1. Prove the following identities. Note how formulae (a)-(c) "tensorize" vector operations
   (a) $\mathbf{a} \cdot \mathbf{b} = \mathrm{tr}\,(\mathbf{a} \otimes \mathbf{b})$
   (b) $\mathbf{a} \times \mathbf{b} = -\,2\,\mathrm{ax}\{\mathrm{sk}\,(\mathbf{a} \otimes \mathbf{b})\}$; the operation '<u>ax</u>' was defined in lectures. The operation '<u>sk</u>' computes the skew-symmetric tensor associated with a tensor A by the formula $\mathrm{sk}(\mathsf{A}) = (\mathsf{A} - \mathsf{A}^T)/2$; easy to check that sk(A) is skew-symmetric.
   (c) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathrm{tr}\,(\mathbf{a} \otimes \mathbf{c})\,\mathbf{b} - \mathrm{tr}\,(\mathbf{a} \otimes \mathbf{b})\,\mathbf{c}$
   (d) $\mathbf{a} \cdot (\mathsf{A} \cdot \mathbf{b}) = (\mathsf{A}^T \cdot \mathbf{a}) \cdot \mathbf{b}$; this equality is often used as a definition of A^T.
   (e) $\mathbf{a} \otimes (\mathsf{A} \cdot \mathbf{b}) = (\mathbf{a} \otimes \mathbf{b}) \cdot \mathsf{A}^T$;

(a) $\underline{a} \cdot \underline{b}$ $\quad \{\mathcal{E}, O, \hat{e}_i\}$

$\underline{a} = a_i \hat{e}_i \qquad \underline{b} = b_j \hat{e}_j$

$\underline{a} \cdot \underline{b} = a_i b_j \, \hat{e}_i \cdot \hat{e}_j = a_i b_i$ —— LHS

$\underline{\text{RHS}}$: $\mathrm{tr}\,(\underline{a} \otimes \underline{b}) = \mathrm{tr}\,(a_i \hat{e}_i \otimes b_j \hat{e}_j)$

$= \mathrm{tr}\,(a_i b_j \; \hat{e}_i \otimes \hat{e}_j)$

$= a_i b_i = \text{LHS}$

(b) $\underline{a} \times \underline{b} = -2\,\mathrm{ax}(\mathrm{sk}\,(\underline{a} \otimes \underline{b}))$

$\underline{\text{LHS}}$: $\underline{a} \times \underline{b} = \epsilon_{ijk}\, a_i b_j\, \hat{e}_k$ $\quad$ in $\{\mathcal{E}, O, \hat{e}_i\}$

$\underline{\text{RHS}}$: $\underline{a} \otimes \underline{b} = a_i b_j \; \hat{e}_i \otimes \hat{e}_j =: \underline{\underline{A}}$

$\mathrm{sk}\,(\underline{a} \otimes \underline{b}) = \mathrm{sk}\,(\underline{\underline{A}}) = \frac{1}{2}\,(\underline{\underline{A}} - \underline{\underline{A}}^T)$

$= \frac{1}{2}\left(\underline{a} \otimes \underline{b} - (\underline{a} \otimes \underline{b})^T\right)$

$(\underline{a} \otimes \underline{b})^T = (a_i b_j \; \hat{e}_i \otimes \hat{e}_j)^T = a_j b_i \; \hat{e}_i \otimes \hat{e}_j$

$\mathrm{sk}(\underline{a} \otimes \underline{b}) = \frac{1}{2}\left\{ a_i b_j \; \hat{e}_i \otimes \hat{e}_j - a_j b_i \; \hat{e}_i \otimes \hat{e}_j \right\}$

$$= \frac{1}{2}(a_i b_j - a_j b_i)\, \hat{e}_i \otimes \hat{e}_j$$

Is $sk(\underline{a} \otimes \underline{b})$ skew symmetric? Yes.

$$\{sk(\underline{a} \otimes \underline{b})\}^T = \frac{1}{2}(a_j b_i - a_i b_j)\, \hat{e}_i \otimes \hat{e}_j$$

$$= -\, sk(\underline{a} \otimes \underline{b})$$

Call $\underline{\underline{W}} = \frac{1}{2}(a_i b_j - a_j b_i)\, \hat{e}_i \otimes \hat{e}_j$

$$ax(\underline{\underline{W}}) = -\frac{1}{2} \epsilon_{ijk} W_{ij}\, \hat{e}_k \quad \text{(from lecture)}$$

