Lecture 10

Rigid body kinematics: Kinematics in a rotating CS; Velocity analysis.

1-7 September, 2021

Relative time derivative

I. Vector $\mathbf{u}(t)$ is observed in two CS :

Primary $\{\mathscr{E}_0, O, \hat{\mathbf{E}}_i\}$; *Secondary* $\{\mathscr{E}(t), G, \hat{\mathbf{e}}_i(t)\}$

1. Secondary rotates *w.r.t.* primary: $\boldsymbol{\omega}_{\mathscr{E}/\mathscr{E}_0}$
2. $\mathscr{E}_0$ observer measures rate $\dot{\mathbf{u}}(t)$.
3. $\mathscr{E}(t)$ observer measures rate $\mathring{\mathbf{u}}(t)$.

Then, $$\dot{\mathbf{u}}(t) = \mathring{\mathbf{u}}(t) + \boldsymbol{\omega}_{\mathscr{E}/\mathscr{E}_0} \times \mathbf{u}(t) .$$

II. **Remark 1**: $\mathscr{E}_0$ could also be rotating. Then $\dot{\mathbf{u}}(t)$ is the rate of change of $\mathbf{u}(t)$ *w.r.t.* rotating observer in $\mathscr{E}_0$.

1. Then $\dot{\mathbf{u}}(t)$ will be *different* from rate of change of $\mathbf{u}(t)$ *w.r.t.* to non-rotating CS.

III. **Remark 2**: If $\mathscr{E}(t)$ does not rotate *w.r.t.* $\mathscr{E}_0$, then $\boldsymbol{\omega}_{\mathscr{E}/\mathscr{E}_0} = 0$ and $\dot{\mathbf{u}}(t) = \mathring{\mathbf{u}}(t)$.

Angular acceleration

$\{\mathscr{E}(t), G, \hat{\mathbf{e}}_i(t)\}$ rotates *w.r.t.* $\{\mathscr{E}_0, O, \hat{\mathbf{E}}_i\}$ at $\boldsymbol{\omega}_{\mathscr{E}/\mathscr{E}_0}(t)$

$$\text{Thus, } \dot{\mathbf{u}}(t) = \mathring{\mathbf{u}}(t) + \boldsymbol{\omega}_{\mathscr{E}/\mathscr{E}_0} \times \mathbf{u}(t)$$

I. **Angular acceleration** of $\mathscr{E}$ *w.r.t.* $\mathscr{E}_0$ is

$$\boldsymbol{\alpha}_{\mathscr{E}/\mathscr{E}_0} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{\omega}_{\mathscr{E}/\mathscr{E}_0}\right) =: \dot{\boldsymbol{\omega}}_{\mathscr{E}/\mathscr{E}_0},$$

with time differentiation *w.r.t.* $\mathscr{E}_0$.

II. Set $\mathbf{u}(t) = \boldsymbol{\omega}_{\mathscr{E}/\mathscr{E}_0}$ to get $\dot{\boldsymbol{\omega}}_{\mathcal{E}/\mathcal{E}_0} = \mathring{\boldsymbol{\omega}}_{\mathcal{E}/\mathcal{E}_0}$

III. **Application**. Rigid body $\mathscr{B}$ with BFCS $\{\mathscr{E}(t), G, \hat{\mathbf{e}}_i(t)\}$ is rotating with angular velocity $\boldsymbol{\omega}_{\mathscr{E}/\mathscr{E}_0}$ *w.r.t.* $\{\mathscr{E}_0, O, \hat{\mathbf{E}}_i\}$.

1. Angular acceleration of $\mathscr{B}$ measured by observers in $\mathscr{E}$ and $\mathscr{E}_0$ <u>is the *same*</u> !
2. $[\boldsymbol{\alpha}_{\mathscr{E}/\mathscr{E}_0}(t)]_{\mathscr{E}} = \left[\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\omega}_{\mathscr{E}/\mathscr{E}_0}\right]_{\mathscr{E}} = \frac{\mathrm{d}}{\mathrm{d}t}[\boldsymbol{\omega}_{\mathscr{E}/\mathscr{E}_0}]_{\mathscr{E}}$

Velocity analysis

Relating velocity of a point P in two CS.

