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Unit - 02

Eigen values and Eigenvectors

Eigen values

Let $A \in M_n(F)$, $\lambda \in F$ is called eigen value of matrix A if $\exists$ $x \neq 0 \in \mathbb{R}^n$ s.t.

$Ax = \lambda x$

$x \rightarrow$ Eigen vector

The value of x corresponding of λ is called eigen vector

Example: $A = \begin{pmatrix} 1 & 2 \\ 0 & -4 \end{pmatrix}$, $\lambda = 1$

Take $x = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{R}^2$

$$Ax = \begin{pmatrix} 1 & 2 \\ 0 & -4 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= x$$

$$= 1 . x$$

$$= \lambda x$$

$$Ax = \lambda x$$

$\therefore \lambda = 1$ is an EV and x is called eigen vector.

eigen vector

Let $A \in M_n(F)$, a non-zero vector $x \neq 0 \in \mathbb{R}^n$ is eigen vector of A if $\exists$ a scalar $\lambda \in F$ s.t.

eigen vector

$Ax = \lambda x$

Suppose λ is an eigen-value of A, $\exists$ $x \neq 0 \in \mathbb{R}^n$ s.t.

$Ax = \lambda x$

$Ax - \lambda x = 0$

eigen

$(A - \lambda I)x = 0.$

This has infinite solutions only if

$|A - \lambda I| = 0$

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

$$\lambda I = \begin{pmatrix} \lambda & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix}$$

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$(-1)^n [\lambda^n - c_1 \lambda^{n-1} + c_2 \lambda^{n-2} + \dots + (-1)^n c_n] = 0$

$P(\lambda) = 0$
characteristic polynomial

The roots of the characteristic polynomial are called eigen values of A

The pair (λ, x) is called eigen pair.

Important points

* Let $A \in M_3(\mathbb{R})$

• The characteristic polynomial of A is

$P_3(\lambda) = \lambda^3 - tr(A)\lambda^2 + (M_{11} + M_{22} + M_{33})\lambda - |A| = 0$

only valid for 3×3

• Let $A \in M_2(\mathbb{R})$

$P_2(\lambda) = \lambda^2 - tr(A)\lambda + |A| = 0$

• $tr(A) = \lambda_1 + \lambda_2 + \lambda_3$

$|A| = \lambda_1 \lambda_2 \lambda_3$

Q. $A = \begin{pmatrix} 1 & -3 & 3 \\ -2 & 0 & 2 \\ 1 & -1 & 3 \end{pmatrix}$ Find eigenvalues and eigenvectors

Sol $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & -3 & 3 \\ -2 & -\lambda & 2 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - tr(A)\lambda^2 + (M_{11} + M_{22} + M_{33})\lambda - |A| = 0$$

$tr(A) = 1 + 0 + 3$
$= 4$

$M_{11} = 0 + 2$
$= 2$

$M_{22} = 3 - 3$
$= 0$

$M_{33} = 0 - 6$
$= -6$

$|A| = 1(0+2) + 3(-6-2) + 3(2-0)$
$= 2 - 24 + 6$
$= -16$

$$\lambda^3 - (4)\lambda^2 + (-4)\lambda - (-16) = 0$$

$$\lambda^3 - 4\lambda^2 - 4\lambda + 16 = 0$$

$$\lambda^2(\lambda-4) - 4(\lambda-4) = 0$$

$$(\lambda^2-4)(\lambda-4) = 0$$

$$(\lambda-4)(\lambda+2)(\lambda-2) = 0$$

$$\lambda = 2, -2, 4$$

$\lambda = 2$

$Ax = (-2)x$

$(A+2I)x = 0$

$$\begin{pmatrix} 3 & -3 & 3 \\ -2 & 2 & 2 \\ 1 & -1 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$R_1 \leftrightarrow R_3$

$R_2 \to R_2 + 2R_1$

$R_3 \to R_3 - 3R_1$

$R_3 = R_3 + R_2$

$$\begin{pmatrix} 1 & -1 & 5 \\ 0 & 0 & 12 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\rho(A) = 2$
$= r$

$n - r = 3 - 2$

— arbitrary constant

$x_3 = 0$

$x_1 - x_2 + 5x_3 = 0$

$x_1 = x_2$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$x = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \therefore EV = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$x_2 \in \mathbb{R} - \{0\}$

For $\lambda = 2$

$Ax = 2x$

$(A - 2I)x = 0$

$$\begin{pmatrix} -1 & -3 & 3 \\ -2 & -2 & 12 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} -1 & -3 & 3 \\ 0 & 4 & -4 \\ 0 & -4 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + R_1 \end{matrix}$$

$R_3 \to R_3 + R_2$

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$$\begin{pmatrix} -1 & -3 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$n - r = 3 - 2$
$= 1$

$-x_1 - 3x_2 + 3x_3 = 0$
$x_2 - x_3 = 0$

Let $x_3 = k$

$x_2 = k$

$-x_1 - \cancel{3k} + \cancel{3k} = 0$
$x_1 = 0$

$\therefore x = \begin{pmatrix} 0 \\ k \\ k \end{pmatrix}$

$= k \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \quad , k \neq 0$

For $\lambda = 4$
$(A - 4I)x = 0$
$x = k \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

Que 2 $B = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 3 \end{bmatrix}$. Find the eigenvalues

$\lambda_1 + \lambda_2 + \lambda_3 = 5$

$\lambda_1 \lambda_2 \lambda_3 = -8$

$\lambda_1 = 4$

$\lambda_2 + \lambda_3 = 1$

$\lambda_2 \lambda_3 = -2$

$\lambda_3 = 1 - \lambda_2$

$(1-\lambda_2)\lambda_2 = -2$

$\lambda_2 - \lambda_1^2 = -2$

$\lambda_2 = 2$

$\lambda_2 = -1$

$\lambda_3 = 1-2 = -1$

$\lambda_3 = 1-(-1) = 2$

$\therefore \lambda = -1, 2, 4$

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Properties of Eigen Values and Eigen Vectors :

