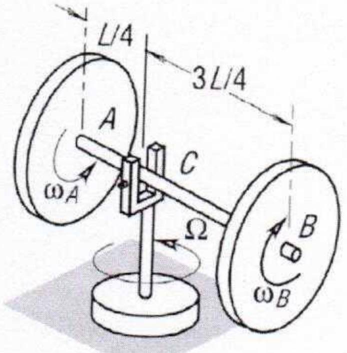
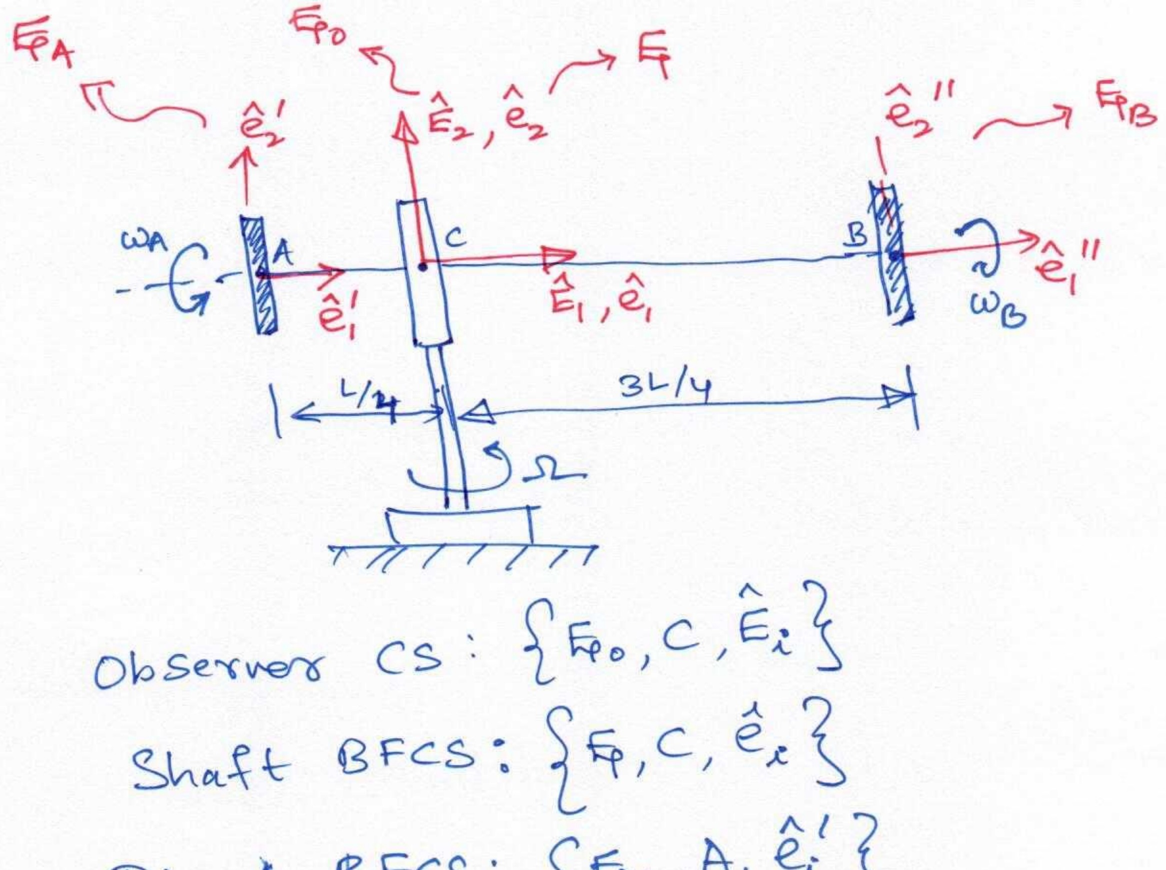
ESO 209 - Tutorial 11

Problem 4

(4) Identical disks A and B spin at the constant rates ω_A and ω_B, respectively, about shaft AB, which is horizontal. The entire system precesses about the vertical axis at the constant rate Ω. Determine the relationship between the spin rates ω_A and ω_B for which this motion can occur without application of a torque acting about the axis of pin C.





