16 10 25

Unit - 4

Scaling

Let $A = \begin{pmatrix} 10 & 10^6 \\ 1 & 1 \end{pmatrix}$, $b = \begin{bmatrix} 10^6 \\ 2 \end{bmatrix}$

and solve $Ax = b$

suppose D be a diagonal matrix, such that the $D^{-1}A$ will convert the large numbers / entries in a matrix to small entries, so that computational time reduce and generate output quickly.
Such a process is called "scaling"

Let $\hat{A} = D^{-1}A$ and $\hat{b} = D^{-1}b$

→ look for solution of the system
$\hat{A}\hat{x} = \hat{b}$

Example: $D = \begin{pmatrix} 10^6 & 0 \\ 0 & 1 \end{pmatrix}$ $\qquad$ $D = \begin{pmatrix} 1 & 0 \\ 0 & 10^6 \end{pmatrix}$

$D^{-1} = \begin{pmatrix} \frac{1}{10^6} & 0 \\ 0 & 1 \end{pmatrix}$ $\qquad$ $D^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{10^6} \end{pmatrix}$

$= \begin{pmatrix} 10^{-6} & 0 \\ 0 & 1 \end{pmatrix}$ $\qquad$ $= \begin{pmatrix} 1 & 0 \\ 0 & 10^{-6} \end{pmatrix}$

$D^{-1}A = ($

PAGE NO.
DATE / /

$$\begin{pmatrix} 10^{-6} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 10 & 10^{6} \\ 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 10^{-5} & 1 \\ 1 & 1 \end{pmatrix}$$

$D^{-1}b$ for (ii) similarly

$$D^{-1}b = \begin{pmatrix} 10^{-6} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 10^{6} \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Now $\hat{a}\hat{x} = \hat{b}$

$$\begin{pmatrix} 10^{-5} & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} = \begin{pmatrix} 10^{-5} & 1 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 10^{-5} & -1 \\ -1 & 10^{-5} \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$|\hat{a}| = 10^{-5} - 1$$

$$\begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} = \frac{1}{10^{-5}-1} \begin{pmatrix} 1 & -1 \\ -1 & 10^{-5} \end{pmatrix}_{2\times 2} \begin{pmatrix} 1 \\ 2 \end{pmatrix}_{2\times 1}$$

$$= \frac{1}{10^{-5}-1} \begin{pmatrix} 1-2 \\ -1+2\times 10^{-5} \end{pmatrix}$$

$$= \frac{1}{10^{-5}-1}\begin{pmatrix} -1 \\ -1+2\times10^{-5} \end{pmatrix}$$

Estimating condition number

Hager's algorithm for estimating condition number:

if $A \in M_n(\mathbb{R})$ and A is invertible

$Ax = b$

$$\|A^{-1}\|_1 = \max_{x \neq 0 \in \mathbb{R}} \left\{ \frac{\|A^{-1}x\|}{\|x\|} \right\}$$

$$\|A^{-1}\|_1 = \max_{\|x\| = 1} \left\{ \|A^{-1}x\| \right\}$$

Algorithm:

Step 1 Choose a vector v such that $\|v\| = 1$ (1-norm).

Step 2 $K = 1, 2, \ldots$ (iterations)

(a): solve $A u_k = v_k$ for u

(b): consider $\omega = \mathrm{sign}(u)$ defined by

$$\omega_i = \begin{cases} 1 & \text{if } u_i \geq 0 \\ -1 & \text{if } u_i < 0 \end{cases}$$

(sign $\bar{u}$)

Example: $u = \begin{pmatrix} -2 \\ 5 \end{pmatrix} \rightarrow \omega = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

PAGE NO.
DATE: / /

(c): Solve $a^T x = \omega$ for x
(d): Calculate $\|x\|_\infty$ and check $\|x\|_\infty \leq \|x\|_1$ or not

If condition satisfies, then solution ends, (approx. sol)

(e): If not i.e., $\|x\|_\infty > \|x\|_1$ then choose $v = e_j$ and repeat the loop.

where e_j is the j^{th} index for $\|x\|_\infty = |x_j|$

(or) $e_j = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \end{pmatrix}$ j^{th} component of $\mathbb{R}^n$.

(standard basis)

Example:
$x = \begin{pmatrix} -7 \\ 4 \end{pmatrix}$

$\|x\|_\infty = 7 = |x_1|$ (max abs row sum)
$v = e_1$

If $x = \begin{pmatrix} 4 \\ -10 \end{pmatrix}$

$\|x\|_\infty = 10$
$v = e_2$

Example 1. $A = \begin{pmatrix} 1 & -10 & 0 & 0 \\ 0 & 1 & -10 & 0 \\ 0 & 0 & 1 & -10 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{4\times 4}$

$\|A\|_1$

Sol)

$A^{-1} = \begin{pmatrix} 1 & 10 & 10^2 & 10^3 \\ 0 & 1 & 10 & 10^2 \\ 0 & 0 & 1 & 10 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Find approx value of $K(A)$ using Hager's algorithm

Sol. $\|A\|_1 = 11$ (max abs col sum)

$\|A^{-1}\|_1 = 10^3 + 10^2 + 10 + 1$
$= 1111$

11×1111 $\qquad \|A\|_1 \|A^{-1}\|_1$

12221

(Exact value)

Now use Hager's algorithm

Step 1. $v = \frac{1}{4}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

1st iteration

(a): $Au = v$

$u = A^{-1} v$

$$= \begin{pmatrix} 1 & 10 & 10^2 & 10^3 \\ 0 & 1 & 10 & 10^2 \\ 0 & 0 & 1 & 10 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{4\times4} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}_{4\times1} \frac{1}{4}$$

$$= \frac{1}{4} \begin{pmatrix} 1+10+10^2+10^3 \\ 1+10+10^2 \\ 1+10 \\ 1 \end{pmatrix}$$

$$= \frac{1}{4} \begin{pmatrix} 1111 \\ 111 \\ 11 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 277.75 \\ 27.75 \\ 2.75 \\ 0.25 \end{pmatrix}$$

$\|u\|_1 = 308.5$

(b) a) $w = \text{sign}(u)$

$$= \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

(c) $A^T x = \omega$

$x = (A^{-1})^T \omega$

$= \begin{pmatrix} 1 & 10 & 10^2 & 10^3 \\ 0 & 1 & 10 & 10^2 \\ 0 & 0 & 1 & 10 \\ 0 & 0 & 0 & 1 \end{pmatrix}^T \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} 1 \\ 11 \\ 111 \\ 1111 \end{pmatrix}$