Let $A \in M_n(R)$ and λ is eigen value of A

(i) $\alpha\lambda$ is eigen value of αA

$Ax = \lambda x$

$\alpha(Ax) = \alpha(\lambda x)$

$(\alpha A)x = (\alpha\lambda)x$

(ii) Eigenvalue of A^m is λ^m

$Ax = \lambda x$

$A^2 x = A(\lambda x)$

$= \lambda(Ax)$

$= \lambda(\lambda x)$

$= \lambda^2 x$

Similarly,

$A^m x = \lambda^m x$

By induction

(iii) The eigen value of $A+kI$ is $\lambda + kI$

$Ax = \lambda x$

$(A+kI)x = (\lambda + kI)x$

$Ax + kIx = \lambda x + Ikx$

$(A+kI)x = (\lambda + k)x$

$Ax = \lambda x$

$Ax + Kx = \lambda x + Kx$

$Ax + (KI)x = \lambda x + (KI)x$

$(A+KI)x = (\lambda + k)x$

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(iv) The eigen value of A^{-1} is $\frac{1}{\lambda}$

$Ax = \lambda x$
$A^{-1}(Ax) = A^{-1}(\lambda x)$
$(A^{-1}A)x = \lambda(A^{-1}x)$
$Ix = \lambda(A^{-1}x)$
$\lambda A^{-1}x = x$
$A^{-1}x = \frac{x}{\lambda}$

$A^{-1}x = \frac{1}{\lambda} \cdot x$

(v) Eigen value of (A) adj A

$Ax = \lambda x$
$(\text{adj}A)Ax = \text{adj}A(\lambda x)$
$(\text{adj}A \cdot A)x = \lambda(\text{adj}A x)$
$|A| I x = \lambda(\text{adj}A\, x)$
$\text{adj}A\, x = \frac{|A|}{\lambda} x$

$(\text{adj}A)x = \frac{|A|}{\lambda} \cdot x$

(vi) Eigen values of A and A^T are same

$(A - \lambda I)x = 0$
$(A - \lambda I)^T = A^T - \lambda I$
$|(A - \lambda I)^T| = |A - \lambda I|$

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(vii) The eigen value of e^{A} is e^{λ}

(viii) The eigen value of $f(A)$ is $f(\lambda)$

Example 1: If A is a singular matrix of order 2 having Trace 3, then find the eigen value of $(A^T)^2$

Sol. $\because$ Singular value

$\therefore$ 1 eigen value $= 0$

Trace $= 3$

$\therefore$ 2nd eigen value $= 3$

0, 3

$A^2 = 0^2, 3^2$

$= 0, 9$

$(A^2)^T = 0, 9$

$(A^T)^2 = 0, 9$

Example 2: If A is a 4×4 matrix with eigen value $(1, -1, 2, -2)$, find the values of determinant of the matrix $B = 2A + A^{-1} - I$

Sol. $B = 2A + A^{-1} - I$

$\lambda_1 = 2.1 + \cancel{1} - \cancel{1}$

$= 2$

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$\lambda_2 = 2 \cdot 1 + (-1) - 1$
$= -4$

$\lambda_3 = 4 + \frac{1}{2} - 1$

$= \frac{7}{2}$

$\lambda_4 = -4 + \frac{1}{-2} - 1$

$= -5 - \frac{1}{2}$

$= \frac{-11}{2}$

$\lambda_1 \lambda_2 \lambda_3 \lambda_4 = |B|$

$|B| = 2 \cdot \cancel{-4} \cdot \frac{7}{\cancel{2}} \cdot \frac{-11}{\cancel{2}}$

$= 2 \cdot 7 \cdot 7$

$= 154$

2 being
$r)^2$

e values
erminant

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Note:

* The eigen values of triangular matrix are diagonal elements of that matrix (both upper and lower triangular matrix)

* Let $A \in M_n(\mathbb{R})$ having $\lambda_1, \lambda_2, \dots, \lambda_k$ distinct eigen values. Let $v_1, v_2, \dots, v_k$ are the eigen vectors w.r.t $\lambda_1, \lambda_2, \dots, \lambda_k$ respectively. Then $\{v_1, v_2, \dots, v_k\}$ is linearly independent

Proof:

(Induction)

For $i=1$, v_1 is the eigen vector w.r.t λ_1

$v_1 \neq 0$

$\therefore \{v_1\}$ is LI.

Assume that theorem is true for $i = r$ $(r < k)$

$\{v_1, v_2, \dots v_r\}$ is LI

We have to prove $\{v_1, v_2, \dots v_r, v_{r+1}\}$ is LI

Consider $c_1v_1 + c_2v_2 + \dots + c_rv_r + c_{r+1}v_{r+1} = 0$ ——(1)

$A(c_1v_1) + A(c_2v_2) + \dots + A(c_rv_r) + A(c_{r+1}v_{r+1}) = 0$

$c_1(Av_1) + c_2(Av_2) + \dots + c_r(Av_r) + c_{r+1}(Av_{r+1}) = 0$

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$c_1(\lambda_1 v_1) + c_2(\lambda_2 v_2) + \dots + c_r(\lambda_r v_r) + c_{r+1}(\lambda_{r+1} v_{r+1}) = 0$ —— (ii)

Now $(\lambda_{r+1}) \times 1 \longrightarrow$ (i)

$c_1(\lambda_{r+1} - \lambda_1) v_1 + c_2(\lambda_{r+1} - \lambda_2) v_2 + \dots + c_r(\lambda_{r+1} - \lambda_r) v_r = 0$

$c_i(\lambda_{r+1} - \lambda_i) = 0$

$c_i = 0$

$\therefore \{v_1, v_2, \dots, v_r, v_{r+1}\}$ is LI

By induction $\{v_1, v_2, \dots, v_r, v_{r+1}, \dots, v_k\}$ is LI

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Cayley-Hamilton Theorem

Let $A \in M_n(\mathbb{R})$. Then A satisfies its own characteristic equation.

