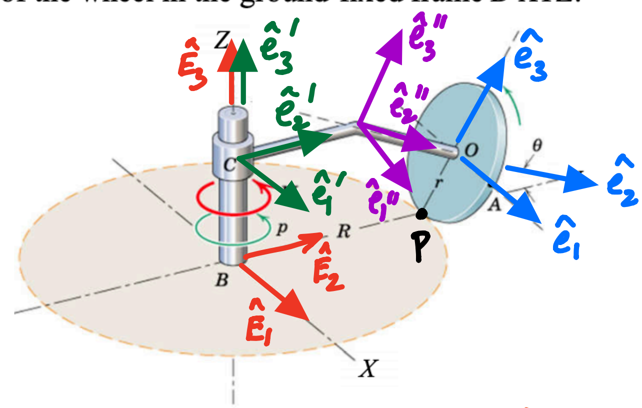
(1) A wheel of radius r, connected to a bent shaft OC, rolls without slipping as shown. The shaft rotates about the vertical shaft BC at a constant angular rate of p rad/sec. Determine the following kinematic quantities:

(a) Angular velocity of the wheel in the ground-fixed frame B-XYZ.

(b) Angular acceleration of the wheel in the ground-fixed frame B-XYZ.



Rotation flowchart:

BFCS of arm

Also, BFCS of arm as $\theta = \text{const.}$

$$\{\mathcal{E}_0, \hat{E}_i\} \xrightarrow{\underline{\underline{R}}(\hat{E}_3,\, \varphi(t))} \{\mathcal{E}', \hat{e}_i'\} \xrightarrow{\underline{\underline{R}}(\hat{e}_1',\, \theta)} \{\mathcal{E}'', \hat{e}_i''\}$$

GROUND frame

BFCS of wheel

$$\{\mathcal{E}, \hat{e}_i\} \xleftarrow{\underline{\underline{R}}(\hat{e}_2'',\, \psi(t))} \{\mathcal{E}'', \hat{e}_i''\}$$

VELOCITY ANALYSIS

Angular velocity: $\underline{\omega}_{\mathcal{E}/\mathcal{E}_0} = \underline{\omega}_{\mathcal{E}/\mathcal{E}''} + \underline{\omega}_{\mathcal{E}''/\mathcal{E}'} + \underline{\omega}_{\mathcal{E}'/\mathcal{E}_0}$

$$= \dot{\psi}\,\hat{e}_2'' + \cancel{\dot{\theta}}^{\,0}\,\hat{e}_1' + \dot{\varphi}\,\hat{E}_3$$

$$\therefore \underline{\omega}_{\mathcal{E}/\mathcal{E}_0} = \underline{\omega}^W = \dot{\psi}\,\hat{e}_2'' + p\,\hat{E}_3 \quad \text{——} \quad (*)$$

as $\dot{\varphi} = p$ is given.

To solve for the angular velocity of the wheel, we will write the velocity of point on wheel (i.e. P_W) which is in contact with the ground in TWO different ways.

1^{st} way: Because of rolling condition $\underline{v}^{P_W} = \underline{v}^{P_G} = 0$. (1)

Point on ground in contact with wheel

2^{nd} way: Observe P_W while sitting on CS attached to arm with origin at O.

$$\underline{v}^{P} = \underline{v}^{G} + \underline{\omega}_{\mathcal{E}/\mathcal{E}_0} \times \underline{r}^{P/G} + \underline{v}^{P}_{rel}$$

$P \rightarrow P_W$

$\mathcal{E} \rightarrow \mathcal{E}''$

$\mathcal{E}_0 \rightarrow \mathcal{E}_0$

$G \rightarrow O$

Then, $\underline{v}^{P_W} = \underline{v}^{O} + \underline{\omega}_{\mathcal{E}''/\mathcal{E}_0} \times \underline{r}^{P/O} + \underline{v}^{P_W}_{rel}$ —— (2)

Use $\mathcal{E}''$ only because its orientation aligns with the disc.