$\|x\|_\infty = 1111$

$\|x\|_\infty \geq \|u\|_1$

2nd iteration

(a) choose $v = e_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ (4th row max)

(b) $Av = v$

$u = A^{-1} v$

$= \begin{pmatrix} 1 & 10 & 10^2 & 10^3 \\ 0 & 1 & 10 & 10^2 \\ 0 & 0 & 1 & 10 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

DATE:

$$= \begin{pmatrix} 1000 \\ 100 \\ 10 \\ 1 \end{pmatrix}$$

$\|x\|_1 = 1111$

(c) $w = \text{sign}(u)$

$$= \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

(d) $A^T x = w$

$$x = \begin{pmatrix} 1 \\ 11 \\ 111 \\ 1111 \end{pmatrix}$$

$\|x\|_\infty = 1111$

(d) $\|x\|_\infty = \|u\|_1$

Hence $\|A^{-1}\|_1 = \|u\|_1$

Note

$\|A^{-1} e_j\| \geq |w^T A^{-T} e_j|$

$\|A\| \, \|A^{-1}\| = 12221$

$$K^{H}(A) \simeq \|A\|_1 \, \|A^{-1}\|_1^{H}$$
$$\simeq 11 \times 1111$$
$$\simeq 12221$$

Good

$\therefore K(A) \cong K^{H}(A)$

$S = \{ x \in \mathbb{R}^n \mid \|Ax - b\|^2 \text{ is minimum} \}$

PAGE NO
DATE

QR factorisation or decomposition

for any rectangular matrix

Case I

Suppose $A \in M_{m \times n}(\mathbb{R})$ and columns are LI or $\rho(A) = n$

Here $A = (a_1 \; a_2 \; \dots \; a_n)$, where $a_j \in \mathbb{R}^m$

i.e., $a_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}$

Let $Q = (q_1 \; q_2 \; \dots \; q_n)$, $q_j \in \mathbb{R}^m$

and q_j is orthonormal vector generated from G-S process
(Gram-Schmidt)

Now from G-S process, we have
$\text{span}\{a_1\} = \text{span}\{q_1\}$

$a_1 = r_{11} q_1$

If $x \in \text{span}(u)$
$x = \alpha u$

Similarly,

$\text{span}\{a_1, a_2\} = \text{span}\{q_1, q_2\}$

$a_2 = r_{12} q_1 + r_{22} q_2$

$\vdots$

$\text{span}\{a_1, a_2, \dots a_n\} = \text{span}\{q_1, q_2, \dots q_n\}$

$a_n = r_{1n} q_1 + r_{2n} q_2 + \dots + r_{nn} q_n$

$r_{ii} \neq 0 \quad \because$ LI set

Now consider

$$R = \begin{pmatrix} r_{11} & r_{12} & \dots & r_{1n} \\ 0 & r_{22} & \dots & r_{2n} \\ \dots & & & \dots \\ 0 & 0 & \dots & r_{nn} \end{pmatrix}_{n \times n}$$

(Upper triangular matrix)

$$(q_1 \; q_2 \dots q_n) \cdot \begin{pmatrix} r_{11} & r_{12} & \dots & r_{1n} \\ 0 & r_{22} & \dots & r_{2n} \\ \dots & & & \dots \\ 0 & & \dots & r_{nn} \end{pmatrix}$$

$= (r_{11} q_1 \quad r_{12} q_1 + r_{22} q_2 \ldots r_{1n} q_1 + r_{2n} q_2 + \ldots + r_{nn} q_n)$

$= (a_1 \quad a_2 \ldots a_n)$

$= A$

$\therefore A = QR$

where $Q^T Q = I_n$

R = upper triangular matrix, $r_{ii} \neq 0$

$Q = m \times n$

$R = n \times n$

$QQ^T \neq I$

The process is called QR factorisation or decomposition of a matrix A using G-S process.

QQ^T

$$q_1 = \begin{pmatrix} a_{11}^* \\ a_{21}^* \\ \vdots \\ a_{m1}^* \end{pmatrix}$$

Similarly

$$q_2 = \begin{pmatrix} a_{12}^* \\ a_{22}^* \\ \vdots \\ a_{m2}^* \end{pmatrix}$$

$$QQ^* = \begin{pmatrix} a_{11}^* & a_{12}^* & \cdots & a_{1n}^* \\ \vdots & \vdots & & \vdots \\ a_{m1}^* & a_{m2}^* & \cdots & a_{mn}^* \end{pmatrix} \begin{pmatrix} a_{11}^* & \cdots & a_{m1}^* \\ \vdots & & \vdots \\ a_{1n}^* & \cdots & a_{mn}^* \end{pmatrix}$$

PAGE NO.
DATE: 17 10 2025

Theorem

If $A \in M_{m\times n}(\mathbb{R})$, $m \geq n$, $\rho(A) = n$, then $\exists\, Q \in M_{m\times n}$ orthogonal matrix and $R \in M_{n\times n}(\mathbb{R})$ is upper triangular matrix such that

(i) $A = QR$
(ii) $Q^TQ = I_n$
(iii) R is an invertible matrix with $r_{ii} \neq 0$ and $r_{ii} > 0$.

Note:
The above theorem is valid for square matrices as well with $Q^TQ = QQ^T = I$

Also called QR-factorisation or decomposition

$\rho(A) = n \iff$ columns of A are L.I.

Theorem
(Generalised QR-factorisation)

Suppose $A \in M_{m\times n}$ and $\rho(A) = r < n$, $m \geq n$. Then $\exists$ an orthogonal matrix $Q \in M_{m\times r}$ and an upper triangular matrix $R \in M_{r\times n}$ such that

(i) $A = QR$
(ii) $Q^TQ = I_r$

Note: If $A = QR$

$$\Rightarrow Q^T A = Q^T(QR)$$
$$= (Q^TQ)R$$
$$= IR$$
$$= R$$

$$\therefore R = Q^T A$$

Example 1 Find QR-decomposition of a matrix

$$A = \begin{pmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{pmatrix}_{4\times 3}$$