Observer CS : $\{F_{\Phi 0}, C, \hat{E}_i\}$

Shaft BFCS : $\{F_\Phi, C, \hat{e}_i\}$

Disc A BFCS : $\{F_{\Phi A}, A, \hat{e}_i'\}$

Disc B BFCS : $\{F_{\Phi B}, B, \hat{e}_i''\}$

Assuming mass of each disc is m, and radius of each disc is R.

Using AMB about Point C

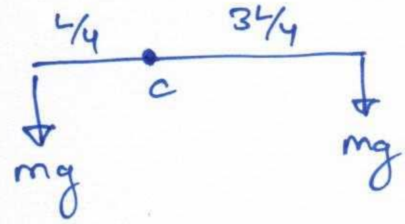
~~Since~~ General AMB eqn

$$\underline{M}^P = \underline{r}^{G/P} \times m\underline{a}^G + \underline{\omega}^{\mathcal{B}} \times (\underline{\underline{I}}^G \cdot \underline{\omega}^{\mathcal{B}}) + \underline{\underline{I}}^G \cdot \underline{\alpha}^{\mathcal{B}}$$

The LHS is sum of moments about C

RHS is rate of change of angular momentum about C

= rate of change of $\underline{H}^A$ + rate of change of $\underline{H}^B$



$$\therefore \quad \underline{LHS} = mg\frac{L}{4}\hat{E}_3 - mg\frac{3L}{4}\hat{E}_3$$

($\because$ it is given no other torque to be applied at C)

$$\boxed{LHS = -mg\frac{L}{2}\hat{E}_3}$$

Now in order to find the RHS we need the various entities:

For disc A

$\to \Omega\hat{E}_2$

$$\underline{\omega}_{E_{\phi A}/E_{\phi 0}} = \underline{\omega}_{E_{\phi A}/E_\phi} + \underline{\omega}_{E_\phi/E_{\phi 0}}$$

$$\boxed{\underline{\omega}_{E_{\phi A}/E_{\phi 0}} = \omega_A\hat{e}_1 + \Omega\hat{E}_2}$$

$$\underline{a}^G = \underline{a}^A = \underline{a}^C + \underline{\omega}_{E_\phi/E_{\phi 0}} \times (\underline{\omega}_{E_\phi/E_{\phi 0}} \times \underline{r}^{A/C}) + \underline{\alpha}_{E_\phi/E_{\phi 0}} \times \underline{r}^{A/C}$$

$\underline{a}^C = 0$

$\underline{\omega}_{E_\phi/E_{\phi 0}} = \Omega\hat{E}_2 \Rightarrow \underline{\alpha}_{E_\phi/E_{\phi 0}} = \frac{d}{dt}(\Omega\hat{E}_2) = \underline{0}$

$\because$ both Ω & $\hat{E}_2$ are constant

$$\underline{\omega}_{E_{eA}/E_{e0}} = \omega_A \hat{e}_1 + \Omega \hat{E}_2 = \omega_A \hat{E}_1 + \Omega \hat{E}_2$$

(at instant of interest)

$$\underline{\alpha}_{E_{eA}/E_{e0}} = \frac{d}{dt}\left(\omega_A \hat{e}_1 + \Omega \hat{E}_2\right)$$

$$= \omega_A \frac{d}{dt}(\hat{e}_1) \qquad \because \frac{d\Omega}{dt}=0,\ \frac{d\hat{E}_2}{dt}=0,\ \frac{d\omega_A}{dt}=0$$

$$= \omega_A\left(\underline{\omega}_{E_e/E_{e0}} \times \hat{e}_1\right)$$

$$= \omega_A\left(\Omega \hat{E}_2 \times \hat{e}_1\right)$$

$$= \omega_A\left(\Omega \hat{E}_2 \times \hat{E}_1\right) = -\omega_A \Omega \hat{E}_3$$

$$\therefore \boxed{\underline{\alpha}_{E_{eA}/E_{e0}} = -\omega_A \Omega \hat{E}_3}$$

