly the maps that are invertible up to weak homotopy? Can the functor h described above also be viewed as the functor that quotients by $\sim_w$? These questions are important because the morphism sets $\mathrm{Hom}_{\mathsf{Top}[\mathcal{W}^{-1}]}(X, Y)$ are difficult to understand, whereas we have a simple description of quotient categories. The answer to these questions is 'no', but we can get a simple description of $\mathsf{Top}[\mathcal{W}^{-1}]$ regardless.

The cellular approximation functor Γ takes weak equivalences to weak equivalences, and is naturally weakly equivalent to the identity functor. This implies that it induces an endofunctor on $\mathsf{Top}[\mathcal{W}^{-1}]$ that is naturally isomorphic to the identity. Therefore, the full image of this endofunctor, which is $CW[\mathcal{W}^{-1}]$, is equivalent to $\mathsf{Top}[\mathcal{W}^{-1}]$. Whitehead's theorem then says that this is equivalent to $CW[\mathcal{H}^{-1}]$. Thus,

$$\mathrm{Hom}_{\mathsf{Top}[\mathcal{W}^{-1}]}(X, Y) \cong \mathrm{Hom}_{\mathsf{Top}}(\Gamma X, \Gamma Y)/\sim$$

In homotopy theory, we often want to treat maps as isomorphisms even when they are not, and having results similar to the above in more general settings would be convenient to study the resulting categories. As it was discovered, the notion of a homotopy and the proofs of the cellular approximation and Whitehead theorems can be expressed using Serre cofibrations and Serre fibrations, using there lifting and factorization properties. This led Quillen to define model categories in his book [Qui]. A model category is a category with three distinguished classes of morphisms called weak equivalences, cofibrations, and fibrations, which satisfy enough properties for an analogous discussion to work. In the following section, we recall what features of these classes allowed us to have this discussion in the first place.

2.1 Classical homotopy theory

We use notation for lifting calculus borrowed from [Rez, Section 15], in which we write $f \boxslash g$ for a pair of maps f, g in a category $\mathcal{M}$ if for every commutative square

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "f"'] & \bullet \arrow[d, "g"] \\ \bullet \arrow[r] \arrow[ur, dashed] & \bullet \end{tikzcd}$$

the indicated lift exists and makes the diagram commute. For a class of maps $\mathcal{S} \subset \mathcal{M}$, we define its left and right lifting complements as

$$^{\boxslash}\mathcal{S} := \{f \in \mathcal{M} \mid f \boxslash s \ \forall s \in \mathcal{S}\}, \qquad \mathcal{S}^{\boxslash} := \{g \in \mathcal{M} \mid s \boxslash g \ \forall s \in \mathcal{S}\}$$

respectively.

Recall that in the category Top of (compactly generated, weak Hausdorff) topological spaces, we have the class of weak homotopy equivalences $\mathcal{W}$, and the class of Serre fibrations is defined as

$$\mathcal{F} := \{i_0 : D^n \hookrightarrow D^n \times I \mid n \geq 0\}^{\boxslash}$$

where $i_t : X \hookrightarrow X \times I$ will always denote the inclusion $x \mapsto (x, t)$. Define the class of Serre cofibrations $\mathcal{C}$ as the class of retracts of relative cell complexes. The motivation for this comes from the following 2 properties, which are analogous to those of Hurewicz (co)fibrations. The proofs of these and their Hurewicz analogues may be found in [MP, Section 17.1 and 17.2].

Proposition 2.1. Any map $f : X \to Y$ can be factored in two ways:

(i) $p \circ \widetilde{i}$ where $p \in \mathcal{F}$ and $\widetilde{i} \in \mathcal{W} \cap \mathcal{C}$.

(ii) $\widetilde{p} \circ i$ where $\widetilde{p} \in \mathcal{W} \cap \mathcal{F}$ and $i \in \mathcal{C}$.

Moreover, these factorizations can be made functorial, in the sense that they both give functors $\mathrm{Map}(\mathsf{Top}) \to \mathrm{Map}(\mathsf{Top}) \times \mathrm{Map}(\mathsf{Top})$.

Proposition 2.2. (i) $\mathcal{C} = {}^{\boxslash}(\mathcal{W} \cap \mathcal{F})$ and $\mathcal{C}^{\boxslash} = \mathcal{W} \cap \mathcal{F}$.

(ii) $(\mathcal{W} \cap \mathcal{C})^{\boxslash} = \mathcal{F}$ and $\mathcal{W} \cap \mathcal{C} = {}^{\boxslash}\mathcal{F}$.

Proposition 2.1 leads immediately to a weaker version cellular approximation theorem. It is weaker in the sense that the cell complexes need not be constructed by attaching cells in increasing order of dimension. We will see that the discussion in the previous section holds because Whitehead's theorem still holds. The standard CW approximation theorem can also be deduced from this using transfinite induction and the fact that there are no homotopically non-trivial maps from a lower dimensional sphere to a higher dimensional one.

Theorem 2.3 (Cellular approximation)**.** There exists an endofunctor Γ on Top which maps each space to a cell complex, and a natural weak equivalence $\eta : \Gamma \to 1_{\mathsf{Top}}$.

Proof. We have a functor $\mathsf{Top} \to \mathrm{Map}(\mathsf{Top})$ which maps $X \mapsto (\varnothing \to X)$. Compose this with the second functorial factorization of Proposition 2.1, and denote the image of X by

$$\varnothing \to \Gamma X \xrightarrow{\sim} X$$

This gives a definition of both Γ and η. By the remark made in that proof, we can ensure that ΓX is a cell complex. $\square$

Theorem 2.4 (Whitehead)**.** A map $f : X \to Y$, where X and Y are cell complexes, is a weak equivalence if and only if it is a homotopy equivalence.

Proof. We know the backward implication to be true. For the forward implication, the idea is to use the Yoneda lemma on $\mathsf{Top}_c/\sim$, where Top_c is the full subcategory of cell complexes and $\sim$ is the homotopy relation. So it suffices to show that for any space Z, the map

$$(\mathsf{Top}/\sim)(f, Z) : (\mathsf{Top}/\sim)(Y, Z) \to (\mathsf{Top}/\sim)(X, Z)$$

is a bijection. It is enough to prove that the following two lifts exist whenever the square commutes ($ev_t : U^I \to U$ denotes evaluation at t).

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & Z^I \arrow[d, "(ev_0, ev_1)"] \\ Y \arrow[r] \arrow[ur, dotted] & Z \times Z \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & Z \\ Y \arrow[ur, dotted] & \end{tikzcd}$$

These lifts signify injectivity and surjectivity respectively (The first says that if there are two maps $Y \to Z$ such that their precomposites with f are homotopic, then they are homotopic). Alternatively, we could show that $(\mathsf{Top}_c/\sim)(Z, f)$ is a bijection, for which it suffices to show that the following lifts exist

$$\begin{tikzcd} Z \sqcup Z \arrow[r] \arrow[d, "i_0+i_1"'] & X \arrow[d, "f"] \\ Z \times I \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd} \qquad \begin{tikzcd} & X \arrow[d, "f"] \\ Z \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

In the first two diagrams, the maps on the right side are fibrations, and in the latter two diagrams, the maps on the left side are cofibrations. By Proposition 2.2, we need f to also be a cofibration or a fibration. We can ensure this by applying either factorization of Proposition 2.1 to f, as they will both factor f as $p \circ i$ with $p \in \mathcal{W} \cap \mathcal{F}$ and $i \in \mathcal{W} \cap \mathcal{C}$. The two cases above show that p and i are both homotopy equivalences, therefore their composite is a homotopy equivalence. □

2.2 Model Categories

The previous section proves all the claims about the homotopy category in the introduction to this chapter. Thus, similar claims are true in any category in which there is a notion of weak equivalences and (co)fibrations satisfying similar properties. This leads to the notion of a model category.

Definition 2.5. A model category is a category $\mathcal{M}$ along with subcategories $\mathcal{W}, \mathcal{C}, \mathcal{F}$ called the subcategories of weak equivalences, cofibrations, and fibrations respectively, satisfying the following axioms:

(A1) $\mathcal{M}$ is bicomplete, i.e. each small diagram in $\mathcal{M}$ has a limit and a colimit.

(A2) If f and g are morphisms of $\mathcal{M}$ such that $g \circ f$ is defined, and any two of f, g, gf are in $\mathcal{W}$, then so is the third.

(A3) $\mathcal{W}$ is closed under retracts in the category of maps in $\mathcal{M}$, i.e. for any commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] \arrow[rr, bend left, "1"] & X \arrow[r] \arrow[d, "g"] & A \arrow[d, "f"] \\ B \arrow[r] \arrow[rr, bend right, "1"'] & Y \arrow[r] & B \end{tikzcd}$$

if $g \in \mathcal{W}$, then $f \in \mathcal{W}$.

(A4) (i) $\mathcal{C} = {}^{\boxslash}(\mathcal{W} \cap \mathcal{F})$ and $\mathcal{C}^{\boxslash} = \mathcal{W} \cap \mathcal{F}$.

(ii) $(\mathcal{W} \cap \mathcal{C})^{\boxslash} = \mathcal{F}$ and $\mathcal{W} \cap \mathcal{C} = {}^{\boxslash}\mathcal{F}$.

(A5) Any map f can be functorially factored in two ways:

(i) $p \circ \widetilde{i}$ where $p \in \mathcal{F}$ and $\widetilde{i} \in \mathcal{W} \cap \mathcal{C}$.

(ii) $\widetilde{p} \circ i$ where $\widetilde{p} \in \mathcal{W} \cap \mathcal{F}$ and $i \in \mathcal{C}$.

The original axioms given by Quillen did not include functoriality of the factorizations. For certain subcategories of $\operatorname{Map}\mathcal{M}$, these factorizations can be made functorial with some effort, so none of the theory later on in this chapter breaks. However, it is more convenient to assume factorizations to be functorial and does not result in any loss of generality in practice, so it was later added to the axioms. Another such difference is that Quillen only required the existence of finite (co)limits, but we now require that all small (co)limits exist.

These axioms are self dual in the sense that if $\mathcal{M}$ is a model category, then so is $\mathcal{M}^{\mathsf{op}}$ with the same $\mathcal{W}$, and swapped $\mathcal{C}$ and $\mathcal{F}$. The rest of the section generalizes everything said in Section 2.1, and because of this self duality, all of definitions and theorems come in pairs. All details for this section can be found in [Hov, Chapter 1]. Let $\mathcal{W} \subset \mathcal{M}$ be categories, and call the maps of $\mathcal{W}$ weak equivalences.

Definition 2.6. The *homotopy category* of $(\mathcal{M}, \mathcal{W})$ is a category $\mathcal{M}[\mathcal{W}^{-1}]$ along with a functor $\ell : \mathcal{M} \to \mathcal{M}[\mathcal{W}^{-1}]$ that maps weak equivalences to isomorphisms, such that for every category $\mathcal{D}$ and functor $\gamma : \mathcal{M} \to \mathcal{D}$ that maps weak equivalences to isomorphisms, there exists a unique functor $\hat{\gamma} : \mathcal{M}[\mathcal{W}^{-1}] \to \mathcal{D}$ such that γ factors as the composite

$$\mathcal{M} \xrightarrow{\ell} \mathcal{M}[\mathcal{W}^{-1}] \xrightarrow{\hat{\gamma}} \mathcal{D}$$

If the subcategory $\mathcal{W}$ is clear from the context, we may write $\operatorname{Ho}\mathcal{M}$ for the homotopy category.

There is a standard construction for $\operatorname{Ho}\mathcal{M}$ but it is not easy to work with, and it can turn a locally small $\mathcal{M}$ to a non-locally small $\operatorname{Ho}\mathcal{M}$. However, when $(\mathcal{M}, \mathcal{W})$ extends to a model structure, this gets resolved. For the rest of this section, let $(\mathcal{M}, \mathcal{W}, \mathcal{C}, \mathcal{F})$ be a model category.

Definition 2.7.
(i) An object $X \in \mathcal{M}$ is *cofibrant* if the map from the initial object $\varnothing$ to X is a cofibration. $\mathcal{M}_c$ is the full subcategory of cofibrant objects. Dually, an object $X \in \mathcal{M}$ is *fibrant* if the map from X to the terminal object $*$ is a fibration. $\mathcal{M}_f$ is the full subcategory of fibrant objects.

(ii) An object $X \in \mathcal{M}$ is *bifibrant* if it is both cofibrant and fibrant. $\mathcal{M}_{cf}$ is the full subcategory of bifibrant objects.

(iii) A (co)fibration is called acyclic if it is also a weak equivalence.

(iv) A *cofibrant replacement* for $X \in \mathcal{M}$ is an cofibrant object X_c with a weak equivalence $X_c \to X$. Dually, a *fibrant replacement* of X is a fibrant object X_f with a weak equivalence $X \to X_f$.

(v) A *cylinder object* for $X \in \mathcal{M}$ is an object $\operatorname{Cyl}(X)$ along with maps

$$X \sqcup X \xrightarrow{i} \operatorname{Cyl}(X) \xrightarrow{w} X$$

such that $i \in \mathcal{C}$, $w \in \mathcal{W}$, and the composite is the fold map. Dually, a *path object* for $X \in \mathcal{M}$ is an object $\mathrm{Path}(X)$ along with maps

$$X \xrightarrow{w} \mathrm{Path}(X) \xrightarrow{p} X \times X$$

such that $i \in \mathcal{C}$, $w \in \mathcal{W}$, and the composite is the diagonal map.

(vi) A *left homotopy* between maps $f, g : X \to Y$ is a choice of object $\mathrm{Cyl}(X)$ along with a map $\mathrm{Cyl}(X) \to Y$ such that the composite

$$X \sqcup X \xrightarrow{i} \mathrm{Cyl}(X) \to Y$$

is $f + g$. Dually, a *right homotopy* between maps $f, g : X \to Y$ is a choice of object $\mathrm{Path}(Y)$ along with a map $X \to \mathrm{Path}(Y)$ such that the composite

$$X \to \mathrm{Path}(Y) \xrightarrow{p} Y \times Y$$

is (f, g).

(vii) $f, g : X \to Y$ are *homotopic*, denoted $f \sim g$, if there exists a left homotopy as well as a right homotopy between them.

Note that functorial (co)fibrant replacements can be constructed by applying (A5) to the maps $\varnothing \to X \to *$. These will be called the *canonical (co)fibrant replacements*. These have the special feature that the weak equivalence between the object and its replacement is also a fibration or a cofibration. This can be used to compare two different replacements directly through a weak equivalence rather than through a zigzag of them.

Lemma 2.8. Let Q, R denote the canonical cofibrant and fibrant replacements functors respectively. Suppose we have cofibrant and fibrant replacements X_c and X_f respectively of $X \in \mathcal{M}$. Then there exist weak equivalences $X_c \to QX \to RQX$ and $QRX \to RX \to X_f$.

Proof. We prove this only in the case of X_c. The weak equivalence $QX \to RQX$ is clear. For $X_c \to QX$, take the lift

$$\begin{tikzcd}
\varnothing \arrow[r] \arrow[d] & QX \arrow[d] \\
X_c \arrow[r] \arrow[ur, dashed] & X
\end{tikzcd}$$

and apply (A2) to prove that the lift is a weak equivalence. □

Similarly, $\mathrm{Cyl}(X)$ and $\mathrm{Path}(X)$ always exist using (A5), but are only defined up to weak equivalence. This notion of homotopy does not behave as expected in all cases, but the following proposition shows that it is well behaved whenever the domains of all involved maps are cofibrant, and all the codomains are fibrant.

Proposition 2.9. (i) Let $X \in \mathcal{M}$ be cofibrant and $Y \in \mathcal{M}$ be fibrant. If there exists either a left homotopy or a right homotopy between $f, g : X \to Y$, then $f \sim g$. Moreover, any choice of cylinder (path) object for X admits a left (right) homotopy through it.

(ii) For X cofibrant and Y fibrant, $\sim$ is an equivalence relation on $\mathcal{M}(X, Y)$.

(iii) For X, Y cofibrant, Y, Z fibrant, and homotopic pairs of maps

$$f_1, g_1 : X \to Y, \qquad f_2, g_2 : Y \to Z$$

we have $f_2 f_1 \cong g_2 g_1$.

(iv) A functor $\mathcal{M}_{cf} \to \mathcal{D}$ for any category $\mathcal{D}$ maps homotopy equivalences to isomorphisms if and only if it respects $\sim$.

Proof. For (i)-(iii), see [Hov, Proposition 1.2.5]. For (iv), see [Hov, Corollary 1.2.9]. $\square$

Theorem 2.10. There is a functor $\Gamma : \mathcal{M} \to \mathcal{M}_{cf}$ connected to $1_{\mathcal{M}}$ via some zigzag of weak equivalences. In particular, there is a functor $Q : \mathcal{M} \to \mathcal{M}$ with natural weak equivalences

$$\Gamma \xleftarrow{\alpha} Q \xrightarrow{\eta} 1_{\mathcal{M}}$$

The inclusion $\mathcal{M} \to \mathcal{M}_{cf}$ is omitted from the notation.

Proof. Let Q denote the cofibrant replacement functor obtained by factoring $\varnothing \to X$ and let R denote the fibrant replacement functor obtained by factoring $X \to *$. Then there is a natural acyclic cofibration $\eta : 1_{\mathcal{M}} \to R$. Take $\Gamma = RQ$ and $\alpha = \varepsilon Q$. $\square$

Dually, we could have taken $\Gamma = QR$, in which case both weak equivalences would need to be flipped. In either case, we have

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "\Gamma"] \arrow[d] & \mathcal{M}_{cf} \arrow[d] \\
\mathcal{M}[\mathcal{W}^{-1}] \arrow[r, dotted] & \mathcal{M}_{cf}[(\mathcal{W} \cap \mathcal{M}_{cf})^{-1}]
\end{tikzcd}$$

where the dotted arrow is induced by the Γ using the universal property of $\mathcal{M}[\mathcal{W}^{-1}]$. The universal property applies because Γ maps weak equivalences to weak equivalences, which may be checked using (A2) and the natural weak equivalences of Theorem 2.10. The inclusion of $\mathcal{M}_{cf}$ into $\mathcal{M}$ induces a functor the other way. These two functors form an equivalence of categories, where the relevant natural isomorphisms come from the natural weak equivalences of Theorem 2.10. We have proved

Corollary 2.11. The bifibrant replacement Γ induces an equivalence of categories
$$\operatorname{Ho}\mathcal{M} \simeq \operatorname{Ho}\mathcal{M}_{cf}$$
so for any $X, Y \in \mathcal{M}$, we have an isomorphism
$$(\operatorname{Ho}\mathcal{M})(X, Y) \cong (\operatorname{Ho}\mathcal{M})(\Gamma X, \Gamma Y)$$

A generalization of Whitehead's theorem describes $\mathcal{M}(\Gamma X, \Gamma Y)$.

Theorem 2.12. Let $X, Y \in \mathcal{M}$ be bifibrant, and $f : X \to Y$. Then f is a weak equivalence if and only if it is a homotopy equivalence.

Proof. The forward implication is just as in Theorem 2.4. This time, the backward implication requires some work. This is where (A3) is used. See [Hov, Proposition 1.2.8]. □

Thus a functor on $\mathcal{M}_{cf}$ maps weak equivalences to isomorphisms if and only if it maps homotopy equivalences to isomorphisms if and only if it respects the homotopy relation, proving that $\operatorname{Ho}\mathcal{M} \simeq \operatorname{Ho}\mathcal{M}_{cf} \simeq \mathcal{M}_{cf}/\sim$.

Theorem 2.13 (Fundamental theorem)**.** Let $\mathcal{M}$ be a model category with a bifibrant replacement functor Γ. Then $\operatorname{Ho}\mathcal{M}$ has the same objects as $\mathcal{M}$ with morphism sets
$$\operatorname{Ho}\mathcal{M}(X, Y) \cong \mathcal{M}(\Gamma X, \Gamma Y)/\sim \; \cong \mathcal{M}(X_c, Y_f)/\sim$$
where X_c is any cofibrant replacement of X and Y_f is any fibrant replacement of Y.

Proof. Only the last isomorphism is new. It follows from [Hov, Proposition 1.2.5(iv)] applied to the weak equivalences of Lemma 2.8. □

2.3 Functors of Model Categories

The purpose of a model category is to aid in the study of its homotopy category, so a functor of model categories should be defined such that it also induces a functor of the homotopy categories. Let $F : \mathcal{M} \to \mathcal{N}$ be an ordinary functor of model categories. To induce a functor $\operatorname{Ho} F : \operatorname{Ho}\mathcal{M} \to \operatorname{Ho}\mathcal{N}$, it needs to at least map homotopic maps of bifibrant objects to homotopic maps. This leads to two kinds of functors, ones that preserve left homotopies and ones that preserve right homotopies. It is important to note that most functors in practice do not preserve weak equivalences, which is why we need a more general notion of a functor.

Let $f, g : X \to Y$ be maps between bifibrant objects in $\mathcal{M}$, and $H : \mathrm{Cyl}(X) \to Y$ a left homotopy between them. By a functor $F : \mathcal{M} \to \mathcal{N}$ preserving the left homotopy H, we mean that FH is a left homotopy between $Ff, Fg : FX \to FY$. For this, $F\,\mathrm{Cyl}(X)$ needs to be a cylinder object for FX. Thus F should preserve the coproduct $X \sqcup X$, the cofibration $X \sqcup X \to \mathrm{Cyl}(X)$, and the weak equivalence $\mathrm{Cyl}(X) \to X$. A dual discussion could be carried out for right homotopies. Since preserving (co)products is involved, it is more convenient, and usually sufficient, to deal with adjoint pairs. Recall that Q and R denote the cofibrant and fibrant replacements respectively.

Definition 2.14. An adjunction $F : \mathcal{M} \leftrightarrows \mathcal{N} : U$ is a *Quillen adjunction* if the left adjoint F preserves cofibrations and the right adjoint U preserves fibrations.

This ensures that the other conditions for F and U to preserve left and right homotopies respectively are satisfied (see [Rie, Lemma 11.3.11 and 11.3.14]; note that $\mathrm{Cyl}(X)$ is cofibrant as it receives a cofibration from a cofibrant object). In fact, this gives the following stronger result. Ken Brown's lemma (see [Rie, Lemma 11.3.14]) implies that F preserves weak equivalences between cofibrant objects, so $F \circ Q$, where Q is a functorial cofibrant replacement, induces a functor $\mathrm{Ho}(FQ) : \mathcal{M} \to \mathcal{N}$. Similarly, $U \circ R$ induces $\mathrm{Ho}(UR) : \mathcal{N} \to \mathcal{M}$ where R is a functorial fibrant replacement. These are called the *total left and right derived functors* of F and U, and denoted $\mathbb{L}F$ and $\mathbb{R}U$, respectively. This is a natural definition because of [Rie, Theorem 2.2.8], which says that $\mathbb{L}F = \mathrm{Ran}_\ell F$ and $\mathbb{R}U = \mathrm{Lan}_\ell U$, where ℓ is the functor to the homotopy category (see Definition 2.6). In this sense, these are the universal homotopical approximations of F and U.

Theorem 2.15. The total derived functors of the Quillen adjunction of Definition 2.14 form an adjunction $\mathbb{L}F \dashv \mathbb{R}G$.

Proof. See [Hov, Lemma 1.3.10]. □

We call this the derived adjunction of $F \dashv U$. Now we consider equivalences of model categories. They should induce equivalences on the homotopy categories. Since any categorical equivalence can be upgraded to an adjoint equivalence ([Lan, Theorem IV.4.1]), we will not lose much generality by defining them to be Quillen adjunctions satisfying the necessary conditions.

Definition 2.16. A *Quillen equivalence* is a Quillen adjunction $F : \mathcal{M} \leftrightarrows \mathcal{N} : U$, such that for every cofibrant $X \in \mathcal{M}$ and fibrant $Y \in \mathcal{N}$, the bijection

$$\mathcal{M}(FX, Y) \cong \mathcal{N}(X, UY)$$

restricts to a bijection of weak equivalences.

Theorem 2.17. The derived adjunction of a Quillen adjunction is an adjoint equivalence if and only if it is a Quillen equivalence.

Proof. See [Hov, Proposition 1.3.13]. □

2.4 Constructing Model Structures

Out of all the axioms, (A5) is the most difficult to verify. For it, we need the small object argument, which is summarized below. A detailed exposition can be found in [Hov, Section 2.1]. First, we have the following generalization of a relative cell complex.

Definition 2.18. Let $\mathcal{I}$ be a class of maps in a category $\mathcal{M}$. A *relative $\mathcal{I}$-cell complex* is a map that can be constructed as a transfinite composite of pushouts of coproducts of maps in $\mathcal{I}$. The class of relative $\mathcal{I}$-cell complexes is denoted by $\overline{\mathcal{I}}$. $X \in \mathcal{M}$ is called a *cellular object* if $(\varnothing \to X) \in \mathcal{I}$.

Definition 2.19. Let $\mathcal{I}$ be a class of maps in a category $\mathcal{M}$. An object $A \in \mathcal{M}$ is *small relative to* $\mathcal{I}$ if there exists an ordinal κ such that for all regular cardinals $\gamma > \kappa$ and directed systems $(B_\beta)_{\beta<\gamma}$ indexed by γ and containing only the maps from $\mathcal{I}$, the natural map below is a bijection

$$\operatorname{colim}_\beta \operatorname{Hom}(A, B_\beta) \to \operatorname{Hom}(A, \operatorname{colim}_\beta B_\beta)$$

Then A is small in the sense that a map to colimit of the directed system must be factor through some stage in the system itself.

Theorem 2.20 (Small object argument)**.** Let $\mathcal{I}$ be a set of maps in a category $\mathcal{M}$ such that each of its domains is small relative to $\overline{\mathcal{I}}$. Then any map $f \in \mathcal{M}$ factors functorially as a map in $\overline{\mathcal{I}}$ followed by a map in $\mathcal{I}^{\boxslash}$.

Proof. The idea is to factor f as $p \circ i$ through some $i \in \overline{\mathcal{I}}$ in a way that 'formally adjoins solutions' to all the lifting problems involving f to the factor p. This factor now may have new possible lifting problems of its own so we might need to continue this process (transfinitely) inductively. The smallness hypothesis on the domains of $\mathcal{I}$ then ensures that there is a large enough ordinal κ such that any lifting problems against $\mathcal{I}$ involving the second factor at the κ-th stage factor through some earlier stage and hence have a solution in the next stage. See [Hov, Theorem 2.1.14] for details. □

Since $\overline{\mathcal{I}}$ is contained in ${}^{\boxslash}(\mathcal{I}^{\boxslash})$ (use (A4)), applying it to a set $\mathcal{I}$ of 'generating cofibrations' and a set $\mathcal{J}$ of 'generating acyclic cofibrations' would give us the desired factorizations.