Sol $$A = \begin{pmatrix} 1 & -1 & 4 \\ 0 & 5 & -6 \\ 0 & 5 & -2 \\ 0 & 0 & -4 \end{pmatrix} \qquad \begin{aligned} R_2 &\to R_2 - R_1 \\ R_3 &\to R_3 - R_1 \\ R_4 &\to R_4 - R_1 \end{aligned}$$

$$= \begin{pmatrix} 1 & -1 & 4 \\ 0 & 5 & -6 \\ 0 & 0 & 4 \\ 0 & 0 & 4 \end{pmatrix} \qquad \begin{aligned} R_3 &\to R_3 - R_2 \\ R_4 &\to \frac{R_4}{-1} \end{aligned}$$

$$= \begin{pmatrix} 1 & -1 & 4 \\ 0 & 5 & -6 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix} \qquad R_4 \to R_4 - R_3$$

PAGE NO.
DATE: / /

$\therefore \rho(A) = 3 =$ no of columns

$u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} -1 \\ 4 \\ 4 \\ -1 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 4 \\ -2 \\ 2 \\ 0 \end{pmatrix}$ are LI

$v_1 = \dfrac{w_1}{\|w_1\|}$

$w_1 = u_1$

$\|w_1\| = \sqrt{1^2+1^2+1^2+1^2}$

$= \sqrt{4}$

$= 2$

$v_1 = \dfrac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

$w_2 = u_2 - \langle u_2 , v_1 \rangle v_1$

$= \begin{pmatrix} -1 \\ 4 \\ 4 \\ -1 \end{pmatrix} - \dfrac{1}{2}\begin{pmatrix} -1 \\ + \\ 4 \\ + \\ 4 \\ + \\ -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}\dfrac{1}{2}$

$= \begin{pmatrix} -1 \\ 4 \\ 4 \\ 1 \end{pmatrix} - \dfrac{1}{4}\,\cancel{6}^{3}\begin{pmatrix} 1 \\ 1 \\ \\ \end{pmatrix}$

$$= \begin{pmatrix} -1 \\ 4 \\ 4 \\ -1 \end{pmatrix} - 3/2 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -5/2 \\ 5/2 \\ 5/2 \\ -5/2 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} -5 \\ 5 \\ 5 \\ -5 \end{pmatrix}$$

$$\|v_2\| = \sqrt{\frac{25}{4} + \frac{25}{4} + \frac{25}{4} + \frac{25}{4}}$$

$$= \sqrt{\frac{100}{4}}$$

$$= \sqrt{19} = \sqrt{25} = 5$$

$$\theta_2 = \frac{1}{2\sqrt{19}} \begin{pmatrix} -5 \\ 5 \\ 5 \\ -5 \end{pmatrix}$$

$$= \frac{1}{10} \begin{pmatrix} -5 \\ 5 \\ 5 \\ -5 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \end{pmatrix}$$

Similarly,

$$v_3 = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

$$w_3 = u_3 - \langle v_3, u_2 \rangle u_2 - \langle v_3, u_1 \rangle u_1$$

Let $Q = (v_1 \;\; v_2 \;\; v_3)$

$$= \frac{1}{2}\begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & -1 \end{pmatrix}$$

$$Q^T = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$$

$$Q^T Q = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & -1 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

$$= I_3$$

Now $R = Q^T A$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix}_{3\times4} \begin{pmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{pmatrix}_{4\times3}$$

$$= \frac{1}{2}\begin{pmatrix} 4 & 6 & 4 \\ 0 & 10 & -4 \\ 0 & 0 & 8 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 3 & 2 \\ 0 & 5 & -2 \\ 0 & 0 & 4 \end{pmatrix}$$

$\therefore A = QR$

$$= \frac{1}{2}\begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & -1 \end{pmatrix}_{4\times3} \begin{pmatrix} 2 & 3 & 2 \\ 0 & 5 & -2 \\ 0 & 0 & 4 \end{pmatrix}_{3\times3}$$

$$= \frac{1}{2}\begin{pmatrix} 2 & -2 & 8 \\ 2 & 8 & -4 \\ 2 & 8 & 4 \\ 2 & -2 & 0 \end{pmatrix}$$

DATE:

$$= \begin{pmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{pmatrix}_{4\times 3}$$

Ques. 2. $A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ -1 & 0 & -2 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 1 \end{pmatrix}_{4\times 4}$

Sol. $A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 \end{pmatrix}$ $\quad R_2 \to R_2 + R_1$, $R_3 \to R_3 - R_1$, $R_4 \to R_4 - R_1$

$$= \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \quad R_4 \to R_4 + R_2$$

$$= \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad R_3 \leftrightarrow R_4$$

$\therefore \rho(A) = 3$
< 4

No of columns = 3.

21 10 2025

Take 1, 2, 4 column

$$u_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix} \qquad u_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad u_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$v_1 = \frac{w_1}{\|w_1\|}$$

$$\|w_1\| = \sqrt{1+1+1+1}$$
$$= 2$$

$$v_1 = \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$$

$$w_2 = u_2 - \langle u_2, v_1 \rangle v_1$$

$$= \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{2}(1+1)\frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$$

DATE: / /

$$= \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{pmatrix}$$

$$v_2 = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}$$

$$v_3 = \frac{w_3}{\|w_3\|}$$

$$w_3 = u_3 - \langle u_3 . v_1 \rangle v_1 - \langle u_3 . v_2 \rangle v_2$$

$$= \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} - \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \cdot \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix} \right\} \cdot \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} - \left[\right.$$

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \cdot \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right\} \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right\}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \left[\frac{1}{2} \left[0 - 1 + 1 \right] \left[v_1 \right] \right.$$

$$- \frac{1}{2} \left[1 - 1 \right] (v_2)$$

$$= \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\|u_3\| = \sqrt{2}$$

$$e_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1/2 & 1/2 & 0 \\ -1/2 & 1/2 & 1/\sqrt{2} \\ 1/2 & 1/2 & 0 \\ 1/2 & -1/2 & 1/\sqrt{2} \end{bmatrix}_{4\times 3}$$

$$Q^T = \begin{bmatrix} 1/2 & -1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1/2 & -1/2 \\ 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}_{3\times 4}$$

$$R = Q^T A$$

$$= Q^T A = \begin{bmatrix} 1/2 & -1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1/2 & -1/2 \\ 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}_{3\times 4} \begin{bmatrix} 1 & 1 & 1 & 0 \\ -1 & 0 & -2 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 1 \end{bmatrix}$$

$$Q^TQ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= I_3$$

$$R = Q^TA$$
$$= 3\times4 \quad 4\times4$$
$$= 3\times4$$

$$= Q^TA$$

$$= \begin{pmatrix} 1/2 & -1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1/2 & -1/2 \\ 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}_{3\times4} \begin{pmatrix} 1 & 1 & 1 & 0 \\ -1 & 0 & -2 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 1 \end{pmatrix}_{4\times4}$$

$$= \begin{pmatrix} 1/2 + 1/2 + 1 & 1/2 + 1/2 & 1/2 + 1 + 1/2 + 1 & -1/2 + 1/2 \\ 1/2 - 1/2 + 1/2 - 1/2 & 1/2 + 1/2 & 1/2 - 1 + 1/2 - 1 & 1/2 - 1/2 \\ -1/\sqrt{2} + 1/\sqrt{2} & \cancel{1/\sqrt{2}}\ 0 & \frac{-2}{\sqrt{2}} + \frac{2}{\sqrt{2}} & 1/\sqrt{2} + 1/\sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & \sqrt{2} \end{pmatrix}.$$

