Lecture 5

Rotation tensor; Change of coordinates; Rigid body kinematics: Euler's theorem

11-18 August, 2021

Review of L4

I. **Orthogonal tensor**: $\left|\mathsf{Q}\cdot\mathbf{a}\right| = |\mathbf{a}|$, any $\mathbf{a}$.

1. $\mathsf{Q}^{-1} = \mathsf{Q}^T$
2. $\det(\mathsf{Q}) = \pm 1$
3. $\lambda_3 = \pm 1,\ \lambda_{1,2} = a \pm i\,b$
4. One *real* principal vector $\hat{\mathbf{e}}_3$ for λ_3.
5. $(\mathsf{Q}\cdot\mathbf{a})\cdot(\mathsf{Q}\cdot\mathbf{b}) = \mathbf{a}\cdot\mathbf{b}$
   i. Q preserves relative orientation.
6. Plane *normal* to $\hat{\mathbf{e}}_3$ is *invariant* under Q
7. $\det(\mathsf{Q}) = \lambda_3(a^2+b^2) = \lambda_3 = \pm 1$

Q *either* <u>rotates</u> objects about $\hat{\mathbf{e}}_3$
or <u>reflects</u> them about invariant plane.

II. **Rotation tensor**: *Orthogonal* tensor R with $\det(\mathsf{R}) = +1 \Leftrightarrow \lambda_3 = +1,\ \lambda_{1,2} = a \pm i\,b$

1. *Rotation angle,* $\theta = \arctan(b/a)$.
2. *Axis of rotation* is along $\hat{\mathbf{e}}_3$.

R <u>rotates</u> objects about $\hat{\mathbf{e}}_3$ by θ.

Rotation tensor: Applications I

I. **Example**. CCS $\{\mathscr{E}, O, \hat{\mathbf{e}}_i\}$, $\{\mathscr{E}_1, O, \hat{\mathbf{e}}'_i\}$:

1. $\hat{\mathbf{e}}'_j = (\hat{\mathbf{e}}'_j \cdot \hat{\mathbf{e}}_i)\,\hat{\mathbf{e}}_i =: R_{ij}\,\hat{\mathbf{e}}_i \implies R_{ij} = \hat{\mathbf{e}}'_j \cdot \hat{\mathbf{e}}_i.$
2. $\mathsf{R} = R_{ij}\,\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j = (\hat{\mathbf{e}}'_j \cdot \hat{\mathbf{e}}_i)\,\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j$
   is a *rotation* tensor. Prove
3. $\hat{\mathbf{e}}'_i = \mathsf{R} \cdot \hat{\mathbf{e}}_i \implies \hat{\mathbf{e}}_i = \mathsf{R}^T \cdot \hat{\mathbf{e}}'_i$

   R *rotates* $\hat{\mathbf{e}}_i$ to $\hat{\mathbf{e}}'_i$; R^T *rotates* $\hat{\mathbf{e}}'_i$ *back* to $\hat{\mathbf{e}}_i$.

II. **Implication**: $\{\mathscr{E}, O, \hat{\mathbf{e}}_i\} \underset{\mathsf{R}^T}{\overset{\mathsf{R}}{\rightleftarrows}} \{\mathscr{E}_1, O, \hat{\mathbf{e}}'_i\}$

Any two *Cartesian* CS are related by a *rotation* tensor R, given by

$$\mathsf{R} = R_{ij}\,\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j = (\hat{\mathbf{e}}'_j \cdot \hat{\mathbf{e}}_i)\,\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j\ !$$

Rotation tensor: Applications II

I. **Multiplication**: R_1, R_2 rotation tensors.

1. $\mathsf{R} = \mathsf{R}_2 \cdot \mathsf{R}_1$ is also a rotation tensor.
   i. NO commutation: $\mathsf{R}_2 \cdot \mathsf{R}_1 \neq \mathsf{R}_1 \cdot \mathsf{R}_2$.
2. Let $\hat{\mathbf{e}}_i' = \mathsf{R}_1 \cdot \hat{\mathbf{e}}_i$ and $\hat{\mathbf{e}}_i'' = \mathsf{R}_2 \cdot \hat{\mathbf{e}}_i'$, then

$$\hat{\mathbf{e}}_i'' = \mathsf{R}_2 \cdot \mathsf{R}_1 \cdot \hat{\mathbf{e}}_i = \mathsf{R} \cdot \hat{\mathbf{e}}_i \,.$$

Successive rotations of $\hat{\mathbf{e}}_i$.