Theorem 2.21. Let $\mathcal{M}$ be a category, $\mathcal{W}$ a subcategory, and $\mathcal{I}$ and $\mathcal{J}$ sets of maps. Let $\mathcal{F} = \mathcal{J}^{\boxslash}$ and $\mathcal{C} = {}^{\boxslash}(\mathcal{I}^{\boxslash})$. If these definitions satisfy the model category axioms (A1) to (A4), and the domains of $\mathcal{I}, \mathcal{J}$ are small relative to $\overline{\mathcal{I}}, \overline{\mathcal{J}}$ respectively, then $(\mathcal{M}, \mathcal{W}, \mathcal{C}, \mathcal{F})$ is a model category.

Proof. This is direct from the the small object argument. □

Such a model category is called a *cofibrantly generated model category*. The small object argument says that not only do we have a cofibrant replacement, but we can in fact have a functorial cellular approximation. In fact, every cofibration is a retract of a relative $\mathcal{I}$-cell.

Proposition 2.22. Let $(\mathcal{M}, \mathcal{W}, \mathcal{C}, \mathcal{F})$ be a cofibrantly generated model category with generating sets $\mathcal{I}$ and $\mathcal{J}$. Then $\mathcal{C}$ is exactly the subcategory of retracts (in the category of maps in $\mathcal{M}$) of the maps in $\overline{\mathcal{I}}$ and $\mathcal{C} \cap \mathcal{W}$ is the subcategory of retracts of the maps in $\overline{\mathcal{J}}$.

Proof. For any $f \in \mathcal{C}$, factor it as a map in $\overline{\mathcal{I}}$ followed by one in $\mathcal{I}^{\boxslash}$. Show that f is then a retract of the former. See [Hov, Lemma 1.1.9] for the last step. □

This justifies the definition of Serre cofibrations given before Proposition 2.1.

Given sets of maps $\mathcal{W}, \mathcal{I}, \mathcal{J}$ in $\mathcal{M}$, the hardest part of checking the hypotheses of Theorem 2.21 is (A4) and that the small object argument can be applied. Since $\mathcal{C}$ and $\mathcal{F}$ are already defined using some lifting properties, the number of checks to be made for (A4) can be significantly reduced. See [Hov, Theorem 2.1.19] for a precise statement.

Example 2.23. As discussed in the introduction to this chapter, Top has two model structures of interest. In the *Strøm model structure*, $\mathcal{W}$ is the subcategory of homotopy equivalences, $\mathcal{C}$ the Hurewicz cofibrations, and $\mathcal{F}$ the Hurewicz fibrations. Then every object is bifibrant, a cylinder object for X is $X \times I$, and a path object for Y is Y^I. This model structure is not cofibrantly generated, and the verification of the axioms is done using point-set level characterizations of the cofibrations. On the other hand, the *Quillen model structure* with $\mathcal{W}$ weak equivalences, and $\mathcal{C}$ and $\mathcal{F}$ the Serre (co)fibrations, has generating sets

$$\mathcal{I} = \left\{S^n \hookrightarrow D^{n+1} \,\middle|\, n \geq 0\right\}, \qquad \mathcal{J} = \{i_0 : D^n \hookrightarrow D^n \times I \mid n \geq 0\}$$

All objects are fibrant, but only the retracts of cell complexes are cofibrant. The same choices of cylinder and path objects work.

Example 2.24. The suspension-loop adjunction on Top_* is a Quillen adjunction and hence it descends to an adjunction on $\operatorname{Ho}\mathsf{Top}_*$. The total left derived functor $\mathbb{L}\Sigma$ of the suspension functor is computed by cofibrant replacement and then ordinary suspension - hence it coincides with the suspension on cofibrant objects, and in particular $(\mathbb{L}\Sigma)S^n = S^{n+1}$.

We will use two other ways to produce model structures from existing ones. The first is to transfer a model structure along an adjunction. The other, Bousfield localization, is treated in the next section.

Theorem 2.25. (Transferred model structure) Let $(\mathcal{M})$ be a cofibrantly generated model category with generating sets $\mathcal{I}$ and $\mathcal{J}$, $\mathcal{N}$ any bicomplete category, and suppose there is an adjunction $F : \mathcal{M} \leftrightarrows \mathcal{N} : U$. If

(i) $F\mathcal{I}$ and $F\mathcal{J}$ permit the small object argument, and

(ii) U takes relative $F\mathcal{J}$-cell complexes to weak equivalences in $\mathcal{M}$,

then there exists a cofibrantly generated model structure on $\mathcal{N}$ with generating sets $F\mathcal{I}$ and $F\mathcal{J}$, weak equivalences $U^{-1}\mathcal{W}$ and fibrations $U^{-1}\mathcal{F}$.

Proof. Check that $F\mathcal{I}$, $F\mathcal{J}$ and $U^{-1}\mathcal{W}$ satisfy the conditions for Theorem 2.21 (the refined statement cited in the paragraph after the theorem is useful here). Then, via the adjunction, $(F\mathcal{J})^{\boxslash} = U^{-1}\mathcal{F}$ is equivalent to $\mathcal{J}^{\boxslash} = \mathcal{F}$ so the fibrations are as stated. □

Remark 2.26. With the transferred model structure on $\mathcal{N}$ in the above scenario, the adjunction $F \dashv U$ becomes a Quillen adjunction, but not necessarily a Quillen equivalence. Also note that there may exist other model structures on $\mathcal{N}$ for which this adjunction is a Quillen adjunction, so the transferred model structure is not the unique one with this property.

Corollary 2.27 (Projective model structure)**.** Let $\mathcal{M}$ be a cofibrantly generated model category with generating sets $\mathcal{I}$ and $\mathcal{J}$ and let $\mathscr{D}$ be a small category. Then the functor category $\mathcal{M}^{\mathscr{D}}$ has a cofibrantly generated model structure in which the weak equivalences and fibrations are levelwise. The generating sets are given by the images of the generating sets under the left adjoints to the evaluation-at-d functors $\mathcal{M}^{\mathscr{D}} \to \mathcal{M}$ for each $d \in \mathscr{D}$ (which exist because $\mathscr{D}$ is small and $\mathcal{M}$ is cocomplete).

Proof. Let $|\mathscr{D}|$ be the discrete category on the objects of $\mathscr{D}$ with the inclusion $i : |\mathscr{D}| \to \mathscr{D}$. Then the product category $\mathcal{M}^{|\mathscr{D}|}$ has a product model structure in which a map is in $\mathcal{W}$, $\mathcal{C}$ or $\mathcal{F}$ if and only if each of its coordinates is. This structure is cofibrantly generated: a generating (acyclic) cofibration is a map which is identity on the initial object in all but one coordinate, and is in $\mathcal{I}$ (resp. $\mathcal{J}$) in this coordinate. The precomposition functor $i^* : \mathcal{M}^{\mathscr{D}} \to \mathcal{M}^{|\mathscr{D}|}$ has a left adjoint given by Lan_i. This adjunction satisfies the hypotheses of Theorem 2.25 and then the transferred model structure on $\mathcal{M}^{\mathscr{D}}$ is as described. □

Remark 2.28. In our applications, we will need the above theorem in the enriched setting. This means that $\mathscr{D}$ will be a $\mathcal{V}$-category and $\mathcal{M}$ a model $\mathcal{V}$-category for some closed symmetric monoidal model category $\mathcal{V}$. This requires defining enriched model and monoidal categories which we avoid. The idea in both cases is that we want this additional structure on a model category in a way that it is compatible with its homotopy theoretic structure and descends to a similar structure on its model category. For the precise statement of the above theorem as we will be applying it, see [nLa, Theorem 3.1].

2.5 Left Bousfield Localization

Sometimes we would have a model structure on some category $\mathcal{M}$ but its weak equivalences would not be the ones we want. Often, we will just want to add more weak equivalences to the already existing ones. This is called localization because it is defined by a universal property similar to that of localization of rings. Since functors of model categories are of two kinds (left and right Quillen), we get two kinds of localizations. All localizations that show up for us would be left localizations so we restrict our attention to those.

Definition 2.29. Let $\mathcal{S}$ be class of maps in a model category $\mathcal{M}$. A *left localization of* $\mathcal{M}$ at $\mathcal{S}$ is a model category $L_{\mathcal{S}}\mathcal{M}$ and a left Quillen functor $\ell_{\mathcal{S}} : \mathcal{M} \to L_{\mathcal{S}}\mathcal{M}$ taking maps in $\mathcal{S}$ to weak equivalences such that for any model category $\mathcal{N}$ and left Quillen functor $F : \mathcal{M} \to \mathcal{N}$ taking maps in $\mathcal{S}$ to weak equivalences, there exists a unique left Quillen functor making the following commute

$$\begin{tikzcd} \mathcal{M} \arrow[r, "\ell_{\mathcal{S}}"] \arrow[dr, "F"'] & L_{\mathcal{S}}\mathcal{M} \arrow[d, dotted, "\exists!"] \\ & \mathcal{N} \end{tikzcd}$$

Now suppose we have model categories $(\mathcal{M}, \mathcal{W}, \mathcal{C}, \mathcal{F})$ and $(\mathcal{M}', \mathcal{W}', \mathcal{C}', \mathcal{F}')$ with the same underlying category $\mathcal{M} = \mathcal{M}'$, the same cofibrations $\mathcal{C} = \mathcal{C}'$, and with $\mathcal{W} \subseteq \mathcal{W}'$. We claim that then $\mathcal{M}'$ along with the identity functor $\mathcal{M} \to \mathcal{M}'$ is a left localization of $\mathcal{M}$ at $\mathcal{W}'$. Clearly it is a left adjoint and preserves cofibrations and acyclic cofibrations so it is left Quillen. It also takes maps in $\mathcal{W}'$ to $\mathcal{W}'$. Then given any functor $F : \mathcal{M} \to \mathcal{N}$ with the same properties, F itself works as the unique witness for the universal property above. This special case of left localization is called a *left Bousfield localization*. From here on, $L_{\mathcal{S}}\mathcal{M}$ will only refer to a left Bousfield localization.

If the left Bousfield localization of $(\mathcal{M}, \mathcal{W}, \mathcal{C}, \mathcal{F})$ at $\mathcal{S}$ exists, then its subcategory of weak equivalences will contain $\mathcal{S} \cup \mathcal{W}$ but could be much larger. The weak equivalences in this localized model structure are called the $\mathcal{S}$-local equivalences. We want to know not only whether the left Bousfield localization exists, but also what the $\mathcal{S}$-local equivalences are (it will not be good to end up with a model structure with too many weak equivalences).

The characterization of the $\mathcal{S}$-local equivalences in our cases of concern requires talking about (co)simplicial resolutions and homotopy function complexes in model categories. This is out of the scope of this thesis, but details can be found in [Hir, Chapter 3]. We summarize the general idea here along with the assumptions that we will be making.

The left Bousfield localization of $\mathcal{M}$ at $\mathcal{S}$ exists whenever $\mathcal{S}$ is a (small) set in a *left proper* and *cellular* model category. The idea is to find the cofibrant generating sets for $L_{\mathcal{S}}\mathcal{M}$ given such sets $\mathcal{I}$ and $\mathcal{J}$ for $\mathcal{M}$ (cellular model categories are cofibrantly generated). Since the cofibrations do not change, $\mathcal{I}$ need not change. $\mathcal{J}$ needs to be replaced with a larger set. A naive choice for this would be all cofibrations that are $\mathcal{S}$-local equivalences, but this would usually be a proper class. $\mathcal{S}$ being a set and cellularity are combinatorial conditions that enable cutting down this into a generating set. Left properness is then needed to show that this is indeed a model structure.

This is all applicable for us because the Quillen model structure on Top is left proper and and cellular. Both of these properties are preserved under the model categorical operations of turning a category into a pointed one, so Top_* also has these properties. The projective model structure on the category of functors into a left proper and cellular model category also has these properties. Lastly, left Bousfield localization itself also preserves these properties. All model structures in this thesis will be constructed from the Quillen model structure using these constructions, so all left Bousfield localizations will exist.

However, in our applications, $\mathcal{S}$ will not be a set. This is fixed by finding a 'generating set' within the class $\mathcal{S}$ and then applying the above theorem. We will simply assume that the left Bousfield localizations that we want exist and that their $\mathcal{S}$-local equivalences are exactly the ones that we need.

3 More Categorical Background

The goal of this chapter is to introduce the nerve-realization adjunction and Day convolution from enriched category theory and the results of [MMSS] on modules over diagrams. The nerve-realization adjunction will be used to describe the weak equivalences in Theorems 4.12, 5.7 and 6.7. Day convolution will be used to define the smash product of spectra and the box product of Mackey functors. The results on modules in diagram categories motivate the definitions of orthogonal and symmetric spectra in Section 4.3. A reader willing to accept these results may skip this chapter or come back to it later.

We assume the definitions of enriched categories, functors, natural transformations and (co)end calculus. For an introduction to these, see [Rie, Chapter 1, 3]. The enriching category will always be denoted $(\mathcal{V}, \odot)$ and it will always be assumed to be closed symmetric monoidal and cocomplete.

3.1 Nerve-Realization Adjunction and Day Convolution

The starting point of this discussion is the enriched Yoneda lemma. $\mathrm{Nat}(F, G)$ denotes the object of natural transformations $F \to G$.

Theorem 3.1 (Yoneda Lemma)**.** Let $\mathcal{M}$ be a $\mathcal{V}$-category. Let $F : \mathcal{M}^{\mathsf{op}} \to \mathcal{V}$ be a $\mathcal{V}$-functor and $m \in \mathcal{M}$. Then there is a natural isomorphism in $\mathcal{V}$

$$\mathrm{Nat}(\mathcal{M}(-, m), F) \cong F(m) \qquad \square$$

This tells us that $\mathcal{M}(-, m)$ is *freely generated by the identity element in* $\mathcal{M}(m, m)$, in the sense that defining a map from this functor to F is equivalent to describing just the image of this generator. Our algebraic intuition says then that any functor should have a description using generators and relations. The most straightforward way to construct such a description would be to take every 'element of the functor' as a generator. Thus for each $m \in \mathcal{M}$, we want '$F(m)$ many' copies of $\mathcal{M}(-, m)$. This gives a description of F as a colimit of representable functors indexed by the comma category $\mathcal{M} \downarrow F$ where $\mathcal{M}$ is treated as a subcategory of the presheaf category $\widehat{\mathcal{M}}$ via the Yoneda embedding (see [Lan, Theorem III.7.1] for a proof for ordinary categories). This can be packaged nicely with a coend.

Theorem 3.2 (Coyoneda Lemma)**.** Let $\mathcal{M}$ be a $\mathcal{V}$-category and $X : \mathcal{M}^{\mathsf{op}} \to \mathcal{V}$ a $\mathcal{V}$-functor. Then there is a canonical isomorphism

$$X \cong \int^{m \in \mathcal{M}} X(m) \odot \mathcal{M}(-, m) \qquad \square$$

This discussion says that every presheaf is canonically a colimit of representable presheaves or that the representable presheaves are *dense* in the presheaf category. This allows us to extend functors on $\mathcal{M}$ to ones that preserve colimits on $\widehat{\mathcal{M}}$, and gives us a universal property of the presheaf category.

Theorem 3.3. Let $\mathcal{M}, \mathcal{N}$ be $\mathcal{V}$-categories and $F : \mathcal{M} \to \mathcal{N}$ a $\mathcal{V}$-functor. If $\mathcal{M}$ is small and $\mathcal{N}$ is cocomplete, then up to natural isomorphism, there exists a unique $\mathcal{V}$-functor $\overline{F} : \overline{\mathcal{M}} \to \mathcal{N}$ which has a right adjoint and makes the following diagram commute, where y is the Yoneda embedding into the presheaf category $\widehat{\mathcal{M}}$.

$$\begin{tikzcd} \mathcal{M} \arrow[r, "y"] \arrow[dr, "F"'] & \widehat{\mathcal{M}} \arrow[d, "\overline{F}"] \\ & \mathcal{N} \end{tikzcd}$$

Proof sketch. Define $\overline{F}(\mathcal{M}(-, m)) = Fm$ and extend using Theorem 3.2. That is, for $X \in \mathcal{M}$, define

$$\overline{F}X = \int^{m \in \mathcal{M}} X(m) \odot F(m)$$

This can also be described as the left Kan extension of F along the Yoneda embedding. The right adjoint to this is the functor

$$\begin{aligned} \mathbb{N}F : \mathcal{N} &\to \widehat{\mathcal{M}} \\ n &\mapsto \mathcal{N}(F-, n) \end{aligned}$$

The adjunction isomorphism for a pair $X \in \widehat{\mathcal{M}}$ and $n \in \mathcal{N}$ can be defined easily when X is representable. Other cases can be reduced to the representable one by writing X as a colimit of representables and using the fact that Hom turns coends in the first coordinate into ends. $\square$

This universal property says that the presheaf category of $\mathcal{M}$ is its *free cocompletion*. It cocompletes $\mathcal{M}$ in the most general way possible: i.e. it is cocomplete and $\mathcal{M}$ embeds into it as a dense subcategory. 'Free' here refers to the fact that any existing colimits may be destroyed, which is necessary because F is not required to be cocontinuous while its extension $\overline{F}$ is.

Definition 3.4. The functor $\mathbb{N}F$ as above is called the *nerve* induced by F and $\overline{F}$ is called the *realization* induced by F. The adjunction above is called the *nerve-realization adjunction* induced by F.

Example 3.5. The nerve-realization adjunction induced by the unenriched inclusion of the discrete category $\mathbb{1}$ on one object into Top as the singleton space is the free-forgetful adjunction between Top and $[\mathbb{1}, \mathsf{Set}] \cong \mathsf{Set}$.

Example 3.6. The nerve-realization adjunction induced by the unenriched inclusion of the simplex category Δ into Top is the familiar adjunction between Top and SSet. The nerve takes a space to its singular simplicial set. The realization is geometric realization. This is a Quillen equivalence.

The nerve may be thought of as the data one gets from probing an arbitrary object of $\mathcal{N}$ by objects in $\mathcal{M}$, which is often a subcategory of 'nice' objects and morphisms sitting in $\mathcal{N}$ via an inclusion F. The realization is then an attempt to best reconstruct an object given a presheaf, which is the form that such probing data usually takes. The above examples show that probing by the singleton space is not enough to capture all homotopical data, whereas probing by the simplices is. However, adding enrichments changes things.

Example 3.7. Treating $\mathbb{1}$ as a Top-category, the same inclusion of $\mathbb{1}$ into Top is a Top-functor. The enriched nerve-realization functor is then an isomorphism of categories. More generally, the $\mathcal{V}$-enriched nerve-realization adjunction induced by the inclusion of the unit into $\mathcal{V}$ is always an equivalence.

Another important application of the coyoneda lemma is Day convolution. Suppose $(\mathcal{M}, +)$ is a symmetric monoidal $\mathcal{V}$-category which is not necessarily closed. We can use this to define a *closed* symmetric monoidal product $\square$ on $\widehat{\mathcal{M}}$. Being closed would imply that $\square$ preserves colimits in both coordinates. This gives us a candidate definition, where we define $\mathcal{M}(-, m) \square \mathcal{M}(-, n) = \mathcal{M}(-, m+n)$ and extend as

$$\begin{aligned}
X \square Y &= \left(\int^{m} X(m) \odot \mathcal{M}(-, n)\right) \square \left(\int^{n} Y(n) \odot \mathcal{M}(-, n)\right) \\
&= \int^{m,n} X(m) \odot Y(n) \odot (\mathcal{M}(-, m) \square \mathcal{M}(-, n)) \\
&= \int^{m,n} X(m) \odot Y(n) \odot (\mathcal{M}(-, m+n))
\end{aligned}$$

Definition 3.8. Let $\mathcal{M}$ be a small symmetric monoidal $\mathcal{V}$-category. The

Day convolution product $X \Box Y$ of $X, Y \in \widehat{\mathcal{M}}$ is defined as the coend

$$X \Box Y = \int^{m,n \in \mathcal{M}} X(m) \odot Y(n) \odot \mathcal{M}(-, m+n)$$

We could also have arrived at the same definition by applying the nerve-realization adjunction to one coordinate of $+$ and then to the other since we want to create a right adjoint to it in each coordinate. From the coend formula for Kan extensions (see [Lan, Theorem X.4.1]), we can infer the following important property of $\Box$.

Proposition 3.9. For $X, Y : \mathcal{M}^{\mathsf{op}} \to \mathcal{V}$, their Day convolution product $X \Box Y$ is precisely the left Kan extension

$$\begin{tikzcd}
\mathcal{M}^{\mathsf{op}} \otimes \mathcal{M}^{\mathsf{op}} \arrow[r, "X \otimes Y"] \arrow[dr, "+"'] & \mathcal{V} \otimes \mathcal{V} \arrow[r, "\odot"] & \mathcal{V} \\
 & \mathcal{M}^{\mathsf{op}} \arrow[ur, dashed, "X \Box Y"'] &
\end{tikzcd}$$

That is, we have an isomorphism

$$\mathrm{Nat}((X \Box Y) \circ +, -) \cong \mathrm{Nat}(\odot \circ (X \otimes Y), -) \qquad \Box$$

This means that defining a natural transformation $X \Box Y \to Z$ is equivalent to defining maps $X(m) \otimes Y(n) \to Z(m+n)$ natural in $m, n \in \mathcal{M}$.

Theorem 3.10 (Day Convolution Theorem)**.** Let $\mathcal{M}$ be a small symmetric monoidal $\mathcal{V}$-category, and $\Box$ the Day convolution product on it. Then $(\widehat{\mathcal{M}}, \Box)$ is a closed symmetric monoidal category.

Proof. To check symmetry and associativity, we can use the universal property above. For example, we can show that

$$\mathrm{Nat}(X \Box (Y \Box Z), -) \cong \mathrm{Nat}((X \Box Y) \Box Z, -)$$

For the unit, first check that $X \Box \mathcal{M}(-, m) \cong X(- + m)$ by checking it for representable X and then extending by coends. Then the unit for $\Box$ is $\mathcal{M}(-, 0)$ where 0 is the unit of $+$. For $\Box$ to be closed, we must have internal hom $[-, -]$ with

$$\begin{aligned} [X, Y](m) &\cong \mathrm{Nat}(\mathcal{M}(-, m), [X, Y]) \\ &\cong \mathrm{Nat}(\mathcal{M}(-, m) \Box X, Y) \\ &\cong \mathrm{Nat}(X(- + m), Y) \end{aligned}$$

Taking this as the definition of the internal hom, the adjunctions are easily checked. $\Box$

3.2 Modules in Diagram Categories

Let $(\mathscr{D}, +)$ be a small symmetric monoidal $\mathcal{V}$-category. Then $(\widehat{\mathscr{D}}, \square)$ is closed symmetric monoidal. Let $\mathbb{S}$ be a monoid object in $\widehat{\mathscr{D}}$ and consider the category $\mathsf{Mod}_{\mathbb{S}}$ of right $\mathbb{S}$-modules. This situation will be of interest in Section 4.3.

The following theorem is a generalization of the case $\mathcal{V} = \mathsf{Ab}$ and $\mathscr{D}$ the terminal category. Then $\widehat{\mathscr{D}} \cong \mathsf{Ab}$ with the same tensor product, so $\mathbb{S}$ is simply a ring. The category $\mathsf{Mod}_{\mathbb{S}}$ is then equivalent to $\widehat{\mathscr{J}}$, where $\mathscr{J}$ is $\mathbb{S}$ viewed as a one object Ab-category. This happens because $\mathscr{J}$ is equivalent to the full subcategory of $\mathsf{Mod}_{\mathbb{S}}$ spanned by the free cyclic $\mathbb{S}$-modules and the maps between these modules capture all possible ways in which elements of $\mathbb{S}$ could act on an element of an arbitrary module.

Theorem 3.11. For any monoid $\mathbb{S}$ as above, there exists a small monoidal $\mathcal{V}$-category $\mathscr{J}$ such that $\mathsf{Mod}_{\mathbb{S}} \cong \widehat{\mathscr{J}}$. Furthermore, if $\mathbb{S}$ is a commutative monoid, then $\mathscr{J}$ has a symmetric monoidal product such that the induced Day convolution product on $\widehat{\mathscr{J}}$ coincides with the usual monoidal product of $\mathbb{S}$-modules over $\mathbb{S}$.

Proof. Construct $\mathscr{J}$ as the full subcategory of $\mathsf{Mod}_{\mathbb{S}}$ spanned by the free modules generated that can be generated by a single element, i.e. the $\mathbb{S}$-modules $\mathscr{D}(-, d) \square \mathbb{S}$ for each $d \in \mathscr{D}$. See [MMSS, Theorem 2.2, Section 23] for details. Note that that paper works instead with copresheaves, so their $\mathscr{J}$ is spanned by the modules $\mathscr{D}(d, -) \square \mathbb{S}$. $\square$

4 Stable Homotopy Theory

Homotopy groups are notoriously difficult to compute. On the other hand, we have some techniques available which allow the computation of (co)homology theories. For example, they are abelian groups in all dimensions, allowing the use of spectral sequences without annoying edge cases. Homology also has the suspension isomorphism $\widetilde{H}_\bullet(Y) \cong \widetilde{H}_{\bullet+1}(\Sigma Y)$ for each $Y \in \mathsf{Top}_*$ which is not present in homotopy. For example,

$$\pi_2(S^1) \cong 0 \qquad \pi_3(S^2) \cong \mathbb{Z}$$

where the former comes from covering theory, and the latter from the long exact sequence of the Hopf fibration $S^3 \to S^2$.

However, there is still a natural comparison map $\pi_n(Y) \to \pi_{n+1}(\Sigma Y)$ by Example 2.24. We are led to the definition of the *stable homotopy groups* of Y

$$\pi_n^s(Y) = \operatorname{colim}_k \pi_{n+k}(\Sigma^k X)$$

These stable homotopy groups then have a suspension isomorphism almost by construction. In fact, they form a generalized homology theory. More generally, for any pair $X, Y \in \mathsf{Top}_*$, one could define the set of *stable homotopy classes of maps*

$$[X, Y]^s = \operatorname{colim}_k [\Sigma^k X, \Sigma^k Y]$$

One could now try to construct a model structure on Top_* such that

$$\operatorname{Ho}\mathsf{Top}_*(X, Y) = [X, Y]^s$$

with respect to this model structure, but it turns out that that is not possible - this category does not have arbitrary coproducts while the homotopy category of a model category does (this is the Spanier-Whitehead category, [BR, Chapter 1]). Instead, we define a larger category Sp where we can still talk about stable homotopy classes of maps with a fully faithful embedding $\mathsf{Top}_* \to \mathsf{Sp}$. For more detailed motivations for stable homotopy groups and spectra, see [Rog, Section 9.1] which describes some of their history, [Mar, Chapter 1] for an axiomatic introduction, or [Greb, Section 1.2] for an introduction through ∞-categorical stabilization. For us, suspension will mean smashing with S^1 on the right, i.e. $\Sigma := - \wedge S^1$.

4.1 Spectra and their Model Structure

Definition 4.1. (i) A *(sequential) spectrum* X is a sequence of spaces $(X_n)_{n\in\mathbb{N}}$ along with maps

$$\sigma_n : \Sigma X_n \to X_{n+1}$$

(ii) A map $f : X \to Y$ of sequential spectra is a sequence of maps $f_n : X_n \to Y_n$ such that for each $n \in \mathbb{N}$, the following square commutes.

$$\begin{tikzcd} \Sigma X_n \arrow[r, "\sigma_n"] \arrow[d, "\Sigma f_n"'] & X_{n+1} \arrow[d, "f_{n+1}"] \\ \Sigma Y_n \arrow[r, "\sigma_n"'] & Y_{n+1} \end{tikzcd}$$

Thus we have a category of spectra, which will be denoted Sp.