PAGE NO:
DATE: / /

$A = QR$

$$= \begin{bmatrix} 1/2 & 1/2 & 0 \\ -1/2 & 1/2 & 1/\sqrt{2} \\ 1/2 & 1/2 & 0 \\ 1/2 & -1/2 & 1/\sqrt{2} \end{bmatrix}_{4\times 3} \begin{bmatrix} 2 & 1 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & \sqrt{2} \end{bmatrix}_{3\times 4}$$

$$= \begin{bmatrix} 1 & 1 & 3/2 - 1/2 & 0 \\ -1 & 0 & -3/2 - 1/2 & 1 \end{bmatrix}$$

PAGE NO.
DATE:

* Solve the given linear system using QR decomposition

Suppose $Ax = b$ be a linear system, where $A \in M_{m \times n}$, $x \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$

Let $A = QR$ — (i) where $Q^TQ = I$ and R is upper triangular matrix with $r_{ii} \neq 0$.

Now from (i) we get

$$(QR)x = b$$
$$Q(Rx) = b$$
$$Q^TQ(Rx) = Q^Tb$$
$$I(Rx) = Q^Tb$$
$$Rx = Q^Tb \quad \text{—— (ii)}$$

eq (ii) is upper triangular system

$$A = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 10 \end{pmatrix}$$

$$u_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$u_2 = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$$

$$u_3 = \begin{pmatrix} 7 \end{pmatrix}$$

Note

$$Q = \begin{bmatrix} 1/\sqrt{14} & \sqrt{\frac{2}{7}} & \frac{3}{\sqrt{14}} \\ \frac{4}{\sqrt{21}} & \frac{1}{\sqrt{21}} & \frac{-2}{\sqrt{21}} \\ \frac{1}{\sqrt{6}} & -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$R = \begin{bmatrix} \sqrt{14} & 16\sqrt{\frac{2}{7}} & \frac{53}{\sqrt{14}} \\ 0 & 3\sqrt{\frac{3}{7}} & \frac{16}{\sqrt{21}} \\ 0 & 0 & \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$Q^T = \begin{bmatrix} \frac{1}{\sqrt{14}} & \frac{4}{\sqrt{21}} & \frac{1}{\sqrt{6}} \\ \sqrt{\frac{2}{7}} & \frac{1}{\sqrt{21}} & -\sqrt{\frac{2}{3}} \\ \frac{3}{\sqrt{14}} & \frac{-2}{\sqrt{21}} & \frac{1}{\sqrt{6}} \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$Q^T R = A$

$Ax = B$

$(QR)x = B$

$Rx = Q^T B$

$QB = Rx$

$$Rx = \begin{bmatrix} \sqrt{19}\,x + 16\sqrt{\frac{2}{7}}\,y + \frac{53z}{\sqrt{14}} \\ 3\sqrt{\frac{3}{7}}\,y + \frac{16z}{\sqrt{21}} \\ \frac{z}{\sqrt{6}} \end{bmatrix}$$

$$QB = \begin{bmatrix} 3\sqrt{\frac{2}{7}} \\ \sqrt{\frac{3}{7}} \\ 0 \end{bmatrix}$$

$\frac{z}{\sqrt{6}} = 0$

$z = 0$

$3\sqrt{\frac{3}{7}}\,y + 0 = \sqrt{\frac{3}{7}}$

$3y = 1$

$y = \frac{1}{3}$

$\sqrt{3}\,\sqrt{19}\,x + 16\sqrt{\frac{2}{7}} = 3\sqrt{\frac{2}{7}} \quad x = -7\sqrt{2}$ ...

PAGE NO.
DATE: 22/10/2025

Theorem:

Suppose $\{u_1, u_2, \ldots, u_n\}$ be a basis of vector space (or) Inner Product space V

Let $v_1 = \dfrac{u_1}{\|u_1\|_2}$ and define

$$v_{k+1} = \frac{u_{k+1} - P_k}{\|u_{k+1} - P_k\|_2}, \quad k = 1, 2, \ldots n-1$$

where $P_k = \langle u_{k+1}, v_1 \rangle v_1 + \langle u_{k+1}, v_2 \rangle v_2 + \ldots$

$$+ \langle u_{k+1}, v_k \rangle v_k$$

is a projection of u_{k+1} onto the subspace of span $\{v_1, v_2, \ldots, v_k\}$

Then $\{v_1, v_2, \ldots, v_n\}$ is an orthonormal set of V and span $\{u_1, u_2, \ldots u_n\}$ = Span $\{v_1, v_2, \ldots v_n\}$

Proof:

$$v_1 = \frac{u_1}{\|u_1\|}$$ and we know that

vector

$$v_{k+1} = \frac{u_{k+1} - P_k}{\|u_{k+1} - P_k\|}, \quad k = 1, 2$$

Note:

$$P_1 = \operatorname{proj}_{\operatorname{span}\{v_1\}} u_2$$

$$= \langle u_2, v_1 \rangle v_1$$

$+ \ldots$

For $k = 1$

$$v_2 = \frac{u_2 - P_1}{\|u_2 - P_1\|}$$

of

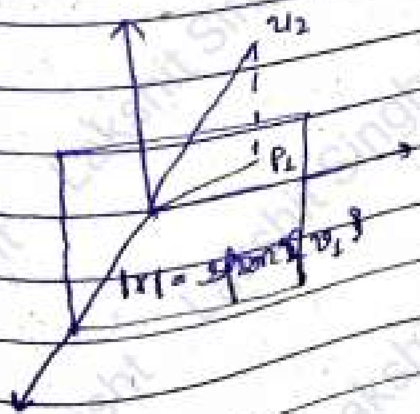
For $k = 2$

$$P_2 = \operatorname{proj}_{\operatorname{span}\{v_1, v_2\}} u_3$$

t of

$v_2, \ldots, v_n\}$

$$v_3 = \frac{u_3 - P_2}{\|u_3 - P_2\|}$$



$P_2 = \langle u_3, v_1 \rangle v_1 + \langle u_3, v_2 \rangle v_2$

QR decomposition through projections

Let $A = (a_1, a_2\ a_3)$, $a_i \in \mathbb{R}^3$, $1 \le i \le 3$ and $\rho(A) = 3$ (or) A has 3 L.I. columns. Suppose $Q = (q_1\ q_2\ q_3)$, $q_i \in \mathbb{R}^3$ be an orthogonal matrix i.e., q_1, q_2 and q_3 are orthonormal basis for $\mathbb{R}^3$. The following procedure can be used to obtain q_1, q_2 and q_3

