Lecture 3

Principal values and Principal vectors; Symmetric and Skew-symmetric tensors; Axial vector

Review of L2

I. **Tensors:** Linear transformation of vectors

1. $\mathsf{A} \cdot \mathbf{a} = \mathbf{b}$ OR $\mathbf{a} \cdot \mathsf{A} = \mathbf{b}$.
2. $\mathsf{C} = (\mathsf{A} + \mathsf{B})$; $\mathsf{D} = \mathsf{A} \cdot \mathsf{B}$.
3. Understood by their operations.

II. **Tensor product $\mathbf{a} \otimes \mathbf{b}$**

1. $(\mathbf{a} \otimes \mathbf{b}) \cdot \mathbf{c} = (\mathbf{c} \cdot \mathbf{b})\mathbf{a}$; $\mathbf{c} \cdot (\mathbf{a} \otimes \mathbf{b}) = (\mathbf{c} \cdot \mathbf{a})\mathbf{b}$
2. $\mathbf{a} \otimes \mathbf{b} = (a_i b_j)\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j$ in $\left\{\mathscr{P}, P, \hat{\mathbf{e}}_i\right\}$.
3. *Tensorial basis* in $\left\{\mathscr{P}, P, \hat{\mathbf{e}}_i\right\}$ is $\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j$.
4. $\mathsf{A} = A_{ij}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j$, with $A_{ij} = \hat{\mathbf{e}}_i \cdot \mathsf{A} \cdot \hat{\mathbf{e}}_j$;

Matrix of A in $\mathscr{P}$: $[\mathsf{A}]_{\mathscr{P}} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$

III. **Tensor algebra**

1. $\mathsf{A}^T = A_{ji}\,\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \implies \left(\mathsf{A}^T\right)_{ij} = A_{ji}$
2. $\mathrm{tr}\,(\mathsf{A}) = A_{ii} = A'_{ii} = \mathrm{tr}\,\left(\mathsf{A}^T\right)$

Principal values, Principal vectors

I. **Definition**: A tensor A has *principal value* λ with associated *principal vector* $\hat{\mathbf{v}}$ if

$$\mathsf{A} \cdot \hat{\mathbf{v}} = \lambda \hat{\mathbf{v}}, \qquad \left|\hat{\mathbf{v}}\right| = 1.$$

1. Same as *eigenvalues* and *eigenvectors*.
2. Any *scaling* of $\hat{\mathbf{v}}$ is a principal vector.
3. Principal pair $\{\lambda, \hat{\mathbf{v}}\}$ *independent* of CS.

II. How to find $\{\lambda, \hat{\mathbf{v}}\}$ for a given A ?

Find $\{\lambda, [\mathbf{v}]_{\mathscr{P}}\}$ for $[\mathsf{A}]_{\mathscr{P}}$ in *any* $\left\{\mathscr{P}, P, \hat{\mathbf{e}}_i\right\}$

III. **Properties**: Real tensor A on 3D vectors:

1. A *always* has ***3*** principal values λ_i :
   - i. *all* real **or** *one* real+complex conjugates
   - ii. *not* all unique; some may repeat;
2. *Real* λ_i has a *real* principal vector $\hat{\mathbf{v}}_i$.
   - i. A *always* has at least ***one*** real $\hat{\mathbf{v}}$.
   - ii. A repeated λ may *not* have as many $\hat{\mathbf{v}}$.
   - iii. $\lambda_i \neq \lambda_j \implies \hat{\mathbf{v}}_i, \hat{\mathbf{v}}_j$ *linearly independent*.
3. A and A^T have *same* λ_i but *not* same $\hat{\mathbf{v}}_i$.

Principal values, Principal vectors

I. Examples

II. **Definition**: <u>If</u> A has *three independent* principal vectors $\hat{\mathbf{v}}_i$, $i = 1\ldots3$, then they define the *principal axes coordinate system* $\left\{\mathscr{P}, O, \hat{\mathbf{v}}_i\right\}$ of A.