Proof

$P_n(\lambda) = |A - \lambda I| = 0$

$P_n(\lambda) = (-1)^n [\lambda^n c_1 \lambda^{n-1} + c_2 \lambda^{n-2} + \dots + (-1)^n c_n] = 0$

$\text{adj}(A - \lambda I) = B_1 \lambda^{n-1} + B_2 \lambda^{n-1} + \dots + B_n I$

$(A - \lambda I) \cdot \text{adj}(A - \lambda I) = |A - \lambda I| I$

$(A - \lambda I) \cdot (B_1 \lambda^{n-1} + B_2 \lambda^{n-2} + \dots + B_n I) = \lambda^n - c_1 \lambda^{n-1} + c_2 \lambda^{n-2} + \dots + (-1)^n c_n$

compare

$-B_1 = I$

$AB_1 - B_2 = -c_1$

$AB_2 - B_3 = c_2$

$\vdots$

$AB_n = (-1)^n c_n$

After adding the above eq

$A^n - c_1 A^{n-1} + c_2 A^{n-2} + \dots + (-1)^n c_n I = 0$

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$\therefore$ A satisfies its own characteristic polynomial $P_n(\lambda)$

Example 1. $A = \begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix}$ verify Cayley-Hamilton theorem and hence find A^{-1}, $adj(A)$ and A^6.

Sol. $P_n(\lambda) = \lambda^2 - tr(A) + |A| = 0$

$\lambda^2 - 7\lambda + 6 = 0$

$A^2 = \begin{pmatrix} 8 & -14 \\ -14 & 29 \end{pmatrix}$

$A^2 - 7A + 6I = 0$

verified

$A^{-1} = \frac{1}{6}\begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix}$

$adj\ A = \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix}$

$A^2 = 7A - 6I$

$A^3 = 7A^2 - 6A \Rightarrow 43A - 42I$

$A^4 = 43A^2 - 42A \Rightarrow 259A - 258I$

$A^5 = 259A^2 - 258A \Rightarrow 1555A - 1554I$

$A^6 = 1555A^2 - 1554A \Rightarrow 9331A - 9330I$

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Algebraic Multiplicity

Suppose $A \in M_n(\mathbb{C})$ or $\mathbb{C}^{n \times n}$. let $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct eigen values of A with λ_1 repeating a_1 times λ_2 repeating a_2 times, ..., λ_k repeating a_k times

$$P_n(\lambda) = (\lambda-\lambda_1)^{a_1} (\lambda-\lambda_2)^{a_2} \ldots (\lambda-\lambda_k)^{a_k}$$

$k \leq n$

Here,

$a_1 + a_2 + \ldots + a_k = n$

In such cases we call $\lambda_1, \lambda_2, \ldots, \lambda_k$ are eigen values of A with algebraic multiplicites $a_1, a_2, \ldots a_k$. It is denoted with "algebraic multiplicity of λ_i, $1 \leq i \leq k$

i.e., $AM(\lambda_i) = a_i$, $1 \leq i \leq k$

e.g. Let $A \in M_7(\mathbb{R})$

$$P_7(\lambda) = (\lambda-1)^2 + (\lambda+2)^3 (\lambda-7)^1 (\lambda+10)^1$$

$AM(\lambda=1) = 2$
$AM(\lambda=-2) = 3$
$AM(\lambda=7) = 1$
$AM(\lambda=10) = 1$

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Geometric Multiplicity

The geometric multiplicity of an eigen value λ_i is defined by

$GM(\lambda = \lambda_i)$ = No. of LI eigen vectors w.r.t. eigen value λ_i

(or)

$= g_i = \dim N(A - \lambda_i I)$

(or)

$\Rightarrow g_i = n - r \longrightarrow$ rank of $(A - \lambda_i I)$

$n \downarrow$ order of matrix

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Note :

$* \ 1 \le g_m \le a_m$

Similar matrices

Let $A, B \in M_n(\mathbb{R})$
$A \sim B$ then

(i) $\det(A) = \det(B)$
(ii) $Tr(A) = Tr(B)$
(iii) Eigen values of both A and B are same
(iv) Eigen vectors may or may not be same

Diagonalization

Let $A \in M_n(\mathbb{R})$. A matrix A is said to be diagonalizable over the field R if $\exists$ an invertible matrix or modal matrix P such that

$D = P^{-1}AP$,

D is diagonal matrix

A is diagonalizable $\Leftrightarrow$ $A \sim D$

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Example: $A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{bmatrix}$

Sol $f_3(\lambda) = \lambda^3 - tr(A)\lambda^2 + (m_{11} + m_{22} + m_{33})\lambda - |A| = 0$

$\lambda^3 - 5\lambda^2 + (0+0+0)\lambda - 0 = 0$

$\lambda^3 - 5\lambda^2 = 0$

$\lambda^2(\lambda - 5) = 0$

$\lambda = 0, 0, 5$

$AM(\lambda = 0) = 2$

$AM(\lambda = 5) = 1$

For $\lambda = 0$

$(A - 0I)X = 0$

$AX = 0$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

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$x_1 - 2x_3 = 0$

$x_1 = 2x_3$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} 2x_3 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$N\{Ax\} = \text{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$\dim(\lambda = 0) = 2$

For $\lambda = 5$

$(A - 5I)x = 0$

$$\begin{pmatrix} -4 & 0 & -2 \\ 0 & -5 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$-4x_1 - 2x_3 = 0$

$-5x_2 = 0$

$x_2 = 0$

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$x_3 = -2x_1$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= \begin{pmatrix} x_1 \\ 0 \\ -2x_1 \end{pmatrix}$$

$$= x_1 \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$$

$\therefore N(A-5I) = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}\right\}$

$\therefore$ Eigen vector w.r.t $\lambda = 5$ is $\left\{\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}\right\}$

$gm(\lambda = 5) = 1$

$$P = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -2 \end{bmatrix}$$

$|P| = -2(-2) + 1(1)$
$= 4 + 1$
$= 5$

Since P is an orthogonal matrix

$P^{-1} = P'$

$$P^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & -2 \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{pmatrix} \begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & -2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & -10 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 25 \end{pmatrix}$$

Let $x_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$x_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$

$x_3 = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$

$v_1 = \dfrac{x_1}{\|x_1\|}$, $v_2 = \dfrac{x_2}{\|x_2\|}$, $v_3 = \dfrac{x_3}{\|x_3\|}$

$u = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$v = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$

$w = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$

$P = \frac{1}{\sqrt{5}} \begin{pmatrix} 0 & 2 & 1 \\ \sqrt{5} & 0 & 0 \\ 0 & 1 & -2 \end{pmatrix}$