But $W_{ij} = \frac{1}{2}(a_i b_j - a_j b_i)$

$$ax(\underline{\underline{W}}) = -\frac{1}{2} \epsilon_{ijk} \cdot \frac{1}{2}(a_i b_j - a_j b_i)\, \hat{e}_k$$

$$= -\frac{1}{2} \epsilon_{ijk} \cdot \frac{1}{2}(a_i b_j + a_i b_j)\, \hat{e}_k$$

$\epsilon_{ijk} a_j b_i \hat{e}_k$
$= -\epsilon_{ijk} a_i b_j \hat{e}_k$

$$= -\frac{1}{2} \epsilon_{ijk} a_i b_j\, \hat{e}_k$$

$$RHS = -2\, ax(sk(\underline{a} \otimes \underline{b})) = -\cancel{2}\left(-\frac{1}{\cancel{2}} \epsilon_{ijk} a_i b_j\, \hat{e}_k\right)$$

$$= \epsilon_{ijk} a_i b_j\, \hat{e}_k$$

$$= LHS \quad ✓.$$

(c) $\underline{a} \times (\underline{b} \times \underline{c}) = \mathrm{tr}(\underline{a} \otimes \underline{c})\, \underline{b} - \mathrm{tr}(\underline{a} \otimes \underline{b})\, \underline{c}$

$\quad \parallel$

$(\underline{a} \cdot \underline{c})\underline{b} - (\underline{a} \cdot \underline{b})\underline{c}$ // from problem (a)

(d) $\underline{a} \cdot (\underline{\underline{A}} \cdot \underline{b}) = (\underline{\underline{A}}^T \cdot \underline{a}) \cdot \underline{b}$ (to show)

LHS

$$\underline{\underline{A}} \cdot \underline{b} = A_{ij} b_j \hat{e}_i \quad \text{in } \{\mathcal{E}, O, \hat{e}_i\}$$

$$\underline{a} \cdot (\underline{\underline{A}} \cdot \underline{b}) = a_k \hat{e}_k \cdot (A_{ij} b_j \hat{e}_i)$$

$$= a_k A_{ij} b_j \delta_{ki} = a_i A_{ij} b_j$$

RHS

$$\underline{\underline{A}}^T \cdot \underline{a} = A_{ji}\, \hat{e}_i \otimes \hat{e}_j \cdot a_k \hat{e}_k = A_{ji} \hat{e}_i a_k \delta_{jk}$$

$$= A_{ji} a_j \hat{e}_i$$

$$(\underline{\underline{A}}^T \cdot \underline{a}) \cdot \underline{b} = (A_{ji} a_j \hat{e}_i) \cdot b_k \hat{e}_k = A_{ji} a_j b_k \delta_{ik}$$

$$= A_{ji} a_j b_i \overset{i \leftrightarrow j}{=\!=} A_{ij} a_i b_j$$

$$= \text{LHS}.$$

(e) $\underbrace{\underline{a} \otimes (\underline{\underline{A}} \cdot \underline{b})}_{\text{tensor}} \stackrel{?}{=} \underbrace{(\underline{a} \otimes \underline{b}) \cdot \underline{\underline{A}}^T}_{\text{tensor}}$

$\underline{\text{LHS}}$: $\underline{a} \otimes (\underline{\underline{A}} \cdot \underline{b}) = a_k \hat{e}_k \otimes (A_{ij} b_j \hat{e}_i)$

$= a_k A_{ij} b_j \hat{e}_k \otimes \hat{e}_i$ in $\{\mathcal{E}, O, \hat{e}_i\}$

$\underline{\text{RHS}}$: $(\underline{a} \otimes \underline{b}) \cdot \underline{\underline{A}}^T$

$= (a_i b_j \hat{e}_i \otimes \hat{e}_j) \cdot (A_{mn} \hat{e}_n \otimes \hat{e}_m)$

[Use identity : $(\underline{a} \otimes \underline{b}) \cdot (\underline{c} \otimes \underline{d}) = (\underline{b} \cdot \underline{c})\, \underline{a} \otimes \underline{d}$]