(iii) The *nth (stable) homotopy group* of a spectrum X is

$$\pi_n(X) := \operatorname{colim}_k \pi_{n+k} X_k$$

where the colimit is over the diagram in which the map $\pi_{n+k}X_k \to \pi_{n+k+1}X_{k+1}$ is the composite

$$[S^{n+k}, X_k] \xrightarrow{\mathbb{L}\Sigma} [S^{n+k+1}, \Sigma X_k] \xrightarrow{(\sigma_k)_*} [S^{n+k+1}, X_{k+1}]$$

This definition also makes sense for $n < 0$ by starting the diagram from $k \geq |n|$. Thus we have functors $\pi_n : \mathsf{Sp} \to \mathsf{Ab}$ for every $n \in \mathbb{Z}$.

(iv) A map of spectra $f : X \to Y$ is a *stable equivalence* if $\pi_n(f)$ is an isomorphism for all $n \in \mathbb{Z}$.

(v) For X a based space, the suspension spectrum of X, denoted $\Sigma^\infty X$, is defined by

$$(\Sigma^\infty X)_n = X \wedge S^n$$

with structure maps the isomorphisms $\sigma_n : X \wedge S^n \wedge S^1 \to X \wedge S^{n+1}$. This is left adjoint to the functor $\Omega^\infty : \mathsf{Sp} \to \mathsf{Top}_*$ that takes the zeroth space of a given spectrum.

Then, for any pointed space X, its stable homotopy groups coincide with the homotopy groups of its suspension spectrum. Our goal is to study spectra up to homotopy, which would subsume the study of spaces up to stable homotopy since Σ^∞ is fully faithful. Thus, we want a model structure on spectra in which the weak equivalences are the stable equivalences. In this section, we construct this model structure on spectra.

The definition of a spectrum is very similar to that of a graded module over a graded ring. By composing structure maps as

$$\Sigma^k X_n \xrightarrow{\Sigma^{k-1}\sigma_n} \Sigma^{k-1} X_{n+1} \xrightarrow{\Sigma^{k-2}\sigma_{n+1}} \cdots \xrightarrow{\sigma_{n+k-1}} X_{n+k}$$

gives us a k-fold structure map

$$\sigma_{n,k} : X_n \wedge S^k \to X_{n+k}$$

so we could have equally defined a spectrum as a sequence of spaces with structure maps $X_n \wedge S^k \to X_{n+k}$ for every $n, k \in \mathbb{N}$ satisfying the composition condition described above. Moreover, the sphere spectrum $\mathbb{S} := \Sigma^\infty S^0$ is analogous to a graded ring as its k-fold structure maps are

$$S^k \wedge S^n \xrightarrow{\sim} S^{k+n}$$

which makes it look like a graded ring.

To make this precise, let $\mathscr{D}_{\mathbb{N}}$ denote the discrete category with objects the natural numbers. This can be viewed as a category enriched over Top_*. It has a symmetric monoidal product given by natural number addition.

Definition 4.2. A *sequential space* X is a sequence $(X_n)_{n\in\mathbb{N}}$ of pointed spaces. Equivalently, it is a presheaf $X : \mathscr{D}_{\mathbb{N}} \to \mathsf{Top}_*$.

Theorem 3.10 then gives us a closed symmetric monoidal product on the category of sequential spaces defined as

$$(X \wedge Y)_n = \int^{i,j} X_i \wedge Y_j \wedge \mathscr{D}_{\mathbb{N}}(n, i+j) = \bigvee_{i+j=n} X_i \wedge Y_j$$

The unit of this product is $\mathbb{1}$, with $\mathbb{1}_0 = S^0$ and $\mathbb{1}_n = *$ for $n \in \mathbb{N}_{>0}$. The sphere sequential spectrum $\mathbb{S}$ is then exactly a monoid in this category and spectrum can then be viewed as a left module over this monoid. Thus Sp is equivalent to $\mathsf{Mod}_{\mathbb{S}}$.

We can now apply Theorem 3.11 to find that there exists a Top_*-category $\mathscr{J}_{\mathbb{N}}$ such that $\mathsf{Mod}_{\mathbb{S}} \cong \widehat{\mathscr{J}_{\mathbb{N}}}$. Then Corollary 2.27 gives us a cofibrantly generated model structure called the *projective model structure* on this category in which the weak equivalences are the levelwise weak equivalences of Top_*. The corollary also describes the generating cofibrations $\mathcal{I}$ as the maps $S(m-1, n) \hookrightarrow D(m, n)$ where

$$(S(m,n))_k = \begin{cases} * & \text{if } k < n \\ S^{m-n+k} & \text{if } k \geq n \end{cases}$$

$$(D(m,n))_k = \begin{cases} * & \text{if } k < n \\ D^{m-n+k} & \text{if } k \geq n \end{cases}$$

and the inclusion is the obvious levelwise inclusion. This lets us define CW-spectra. Say that $D(m,n)$ has dimension $m-n$ and define a CW spectrum to be a spectrum obtained by attaching these disks in increasing order of dimension. This also admits a point-set level description: A CW spectrum is a spectrum in which every space is a pointed CW complex and the structure maps are inclusions of subcompexes (see [Ada, Section III.2]).

However, this is definitely not the model structure we want on spectra because there are more stable equivalences than there are levelwise weak equivalences of spectra. For example, the levelwise inclusion $S(0,1) \hookrightarrow S(0,0) = \mathbb{S}$ is a stable equivalence since the colimit defining the stable homotopy groups is not affected by a finite number of changes in the spectra. Therefore, we want to change the model structure by expanding its subcategory of weak equivalences. We achieve this using left Bousfield localization at the class of stable equivalences.

Theorem 4.3 (Stable model structure)**.** There exists a cofibrantly generated model structure on Sp in which the weak equivalences are the stable equivalences and the cofibrations are the cofibrations in the projective model structure. The fibrant objects in this model structure are the Ω-spectra, which are spectra X in which the adjoints of the structure maps $X_n \to \Omega X_{n+1}$ are weak equivalences.

Proof. See [HHRb, Section 7.3]. □

Since we have the same generating cofibrations and more weak equivalences, CW-spectra are cellular objects and $I_+ \wedge X$, defined by $(I_+ \wedge X)_n = I_+ \wedge X_n$, is a cylinder object for any spectrum X, so left homotopies are exactly the compatible levelwise homotopies.

4.2 Stable Homotopy Category

The first thing to check is whether we achieved our goal of constructing a model category in which homotopy classes of maps are the stable homotopy classes.

Proposition 4.4. Let $A, B \in \mathsf{Top}_*$ with A a CW complex with finitely many cells. Then $\mathcal{SH}(\Sigma^\infty A, \Sigma^\infty B) \cong [A,B]^s$ with the right side as defined in the introduction of this chapter.

Proof. See [BR, Lemma 5.1.2]. □

Recall the spectra $S(m,n)$ from the discussion preceeding Theorem 4.3. We say that $S(m,n)$ has dimension m. It can be checked using similar methods that there are no non-trivial maps in $\mathcal{SH}$ from a lower dimensional sphere to a higher dimensional one. This implies that we can turn our cellular approximation theorem into a CW approximation theorem where the CW spectra are defined as relative $\mathcal{I}$-cells in which the cells are attached in a non-decreasing order of dimension. These CW spectra can be defined as spectra in which each space is a pointed CW complex and the structure maps are inclusions of subcomplexes.

Define suspension and loop functors $\Sigma, \Omega : \mathsf{Sp} \to \mathsf{Sp}$ by applying the ordinary suspension or loop functors levelwise.

Theorem 4.5. Suspension and loop form a Quillen equivalence $\Sigma \dashv \Omega$ on Sp.

Proof. See [BR, Theorem 2.3.14]. □

We can also define the smash product of a space with a spectrum and a spectrum of maps from a space to a spectrum levelwise. That is, if X is a space and E a spectrum, we define $(X \wedge E)_n = X \wedge E_n$ and $(\mathrm{Map}(X, E))_n = \mathrm{Map}(X, E_n)$ with the obvious structure maps. Then $X \wedge -$ is left adjoint to $\mathrm{Map}(X, -)$.

Being a colimit of maps between suspensions, $[\Sigma^\infty A, \Sigma^\infty B]$ is an abelian group for any $A, B \in \mathsf{Top}_*$. In fact, $\mathcal{SH}$ is an Ab-enriched category. This follows from the fact that Σ^2 is both an auto-equivalence and an abelian cogroup functor up to stable homotopy - there is a natural multiplication $\Sigma^2 X \to \Sigma^2 X \vee \Sigma^2 X$ (suspension of the stable pinch map or equivalently the stable pinch map for ΣX) that makes $\Sigma^2 X$ into an abelian cogroup object in $\mathcal{SH}$. Thus every object $X \in \mathcal{SH}$ is naturally an abelian cogroup object making $\mathcal{SH}(X, -)$ an abelian group object and hence Ab-valued.

In particular, any finite coproducts that exist in $\mathcal{SH}$ are also products and vice versa. While homotopy categories rarely contain (co)limits, [Ram] proves that products and coproducts in the homotopy category of a model category always exist and can be computed at the point-set level. The Ab-enrichment of $\mathcal{SH}$ then proves that finite products and coproducts of spectra coincide up to homotopy.

We can define homotopy (co)fiber of a map of spectra levelwise. Since Σ and Ω are equivalences, we are able to continue the homotopy (co)fiber sequence of a map in the 'wrong' direction too. We get the expected property that the homotopy groups of a fiber sequence give a long exact sequence, but we also get the same property for cofiber sequences (see [Rog, Proposition 9.3.7]). A consequence of this and the five lemma is that a sequence is a fiber sequence if and only if it is a cofiber sequence. Another important property of cofiber sequences is that they are preserved by the smash product with a fixed spectrum. This is because homotopy cofibers can be described as pushouts and the smash product has a right adjoint. These properties of cofiber sequences allow us to build generalized homology and cohomology theories given any spectrum.

Theorem 4.6. Let E be a spectrum. For each $n \in \mathbb{Z}$, define

$$E^n(X) = \mathcal{SH}(\Sigma^\infty X, S^n \wedge E)$$
$$E_n(X) = \mathcal{SH}(S^n, X \wedge E) \cong \pi_n(X \wedge E)$$

Then $E_\bullet$ is a generalized homology theory and $E^\bullet$ is a generalized cohomology theory.

Proof. The homotopy invariance of $E^\bullet$ and $E_\bullet$ follows from the fact that Σ^∞ takes weak equivalences to stable equivalences. The remaining axioms (additivity, exactness for any cofiber sequence, suspension isomorphism) follow from the preceeding discussion and the properties of the functors used in these definitions. □

Perhaps a more surprising result is that all generalized cohomology theories are built this way.

Lemma 4.7 (Brown representability)**.** A contravariant functor $K : (\mathrm{Ho}\,\mathsf{CW}_*^c)^{\mathsf{op}} \to \mathsf{Set}$ from the homotopy category of pointed connected CW complexes is representable if and only if it converts wedge sums to direct products and satisfies the Mayer-Vietoris axiom: For any CW complex X, subcomplexes A_1, A_2 that cover X, and elements $a_i \in KA_i$ with the same restriction to $K(A_1 \cap A_2)$, there exists some $x \in KX$ such whose restriction to KA_i is a_i.

Proof. The harder part is to prove that a functor satisfying these conditions is representable. This is prove by constructing the representing CW complex Z cell by cell. The conditions imply that it is enough to ensure that $\mathrm{Ho}\,\mathsf{CW}_*^c(X, Z) \cong KX$ when X is a sphere, thus we just need $\pi_n Z \cong KS^n$. See [Put] for details. □

Corollary 4.8. For any generalized cohomology theory $E^\bullet$, there exists an Ω-spectrum Z such that $Z^\bullet \cong E^\bullet$.

Proof. Apply Lemma 4.7 to each functor E^n for $n \geq 0$ to get a representing CW complex Z_n. The suspension axiom gives a natural isomorphism $[X, Z_n] \cong [\Sigma X, Z_{n+1}] \cong [X, \Omega Z_{n+1}]$, which, by the Yoneda lemma, must correspond to a homotopy equivalence $Z_n \to \Omega Z_{n+1}$. These spaces define an Ω-spectrum Z, and to check that this has the correct value on each space, it is enough to check it on CW complexes. Since suspension spectra of CW complexes are cofibrant and Ω-spectra are fibrant, this can be checked by computing the levelwise homotopy classes and using the fact that Σ^∞ is left adjoint to taking the 0th space. The abelian group structure on both must be the same because the suspension isomorphism being a group homomorphism implies that the two group structures satisfy the interchange identity, allowing use of the Eckmann-Hilton argument. □

A similar result holds for generalized homology theories satisfying a mild condition (see [Ada, Remark III.6.5]).

We can also show the existence of Eilenberg-Maclane spectra. An Eilenberg-Maclane spectrum HA for an abelian group A is a spectrum such that $\pi_0(HA) \cong A$ and $\pi_n(HA) = 0$ otherwise. One construction of these is as $(HA)_n = K(A, n)$ with structure maps the adjoints of the weak equivalences $K(A, n) \to \Omega K(A, n+1)$ and it is an easy check that the homotopy groups are as claimed. The Eilenberg-Maclane spectra for other indices can be constructed as suspensions or loops of HA.

Proposition 4.9. For any $A \in \mathsf{Ab}$, $HA_\bullet \cong H_\bullet(-; A)$ and $HA^\bullet \cong H^\bullet(-; A)$.

Proof. Check that these theories agree on all spheres and hence on CW complexes. □

4.3 Smash Products and Orthogonal Spectra

Lemma 4.7 told us that spectra represent cohomology thoeries as contravariant functors into the category of graded abelian groups, but many of our favourite cohomology theories have extra multiplicative structure. For example, ordinary cohomology with coefficients in a ring is valued in graded rings which are monoid objects in the category of graded abelian groups. We would like such cohomology theories to be represented by ring spectra - spectra that are monoid objects in Sp under some symmetric monoidal product on Sp. In [Ada, Section III.4], Adams constructs a symmetric smash product on $\mathcal{SH}$ with much effort which achieves this but this is not ideal because then we cannot exploit it at the point-set level of Sp.

Recognizing spectra as right $\mathbb{S}$-modules tells us that the reason it is so difficult to construct a good monoidal product on Sp is that $\mathbb{S}$ is not a commutative monoid, so the general machinery for smashing left modules over it (see [Lan, Exercise VII.4.6]) fails. This can be fixed by adding more symmetries of (the spaces that appear in) $\mathbb{S}$ to the domain $\mathscr{D}_{\mathbb{N}}$. The category of $\mathbb{S}$-modules that we get then will not be equivalent to Sp, but with an appropriate model structure, it will be Quillen equivalent. Here is a sketch of how this works for the interested reader, concluding in Definition 4.11.

Let $\Sigma(n)$ denote the symmetric group on n letters and $O(n)$ the nth orthogonal group. Let $\mathscr{D}_\Sigma$ be the Top_*-category with objects the natural numbers and Hom spaces $\mathscr{D}_\Sigma(n, n) = (\Sigma(n))_+$. Let $\mathscr{D}_{\mathbf{O}}$ be the Top_*-category with objects the natural numbers and Hom spaces $\mathscr{D}_{\mathbf{O}}(n, n) = (O(n))_+$. All other Hom spaces are trivial.

Definition 4.10. A *symmetric space* is a presheaf on $\mathscr{D}_\Sigma$ and an *orthogonal space* is a presheaf on $\mathscr{D}_{\mathbf{O}}$. Equivalently, a symmetric/orthogonal space is a sequence $(X_n)_{n\in\mathbb{N}}$ of pointed spaces with an action of the nth symmetric/orthogonal group acting on X_n.

The embeddings $\Sigma(m)\times\Sigma(n) \to \Sigma(m+n)$ and $O(m)\times O(n) \to O(m+n)$ allow us to extend the monoidal structure on $\mathscr{D}_\mathbb{N}$ to those on $\mathscr{D}_\Sigma$ and $\mathscr{D}_{\mathbf{O}}$. Day convolution then makes the categories of symmetric and orthogonal spaces into closed symmetric monoidal categories. $\mathbb{S}$ then extends to a symmetric space $\mathbb{S}_\Sigma$ and an orthogonal space $\mathbb{S}_{\mathbf{O}}$ with the actions being the one-point compactifications of the usual actions of $\Sigma(n)$ and $O(n)$ on $\mathbb{R}^n$. Both of these are still monoids. We sketch this for the symmetric case. $S \wedge S$ is described as the coend

$$(S \wedge S)_n = \int_{i,j} S^i \wedge S^j \wedge \mathscr{D}_\Sigma(n, i+j)$$

The multiplication $\mathbb{S} \wedge \mathbb{S} \to \mathbb{S}$ is the one induced by the maps

$$S^i \wedge S^j \wedge \mathscr{D}_\Sigma(n, i+j) \to S^n$$

The domain is trivial when $n \neq i+j$. If $n = i+j$, this is the composite

$$S^i \wedge S^j \wedge (\Sigma(n))_+ \cong S^n \wedge (\Sigma(n))_+ \to S^n$$

where the second map is the action map. This is a commutative monoid because in the diagram

$$\begin{tikzcd} S^i \wedge S^j \wedge (\Sigma(i+j))_+ \arrow[rr, "\tau"] \arrow[dr] & & S^j \wedge S^i \wedge (\Sigma(j+i))_+ \arrow[dl] \\ & S^{i+j} & \end{tikzcd}$$

the twist map τ exchanges the two spheres and also precomposes the permutation with the permutation

$$1, 2, \ldots, n \mapsto i+1, \ldots, n, 1, \ldots, i$$

and both of these changes cancel out. Therefore, $\mathbb{S}_\Sigma$ is a commutative monoid in the presheaf category $\mathscr{D}_\Sigma$ which gives us a closed symmetric monoidal category of modules over it. Similar arguments can be made for $\mathbb{S}_{\mathbf{O}}$. We define a symmetric spectrum as a right module over $\mathbb{S}_\Sigma$ and an orthogonal spectrum as a right module over $\mathbb{S}_{\mathbf{O}}$. From this point on, we talk only about orthogonal spectra because of Remark 4.13.

Definition 4.11. An orthogonal spectrum is a sequence of spaces $(X_n)_{n\in\mathbb{N}}$

with an action of $O(n)$ on X_n and $O(n) \times O(k)$-equivariant structure maps

$$\sigma_{n,k} : X_n \wedge S^k \to X_{n+k}$$

such that $\sigma_{n,0} = \mathrm{id}_{X_n}$ and the composite

$$X_n \wedge S^k \xrightarrow{\sigma_{n,k} \wedge S^l} X_{n+k} \wedge S^l \xrightarrow{\sigma_{n+k,l}} X_{n+k+l}$$

is the map $\sigma_{n,k+l}$ for each $n, k, l \in \mathbb{N}$.

A map of orthogonal spectra is a map between them as $\mathbb{S}_{\mathbf{O}}$-modules, which amounts to a map of the underlying spectra such that the levelwise maps are $O(n)$-equivariant. The category of orthogonal spectra is denoted $\mathsf{Sp}_{\mathbf{O}}$.

Theorem 3.11 now gives us a Top_*-category $\mathscr{J}_{\mathbf{O}}$ such that $\mathsf{Sp}_{\mathbf{O}}$ is equivalent to the category of presheaves on $\mathscr{J}_{\mathbf{O}}$. There is a natural inclusion $\mathscr{J}_{\mathbb{N}} \to \mathscr{J}_{\mathbf{O}}$ which induces, by precomposition, a forgetful functor $\mathsf{Sp}_{\mathbf{O}} \to \mathsf{Sp}_{\mathbb{N}}$. This has a left adjoint given by left Kan extension since $\mathscr{J}_{\mathbb{N}}$ is small and Top_* is cocomplete. By composing left adjoints, this proves that once again we have a left adjoint to the functor

$$\begin{aligned} \Omega^\infty : \mathsf{Sp}_{\mathbf{O}} &\to \mathsf{Top}_* \\ X &\mapsto X_0 \end{aligned}$$

which we call the *suspension spectrum functor* and denote by Σ^∞.

We finally have a closed symmetric monoidal category of spectra. The smash product of spectra is defined by Day convolution (Theorem 3.10) as the coend

$$(X \wedge Y)_k = \int^{m,n} X_m \wedge Y_n \wedge \mathscr{J}_{\mathbf{O}}(k, m+n)$$

The closed structure is given by internal homs

$$\mathrm{Map}(X, Y)_n = \mathsf{Sp}_{\mathbf{O}}(X(-+n), Y)$$

For $n = 0$ in this, we get the mapping space of maps $X \to Y$ as diagrams in Top_*, and we denote this by $\mathrm{Map}_0(X, Y)$. Then the functor $- \wedge E : \mathsf{Top}_* \to \mathsf{Sp}_{\mathbf{O}}$ is left adjoint to $\mathrm{Map}_0(E, -)$. A consequence of these adjunctions is the natural isomorphism

$$\Sigma^\infty X \wedge E \cong X \wedge E$$

which can be verified by checking the maps out of both. As it does not change meaning, we omit Σ^∞ from all our notation unless we need to emphasize that the objects are spaces rather than spectra.

All the proofs about sequential spectra from Section 4.1 now work here and we are able to define a stable model structure on orthogonal spectra. A map of orthogonal spectra is classed a stable equivalence if it induces an isomorphism on all stable homotopy groups, which are defined just as in Definition 4.1.

Theorem 4.12. There exists a model structure called the *stable model structure* on $\mathsf{Sp}_{\mathbf{O}}$ in which the weak equivalences are the stable equivalences of orthogonal spectra. The free-forgetful adjunction between Sp and $\mathsf{Sp}_{\mathbf{O}}$ described above is then a Quillen equivalence.

Proof. See [HHRb, Section 7.3] for the first statement and [MMSS, Section 10] for the second one. The Quillen equivalence can be briefly described as follows. The category $\mathscr{D}_{\mathbb{N}}$ embeds into $\mathscr{D}_{\mathbf{O}}$, which extends to an embedding of $\mathscr{J}_{\mathbb{N}}$ into $\mathscr{J}_{\mathbf{O}}$. From an orthogonal spectrum, we get a sequential one by restriction to this subcategory (equivalently, by forgetting the orthogonal group actions), and we go the other way using left Kan extension along this embedding. Checking that this is a Quillen equivalence reduces to the fact that the map

$$O(n)_+ \wedge_{O(k-1)} S^n \to O(n)_+ \wedge_{O(k)} S^n$$

is a $(2n+k-1)$-equivalence (and hence a stable homotopy equivalence) for all $k \leq n \in \mathbb{N}$. □

Remark 4.13. Symmetric spectra are conceptually than orthogonal spectra because adding the symmetric group actions is the least invasive way to fix the non-commutativity of $\mathbb{S}$ as a monoid. However, their homotopy theory is not as well behaved. If we want Sp_{Σ} to be Quillen equivalent to Sp, then the definition of stable equivalences will not be the expected one and will instead have to rely on cohomology. The reason we prefer orthogonal spectra is that their stable equivalences are exactly those of their underlying sequential spectra. In many cases, orthogonal spectra also add some features that are not present in sequential spectra, as we see in the following results. They are also the only kind of spectra out of these that generalize easily to the equivariant case (see the introduction in Section 6.1) From this point on, 'spectrum' always refers to an orthogonal spectrum.

Define the shift functor $\mathrm{sh} : \mathsf{Sp}_{\mathbf{O}} \to \mathsf{Sp}_{\mathbf{O}}$ as the functor that takes a spectrum X to the spectrum with nth space X_{n+1}, with $O(n)$ acting via the inclusion $O(n) \to O(n+1)$ and shifted structure maps. Viewing spectra as functors $\mathscr{J}_{\mathbf{O}} \to \mathsf{Top}_*$, this is exactly precomposition by the functor $-+1 : \mathscr{J}_{\mathbf{O}} \to \mathscr{J}_{\mathbf{O}}$, where $+$ is the monoidal product on $\mathscr{J}_{\mathbf{O}}$.

Proposition 4.14. There is a natural stable equivalence $\Sigma \to \mathrm{sh}$.

Proof. The map $\Sigma X \to \mathrm{sh}\, X$ is given by the structure maps $\Sigma X_n \to X_{n+1}$ of X. That it is a stable equivalence is a direct consequence of the suspension isomorphism for homotopy groups. □

This feature is exclusive to orthogonal spectra. For symmetric spectra, this is not a stable equivalence, whereas for sequential spectra, it is not even a map of spectra because of the twist map τ that shows up in the structure maps of ΣX:

$$\begin{tikzcd}
X_n \wedge S^1 \wedge S^1 \arrow[r, "X_n \wedge \tau"] \arrow[d, "\sigma_n \wedge S^1"'] & X_n \wedge S^1 \wedge S^1 \arrow[r, "\sigma_n \wedge S^1"] & X_{n+1} \wedge S^1 \arrow[d, "\sigma_{n+1}"] \\
X_{n+1} \wedge S^1 \arrow[rr, "\sigma_{n+1}"'] & & X_{n+2}
\end{tikzcd}$$

The shift gives us a clearer picture of both the suspension and its inverse. Indeed, we can define an essential inverse sh^{-1} by shifting the other way, with the action of $O(n)$ on X_{n-1} induced up from that of $O(n-1)$ (see Proposition 5.2). That is,

$$(\mathrm{sh}^{-1} X)_n = \mathrm{Ind}_{O(n-1)}^{O(n)} X_{n-1}$$

for $n \geq 1$ and $(\mathrm{sh}^{-1} X)_0 = *$. These functors actually form a Quillen equivalence

$$\mathrm{sh}^{-1} \dashv \mathrm{sh}$$

An advantage of sh^{-1} over Ω is that it takes the generating cofibrations of orthogonal spectra to generating cofibrations and hence preserves orthogonal CW spectra. Indeed, the generating cofibrations of $\mathsf{Sp}_{\mathbf{O}}$ are precisely the negative shifts of the inclusion $S^0 \to D^1$, which is a consequence of the fact that $\mathrm{sh}^{-n} \circ \Sigma^\infty$ is the left adjoint of the evaluation of spectra at n because the latter can be written as $\Omega^\infty \circ \mathrm{sh}^n$. The description of the generating cofibrations in the projective model structure says exactly that they are the images of the generating cofibrations of the objects under the left adjoint of each evaluation functor.

In any closed symmetric monoidal category, we can define the *(weak) dual* of an object X as the internal hom from X to the monoidal unit. If this dual satisfies some additional conditions, we call it a strong dual and X dualizable. A dualizable object X is *invertible* if its smash product with its dual is the monoidal unit. This implies that the functor $- \wedge X$ has an essential inverse given by $- \wedge DX$ where DX is the dual. Now Theorem 4.5 tells us that $- \wedge S^n$ is an equivalence $\mathcal{SH} \to \mathcal{SH}$, and we can show that S^n is in fact invertible. However, the internal hom construction rarely gives a CW spectrum and is hard to use. Instead, using the shift operators gives us weakly equivalent models which are also CW spectra. We define $S^{-n} = \mathrm{sh}^{-n} \mathbb{S}$ and this is indeed the inverse of S^n in $\mathcal{SH}$ as $- \wedge \mathrm{sh}^{-n} \mathbb{S} \cong \mathrm{sh}^{-n}$. Using this model of S^{-n}, we also find that $\pi_{-n}(X) \cong \mathcal{SH}(S^{-n}, X)$ (when X is fibrant, we can explicitly compute the left homotopy classes of maps).

