SOLUTION TO PROBLEM 3 OF TUTORIAL 6

Given, a disc mounted on a shaft of an inclined motor (rotating at a speed of $\omega_0 = 180$ rpm about its axis) which in turn is mounted on a turntable (rotating at a speed of $N = 60$ rpm about the ground fixed Z axis) as shown in Fig. 1. The angle between the turn-table rotational plane and the axis of the shaft has been specified to be fixed at $\gamma = 30^o$. Also $AC = 0.125$ m and $OC = 0.25$ m.

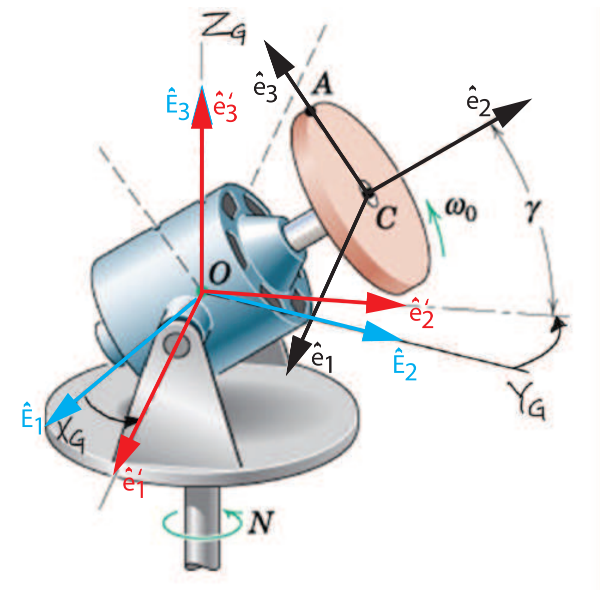

Figure 1: Various coordinate systems involved in the solution of problem 3 of tutorial 6.

1. We first need to find the absolute angular velocity and angular acceleration of the disc in the ground fixed frame at the instant when the motor frame $\mathcal{E}'$ coincides with the ground frame $\mathcal{E}_0$.
2. We further need to find the linear velocity and acceleration of the point A in the ground frame at the same instant.

In evaluating the solution to this problem, we will consider three reference frames, the ground fixed reference frame $\mathcal{E}_0$, the motor fixed reference frame $\mathcal{E}'$ and the disc fixed frame $\mathcal{E}$.
It has been stated in the problem that we are interested in the instant when the frames $\mathcal{E}_0$ and $\mathcal{E}'$ coincide. Accordingly, we have

$$\hat{E}_1 = \hat{e_1}', \quad \hat{E}_2 = \hat{e_2}', \quad \hat{E}_3 = \hat{e_3}'.$$

Also, for convenience, we choose the basis vector $\hat{e_1}$ of the disc fixed frame to be parallel to the basis vector $\hat{e_1}'$ of the motor fixed frame so that $\hat{e_1} = \hat{e_1}' = \hat{E}_1$ and the transformation from the reference frame $\mathcal{E}'$ to $\mathcal{E}$ is a simple two-dimensional rotation by an angle γ about the $\hat{e_1}'$ axis. Accordingly, one can write

$$\hat{e_2} = \cos\gamma \hat{e_2}' + \sin\gamma \hat{e_3}' = \cos\gamma \hat{E}_2 + \sin\gamma \hat{E}_3 \,,$$

$$\hat{e_3} = -\sin\gamma \hat{e_2}' + \cos\gamma \hat{e_3}' = -\sin\gamma \hat{E}_2 + \cos\gamma \hat{E}_3 \,.$$