$P^T = \frac{1}{\sqrt{5}} \begin{pmatrix} 0 & \sqrt{5} & 0 \\ 2 & 0 & 1 \\ 1 & 0 & -2 \end{pmatrix}$

$P^{-1} = P^T$ (orthogonal)

$P^{-1} = \frac{1}{\sqrt{5}} \begin{bmatrix} 0 & \sqrt{5} & 0 \\ 2 & 0 & 1 \\ 1 & 0 & -2 \end{bmatrix}$

$P^{-1}AP = \frac{1}{\sqrt{5}} \begin{bmatrix} 0 & \sqrt{5} & 0 \\ 2 & 0 & 1 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{bmatrix} \begin{bmatrix} 0 & 2 & 1 \\ \sqrt{5} & 0 & 0 \\ 0 & 1 & -2 \end{bmatrix} \frac{1}{\sqrt{5}}$

$= \frac{1}{5} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 25 \end{bmatrix}$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$= 0$

$D = P^{-1} A P$

Hence A is diagonalizable.

$AM(\lambda) = GM(\lambda)\ \forall\ \lambda$

• Theorems on diagonalizability

* A is diagonalizable

(i) iff $AM(\lambda) = GM(\lambda)\ \forall\ \lambda$
(ii) iff A has n distinct eigen values
(iii) iff A has n linearly independent eigen vectors

* Symmetric matrix is always diagonalizable

Example 1: $A = \begin{bmatrix} -3 & -1 & 1 \\ 7 & -5 & 1 \\ 6 & -6 & 2 \end{bmatrix}$

Given that eigen values of A are $2, -4, x$

(i) Find x.

Also verify whether matrix A is diagonalisable or not.

Sol. $2 - 4 + x = \text{Tr } A$

$-2 + x = 0$

$x = 2$

$AM(\lambda_i) = 1$

$(A - 2I)x = 0$

$$\begin{bmatrix} 1 & -1 & 1 \\ 7 & -7 & 1 \\ 6 & -6 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_3 = 0$

$x_2 = x_1 + x_3$

$N(A - 2I) = n - r$

$= 3 - 2$

$= 1$

$am(2) \neq gm(2)$

$\therefore$ A is not diagonalisable

$A = PDP^{-1}$

$A^n = PD^nP^{-1}$

$e^A = 1 + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots$

$P^{-1}e^AP = P^{-1}P + P^{-1}AP + \frac{P^{-1}A^2P}{2!} + \frac{P^{-1}A^3P}{3!} + \dots$

$= I + D + \frac{D^2}{2!} + \frac{D^3}{3!}$

$= e^D$

$\therefore \boxed{e^A = Pe^DP^{-1}}$

Special Matrix

(i) Complex matrix
(ii) Symmetric matrix
(iii) Skew-symmetric matrix
(iv) Orthogonal matrix
(v) Hermitian matrix
(vi) Skew-hermitian matrix
(vii) Unitary matrix

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Spectral Decomposition

For any symmetric matrix A $\exists$ an invertible matrix Q such that

$$A = Q D Q^T$$

- Q is orthogonal matrix (eigen)
- D is diagonal matrix

Singular Value Decomposition

Let $A \in M_{m \times n}$, then
$$A = U \Sigma V^T$$

- U is $(u_{ij})_{m \times m}$ and orthogonal
- Σ is diagonal matrix containing singular values
- V is $(v_{ij})_{n \times n}$ and orthogonal

Gram Matrix

If $A \in M_{m \times n}$, then the Gram matrix of A is $A^T A$

Suppose the eigen values of G_M are $\lambda_1, \lambda_2, \ldots, \lambda_n$

Note:

$A^T A$ is always +ve semi definite matrix

Now
$\lambda_i \geq 0$
$i = 1, 2, \ldots, n$

and $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$

we define
$\sigma_i = \sqrt{\lambda_i}$ such that

$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n$

σ_i's are called the singular values of a matrix A or gram matrix

Now
$A = U \Sigma V^T$

$U = (u_1, u_2, \ldots, u_m)$
where,
u_i is the e-vector corresponding to e-value λ_i of $A A^T$

and $V = (v_1, v_2, \ldots, v_n)$
where,
$v_i \rightarrow A^T A$

$$\Sigma = \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ 0 & & & \sigma_r \end{bmatrix}$$

$$A = (u_1\ u_2\ \dots\ u_m) \left(\begin{array}{cccc|c} \sigma_1 & & & & \\ & \sigma_2 & & & 0 \\ & & \ddots & & \\ & & & \sigma_r & \\ \hline & 0 & & & 0 \end{array} \right) \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_n^T \end{bmatrix}$$

$m \times m$ $\quad$ $m \times n$ $\quad$ $n \times n$

$$= \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \dots + \sigma_r u_r v_r^T$$

Example 1. Find the SVD of $A = \begin{bmatrix} 0 & 1 & 1 \\ \sqrt{2} & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

Sol. $A^T = \begin{bmatrix} 0 & \sqrt{2} & 0 \\ 1 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

$$A^TA = \begin{bmatrix} 0 & \sqrt{2} & 0 \\ 1 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ \sqrt{2} & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2\sqrt{2} & 0 \\ 2\sqrt{2} & 1+4+1 & 1+1 \\ 0 & 1+1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2\sqrt{2} & 0 \\ 2\sqrt{2} & 6 & 2 \\ 0 & 2 & 2 \end{bmatrix}$$

$$\lambda^3 - 10\lambda^2 + [(12-4) + (4-0) + (12-8)]\lambda -$$

$$[2(12-4) - 2\sqrt{2}(4\sqrt{2})] = 0$$

$$\lambda^3 - 10\lambda^2 + (8+4+4)\lambda - [16-16] = 0$$

$$\lambda^3 - 10\lambda^2 + 16\lambda = 0$$

$$\lambda(\lambda^2 - 10\lambda + 16) = 0$$

$\lambda = 0$

$$\lambda^2 - 10\lambda + 16 = 0$$

$$\lambda = \frac{10 \pm \sqrt{100-64}}{2}$$

$$= \frac{10 \pm 6}{2}$$

$$\lambda = 8, 2$$

$\lambda = 0, 2, 8$

For $\lambda = 0$

$(A' - \lambda I)x = 0$

$A'x = 0$

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$$\begin{bmatrix} 2 & 2\sqrt{2} & 0 \\ 2\sqrt{2} & 6 & 2 \\ 0 & 2 & 2 \end{bmatrix}$$