Note :

$a_1 = r_{11} q_1$
$a_2 = r_{12} q_1 + r_{22} q_2$
$a_3 = r_{13} q_1 + r_{23} q_2 + r_{33} q_3$

$\langle a_1, a_1 \rangle = \langle r_{11} q_1, r_{11} q_1 \rangle$
$\|a_1\|^2 = |r_{11}|^2 \langle q_1, q_1 \rangle$
$= |r_{11}|^2 \|q_1\|^2$
$= |r_{11}|^2 (1)$
$= |r_{11}|^2$

$\|a_1\| = |r_{11}|$

Similarly for others

Step 1 :

let $r_{11} = \|a_1\|$

Now from the previous theorem, then

$$q_1 = \frac{a_1}{\|a_1\|}$$

$$= \frac{a_1}{r_{11}}$$

Step 2:

Now let $r_{12} = \langle a_2, q_1 \rangle$ and

$P_1 = r_{12}\, q_1$ $\qquad\qquad P_1 = \langle a_2, a_1 \rangle q_1$

$= \text{proj}_{\text{span}\{q_1\}}\, u_2$

Now $r_{22} = \|a_2 - P_1\|$, then

$$q_2 = \frac{a_2 - P_1}{r_{22}}$$

$$= \frac{1}{r_{22}}(a_2 - P_1)$$

Step 3:

let $r_{13} = \langle a_3, q_1 \rangle$ and $r_{23} = \langle a_3, q_2 \rangle$

Define $P_2 = r_{13}\, q_1 + r_{23}\, q_2 = \text{proj}_{\text{span}\{q_1, q_2\}}\, u_3$

and we can obtain

$r_{33} = \|a_3 - P_2\|$

$$q_3 = \frac{a_3 - P_2}{r_{33}}$$

$$= \frac{1}{r_{33}}(a_3 - P_2)$$

Theorem

Let $A \in M_{m \times n}(\mathbb{R})$, $m \geq n$ and $\rho(A) = n$ (or) columns of A are LI. Then $\exists$ an orthogonal matrix $Q \in M_{m \times n}(\mathbb{R})$ and upper triangular matrix $R \in M_{n \times n}(\mathbb{R})$ such that $A = QR$ where $Q^T Q = I_n$

Sol. $A = \begin{pmatrix} 1 & -2 & -1 \\ 2 & 0 & 1 \\ 2 & -4 & 2 \\ 4 & 0 & 0 \end{pmatrix}_{4\times 3}$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 2R_1$

$R_4 \to R_4 - 4R_1$

$A = \begin{pmatrix} 1 & -2 & -1 \\ 0 & 4 & 3 \\ 0 & 0 & 4 \\ 0 & 8 & 4 \end{pmatrix}$

$R_4 \to R_4 - 2R_2$

$= \begin{pmatrix} 1 & -2 & -1 \\ 0 & 4 & 3 \\ 0 & 0 & 4 \\ 0 & 0 & 2 \end{pmatrix}$

$R_4 \to R_4 - \frac{1}{2} R_3$

$= \begin{pmatrix} 1 & -2 & -1 \\ 0 & 4 & 3 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix}$

$\rho(A) = 3$

= No of LI columns

Let $A = QR$ $\qquad A = (a_1 \; a_2 \; a_3)$

$r_{11} = \|a_1\|$

$$a_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \\ 4 \end{pmatrix}$$

$$\|a_1\| = \sqrt{1+4+4+16}$$
$$= \sqrt{25}$$
$$= 5$$

$\therefore r_{11} = 5$

$$q_1 = \frac{a_1}{r_{11}}$$
$$= \frac{1}{5}\begin{pmatrix} 1 \\ 2 \\ 2 \\ 4 \end{pmatrix}$$

$P_1 = r_{11} q_1$

$r_{12} = \langle a_2, q_1 \rangle$

$$= \begin{pmatrix} -2 \\ 0 \\ -4 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1/5 \\ 2/5 \\ 2/5 \\ 4/5 \end{pmatrix}$$

$$= -\frac{2}{5} - \frac{8}{5}$$

$$= -2$$

$$P_1 = -2 \cdot \frac{1}{5} \begin{pmatrix} 1 \\ 2 \\ 2 \\ 4 \end{pmatrix}$$

$$= \frac{-2}{5} \begin{pmatrix} 1 \\ 2 \\ 2 \\ 4 \end{pmatrix}$$

$$= \frac{4}{5}\begin{bmatrix}-2\\1\\-4\\2\end{bmatrix}$$

$$r_{22} = \sqrt{\frac{64}{25}+\frac{16}{25}+\frac{256}{25}+\frac{64}{25}}$$

$$= \sqrt{\frac{400}{25}}$$

$$= \sqrt{\frac{\cancel{400}^{4}\times 4}{\cancel{25}}}$$

$$= 4$$

$$q_2 = \frac{1}{r_{22}}\left(a_2 - p_1\right)$$

$$= \frac{1}{\cancel{4}}\,\frac{\cancel{4}}{5}\begin{bmatrix}-2\\1\\ \vdots\end{bmatrix}$$

$$= \begin{pmatrix} -1 \\ 1 \\ 2 \\ 0 \end{pmatrix} \cdot \frac{1}{5} \begin{pmatrix} 1 \\ 2 \\ 2 \\ 4 \end{pmatrix}$$

$$= \frac{1}{5} \begin{pmatrix} -1 \\ 2 \\ 4 \\ 0 \end{pmatrix}$$

$$= \frac{1}{\cancel{5}} \cdot \cancel{5}$$

$$= 1$$

$$(\text{or}) = \frac{-1}{5} + \frac{2}{5} + \frac{4}{5} + 0$$

$$= 1$$

$$x_{23} = \langle a_3, q_2 \rangle$$

$$= \begin{pmatrix} -1 \\ 1 \\ 2 \\ 0 \end{pmatrix} \cdot \frac{1}{5} \begin{pmatrix} -2 \\ 1 \\ -4 \\ 2 \end{pmatrix}$$

