Tutorial 3, Question 3

given two CS, $\{\mathcal{E}_0, O, \hat{E}_i\}$

$\{\mathcal{B}, G, \hat{e}_i\}$

$\mathcal{B}$ is fixed to the cube.

Position vector of G w.r.t. O,

$$r_{G/O} = \hat{E}_1 + 2\hat{E}_2 + 3\hat{E}_3$$

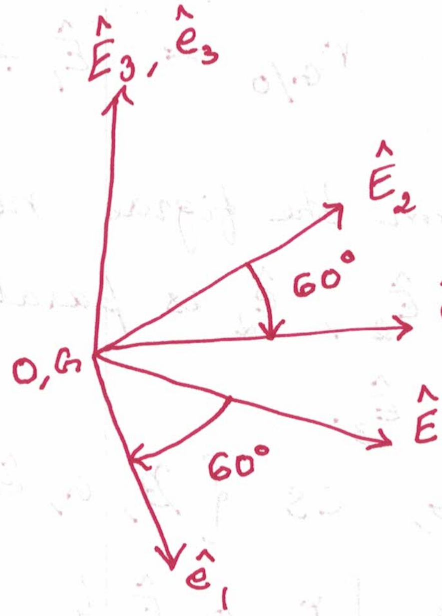
From the figure note that $\hat{e}_1$ is parallel to $\hat{E}_2$, $\hat{e}_2$ is parallel to $\hat{E}_1$ and $\hat{e}_3$ is parallel to $\hat{E}_3$.

So, CS $\{\mathcal{B}, G, \hat{e}_i\}$ can be obtained from CS $\{\mathcal{E}_0, O, \hat{E}_i\}$ by doing a negative rotation of $90°$ about $\hat{E}_3$. Negative means by right hand thumb rule, if you curl your fingers in the direction of rotation, the thumb will be pointing towards negative $\hat{E}_3$ direction.

Now, it is given that the cube (hence the CS $\{\mathcal{B}, G, \hat{e}_i\}$) is rotated about $\hat{e}_3$

in a counter-clockwise direction, which is a positive rotation about $\hat{e}_3$. The rotation angle is 30°.

Thus, after the cube's rotation, the CS $\{B, G, \hat{e}_i\}$ is at an angle 60° with the CS $\{\xi_0, O, \hat{E}_i\}$ (as shown in the figure)



This figure has been obtained by translating the CS $\{B, G, \hat{e}_i\}$ such that G coincides ~~with~~ with O; in order to show the rotation angle between the two CS.

Note that this corresponds to a negative rotation of CS ~~$\{B, G\}$~~ $\{\xi_0, O, \hat{E}_i\}$ by 60° about $\hat{E}_3$.

a) Now defining rotation tensor R such that

$$\{\mathcal{E}_0, O, \hat{E}_i\} \underset{R^T}{\overset{R}{\rightleftarrows}} \{\mathcal{B}, G, \hat{e}_i\}$$

$$R = R_{ij}\, \hat{E}_i \otimes \hat{E}_j$$

where $R_{ij} = \hat{e}_j \cdot \hat{E}_i$

$$\hat{E}_1 \otimes \hat{E}_2 \implies R_{12} = \hat{e}_2 \cdot \hat{E}_1$$

$$= |\hat{e}_2|\,|\hat{E}_1| \cos 30$$

$$= \cos 30 = \sin 60$$

$$\hat{E}_2 \otimes \hat{E}_1 \implies R_{21} = \hat{e}_1 \cdot \hat{E}_2$$

$$= 1.1.\cos(90+60)$$

$$= -\sin 60$$

$$\hat{E}_1 \otimes \hat{E}_1 \implies R_{11} = \hat{e}_1 \cdot \hat{E}_1$$

$$= \cos 60$$

$$\hat{E}_2 \otimes \hat{E}_2 \implies R_{22} = \hat{e}_2 \cdot \hat{E}_2$$

$$= \cos 60$$

$$\hat{E}_3 \otimes \hat{E}_3 \implies R_{33} = \hat{e}_3 \cdot \hat{E}_3$$

$$= 1$$

$$\hat{E}_1 \otimes \hat{E}_3 \Longrightarrow R_{13} = \hat{e}_3 \cdot \hat{E}_1$$
$$= 0$$

Similarly, $R_{31} = R_{23} = R_{32} = 0$

Thus, the rotation tensor is

$$\cancel{R = \sin 60 \{ \hat{E}_2}$$

$$R = \sin 60 \cdot \hat{E}_1 \otimes \hat{E}_2 - \sin 60\, \hat{E}_2 \otimes \hat{E}_1$$
$$+ \cos 60 \cdot \hat{E}_1 \otimes \hat{E}_1 + \cos 60\, \hat{E}_2 \otimes \hat{E}_2 + \hat{E}_3 \otimes \hat{E}_3$$

In matrix form,

$$[R]_{\mathcal{E}_0} = \begin{bmatrix} \cos 60 & \sin 60 & 0 \\ -\sin 60 & \cos 60 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

b) The location vector to be found is

$$[r_{O/G}]_{\mathcal{B}}$$

Note that the vector $r_{O/G} = -r_{G/O}$

$$r_{O/G} = -\left(\hat{E}_1 + 2\hat{E}_2 + 3\hat{E}_3\right)$$

This is in CS $\{\mathcal{E}_0, O, \hat{E}_i\}$

$$[r_{O/G}]_{\mathcal{B}} = [R]^T_{\mathcal{E}_0} [r_{O/G}]_{\mathcal{E}_0}$$

$$= \begin{bmatrix} \cos 60 & -\sin 60 & 0 \\ \sin 60 & \cos 60 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ -2 \\ +3 \end{bmatrix}$$

$$= (-\cos 60 + 2\sin 60)\,\hat{e}_1$$

$$+ (-\sin 60 - 2\cos 60)\,\hat{e}_2$$

$$-3\,\hat{e}_3$$

$$= \left(\sqrt{3} - \frac{1}{2}\right)\hat{e}_1 - \left(1 + \frac{\sqrt{3}}{2}\right)\hat{e}_2 - 3\,\hat{e}_3$$