$\sigma_1 = \sqrt{8}$

$= 2\sqrt{2}$

$\sigma_2 = \sqrt{2}$

$\sigma_3 = \sqrt{0}$

$= 0$

$(A - 8I)x = 0$

$$\left\{\begin{bmatrix} 2 & 2\sqrt{2} & 0 \\ 2\sqrt{2} & 6 & 2 \\ 0 & 2 & 2 \end{bmatrix} - \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}\right\} x = 0$$

$$\begin{bmatrix} -6 & 2\sqrt{2} & 0 \\ 2\sqrt{2} & -2 & 2 \\ 0 & 2 & -6 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$-6x_1 = 0 - 2\sqrt{2}x_2$

$2x_2 - 6x_3 = 0$

$2\sqrt{2}x_1 - 2x_2 + 2x_3 = 0$

$x_1 = 0 - 2\sqrt{2}x_2$

$x_2 = x_3$

$+6x_1 = +2\sqrt{2}x_2$

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$\sqrt{3}x_1 = \sqrt{5}x_2$

$2\sqrt{2}x_1 - 2x_2 - 2x_3 = 0$

$2\sqrt{2}x_1 = 4x_2$

$2\sqrt{2}\left(\frac{\sqrt{5}x_2}{\sqrt{3}}\right) = 4x_2$

$\frac{2}{\sqrt{3}}x_2 = 4x_2$

$\frac{2}{\sqrt{3}} = 4$, not possible

Only possible if $x_2 = 0$

$x_2 = x_3$

$x_3 = 0$

$x_1 = \frac{\sqrt{5}}{3}x_2$

$= 0$

0 solution

But

Rank$(A') = 2$

$A = \{ u_1 \quad u_2 \quad u_3 \}$

$= \left\{ \frac{u_1}{\|u_1\|} \quad \frac{u_2}{\|u_2\|} \quad \frac{u_3}{\|u_3\|} \right\}$

$B = A^T A$

$U = A A^T$

$\Sigma = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Note:

$A v_j = \sigma_j u_j$

$B = AA^T$

$= \begin{pmatrix} 0 & 1 & 1 \\ \sqrt{2} & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 1 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 2 & 2 & 2 \\ 2 & 6 & 2 \\ 2 & 2 & 2 \end{pmatrix}$

Eigen values $= 2, 8, 0$

Let $\lambda_1 = 8$

$\lambda_2 = 2$

$\lambda_3 = 0$

$\sigma_1 = 2\sqrt{2}$

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$x_1 = 5$
$x_2 = 3$

For $\lambda = 8$
$(A - 8I)x = 0$

$$\begin{pmatrix} -6 & 2 & 2 \\ 2 & -2 & 2 \\ 2 & 2 & -6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 1 \\ 0 & -2 & 4 \\ 0 & 2 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\begin{pmatrix} 1 & -1 & 1 \\ 0 & -2 & 4 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$x_1 - x_2 + x_3 = 0$

$-2x_2 + 4x_3 = 0$
$x_2 = 2x_3$

$x_1 - x_3 = 0$
$x_1 = x_3$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$x = \begin{pmatrix} [illegible] \end{pmatrix}$

$(A - I)x = 0$

$$\begin{pmatrix} [illegible] & [illegible] & [illegible] \\ [illegible] & 1 & [illegible] \\ [illegible] & [illegible] & [illegible] \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

$x_1 + 2x_2 + x_3 = 0$

$x_2 + x_3 = 0.$

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x_1 [illegible] $x_3) = 0$

$x_1 = $ [illegible] x_3

$x_2 = x_3$

$$x = \begin{pmatrix} x_3 \\ -x_3 \\ x_3 \end{pmatrix}$$

$$= -x_3 \begin{pmatrix} -1 \\ +1 \\ 0 \end{pmatrix}$$

For $\lambda = 0$

$Ax = 0$

$$\begin{pmatrix} 2 & 2 & 2 \\ 2 & 6 & 2 \\ 2 & 2 & \cancel{8} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$x_1 + x_2 + x_3 = 0$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$2x_2 = 0$

$x_1 = -1$

$x = \begin{pmatrix} -x \\ 0 \\ x \end{pmatrix}$

$= x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$u_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

$u_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

$u_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$u = \begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix}$

$= \begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$

$\frac{u_1}{\|u_1\|} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

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$$\frac{u_2}{\|u_2\|} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$\frac{u_3}{\|u_3\|} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$U = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

for λ

$$v_1 = \begin{pmatrix} \sqrt{2} \\ \sqrt{3} \\ 1 \end{pmatrix}$$

$$v_2 = \begin{pmatrix} -1 \\ 0 \\ \sqrt{2} \end{pmatrix}$$

for $\lambda = 2$

$$\begin{pmatrix} 0 & 2\sqrt{2} & 0 \\ 2\sqrt{2} & 4 & 2 \\ 0 & 2 & 0 \end{pmatrix}$$

$\Rightarrow$

$$\begin{pmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \sqrt{2} & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\sqrt{2}x_1 + x_3 = 0$$

$$x_1 = \frac{-x_3}{\sqrt{2}}$$

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$$\begin{pmatrix} 2\sqrt{5} & [illegible] & 0 \\ 0 & [illegible] & [illegible] \end{pmatrix}$$

$$\begin{pmatrix} [illegible] & \sqrt{5} & 0 \\ [illegible] & 5 & 1 \\ 0 & 1 & [illegible] \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & \sqrt{2} & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$x_1 + \sqrt{2}x_2 = 0$$
$$x_2 + x_3 = 0$$

$$v_3 = \begin{pmatrix} \sqrt{2} \\ -1 \\ 1 \end{pmatrix}$$

$$v_1 = \begin{pmatrix} \sqrt{2} \\ 3 \\ 1 \end{pmatrix} , \quad v_2 = \begin{pmatrix} -1 \\ 0 \\ \sqrt{2} \end{pmatrix} , \quad v_3 = \begin{pmatrix} \sqrt{2} \\ -1 \\ 1 \end{pmatrix}$$

$$V = \begin{pmatrix} \frac{v_1}{\|v_1\|} & \frac{v_2}{\|v_2\|} & \frac{v_3}{\|v_3\|} \end{pmatrix}$$

$$= \left(\frac{1}{\sqrt{12}} \begin{pmatrix} \sqrt{2} \\ 3 \\ 1 \end{pmatrix} \quad \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ 0 \\ \sqrt{2} \end{pmatrix} \quad \frac{1}{\sqrt{4}} \begin{pmatrix} \sqrt{2} \\ -1 \\ 1 \end{pmatrix} \right)$$