- $\underline{v}^{P_W}_{rel}$ = velocity of P_W as seen by an observer sitting at O and rotating with the arm

$$= \underline{\omega}_{\mathcal{E}/\mathcal{E}''} \times \underline{r}^{P/O} = -\dot{\psi}\, \hat{e}_2'' \times r\, \hat{e}_3'' = -\dot{\psi} r\, \hat{e}_1''. \quad (3)$$

- $\underline{\omega}_{\mathcal{E}''/\mathcal{E}_0} \times \underline{r}^{P/O} = \underline{\omega}_{\mathcal{E}'/\mathcal{E}_0} \times \underline{r}^{P/O} = -\phi\, \hat{E}_3 \times r\, \hat{e}_3'' = \phi r \sin\theta\, \hat{e}_1''.$ (4)

same as $\hat{e}_3'$

- $\underline{v}^{O} = \cancelto{0}{\underline{v}^{C}} + \underline{\omega}_{\mathcal{E}'/\mathcal{E}_0} \times \underline{r}^{O/C} = \phi \hat{E}_3 \times \left(-|\underline{r}^{B/C}|\hat{E}_3 + R\hat{e}_2' + r\hat{e}_3''\right)$

$$= -\phi\left(R + r\sin\theta\right)\hat{e}_1''. \quad (5)$$

Same as $\hat{e}_1'$

Combining (2)–(5):

$$\underline{v}^{P_W} = \left\{ -p\left(R + \cancel{r\sin\theta}\right) + \cancel{p r \sin\theta} - \dot{\psi} r \right\} \hat{e}_1''. \qquad (2')$$

Finally, comparing (1) & (2'):

$$0 = pR + \dot{\psi} r \quad \Rightarrow \quad \boxed{\dot{\psi} = -pR/r} \qquad (6)$$

Replacing (6) in (*) $\Rightarrow$

$$\underline{\omega}^W = p\left(\hat{E}_3 - \frac{R}{r}\hat{e}_2''\right) \qquad (7a)$$

$$= p\left(\hat{E}_3 - \frac{R}{r}\cos\theta\, \hat{e}_2' + \frac{R}{r}\sin\theta\, \hat{e}_3'\right) \qquad (7b)$$

at the current moment $\hat{e}_i' = \hat{E}_i$

$$= -p\frac{R}{r}\cos\theta\, \hat{E}_2 + p\left(1 + \frac{R}{r}\sin\theta\right)\hat{E}_3 \qquad (7c)$$

ACCELERATION ANALYSIS

METHOD 1

To find $\underline{\alpha}^W$ we can differentiate (7a):

$$\underline{\alpha}^W = \frac{d\underline{\omega}^W}{dt} = -\frac{R}{r} p \frac{d\hat{e}_2''}{dt} = -\frac{R}{r} p\, \underline{\omega}^{e''/k_0} \times \hat{e}_2''$$

$$= -p^2 \frac{R}{r} \hat{E}_3 \times \hat{e}_2'' = p^2 \frac{R}{r}\cos\theta\, \hat{e}_1'' = p^2 \frac{R}{r}\cos\theta\, \hat{e}_1'$$

$$\therefore \quad \underline{\alpha}^W = p^2 \frac{R}{r}\cos\theta\, \hat{E}_1 \quad \text{(at this moment)}$$

The method above has the drawback that one has to be careful about which of the formulae for $\underline{\omega}^W$ — (7a), (7b) or (7c) — one uses to differentiate.

We can NOT use (7c), WHY?

but either of (7a) or (7b) is okay. WHY?

Thus, I will now use the second rolling condition which relates accelerations of P_W & P_G to describe another way to find $\underline{a}^W$. While the method is longer, it is safer (less error prone).

METHOD 2

Need 3 ways as 1st way is inconclusive

Write the acceleration of P_W in THREE different ways.

