5.15. Slutsky Equation

The Slutsky equation shows the relationship between the price effect and the income and substitution effects mathematically. The equation can be deduced as follows :

The first order conditions of utility maximisation require

$$f_1 - \lambda p_1 = 0 \quad \cdots\cdots(1)$$

$$f_2 - \lambda p_2 = 0 \quad \cdots\cdots(2)$$

$$y - p_1 q_1 - p_2 q_2 = 0 \quad \cdots\cdots(3)$$

Taking the total differential of equations (1)-(3) we get

$$f_{11} dq_1 + f_{12} dq_2 - p_1 d\lambda = \lambda dp_1 \quad \cdots\cdots(4)$$

$$f_{21} dq_1 + f_{22}\, dq_2 - p_2 d\lambda = \lambda dp_2 \quad \cdots\cdots(5)$$

$$-p_1\, dq_1 - p_2\, dq_2 = -\, dy + q_1\, dp_1 + q_2\, dp_2 \quad \cdots\cdots(6)$$

In matrix form

$$\begin{bmatrix} f_{11} & f_{12} & -p_1 \\ f_{21} & f_{22} & -p_2 \\ -p_1 & -p_2 & 0 \end{bmatrix} \begin{bmatrix} dq_1 \\ dq_2 \\ d\lambda \end{bmatrix} = \begin{bmatrix} \lambda dp_1 \\ \lambda dp_2 \\ -dy + q_1\, dp_1 + q_2 dp_2 \end{bmatrix}$$

We can solve these three equations for dq_1, dq_2 and $d\lambda$ by the help of Cramer's rule. Let D represent the determinant of the coefficient matrix and let D_{ij} be the cofactor of the (i,j) th element of the coefficient matrix.

Then

$$dq_1 = \frac{\lambda dp_1 D_{11} + \lambda dp_2 . D_{21} + (-dy + q_1 dp_1 + q_2 dp_2).D_{31}}{D} \quad \cdots\cdots(7)$$

and

$$dq_2 = \frac{\lambda dp_1 D_{12} + \lambda dp_2 . D_{22} + (-dy + q_1 dp_1 + q_2 dp_2).D_{32}}{D} \quad \cdots\cdots(8)$$

Consider now equation (7). Suppose that p_1 and p_2 do not change and only y changes. Then $dp_1 = dp_2 = 0$.

$$\text{and } \delta q_1 = \frac{-\delta y . D_{31}}{D}$$

$$\therefore \left(\frac{\delta q_1}{\delta y}\right)_{\text{prices = const}} = -\frac{D_{31}}{D} \quad \cdots\cdots(9)$$

This equation shows the effect of change in income on the quantity demanded of Q_1, prices remaining the same. This gives us the income effect of change in income on the demand for Q_1.

Consider now the substitution effect. In the Hicksian measure of the substitution effect, total utility remains the same. Hence dU=0.

i.e. $f_1 dq_1 + f_2\, dq_2 = 0$. But from the first order conditions of utility maximisation

$$-\frac{dq_2}{dq_1} = \frac{f_1}{f_2} = \frac{p_1}{p_2}$$

$$\therefore\ p_1 dq_1 = -p_2 . dq_2$$

Hence it follows that $p_1 dq_1 + p_2 dq_2 = 0$. Therefore from (6) $-dy + q_1\, dp_1 + q_2 dp_2 = 0$ when utility remains constant. Now assuming that $dp_2 = 0$ and $-dy + q_1 dp_1 + q_2 dp_2 = 0$ we get from (7)

$$\delta q_1 = \frac{\lambda \delta p_1 D_{11}}{D}$$

$$\therefore\ \left(\frac{\delta q_1}{\delta p_1}\right)_{U=\text{const}} = \frac{\lambda D_{11}}{D} \quad \text{.....(10)}$$

This equation gives the Hicksian substitution effect of the fall in the price of Q_1.

On the other hand if price of Q_1only changes, while price of $Q_2$and income remain the same, then $dp_2 = 0$, $dy = 0$ and from (7) we get

$$\delta q_1 = \frac{\lambda D_{11} . \delta p_1 + q_1 \delta p_1 . D_{31}}{D}$$

$$\therefore \frac{\delta q_1}{\delta p_1} = \frac{\lambda D_{11}}{D} + q_1 . \frac{D_{31}}{D} \quad \text{......(11)}$$

Now combining (9), (10) and (11) we get

own effect

$$\frac{\delta q_1}{\delta p_1} = \left(\frac{\delta q_1}{\delta p_1}\right)_{U=\text{const}} - q_1 \left(\frac{\delta q_1}{\delta y}\right)_{\text{prices}=\text{const}} \quad \text{..... (12)}$$

This equation is known as the Slutsky equation. The left hand side of this equation represents the price effect. The first term on the right hand side represents the substitution effect while the second term represents the income effect. Thus the Slutsky equation shows that the price effect is the resultant of the income effect and the substitution effect.