The concept of duality leads to some important theorems in homotopy theory. For example, given a compact manifold M, we embed it into some $\mathbb{R}^n$ and hence into S^n using Whitney's theorem and define $DM = \mathrm{sh}^{-(n-1)} \Sigma^\infty(S^n \setminus M)$. Then DM is well defined up to stable equivalence and is the strong dual of M. This is called *Alexander duality* and it leads to a proof of Poincare duality by identifying

$\Sigma^n D(M_+)$ with the Thom space of the normal bundle of the embedding $M \hookrightarrow \mathbb{R}^n$ and using the Thom isomorphism theorem. See [Wic, Lecture 10]. An important general fact about duality in a closed symmetric monoidal category is that if the dual of X exists, it is given by the internal hom from X to the monoidal unit, making dualization a functor on the subcategory of dualizable objects.

5 Equivariant Homotopy Theory

Given a group G, a G-space is a topological space with an action of G on it. We will mostly be concerned with the case of $G = C_2$, the cyclic group with 2 elements, but the theory will be developed in the general setting. The point of studying these spaces is highlighted in the introduction of [Grea]. It is also summarized in Section 7.3.

Throughout this thesis, we will restrict our attention to actions of a finite group G with identity $e \in G$.

5.1 Category of G-spaces

A (left) G-space is a topological space along with a continuous map

$$G \times X \to X$$

usually written with juxtaposition $(g, x) \mapsto gx$, satisfying the usual action axioms

$$g_1(g_2x) = (g_1g_2)x, \qquad ex = x$$

We then have the category Top^G of G-spaces in which the morphisms are the G-maps, i.e. maps $f : X \to Y$ such that

$$f(gx) = g(f(x)) \qquad \forall g \in G, x \in X, y \in Y$$

A right G-space is defined similarly. A G-space will be a left G-space unless otherwise specified.

The category of pointed G-spaces Top_*^G is defined similarly, where the action of G is required to fix the basepoint. For the rest of this chapter, a map will be a G-map unless otherwise specified.

Both Top^G and Top_*^G are bicomplete. They can be viewed as the categories of functors from a category $\mathcal{B}G$ to Top and Top_* respectively, where $\mathcal{B}G$ is G viewed as a category with one object. As such, a limit or colimit of a diagram in one of these categories is computed as it is in the non-equivariant case and then endowed with the unique G-action compatible with the actions of all the G-spaces in the diagram.

In particular, G acts diagonally on a product of (pointed) G-spaces, and hence also on its quotient, the smash product. This makes these categories symmetric monoidal. We also have an internal hom. For G-spaces X, Y, this is defined as the set $\mathsf{Top}(X, Y)$ of all (not necessarily equivariant) maps with its usual compact-open topology and G-action defined by

$$(gf)(x) = g(f(g^{-1}x))$$

for all $g \in G$, $f \in \mathsf{Top}(X, Y)$ and $x \in X$. In the pointed case, we take the pointed space of pointed maps with the same action. We denote these internal homs by $\mathrm{Map}(X, Y)$ and $\mathrm{Map}_*(X, Y)$ respectively.

Given any space X, we can view it as a G-space by endowing it with the trivial G-action. This gives a functor

$$\mathrm{Triv}_G : \mathsf{Top} \to \mathsf{Top}^G$$

We also have restriction functors for every subgroup $H \leq G$

$$\mathrm{Res}^G_H : \mathsf{Top}^G \to \mathsf{Top}^H$$

that forget the actions of all elements not in H. Both of these are part of important adjunctions as follows.

Definition 5.1. The *product over H* of a right H-space X with a left H-space Y is

$$X \times_H Y := (X \times Y)/\sim$$

where $\sim$ is the equivalence relation generated by $(gh, y) \sim (x, hy)$ for all $x \in X, y \in Y, h \in H$.

A smash product over H when X and Y are pointed is defined as a similar quotient of the smash product. Note that the (smash) product over H is a non-equivariant space. However, if X also has a compatible left action of a group G, then G has a left action on $X \times_H Y$ by acting on X. If $H \leq G$, then G has a left G-action and a right action of (in particular) H. This is analogous to tensor products of right R-modules with left R-modules to get abelian groups.

Proposition 5.2. (i) Triv_G has a right adjoint called the *fixed point functor*, $(-)^G : \mathsf{Top}^G \to \mathsf{Top}$. This functor takes a G-space X to its subspace of fixed points X^G.

(ii) For each $H \leq G$, Res^G_H has a left adjoint called *induction*

$$\mathrm{Ind}^G_H := G \times_H - : \mathsf{Top}^H \to \mathsf{Top}^G$$

When X has a trivial H-action, then $G \times_H X \cong G/H \times X$. All of these definitions and results can again be adapted to pointed G-spaces. In this case, $\mathrm{Ind}^G_H X = (G)_+ \wedge_H X$.

5.2 Homotopy theory of G-spaces

A G-homotopy between $f, g : X \to Y$ is a G-map $H : X \times I \to Y$ where G acts trivially on $I = [0,1] \in \mathbb{R}$ with

$$H(-,0) = f, \qquad H(-,1) = g$$

G-homotopy equivalences can then be defined in the usual way. An analogous definition can be made for the pointed case. Our objective is to study the homotopy theory of Top^G and Top^G_*, which we do by putting model structures on both of these categories.

Viewing Top^G as the category of functors $\mathcal{B}G \to \mathsf{Top}$ gives us a projective model structure on Top^G via Corollary 2.27, but a weak equivalence in this model structure is a weak homotopy equivalence of the underlying non-equivariant spaces, so this is not the model structure we want. To get at the right model structure, we begin by identifying the G-CW complexes and then proving a version of Whitehead's theorem that tells us what G-weak equivalences should be.

Our notion of G-CW complexes should be broad enough that it captures every G-space that we would care about, such as G-manifolds or spaces constructed from orthogonal representations of G. There are many equivalent versions which differ in their choices of cells, but the one below is the most commonly used for its simplicity. It is to be thought of as a CW-complex with a G-action that only permutes cells. In the following, S^n and D^n are regarded as G-spaces with trivial actions (i.e. we are suppressing Triv_G from our notation).

Definition 5.3. A *G-CW complex* is a G-space X with a filtration $X_0 \subset X_1 \subset X_2 \subset \dots$ where X_{-1} is empty and X_n is obtained from X_{n-1} by attaching the induced up n-disks $G/H \times D^n$ along maps on their boundaries $G/H \times S^{n-1} \to X_{n-1}$. We call this an n-dimensional G-cell of type G/H.

We also have, for any such G-cell, the adjoint $S^{n-1} \to X^H_{n-1}$ of the attaching map obtained by applying both the adjunctions of Proposition 5.2. Indeed, a G-map $G/H \times S^{n-1} \cong G \times_H S^{n-1} \to X_{n-1}$ is equivalent to an H-map $S^{n-1} \to X$, which is then equivalent to a non-equivariant map $S^{n-1} \to X^H$. This is usually how we will describe the attaching map. We could also take the union of all the G-orbits used to go from X_{n-1} to X_n to get some G-set K, and then X_n is the pushout

$$\begin{tikzcd}
K \times S^{n-1} \arrow[r, hook] \arrow[d] & K \times D^n \arrow[d] \\
X_{n-1} \arrow[r] \arrow[ur, phantom, "\ulcorner"] & X_n
\end{tikzcd}$$

Every G-manifold has a G-CW complex structure, justifying this definition to some extent (see [Ill]). As with any CW construction, May's HELP Lemma [Mayb, Section 10.3] gives us the following version of Whitehead's theorem for these G-CW complexes. This is spelled out in greater detail in [Blu, Theorem 1.2.11]. The idea, similar to that of the the non-equivariant case, is that G-CW complexes are built from attaching cones along $G/H \times S^n$, so probing X by these suffices to infer its homotopy type. But again, maps of this kind are equivalent to maps $S^n \to X^H$ by Proposition 5.2. Of course, the model theoretic proof of Whitehead's theorem cannot be applied since we have not yet constructed the correct model structure on G-spaces.

Theorem 5.4. Let $f : X \to Y$ be a map of G-CW complexes. Then f is a homotopy equivalence if and only if $f^H : X^H \to Y^H$ is a weak equivalence for every $H \leq G$. □

Definition 5.5. A map $f : X \to Y$ of G-spaces is a weak equivalence if f^H is a weak equivalence for every $H \leq G$.

This leads to the following observation. Let $\mathcal{O}_G$ be the full Top-subcategory of Top^G spanned by the G-orbits G/H with the discrete topology (the topological enrichment here will not matter as each morphism set in $\mathcal{O}_G$ has the discrete topology). Then a map $X \to Y$ is a weak equivalence if and only if the nerve induced by the inclusion $\mathcal{O}_G \to \mathsf{Top}^G$ maps it to a weak equivalence in the presheaf category $\widehat{\mathcal{O}_G}$ with the projective model structure. It can be checked that this nerve-realization adjunction satisfies the criteria of Theorem 2.25, so it may be used to induce a model structure on Top^G with the correct weak equivalences. We have thus the following description of the model structure on Top^G.

Theorem 5.6. There is a cofibrantly generated model structure on Top^G with generating sets

$$\mathcal{I} = \left\{G/H \times S^n \hookrightarrow G/H \times D^{n+1} \,\middle|\, n \in \mathbb{N}, H \leq G\right\}$$
$$\mathcal{J} = \{G/H \times D^n \hookrightarrow G/H \times D^n \times I \mid n \in \mathbb{N}, H \leq G\}$$

where each inclusion is the standard one. $f : X \to Y$ is a weak equivalence (resp. fibration) if and only if f^H is a weak equivalence (resp. fibration) for each $H \leq G$. □

This also justifies our definition of G-CW complexes as it gives us a G-CW approximation theorem (see the paragraph preceeding Theorem 2.3). We will write

$$[X, Y]^G := \operatorname{Ho} \mathsf{Top}^G(X, Y), \qquad [X, Y]^G_* := \operatorname{Ho} \mathsf{Top}^G_*(X, Y)$$

for (pointed) G-spaces X, Y.

Generally, a transferred model structure need not be Quillen equivalent to the model structure it was transferred from, but we do have this result in this case.

Theorem 5.7 (Elmendorf)**.** The topological nerve-realization adjunction

$$\begin{tikzcd}
\mathcal{O}_G \arrow[r, "y"] \arrow[dr, hook] & \widehat{\mathcal{O}_G} \arrow[d, shift right=2] \arrow[d, phantom, "\dashv"] \\
& \mathsf{Top}^G \arrow[u, shift right=2]
\end{tikzcd}$$

is a Quillen equivalence.

Proof. See [Blu, Section 1.3]. □

Example 5.8. Let $G = C_6$. We think of this as $\mathbb{Z}/6\mathbb{Z}$ under addition. Then $\mathcal{O}_G$ is the category

$$\begin{tikzcd}
& C_6/C_6 & \\
C_6/3C_6 \arrow[ur] & & C_6/2C_6 \arrow[ul] \\
& C_6/e \arrow[uu] \arrow[ul, shift right=2, "1"'] \arrow[ul] \arrow[ul, shift left=2, "2"] \arrow[ur, shift left=1] \arrow[ur, shift right=1, "1"'] &
\end{tikzcd}$$

where the identity maps and the endomorphisms have been omitted. The arrow labelled i is the one that maps the equivalence class 0 to the equivalence class of i. The unlabelled arrows are to be interpreted as being labelled by 0. The nerve $\mathsf{Top}^G \to \widehat{\mathcal{O}_G}$ takes a space X to the presheaf

$$\begin{tikzcd}
& X^{C_6} \arrow[dl] \arrow[dr] \arrow[dd] & \\
X^{3C_6} \arrow[dr, shift left=2, "1"] \arrow[dr] \arrow[dr, shift right=2, "2"'] & & X^{2C_6} \arrow[dl, shift right=1, "1"'] \arrow[dl, shift left=1] \\
& X &
\end{tikzcd}$$

where each map is the inclusion followed by the action of the appropriate element.

Everything done till now can also be adapted to pointed G-spaces. In this case, we change the enrichment of $\mathcal{O}_G$ to be over Top_* by adding a disjoint basepoint to each Hom space and use the nerve-realization adjunction induced by the composite

$$\mathcal{O}_G \hookrightarrow \mathsf{Top}^G \xrightarrow{(-)_+} \mathsf{Top}_*^G$$

with the understanding that the newly added basepoints are mapped to the basepoints of the Hom spaces in Top^G, i.e. the constant maps. It can be checked that the nerve of a pointed G-space has a description similar to the one in Example 5.8, except that now all the spaces appearing in the diagram must be pointed. More succinctly, the above is a case of free change of enrichments, as $(-)_+$ is the free functor $\mathsf{Top} \to \mathsf{Top}_*$, and hence the form that the diagrams take does not change. The pointed version of Elmendorf's theorem then holds and tells us the structure of the homotopy invariants.

Definition 5.9. A *G-coefficient system* is a functor $\mathcal{O}_G^{\mathsf{op}} \to \mathsf{Ab}$. More generally, for a category $\mathcal{C}$, a $\mathcal{C}$-valued G-coefficient system is a functor $\mathcal{O}_G^{\mathsf{op}} \to \mathcal{C}$.

Definition 5.10. Let X be a pointed G-space, and let $\widetilde{X} : \mathcal{O}_G \to \mathsf{Top}_*$ be its nerve as described above. Define the *nth homotopy coefficient system of X* to be the composite

$$\mathcal{O}_G \xrightarrow{\widetilde{X}} \mathsf{Top}_* \xrightarrow{\pi_n} \mathsf{Ab}$$

Of course, this is only a G-coefficient system for $n \geq 2$ since π_0 and π_1 are Set and Grp valued respectively. In those cases, we instead have Set- or Grp-valued G-coefficient systems. In the stable case, we will not have to worry about these edge cases.

Unpacking the definition, this says that the coefficient system $\pi_n(X)$ has values

$$\pi_n(X)(G/H) \cong \pi_n(X^H) \cong [S^n, X^H]_*^e \cong [(G/H)_+ \wedge S^n, X]_*^G$$

for each $H \leq G$. It is clear that these homotopy coefficient systems together form a complete set of invariants on the category of presheaves on $\mathcal{O}_G$, i.e. a map f of such presheaves is a weak equivalence if and only if $\pi_n(f)$ is a weak equivalence for every $n \geq 0$ and every compatible choice of basepoints in the domain of f. Since G-CW complexes are realizations of cofibrant presheaves on $\mathcal{O}_G$ (because the G-cell inclusions are) and every G-space is fibrant, a map of G-CW complexes is a weak equivalence if and only if its image under each π_n is an isomorphism. This justifies our definition of the homotopy coefficient systems.

One could also then define the nth equivariant (co)homology of a G-space X as the composite

$$\mathcal{O}_G \xrightarrow{\widetilde{X}} \mathsf{Top} \xrightarrow{H_n \text{ or } H^n} \mathsf{Ab}$$

where $\widetilde{X}$ is the nerve of X. This is not the right definition of these functors because it lacks one of the most important features of non-equivariant (co)homology - Poincaré duality. We defer equivariant (co)homology and Eilenberg-Maclane spaces to the next chapter.

6 Equivariant Stable Homotopy Theory

We now wish to stabilize the equivariant homotopy theory. One could again invert the suspension functor $- \wedge S^1 : \mathsf{Top}_*^G \to \mathsf{Top}_*^G$ by taking spectrum objects in Top_*^G, i.e. sequences $(X_n)_{n\in\mathbb{N}}$ in Top_*^G with structure maps $X_n \wedge S^1 \to X_{n+1}$. Such spectra are called the naive G-spectra. This is not the stability we want.

If we want some of our most important theorems, such as Poincare duality, to generalize to the equivariant setting, then we need to be able to embed G-manifolds equivariantly into spheres. However, since the spheres S^n have trivial G-action, this will be impossible for any manifolds with non-trivial action. However, an equivariant version of Whitney's theorem says that every G-manifold embeds into some finite dimensional orthogonal G-representation V and hence into the one-point compactification S^V of V, where G acts trivially on the point at infinity. These spaces S^V are called the G-representation spheres. Therefore, the functors that we want to invert are $\Sigma^V := - \wedge S^V$ rather than the 'trivial' suspensions Σ^n. Since every finite dimensional G-representation is a summand of a multiple of the regular G-representation ρ and $S^V \wedge S^W \cong S^{V\oplus W}$, inverting all Σ^V is equivalent to inverting just Σ^ρ. For us, G-representations will always be finite dimensional and orthogonal. We write $|V|$ for the dimension of V.

6.1 G-spectra

One attempt at a definition of G-spectra could be as follows. A G-spectrum would be a G-space X_V for each G-representation V with compatible structure maps $\sigma_{V,W} : \Sigma^W X_V \to X_{V\oplus W}$. However, this compatibility is hard to state because direct sum is only associative up to coherent natural isomorphisms and not on-the-nose. Therefore, for every G-isomorphism $V \to W$, we should also have an isomorphism $X_V \to X_W$ satisfying some compatibility conditions with the structure maps. Once again, we would realize these objects as right modules over a diagram $\mathbb{S}$ defined by $\mathbb{S}_V = S^V$. $\mathbb{S}$ would be a functor from the Top_*^G-category $\mathscr{D}^G$ with objects the G-representations and the morphism object $\mathscr{D}^G(V,W)$ the space of the G-isomorphisms $V \to W$ with a disjoint basepoint and a trivial G-action. For G the trivial group, this gives a category equivalent to that of orthogonal spectra.

However, $\mathbb{S}$ would not be a commutative monoid. If we try to follow the argument for commutativity in the case of symmetric spectra (preceeding Definition 4.11), we see the problem. The G-isomorphisms $V \oplus W \to W \oplus V$ do not include the linear isomorphisms which 'interchange V and W', i.e. the isomorphisms given by block matrices of the form

$$\left[\begin{array}{c|c} 0 & I^{|V|} \\ \hline I^{|W|} & 0 \end{array}\right]$$

upon a choice of bases of V and W. In order to make $\mathbb{S}$ a commutative monoid, we need the morphism objects to contain all of these isomorphisms.

Define $\mathscr{D}^G_{\mathbf{O}}$ as the Top_*-category with objects the G-representations and morphism objects $\mathscr{D}^G_{\mathbf{O}}(V,W) = (O(V,W))_+$ where $O(V,W)$ denotes the G-space of linear isomorphisms $V \to W$ that preserve the inner product. The action of G on $O(V,W)$ is given by conjugation (similar to the internal hom of G-spaces). We will write $O(V) := O(V,V)$. This has a symmetric monoidal product defined by direct sum and hence its presheaf category is closed symmetric monoidal by Theorem 3.10 ($\mathscr{D}^G_{\mathbf{O}}$ is not small, but it has a small skeleton, which is enough for all the coends used in the proof to exist). $\mathbb{S}$ then extends to a commutative monoid in this category. We define an orthogonal G-spectrum to be a left module over this extended monoid (which we will again denote by $\mathbb{S}$), or equivalently a presheaf on a small category $\mathscr{J}^G_{\mathbf{O}}$ as in Theorem 3.11. Unravelling, we get the following definition.

Definition 6.1. (i) An *(genuine, orthogonal) G-spectrum* consists of a pointed $G \times O(V)$-space X_V for every G-representation V along with G-equivariant action maps

$$\alpha_{V,W} : X_V \wedge O(V,W)_+ \to X_W$$

for G-representations V, W of the same dimension and $G \times O(V) \times O(W)$-equivariant structure maps

$$\sigma_{V,W} : \Sigma^W X_V \to X_{V \oplus W}$$

for G-representations V, W such that:

- For G-representations U, V, W,

$$\begin{tikzcd}
\Sigma^{V\oplus W} X_U \arrow[d, "\sigma_{U,V\oplus W}"'] & \Sigma^W \Sigma^V X_U \arrow[l, "\sim"'] \arrow[r, "\Sigma^W \sigma_{U,V}"] & \Sigma^W X_{U\oplus V} \arrow[d, "\sigma_{U\oplus V,W}"] \\
X_{U\oplus(V\oplus W)} \arrow[rr, "\sim"'] & & X_{(U\oplus V)\oplus W}
\end{tikzcd}$$

  commutes, wherethe bottom map is the action of the canonical associator isomorphism $\alpha : U \oplus (V \oplus W) \to (U \oplus V) \oplus W$.
- The actions are associative and unital: the two maps $X_U \wedge O(U,V)_+ \wedge O(V,W)_+ \to X_W$ built from the action maps coincide, and the identity map in $O(V,V)$ acts by identity on X_V.

- The action maps on X and on $\mathbb{S}$ are compatible, i.e. the following diagram commutes for any G-representations V, V', W, W'.

$$\begin{tikzcd} X_V \wedge S^W \wedge O(V,V')_+ \wedge O(W,W')_+ \arrow[r] \arrow[d] & X_{V'} \wedge S^{W'} \arrow[d] \\ X_{V\oplus W} \wedge O(V\oplus W, V'\oplus W')_+ \arrow[r] & X_{V'\oplus W'} \end{tikzcd}$$

- The composite,

$$X_V \wedge S^0 \xrightarrow{\sim} X_V \xrightarrow{\sim} X_{V\oplus 0}$$

where both maps are (induced by) the canonical isomorphisms, coincides with the structure map $\sigma_{V,0} : \Sigma^0 X_V \to X_{V\oplus 0}$.

(ii) A map $f : X \to Y$ of G-spectra is a collection of $G \times O(V)$-maps $f_V : X_V \to Y_V$ for each G-representation V which commute with the structure and action maps. Thus we have a category of orthogonal G-spectra, which will be denoted $\mathsf{Sp}^G_{\mathbf{O}}$.

Proposition 6.2. For any G-representations V, W of the same dimension, there is a natural G-homeomorphism

$$X_W \cong O(V,W)_+ \wedge_{O(V)} X_V$$

Proof. The map from the right side to the left is given by the action map (precomposed with the twist map), which factors through this smash over $O(V)$ because of the associativity condition on the action. The inverse can be defined by choosing any isomorphism $h : W \to V$ and mapping $x \in X_W$ to the class of $(h^{-1}, \alpha_{W,V}(h,x))$. □

Remark 6.3. From this result and the coherence conditions on the structure and action maps, we can infer that a G-spectrum is completely determined by its restriction to the integer degrees, i.e. define $\mathscr{I}^G_{tr} \subset \mathscr{I}^G_{\mathbf{O}}$ to be the full subcategory spanned by the positive integers, and then $X : (\mathscr{I}^G_{\mathbf{O}})^{\mathsf{op}} \to \mathsf{Top}^G_*$ is determined by its restriction to $(\mathscr{I}^G_{tr})^{\mathsf{op}}$. This gives an equivalence of categories between $\mathsf{Sp}^G_{\mathbf{O}}$ and the so-called *naive orthogonal G-spectra*, which are the presheaves on $\mathscr{I}^G_{tr}$. If we unravel the definition of this structure, we find that this is exactly an orthogonal spectrum with an action of G, or equivalently an 'orthogonal spectrum object' in Top^G_*. Direct sum restricts to a symmetric monoidal product on $\mathscr{I}^G_{tr}$ which induces a product on naive G-spectra. This agrees with the Day convolution product on genuine G-spectra. The stable model structure is easier to define on genuine G-spectra while point-set level constructions are easier on the naive G-spectra because we need to give less data.

Definition 6.4. Let $H \leq G$ be groups. Define the *restriction functor* $\mathrm{Res}^G_H : \mathsf{Sp}^G_{\mathbf{O}} \to \mathsf{Sp}^H_{\mathbf{O}}$ by restricting the action to H:

$$(\mathrm{Res}^G_H X)_n := \mathrm{Res}^{G\times O(n)}_{H\times O(n)} X_n$$

This has a left adjoint called *induction*

$$(\mathrm{Ind}^G_H Y)_n := \mathrm{Ind}^{G\times O(n)}_{H\times O(n)} Y_n$$

We also have $\mathrm{Triv}_G : \mathsf{Sp}_{\mathbf{O}} \to \mathsf{Sp}^G_{\mathbf{O}}$ that takes a spectrum to itself with the trivial G-action with the orthogonal groups acting the same way

$$(\mathrm{Triv}_G X)_n := \mathrm{Triv}_G X_n$$

This has a right adjoint $(-)^G$ which takes fixed points at each level under the restricted action of the orthogonal groups

$$(X^G)_n := (X_n)^G$$

For X a based G-space, the *suspension G-spectrum* of X is the image of X under the left adjoint Σ^∞ of the functor $\Omega^\infty : \mathsf{Sp}^G_{\mathbf{O}} \to \mathsf{Top}^G_*$ that evaluates a G-spectrum at the zero representation.

We can define shift operators on G-spectra too. For any G-representation V,

$$\mathrm{sh}^V X := X_{V\oplus -}$$

This has a left adjoint sh^{-V}. The easiest way to see this is to notice that sh^V is the nerve functor of Definition 3.4 induced by $W \mapsto \mathscr{J}^G_{\mathbf{O}}(-, V \oplus W)$, and hence has the realization as its left adjoint.

We get projective model structures on genuine G-spectra by viewing them as presheaves on $\mathscr{J}^G_{\mathbf{O}}$. The generating cofibrations in this model structure can be described as in Chapter 4; they will be the inclusions $S(H, m-1, V) \hookrightarrow D(H, m, V)$ where

$$S(H, m, V) = \mathrm{sh}^{-V}\Sigma^\infty((G/H)_+ \wedge S^m), \qquad D(H, m, V) = \mathrm{sh}^{-V}\Sigma^\infty((G/H)_+ \wedge D^m)$$

However, the fact that every G-representation V admits a G-CW structure implies that the inclusion $S(H, m-1, V) \hookrightarrow D(H, m, V)$ is always a relative $\mathcal{S}$-cell where $\mathcal{S}$ denotes the set of inclusions $S(H, m-1, n) \hookrightarrow D(H, m, n)$ for $m, n \in \mathbb{N}$. (ADD details). Therefore, we can reduce our generating cofibrations to $\mathcal{S}$, which are also exactly the generating cofibrations in the projective model structure of the naive G-spectra.

Define the dimension of the sphere $S(H, m, n)$ to be $m-n$, and then we see that every map from a lower dimensional sphere to higher dimensional one is trivial up to levelwise homotopies. Therefore, we have a notion of G-CW spectra, which are $\mathcal{S}$-cellular objects in which the cells are attached in dimension order.

We now need to define the stable equivalences. Since we are replacing the ordinary spheres with representation spheres, our homotopy groups must also be indexed by G-representations rather than just the integers and the colimit should be taken over all representation spheres. Since we are doing these things stably, we get negative degree homotopy groups too. Thus, the homotopy groups will be indexed over the class of formal differences $V - W$ where V, W are G-representations. Similar to Definition 5.10, the homotopy groups should actually be richer - they should allow evaluation at every G-orbit.

