ESO 209 - Tutorial 2

Problem 5

Find the principal values λ_i and real principal vectors $\hat{v}_i$ of the second order tensor

$$\underline{\underline{W}} = 2(\hat{e}_1 \otimes \hat{e}_2) - 2(\hat{e}_2 \otimes \hat{e}_1) - 4(\hat{e}_1 \otimes \hat{e}_3) + 4(\hat{e}_3 \otimes \hat{e}_1) - 4(\hat{e}_2 \otimes \hat{e}_3) + 4(\hat{e}_3 \otimes \hat{e}_2)$$

How many principal values are real? Also investigate if all the principal vectors are orthogonal to each other. Find the axial vector $\underline{w}$ of $\underline{\underline{W}}$. How does $\underline{w}$ relate to $\underline{\underline{W}}$'s principal vectors?

Solution

Given the coordinate system $\{\mathcal{S}, P, \hat{e}_i\}$ we write the tensor $\underline{\underline{W}}$ ~~as~~ in the above CS as

$$\underline{\underline{W}} = W_{ij} (\hat{e}_i \otimes \hat{e}_j)$$

~~So~~ The matrix of $\underline{\underline{W}}$ in $\mathcal{S}$, ie $[\underline{\underline{W}}]_{\mathcal{S}} = [W_{ij}]$

Thus from the given expression for $\underline{\underline{W}}$ we can write $[\underline{\underline{W}}]_{\mathcal{S}}$ as

$$[\underline{\underline{W}}]_{\mathcal{S}} = \begin{bmatrix} 0 & 2 & -4 \\ -2 & 0 & -4 \\ 4 & 4 & 0 \end{bmatrix}$$

To find the principal values, we use the relation,

$$\det\left\{[\underline{\underline{W}}]_\beta - \lambda[\underline{\underline{1}}]_\beta\right\} = 0$$

Substituting for $[\underline{\underline{W}}]_\beta$, we get

$$\det\left\{\begin{bmatrix} 0 & 2 & -4 \\ -2 & 0 & -4 \\ 4 & 4 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\right\} = 0$$

$$\det\left\{\begin{bmatrix} -\lambda & 2 & -4 \\ -2 & -\lambda & -4 \\ 4 & 4 & -\lambda \end{bmatrix}\right\} = 0$$

the expression of the determinant is

$$\lambda^3 + 36\lambda = 0$$

$$\Rightarrow \quad \lambda = 0,\ +6i,\ -6i$$

Thus we have one real and two complex principal values.

To find the real principal vector, we use

$$\left\{[\underline{\underline{W}}]_\beta - \lambda_i[\underline{\underline{1}}]_\beta\right\}[\hat{e}_i]_\beta = 0$$

for $\lambda = 0$ (real principal value) the expression reduces to

$$\begin{bmatrix} 0 & 2 & -4 \\ -2 & 0 & -4 \\ 4 & 4 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$$

$$\Rightarrow \quad v_2 = 2v_3, \quad v_1 = -2v_3, \quad v_1 = -v_2$$

we choose $v_3 = 1$, from which we get

$v_2 = 2$, $v_1 = -2$

$\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$ but since the norm of the principal vectors are taken as unity here,

ie $|\hat{v}| = 1$

we get $\hat{v} = \begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$

So this is the real principal vector corresponding to the principal value $\lambda = 0$

There are no other real principal vectors to check orthogonality in the field $\mathbb{R}$.

To find the axial vector $\underline{\omega}$

It is noted that $\underline{\underline{W}}$ is skew-symmetric with

$$[\underline{\underline{W}}]_p = \begin{bmatrix} 0 & 2 & -4 \\ -2 & 0 & -4 \\ 4 & 4 & 0 \end{bmatrix}$$

Using the definition derived in Lecture 3,

$$\underline{\omega} := -\frac{1}{2} \epsilon_{ijk} W_{jk} \hat{E}_i$$

$$= -W_{23}\hat{E}_1 + W_{13}\hat{E}_2 - W_{12}\hat{E}_3$$

In this case,

$$W_{23} = -4, \quad W_{13} = -4, \quad W_{12} = 2$$

$$\therefore \boxed{\underline{\omega} = 4\hat{E}_1 - 4\hat{E}_2 - 2\hat{E}_3}$$

The principal vector is

$$\hat{v}_1 = -\frac{2}{3}\hat{E}_1 + \frac{2}{3}\hat{E}_2 + \frac{1}{3}\hat{E}_3$$

We can see that

$$\underline{\omega} = -6\hat{v}_1$$

Thus, the axial vector is along the principal vector.