$|||^{ly}$

$$\underline{\omega}_{E_{eB}/E_{e0}} = -\omega_B \hat{e}_1 + \Omega \hat{E}_2$$

$$\underline{\alpha}_{E_{eB}/E_{e0}} = \frac{d}{dt}\left(-\omega_B \hat{e}_1 + \Omega \hat{E}_2\right)$$

$$= -\omega_B \frac{d(\hat{e}_1)}{dt}$$

$$= -\omega_B\left(\Omega \hat{E}_2 \times \hat{e}_1\right)$$

$$\Rightarrow \boxed{\underline{\alpha}_{E_{eB}/E_{e0}} = \omega_B \Omega \hat{E}_3}$$

$$\underline{a}^A = \cancelto{0}{\underline{a}^C} + \underline{\omega}^{E_F/E_0} \times \left(\underline{\omega}^{E_F/E_0} \times \underline{r}^{A/C}\right) + \cancelto{0}{\underline{\alpha}^{E_F/E_0}} \times \underline{r}^{A/C}$$

$$= \Omega \hat{E}_2 \times \left(\Omega \hat{E}_2 \times -\frac{L}{4} \hat{E}_1\right)$$

$$= \Omega \hat{E}_2 \times \left(\frac{\Omega L}{4} \hat{E}_3\right)$$

$$\underline{a}^A = \frac{\Omega^2 L}{4} \hat{E}_1$$

Now finding $\dot{\underline{H}}_A$ (RHS terms)

$$\underline{r}^{A/C} \times m\underline{a}^A = -\frac{L}{4}\hat{E}_1 \times m\frac{\Omega^2 L}{4}\hat{E}_1 = \underline{0}$$

$$\underline{\omega}^{E_A/E_0} \times \left(\underline{\underline{I}}_G \cdot \underline{\omega}^{E_A/E_0}\right)$$

$$\left[\underline{\underline{I}}_G\right]_{E_A} = \left[\underline{\underline{I}}_G\right]_{E_0} = \begin{bmatrix} \frac{mR^2}{2} & 0 & 0 \\ 0 & \frac{mR^2}{4} & 0 \\ 0 & 0 & \frac{mR^2}{4} \end{bmatrix}$$

(at this instant)

$$\left[\underline{\omega}^{E_A/E_0}\right]_{E_0} = \begin{bmatrix} \omega_A \\ \Omega \\ 0 \end{bmatrix}$$

$$\left[\underline{\underline{I}}_G\right] \cdot \left[\underline{\omega}\right] = \begin{bmatrix} \frac{mR^2}{2} & 0 & 0 \\ 0 & \frac{mR^2}{4} & 0 \\ 0 & 0 & \frac{mR^2}{4} \end{bmatrix} \begin{bmatrix} \omega_A \\ \Omega \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{mR^2}{2}\omega_A \\ \frac{mR^2}{4}\Omega \\ 0 \end{bmatrix}$$

$$\underline{\omega}_{E_{GA}/E_{G0}} \times \left(\underline{\underline{I}}_G \cdot \underline{\omega}_{E_{GA}/E_{G0}}\right)$$

$$= \left| \begin{bmatrix} \hat{E}_1 & \hat{E}_2 & \hat{E}_3 \\ \omega_A & \Omega & 0 \\ \frac{mR^2\omega_A}{2} & \frac{mR^2\Omega}{4} & 0 \end{bmatrix} \right|$$