$$= \frac{1}{5} [2 + 1 - 8]$$

$$= \frac{-5}{5}$$

$$= -1$$

$$P_2 = \frac{1}{5}\begin{bmatrix}1\\2\\2\\4\end{bmatrix} - \frac{1}{5}\begin{bmatrix}-2\\1\\-4\\2\end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix}+3\\1\\6\\2\end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix}3\\1\\6\\2\end{bmatrix}$$

$$r_{33} = \| a_3 - P_2 \|$$

$$a_3 - P_2 = \begin{bmatrix}-1\\1\\2\\0\end{bmatrix} - \frac{1}{5}\begin{bmatrix}3\\1\\6\\2\end{bmatrix}$$

$$= \begin{bmatrix}-8/5\\4/5\\4/5\\-2/5\end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix}-8\\4\\4\\-2\end{bmatrix}$$

$$= \frac{2}{5}\begin{bmatrix} -4 \\ 2 \\ 2 \\ -1 \end{bmatrix}$$

$$\|a_3 - P_2\| = \sqrt{\frac{64}{25} + \frac{16}{25} + \frac{16}{25} + \frac{4}{25}}$$

$$= \sqrt{\frac{100}{25}}$$

$$= 2$$

$$r_{33} = 2$$

$$q_3 = \frac{a_3 - P_2}{r_{33}}$$

$$= \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{5}\begin{bmatrix} -4 \\ 2 \\ 2 \\ -1 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} -4 \\ 2 \\ 2 \\ -1 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1/5 & -2/5 & -4/5 \\ 2/5 & 1/5 & 2/5 \\ 2/5 & -4/5 & 2/5 \\ 4/5 & 2/5 & 1/5 \end{bmatrix}$$

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -2 & 1 \\ 0 & 4 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$Rx = Q^T b$$

$$Rx = \begin{bmatrix} 5 & -2 & 1 \\ 0 & 4 & -1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} 5x_1 - 2x_2 + x_3 \\ 4x_2 - x_3 \\ 2x_3 \end{bmatrix}$$

$$Q^T b = \left\{ \frac{1}{5} \begin{bmatrix} 1 & 2 & -4 \\ 2 & 1 & \\ 2 & -4 & 2 \\ 4 & 2 & -1 \end{bmatrix}_{4\times 3} \right\}^T \begin{bmatrix} -1 \\ 1 \\ 1 \\ -2 \end{bmatrix}_{4\times 1}$$

$$= \frac{1}{5}\begin{bmatrix} -1 & 2 & 2 & 4 \\ -2 & 1 & -4 & 2 \\ -4 & 2 & 2 & -1 \end{bmatrix}_{3\times 4}\begin{bmatrix} -1 \\ 1 \\ 1 \\ -2 \end{bmatrix}_{4\times 1}$$

$$= \frac{1}{5}\begin{bmatrix} -1+2+2-8 \\ 2+1-4-4 \\ +4+2+2+2 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} -5 \\ -5 \\ 10 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

$5x_1 - 2x_2 + x_3 = -1$

$4x_2 - x_3 = -1$

$2x_3 = 2$

$x_3 = 1$

$4x_2 - \cancel{1} = \cancel{-1}$

$x_2 = 0$

$5x_1 + 1 = -1$

$x_1 = \frac{-2}{5}$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= \begin{pmatrix} -2/5 \\ 0 \\ 1 \end{pmatrix}$$

DATE: 28 / 10 / 2025

Householder QR Factorization

Unit - 4

Householder Matrix

A householder matrix is a matrix of the form
$H = I - 2ww^T$

where w is a unit vector in $\mathbb{R}^n$

Properties

Theorem

Let H be a householder matrix defined by
$H = I - 2ww^T$

where w is a unit vector in $\mathbb{R}^n$. Let S be a subspace of $\mathbb{R}^n$ defined by

$S = \text{span}(w)$.

Note:

$S = \{x \in \mathbb{R}^n \mid x \in \text{span}\{w\}\}$

$S^\perp = \{y \in \mathbb{R}^n \mid \langle x, y \rangle = 0 \ \forall \ x \in S\}$

$\mathbb{R}^n = S \oplus S^\perp$

$x = x_1 + x_2$

$x_1 \in S$ and $x_2 \in S^\perp$

Then the following properties hold true

1. If $x \in S$ then $Hx = -x$

2. If $x \in S^\perp$ then $Hx = x$

3. If $x = x_s + x_n$ where $x_s \in S$ and $x_n \in S^\perp$, then $Hx = -x_s + x_n$

4. H is an involution
i.e., $H^2 = I$

5. H has only eigenvalues ± 1 i.e.,
$eig(H) = \{-1, 1\}$

6. The eigenspace corresponding to the eigenvalue
$\lambda_1 = -1$ of H is
$\mathcal{E}_{\lambda_1} = N(H - \lambda_1 I) = S$ and
$\mathcal{E}_{\lambda_2} = N(H - \lambda_2 I) = S^\perp$

7. $\det(H) = -1$

8. H preserves the length (2-norm) of vectors in $\mathbb{R}^n$ i.e,
$\|Hx\|_2 = \|x\|_2 \quad \forall \; x \in \mathbb{R}^n$

9. H is symmetric and orthogonal.

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PAGE NO.
DATE

Theorem:

Let u and v be two unit vectors in $\mathbb{R}^n$. Let H be the householder matrix defined by

$$H = I - 2ww^T$$

where, $$w = \frac{u-v}{\|u-v\|_2}$$

Then $Hu = v$

Proof:

$$ww^T = \frac{(u-v)(u^T-v^T)}{\|u-v\|_2^2}$$

$$= \frac{u^T(u-v) - (u-v)v^T}{\|u\|^2 + \|v\|^2 - 2(u \cdot v)}$$

$\|u\| = 1$
$\|v\| = 1$

$$ww^T = \frac{1}{2(1-u \cdot v)} [(u-v)u^T - (u-v)v^T]$$

$$H = I - 2ww^T$$

$$H = I - \frac{1}{1-uv} [(u-v)u^T - (u-v)v^T]$$

be

$$Hu = u - \left[\frac{(u-v)v^T u - (u-v)v^T u}{1 - uv} \right]$$

$$= u - \frac{(u-v).1 - (u-v)uv}{1-uv}$$

$$= u - \frac{(u-v)\cancel{(1-uv)}}{\cancel{1-uv}}$$

$$= \cancel{u} - \cancel{u} + v$$

$$= v$$

$\therefore Hu = v$

Corollary

Let x and y be two non-zero vectors in $\mathbb{R}^n$ s.t. $x \neq y$. Define u and v by

$$u = \frac{x}{\|x\|_2} \quad \text{and} \quad v = \frac{y}{\|y\|_2}$$

Let H be householder matrix defined by $H = I - 2ww^T$, where

$$w = \frac{u-v}{\|u-v\|_2}$$

Then $Hx = \mu y$

PAGE NO:
DATE:

where $\mu = \frac{\|x\|_2}{\|y\|_2}$

Proof:

$Hu = v$

$H(\|x\|_2 u) = \|x\|_2 Hu$

$= \|x\|_2 v$

$= \frac{\|x\|_2}{\|y\|_2} y$

$= \mu y$

Hence Proved.

By using above corollary, we can create zeros in a vector.

Que 1. $x = \begin{bmatrix} 4 \\ 5 \\ 3 \\ -2 \end{bmatrix}$

- Find Hx of 0 vector

Sol. $u = \frac{x_0}{\|x_0\|}$

$$= \frac{1}{\sqrt{16+25+9+4}} \begin{bmatrix} 4 \\ 5 \\ 3 \\ -2 \end{bmatrix}$$

$$= \frac{1}{\sqrt{54}} \begin{bmatrix} 4 \\ 5 \\ 3 \\ -2 \end{bmatrix}$$

$$= \frac{1}{3\sqrt{6}} \begin{bmatrix} 4 \\ 5 \\ 3 \\ -2 \end{bmatrix}$$

$$v = -\mathrm{sign}(x_1)\, e_1$$

$$= \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$w = \frac{u - v}{\|u - v\|}$$

$$u - v = \begin{bmatrix} 4/3\sqrt{6} \\ 5/3\sqrt{6} \\ 1/\sqrt{6} \\ -2/3\sqrt{6} \end{bmatrix} - \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$u = \begin{bmatrix} 0.5443 \\ 0.6804 \\ 0.4082 \\ -0.2722 \end{bmatrix}$$

PAGE NO.
DATE:

$$w = \begin{bmatrix} 0.8787 \\ 0.3872 \\ 0.2323 \\ -0.1549 \end{bmatrix}$$

$$H = I - 2ww^T$$

$$= \begin{bmatrix} -0.5543 & -0.6804 & -0.4082 & 0.2722 \\ -0.6804 & 0.7002 & -0.1799 & 0.1199 \\ -0.4082 & -0.1799 & 0.8921 & 0.0719 \\ 0.2722 & 0.1199 & 0.0719 & 0.9520 \end{bmatrix}$$

Note that

$$Hx = \begin{bmatrix} -7.3485 \\ 0.0000 \\ 0.0000 \\ 0 \end{bmatrix}$$

and $\mu = -\text{sign}(x_1)\, \|x\|_2$
$= -7.3485$

Thus $Hx = \mu e_1$

example : let $A = \begin{bmatrix} -5 & 2 & 3 & -5 \\ 5 & -1 & 2 & 9 \\ 0 & -4 & 2 & 1 \\ 1 & 3 & -2 & 4 \end{bmatrix}$

Now we want to find a householder matrix H s.t.

$$HA = \begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{bmatrix}$$

Let $X = \begin{bmatrix} -5 \\ 3 \\ 8 \\ 1 \end{bmatrix}$

Let us define w by

$$w = \frac{u - v}{\|u - v\|_2}$$

$$= \begin{bmatrix} -0.8668 \\ 0.1759 \\ 0.4638 \\ 0.0580 \end{bmatrix}$$

As per the algorithm the required Householder matrix H is given by

$$H = I - 2ww^T$$

$$= \begin{bmatrix} -0.5025 & 0.3015 & 0.8040 & 0.1005 \\ 0.3015 & 0.9395 & -0.1613 & -0.0202 \\ 0.8040 & -0.1613 & 0.5697 & -0.0538 \\ 0.1005 & -0.0202 & -0.0538 & 0.9933 \end{bmatrix}$$

Observe that

$$HA = \begin{bmatrix} 9.9499 & -4.2212 & 6.5025 & 6.4322 \\ 0.0000 & 0.2484 & 2.5012 & 6.7059 \\ 0.0000 & -0.6709 & 3.3365 & -5.1196 \\ 0.0000 & 3.4161 & -1.8529 & 3.2353 \end{bmatrix}$$

which has the desired form.

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Householder QR Factorisation Theorem

Theorem 1

If A is any real $n \times n$ matrix, then there exists an orthogonal matrix Q and an upper triangular matrix R s.t.

$$A = QR$$

where the matrix Q can be expressed as a product of householder matrices.

Proof

This proof is a constructive proof and consist of $(n-1)$ steps.

Step 1:- Find a householder matrix H_1 s.t the matrix $A_1 = H_1 A$ has zeros below the first diagonal entry in its first column, i.e., A has the form

$$A_1 = H_1 A = \begin{bmatrix} * & * & \cdots & * \\ 0 & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & \vdots & & \vdots \\ 0 & * & \cdots & * \end{bmatrix}$$

To get the above form we proceed as follows
Find a householder matrix

$H_1 = I - 2w_1w_1^T$, where $\|w_1\|_2 = 1$

s.t. $H_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} = \begin{bmatrix} * \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

Then the product $A_1 = H_1 A$ will have zeros below the first diagonal entry in its first column. Now, we start working with $A_1 = H_1 A$ in place of A i.e. A_1. We can write A_1 as

$$A_1 = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Step 2 = Find a householder matrix H_2 s.t. the matrix
$A_2 = H_2 A_1 = (H_2 H_1) A$

has zeros below the first and second diagonal entries in its first two columns, i.e., A_2 has the form

$A_2 = H_2 H_1$

$$= \begin{bmatrix} * & * & * & \cdots & * \\ 0 & * & * & \cdots & * \\ 0 & 0 & * & \cdots & * \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & * & \cdots & * \end{bmatrix} \longrightarrow (i)$$

For this we proceed as follows.