1. Generally, $\hat{\mathbf{v}}_i \cdot \hat{\mathbf{v}}_j \neq \delta_{ij}$, i.e. $\left\{\mathscr{P}, O, \hat{\mathbf{v}}_i\right\}$is *not* always a Cartesian CS.

III. **Properties**: A has principal pairs $\{\lambda_i, \hat{\mathbf{v}}_i\}$, $i = 1,2,3$ and principal CS $\left\{\mathscr{P}, O, \hat{\mathbf{v}}_i\right\}$:

1. Use *similarity transformation* to compute

$$[\mathsf{A}]_{\mathscr{P}} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}.$$

A is *diagonalized* in its principal CS!

2. Thus, $\mathsf{A} = \sum_{i=1}^{3} \lambda_i \hat{\mathbf{v}}_i \otimes \hat{\mathbf{v}}_i$.

Symmetric tensor

I. **Definition**: S is *symmetric* if $\mathsf{S} = \mathsf{S}^T$.

II. **Properties**: If S is symmetric with principal values S_i, $i = 1\ldots3$:

1. S_i are *always* real;
2. There exist ***three*** principal vectors $\hat{\mathbf{e}}_i$.
   i. $\hat{\mathbf{e}}_i$ are *always* <u>*orthogonal*</u>.
   ii. *Principal CS* $\left\{\mathscr{P}, C, \hat{\mathbf{e}}_i\right\}$ is *Cartesian*.
3. $\mathsf{S} = \displaystyle\sum_{i=1}^{3} S_i \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_i$ and $[\mathsf{S}]_{\mathscr{P}} = \begin{bmatrix} S_1 & 0 & 0 \\ 0 & S_2 & 0 \\ 0 & 0 & S_3 \end{bmatrix}$.

Symmetric S is <u>*always*</u> *diagonalizable*.

Action of S is a <u>*stretch*</u> by S_i along $\hat{\mathbf{e}}_i$.

III. **Definition**: S is *positive definite* if S is symmetric and $\lambda_i > 0$, $i = 1\ldots3$. Example

Skew-symmetric tensor

I. **Definition**: W is *skew-symmetric* if $\mathsf{W} = -\mathsf{W}^T$ (also called *anti*-symmetric)

II. In *any* CS $\left\{\mathscr{E}, O, \hat{\mathbf{E}}_i\right\}$: $[\mathsf{W}]_{\mathscr{E}} = \begin{bmatrix} 0 & W_{12} & W_{13} \\ -W_{12} & 0 & W_{23} \\ -W_{13} & -W_{23} & 0 \end{bmatrix}$

1. W has *three* independent components.
2. W is naturally *equivalent* to a vector $\mathbf{w}$.
3. $\mathsf{W} \cdot \mathbf{a} \equiv$ *operation* between $\mathbf{w}$ and $\mathbf{a}$ which outputs a *vector*.

III. **Property**: $\mathsf{W} \cdot \mathbf{a} = \mathbf{w} \times \mathbf{a}$, *if* we define

$$\mathbf{w} := -\frac{1}{2}\epsilon_{ijk} W_{jk} \hat{\mathbf{E}}_i = -W_{23}\hat{\mathbf{E}}_1 + W_{12}\hat{\mathbf{E}}_2 - W_{12}\hat{\mathbf{E}}_3$$

IV. **Remarks**:

1. $\mathbf{w}$ is the *axial vector* of W: $\mathbf{w} = \mathrm{ax}(\mathsf{W})$.
2. Any vector $\mathbf{w}$ gives a skew-symmetric tensor: $\mathsf{W} = -\epsilon_{ijk} w_i \hat{\mathbf{E}}_j \otimes \hat{\mathbf{E}}_k =: \mathrm{asym}(\mathbf{w})$.

Example


The 5th Wave
By Rich Tennant
©RICHTENNANT
Don't worry, I'm an expert in vectors and tensor algebra.
I'd still rather have a magician doing this.