Problem 6

Find the principal values λ_i and principal vectors $\hat{v}_i$ of the second order tensor

$$\underline{\underline{T}} = \hat{e}_1 \otimes \hat{e}_1 + 2(\hat{e}_1 \otimes \hat{e}_3) + \hat{e}_2 \otimes \hat{e}_2 + 2(\hat{e}_3 \otimes \hat{e}_1) - 2(\hat{e}_3 \otimes \hat{e}_3)$$

Comment on the nature of the principal values and vectors. Confirm that $\hat{v}_i$ are independent, so that we can define a CS β. Is β cartesian? Express $\underline{\underline{T}}$ in terms of the unit tensorial basis by $\hat{v}_i \otimes \hat{v}_j$ in β and also find $[\underline{\underline{T}}]_\beta$

Solution

Considering the given coordinate system $\{\delta, S, \hat{e}_i\}$, we can write the tensor $\underline{\underline{T}}$ as

$$\underline{\underline{T}} = T_{ij}(\hat{e}_i \otimes \hat{e}_j)$$

Thus the matrix of $\underline{\underline{T}}$ in $\mathcal{S}$ is (from given expression)

$$[\underline{\underline{T}}]_{\mathcal{S}} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & -2 \end{bmatrix}$$

Note that the matrix is symmetric.

we find the principal values from

$$\det\left\{ [\underline{T}]_{\mathcal{S}} - \lambda [\underline{1}]_{\mathcal{S}} \right\} = 0$$

$$\det\left\{ \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & -2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right\} = 0$$

which gives the characteris equation

$$(1-\lambda)(\lambda^2+\lambda-6) = 0$$

which gives the principal values,

$$\lambda_1 = 1 \quad \lambda_2 = -3 , \quad \lambda_3 = 2$$

Note that they are distinct and real

The corresponding principal vectors can then be determined as

$$[\hat{v}_1]_{\mathcal{S}} = \begin{bmatrix} 0 \\ +1 \\ 0 \end{bmatrix} \quad [\hat{v}_2]_{\mathcal{S}} = \begin{bmatrix} -0.4472 \\ 0 \\ 0.8944 \end{bmatrix} \quad [\hat{v}_3]_{\mathcal{S}} = \begin{bmatrix} 0.8944 \\ 0 \\ 0.4472 \end{bmatrix}$$

Note that they are normalized such that $|v| = 1$, the numbering is chosen such that they form a right-handed system

It can be easily checked that

$$\hat{v}_1 \cdot \hat{v}_2 = 0$$

$$\hat{v}_1 \cdot \hat{v}_3 = 0$$

$$\hat{v}_2 \cdot \hat{v}_3 = 0$$

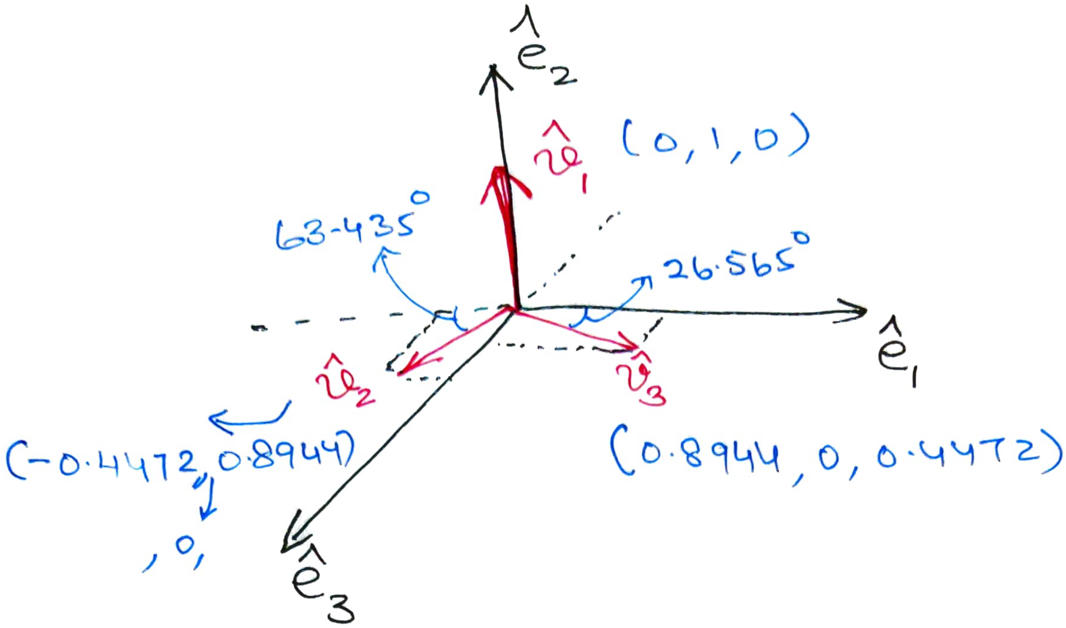
Thus they form an orthogonal basis and we can define a coordinate system β with these vectors as the basis. Thus we have