$= a_i b_j A_{mn} \delta_{jn} \hat{e}_i \otimes \hat{e}_m$

$= a_i b_j A_{mj} \hat{e}_i \otimes \hat{e}_m$

$i \to k \quad m \to i$: $a_k b_j A_{ij} \hat{e}_k \otimes \hat{e}_i$

$\therefore$ LHS = RHS ✓

2. Find the principal values λ_i and principal vectors $\hat{\mathbf{v}}_i$ of the second order tensor $\mathsf{D} = 6(\hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_1) + \hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_2 + 9(\hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_3) + 2(\hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_2) + 2(\hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_3) + \hat{\mathbf{e}}_3 \otimes \hat{\mathbf{e}}_3$. Confirm that $\hat{\mathbf{v}}_i$ are independent, so that we can define the principal CS $\mathscr{P}$ of D. Is $\mathscr{P}$ Cartesian? Express D in terms of the unit tensorial basis by $\hat{\mathbf{v}}_i \otimes \hat{\mathbf{v}}_j$ in $\mathscr{P}$ and also find $[\mathsf{D}]_{\mathscr{P}}$.

In $\{\mathcal{E}, \mathcal{O}, \hat{e}_i\}$

$$[\underline{\underline{D}}]_{\mathcal{E}} = \begin{bmatrix} 6 & 1 & 9 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$D_{ij} = \hat{e}_i \cdot \underline{\underline{D}} \cdot \hat{e}_j$$

$$\Rightarrow D_{21} = \hat{e}_2 \cdot \underline{\underline{D}} \cdot \hat{e}_1 = 0$$

or $D_{33} = \hat{e}_3 \cdot \underline{\underline{D}} \cdot \hat{e}_3 = 1$

Eigenvalues/principal values of $[\underline{\underline{D}}]_{\mathcal{E}}$

$$\det [\underline{\underline{D}} - \lambda \underline{\underline{1}}]_{\mathcal{E}} = \begin{vmatrix} 6-\lambda & 1 & 9 \\ 0 & 2-\lambda & 2 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(2-\lambda)(6-\lambda) = 0$$

$$\therefore \lambda_1 = 1, \quad \lambda_2 = 2, \quad \lambda_3 = 6$$

We have 3 different real principal values.

Eigenvectors / principal vectors

for $\lambda_1 = 1$: $\left(\underline{\underline{D}} - \lambda_1 \underline{\underline{1}}\right) \cdot \hat{v}_1 = 0 \qquad |\hat{v}_1| = 1$

in $\mathcal{E}$

$$\begin{bmatrix} 5 & 1 & 9 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

we will get $[\hat{v}_1]_{\mathcal{E}} = \frac{1}{\sqrt{174}} \begin{bmatrix} -7 \\ -10 \\ 5 \end{bmatrix}$

or $\boxed{\hat{v}_1 = \frac{1}{\sqrt{174}}\left(-7\hat{e}_1 - 10\hat{e}_2 + 5\hat{e}_3\right)}$

Similarly for $\lambda_2 = 2$: $\boxed{\hat{v}_2 = \frac{\hat{e}_1 - 4\hat{e}_2}{\sqrt{17}}}$

and for $\lambda_3 = 6$: $\boxed{\hat{v}_3 = \hat{e}_1}$

Clearly, $\hat{v}_i$ are linearly independent.

define the principal CS

$\{ \mathcal{P}, O, \hat{v}_i \}$

in $\mathcal{P}$: $\underline{\underline{D}} = \sum_{i=1}^{3} \lambda_i \, \hat{v}_i \otimes \hat{v}_i$

$= \hat{v}_1 \otimes \hat{v}_1 + 2 \hat{v}_2 \otimes \hat{v}_2 + 6 \hat{v}_3 \otimes \hat{v}_3$

$[\underline{\underline{D}}]_{\mathcal{P}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix}$

cannot get by $\hat{v}_i \cdot \underline{\underline{D}} \cdot \hat{v}_j$

You check.

Finally $\mathcal{P}$ is NOT Cartesian

$\Rightarrow$ Cannot find components of $[\underline{\underline{D}}]_{\mathcal{P}}$

by using $\hat{v}_i \cdot \underline{\underline{D}} \cdot \hat{v}_j$!