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$$\begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{4}} \\ \frac{2}{\sqrt{6}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{\sqrt{3}}{\sqrt{12}} & \frac{1}{2\sqrt{3}} \end{pmatrix}$$

Now

$A = U\Sigma V^T$

$$\begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 2\sqrt{3} & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{3}{\sqrt{12}} & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Date :

Case (I)

If

$AA^T \;\begin{cases} r(A) \le \min(m, n) \\ r(A) \le \min\{5, 3\} \\ r(A) = 3 \end{cases}$

$A_{m \times n} = A_{5 \times 3}$

$$\Sigma = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{5 \times 3}$$

$\lambda = \lambda_1, \lambda_2, \lambda_3, 0, 0$

If $r = 3$

$\lambda_1, \lambda_2, 0, 0, 0$

$$\Sigma = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{5 \times 3}$$

Case (II)

$$u_i = \frac{A v_i}{\sigma_i}$$

$i = 1, 2, \ldots, r$

$f(A) = r$

$$v_i = \frac{A^T u_i}{\sigma_i}$$

$A = U \Sigma V^T$

$AV = U \Sigma$

$A^T = V \Sigma^T U^T$

If $m < n$

$A_{m \times n} = A_{3 \times 5}$

$r(A) = 2$

$$\Sigma = \begin{pmatrix} \sigma_1 & 0 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Note

* The non-zero eigen values of both AA^T and A^TA are same

AA^T has $m-n$ zero eigen values

* Let $v_1, v_2, \ldots, v_n$ be orthogonal vector of $\mathbb{R}^n$.
  Then $Av_1 = \sigma_1 u_1$
  $Av_2 = \sigma_2 u_2$
  $\vdots$
  $Av_n = \sigma_n u_n$

i.e., $u_1 = \dfrac{Av_1}{\sigma_1}$

$u_2 = \dfrac{Av_2}{\sigma_2}$

$\vdots$

$u_n = \dfrac{Av_n}{\sigma_n}$

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Special Matrices

- Complex matrix

$A \in M_{m \times n}(\mathbb{C})$

A matrix of order $n \times n$ is said to be complex matrix if atleast one of the element of A is complex (or) any element of A is complex number

ex $A = \begin{bmatrix} 2 & 5 \\ i & 3 \end{bmatrix}$ (a complex matrix)

if $C = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, ~~not~~ (a complex matrix)
$-1+i0$

- Symmetric matrix

Let $A \in M_n(\mathbb{R})$. The matrix A is said to be symmetric, if $A = A^T$ (or) $a_{ij} = a_{ji}$ $\forall$ i, j
$i \leq j$, $j \geq i$

ex $A = \begin{bmatrix} -4 & 5 \\ 5 & 0 \end{bmatrix}$

Properties:

Let $A \in M_n(\mathbb{R})$ be a symmetric matrix, then

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1. The eigen values of A are always real.

Let λ be an eigen value of A.

$Ax = \lambda x$

$\bar{A}\bar{x} = \bar{\lambda}\bar{x}$

$(\bar{A}\bar{x})^T = (\bar{\lambda}\bar{x})^T$

$\bar{x}^T\bar{A}^T = \bar{\lambda}\bar{x}^T$

$\bar{x}^T A^T = \bar{\lambda}\bar{x}^T$

$\bar{x}^T A = \bar{\lambda}\bar{x}^T$

$\bar{x}^T A x = \bar{\lambda}\bar{x}^T x$

$\bar{x}^T \lambda x = \bar{\lambda}\bar{x}^T x$

$\bar{x}^T x(\lambda - \bar{\lambda}) = 0$

$\lambda = \bar{\lambda}$

2. A is diagonalisable

3. A has n LI eigen vectors

- Skew-symmetric matrix

Let $A \in M_n(\mathbb{R})$. A is called skew-symmetric if $A = -A^T$ or $a_{ij} = -a_{ji} \ \forall \ i, j$

eg. $A = \begin{bmatrix} 0 & -4 \\ 4 & 0 \end{bmatrix}$

$B = \begin{bmatrix} 0 & 2 & 8 \\ -2 & 0 & -2 \\ -8 & 2 & 0 \end{bmatrix}$

Properties of skew symmetric matrix

1. The diagonal elements of skew symmetric matrix are always zero.

$a_{ij} = -a_{ji}$

for diagonal elements $a_{ii} = -a_{ii}$ i = j

$2a_{ii} = 0$

$a_{ii} = 0$

2. The eigen values of skew-symmetric matrix are either zero (or) purely Imaginary

Proof :

Let $A \in M_n(\mathbb{R})$ be skew symmetric matrix i.e., $A = -A^T$

$AX = \lambda X$

$\bar{A}\bar{x} = \bar{\lambda}\bar{x}$

$(\bar{A}\bar{x})^T = (\bar{\lambda}\bar{x})^T$

$\bar{x}^T \bar{A}^T = \bar{\lambda}\bar{x}^T$

$\bar{x}^T A^T = \bar{\lambda}\bar{x}^T$

$\bar{x}^T(-A) = \bar{\lambda}\bar{x}^T$

$-\bar{x}^T A = \bar{\lambda}\bar{x}^T$

$-\bar{x}^T A x = \bar{\lambda}\bar{x}^T x$

$-\bar{x}^T \lambda x = \bar{\lambda}\bar{x}^T x$

$\bar{x}^T x(-\lambda + (-\bar{\lambda})) = 0$

$\bar{\lambda} = -\lambda$

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$I + AB = 0$

$\Rightarrow A(BX) = 0$

$ABX = 0$

5. Determinant of odd order skew-symmetric matrix is 0.

$A = -A^T$

$|A| = |-A^T|$

$= (-1)^n |A^T|$

$= (-1)^n |A|$

$(1 - (-1)^n)|A| = 0$

if n = odd

$|A| = 0$

- Orthogonal matrix

Let $A \in M_n(\mathbb{R})$

A is called orthogonal if $AA^T = A^T A = I$

All orthogonal matrices are non-singular

Ex. $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$A^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