$$= -\frac{mR^2}{4}\omega_A\Omega\hat{E}_3$$

$$\left[\underline{\underline{I}}_G\right] \cdot \left[\underline{\alpha}_{E_{GA}/E_{G0}}\right] = \begin{bmatrix} \frac{mR^2}{2} & 0 & 0 \\ 0 & \frac{mR^2}{4} & 0 \\ 0 & 0 & \frac{mR^2}{4} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -\omega_A\Omega \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ -\frac{mR^2}{4}\omega_A\Omega \end{bmatrix} \equiv -\frac{mR^2}{4}\omega_A\Omega\hat{E}_3$$

$$\therefore \; \underline{\dot{H}}_A = -\frac{mR^2}{4}\omega_A\Omega\hat{E}_3 - \frac{mR^2}{4}\omega_A\Omega\hat{E}_3$$

$$= -\frac{mR^2}{2}\omega_A\Omega\hat{E}_3$$

We find $\underline{\dot{H}}_B$ similarly.

$$\dot{\underline{H}}_B = \underline{\omega}_{E_{GB}/E_{G0}} \times \left(\underline{\underline{I}}_G \cdot \underline{\omega}_{E_{GB}/E_{G0}}\right) + \underline{\underline{I}}_G \cdot \underline{\alpha}_{E_{GB}/E_{G0}}$$

$$\left[\underline{\underline{I}}_G\right] \cdot \left[\underline{\omega}_{E_{GB}/E_{G0}}\right] = \begin{bmatrix} \frac{mR^2}{2} & 0 & 0 \\ 0 & \frac{mR^2}{4} & 0 \\ 0 & 0 & \frac{mR^2}{4} \end{bmatrix} \begin{bmatrix} -\omega_B \\ \Omega \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{mR^2}{2}\omega_B \\ \frac{mR^2}{4}\Omega \\ 0 \end{bmatrix}$$

$$\underline{\omega}_{E_{GB}/E_{G0}} \times \left(\underline{\underline{I}}_G \cdot \underline{\omega}_{E_{GB}/E_{G0}}\right) = \left| \begin{matrix} \hat{E}_1 & \hat{E}_2 & \hat{E}_3 \\ -\omega_B & \Omega & 0 \\ -\frac{mR^2}{2}\omega_B & \frac{mR^2}{4}\Omega & 0 \end{matrix} \right|$$

$$= \frac{mR^2}{4}\omega_B \Omega \hat{E}_3$$

$$\left[\underline{\underline{I}}_G\right] \cdot \left[\underline{\alpha}_{E_{GB}/E_{G0}}\right] = \begin{bmatrix} \frac{mR^2}{2} & 0 & 0 \\ 0 & \frac{mR^2}{4} & 0 \\ 0 & 0 & \frac{mR^2}{4} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \omega_B \Omega \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ \frac{mR^2}{4}\omega_B \Omega \end{bmatrix}$$

$$\Rightarrow \dot{\underline{H}}_B = \frac{mR^2}{4}\omega_B \Omega \hat{E}_3 + \frac{mR^2}{4}\omega_B \Omega \hat{E}_3$$

$$= \frac{mR^2}{2}\omega_B \Omega \hat{E}_3$$

Now substituting in the ~~AB~~ AMB about C

$$\underbrace{-\frac{mgL}{2}\hat{E}_3}_{\sum \underline{M}\ \cancel{\text{about}}} = \underbrace{-\frac{mR^2}{2}\omega_A\Omega\hat{E}_3}_{\dot{\underline{H}}_A} + \underbrace{\frac{mR^2}{2}\omega_B\Omega\hat{E}_3}_{\dot{\underline{H}}_B}$$

$$\Rightarrow \quad -\frac{mgL}{2} = \frac{mR^2\Omega}{2}(-\omega_A+\omega_B)$$

$$\Rightarrow \quad (-\omega_A+\omega_B) = \frac{-gL}{R^2\Omega}$$

$$\Rightarrow \quad \boxed{\omega_B = \omega_A - \frac{gL}{R^2\Omega}}$$

This is the relation between ω_A & ω_B if no torque is applied at point C.