II. **Successive rotations of CS**: Given CCS

$$\left\{\mathscr{E}, O, \hat{\mathbf{e}}_i\right\}, \left\{\mathscr{E}_1, O, \hat{\mathbf{e}}_i'\right\}, \left\{\mathscr{E}_2, O, \hat{\mathbf{e}}_i''\right\}.$$

1. Can find rotation tensors R_1 and R_2:

$$\left\{\mathscr{E}, O, \hat{\mathbf{e}}_i\right\} \underset{\mathsf{R}_1^T}{\overset{\mathsf{R}_1}{\rightleftarrows}} \left\{\mathscr{E}_1, O, \hat{\mathbf{e}}_i'\right\} \underset{\mathsf{R}_2^T}{\overset{\mathsf{R}_2}{\rightleftarrows}} \left\{\mathscr{E}_2, O, \hat{\mathbf{e}}_i''\right\}$$

2. Then, with $\mathsf{R} = \mathsf{R}_2 \cdot \mathsf{R}_1$:

$$\left\{\mathscr{E}, O, \hat{\mathbf{e}}_i\right\} \underset{\mathsf{R}^T}{\overset{\mathsf{R}}{\rightleftarrows}} \left\{\mathscr{E}_2, O, \hat{\mathbf{e}}_i''\right\}$$

III. **Addition**: $\mathsf{R}_2 + \mathsf{R}_1$ is *not* a rotation tensor.

Coordinate transformation

I. **Requirement**: In CCS $\{\mathscr{E}, O, \hat{\mathbf{e}}_i\}$: $[\mathbf{a}]_{\mathscr{E}}$, $[\mathsf{A}]_{\mathscr{E}}$ known. Given another CCS, $\{\mathscr{E}_1, O, \hat{\mathbf{e}}_i'\}$, compute $[\mathbf{a}]_{\mathscr{E}_1}$ and $[\mathsf{A}]_{\mathscr{E}_1}$.

II. **Solution**: We can find *rotation* tensor R:

$$\{\mathscr{E}, O, \hat{\mathbf{e}}_i\} \underset{\mathsf{R}^T}{\overset{\mathsf{R}}{\rightleftarrows}} \{\mathscr{E}_1, O, \hat{\mathbf{e}}_i'\}, \text{ i.e. } \hat{\mathbf{e}}_i' = \mathsf{R} \cdot \hat{\mathbf{e}}_i$$

with $\mathsf{R} = R_{ij}\,\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j$ and $R_{ij} = \hat{\mathbf{e}}_j' \cdot \hat{\mathbf{e}}_i$.

Then, we will find:

1. $[\mathbf{a}]_{\mathscr{E}_1} = [\mathsf{R}]_{\mathscr{E}}^T [\mathbf{a}]_{\mathscr{E}} \Leftrightarrow a_i' = R_{ji} a_j$.
2. $[\mathsf{A}]_{\mathscr{E}_1} = [\mathsf{R}]_{\mathscr{E}}^T [\mathsf{A}]_{\mathscr{E}} [\mathsf{R}]_{\mathscr{E}} \Leftrightarrow A_{ij}' = R_{ki} A_{kl} R_{lj}$.

III. **Fact**: $\mathsf{R} = R_{ij}'\,\hat{\mathbf{e}}_i' \otimes \hat{\mathbf{e}}_j'$ in $\{\mathscr{E}_1, O, \hat{\mathbf{e}}_i'\}$.

1. $[\mathsf{R}]_{\mathscr{E}_1} = [\mathsf{R}]_{\mathscr{E}} \Longleftrightarrow R_{ij}' = R_{ij} = \hat{\mathbf{e}}_j' \cdot \hat{\mathbf{e}}_i$!
2. Lessens work and confusion.

End of Math preliminaries!

I think.

Its OK to like tensors.

But you may start getting angry at people who don't.

Rigid body dynamics

Wolfgang Pauli (left), Neils Bohr and a Tippie Top

Rigid body

I. **Definition**: A body in which the distance between two *material points* remains *fixed*: $|\mathbf{r}_A - \mathbf{r}_B| =$ constant, for *any* two points.