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Properties of orthogonal matrix

1. The determinant of orthogonal matrix is either 1 or (-1)

$AA^T = I$

$|AA^T| = |I|$

$|A||A^T| = 1$

$|A||A| = 1$

$|A|^2 = 1$

$|A| = \pm 1$

2. The eigen values of orthogonal matrix → unit modulus i.e. $|\lambda| = 1$

$Ax = \lambda x$

$\bar{A}\bar{x} = \bar{\lambda}\bar{x}$

$\bar{A}^T \bar{x}^T = \bar{\lambda}\bar{x}^T$

$\bar{x}^T \bar{A}^T = \bar{\lambda}\bar{x}^T$

$\bar{x}^T A^{-1} = \bar{\lambda}\bar{x}^T$

$\bar{x}^T A^{-1} x = \bar{\lambda}\bar{x}^T x$

$\bar{x}^T \frac{1}{\lambda} x = \bar{\lambda}\bar{x}^T x$

$\left(\frac{1}{\lambda} - \bar{\lambda}\right) = 0$

$\bar{\lambda} = \frac{1}{\lambda}$

3. inverse of orthogonal matrix is also orthogonal

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$AA^T = I$

$AA^T = AA^{-1}$

$A^T = A^{-1}$

Let $A^{-1} = B$

$BB^T = A^{-1}(A^{-1})^T$

$= A^{-1}(A^T)^{-1}$

$= A^{-1}A$

$= I$

- Hermitian matrix

Let $A \in M_n(\mathbb{C})$, A is said to be Hermitian matrix if

$A = A^*$

$A^* = \overline{A}^T$

or $a_{ij} = \overline{a_{ji}}$ $\forall$ i, j

ex. $A = \begin{pmatrix} 3 & -1+i & -2i \\ i-1 & -1 & 5+\sqrt{7}i \\ 2i & 5-\sqrt{7}i & 6 \end{pmatrix}$

Properties

1. The diagonal elements of HM are always real.

$a_{ii} = \overline{a_{ii}}$

$a_{ii} - \overline{a_{ii}} = 0$

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$2\,Im(a_{ii}) = 0$
$Im(a_{ii}) = 0$

2. The eigen values of HM are always real

3. Inverse of a HM is also a HM

4. Product of two HM A and B is hermitian iff AB = BA

- Skew Hermitian matrix

Let $A \in M_n(\mathbb{C})$,
A is said to be skew-hermitian matrix if $A = -A^*$
or
$a_{ij} = -\overline{a_{ji}} \ \forall \ i, j$

ex - $A = \begin{bmatrix} 2i & -2+i \\ 2+i & 0 \end{bmatrix}$

Properties of skew-hermitian matrix

* The diagonal elements of skew-hermitian matrix are either zero (or) purely imaginary.

$a_{ij} = -\overline{a_{ji}}$

$a_{ii} = -\overline{a_{ii}}$

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$a_{ii} + \bar{a}_{ii} = 0$

$2\,Re\,(a_{ii}) = 0$

$Re\,(a_{ii}) = 0$

* The eigen value of skew hermitian matrix are either zero or purely imaginary

* If A is skew hermitian matrix, then iA and -iA are hermitian matrix

let $B = iA$

$B = iA$

$= i(-A^*)$

$= -i(A^*)$

$= -B^*$

$\bar{B} = -i\bar{A}$

$\bar{B}^T = (-i\bar{A})^T$

$= -i\bar{A}^T$

$= -iA^*$

$= iA$

$= B$

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Unitary matrix

Let $A \in M_n(\mathbb{C})$.
A matrix is said to be unitary if
$AA^* = A^*A = I$

ex. $A = \frac{1}{\sqrt{7}} \begin{pmatrix} 1+2i & 1+i \\ 1-i & -1+2i \end{pmatrix}$

Properties of unitary matrix

1. The modulus of determinant of unitary matrix is 1.

$AA^* = I$
$|AA^*| = |I|$
$|A|^2 = 1$
$|A| = \pm 1$
$|A| = 1$

2. Eigen values of unitary matrix are having unit modulus.

$Ax = \lambda x$
$\bar{A}\bar{x} = \bar{\lambda}\bar{x}$
$\bar{x}^T\bar{A}^T = \bar{\lambda}\bar{x}^T$
$\bar{x}^T A^* = \bar{\lambda}\bar{x}^T$
$\bar{x}^T A^* A = \bar{\lambda}\bar{x}^T A$
$\bar{x}^T I = \bar{\lambda}\bar{x}^T A$
$\bar{x}^T x = \bar{\lambda}\bar{x}^T A x$

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$$\bar{x}^T x = \bar{\lambda} \bar{x}^T \lambda x$$

$$= \bar{x}^T \lambda \bar{\lambda} x$$

$$\lambda \bar{\lambda} = 1$$

$$|\lambda|^2 = 1$$

$$\lambda = 1$$

Positive - definite matrix

Let $A \in M_n(\mathbb{R})$

The matrix "A" is said to be positive definite if A is symmetric matrix and $x^T A x > 0 \quad \forall \ x \neq 0$

$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$x^T = (x_1 \ \ x_2)$

$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$x^T A x = (x_1 \ \ x_2) \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$= (x_1 \ \ x_2) \begin{pmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{pmatrix}$

$= a_{11}x_1^2 + a_{12}x_1x_2 + a_{21}x_1x_2 + a_{22}x_2^2$

$= a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2$

$\forall \ x \in \mathbb{R}^2$

called quadratic form

ex. $A = \begin{bmatrix} 2 & 4 \\ 4 & 3 \end{bmatrix}$

Sol $2x_1^2 + 3x_2^2 + 8x_1x_2$

Q. $q(A) = -x_1^2 + 2x_2^2 + \sqrt{3}x_3^2 + 4x_1x_2 + 6x_2x_3 + 5x_1x_3$

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$$A = \begin{bmatrix} -1 & & [illegible] \\ 2 & 2 & 3 \\ [illegible] & 3 & [illegible] \end{bmatrix}$$

Leading principal minors

Let $A \in M_n(\mathbb{R})$.
The minors of the form

$$|a_{11}|, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \ldots$$

are called leading principal minors (or) principal minors of a matrix A.

