SOLUTION TO PROBLEM 3 OF TUTORIAL 2

Given:

$$\underline{r} = x_1\,\widehat{e}_1 \,+\, x_2\,\widehat{e}_2 \,+\, x_3\,\widehat{e}_3\,,$$

and

$$\underline{\underline{I}} = 6\,(\widehat{e}_1 \otimes \widehat{e}_1) + 2\,(\widehat{e}_2 \otimes \widehat{e}_2) + 2\,(\widehat{e}_3 \otimes \widehat{e}_3)\ .$$

We need to show that the equation $\underline{r} \cdot \underline{\underline{I}} \cdot \underline{r} = 1$ represents an ellipsoid.
Further we need to find the lengths of the semi-major axes of the ellipsoid thus formed.

We first evaluate the quantity

$$\begin{aligned}\underline{\underline{I}} \cdot \underline{r} &= [6\,(\widehat{e}_1 \otimes \widehat{e}_1) + 2\,(\widehat{e}_2 \otimes \widehat{e}_2) + 2\,(\widehat{e}_3 \otimes \widehat{e}_3)] \cdot [x_1\widehat{e}_1 \,+\, x_2\,\widehat{e}_2 \,+\, x_3\,\widehat{e}_3] \\ &= 6\,x_1\,\widehat{e}_1 \,+\, 2\,x_2\,\widehat{e}_2 \,+\, 2\,x_3\,\widehat{e}_3\,.\end{aligned}$$

In the above, we have made use of the fact that

$$(\underline{a} \otimes \underline{b}) \cdot \underline{c} = (\underline{c} \cdot \underline{b})\,\underline{a}$$

and $\widehat{e}_i, i = 1, 2, 3$ are orthogonal to each other.

Accordingly, the quantity $\underline{r} \cdot \underline{\underline{I}} \cdot \underline{r}$ can be evaluated as

$$\underline{r} \cdot \underline{\underline{I}} \cdot \underline{r} = 6x_1^2 + 2x_2^2 + 2x_3^2\,.$$

Therefore, the equation $\underline{r} \cdot \underline{\underline{I}} \cdot \underline{r} = 1$ takes the form

$$6x_1^2 + 2x_2^2 + 2x_3^2 = 1$$

which is the equation of an ellipsoid.

Comparing the above equation with the standard form for an ellipsoid, the lengths of the semi-major axes are $\dfrac{1}{\sqrt{6}}$, $\dfrac{1}{\sqrt{2}}$ and $\dfrac{1}{\sqrt{2}}$.