1st way: Second rolling condition ⇒

$$\underline{a}^{P_W} \cdot \hat{t} = \underline{a}^{P_G} \cdot \hat{t} = 0 \quad \text{as } \underline{a}^{P_G} = \underline{0}.$$

Here $\hat{t}$ is the tangent to the contact path, which is a circle of radius R ⇒ $\hat{t} = -\hat{e}_1'$

$$\therefore \quad \underline{a}^{P_W} \cdot \hat{e}_1' = 0 \qquad \text{——— (8)}$$

$$\Rightarrow \quad \underline{a}^{P_W} = a_2 \hat{e}_2' + a_3 \hat{e}_3'.$$

This is the most that we can obtain from the condition of pure rolling.

2nd way: As for velocity analysis, we observe P_W's acceleration while sitting on a CS attached to the arm and centred at O.

Then we use the FIVE-term acceleration formula:

$$\underline{a}^P = \underline{a}^G + \underline{\omega}_{\varepsilon/\varepsilon_0} \times (\underline{\omega}_{\varepsilon/\varepsilon_0} \times \underline{r}^{P/G}) + \underline{\alpha}_{\varepsilon/\varepsilon_0} \times \underline{r}^{P/G}$$
$$+ 2\,\underline{\omega}_{\varepsilon/\varepsilon_0} \times \underline{v}^P_{rel} + \underline{a}^P_{rel}$$

In the present case, we will have

$$P \longrightarrow P_W, \quad \varepsilon \longrightarrow \varepsilon'', \quad \varepsilon_0 \longrightarrow \varepsilon_0, \quad G \longrightarrow O$$

$$\underline{a}^{P_W} = \underline{a}^O + \underline{\omega}_{\varepsilon''/\varepsilon_0} \times (\underline{\omega}_{\varepsilon''/\varepsilon_0} \times \underline{r}^{P/O}) + \underline{\alpha}_{\varepsilon''/\varepsilon_0} \times \underline{r}^{P/O}$$
$$+ 2\,\underline{\omega}_{\varepsilon''/\varepsilon_0} \times \underline{v}^{P_W}_{rel} + \underline{a}^{P_W}_{rel} \qquad \text{——— } (\dagger)$$

• $\underline{a}^{P_W}_{rel}$ = acceleration of P^W as seen by an observer sitting at O and rotating with the arm

$$= \underline{\omega}_{\mathcal{E}/\mathcal{E}''} \times (\underline{\omega}_{\mathcal{E}/\mathcal{E}''} \times \underline{r}^{P/O}) + \underline{\alpha}_{\mathcal{E}/\mathcal{E}''} \times \underline{r}^{P/O}$$

$$= -\dot{\psi}\, \hat{e}_2'' \times (\dot{\psi}\, \hat{e}_2'' \times r\, \hat{e}_3'') + \ddot{\psi}\, \hat{e}_2'' \times (-r\hat{e}_3'')$$

$$= -\dot{\psi}^2 r\, \hat{e}_3'' - \ddot{\psi} r\, \hat{e}_1'' \qquad \text{———} \qquad (9)$$

• $$2\,\underline{\omega}_{\mathcal{E}''/\mathcal{E}_0} \times \underline{v}^{P_W}_{rel} = 2p\hat{E}_3 \times \overbrace{(-\dot{\psi} r\, \hat{e}_1'')}^{\text{from (3)}}$$

$$= -2p\dot{\psi} r\, \hat{e}_2' \qquad \text{———} \qquad (10)$$

• $\underline{\alpha}_{\mathcal{E}''/\mathcal{E}_0} \times \underline{r}^{P/O} = 0$ as $\underline{\alpha}_{\mathcal{E}''/\mathcal{E}_0} = 0$ (11)

• $$\underline{\omega}_{\mathcal{E}''/\mathcal{E}_0} \times (\underline{\omega}_{\mathcal{E}''/\mathcal{E}_0} \times \underline{r}^{P/O}) = p\hat{E}_3 \times \{p\hat{E}_3 \times (-r\hat{e}_3'')\}$$

$$= +p^2 r \sin\theta\, \hat{E}_3 \times \hat{e}_1'' = p^2 r \sin\theta\, \hat{e}_2'. \qquad (12)$$