Definition :

Let $A \in M_n(\mathbb{R})$ be a symmetric matrix. The matrix A is called positive definite matrix if all leading principal minors are positive

ie, $|a_{11}| > 0$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} > 0 \ldots$$

(ii) $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

$\Delta_1 = |2|$
$= 2$

$\Delta_2 = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}$
$= 4 - 1$
$= 3$

$\Delta_3 = \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix}$
$= 2(3) + 1(-2) + 0$
$= 6 - 2$
$= 4$

$\therefore$ PD matrix

Definition

Let $A \in M_n(\mathbb{R})$ be a symmetric matrix. A is called positive definite iff all eigen values of A are positive

i.e, $\Delta_i > 0$ $\quad i = 1, 2, 3, \ldots, n$

ex. $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$

$$P_3(\lambda) = \lambda^3 - tr(A)\lambda^2 + (M_{11} + M_{22} + M_{33})\lambda - |A| = 0$$

$$= \lambda^3 - 6\lambda^2 + (3+4+3)\lambda - 4 = 0$$

$$= \lambda^3 - 6\lambda^2 + 10\lambda - 4 = 0$$

$$= (\lambda - 2)(\lambda^2 - 4\lambda + 2) = 0$$

$\lambda = 2$

$\lambda = 2 \pm \sqrt{2}$

$\lambda_i > 0$

$\therefore$ PD matrix.

Definition

Let $A \in M_n(\mathbb{C})$

The matrix A is called positive definite matrix, if A is hermitian and $x^* A x > 0 \ \forall \ x \neq 0 \in \mathbb{C}^n$

Note :

Other definitions using leading principal minors of eigen values hold for hermitian matrix

Positive semi-definite :

A hermitian matrix 'A' of order n is called PSD if $x^* A x \geq 0 \ \forall\, x (\neq 0) \in \mathbb{C}^n$.

$x^* = \bar{x}^T$

Atleast one eigenvalue must be zero then also PSD

$\lambda = 0, 0, 2$
PSD

$\lambda = 0, 1, 2$
PSD

Results :

* Let $A \in M_{m \times n}(\mathbb{R})$.
  AA^T is a symmetric matrix and it is positive semi definite always

(or)

Let A be any real matrix, then AA^T is always PSD

Proof: Now $(AA^T)^T = (A^T)^T A^T$
$= AA^T$

AA^T and A^TA are symmetric matrices

Let λ be any eigen value of A

$\Rightarrow$ Eigen value of AA^T is $\lambda_i\lambda_i = \lambda_i^2 \geq 0$
i.e., $\lambda_i \geq 0$, $i = 1, 2, 3, \ldots, n$

$\therefore AA^T$ is PSD

* If $A \in M_n(\mathbb{R})$ and A is non-singular, then AA^T is a PD matrix.

Proof

Given $|A| \neq 0$
$|AA^T| = |A||A^T| = |A|^2$
$|A| > 0$

$\therefore AA^T$ is PSD, but A is non-singular matrix, so AA^T is PD matrix

because $\lambda_i \geq 0$, but not $\lambda_i \neq 0$
(or)
If A is singular $\Rightarrow (AA^T) = |A||A^T| = |A|^2 = 0$

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Negative definiteness.

Let $A \in M_n(\mathbb{R})$, a matrix A is said to be negative definite if $x^T A x < 0 \ \forall \ x \neq 0 \in \mathbb{R}^n$ and A is called NSD if $x^T A x \leq 0 \ \forall \ x \in \mathbb{R}^n$

Note :

Let $A \in M_n(\mathbb{R})$ and $\lambda_1, \lambda_2, \dots, \lambda_r$ are distinct eigen values of A, where $1 \leq r \leq n$
then

S.No	Nature of A	Eigen values
1	PD	$\lambda_j > 0$
2	PSD	$\lambda_i = 0, \lambda_j > 0$
3.	ND	$\lambda_i < 0$
4	NSD	$\lambda_i = 0, \lambda_j < 0$
5	Indefinite	$\lambda_i < 0, \lambda_j > 0$

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SVD

Let $A \in M_{m \times n}(\mathbb{R})$, then the SVD of A is defined by

$A = U \Sigma V^T$

U is $m \times m$ orthogonal matrix

V is $n \times n$ orthogonal matrix

Σ is $m \times n$ diagonal matrix

$U = AA^T$

$V = A^T A$

$Av_i = \sigma_i u_i$

$u_i = \dfrac{Av_i}{\sigma_i}$

$v_i = \dfrac{A^T u_i}{\sigma_i}$ $\quad (A^T u_i = \sigma_i v_i)$

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Properties of SVD

Let $A \in M_{m \times n}(\mathbb{R})$ with $\rho(A) = r$. Assume that $A = U \Sigma V^T$ be a SVD

(i) $\rho(A)$ = no of non-zero singular values of A.
(ii) $R(A)$ = first r columns of U
(iii) $N(A)$ = first $n-r$ columns of V
(iv) $R(A^T)$ = first r columns of V
(v) $N(A^T)$ = first $n-r$ columns of U

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that A

of A

$Av_i = \sigma_i u_i$

$v \to A^T A$

$u \to A A^T$

$A_{m \times n} = U \Sigma V^T$

$$= \begin{bmatrix} \underbrace{u_1 \; u_2 \; \dots \; u_r}_{} & u_{r+1} \; \dots \; u_m \end{bmatrix} \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & 0 \\ & & \ddots & \\ & & & \sigma_r \\ & & & & 0 \end{bmatrix} \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_n^T \end{bmatrix}$$

$v = \{v_1 \; v_2 \; \dots \; v_n\}$

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