Definition 6.5. (i) For G-representations V_1, V_2, the *$(V_1 - V_2)$th homotopy group of $X \in \mathsf{Sp}^G_{\mathsf{O}}$ at a G-orbit G/H* is

$$\underline{\pi}_{V_1 - V_2}(X)(G/H) := \operatorname*{colim}_{\overrightarrow{G\text{-rep } W \geq V}} [(G/H)_+ \wedge S^{(W - V_2)\oplus V_1}, X_W]^G_*$$

where the colimit is over G-representations that contain V_2, which form a directed set under inclusion. $W - V_2$ denotes the orthogonal complement of V_2 in W.

(ii) A map of G-spectra $f : X \to Y$ is a *stable equivalence* if $\underline{\pi}_{V-W}(f)(G/H)$ is an isomorphism for all G-representations V, W and $H \leq G$.

Everything said about orthogonal spectra in Chapter 4 now applies. We take the left Bousfield localization of the model structure at the stable equivalences to get the stable model structure. The set of generating cofibrations is again $\mathcal{S}$ as described above. Define the *G-equivariant stable homotopy category* as $\mathcal{SH}^G := \operatorname{Ho} \mathsf{Sp}^G$.

As for non-equivariant orthogonal spectra, we have a natural weak equivalence $\Sigma^V \to \mathrm{sh}^V$. If V is a non-trivial representation, then sh^{-V} does not take generating cofibrations to generating cofibrations. However, it does take them to relative $\mathcal{S}$-cells, which implies that it preserves G-CW spectra.

Day convolution gives us a closed symmetric monoidal product, which we again call the smash product $\wedge$. We have an internal hom, which is a G-spectrum $\mathrm{Map}_G(X, Y)$. The Vth pointed G-space in this G-spectrum is the space of (non-equivariant) spectrum maps $\mathrm{sh}^V X = X(- + V) \to Y$ with G acting on it by conjugation (see Remark 6.3 to understand 'spectrum maps'). Taking the fixed point spectrum of this, we get the spectrum $\mathrm{Map}^G(X, Y)$ of maps from $X \to Y$, giving a spectral enrichment on $\mathsf{Sp}^G_{\mathsf{O}}$. More explicitly, the nth space of $\mathrm{Map}^G(X, Y)$ is the space of equivariant maps $X(- + n) \to Y$, which is the subspace of compatible maps in the product of mapping spaces $X(k+n) \to Y(k)$.

6.2 Mackey Functors

Similar to the unstable case, we have more structure on the homotopy groups of G-spectra and the following stable analogue of Elmendorf's theorem tells us what this structure should be. As in Theorem 5.7, we probe each spectrum by the G-orbits. This probing will be spectrally enriched. However, there is a simplification here. A spectrally enriched functor preserves all finite coproducts so instead of probing with just the G-orbits, we may probe with all finite G-sets without making our presheaf category any larger. This is a simplification because the subcategory of the finite G-sets will then have finite colimits, making its homotopy category additive rather than just pre-additive.

Definition 6.6. The *spectral Burnside category* $\mathscr{D}$ is the full spectrally enriched subcategory of $\mathsf{Sp}_{\mathbf{O}}^G$ spanned by the fibrant replacements of $\Sigma^\infty T_+$, the suspension spectra of finite G-sets T.

Theorem 6.7. The spectral nerve-realization adjunction

$$\mathbb{R}\mathscr{D} : \widehat{\mathscr{D}} \rightarrow \mathsf{Sp}_{\mathbf{O}}^G : \mathbb{N}\mathscr{D}$$

is a Quillen equivalence.

Proof. This is a special case of [GM, Theorem 1.36]. □

As in Section 5.2, the 0th homotopy functor evaluated at $X \in \mathsf{Sp}_{\mathbf{O}}^G$ should be defined as the composite

$$\mathscr{D} \xrightarrow{\widetilde{X}} \mathsf{Sp}_{\mathbf{O}} \xrightarrow{\pi_0} \mathsf{Ab}$$

where $\widetilde{X} \in \widehat{\mathscr{D}}$ is the nerve of X. The evaluation of this composite at a finite G-set T is

$$\begin{aligned} \underline{\pi}_0(X)(T) &= \pi_0(\widetilde{X}(T)) \\ &= \mathcal{SH}(S^0, \mathrm{Map}^G(\Sigma^\infty T_+, X)) \\ &\cong \mathcal{SH}^G(T_+ \wedge \Sigma^\infty S^0, X) \end{aligned}$$

The fibrant replacement of Definition 6.6 will not matter since that does not affect the set of maps in $\mathcal{SH}^G$. More generally, we will want to define the $(V_1 - V_2)$th homotopy functor evaluated X to be a functor $\mathscr{D} \rightarrow \mathsf{Ab}$ given by

$$\underline{\pi}_{V_1 - V_2}(X) = \mathcal{SH}^G(-_+ \wedge S^{V_1}, S^{V_2} \wedge X)$$

where $-$ takes values in finite G-sets. This functor is additive as $\wedge$ distributes over $\vee$ and $\mathcal{SH}^G$ is additive. It also factors through $\mathrm{Ho}\,\mathscr{D}$ because it takes hom sets at the homotopy level. Therefore, the homotopy functors on $\mathsf{Sp}_{\mathbf{O}}^G$ should be valued in additive functors $\mathrm{Ho}\,\mathscr{D} \to \mathsf{Ab}$. Therefore we need to understand the category $\mathrm{Ho}(\mathscr{D})$.

Definition 6.8. A *span* $X \to Y$ in a category $\mathcal{C}$ is a diagram

$$X \leftarrow Z \to Y$$

in $\mathcal{C}$. We call Z the *spanning object*. An isomorphism of spans from $X \to Y$ is an isomorphism of spanning objects making the following diagram commute.

$$\begin{tikzcd} & Z_1 \arrow[dl] \arrow[dr] \arrow[dd, "\cong"] & \\ X & & Y \\ & Z_2 \arrow[ul] \arrow[ur] & \end{tikzcd}$$

If $\mathcal{C}$ has finite coproducts, the isomorphism classes of spans $X \to Y$ form an abelian monoid where the addition is defined by taking coproducts of the spanning objects.

Definition 6.9. The *(algebraic) Burnside category* $\mathscr{B}_G$ is the additive category in which the objects are the finite G-sets and the abelian group $\mathscr{B}_G(X, Y)$ is the Grothendieck group (ADD) of the abelian monoid of isomorphism classes of spans of finite G-sets. The composition of spans is defined by pushouts of spans and then extended to the negatives.

$$Y \leftarrow U_2 \to Z \quad \circ \quad X \leftarrow U_1 \to Y \quad = \quad X \leftarrow U_1 \cup_Y U_2 \to Z$$

This is an additive category because it has finite biproducts which are given by disjoint union.

Proposition 6.10. There is an equivalence of additive categories $\mathscr{B}_G \to \mathrm{Ho}(\mathscr{D})$.

Proof. On an object X, this equivalence is defined as

$$X \mapsto \Sigma^\infty X_+$$

On a basis element $X \leftarrow U \to Y$ of $\mathscr{B}_G$, this is defined to be the composite

$$\Sigma^\infty X_+ \to \Sigma^\infty U_+ \to \Sigma^\infty Y_+$$

where the second map is the obvious one, and the first one is the transfer map associated to the map $X \leftarrow U$ constructed via the Pontryagin-Thom map. For details, see [Maya, Section IX.3]. □

Definition 6.11. A *G-Mackey functor* is an additive functor $\mathscr{B}_G^{\mathsf{op}} \to \mathsf{Ab}$.

A Mackey functor is typically denoted by a symbol with an underline. The following definition extends (i) of Definition 6.5.

Definition 6.12. For $X \in \mathsf{Sp}_{\mathsf{O}}^G$ and G-representations V_1, V_2, the $(V_1 - V_2)$th homotopy Mackey functor is defined on a G-set $T \in \mathscr{B}_G$ as

$$(\underline{\pi_V X})(T) :=\cong [T_+ \wedge S^{V_1}, S^{V_2} \wedge X]$$

Remark 6.13. Mackey functors are the result of us inverting the functor Σ^V for every G-representation V. If, instead, we only inverted the ordinary suspensions, we would get coefficient systems. Indeed, it can be checked that in that case, the homotopy category of the full subcategory of the naive suspension G-spectra of the orbits would be exactly the free Ab-enrichment of $\mathcal{O}_G$, so contravariant additive functors from it to Ab would be equivalent to coefficient systems.

The rest of this section is devoted to the structure and examples of Mackey functors. Any map $X \to Y$ in $\mathscr{B}_G$ can be factored as

$$X \leftarrow U \to Y \quad = \quad X \leftarrow U = U = U \to Y$$

i.e. as a composite of a purely 'contravariant part' and a purely 'covariant part'. Let $\mathcal{F}_G$ denote the category of finite G-sets.

Lemma 6.14. The data of a Mackey functor $\underline{M} : \mathscr{B}_G^{\mathsf{op}} \to \mathsf{Ab}$ is equivalent to a pair of functors $M_* : \mathcal{F}_G \to \mathsf{Ab}$ and $M^* : \mathcal{F}_G^{\mathsf{op}} \to \mathsf{Ab}$ that agree on objects and convert disjoint unions to direct sums such that for every pullback square as on the left,

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "i"] \\ C \arrow[r, "h"'] & D \end{tikzcd} \qquad \begin{tikzcd} MA \arrow[d, "M_* g"'] & MB \arrow[l, "M^* f"'] \arrow[d, "M_* i"] \\ MC & MD \arrow[l, "M^* h"] \end{tikzcd}$$

the square on the right commutes.

Proof. Use the decomposition described above. For example, given such a pair of functors, define $\underline{M}(X \leftarrow U \to Y) = M_*(U \to Y) \circ M^*(X \leftarrow U)$. Functoriality is equivalent to the condition on pullbacks and additivity is equivalent to the condition on disjoint unions. $\square$

For a map $f \in \mathcal{F}_G$, $M_* f$ is called the *transfer map* induced by f, and $M^* f$ is called the *restriction map* induced by f. It can be checked, directly through the construction of $\mathscr{B}_G$ or using the universal property described above, that $\mathscr{B}_G^{\mathsf{op}} \cong \mathscr{B}_G$. Any Mackey functor $\underline{M}$ is completely determined by its restriction to the orbits because of the condition on disjoint unions. This means that we can represent Mackey functors by their *Lewis diagrams*, which are diagrams that show the image of each G-orbit and all the transfer and restriction maps. For example, a C_2-Mackey functor $\underline{M}$ is completely determined by the data

$$\begin{tikzcd}
\underline{M}(C_2/C_2) \arrow[d, bend right, "\mathrm{Res}"'] \\
\underline{M}(C_2/e) \arrow[u, bend right, "\mathrm{Tr}"'] \arrow[loop below]
\end{tikzcd}$$

The transfers and restrictions induced by the identity maps are omitted as they will be identity. The indicated restriction and transfer are the ones induced by the unique map $C_2/e \to C_2/C_2$. The looped arrow is both the restriction and transfer induced by the unique non-identity endomorphism on C_2/e. These coincide because for any G-orbit X and map $X \to X$ (which must always be an isomorphism), there is an isomorphism of spans

$$\begin{tikzcd}
 & X \arrow[dl, "="'] \arrow[dr, "f"] \arrow[dd, "f"] & \\
X & & X \\
 & X \arrow[ul, "f^{-1}"] \arrow[ur, "="'] &
\end{tikzcd}$$

In our case, $X = C_2/e$ and $f = f^{-1}$ is the swap map, so that the transfer induced by it is $\underline{M}$ applied to the span at the top and the restriction is $\underline{M}$ applied to the span at the bottom. For other groups, the transfer induced by f is the restriction induced by f^{-1}, but the information of one is found in the other. This gives an action of the *Weyl group* of H

$$WH := \mathrm{Aut}(G/H) \cong NH/H$$

on $\underline{M}(G/H)$ for every $H \leq G$, where NH denotes the normalizer of H in G.Thus, the arrows in a Lewis diagram are of three forms: the transfers and restrictions along non-endomorphisms and Weyl group actions on each object.

Example 6.15. For any mackey functor $\underline{M}$ and finite G-set X, we can produce a *precomposite Mackey functor* $\underline{M}_X$ as $\underline{M}_X(-) = \underline{M}(X \times -)$. This is a Mackey functor because $X \times -$ preserves pullbacks.

Example 6.16. For each $X \in \mathscr{B}_G$, we have the Mackey functor $\underline{A}_X = \mathscr{B}_G(-, X)$ represented by X. For $*$ the terminal G-set, we write $\underline{A}_* = \underline{A}$ and call this the *Burnside Mackey functor*. The notation here is consistent with that of the last example because $\mathscr{B}_G(X \times -, Y) \cong \mathscr{B}_G(-, X \times Y)$ and $X \times * \cong X$.

Example 6.17. Let N be a $\mathbb{Z}[G]$-module. Its *fixed point Mackey functor* $\underline{N}$ is defined by

$$\begin{aligned}
\underline{N}(X) &= \mathrm{Hom}_{\mathbb{Z}[G]}(X, N) \\
N^*(f : X \to Y) &= (q \mapsto q \circ f) \\
N_*(f : X \to Y) &= \left(p \mapsto \left(y \mapsto \sum_{x \in f^{-1}(y)} p(x) \right) \right)
\end{aligned}$$

i.e. $N^* = \mathrm{Hom}_{\mathbb{Z}[G]}(-, N)$ and transfers are defined by 'summing over fibers'. The terminology comes from the fact that

$$\underline{N}(G/H) \cong N^H$$

Example 6.18. Let N be an abelian group. The *constant N Mackey functor $\underline{N}$* is the fixed point Mackey functor of N regarded as a $\mathbb{Z}[G]$-module where G acts trivially. The restriction of the contravariant part N^* to the G-orbits is then the constant N functor, hence the name.

Example 6.19. Let N be a $\mathbb{Z}[G]$-module. Its *fixed quotient Mackey functor $\underline{N}^{\mathsf{op}}$* is defined by

$$\begin{aligned}
N_* &= \mathbb{Z}[-] \otimes_{\mathbb{Z}[G]} N \\
N^*(f : X \to Y) &= \left(y \otimes n \mapsto \sum_{x \in f^{-1}(y)} x \otimes n \right)
\end{aligned}$$

where, for a G-set X, $\mathbb{Z}[X]$ denotes the $\mathbb{Z}[G]$-module with basis X. The terminology comes from the fact that

$$\underline{N}(G/H) \cong N/\sim$$

where $\sim$ is the equivalence relation generated by $n \sim hn$ for all $h \in H$, $n \in N$.

Example 6.20. Let N be an abelian group. The *coconstant N Mackey functor $\underline{N}^{\mathsf{op}}$* is the fixed quotient Mackey functor of N regarded as a $\mathbb{Z}[G]$-module where G acts trivially. The restriction of the covariant part N_* to the G-orbits is then the constant N functor, hence the name.

The Lewis diagrams of some of these can be drawn for $G = C_2$ as follows. We assign special symbols to some of them so that labelling figures with them is easier later on. $\mathbb{Z}_-$ denotes the $\mathbb{Z}[C_2]$-module $\mathbb{Z}$ with the sign action of C_2,

$$X = \begin{bmatrix} 1 & 2 \end{bmatrix}, \qquad Y = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and s denotes the map on $\mathbb{Z}^2$ that swaps the two copies of $\mathbb{Z}$.

Description	$\underline{A}$	$\underline{A}_{C_2/e}$	$\underline{\mathbb{Z}}$	$\underline{\mathbb{Z}}^{\mathsf{op}}$	$\underline{\mathbb{Z}_-}$	$\underline{\mathbb{Z}_-}^{\mathsf{op}}$
Symbol			$\square$	$\overline{\square}$	$\odot$	$\overline{\odot}$
Lewis Diagram	$\begin{array}{c}\mathbb{Z}^2\\ X\downarrow\uparrow Y\\ \mathbb{Z}\\ \circlearrowleft\\ 1\end{array}$	$\begin{array}{c}\mathbb{Z}\\ \Delta\downarrow\uparrow\nabla\\ \mathbb{Z}^2\\ \circlearrowleft\\ s\end{array}$	$\begin{array}{c}\mathbb{Z}\\ 1\downarrow\uparrow 2\\ \mathbb{Z}\\ \circlearrowleft\\ 1\end{array}$	$\begin{array}{c}\mathbb{Z}\\ 2\downarrow\uparrow 1\\ \mathbb{Z}\\ \circlearrowleft\\ 1\end{array}$	$\begin{array}{c}0\\ 0\downarrow\uparrow 0\\ \mathbb{Z}\\ \circlearrowleft\\ -1\end{array}$	$\begin{array}{c}\mathbb{Z}/2\mathbb{Z}\\ 0\downarrow\uparrow 1\\ \mathbb{Z}\\ \circlearrowleft\\ -1\end{array}$

Figure 6.1: Examples of C_2-Mackey functors; Note that $\underline{A}_{C_2/e} \cong \underline{\mathbb{Z}}_{C_2/e}$

Maps of Mackey functors are natural transformations between them. The category of Mackey functors, denoted $\mathscr{M}_G$, is then a category of Ab-valued additive functors and hence an abelian category in which direct sums, kernels and cokernels are computed pointwise. Moreover, Cartesian product induces a symmetric monoidal product on $\mathscr{B}_G$, which gives the category of Mackey functors a closed symmetric monoidal structure via Theorem 3.10.

Definition 6.21. The *box product* $\square$ is the Day convolution product induced on the category of G-Mackey functors by the Cartesian product on $\mathscr{B}_G$. The internal hom of Mackey functors is denoted $\underline{\mathrm{Hom}}(-,-)$.

Proposition 6.22. Let X, Y be finite G-sets, N, P modules over $\mathbb{Z}[G]$, and $\underline{M}, \underline{M}_1, \underline{M}_2 \in \mathscr{M}_G$. There are natural isomorphisms

(i) $\underline{A}_X \oplus \underline{A}_Y \cong \underline{A}_{X \amalg Y}$.

(ii) $\underline{A}_X \square \underline{M} \cong \underline{M}_X$.

(iii) $\underline{\mathrm{Hom}}(\underline{A}_X, \underline{M}) \cong \underline{M}_X$

(iv) $(\underline{M}_1 \square \underline{M}_2)(G/e) \cong \underline{M}_1(G/e) \otimes \underline{M}_2(G/e)$.

Proof. (i) Applying the universal property of representable functors,

$$\mathrm{Nat}(A_{X\amalg Y}, -) \cong -(X \amalg Y) \cong -(X) \oplus -(Y) \cong \mathrm{Nat}(A_X, -) \oplus \mathrm{Nat}(A_Y, -)$$

(ii) As mentioned in the proof of Theorem 3.10, this is true when $\underline{M}$ is representable and follows in the general case by applying Theorem 3.2.

(iii) $\underline{\mathrm{Hom}}(\underline{A}_X, \underline{M})(Y) \cong \mathrm{Hom}(\underline{A}_{X\times Y}, \underline{M}) \cong \underline{M}(X \times Y) \cong \underline{M}_X(Y)$.

(iv) Check that this is true for the functors $\underline{A}_X$ using their description as representables and then extend the result using Theorem 3.2.

$\square$

6.3 Equivariant Homology and Cohomology

Equivariant homology theory was deferred to this chapter because it is stable and hence should be valued in Mackey functors rather than coefficient systems even for G-spaces. We want our homology functor to be a generalized equivariant homology theory. That is, it should be a collection of functors $\widetilde{E}_V : \mathrm{Ho}\,\mathsf{Top}_*^G$ which takes wedge sums to direct sums, takes cofiber sequences to exact sequences, and has a suspension isomorphism

$$E_\bullet(-) \cong E_{\bullet+V}(\Sigma^V -)$$

for every $V \in RO(G)$. We first define an integer-graded cellular homology theory, and later extend it to an $RO(G)$-graded one.

Let X be a G-CW complex. Define its nth chain Mackey functor with coefficients to be the free Mackey functor generated by its n-cells

$$\underline{C}_n(X) = \bigoplus_{\substack{\sigma \text{ an } n\text{-cell} \\ \text{of type } G/H}} \underline{A}_{G/H}\{\sigma\}$$

where $\underline{A}_{G/H}\{\sigma\}$ is a copy of $\underline{A}_{G/H}$ freely generated by $\sigma \in \underline{A}_{G/H}(G/H)$. The boundary map

$$d_n : \underline{C}_n(X) \to \underline{C}_{n-1}(X)$$

on the summand corresponding to σ is described as follows, generalizing the definition of the boundary operators in non-equivariant cellular homology. Let

$$\phi : G/H \times S^{n-1} \to X_{n-1}$$

denote the attaching map of σ. Composing this with the quotient $X_{n-1} \to X_{n-1}/X_{n-2}$ we get a map from $G/H \times S^{n-1}$ to a wedge of $(G/K)_+ \wedge S^{n-1}$ for various orbits G/K. Each wedge summand here corresponds to an $(n-1)$-cell τ of type G/K. Define $\phi_\tau : G/H \times S^{n-1} \to (G/K)_+ \wedge S^{n-1}$ to be the composite

$$G/H \times S^{n-1} \xrightarrow{\phi} X^{n-1} \to X^{n-1}/X^{n-2} \to (G/K)_+ \wedge S^{n-1}$$

where the last map is the one that collapses all summands other than the one corresponding to τ. We call ϕ_t the *collapsed adjoint attaching map*. This map is G-homotopic to a basepoint-preserving map, which makes it factor through $(G/H)_+ \wedge S^{n-1}$. Therefore it represents a stable homotopy class of maps $G/H \to G/K$ and hence is a morphism in $\mathrm{Ho}\,\mathscr{D} \simeq \mathscr{B}_G$. We define the boundary operator

$$\underline{A}_{G/H}\{\sigma\} \to \bigoplus_{\substack{\tau \text{ an } (n-1)\text{-cell} \\ \text{of type } G/K}} \underline{A}_{G/K}\{\tau\}$$

to be the sum of the maps $\underline{A}_{G/H}\{\sigma\} \to \underline{A}_{G/K}\{\tau\}$ induced by ϕ_τ. This is well defined because all but finitely many of these ϕ_τ must be nullhomotopic by the compactness of S^{n-1}. Here, the 'map induced by $G/H \to G/K$' refers to the natural transformation $\underline{A}_{G/H} \to \underline{A}_{G/K}$ defined levelwise as post-composition by the map of $\operatorname{Ho}\mathscr{D}$ represented by ϕ_τ.

We can also define homology and cohomology with arbitrary coefficients.

Definition 6.23. Let X be a G-CW complex, $\underline{M} \in \mathscr{M}_G$, and $\underline{C}_\bullet(X)$ the chain complex of Mackey functors described above. The *cellular chain complex for X with coefficients in $\underline{M}$* is

$$\underline{C}_\bullet(X; \underline{M}) := \underline{C}_\bullet(X) \,\Box\, \underline{M}$$

The *nth homology Mackey functor for X with coefficients in $\underline{M}$*, denoted by $\underline{H}_n(X; \underline{M})$, is the nth homology of this chain complex.

The *cellular cochain complex for X with coefficients in $\underline{M}$* is

$$\underline{C}^\bullet(X; \underline{M}) := \underline{\operatorname{Hom}}(\underline{C}_\bullet(X), \underline{M})$$

The *nth cohomology Mackey functor for X with coefficients in $\underline{M}$*, denoted by $\underline{H}^n(X; \underline{M})$, is the nth cohomology of this cochain complex.

The hard part of any computation here would be to figure out the boundary or coboundary maps. For this, we need to figure out the map of Mackey functors induced by ϕ_τ as a representative of a map in $\operatorname{Ho}\mathscr{D}$. Since we have a more algebraic description $\mathscr{B}_G$ of $\operatorname{Ho}\mathscr{D}$, the first step is to figure out what map ϕ_τ represents in $\mathscr{B}_G$. Because of Remark 6.13 and the fact that ϕ_τ are maps between trivial suspensions of G-orbits (trivial in the sense that we are smashing by the trivial G-representation), we expect the map in $\mathscr{B}_G$ corresponding to ϕ_τ to be a linear combination of spans of the type $G/H = G/H \to G/K$. This is indeed what we get.

A map $(G/H)_+ \wedge S^n \to (G/K)_+ \wedge S^n$ is equivalent to an H-map $S^n \to (G/K)_+ \wedge S^n$, which is equivalent to a non-equivariant map $S^n \to ((G/K)_+ \wedge S^n)^H$. The codomain here is always a wedge sum of spheres. From non-equivariant theory, the (stable or unstable) homotopy class of such a map is completely determined by the degrees of its composites with the maps that collapse all but one wedge summand and the addition of stable homotopy classes of such maps can be carried out by simply adding these degrees. This allows us to decompose the stable homotopy class of ϕ_τ above into a sum of maps for which all but one of these degrees are zero and the non-zero degree is ± 1.

Now take one of these summands $f : S^n \to (G/K)_+ \wedge S^n$ as above and suppose that it has degree 1 onto the wedge summand corresponding to the coset gK. Hence this is homotopic to the map which maps homeomorphically onto that wedge summand. Viewing f again as a map $(G/H)_+ \wedge S^n \to (G/K)_+ \wedge S^n$, we see that it is now an n-fold suspension of a map $(G/H)_+ \to (G/K)_+$. Under the equivalence $\operatorname{Ho}\mathscr{D} \simeq \mathscr{B}_G$, this corresponds to the span $G/H = G/H \to G/K$, where the map on the right is the one above. Thus, we have decomposed the stable homotopy class represented by ϕ_τ into a sum of spans.

Now we need to figure out what map of Mackey functors is induced by said spans. In the case of homology with coefficients in $\underline{A}$, this span is supposed to induce a map from $\underline{A}_{G/H}(X) \cong \underline{A}(G/H \times X)$ to $\underline{A}_{G/K}(X) \cong \underline{A}(G/K \times X)$ for each X. Using this and the representable functor description of $\underline{A}_{G/H}$, we can view the induced map either as post-composition by the given span or the functor $\underline{A}$ applied to it. In the second case, this is exactly the description of the transfer map induced by $G/H \to G/K$.

For the cellular chain complex with coefficients in some $\underline{M} \in \mathscr{M}_G$, we need to take the box product with $\underline{M}$. Using the natural isomorphisms of Proposition 6.22, we get a commutative square

$$\begin{array}{ccc} (\underline{A}_{G/H} \square \underline{M})(X) & \longrightarrow & (\underline{A}_{G/K} \square \underline{M})(X) \\ \wr\| & & \wr\| \\ \underline{M}(G/H \times X) & \xrightarrow[tr]{} & \underline{M}(G/K \times X) \end{array}$$

which again identifies the map that we are concerned with as the transfer map induced by the given map of orbits. Therefore, the boundary maps will always be linear combinations of transfers. Lastly, for the cellular cochain complex, we again have natural isomorphisms

$$\begin{array}{ccc} \underline{\mathrm{Hom}}(\underline{A}_{G/H}, \underline{M})(X) & \longleftarrow & \underline{\mathrm{Hom}}(\underline{A}_{G/K}, \underline{M})(X) \\ \wr\| & & \wr\| \\ \underline{M}(G/H \times X) & \xleftarrow[res]{} & \underline{M}(G/K \times X) \end{array}$$

identifying the map with the map induced by the span $G/K \leftarrow G/H = G/H$, which is exactly the restriction induced by $G/H \to G/K$. Therefore, the coboundary maps will always be linear combinations of restrictions.