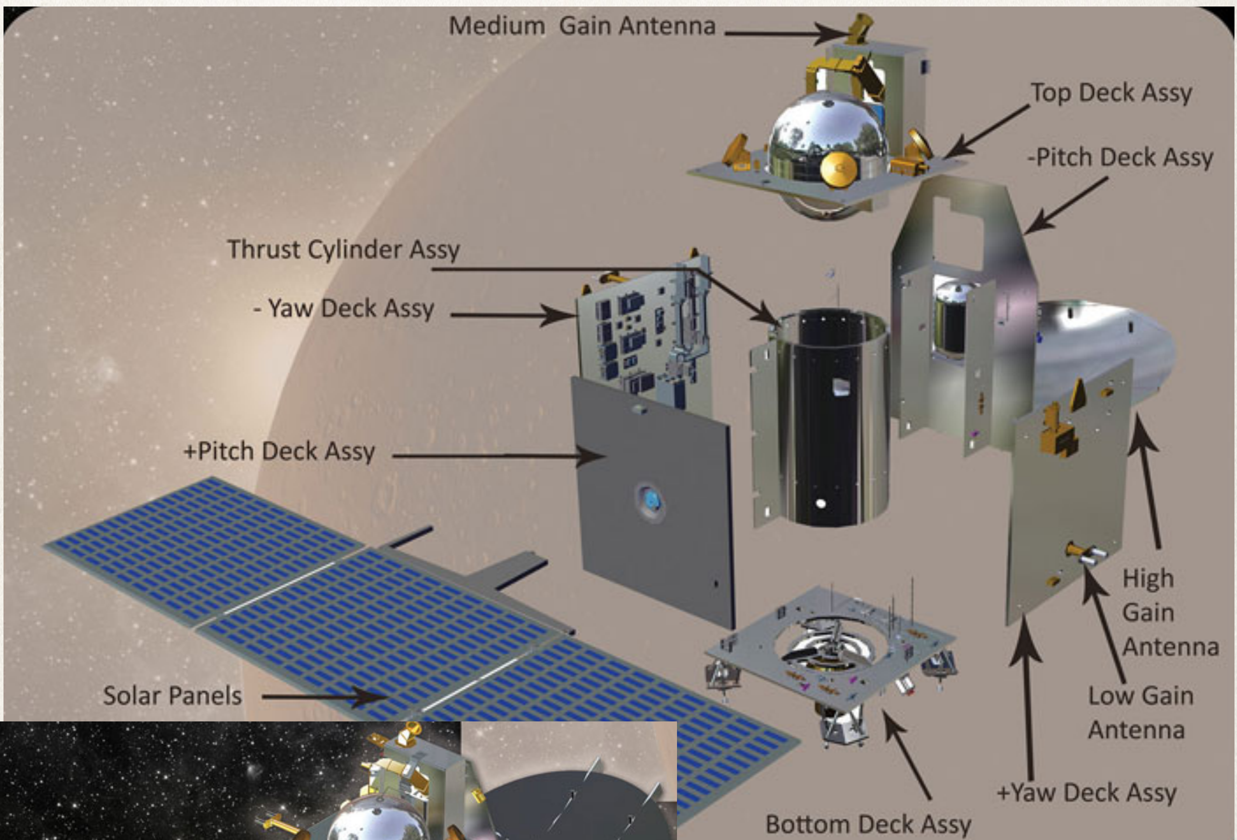
II. **Implications**:

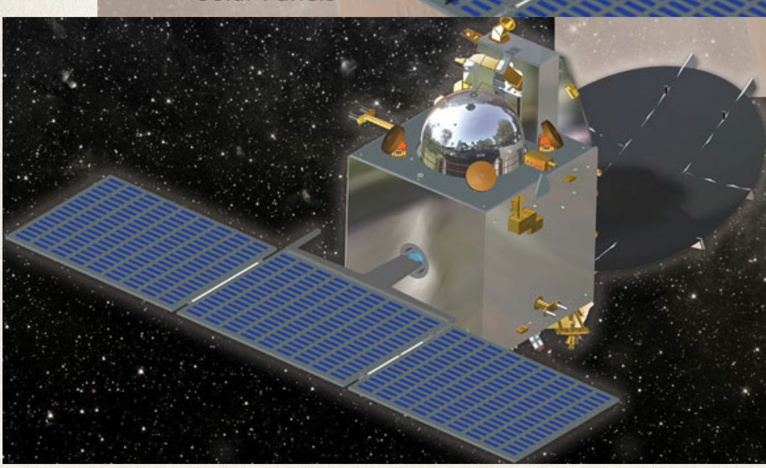
1. Angle between intersecting *material lines* remains fixed.
2. Orientation and location of a rigid body is fixed by the position of *three <u>non</u>-collinear* material points.
3. Three *non*-collinear material points define a CS attached to rigid body.

