Lecture 7

Rigid body kinematics: Euler angle sequence

18-24 August, 2021

Euler angles

I. *Any* rotation tensor R requires <u>*three*</u> independent pieces of information: $\hat{\mathbf{n}}$, θ.

II. **Euler angle representation**. *Any* R may be represented as <u>*three*</u> *successive rotations* θ_i about *three* <u>*known*</u> *axes* $\hat{\mathbf{a}}_i$, $i = 1 \ldots 3$:

1. $\mathsf{R} = \mathsf{R}_3\left(\hat{\mathbf{a}}_3, \theta_3\right) \cdot \mathsf{R}_2\left(\hat{\mathbf{a}}_2, \theta_2\right) \cdot \mathsf{R}_1\left(\hat{\mathbf{a}}_1, \theta_1\right)$.
2. Rotation axes can <u>*not*</u> be parallel.
3. ***Euler angle sequence***: $\left\{\hat{\mathbf{a}}_i, \theta_i\right\}, i = 1 \ldots 3$.
4. <u>*Infinite*</u> Euler angle sequences possible.

III. **Equivalently**, a rigid body can go from one orientation to another through <u>*three*</u> *successive rotations* θ_i about *axes* $\hat{\mathbf{a}}_i$.

IV. **Equivalently**, in order to take a rigid body from one orientation to another, rotate its BFCS through <u>*three*</u> *successive rotations* θ_i *about axes* $\hat{\mathbf{a}}_i$.

Euler angles

Example Relate CCS $\{\mathscr{E}_0, G, \hat{\mathbf{E}}_i\}$ and $\{\mathscr{E}, G, \hat{\mathbf{e}}_i\}$

1. *Application*: Relating orientations of a rigid body, or global CS and BFCS.
2. Can find R: $\{\mathscr{E}_0, G, \hat{\mathbf{E}}_i\} \xrightarrow{\mathsf{R}} \{\mathscr{E}, G, \hat{\mathbf{e}}_i\}$.
3. Find an Euler angle sequence for R.

I. ***Step* 0**. Select an Euler angle sequence.

Use ***z-x-z*** or ***3-1-3* Euler angle sequence**:

1. ***Step* 1**. Set $\{\hat{\mathbf{a}}_1, \theta_1\} = \{\hat{\mathbf{E}}_3, \varphi\}$.
   i. Find CCS $\{\mathscr{E}', G, \hat{\mathbf{e}}_i'\}$: $\hat{\mathbf{e}}_i' = \mathsf{R}_\varphi(\hat{\mathbf{E}}_3, \varphi) \cdot \hat{\mathbf{E}}_i$
2. ***Step* 2**. Set $\{\hat{\mathbf{a}}_2, \theta_2\} = \{\hat{\mathbf{e}}_1', \theta\}$.
   i. Find CCS $\{\mathscr{E}'', G, \hat{\mathbf{e}}_i''\}$: $\hat{\mathbf{e}}_i'' = \mathsf{R}_\theta(\hat{\mathbf{e}}_1', \theta) \cdot \hat{\mathbf{e}}_i'$
3. ***Step* 3**. Set $\{\hat{\mathbf{a}}_3, \theta_2\} = \{\hat{\mathbf{e}}_3'', \psi\}$.
   i. Find CCS $\{\mathscr{E}, G, \hat{\mathbf{e}}_i\}$: $\hat{\mathbf{e}}_i = \mathsf{R}_\psi(\hat{\mathbf{e}}_3'', \psi) \cdot \hat{\mathbf{e}}_i''$

$$\{\mathscr{E}_0, \hat{\mathbf{E}}_i\} \xrightarrow{\mathsf{R}_\varphi} \{\mathscr{E}', \hat{\mathbf{e}}_i'\} \xrightarrow{\mathsf{R}_\theta} \{\mathscr{E}'', \hat{\mathbf{e}}_i''\} \xrightarrow{\mathsf{R}_\psi} \{\mathscr{E}, \hat{\mathbf{e}}_i\}$$

$$\mathsf{R} = \mathsf{R}_\psi(\hat{\mathbf{e}}_3'', \psi) \cdot \mathsf{R}_\theta(\hat{\mathbf{e}}_1', \theta) \cdot \mathsf{R}_\varphi(\hat{\mathbf{E}}_3, \varphi)$$

Euler angles

Aim. Find φ, θ, ψ for $\{\mathscr{E}_0, G, \hat{\mathbf{E}}_i\} \xrightarrow{\mathsf{R}} \{\mathscr{E}, G, \hat{\mathbf{e}}_i\}$.

I. Compute $[\mathsf{R}]_{\mathscr{E}_0}$ directly: $R_{ij} = \hat{\mathbf{e}}_j \cdot \hat{\mathbf{E}}_i$.

II. Compute $[\mathsf{R}]_{\mathscr{E}_0}$ for Euler angle sequence:

1. $[\mathsf{R}]_{\mathscr{E}_0} = [\mathsf{R}_\psi(\hat{\mathbf{e}}_3'')]_{\mathscr{E}_0} [\mathsf{R}_\theta(\hat{\mathbf{e}}_1')]_{\mathscr{E}_0} [\mathsf{R}_\varphi(\hat{\mathbf{E}}_3)]_{\mathscr{E}_0}$
   i. *Don't* know $[\mathsf{R}_\psi(\hat{\mathbf{e}}_3'')]_{\mathscr{E}_0}$ and $[\mathsf{R}_\theta(\hat{\mathbf{e}}_1')]_{\mathscr{E}_0}$!
2. $[\mathsf{R}]_{\mathscr{E}_0} = [\mathsf{R}_\varphi(\hat{\mathbf{E}}_3)]_{\mathscr{E}_0} [\mathsf{R}_\theta(\hat{\mathbf{e}}_1')]_{\mathscr{E}'} [\mathsf{R}_\psi(\hat{\mathbf{e}}_3'')]_{\mathscr{E}''}$
   i. Ordering is *reversed*. See also *Tut. 3.5*.
3. Each rotation about a coordinate axis:

$$[\mathsf{R}_\varphi]_{\mathscr{E}_0} = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad [\mathsf{R}_\theta]_{\mathscr{E}'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix},$$

$$[\mathsf{R}_\psi]_{\mathscr{E}''} = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