1. Velocity of P is found to be $\mathbf{v}^P_{\mathscr{E}}\ (=: \mathbf{v}^P_{rel})$ by an observer rotating with rigid body $\mathscr{B}$ with BFCS $\{\mathscr{E}(t), G, \hat{\mathbf{e}}_i(t)\}$.

2. $\mathscr{B}$ rotates at $\boldsymbol{\omega}_{\mathscr{B}} := \boldsymbol{\omega}_{\mathscr{E}/\mathscr{E}_0}$ *w.r.t.* $\{\mathscr{E}_0, O, \hat{\mathbf{E}}_i\}$

Find velocity of P *w.r.t.* $\mathscr{E}_0$, *i.e.* $\mathbf{v}^P_{\mathscr{E}_0} =: \mathbf{v}^P$.

Case 1: $O \equiv G$, *i.e.* $\mathscr{E}_0$, $\mathscr{E}$ have *same* origin:

$$\mathbf{v}^P = \mathbf{v}^P_{rel} + \boldsymbol{\omega}_{\mathscr{B}} \times \mathbf{r}^{P/G}.$$

Case 2: $\mathscr{E}_0$ and $\mathscr{E}$ have *different* origins.

a. $\mathscr{E}$'s origin is at $\mathbf{r}^{G/O}(t)$ *w.r.t.* O, and

b. $\mathscr{E}$'s origin has velocity $\mathbf{v}^G_{\mathscr{E}_0} =: \mathbf{v}^G$; then

$$\mathbf{v}^P = \mathbf{v}^P_{rel} + \boldsymbol{\omega}_{\mathscr{B}} \times \mathbf{r}^{P/G} + \mathbf{v}^G.$$

Application

I. **Example 1**. *Relating velocities of two points on the same rigid body*. Let A and B be points on a rigid body $\mathscr{B}$, which has angular velocity $\boldsymbol{\omega}_{\mathscr{B}}$ *w.r.t.* $\{\mathscr{E}_0, O, \hat{\mathbf{E}}_i\}$. Let $\mathbf{v}^A$ be velocity of A in $\mathscr{E}_0$.

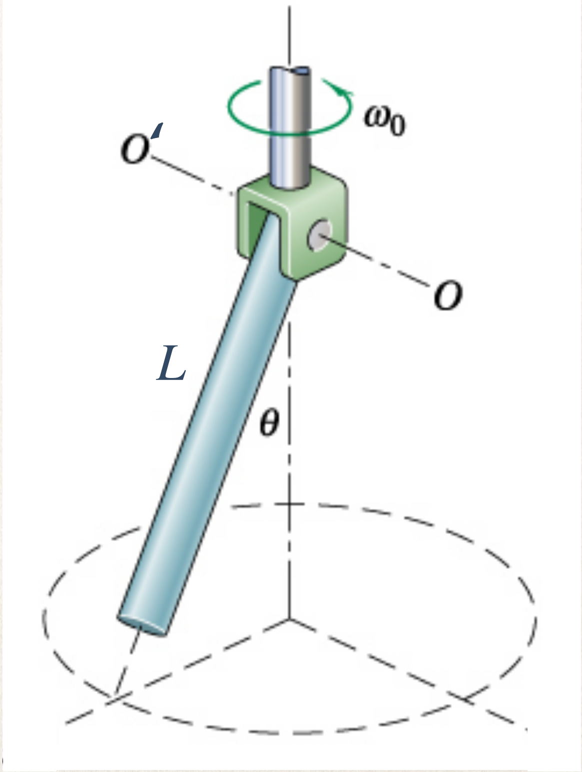
Find $\mathbf{v}^B$.

Answer: $$\mathbf{v}^B = \mathbf{v}^A + \boldsymbol{\omega}_{\mathscr{B}} \times \mathbf{r}^{B/A} .$$

II. **Example 2**. Find velocity of the rod's end.

Answer:

$\omega_0 L \sin\theta$ towards $\boldsymbol{O}$.



"On the other hand, if I die next week then this isn't a midlife crisis."

Don't panic, its only the mid-sem.

End-sem is still to come.

Good luck!