Find a householder matrix

$\hat{H}_2 = I_{n-1} - 2\hat{\omega}_2 \hat{\omega}_2^T$, where $\|\hat{\omega}_2\|_2 = 1$

s.t

$$\hat{H}_2 \begin{bmatrix} a_{22} \\ a_{32} \\ \vdots \\ a_{n2} \end{bmatrix} = \begin{bmatrix} * \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Next we define H_2 by

$$H_2 = \begin{bmatrix} I_1 & 0 \\ 0 & \hat{H}_2 \end{bmatrix}$$

and $\omega_2 = \begin{bmatrix} 0 \\ \hat{\omega}_2 \end{bmatrix} \in \mathbb{R}^n$

Observe that

$\|\omega_2\|_2 = \|\hat{\omega}_2\|_2 = 1$

and $I - 2\omega_2 \omega_2^T = \begin{bmatrix} I & 0 \\ 0 & I_{n-1} - 2\hat{\omega}_2 \hat{\omega}_2^T \end{bmatrix}$

$$= \begin{bmatrix} I_1 & 0 \\ 0 & \hat{H}_2 \end{bmatrix}$$

$$= H_2$$

Step i : In this step we find a householder matrix H_i s.t.

$A_i = H_i A_{i-1} = \dots = (H_i H_{i-1} \dots H_2 H_1) A$ has zeros below the first i diagonal entries in its first i columns.

$$\hat{H}_i = I_{n-(i-1)} - 2\hat{w}_i \hat{w}_i^T, \quad \text{where } \|\hat{w}_i\|_2 = 1$$

$$\hat{H}_i \begin{bmatrix} a_{ii} \\ a_{i+1,i} \\ \vdots \\ a_{ni} \end{bmatrix} = \begin{bmatrix} * \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} I_{i-1} & 0 \\ 0 & \hat{H}_i \end{bmatrix}$$

$$= H_i$$

Thus, H_i is a householder matrix, and H_i preserves the zeros already created in the first $i-1$ steps and also the product $A_i = H_i A_{i-1}$ has zeros below the diagonal entries in its i^{th} column. Store A_i over A i.e., $A \leftarrow A_i$

Observe that the matrix $A_{n-1} = H_{n-1} A_{n-2}$ obtained at the end of step $(n-1)$ is an upper triangular matrix R:

$$R = A_{n-1} = H_{n-1} A_{n-2} = \dots = H_{n-1} H_{n-2} \dots H_2 H_1 A$$
— (ii)

Define Q by
$$Q = H_1 H_2 \dots H_{n-1}$$

Since each householder matrix H_i is symmetric we have

$$Q^T = H_{n-1}^T H_{n-2}^T \dots H_2^T H_1^T \quad \text{— (iii)}$$
$$= H_{n-1} H_{n-2} \dots H_2 H_1$$

[redacted]

PAGE NO.
DATE:

$\because$ Each householder matrix H_i is orthogonal, Q is orthogonal. This completes the proof.

Example: Let $A = \begin{bmatrix} 17 & 14 & 10 & -11 \\ -10 & 15 & 8 & 17 \\ 9 & 16 & -7 & 5 \\ -6 & 8 & 19 & -15 \end{bmatrix}$

Sol. In this example, we use the previous theorem to find an orthogonal matrix Q and an upper triangular matrix R so that

$A = QR$

The process will take total $4-1 = 3$ steps.

Step 1. Here we find H_1 so that

$$A_1 = H_1 A = \begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{bmatrix}$$

It is easy to see that H_1 is defined by

$H_1 = I_4 - 2w_1 w_1^T$

where,

$$w_1 = \begin{bmatrix} 0.9032 \\ -0.2914 \\ 0.2622 \\ -0.1748 \end{bmatrix}$$

Step 2 Now we find H_2 so that $A_2 = H_2 A_1$ has the form

$$A_2 = H_2 A_1$$

$$= H_2 \begin{bmatrix} -19.0000 & -6.0000 & 7.2105 & 8.7895 \\ 0.0000 & 21.4516 & 8.8998 & 10.6163 \\ 0.0000 & 10.1935 & -7.8098 & 10.7453 \\ 0.0000 & 11.8710 & 19.5599 & -18.8302 \end{bmatrix}$$

to

$$= \begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{bmatrix}$$

First, we find $\hat{H}_2$ so that

$$\hat{H}_2 \begin{bmatrix} 21.4516 \\ 10.1935 \\ 11.8710 \end{bmatrix} = \begin{bmatrix} * \\ 0 \\ 0 \end{bmatrix}$$

Observe that

$$\hat{H}_2 = I_3 - 2\hat{w}_2 \hat{w}_2^T$$

where $\hat{w}_2 = \begin{bmatrix} 0.9508 \\ 0.2019 \\ 0.2351 \end{bmatrix}$

Step 3 Now we find H_3 so that $A_3 = H_3 A_2$ for the form

$A_3 = H_3 A_2$

$$= H_3 \begin{bmatrix} -19.0000 & -6.0000 & 7.2105 & 8.7895 \\ 0.0000 & -26.5518 & -12.9280 & -4.2836 \\ 0.0000 & 0 & -12.4950 & 7.5813 \\ 0.0000 & 0 & 14.1420 & -22.5199 \end{bmatrix}$$

$$= \begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix}$$

First we find $\hat{H}_3$ so that

$\hat{H}_3 = I_2 - 2\hat{w}_3 \hat{w}_3^T$

where $\hat{w}_3 = \begin{bmatrix} -0.9112 \\ 0.4119 \end{bmatrix}$

Thus H_3 is defined by

$H_3 = \begin{bmatrix} I_2 & 0 \\ 0 & \hat{H}_3 \end{bmatrix}$

i.e. $H_3 = I_4 - 2w_3 w_3^T$, where

$w_3 = \begin{bmatrix} 0 \\ \hat{w}_3 \end{bmatrix} \in \mathbb{R}^4$

$\therefore$ we have

$R = A_3 = H_3 A_2$

$$= \begin{bmatrix} -19.0000 & -6.0000 & 7.2105 & 8.7895 \\ 0.0000 & -26.5518 & -12.9280 & -4.2836 \\ 0.0000 & 0 & 18.8381 & -21.9106 \\ 0.0000 & 0 & 0.0000 & -9.1826 \end{bmatrix}$$

and

$$Q = H_1 H_2 H_3$$

$$= \begin{bmatrix} -0.6316 & -0.3846 & 0.5087 & -0.4410 \\ 0.5265 & -0.6839 & -0.2461 & -0.4413 \\ -0.4737 & -0.4956 & -0.5304 & 0.4988 \\ 0.3158 & -0.3727 & 0.6320 & 0.6017 \end{bmatrix}$$