We extend these to $RO(G)$-graded cohomology theories using duality (briefly discussed at the end of Section 4.3). The cellular chains and cochains can be defined not only for G-CW complexes, but also for any G-CW spectrum (see [Maya, Section XII.7]). We need to define $\underline{H}_V(-;\underline{M})$ for any $V \in RO(G)$. Let $V = W_1 - W_2$ where $W_i \in \mathcal{R}_G$. Let S^{-W_2} be the dual of S^{W_2} constructed as a G-CW spectrum. Define $S^V := S^{W_1 - W_2}$ (which does not depend on the choice of W_1, W_2). Then for any G-CW complex (or spectrum) X, we define

$$\underline{C}_{n-V}(X;\underline{M}) := \underline{C}_n(S^V \wedge X;\underline{M})$$

ensuring that the suspension isomorphism will hold. Cohomology is extended in the same way. Another way to do this extension is demonstrated in Proposition 6.24 and the discussion preceeding it.

An equivariant version of the Brown representability theorem also holds and implies a version of Corollary 4.8 for any $RO(G)$-graded cohomology theory. In this case, it is more complicated to infer the structure maps from the suspension isomorphism than in Corollary 4.8 because the suspension isomorphisms only give us weak equivalences of the representing objects uniquely up to homotopy, whereas the definition of a G-spectrum requires these to be on-the-nose. See [Maya, Theorem XIII.3.1] for the proof.

We then have, corresponding to reduced cellular cohomology theory with coefficients in $\underline{M}$, a representing G-spectrum $H\underline{M}$. We call this the *Eilenberg-Maclane G-spectrum* for the Mackey functor $\underline{M}$. Computing its Vth homotopy Mackey functor would then give us the 0th reduced cohomology Mackey functor of S^V, or equivalently the $-V$th reduced cohomology Mackey functor of S^0, with coefficients in $\underline{M}$. Therefore,

$$\underline{\pi}_V(H\underline{M}) = \begin{cases} \underline{M} & \text{if } V = 0 \\ 0 & \text{otherwise} \end{cases}$$

It can also be checked that $\underline{H}_\bullet(-;\underline{M}) \cong \mathcal{SH}^G(S^\bullet, - \wedge H\underline{M})$ by checking this on the representation spheres.

6.4 Homology of a Point

We compute the $RO(C_2)$-graded homology C_2-Mackey functors for a point treated as a space with trivial C_2-action with coefficients in the constant $\mathbb{Z}$ Mackey functor $\underline{\mathbb{Z}}$. This computation will be used in the final computation of this thesis in Section 8.2. Every orthogonal C_2-representation can be written as a direct sum $p + q\sigma$ of p copies of the trivial representation and q copies of the sign representation. Therefore $RO(C_2)$ is the abelian group

$$\{p + q\sigma \mid p, q \in \mathbb{Z}\} \cong \mathbb{Z}^2$$

We have the following isomorphisms, where $*$ denotes the singleton space.

$$\begin{aligned} \underline{H}_{p+q\sigma}(*) &\cong \underline{\widetilde{H}}_{p+q\sigma}(S^0) \\ &\cong [S^{p+q\sigma}, S^0 \wedge H\underline{\mathbb{Z}}] \\ &\cong [S^p, S^{-q\sigma} \wedge H\underline{\mathbb{Z}}] \\ &\cong [S^{q\sigma}, S^{-p} \wedge H\underline{\mathbb{Z}}] \end{aligned}$$

Therefore, we have converted the computation of one $RO(G)$-graded homology into many integer-graded computations of homologies and cohomologies depending on the sign of q.

Proposition 6.24. Let $*$ denote the terminal C_2-space. Then

$$\begin{aligned}\underline{H}_{p+q\sigma}(*) &\cong \pi_{p+q\sigma}(H\underline{\mathbb{Z}}) \\ &\cong \widetilde{\underline{H}}_p(S^{-q\sigma}) \\ &\cong \widetilde{\underline{H}}^{-p}(S^{q\sigma})\end{aligned}$$

To compute these using cellular chains, we need C_2-CW complex structures on $S^{n\sigma}$ for every $n \geq 0$.

There will be two 0-cells, one representing 0 and one the point at infinity, both of type C_2/C_2. This gives us $S^{0\rho} = S^0$. Then we attach a 1-cell of type C_2/e along the attaching map $C_2/e \times S^0 \to S^0$ given by projection onto the second coordinate. These represent the positive and negative halves of the same axis. Continuing this way, we will end up with a structure in which there are two 0-cells of type C_2/C_2 and one i-cell of type C_2/e for each $i \in \{1, \ldots, n\}$. Therefore, the cellular chain complex with coefficients in $\underline{\mathbb{Z}}$ would be

$$\begin{array}{ccccccccccc} 0 & & 1 & & 2 & & \cdots & & n-1 & & n \\ \underline{\mathbb{Z}} \oplus \underline{\mathbb{Z}} & \xleftarrow[d_1]{} & \underline{\mathbb{Z}}_{C_2/e} & \xleftarrow[d_2]{} & \underline{\mathbb{Z}}_{C_2/e} & \longleftarrow & \cdots & \longleftarrow & \underline{\mathbb{Z}}_{C_2/e} & \xleftarrow[d_n]{} & \underline{\mathbb{Z}}_{C_2/e} \end{array}$$

To compute the boundary maps, we will need to compute degrees of the attaching maps. The 1-cell has the adjoint attaching map $S^0 \to S^0$, where the second copy of S^0 is really to be thought of as two 0-cells $D_0 \sqcup D_0$. Quotienting by the -1-skeleton (which is empty and hence adds a disjoint point), we get $S^0 \vee S^0$. The composites with the two collapse maps then have degrees 1 and -1. Therefore the boundary map will be the transfer induced by $C_2/e \to C_2/C_2$ to the first summand and the negative of the same to the second summand.

For $n > 1$, we have the collapsed adjoint attaching map $S^{n-1} \to (C_2/e)_+ \wedge S^{n-1}$ for the n-cell. This has the degree assignment

$$\begin{aligned}\{e\} &\mapsto 1 \\ \{t\} &\mapsto (-1)^n\end{aligned}$$

Since the map onto the summand corresponding to e is homotopic to the identity and the map onto the other summand is homotopic to the antipodal map. Therefore the boundary map will be the transfer induced by the identity map on C_2/e plus or minus the transfer induced by the switch map on the same.

The chain complex for coefficients in $\underline{\mathbb{Z}}$, expanded out as Lewis diagrams, looks like

$$
\begin{tikzcd}
0 & 1 & 2 & \cdots & n-1 & n \\
\mathbb{Z}\oplus\mathbb{Z} \arrow[d, bend right, "1"'] & \mathbb{Z} \arrow[l, "{(2,-2)}"'] \arrow[d, bend right, "\Delta"'] & \mathbb{Z} \arrow[l, "0"'] \arrow[d, bend right, "\Delta"'] & \cdots \arrow[l, "2"'] & \mathbb{Z} \arrow[l] \arrow[d, bend right, "\Delta"'] & \mathbb{Z} \arrow[l, "1+(-1)^n"'] \arrow[d, bend right, "\Delta"'] \\
\mathbb{Z}\oplus\mathbb{Z} \arrow[u, bend right, "2"'] \arrow[loop below, "1"] & \mathbb{Z}\oplus\mathbb{Z} \arrow[l, "A"] \arrow[u, bend right, "\nabla"'] \arrow[loop below, "s"] & \mathbb{Z}\oplus\mathbb{Z} \arrow[l, "B_{-1}"] \arrow[u, bend right, "\nabla"'] \arrow[loop below, "s"] & \cdots \arrow[l, "B_1"] & \mathbb{Z}\oplus\mathbb{Z} \arrow[l] \arrow[u, bend right, "\nabla"'] \arrow[loop below, "s"] & \mathbb{Z}\oplus\mathbb{Z} \arrow[l, "B_{(-1)^n}"] \arrow[u, bend right, "\nabla"'] \arrow[loop below, "s"]
\end{tikzcd}
$$

where Δ denotes the diagonal map $\mathbb{Z} \to \mathbb{Z}\oplus\mathbb{Z}$, ∇ denotes the fold map, s denotes map that swaps the two copies, and

$$
A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}
$$

$$
B_\varepsilon = \begin{bmatrix} 1 & \varepsilon \\ \varepsilon & 1 \end{bmatrix}
$$

To figure these out, one has to use an explicit isomorphism $C_2/e \times C_2/e \cong C_2/e \sqcup C_2/e$ and apply additivity of the Mackey functor $\underline{\mathbb{Z}}$ to find the bottom row of groups along with all the maps in and out of all of them.

Similarly, we find that the cochain complex of $S^{n\rho}$ with coefficients in $\underline{\mathbb{Z}}$ is

$$
\begin{tikzcd}
0 & 1 & 2 & \cdots & n-1 & n \\
\mathbb{Z}\oplus\mathbb{Z} \arrow[r, "{[1\ \ -1]}"] \arrow[d, bend right, "1"'] & \mathbb{Z} \arrow[r, "0"] \arrow[d, bend right, "\Delta"'] & \mathbb{Z} \arrow[r, "2"] \arrow[d, bend right, "\Delta"'] & \cdots \arrow[r] & \mathbb{Z} \arrow[r, "1+(-1)^n"] \arrow[d, bend right, "\Delta"'] & \mathbb{Z} \arrow[d, bend right, "\Delta"'] \\
\mathbb{Z}\oplus\mathbb{Z} \arrow[r, "A^T"'] \arrow[u, bend right, "2"'] \arrow[loop below, "1"] & \mathbb{Z}\oplus\mathbb{Z} \arrow[r, "B_{-1}"'] \arrow[u, bend right, "\nabla"'] \arrow[loop below, "s"] & \mathbb{Z}\oplus\mathbb{Z} \arrow[r, "B_1"'] \arrow[u, bend right, "\nabla"'] \arrow[loop below, "s"] & \cdots \arrow[r] & \mathbb{Z}\oplus\mathbb{Z} \arrow[r, "B_{(-1)^n}"'] \arrow[u, bend right, "\nabla"'] \arrow[loop below, "s"] & \mathbb{Z}\oplus\mathbb{Z} \arrow[u, bend right, "\nabla"'] \arrow[loop below, "s"]
\end{tikzcd}
$$

where A^T denotes the transpose of A.

An important feature to note is that the bottom row of these diagrams are exactly the chain and cochain complexes for $S^{n\sigma}$ treated as a non-equivariant space. This is a consequence of the fact that a C_2-map out of $C_2/e \times S^n$ is equivalent to a non-equivariant map out of S^n.

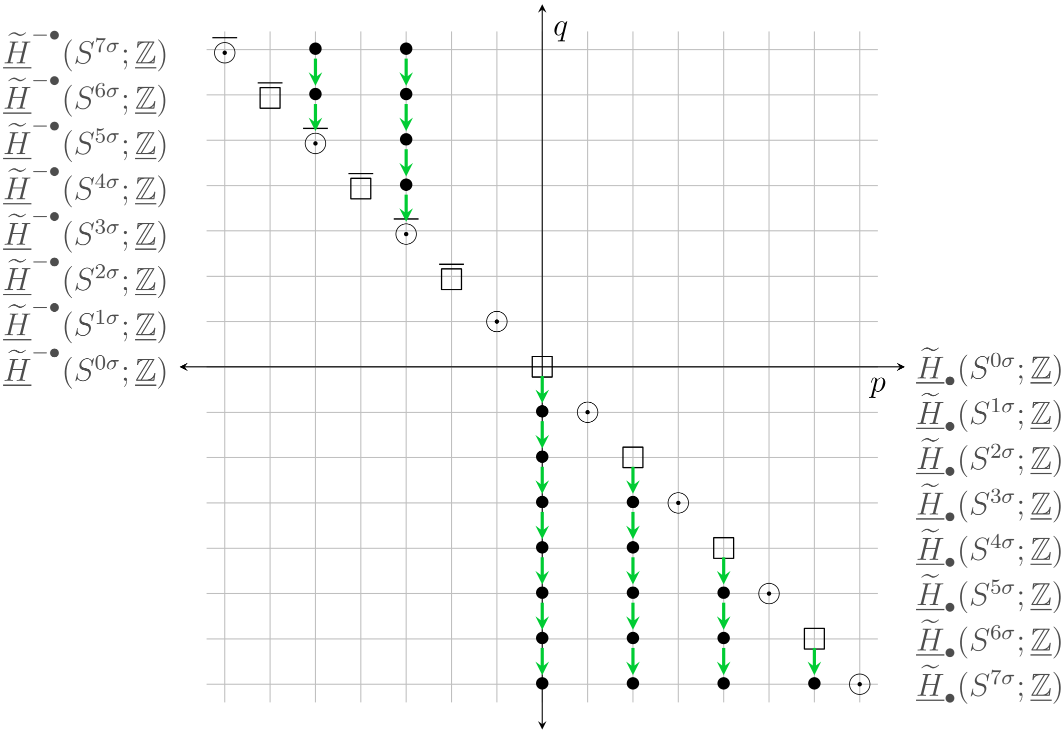
Now recall that we need to compute the reduced homology and cohomology for these, since our aim was to compute the unreduced versions for a point. The reduced theories, as for any generalized cohomology theories, are defined by taking the quotient by the split inclusion

$$
\underline{H}_\bullet(*) \to \underline{H}_\bullet(X)
$$

induced by any inclusion $* \to X$. Let this be the inclusion as either 0 or the point at ∞, making this a cellular inclusion. Then the effect on chain and cochain complexes is of killing the $\underline{\mathbb{Z}}$ summand corresponding to that point.

Finally, computing the homologies and cohomologies of the above complexes and plotting them according to Proposition 6.24, we get the following theorem.

Theorem 6.25. Let $*$ be the terminal C_2-space. Then its homology $\underline{H}_{p+q\sigma}(*;\underline{\mathbb{Z}})$ with coefficients in $\underline{\mathbb{Z}}$ is as shown in the following grid of Mackey functors.



See Fig. 6.1 for the meanings of the symbols. The green arrows describe part of the multiplicative structure and are explained in Section 6.5.

Again, note that evaluating all the Mackey functors in the grid at C_2/e, the bottom level of the Lewis diagrams, we are left with $\mathbb{Z}$ on each lattice point on the anti-diagonal as we would expect from the non-equivariant case.

6.5 Multiplicative Structure

Similar to the cross and cup products in non-equivariant (co)homology theory with coefficients in a ring, we get products on equivariant cohomology with coefficients in a monoid objects in $\mathscr{M}_G$, which is a Mackey functor $\underline{M}$ with maps $\underline{A} \to \underline{M}$ and $\underline{M} \mathbin{\Box} \underline{M} \to \underline{M}$ satisfying the appropriate associativity and identity conditions. Such a Mackey functor is called a *Green functor*.

This product can be described on cellular cohomology as follows. We first describe a cross product map

$$\underline{H}^{m-V}(X;\underline{M}) \,\square\, \underline{H}^{n-W}(X;\underline{M}) \to \underline{H}^{m+n-V-W}(X \times X;\underline{M})$$

and then compose it with the map induced by a cellular approximation of the diagonal $X \to X \times X$. This map above exists for both homology and cohomology and can be described at the level of (co)chain complexes. For homology, let

$$\begin{aligned} x &\in \underline{C}_m(S^V \wedge X;\underline{M})(G/H) \\ y &\in \underline{C}_n(S^W \wedge X;\underline{M})(G/K) \end{aligned}$$

be m- and n-cells of types G/H and G/K respectively. The G-CW structures on $S^V \wedge X$ and $S^W \wedge X$ give us a product G-CW structure on $S^{V+W} \wedge X \wedge X$, which contains a cell $x \wedge y$ of type $G/H \times G/K$. This product may not be an orbit; regardless, $x \wedge y$ is a union of cells and by the additivity of Mackey functors, the sum of those cells is of type $G/H \times G/K$. We extend this product to the case where x and y are not cells using the fact that the cells freely generate the chain Mackey functors to get a product map

$$\underline{C}_{m-V}(X;\underline{M}) \,\square\, \underline{C}_{n-W}(X;\underline{M}) \to \underline{C}_{m+n-V-W}(X;\underline{M} \,\square\, \underline{M})$$

Composing this with the map induced on the chains by $\underline{M} \,\square\, \underline{M} \to \underline{M}$, we land in $\underline{C}_{m+n-V-W}(X;\underline{M})$ which then gives us the cross product on homology. A similar construction is done for cohomology giving us the desired product.

In the case of the homology of a point, part of this multiplicative structure is indicated using the green arrows in Theorem 6.25. They indicate all the cases in which multiplication by the generator of $\underline{H}_{-\sigma}(*;\underline{\mathbb{Z}})$ takes the generator of the source of the arrow to the unique non-zero element at the top level of the target (notice that each Mackey functor that appears as a source of a green arrow can be generated by a single element in the top level of its Lewis diagram).

Therefore, in each occurrence of the green arrow, the source of the arrow boxed with $\underline{F}$ is $\underline{F}$ and the multiplication map from this box product to the target of the arrow being non-zero completely describes the multiplication.

7 Vector Bundles and Topological K-Theory

A vector bundle is a structure on a space that generalizes the tangent and normal bundles on manifolds. To say that B has a vector bundle structure over it is to say that each point of B has a vector space over it, which 'varies continuously' over B. For us, vector bundles will be used as homotopy invariants of spaces.

7.1 Vector Bundles as Homotopy Invariants

Definition 7.1. A *real vector bundle* over a space B is a map $p : E \to B$ of spaces such there exists an open cover $(U_\alpha)_{\alpha \in \Lambda}$ of B with homeomorphisms $\phi_\alpha : p^{-1}(U_\alpha) \cong U_\alpha \times \mathbb{R}^{n_\alpha}$ for some $n_\alpha \in \mathbb{N}$ satisfying the following conditions

(i) The triangle

$$\begin{tikzcd} p^{-1}(U_\alpha) \arrow[rr, "\phi"] \arrow[dr, "p"'] & & U_\alpha \times \mathbb{R}^{n_\alpha} \arrow[dl, "\pi_1"] \\ & U_\alpha & \end{tikzcd}$$

commutes for each $\alpha \in \Lambda$, where π_1 denotes the first projection.

(ii) For each pair $\alpha, \beta \in \Lambda$, the *transition map*

$$(\phi_\beta \circ \phi_\alpha^{-1}) \mid_{(U_\alpha \cap U_\beta) \times \mathbb{R}^{n_\alpha}} : (U_\alpha \cap U_\beta) \times \mathbb{R}^{n_\alpha} \to (U_\alpha \cap U_\beta) \times \mathbb{R}^{n_\beta}$$

is a linear isomorphism on each fiber $\{b\} \times \mathbb{R}^{n_\alpha}$ for $b \in U_\alpha \cap U_\beta$.

We call B the *base space*, E the *total space*, and p the *projection map* of the vector bundle. (i) is called the *local triviality* condition and each U_α is called a *trivializing neighbourhood*.

A *complex vector bundle* is defined similarly, except each occurrence of $\mathbb{R}^n$ is replaced with $\mathbb{C}^n$. Vector bundles locally look like projection maps from products, so we think of them as 'twisted products' with $\mathbb{R}^n$ (for example, the Mobius strip, if thought of as a strip of infinite width, is a real vector bundle over S^1). The condition on the transition maps implies that the n_α are constant over each connected component. If they are all equal to some $n \in \mathbb{N}$, we say that the bundle is n-dimensional. In the following, unless otherwise specified, every statement is about both real and complex vector bundles and k denotes either $\mathbb{R}$ or $\mathbb{C}$.

Each fiber $p^{-1}(b)$ in a bundle $p : E \to B$ is canonically a vector space. This comes from choosing a trivializing neighbourhood U_α containing b and transferring the vector space structure along the homeomorphism $\phi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times k^{n_\alpha}$. The condition on the transition maps ensures that this structure does not depend on the U_α chosen.

A morphism of vector bundles $p_1 \to p_2$ over B with total spaces E_1, E_2 respectively is a map $f : E_1 \to E_2$ which respects the projection maps in the sense that

$$p_2 f = p_1$$

and restricts to a linear map on each fiber. This is called an injection/surjection/isomorphism if f is one.

The following examples describe some important constructions of vector bundles. For the detailed constructions and proofs, see [MS, Section 3].

Example 7.2. The projection $\pi_1 : B \times k^n \to B$ is a vector bundle over B with trivializing cover $\{B\}$. This is called the *trivial vector bundle* over B of dimension n.

Example 7.3. Let $p : E \to B$ be a vector bundle and $f : B' \to B$ any map. Then the *pullback bundle* of p along f is the map $f^*p : E' \to B'$ in the pullback diagram

$$\begin{tikzcd} E' \arrow[r] \arrow[d, "f^*p"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\ B' \arrow[r, "f"'] & B \end{tikzcd}$$

It can be checked that this is always a vector bundle. In particular, if f is an inclusion of a subspace, then $E' = p^{-1}B'$ and f^*p is called the *restriction* of p to B', denoted $p|_{B'}$.

The direct sum and tensor product of vector spaces extend to give us these operations on vector bundles over B by applying these constructions fiberwise. In the following, let $p_i : E_i \to B$ be vector bundles for $i = 1, 2$.

Example 7.4. There exists a vector bundle $p_1 \oplus p_2$ over B with inclusions $p_i \to p_1 \oplus p_2$ such that each fiber $(p_1 \oplus p_2)^{-1}(b)$ is a direct sum of the images of the fibers $p_i^{-1}(b)$ under these inclusions. This is both the product and the coproduct in the category of vector bundles over B and is called the *Whitney sum* of p_1 and p_2.

Example 7.5. There exists a vector bundle $p_1 \otimes p_2$, called the *tensor product* of p_1 and p_2, with a map $f : p_1 \oplus p_2 \to p_1 \otimes p_2$ commuting with the projection maps and bilinear on each fiber (and hence not a map of vector bundles) such that for any vector bundle q over B and any map $g : p_1 \oplus p_2 \to q$ commuting with the projections and bilinear on each fiber, there is a unique factoring of g through f via a map of vector bundles. The fibers of $p_1 \otimes p_2$ are the tensor products of the fibers of p_i.

Let $\mathrm{Vect}_k X$ denote the set of isomorphism classes of vector bundles over X. Example 7.3 says that any map $f : X \to Y$ induces a map of sets $f^* : \mathrm{Vect}_k Y \to \mathrm{Vect}_k X$. Since a pullback of a pullback is the same as a pullback along the composite, Vect_k is a functor $\mathsf{Top}^{\mathsf{op}} \to \mathsf{Set}$. Moreover, f^* preserves the Whitney sum of vector bundles, which can be proved by checking that the Whitney sum satisfies the universal property of the appropriate pullback. The Whitney sum makes $\mathrm{Vect}_k X$ into a commutative monoid and hence Vect_k is a functor $\mathsf{Top}^{\mathsf{op}} \to \mathsf{CMon}$, the category of commutative monoids.

Proposition 7.6. Let $f, g : X \to Y$ be homotopic maps of paracompact topological spaces. Then $f^* = g^* : \mathrm{Vect}_k Y \to \mathrm{Vect}_k X$.

Proof. Pull back a vector bundle along a homotopy $H : X \times I \to Y$ to get a bundle over $X \times I$. From functoriality, the restrictions of this bundle to the top and bottom faces of $X \times I$ are the same as the pullbacks of the original bundle along f and g respectively. The paracompactness of X then implies that these two restrictions are isomorphic. See [Hatb, Theorem 1.6] for details. □

Example 7.7. A way to construct vector bundles is using the *clutching construction*. Let X be a CW complex and A_1, A_2 subcomplexes that cover it. Let $p_i : E_i \to A_i$ be vector bundles for $i = 1, 2$ whose restrictions p_i' to the intersection $A_1 \cap A_2$ are isomorphic via a map $\phi : p_1^{-1}(A_1 \cap A_2) \to p_2^{-1}(A_1 \cap A_2)$. We can then glue these bundles along ϕ to get a bundle on X. That is, we define E as the pushout

$$\begin{tikzcd}
p_1^{-1}(A_1 \cap A_2) \arrow[r, "\phi"] \arrow[d] & p_2^{-1}(A_1 \cap A_2) \arrow[r] & E_2 \arrow[d] \\
E_1 \arrow[rr] & & E \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

and the glued vector bundle $p : E \to X$ is described by the natural map from the pushout. We need A_i to be subcomplexes rather than just arbitrary subspaces to ensure that local triviality is satisfied. We could instead have asked that the A_i be open subspaces, in which case the proof of local triviality would be even simpler. The tuple (p_1, p_2, ϕ) is called the *clutching data* of a vector bundle over X. See the argument in [AB, Page 234-235].

This clutching data is often given in the following form. Suppose first that the restrictions of the bundles to $A_1 \cap A_2$ are trivial bundles of dimension n. Fix bundle identifications

$$f_i : p_i^{-1}(A_1 \cap A_2) \cong (A_1 \cap A_2) \times k^n$$

ϕ is then viewed as a bundle automorphism of $(A_1 \cap A_2) \times k^n$. Currying this map, we have a map $A_1 \cap A_2 \to \mathrm{Map}(k^n, (A_1 \cap A_2) \times k^n)$. This is equivalent to two maps $A_1 \cap A_2 \to \mathrm{Map}(k^n, A_1 \cap A_2)$ and $A_1 \cap A_2 \to \mathrm{Map}(k^n, k^n)$. Since bundle isomorphisms are fiberwise linear, the former is redundant and the latter is a map $A_1 \cap A_2 \to \mathrm{GL}_n(k)$. Thus the isomorphism ϕ is completely specified by a continuous map $A_1 \cap A_2 \to \mathrm{GL}_n(k)$, which is called the *clutching function* for gluing p_1 and p_2. In the case where the restrictions to the intersection are non-trivial, we cover the intersection by trivializing neighbourhoods and give clutching functions on each trivialization separately in a compatible way (satisfying an appropriate cocycle condition).

Over paracompact spaces, homotopic clutching functions produce isomorphic vector bundles: the homotopy between them can be used to produce a clutching function for a bundle over $X \times I$, after which we can use the argument in Proposition 7.6.

The clutching construction shows that as a Set-valued functor on pointed connected CW complexes, $\mathrm{Vect}_k : (\mathrm{Ho}\, CW_*^c)^{\mathsf{op}} \to \mathsf{Set}$ satisfies the Mayer-Vietoris axiom of Lemma 4.7. However, Vect_k is not the correct functor for pointed spaces as it does not take wedge sums to direct products. For example, $* \vee * \cong *$, the terminal space, but $\mathrm{Vect}_k(* \vee *) \cong \mathbb{N} \cong \mathrm{Vect}_k(*)$.