III. Equate components of $[\mathsf{R}]_{\mathscr{E}_0}$ to get φ, θ, ψ


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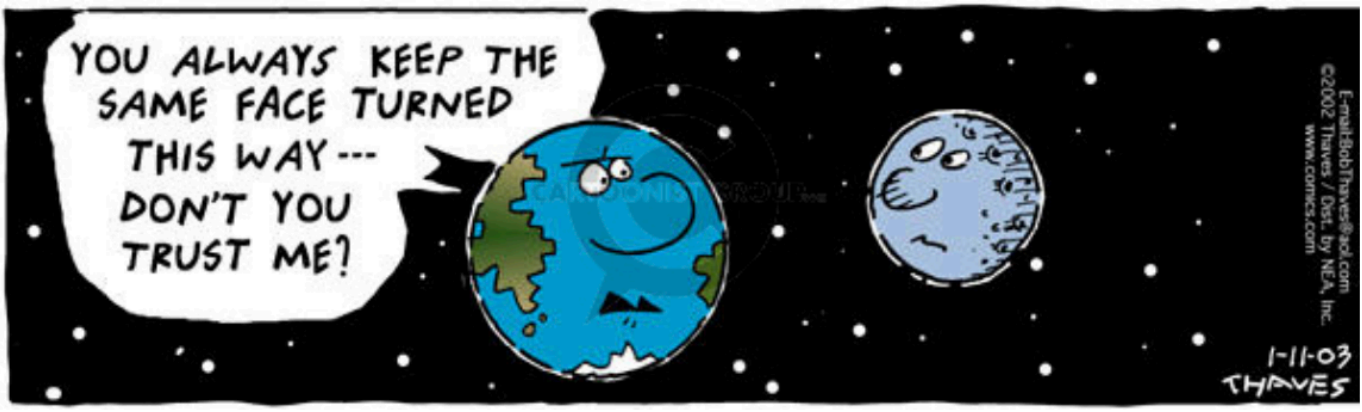
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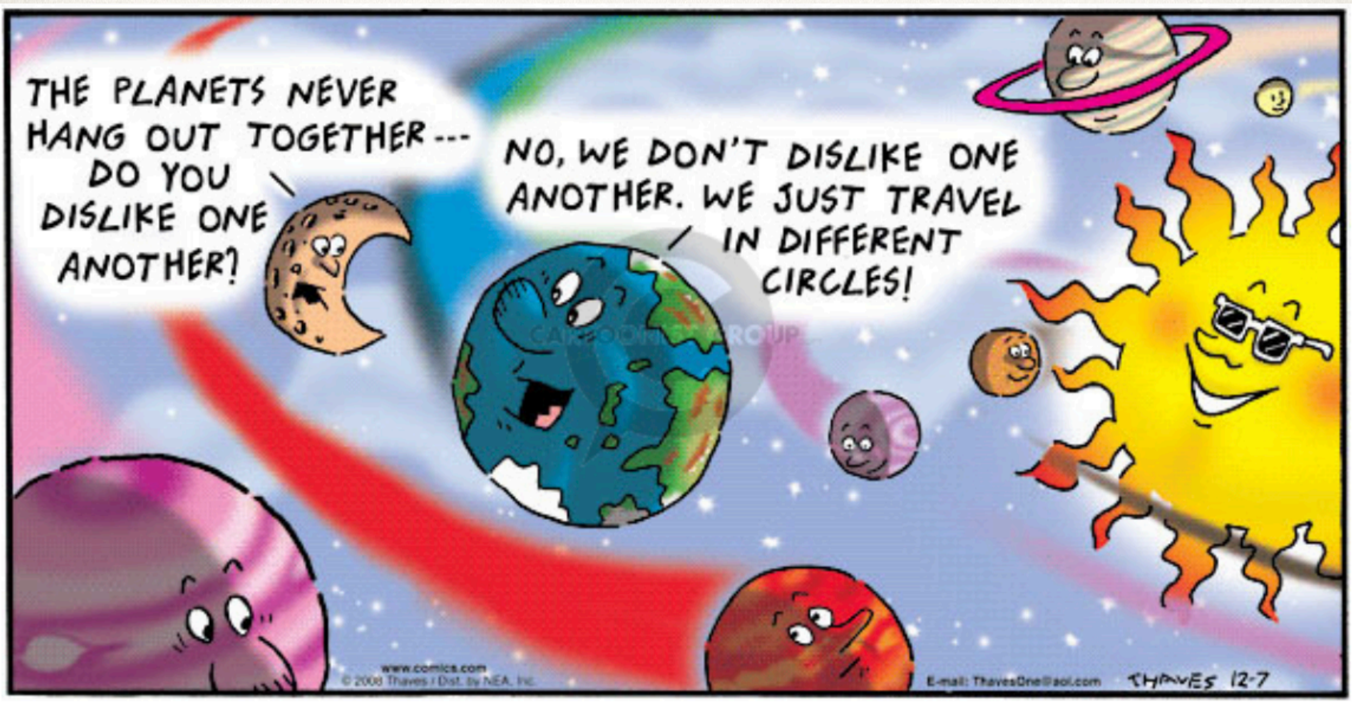
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