• $$\underline{a}^O = \underline{\omega}_{\mathcal{E}'/\mathcal{E}_0} \times (\underline{\omega}_{\mathcal{E}'/\mathcal{E}_0} \times \underline{r}^{O/C}) + \cancelto{0}{\underline{\alpha}_{\mathcal{E}'/\mathcal{E}_0}} \times \underline{r}^{O/C} + \cancelto{0}{\underline{a}^C}$$

(as O & C are on the arm)

$$= p\hat{E}_3 \times \{p\hat{E}_3 \times (-|\underline{r}^{B/C}|\hat{E}_3 + R\hat{e}_2' + r\hat{e}_3'')\}$$

$$= -p^2 R\,\hat{e}_2' - p^2 r \sin\theta\,\hat{e}_2' \quad \text{———} \quad (13)$$

Substituting (9) – (13) in (†):

$$\underline{a}^{P_W} = -p^2 r\left(\frac{R}{r} + \cancel{\sin}\theta\right)\hat{e}_2' + \cancel{p^2 r\sin\theta\,\hat{e}_2'}$$

$$- 2p\dot{\psi} r\,\hat{e}_2' + \dot{\psi}^2 r\,\hat{e}_3'' - \ddot{\psi} r\,\hat{e}_1''$$

$$= -\ddot{\psi} r\hat{e}_1' - (p^2 R + 2p\dot{\psi} r - \dot{\psi}^2 r\sin\theta)\,\hat{e}_2' + \dot{\psi}^2 r\cos\theta\,\hat{e}_3'$$

$\dot{\psi} = -pR/r$

$$= -\ddot{\psi} r\hat{e}_1' - \left(-p^2 R - p^2\frac{R^2}{r}\sin\theta\right)\hat{e}_2' + p^2\frac{R^2}{r}\cos\theta\,\hat{e}_3' \quad (14)$$

3rd way: Because P_W & O are both on the wheel:

$$\underline{a}^{P_W} = \underline{a}^{O} + \underline{\omega}^{W}\times(\underline{\omega}^{W}\times\underline{r}^{P/O}) + \underline{\alpha}^{W}\times\underline{r}^{P/O} \quad (\S)$$

Let $\underline{\alpha}^{W} = \alpha_i\,\hat{e}_i''$.

Then $\underline{\alpha}^{W}\times\underline{r}^{P/O} = r(\alpha_1\hat{e}_2'' - \alpha_2\hat{e}_1'')$ (15)

Substituting from (7a), (13) & (15) into (§):

$\hat{e}_2'' = \hat{e}_2'\cos\theta - \hat{e}_3'\sin\theta$

$\hat{e}_3'' = \hat{e}_2'\sin\theta + \hat{e}_3'\cos\theta$

$$\underline{a}^{P_W} = \cancel{-p^2 R\,\hat{e}_2'} - \cancel{p^2 r\sin\theta\,\hat{e}_2'}$$

$$+ p^2 r\left(\sin\theta + \frac{R}{r}\right)\left(\cancel{\hat{e}_2'} + \frac{R}{r}\hat{e}_3''\right) + r(\alpha_1\hat{e}_2'' - \alpha_2\hat{e}_1'')$$

$$= -\alpha_2 r \hat{e}_1' + \left\{\alpha_1 r \cos\theta + p^2 R\left(\sin\theta + \frac{R}{r}\right)\sin\theta\right\}\hat{e}_2' + \left\{-\alpha_1 r \sin\theta + p^2 R\left(\sin\theta + R/r\right)\cos\theta\right\}\hat{e}_3' \tag{16}$$

Comparing (16) with (14):

$$-\ddot{\psi} r \hat{e}_1' + \left(+p^2 R + p^2 \frac{R^2}{r}\sin\theta\right)\hat{e}_2' + p^2\frac{R^2}{r}\cos\theta\,\hat{e}_3'. \tag{14}$$

and invoking (8) we obtain

$$\ddot{\psi} = \alpha_2 = 0 \qquad \alpha_1 = p^2\frac{R}{r}\cos\theta \tag{17}$$

which matches the answer obtained by method 1.