To fix this, we define a 'reduced' functor $\widetilde{\mathrm{Vect}}_k$ by defining $\widetilde{\mathrm{Vect}}_k(X)$ as the kernel of the map $\mathrm{Vect}_k(X) \to \mathrm{Vect}_k(*)$ induced by the inclusion of the basepoint into X. Equivalently, we get this by quotienting $\mathrm{Vect}_k(X)$ by the submonoid of trivial bundles. This functor then satisfies the conditions for Lemma 4.7 and is hence represented by a CW complex.

Definition 7.8. The *classifying space for real vector bundles BO* is the CW complex representing $\widetilde{\mathrm{Vect}}_{\mathbb{R}}$. The *classifying space for complex vector bundles BU* is the CW complex representing $\widetilde{\mathrm{Vect}}_{\mathbb{C}}$.

The O and U stand for 'orthogonal' and 'unitary' respectively. This is because the transition maps of an n-dimensional vector bundle come from the groups $\mathrm{GL}_n(k)$, but by Gram-Schmidt orthonormalization we may assume that they are elements of $O(n)$ or $U(n)$. $BO(n)$ and $BU(n)$ are defined as spaces that represent the functors taking a CW complex to the set of isomorphism classes of principal $O(n)$- or $U(n)$-bundles on it. By the theory of associated bundles, these functors are naturally isomorphic to the ones that take a CW complex to the set of isomorphism classes of n-dimensional real or complex bundles.

7.2 Real and Complex K-Theory

Since $\widetilde{\mathrm{Vect}}_k$ is representable, it will take cofiber sequences to exact sequences of pointed sets (and hence of commutative monoids). This suggests that we should be able to construct a generalized cohomology theory out of these. The first issue is that $\widetilde{\mathrm{Vect}}_k$ only satisfies the axioms that we need on paracompact spaces and CW complexes. This can be fixed by applying the CW approximation functor Γ before applying $\widetilde{\mathrm{Vect}}_k$ as CW complexes are paracompact. The second issue is that $\widetilde{\mathrm{Vect}}_k$ takes values in commutative monoids instead of abelian groups. This can be fixed by then applying the Grothendieck group functor G to the resulting monoid (REF Grothendieck group). Therefore we define

$$\widetilde{KO}^0 = G \circ \widetilde{\mathrm{Vect}}_{\mathbb{R}} \circ \Gamma : \mathsf{Top}^{op} \to \mathsf{Ab}$$

in the real case and

$$\widetilde{K}^0 = G \circ \widetilde{\mathrm{Vect}}_{\mathbb{C}} \circ \Gamma : \mathsf{Top}^{op} \to \mathsf{Ab}$$

in the complex case. We can infer that these take weak equivalences to isomorphisms, wedge sums to direct products, and cofiber sequences to exact sequences (see [Hatb, Chapter 2] for direct proofs). We could also first define unreduced versions of these functors using Vect_k instead of $\widetilde{\mathrm{Vect}}_k$ and then define the reduced versions by taking the kernels of the maps induced by the inclusion of the basepoint. These definitions are equivalent because the short exact sequence involving the reduced and unreduced functors is always split by the map induced by the projection onto the singleton.

For these to be cohomology theories, we need to define functors $\widetilde{KO}^n, \widetilde{K}^n : \mathsf{Top} \to \mathsf{Ab}$, all satisfying these axioms along with the suspension axiom. We define these functors in the negative degrees using the suspension axiom itself.

$$\widetilde{KO}^{-n}(X) := \widetilde{KO}^0(\Sigma^n X), \qquad \widetilde{K}^{-n}(X) := \widetilde{K}^0(\Sigma^n X)$$

This is almost a cohomology theory. The following major result allows us to extend these definitions to positive degrees.

Theorem 7.9 (Bott periodicity)**.** There are natural isomorphisms

$$\widetilde{KO}^0(\Sigma^8 X) \to \widetilde{KO}^0(X), \qquad \widetilde{K}^0(\Sigma^2 X) \to \widetilde{K}^0(X)$$

Proof. See [AB] or [Hatb, Section 2.2] for proofs of the K statement. The statement about KO is harder and is best proved using KR-theory (Section 7.3). □

We can therefore define $\widetilde{K}^n = \widetilde{K}^{-n}$ to get a cohomology theory called *reduced complex K-theory* or simply K-theory. Similarly extending KO to positive degrees by setting $\widetilde{KO}^n = \widetilde{KO}^{n-8}$, we get a cohomology theory called *reduced real K-theory* or KO-theory.

By Lemma 4.7, there exist spectra KU and KO representing complex and real K-theory respectively. In particular, this means that

$$\pi_n(KU) \cong [S^n, KU] \cong \widetilde{K}^{-n}(S^0) \cong \begin{cases} \widetilde{K}^0(S^0) & \text{if } n \text{ is even} \\ \widetilde{K}^0(S^1) & \text{if } n \text{ is odd} \end{cases}$$

A complex vector bundle over S^0 is exactly a pair of complex vector bundles, whose monoid under addition is $\mathbb{N}^2$. Reducing and taking the Grothendieck group, we get $\mathbb{Z}$. Using clutching functions, one can show that there are no non-trivial complex vector bundles over S^1. Indeed, covering S^1 by two intervals A, B with intersection S^0, any complex vector bundle p over S^1 is glued together from the clutching data $(p|_A, p|_B, \mathrm{id}_{p^{-1}(A\cap B)})$. Since A, B are contractible, p_A and p_B are trivial of some dimension n. Then p is completely determined by a clutching function $A \cap B \simeq S^0 \to \mathrm{GL}_n(\mathbb{C})$. $\mathrm{GL}_n(\mathbb{C})$ is path connected for $n \geq 1$, so any such clutching function is nullhomotopic, and hence p must be trivial.

Therefore, the homotopy groups of KU are $\mathbb{Z}$ in even degrees and 0 in odd degrees. In this way, we can use the periodicity theorem to determine all complex vector bundles over all spheres (up to addition of trivial bundles - a non-trivial bundle and a trivial bundle can sum to a trivial bundle). Similar calculations can be done for KO. We will need the following result in Section 8.2.

Proposition 7.10. $\pi_3(KO) \cong \widetilde{KO}^0(S^3) \cong 0.$

Proof sketch. Cover S^3 by two copies A, B of D^3 intersecting in S^2. As above, since A, B are contractible, any real vector bundle p over S^3 is completely determined by a clutching function $S^2 \to \mathrm{GL}_n(\mathbb{R})$ for some $n \in \mathbb{N}$. The bundle depends only on the homotopy class of this function and $\mathrm{GL}_n(\mathbb{R})$ deformation retracts to $O(n)$ so we can equivalently use a clutching function $S^2 \to O(n)$. We can assume that its image lies in the positive determinant component because if not, we can apply a fiberwise reflection to the trivialization of $p|_A$ and compose the clutching function with multiplication by the same reflection. Therefore, a non-trivial bundle p of dimension n implies the existence of a non-zero element of $\pi_2(SO(n))$. The covering map

$$S^3 \cong SU(2) \to SO(3)$$

shows that $\pi_2(SO(3)) \cong 0$ and the fiber sequence

$$SO(n-1) \to SO(n) \to S^{n-1}$$

shows the same for $SO(n)$ for $n \geq 3$ by induction. Thus, adding a trivial bundle of dimension 3 to any bundle gives the trivial bundle, which completes the proof. □

7.3 KR-Theory

There is a certain feature called a *complex orientation* that K-theory has but KO-theory does not. This makes working with K a lot easier than those with KO. For example, there exists a universal complex oriented cohomology theory MU which allows us to study some aspects of all such theories at once. There is also a correspondence between certain complex oriented cohomology theories and algebraic structures called *formal groups*. There is no good real analogue of this theory.

KR-theory provides, in particular, a way to study real K-theory using complex K-theory. It is a C_2-equivariant cohomology theory constructed from KU using the conjugation action of C_2 on $\mathbb{C}$, and so contains the information of KO in its fixed points. More precisely, it is defined as follows. The action of the non-identity element of C_2 will always be denoted by an overline. Since the action of this element on $\mathbb{C}$ is complex conjugation, this will also be used to denote complex conjugation.

> **Definition 7.11.** Let X be a C_2-space. A *vector bundle with conjugation* over X is a C_2-map $p : E \to X$ which is non-equivariantly a complex vector bundle such that the C_2-action restricted to the fiber of a point b
>
> $$\bar{\cdot} : p^{-1}(b) \to p^{-1}(\bar{b})$$
>
> coincides with complex conjugation. That is, for any $\alpha \in \mathbb{C}$, $v \in p^{-1}(b)$, we have
>
> $$\overline{\alpha v} = \overline{\alpha}\,\overline{v}$$

Atiyah introduced these vector bundles and their associated cohomology theory in his paper [Ati]. There, he called C_2-spaces *real spaces* and such bundles *real vector bundles over a real space*. This is why the associated cohomology theory on pointed C_2-spaces, which we construct in the following, is now often called reduced Real K-theory (with a capital R) or KR-theory. We avoid using the word 'real' to avoid confusion.

We can define Whitney sums and tensor products of such vector bundles as we did before. We are able to mimic much of the theory from the previous sections, except now our cohomology theory should be valued in C_2-Mackey functors.

Now suppose X is a pointed space, which we also treat as a C_2-space with the trivial action. A vector bundle with conjugation over X would be a complex vector bundle in which each fiber has a complex conjugation on it. There is an isomorhism $\widetilde{\mathrm{Vect}}_{C_2}(X) \cong \widetilde{\mathrm{Vect}}_{\mathbb{R}}(X)$ given by taking the fixed points under conjugation, with inverse given by tensoring a given bundle with the trivial 1-dimensional complex vector bundle over $\mathbb{R}$.

We can also study complex vector bundles over X using vector bundles with conjugation using the space $\mathrm{Ind}_e^{C_2} X = (C_2/e)_+ \wedge X$. There is an isomorphism $\widetilde{\mathrm{Vect}}_{C_2}(\mathrm{Ind}_e^{C_2} X) \cong \widetilde{\mathrm{Vect}}_{\mathbb{C}}(X)$. This isomorphism is given by restricting a given bundle on $\mathrm{Ind}_e^{C_2} X$ to the subspace $\{e\} \times X$, with inverse given by inducing up a bundle $E \to X$ by taking $(C_2/e)_+ \wedge E \to (C_2/e)_+ \wedge X$. In this way, the study of these bundles will subsume that of both complex and real vector bundles.

The C_2-cohomology theory $\widetilde{\underline{KR}}^\bullet$ is constructed as follows. For a C_2-space X, define $\mathrm{Vect}_{C_2}(X)$ as the commutative monoid of vector bundles with conjugation over X under Whitney sum. If X is pointed, define $\widetilde{\mathrm{Vect}}_{C_2}(X)$ as the kernel of the map $\mathrm{Vect}_{C_2}(X) \to \mathrm{Vect}_{C_2}(*)$. For a finite G-set T, define a C_2-Mackey functor

$$\widetilde{\underline{KR}}^0(X)(T) = G(\widetilde{\mathrm{Vect}}_{C_2}(\Gamma(T_+ \wedge X)))$$

where Γ is the G-CW approximation functor and G is the Grothendieck group functor. For a map $f : T_1 \to T_2$ of C_2-sets, the restriction along f is given by pullback along $f_+ \wedge X$. For the transfer along f, suppose we have a bundle over $(T_1)_+ \wedge X$. The transferred bundle on $(T_2)_+ \wedge X$ is then defined component by component: if $t \in T_2$, then the bundle over the subspace $\{t\} \times X$ is defined as the Whitney sum of the bundles over $\{s\} \times X$ for all $s \in f^{-1}(t)$ (i.e. summing over the fibers, similar to Example 6.18; in fact, the proof that this is a Mackey functor will be essentially the same).

We define $\widetilde{\underline{KR}}^{-V}(X) = \widetilde{\underline{KR}}^0(\Sigma^V X)$ for each $V \in \mathcal{R}_{C_2}$. To extend this to a $RO(C_2)$-graded cohomology theory, we need a version of the Bott periodicity theorem.

Theorem 7.12 (Bott periodicity)**.** There is a natural isomorphism $\widetilde{\underline{KR}}^{-V} \to \widetilde{\underline{KR}}^{-V-\rho}(X)$ where ρ denotes the regular representation of C_2.

Proof. See [Ati, Theorem 2.3] □

This allows extending $\widetilde{\underline{KR}}^\bullet$ to a cohomology theory since every $V \in RO(C_2)$ can be written as $n\rho - W$ for some $n \in \mathbb{Z}$ and $W \in \mathcal{R}_{C_2}$. This is called (reduced) KR-theory, and its representing C_2-spectrum is denoted KR. The underlying non-equivariant spectrum of this is $\mathrm{Res}_e^{C_2} KR \simeq KU$. Indeed, the stable homotopy class of a non-equivariant map $S^n \to KR$ is equivalent to one of a C_2-map $(C_2/e)_+ \wedge S^n \to KR$. This is equivalent to a vector bundle with conjugation over $(C_2/e)_+ \wedge S^n$, which is equivalent to complex vector bundle over S^n, which is equivalent to the stable homotopy class of a map $S^n \to KU$, proving that $\mathrm{Res}_e^{C_2} KR$ and KU are stably equivalent. Thus, the periodicity theorem for KU can be deduced from the above. Atiyah was also able to give a simple proof of the periodicity theorem for KO by applying the following to spaces with trivial C_2-actions.

Theorem 7.13. There is a natural isomorphism $\underline{KR}^V(X) \to \underline{KR}^{V-8}(X)$.

Proof. See [Ati, Theorem 3.10]. □

This is the periodicity theorem for KR mentioned in the preface.

Lastly, K, KO, and KR are all multiplicative cohomology theories. We sketch this for KR. We begin with a multiplication on the negative degrees. Let $a \in \widetilde{\underline{KR}}^{-V}(X)(S)$ and $b \in \widetilde{\underline{KR}}^{-W}(X)(T)$ for X a pointed C_2-space, $V, W \in \mathcal{R}_{C_2}$, and C_2-sets S, T. These then correspond to formal differences of vector bundles with conjugation. Suppose first that they correspond to actual bundles over the C_2-spaces

$$S_+ \wedge S^V \wedge X \quad \text{and} \quad T_+ \wedge S^W \wedge X$$

respectively. The smash product of these bundles is then a vector bundle with conjugation over

$$S_+ \wedge T_+ \wedge S^{V+W} \wedge X \wedge X \cong (S \times T)_+ \wedge S^{V+W} \wedge X \wedge X$$

We can pull this back along the map induced by the diagonal map $X \to X \wedge X$ to get a bundles over $(S \times T)_+ \wedge S^{V+W} \wedge X$, or an element

$$a \star b \in \widetilde{\underline{KR}}^{-V-W}(X)(S \times T)$$

By extending this to the negatives of the bundles, this describes a natural multiplication

$$\widetilde{\underline{KR}}^{-V}(X)(S) \otimes \widetilde{\underline{KR}}^{-W}(X)(T) \to \widetilde{\underline{KR}}^{-V-W}(X)(S \times T)$$

which, by the universal property of Day convolution (Proposition 3.9), corresponds to a map

$$\widetilde{\underline{KR}}^{-V}(X) \square \widetilde{\underline{KR}}^{-W}(X) \to \widetilde{\underline{KR}}^{-V-W}(X)$$

We also similarly get a multiplication map

$$\widetilde{\underline{KR}}^{-V}(S^0) \square \widetilde{\underline{KR}}^{-W}(X) \to \widetilde{\underline{KR}}^{-V-W}(X)$$

by following the above outline, except that in the last step we pull back the bundle along the isomorphism $X \to S^0 \wedge X$ instead of the diagonal map. Now [Ati] shows that the Bott periodicity isomorphism in Theorem 7.12 is described by multiplication by an element $\beta \in \widetilde{\underline{KR}}^{-\rho}(S^0)(C_2/C_2)$. This gives us the required compatibility of periodicity and multiplication to be able to extend the multiplication to all of KR.

This multiplication on the cohomology theory corresponds to a multiplicative structure on KR described by maps $KR \wedge KR \to KR$ and $S^0 \to KR$ which make KR into a monoid object in $\mathcal{SH}^G$. This will give us the multiplicative structure on the slice spectral sequence of KR in the next chapter.

8 The Slice Spectral Sequence

The goal of this chapter is to compute the homotopy Mackey functors of KR using the slice spectral sequence and prove the gap theorem (REF). This computation will be done by breaking KR apart into simpler pieces using an equivariant analogue of the Postnikov tower.

For a non-equivariant spectrum X, its Postnikov tower is a sequence of spectra

$$\cdots \to P^1X \to P^0X \to P^{-1}X \cdots$$

with maps $X \to P^nX$ commuting with the maps in the sequence such that $\pi_i(P^nX) \cong 0$ for $i \geq n$, each map $X \to P^nX$ induces an isomorphism on π_i for $i \leq n$, and each map $P^nX \to P^{n-1}X$ induces an isomorphism π_i for $i < n$. The spectrum P^nX can be constructed by killing maps from (everything weakly equivalent to) S^i to X for $i > n$ by attaching cells. These properties imply that for each n, we have a homotopy fiber sequence

$$K(\pi_n(X), n) \to P^nX \to P^{n-1}X$$

P^nX is called the n-truncation of X as its homotopy groups have been truncated above degree n. Dually, we can contruct the *n-connected covers* P_nX of X. P_nX has trivial homotopy groups below degree n and has a natural map to X that induces isomorphisms on all homotopy groups at degrees n and above. This can be constructed as the homotopy fiber of the map $X \to P^nX$. These fit together into the *Whitehead tower* of X

$$\begin{tikzcd}
\cdots \arrow[r] & P_1X \arrow[r] \arrow[d] & P_0X \arrow[r] \arrow[d] & P_{-1}X \arrow[r] \arrow[d] & \cdots \\
& K(\pi_1(X), 1) & K(\pi_0(X), 0) & K(\pi_{-1}(X), -1) &
\end{tikzcd}$$

with the indicated homotopy cofibers.

Applying the homotopy functor π_*, we get an unrolled exact couple of graded abelian groups in which $A_s = P_{-s}X$ in the notation from Chapter 1. The inverse limit $A_{-\infty}$ for this exact couple is 0 and the direct limit A_∞ is $\pi_*(X)$. Therefore we can apply Theorem 1.9 to get a spectral sequence converging to π_*X.

However, this spectral sequence will give us no useful information. The E^1 page, under the default grading of Definition 1.6, is given by

$$E^1_{s,t} = \pi_t(\Sigma^{-s}H(\pi_{-s}(X))) \cong \begin{cases} \pi_{-s}(X) & \text{if } t = -s \\ 0 & \text{otherwise} \end{cases}$$

Since the rth differential d^r has bidegree $(-r,-1)$ under this grading, this spectral sequence cannot support any differentials on any page, hence there is no simplification before the E^∞ page. The filtration on $\pi_*(X)$ is also then trivial in the sense that it merely separates out the various degrees without easing the computation of any of them. However, there is an equivariant analogue of the Postnikov tower which does not suffer from this issue.

8.1 Slice Tower and its Spectral Sequence

One obvious equivariant analogue of the Postnikov tower for a G-spectrum X could be constructed by killing maps into X from $(G/H)_+ \wedge S^i \cong \mathrm{Ind}_H^G S^i$ for all $i > n$ and $H \leq G$ to get P^nX, because the ith homotopy Mackey functor of X is zero exactly when all maps from $(G/H)_+ \wedge S^i$ to X are stably nullhomotopic. This would then also have the property that it will be preserved by change-of-group adjunctions. We will call this the *equivariant Postnikov tower*.

However, the resulting spectral sequence would once again suffer from the same problem as the non-equivariant one. This is a consequence of the fact that there are no homotopically non-trivial maps $\mathrm{Ind}_H^G S^m \to \mathrm{Ind}_K^G S^n$ for any $H, K \leq G$ and $n > m$, so that killing the higher homotopy groups does not interfere with the lower ones. This will no longer be true if we include representation spheres.

Moreover, the construction proposed above would not have another good property of the non-equivariant Postnikov tower. Suspending a spectrum X moves its homotopy groups up one degree, so the Postnikov tower simply gets shifted. We do get this property for G-spectra, but only when we suspend by smashing with S^1. It is instead more natural to have this stability property to be with respect to smashing with S^ρ, the representation sphere for the regular G-representation ρ, especially since we wish to use this spectral sequence on KR which is ρ-periodic. Hence we change the subcategory of objects from which we kill all maps. Note that this kind of stability under smashing with S^{ρ_G} does not ensure stability under smashing with every G-representation.

To ensure both stability under smashing with the regular representation and preservation by change-of-group adjunctions, we replace the induced up spheres with the G-spectra

$$S(m,H) := \mathrm{Ind}_H^G S^{m\rho_H}$$

for $m \in \mathbb{Z}$ and $H \leq G$. $S(m,H)$ is called a slice sphere of type (m,H). Define the dimension of a slice sphere to be that of its underlying space (which would be a wedge of spheres)

$$\dim S(m,H) := m|H|$$

We construct P^nX by killing all maps into X from slice spheres of dimension greater than n. We will say that P^nX is the *nth slice section* of X. In the case of the Postnikov sections, we show that this construction is possible by attaching cells in increasing order of dimension and using the fact that attaching a higher dimensional sphere does not affect the lower degree homotopy groups. This argument does not work here. However, we can use the small object argument for this.

Suppose we have some set $\mathcal{A}$ of G-spectra and a G-spectrum $X \in \mathsf{Sp}^G_{\mathbf{O}}$. We wish to kill all maps $A \to X$ up to homotopy where $A \in \mathcal{A}$. Equivalently, we want every map $A \to X$ to extend to a map $CA \to X$ where CA is the cone on A, i.e. we want the map $X \to *$ to be in the right lifting component of the set of the canonical inclusions $A \hookrightarrow CA$:

$$(X \to *) \in \mathcal{J}^{\boxslash} \text{ where } \mathcal{J} = \{A \hookrightarrow CA \mid A \in \mathcal{A}\}$$

If every $A \in \mathcal{A}$ is small relative to $\overline{\mathcal{J}}$, then Theorem 2.20 gives us a factorization of $X \to *$ into maps $X \to P^{\mathcal{A}}X$ and $P^{\mathcal{A}}X \to *$, where the former is in $\overline{\mathcal{J}}$, meaning $P^{\mathcal{A}}X$ is obtained from X by attaching cones of the G-spectra in $\mathcal{A}$, and the latter is in $\mathcal{J}^{\boxslash}$, meaning all maps from $\mathcal{A}$ into it are trivial.

However, the $P^{\mathcal{A}}X$ constructed here will not necessarily be unique up to stable equivalence with these properties. For example, when G is the trivial group (reducing us to the case of non-equivariant homotopy theory), $\mathcal{A}$ contains only S^n, X has trivial nth homotopy group, the $P^{\mathcal{A}}X$ constructed functorially here will attach more copies of D^n to X regardless. This will not affect the nth homotopy group, but the higher homotopy groups could change. Thus $P^{\mathcal{A}}X$ and X will both satisfy the conditions that we need without being stably equivalent. If we want our slice spectral sequence to be unique, we need to ensure this.

Suppose we had two different extensions of $A \to X$ to maps $CA \to X$. They would then glue into a single map $\Sigma A \to X$. If we ensure that this map then extends to a map $C\Sigma A \to X$, then these extensions would be left homotopic. Therefore, the set $\mathcal{A}$ should be closed under suspensions, its objects should be cofibrant, and $\mathcal{J}$ should also include the generating acyclic fibrations. These last two conditions will ensure bifibrancy of the relevant objects, enabling the use of homotopies.

We want to construct the nth slice section by killing maps from the set $\{S(m, H) \mid m|H| > n\}$. We take the closure of this set under suspensions.

$$\mathcal{A}_n := \left\{\Sigma^k S(m, H) \,\middle|\, m|H| > n, k \in \mathbb{N}\right\}$$

Let $\mathcal{J}_n$ denote the set of generating acyclic cofibrations and the canonical inclusions $A \to CA$ for $A \in \mathcal{A}_n$. The above construction applies because the suspensions of the slice spheres are small relative to $\mathcal{A}_n$ (a consequence of the compactness of the spheres; see [Hov, Proposition 2.4.2]). The G-spectrum we get from the above construction is P^nX. If, for a G-spectrum X, all maps $A \to X$ are nullhomotopic for $A \in \mathcal{A}_n$, we say that X is *slice n-truncated*, in analogy with n-truncated spaces which are those for which all maps from S^k are nullhomotopic for $k > n$. P^nX is then universal among slice n-truncated G-spectra receiving maps from X up to homotopy.

Lemma 8.1. Let $X \to P^n X$ be the nth slice section of X. Any map $X \to Y$ where Y is slice n-truncated factors through a map $P^n X \to Y$. This factoring is unique up to left homotopy.

Proof. Such a factoring exists because in the following square, the map on the left is in $\overline{\mathcal{J}_n}$ and the map on the right is in $\mathcal{J}_n^{\boxslash}$.

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ P^n X \arrow[r] \arrow[ur, dotted] & * \end{tikzcd}$$

If we have any other such factoring, we can combine the two into a single map from the pushout $P^n X \cup_X P^n X \to Y$. The cone on this pushout can alternatively be obtained by attaching cones of the suspensions of the G-spectra in $\mathcal{A}_n$ to X along the same attaching maps. (TO ADD) □

Corollary 8.2. For any G-spectrum X and maps $X \to P^n X$, $X \to \widetilde{P}^n(X)$ in $\overline{\mathcal{J}}_n$ to slice n-truncated G-spectra, there is a stable equivalence $P^n X \to \widetilde{P}^n X$.

Proof. (TO ADD) □

Theorem 8.3. For each G-spectrum X, there exists a tower under X

$$\cdots P^1 X \to P^0 X \to P^{-1} X \to \cdots$$

This is called the *slice tower of* X.

Proof. For each $n \in \mathbb{Z}$, apply Lemma 8.1 to get a factoring of $X \to P^{n-1}X$ through $X \to P^n X$. The lemma applies because $P^{n-1}X$ is slice $(n-1)$-truncated and hence also slice n-truncated. □

Dually, we could factor $\varnothing \to X$ as a map $\varnothing \to P_n X$ in $\overline{\mathcal{J}}$ followed by a map $P_n X \to X$ in $\mathcal{J}^{\boxslash}$. $P_n X$ is then called the *nth coslice section* of X.

Definition 8.4. A G-spectrum Y is called *slice n-connected* if the map $\varnothing \to Y$ is in $\overline{\mathcal{J}}$.

To match HHR, I would need to close this under weak equivalences. Should I do that?

Then $P_n X$ is similarly universal among slice n-connected G-spectra mapping to X. This gives us a *coslice tower* of X.

Proposition 8.5. For any G-spectrum X, we have stable equivalences

(i) $P_i P_j X \simeq P_{\max(i,j)} X$.

(ii) $P^i P^j X \simeq P^{\min(i,j)} X$.

(iii) $P^j P_i X \simeq P_i P^j X$.

Proof. (TO ADD) □

Proposition 8.6. The sequence

$$P_i \to X \to P^{i-1} X$$

is a (co)fiber sequence of G-spectra.