$\{\beta, P, \hat{v}_i\}$ which is a Cartesian Coordinate System.

Now let us express $\underline{\underline{T}}$ in terms of the unit tensorial basis $\hat{v}_i \otimes \hat{v}_j$ in β.

ie $\underline{\underline{T}} = T'_{ij} \hat{v}_i \otimes \hat{v}_j$, we need to determine T'_{ij} in this equation.

Since we have $\underline{\underline{T}} = T_{ij} \hat{e}_i \otimes \hat{e}_j$ let us see how we can relate the basis $\hat{v}_i$ with the basis $\hat{e}_i$



So we wish to write the relations as

$$\hat{e}_1 = e_{11}\hat{v}_1 + e_{12}\hat{v}_2 + e_{13}\hat{v}_3$$

$$\hat{e}_2 = e_{21}\hat{v}_1 + e_{22}\hat{v}_2 + e_{23}\hat{v}_3$$

$$\hat{e}_3 = e_{31}\hat{v}_1 + e_{32}\hat{v}_2 + e_{33}\hat{v}_3$$

where $e_{ij} = \hat{e}_i \cdot \hat{v}_j$

Evaluating the direction cosines, we can write,

$$\hat{e}_1 = -0.4472\,\hat{v}_2 + 0.8944\,\hat{v}_3$$

$$\hat{e}_2 = \hat{v}_1$$

$$\hat{e}_3 = 0.8944\,\hat{v}_2 + 0.4472\,\hat{v}_3$$

$$\begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -0.4472 & 0.8944 \\ 1 & 0 & 0 \\ 0 & 0.8944 & 0.4472 \end{bmatrix}}_{M} \begin{bmatrix} \hat{v}_1 \\ \hat{v}_2 \\ \hat{v}_3 \end{bmatrix}$$

Starting from

$$\underline{\underline{T}} = T_{ij}\, \hat{e}_i \otimes \hat{e}_j$$

and substituting for $\hat{e}_i$ and $\hat{e}_j$

$$\underline{\underline{T}} = T_{ij}\left(e_{im}\hat{v}_m\right) \otimes \left(e_{jn}\hat{v}_n\right)$$

$$= T_{ij}\, e_{im}\, e_{jn}\, \hat{v}_m \otimes \hat{v}_n$$

$$\therefore \; T'_{mn} = T_{ij}\, e_{im}\, e_{jn}$$

This can also be written as

$$[T'] = [M]^T [T] [M]$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ -0.4472 & 0 & 0.8944 \\ 0.8944 & 0 & 0.4472 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & -2 \end{bmatrix} \times \begin{bmatrix} 0 & -0.4472 & 0.8944 \\ 1 & 0 & 0 \\ 0 & 0.8944 & 0.4472 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

It is a diagonal matrix with the principal values along the diagonal.