Consider now the direction of these effects. It can be seen that if the second order conditions are to be fulfilled, the determinant D must be positive. Expanding the determinant D we get $D = 2f_{12} p_1 p_2 - p_2^2 f_{11} - p_1^2\, f_{22}$ which must be positive if the second order condition of utility maximisation is satisfied. Further, the Lagrange multiplier λ is equal to the marginal utility of income so that λ is also positive. Now the substitution effect $= \frac{\lambda D_{11}}{D}$. But $D_{11} = \begin{bmatrix} f_{22} & -p_2 \\ -p_2 & 0 \end{bmatrix} = -p_2^2 < 0$, $\lambda > 0$, $D > 0$. This shows that the substitution effect is always negative. The income effect =

$$-q_1 \left(\frac{\delta q_1}{\delta y} \right)_{\text{prices = const,}}$$

$$= \frac{q_1 \cdot D_{31}}{D} = \frac{q_1 \begin{vmatrix} f_{12} & -p_1 \\ f_{22} & -p_2 \end{vmatrix}}{D}$$

$$= \frac{q_1}{D}(-p_2 f_{12} + f_{22} p_1)$$

Though $q_1 > 0$. $D > 0$, but $(-p_2 f_{12} + f_{22} p_1)$ may be $\gtrless 0$. Thus the sign of the income effect is indeterminate. The income effect is positive for a normal commodity but negative for an inferior commodity. Thus if Q_1 is a normal commodity, both the substitution effect and the income effect in (12) have negative signs so that $\frac{\delta q_1}{\delta p_1} < 0$ and the demand curve is downward sloping. But in the case of inferior goods the first term on the right hand side of (12) is negative while the second term is positive. Hence $\frac{\partial q_1}{\partial p_1}$ may be positive or negative. If the second term is positive and stronger than the first term, $\frac{\delta q_1}{\delta p_1} > 0$ and the demand curve is upward rising. This is the case of Giffen goods where the income effect is negative and stronger than the substitution effect.

We can similarly analyse the effect of change in p_2 on Q_2 from (8). We can also get the cross effects of change in the price of one commodity on the quantity demanded of the other from (7) and (8). For example suppose that in (7) we assume $dp_1 = 0$ and $dy = 0$. Then

$$\frac{\delta q_1}{\delta p_2} = \frac{\lambda D_{21}}{D} + q_2 \frac{D_{31}}{D} \quad \text{.....(13)}$$

Similarly assuming that $dp_2 = 0$ and $dy = 0$ we get from (8) cross effects

$$\frac{\delta q_2}{\delta p_1} = \frac{\lambda D_{12}}{D} + q_1 \cdot \frac{D_{32}}{D} \quad \text{.....(14)}$$

The first two terms on the right hand sides are called the cross substitution effects. The cross substitutuin effect measures the substitution effect for one commodity with respect to change in the price of the other commodity. Since D is a symmetric determinant. $D_{21} = D_{12}$. Hence $\frac{\lambda D_{21}}{D} = \frac{\lambda D_{12}}{D}$. This shows that the cross substitution effects are equal. In other words, the substitution effect on Q_1 resulting from a change in p_2 is equal to the substitution effect on Q_2 resulting from a change in p_1.

Substitutes and Complements

The cross substitution effects are used to define substitutability and complementarity between two commodities. The two commodities Q_1 and Q_2 are said to be substitutes if the cross substitution effect is positive. They are said to be complements if the cross substitution effect is negative. Thus when $\frac{\lambda D_{21}}{D} > 0$ the two commodities are substitutes and when $\frac{\lambda D_{21}}{D} < 0$, the two commodities are complements. However it should be noted that in the two good case there can be no complementarity between Q_1 and Q_2. They are always substitutes. For showing complementarity we should assume at least three goods.

5.16. Compensating Variation and Equivalent Variation in Income

There are two income changes corresponding to any price change: compensating variation in income and equivalent variation in income. Suppose that the initial equilibrium position of the consumer is disturbed by a change in the price of one commodity and as a result the consumer moves on to a new indifference curve. Then the compensating variation in income is defined as that amount of income which would compensate the change in price and bring the consumer back to his old indifference curve, though at a different point. The consumer remains on the same indifference curve as before the price change since the change in income compensates the gain or loss which could have accrued due to change in price. The equivalent