SOLUTION TO PROBLEM 4 OF TUTORIAL 2

Given:

$$\underline{\underline{D}} = 6\,(\widehat{e}_1 \otimes \widehat{e}_1) + \widehat{e}_1 \otimes \widehat{e}_2 + 9\,(\widehat{e}_1 \otimes \widehat{e}_3) + 2\,(\widehat{e}_2 \otimes \widehat{e}_2) + 2\,(\widehat{e}_2 \otimes \widehat{e}_3) + \widehat{e}_3 \otimes \widehat{e}_3\,.$$

We need to find the skew-symmetric (or anti-symmetric) part of the second-order tensor $\underline{\underline{D}}$ and its associated axial vector (or the dual vector).

We first note that a general second-order tensor can always be written as a sum of a symmetric and a skew-symmetric tensor in the following manner:

$$\underline{\underline{D}} = \underbrace{\frac{1}{2}\left(\underline{\underline{D}} + \underline{\underline{D}}^T\right)}_{\text{symmetric}} + \underbrace{\frac{1}{2}\left(\underline{\underline{D}} - \underline{\underline{D}}^T\right)}_{\text{skew-symmetric}}\quad.$$

Hence, the skew-symmetric part of the tensor $\underline{\underline{D}}$ is $\frac{1}{2}\left(\underline{\underline{D}} - \underline{\underline{D}}^T\right)$.

Now, we evaluate

$$\underline{\underline{D}}^T = 6\,(\widehat{e}_1 \otimes \widehat{e}_1) + \widehat{e}_2 \otimes \widehat{e}_1 + 2\,(\widehat{e}_2 \otimes \widehat{e}_2) + 9\,(\widehat{e}_3 \otimes \widehat{e}_1) + 2\,(\widehat{e}_3 \otimes \widehat{e}_2) + \widehat{e}_3 \otimes \widehat{e}_3\,.$$

In the above, we have used the fact that

$$(\underline{a} \otimes \underline{b})^T = \underline{b} \otimes \underline{a}\,.$$

Therefore, the skew-symmetric part is

$$\underline{\underline{SSD}} = \frac{1}{2}\left(\underline{\underline{D}} - \underline{\underline{D}}^T\right) = \frac{1}{2}\left[\widehat{e}_1 \otimes \widehat{e}_2 + 9\,(\widehat{e}_1 \otimes \widehat{e}_3) - \widehat{e}_2 \otimes \widehat{e}_1 + 2\,(\widehat{e}_2 \otimes \widehat{e}_3) - 9\,(\widehat{e}_3 \otimes \widehat{e}_1) - 2\,(\widehat{e}_3 \otimes \widehat{e}_2)\right]\,.$$

By definition, the axial vector associated with the skew-symmetric tensor $\underline{\underline{SSD}}$ is

$$\begin{aligned} ax(\underline{\underline{SSD}}) = -\frac{1}{2}\epsilon_{ijk} SSD_{jk}\widehat{e}_i &= -\frac{1}{4}\left[\epsilon_{123}\,2\,\widehat{e}_1 \;+\; \epsilon_{132}\,(-2)\,\widehat{e}_1 \;+\; \epsilon_{213}\,9\,\widehat{e}_2 \;+\; \epsilon_{231}\,(-9)\,\widehat{e}_2 \right.\\ &\qquad \left. +\,\epsilon_{312}\,\widehat{e}_3 \;+\; \epsilon_{321}\,(-1)\,\widehat{e}_3\right]\\ &= -\widehat{e}_1 \;+\; \frac{9}{2}\,\widehat{e}_2 \;-\; \frac{1}{2}\,\widehat{e}_3\,. \end{aligned}$$

Recall that $\epsilon_{123} = \epsilon_{312} = \epsilon_{231} = 1$ and $\epsilon_{132} = \epsilon_{213} = \epsilon_{321} = -1$.