III. **Definition**: <u>*Body Fixed CS*</u> (*BFCS*): *Cartesian* CS <u>attached</u> to the rigid body.

1. Rigid body's motion $\Longleftarrow$ BFCS' motion

Rigid body kinematics





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(maṅgalayAna)

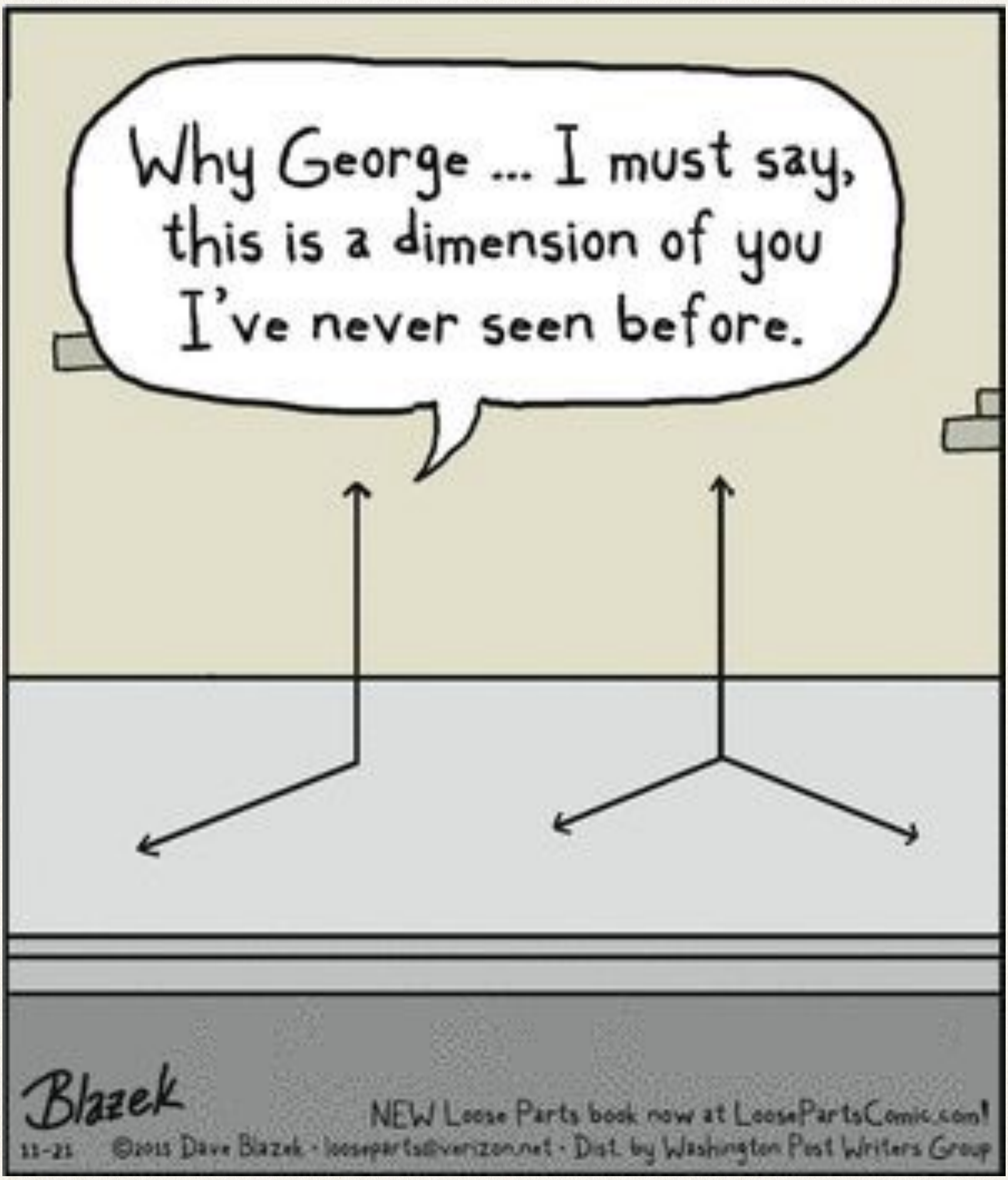
Euler's theorem

I. **Question**: How are two orientations of a *rigid* body related?

II. **Euler's theorem**: *Any* two orientations of a rigid body are linked by a *single rotation about an axis*.

Proof: Let the rigid body's two orientations be labeled '0' and '1'.

1. Construct a BFCS $\{\mathscr{E}, O, \hat{\mathbf{e}}_i\}$ in '0'.
    i. Will become BFCS $\{\mathscr{E}_1, O, \hat{\mathbf{e}}_i'\}$ in '1'.
2. There exists rotation tensor R such that
$$\{\mathscr{E}, O, \hat{\mathbf{e}}_i\} \underset{\mathsf{R}^T}{\overset{\mathsf{R}}{\rightleftarrows}} \{\mathscr{E}_1, O, \hat{\mathbf{e}}_i'\}.$$
3. R's principal values: $\lambda_3 = 1$, $\lambda_{1,2} = a \pm ib$.
4. *Angle of rotation* $\theta = \arctan(b/a)$
5. *Rotation axis:* Principal vector $\hat{\mathbf{e}}_3$ of $\lambda_3 = 1$



2d Dynamics $\ll$ *3d* Dynamics