Proof. (TO ADD) □

Definition 8.7. Define $P_i^j X := P_i P^j X$. P_n^n is called the *nth slice* of X.

Applying the integer-graded homotopy Mackey functor functor $\underline{\pi}_*$ to the slice tower and its homotopy fibers and to the coslice tower and its homotopy cofibers, we get two unrolled exact couples. The stable equivalence in Proposition 8.5 implies that the E_s terms (notation as in Chapter 1) of both are the same and hence they give the same spectral sequence. The exact couple in the coslice case is

$$\begin{tikzcd}
\cdots \arrow[r] & \underline{\pi}_*(P_2 X) \arrow[r] & \underline{\pi}_*(P_1 X) \arrow[r] \arrow[d] & \underline{\pi}_*(P_0 X) \arrow[r] \arrow[d] & \underline{\pi}_*(P_{-1} X) \arrow[r] \arrow[d] & \cdots \\
& & \underline{\pi}_*(P_1^1 X) \arrow[ul, dashed] & \underline{\pi}_*(P_0^0 X) \arrow[ul, dashed] & \underline{\pi}_*(P_{-1}^{-1} X) \arrow[ul, dashed] &
\end{tikzcd}$$

In the notation of Chapter 1, we have

$$A_s = \underline{\pi}_*(P_{-s} X)$$
$$E_s = \underline{\pi}_*(P_{-s}^{-s} X)$$

Remark 8.8. When an exact couple has decreasing indexing, we call it a *cohomological exact couple*, as opposed to the exact couples we saw in Chapter 1 which are called *homological exact couples.* If we do not make the change in indexing by putting negative signs as above, we get *cohomological spectral sequences*, with the only difference being in grading. To keep indexing related confusion to the minimum, we work only with homological indexings.

We draw this spectral sequence with the slice grading of Definition 1.6, meaning that the x-axis will represent $t - s$ and the y-axis will represent t. The differentials d^r then go 1 unit to the left and $r+1$ units up. To make reference to the drawn spectral sequence easier, we will re-index the spectral sequence itself.

Definition 8.9. The *slice spectral sequence* of a G-spectrum X is the spectral sequence (E^r, d^r) associated to the exact couple defined above under the slice grading. Its E^1 page is described as

$$E^1_{s,t} = \underline{\pi}_t(P^{s+t}_{s+t})$$

and its differentials d^r have bidegree $(-1, r+1)$.

The aim of this section is to show that the slice spectral sequence converges to the integer-graded homotopy groups of X. In the following, to say that X is (ordinary) n-connected means that its homotopy Mackey functors vanish in degrees less than n.

Proposition 8.10. (i) A 0-connected G-spectrum X is stably equivalent to a slice 0-connected G-spectrum.

(ii) Smash adds connectivity (TO ADD).

(iii) Stability (TO ADD)

Proof. (i) The -1st slice section $P^{-1}X$ has the property that $P^{-1}X \to *$ is in $\mathcal{J}^{\boxslash}_{-1}$. Consider the set of canonical inclusions of the non-negative dimensional induced up spheres into their cones

$$\mathcal{I}_{-1} := \{(\mathrm{G/H})_+ \wedge \mathrm{S^n} \hookrightarrow (\mathrm{G/H})_+ \wedge \mathrm{D^n} \mid n > 0\}$$

Then $\mathcal{I}_{-1} \subset \mathcal{J}_{-1}$ since $S^0 = S^{0\rho}$ and hence $P^{-1}X \to *$ is in $\mathcal{I}^{\boxslash}_{-1}$. This implies that its homotopy Mackey functors vanish in degrees 0 and above, so the cofiber sequence $P_0X \to X \to P^{-1}X$ implies that P_0X is (TO ADD)

□

8.2 The Slice Spectral Sequence for KR-Theory

Our goal is to compute the homotopy groups of KR, the spectrum representing KR-theory, and proving its gap and periodicity theorems as mentioned in (REF introduction). This will be done by finding the coslice tower for KR and applying the slice spectral sequence. The main tool that will help here is the following cofiber sequence.

Theorem 8.11. There exists a cofiber sequence

$$\Sigma^{\rho} kr \to kr \to H\underline{\mathbb{Z}}$$

where kr denotes the 0-connected cover, or by Proposition 8.10, the 0th coslice section, of KR.

Proof. See [Dug, Section 6]. □

Lemma 8.12. We have a commutative square

$$\begin{tikzcd} P_2(KR) \arrow[r, "\beta"] \arrow[d] & P_0(KR) \arrow[d] \\ \Sigma^{\rho} kr \arrow[r, "\beta"'] & kr \end{tikzcd}$$

where both the vertical maps are weak equivalences.

Lemma 8.13. $P_1 KR \simeq \Sigma^{\rho} kr$.

Proposition 8.14. The coslice tower of KR is

$$\begin{tikzcd} & -2 & -1 & 0 & 1 & 2 & \\ \cdots \arrow[r, "\Sigma^{\rho}\beta"] & \Sigma^{\rho} kr \arrow[r, "="] \arrow[d] & \Sigma^{\rho} kr \arrow[r, "\beta"] \arrow[d] & kr \arrow[r, "="] \arrow[d] & kr \arrow[r, "\Sigma^{-\rho}\beta"] \arrow[d] & \Sigma^{-\rho} kr \arrow[r] \arrow[d] & \cdots \\ & \Sigma^{\rho} H\underline{\mathbb{Z}} & * & H\underline{\mathbb{Z}} & * & \Sigma^{-\rho} H\underline{\mathbb{Z}} & \end{tikzcd}$$

with slice sections as indicated.

We can now use the slice spectral sequence to compute the homotopy Mackey functors of KR. We have

$$E^1_{s,t} = \underline{\pi}_s(P^{s+t}_{s+t} X) \Rightarrow \underline{\pi}_s(X)$$

Since $P^{s+t}_{s+t} = *$ whenever $s+t$ is odd, nothing appears on the odd anti-diagonals. On the $2n$th anti-diagonal ($s+t = 2n$), we have the homotopy Mackey functors of $\Sigma^{n\rho} H\underline{\mathbb{Z}}$. There are isomorphisms

$$\begin{aligned} \underline{\pi}_s(\Sigma^{n\rho} H\underline{\mathbb{Z}}) &\cong \underline{[S^s, S^{n\rho} \wedge H\underline{\mathbb{Z}}]} \\ &\cong \underline{\widetilde{H}}_s(S^{n\rho}) \\ &\cong \underline{H}_{s-n-n\sigma}(*) \end{aligned}$$

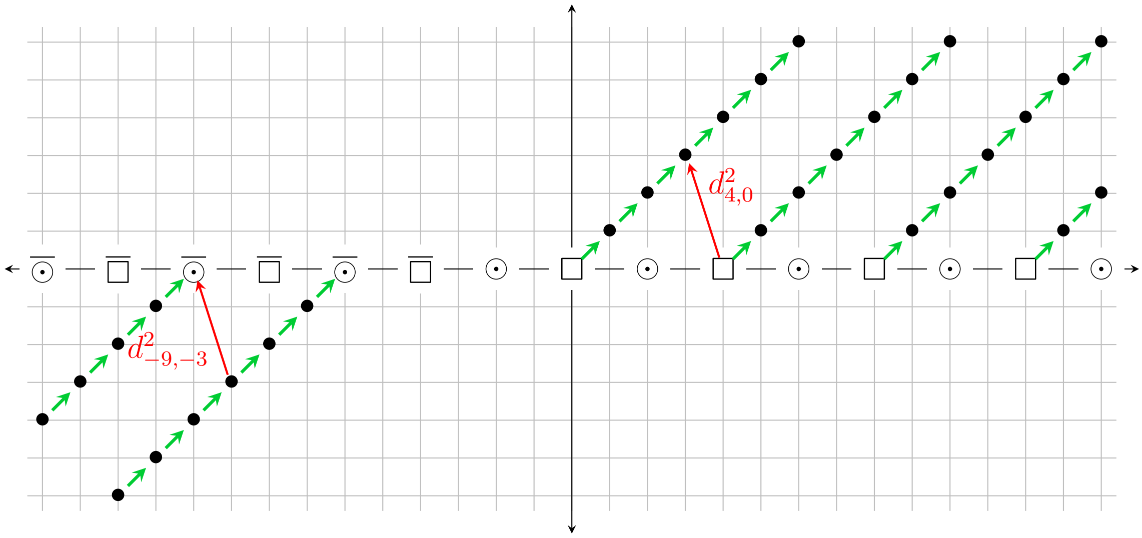


Figure 8.1: The $E^1 \cong E^2$ page of the slice spectral sequence of KR

Therefore, whenever $s + t = 2n$,

$$E^1_{s,t} \cong \underline{H}_{\frac{s-t}{2} - \frac{s+t}{2}\sigma}(*)$$

We have already computed the $RO(G)$-graded homology of a point in Theorem 6.25. We can therefore draw this page as follows (see Fig. 6.1 for notation). The lines and arrows are explained later.

Our next task is to figure out the differentials. The d^1 differentials are easy - they have bidegree $(-1, 2)$ and there is no non-zero pair of Mackey functors on the E^1 page differing by that bidegree. Therefore $d^1 = 0$ which implies that $E^1 \cong E^2$.

Figuring out the d^2 differentials is trickier because they will have bidegree $(-1, 3)$ and we can find many non-zero Mackey functors on this page differing by that bidegree. First, we prove that

$$d^2_{4,0} : E^2_{4,0} \to E^2_{3,3}$$

(labelled in Fig. 8.1) is non-zero. Suppose this were not the case. Then $E^2_{3,3}$ must survive to E^∞. This is because every d^r originating in $E^r_{3,3}$ would land on a zero Mackey functor and every d^r that lands on $E^r_{3,3}$ for $r > 2$ must originate in a zero Mackey functor. This would imply that $\underline{\pi}_3(KR) \neq 0$, and in particular the evaluation of this homotopy Mackey functor at the top level of its Lewis diagram, i.e. at C_2/C_2, would be non-zero. But since S^3 has a trivial C_2-action, a vector bundle with conjugation over $C_2/C_2 \wedge S^3$ is equivalent to a real vector bundle over S^3, and hence

$$\underline{\pi}_3(KR)(C_2/C_2) \cong \pi_3(KO)$$

Proposition 7.10 tells us that this is zero. Therefore, $d^2_{4,0}$ must be surjective at the top level, which determines it to be the map

$$\begin{tikzcd} \mathbb{Z} \arrow[r] \arrow[d, bend right, "1"'] & \mathbb{Z}/2\mathbb{Z} \arrow[d, bend right] \\ \mathbb{Z} \arrow[r] \arrow[u, bend right, "2"'] & 0 \arrow[u, bend right] \end{tikzcd}$$

To deduce the rest of the d^2 differentials, we will need the multiplicative structure of this spectral sequence. This comes from the multiplicative structure of KR. The multiplication $KR \wedge KR$ induces graded multiplication maps

$$P_m KR \wedge P_n KR \to P_{m+n} KR$$

on the coslice tower. These come from the universal property of $P_{m+n}KR$ as in the diagram (ADD diagram) These then induce multiplication maps $P_m^{m+r}KR \wedge P_n^{n+r}KR \to P_{m+n}^{m+n+r}KR$. These are described by viewing P_k^{k+r} as the homotopy cofiber of $P_{k+r+1} \to P_k$ and the diagram (ADD diagram) Then the Cartan-Eilenberg system described by $K_{i,j} = \pi_*(P_i^j)$ has a pairing described by the above multiplication maps, leading to a multiplicative structure on the spectral sequence.

Figuring out the exact multiplicative structure on the E^1 page requires working out this structure. Since the E^1 page is the homology of a point (after some linear transformation), we expect it to have the same multiplicative structure. (ADD)

We can now figure out the differential

$$d^2_{5,1} : E^2_{5,1} \to E^2_{4,4}$$

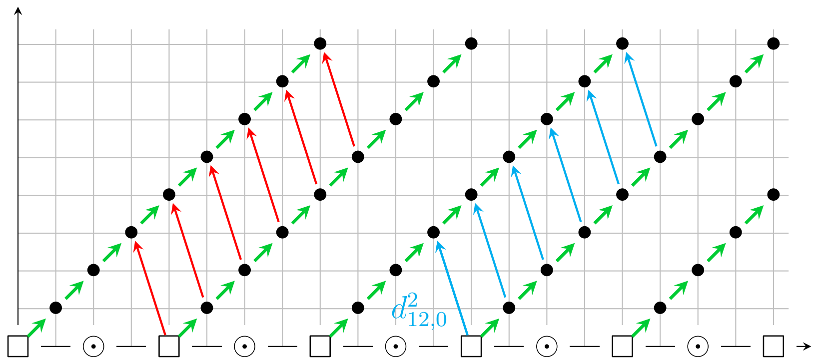
or more generally,

$$d^2_{4+k,k} : E^2_{4+k,k} \to E^2_{3+k,3+k}$$

inductively for $k \geq 1$ using the Leibniz formula. Write the generator of $E^2_{4+k,k}(C_2/e)$ as xy where x is the generator of $E^2_{4+k-1,k-1}(C_2/e)$ and y is the generator of $E^2_{1,1}$. Then we have

$$\begin{aligned} d^2_{4+k,k}(xy) &= (d^2_{4+k-1,k-1}x)y + x(d^2_{1,1}y) \\ &= (d^2_{4+k-1,k-1}x)y \end{aligned}$$

which is the product of the generators of $E^2_{3+k-1,3+k-1}$ and of $E^2_{1,1}$. The multiplicative structure tells us that this is the generator of $E^2_{3+k,3+k}$. The first quadrant then looks like the following, where we have shown that all the red arrows are non-zero differentials.



From this, we can deduce that the differential

$$d^2_{12,0} : E^2_{12,0} \to E^2_{11,3}$$

labelled above, or more generally, the differentials

$$d^2_{4+8k,0} : E^2_{4+8k,0} \to E^2_{3+8k,3}$$

is non-zero for $k \geq 1$ using the 8-periodicity of KR from Theorem 7.13. Suppose that this differential is non-zero for $k = n$. By periodicity, $\pi_3 KO = \pi_{3+8n} KO$, so the Mackey functor $E^2_{3+8n,3}$ must die. We wish to say that the only way this can happen is if it is the target of a non-zero d^2 differential from $E^2_{4+8n,0}$.

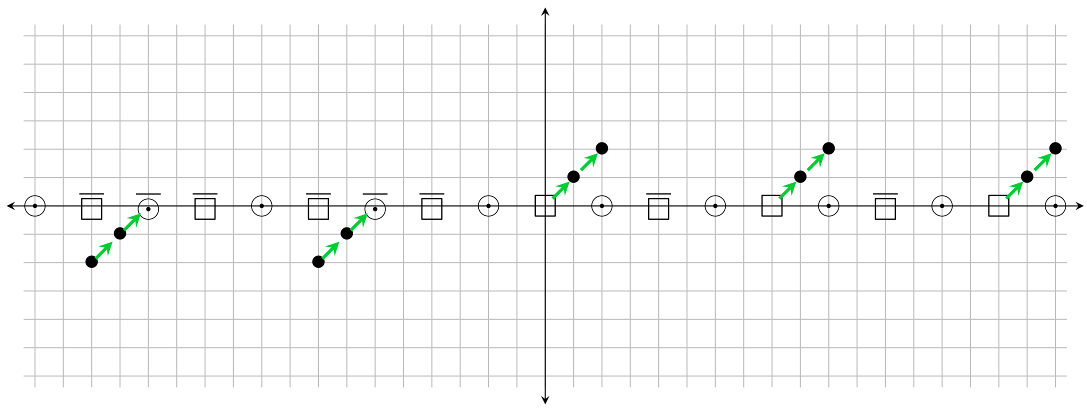
By the same argument using the multiplicative structure as for all the red arrows, we can show that the differentials that are diagonally above and to the right of this differential are also non-zero, as displayed with the cyan arrows in the figure above for the case $k = 1$. This means that in the next page E^3, all the Mackey functors $E^3_{s,t}$ for $t \geq 3$ will die, as either they are the codomain of a surjective d^2 map or the domain of an injective d^2 map. Therefore, $E^3_{3+8n,3}$ must be 0 since it cannot be killed later. This means that it has to be a domain or a codomain of a non-zero d^2 differential. It cannot be a domain of such a differential, because inductively, $E^2_{2+8n,6}$, which would have to be the codomain, is already the domain of an injective differential, and the differentials are supposed to compose to 0. Therefore, the only way this Mackey functor dies is if $d^2_{4+8n,0}$ surjects onto it.

This proves that the non-zero d^2 differentials in the first quadrant are arranged as above. These must be all of the non-zero differentials because there is no room left for any others if they are to satisfy the composing to 0 property. Now we can do something similar for the third quadrant. We first prove that the differential

$$d^2_{-9,-3} : E^2_{-9,-3} \to E^2_{-10,0}$$

must be non-zero (labelled in Fig. 8.1). This is because if it is not, then the Mackey functor $E^2_{-9,-3}$ will survive to the ∞th page, implying $\underline{\pi}_{-9}(KR) \neq 0$. However, this cannot be the case, because the analysis of the first quadrant showed that both the Mackey functors in the column $s = 7$ will get killed, giving $\underline{\pi}_7(KR) = 0$, which contradicts the above because of Theorem 7.13.

Using the Leibniz formula and periodicity, we can show that the d^2 differentials in the second quadrant are arranged with a pattern similar to that of the first quadrant. Here, the multiplication maps go the other way, so the argument will go backwards; assuming that the differential diagonally below and to the left of a non-zero differential is zero, we would get a contradiction by deducing that said non-zero differential must be zero too. Having found all the differentials, we can draw E^3.



From this page onwards, all differentials will have to be zero because there is no pair of non-zero Mackey functors in this grid that differs by bidegree $(-1, r+1)$ for $r \geq 3$. Therefore, this is isomorphic to the E^∞ page of this spectral sequence.

Now we can compute the homotopy Mackey functors of KR. Because of the 8-periodicity, we only need to do this in degrees 0 through 7. For degree 0, the E^∞ page tells us that the there is a filtration on $\underline{\pi}_0(KR)$ whose intersection is 0 and union is $\underline{\pi}_0(KR)$ such that its successive quotients are exactly the Mackey functors $E^\infty_{0,t}$ as t varies. Since this 0th column of the E^∞ page has only one non-zero Mackey functor $\underline{\mathbb{Z}}$, we find that this must be the value of $\underline{\pi}_0(KR)$. Similarly, we find the homotopy Mackey functors of KR in all degrees other than 2, since in the column $s = 2$ of the E^∞ page, there are two non-zero Mackey functors.

This tells us that there is a 2-step filtration on $\underline{\pi}_2(KR)$

$$0 = \underline{M}_0 \subset \underline{M}_1 \subset \underline{M}_2 = \underline{\pi}_2(KR)$$

with successive quotients

$$\frac{\underline{M}_1}{\underline{M}_0} \cong \underline{\mathbb{Z}}_- \qquad \frac{\underline{M}_2}{\underline{M}_1} \cong \underline{F}$$

where we are writing $\underline{F}$ for the Mackey functor being drawn as $\bullet$ (the order of these successive quotients is important). This does not uniquely determine $\underline{\pi}_2(KR)$. Indeed, all this says is that there is a short exact sequence

$$0 \to \underline{\mathbb{Z}}_- \to \underline{\pi}_2(KR) \to \underline{F} \to 0$$

Looking at the Lewis diagrams tells us that there are two Mackey functors that can fit in the middle of this sequence, one being $\underline{\mathbb{Z}}_- \oplus \underline{F}$ and the other being $\underline{\mathbb{Z}}_-^{\mathsf{op}}$. (ADD Lewis diagrams) Once again, the 8-periodicity of KR comes to the rescue, because we have $\underline{\pi}_{-6}(KR) = \underline{\mathbb{Z}}_-^{\mathsf{op}}$ directly by looking at the E^∞ page. In particular, this implies that the above short exact sequence of Mackey functors is not split. We have found all the homotopy Mackey functors of KR.

$$
\begin{aligned}
\underline{\pi}_0(KR) &\cong \underline{\mathbb{Z}} \\
\underline{\pi}_1(KR) &\cong \underline{F} \\
\underline{\pi}_2(KR) &\cong \underline{\mathbb{Z}}_-^{\mathsf{op}} \\
\underline{\pi}_3(KR) &\cong 0 \\
\underline{\pi}_4(KR) &\cong \underline{\mathbb{Z}}^{\mathsf{op}} \\
\underline{\pi}_5(KR) &\cong 0 \\
\underline{\pi}_6(KR) &\cong \underline{\mathbb{Z}}_- \\
\underline{\pi}_7(KR) &\cong 0
\end{aligned}
$$

These can be extended to the $RO(G)$-graded homotopy Mackey functors of KR using periodicity (Theorems 7.12 and 7.13). In particular, we have proved the KR analogue of [HHRa]'s gap theorem of Ξ as mentioned in the preface.

Corollary 8.15 (Gap theorem)**.** The homotopy C_2-Mackey functors of KR in degrees $-1, -2, -3$ evaluated at the top level of their Lewis diagrams, C_2/C_2, are 0. □

Bibliography

[Ada] J.F. Adams. *Stable Homotopy and Generalised Homology*. The University of Chicago Press, 1974 (cit. on pp. 39, 42).

[Ati] M.F. Atiyah. "K-Theory and Reality". In: *The Quarterly Journal of Mathematics* 17.1 (Jan. 1966), pp. 367–386. ISSN: 0033-5606. DOI: 10.1093/qmath/17.1.367. eprint: https://academic.oup.com/qjmath/article-pdf/17/1/367/7295753/17-1-367.pdf. URL: https://doi.org/10.1093/qmath/17.1.367 (cit. on pp. 79–81).

[AB] Michael Atiyah and Raoul Bott. "On the periodicity theorem for complex vector bundles". In: *Acta Mathematica* 112 (1964), pp. 229–247. DOI: 10.1007/BF02391772. URL: https://doi.org/10.1007/BF02391772 (cit. on pp. 75, 77).

[BR] David Barnes and Constanze Roitzheim. *Foundations of Stable Homotopy Theory*. Cambridge University Press, 2020 (cit. on pp. 36, 39, 40).

[Blu] Andrew Blumberg. *The Burnside Category*. Lecture notes, available: https://web.ma.utexas.edu/users/a.debray/lecture_notes/m392c_EHT_notes.pdf. 2017 (cit. on pp. 51, 52).

[Dug] Daniel Dugger. "An Atiyah-Hirzebruch spectral sequence for KR-theory". In: *arXiv: Algebraic Topology* (2003) (cit. on p. 88).

[Grea] J.P.C. Greenlees. "Four Approaches to Cohomology Theories with Reality". In: *arXiv: Algebraic Topology* (2017) (cit. on p. 48).

[Greb] Rok Gregoric. *Spectra Are Your Friends*. Lecture notes, available: https://web.ma.utexas.edu/users/gregoric/Spectra%20Are%20Your%20Friends.pdf. 2019 (cit. on p. 36).

[GM] Bertrand Guillou and Peter May. "Models of G-Spectra as Presheaves of Spectra". In: *arXiv: Algebraic Topology* (2011) (cit. on p. 59).

[Hata] Allen Hatcher. *Spectral Sequences*. Unpublished, available: https://pi.math.cornell.edu/~hatcher/AT/ATch5.pdf. 2004 (cit. on p. 6).

[Hatb] Allen Hatcher. *Vector Bundles and K-Theory*. Unpublished, available: https://pi.math.cornell.edu/~hatcher/VBKT/VB.pdf. 2017 (cit. on pp. 75, 77).

[HHRa] Michael Hill, Michael Hopkins, and Douglas Ravenel. "On the nonexistence of elements of Kervaire invariant one". In: *Annals of Mathematics* 184 (2015) (cit. on pp. 2, 7, 93).

[HHRb] Michael Hill, Michael Hopkins, and Douglas Ravenel. *Equivariant Stable Homotopy Theory and the Kervaire Invariant Problem*. Cambridge University Press, 2021 (cit. on pp. 39, 45).

[Hir] Philip S. Hirschhorn. *Model Categories and their Localizations*. American Mathematical Society, 2002 (cit. on p. 30).

[Hov] Mark Hovey. *Model Categories*. American Mathematical Society, 1999 (cit. on pp. 21, 23–27, 84).

[Ill] Soren Illman. "Equivariant Algebraic Topology". In: (1972) (cit. on p. 50).

[Lan] Saunders Mac Lane. *Categories for the Working Mathematician*. Springer-Verlag, 1998 (cit. on pp. 25, 31, 34, 42).

[MMSS] Michael Mandell, Peter May, Stefan Schwede, and Brooke Shipley. "Model Categories of Diagram Spectra". In: *Proceedings of the London Mathematical Society* 82.2 (2000) (cit. on pp. 31, 35, 45).

[Mar] H.R. Margolis. *Spectra and the Steenrod Algebra*. North-Holland Mathematical Library, 1983 (cit. on p. 36).

[Maya] Peter May. *Equivariant Homotopy and Cohomology Theory*. American Mathematical Society, 1996 (cit. on pp. 60, 67, 68).

[Mayb] Peter May. *A Concise Course in Algebraic Topology* (cit. on p. 50).

[MP] Peter May and Kate Ponto. *More Concise Algebraic Topology*. University of Chicago Press, 2011 (cit. on p. 18).

[MS] John W. Milnor and James D. Stasheff. *Characteristic Classes*. Princeton University Press and University of Tokyo Press, 1974 (cit. on p. 74).

[nLa] nLab authors. *model structure on functors*. `https://ncatlab.org/nlab/show/model+structure+on+functors`. Revision 49. Mar. 2023 (cit. on p. 29).

[Put] Andrew Putnam. *Classifying Spaces and Brown Representability*. Notes, available: `https://www3.nd.edu/~andyp/notes/BrownRepresentability.pdf` (cit. on p. 41).

[Qui] Daniel Quillen. *Homotopical Algebra*. Springer-Verlag, 1967 (cit. on p. 17).

[Ram] Maxime Ramzi. *Products in the homotopy category of a model category*. Mathematics Stack Exchange. Version: 2019-07-25. URL: `https://math.stackexchange.com/q/3303767` (cit. on p. 40).

[Rez] Charles Rezk. *Introduction to Quasicategories*. Lecture notes, available: `https://faculty.math.illinois.edu/~rezk/quasicats.pdf`. 2021 (cit. on p. 18).

[Rie] Emily Riehl. *Categorical Homotopy Theory*. Cambridge University Press, 2014 (cit. on pp. 25, 31).

[Rog] John Rognes. *Spectral Sequences*. Lecture notes, available: https://www.uio.no/studier/emner/matnat/math/MAT9580/v21/dokumenter/spseq.pdf. 2021 (cit. on pp. 8, 12, 15, 16, 36, 40).

[Vak] Ravi Vakil. *Puzzling through exact sequences*. Guest post on 3Blue1Brown's blog. 2021. URL: https://www.3blue1brown.com/blog/exact-sequence-picturebook (cit. on p. 8).

[Wic] Kirsten Graham Wickelgren. *Stable Homotopy Theory*. Lecture notes, available: https://services.math.duke.edu/~kgw/8803_Stable/. 2015 (cit. on p. 47).