1. Now, we know that
$$\underline{\omega}_{\mathcal{E}'/\mathcal{E}_0} = N\hat{E}_3 \,,$$
and
$$\underline{\omega}_{\mathcal{E}/\mathcal{E}'} = \omega_0 \hat{e}_2 \,.$$
Accordingly, the absolute angular velocity of the disc or its attached frame $\mathcal{E}$ is
$$\underline{\omega}_{\mathcal{E}/\mathcal{E}_0} = \underline{\omega}_{\mathcal{E}/\mathcal{E}'} + \underline{\omega}_{\mathcal{E}'/\mathcal{E}_0} = \omega_0 \hat{e}_2 + N\hat{E}_3$$
$$= \omega_0 \cos\gamma \hat{E}_2 + (\omega_0 \sin\gamma + N)\,\hat{E}_3 \,.$$
The absolute angular acceleration of the disc can be found using
$$\underline{\alpha}_{\mathcal{E}/\mathcal{E}_0} = \underline{\alpha}_{\mathcal{E}/\mathcal{E}'} + \underline{\alpha}_{\mathcal{E}'/\mathcal{E}_0} \,. \tag{1}$$
Also, we know that
$$\underline{\alpha}_{\mathcal{E}/\mathcal{E}'}\Big|_{\mathcal{E}_0} = \underline{\alpha}_{\mathcal{E}/\mathcal{E}'}\Big|_{\mathcal{E}'} + \underline{\omega}_{\mathcal{E}'/\mathcal{E}_0} \times \underline{\omega}_{\mathcal{E}/\mathcal{E}'} \,.$$
Since both the turn table and the shaft of the motor are rotating at a steady speed, we have
$$\underline{\alpha}_{\mathcal{E}/\mathcal{E}'}\Big|_{\mathcal{E}'} = 0 \,,$$
$$\underline{\alpha}_{\mathcal{E}'/\mathcal{E}_0} = 0 \,.$$
Substituting the desired values in Eq. (1), we get
$$\underline{\alpha}_{\mathcal{E}/\mathcal{E}_0}\Big|_{\mathcal{E}_0} = 0 + 0 + N\hat{E}_3 \times \omega_0 \hat{e}_2 = N\,\omega_0\,\hat{E}_3 \times \left(\cos\gamma \hat{E}_2 + \sin\gamma \hat{E}_3\right) = -N\,\omega_0\,\cos\gamma\,\hat{E}_1 \,.$$
Now, using the numerical values of $N = 60$ rpm which is equivalent to $N = 2\pi$ rad/sec and $\omega_0 = 180$ rpm which translates to $\omega_0 = 6\pi$ rad/sec and the value of $\gamma = 30^o$, we have the absolute angular velocity and angular acceleration of the disc as
$$\underline{\omega}_{\text{disc}} = 3\sqrt{3}\,\pi\hat{E}_2 + 4\,\pi\hat{E}_3 \quad \text{rad/sec}$$
and
$$\underline{\alpha}_{\text{disc}} = -6\,\sqrt{3}\,\pi^2\,\hat{E}_1 \quad \text{rad/sec}^2 \,.$$

2. We now shift our attention to the linear velocity of the point A. It becomes systematic and simple if we start with the disc fixed reference frame $\mathcal{E}$ whose origin is at the point C so that the velocity of A can be written as
$$\underline{v}^A = \underline{v}^C + \underline{v}^A_{rel} + \underline{\omega}_{\mathcal{E}/\mathcal{E}_0} \times \underline{r}^{A/C} \,,$$
where $\underline{v}^A_{rel}$ is the velocity of A relative to C as measured in the disc fixed frame and $\underline{v}^C$ is the linear velocity of the point C. Since, both A and C are points on the same rigid body, there is no relative motion between them. Hence, $\underline{v}^A_{rel} = 0$.

   Now, $\underline{r}^{A/C} = AC\hat{e}_3$. Accordingly, we have
$$\underline{v}^A = \underline{v}^C + \left(\omega_0 \hat{e}_2 + N\hat{E}_3\right) \times AC\hat{e}_3 = \underline{v}^C + (\omega_0\,AC + N\,AC\,\sin\gamma)\,\hat{E}_1 \,.$$

In order to evaluate $\underline{v}^C$, we now consider the motor fixed frame $\mathcal{E}'$ with its origin at the stationary point O. Hence, we have the relation

$$\underline{v}^C = \underline{v}^C_{rel} + \underline{\omega}_{\mathcal{E}'/\mathcal{E}_0} \times \underline{r}^{C/O}\,.$$

Again, O and C are points on the same rigid body and hence, $\underline{v}^C_{rel} = 0$. Also, $\underline{r}^{C/O} = OC\hat{e_2}$. With that, the velocity of the point C can be evaluated as

$$\underline{v}^C = N\hat{E}_3 \times OC\hat{e}_2 \;=\; -N\,OC\,\cos\gamma\hat{E}_1\,.$$

Substituting this value of $\underline{v}^C$ in the expression for $\underline{v}^A$, we have

$$\underline{v}^A = (\omega_0\,AC + N\,AC\,\sin\gamma - N\,OC\,\cos\gamma)\,\hat{E}_1\,.$$

Using the numerical values specified in the problem, we get

$$\underline{v}^A = \left(0.875 - 0.25\sqrt{3}\right)\pi\hat{E}_1 \;= 1.3885\,\hat{E}_1 \quad \text{m/sec}\,.$$

We finally evaluate the linear acceleration of the point A. Again, we start with the disc fixed reference frame $\mathcal{E}$ whose origin is at the point C so that the acceleration of A can be written as

$$\underline{a}^A = \underline{a}^C + \underline{a}^A_{rel} + 2\,\underline{\omega}_{\mathcal{E}/\mathcal{E}_0} \times \underline{v}^A_{rel} + \underline{\alpha}_{\mathcal{E}/\mathcal{E}_0} \times \underline{r}^{A/C} + \underline{\omega}_{\mathcal{E}/\mathcal{E}_0} \times \left(\underline{\omega}_{\mathcal{E}/\mathcal{E}_0} \times \underline{r}^{A/C}\right)\,,$$

where $\underline{a}^A_{rel}$ is the acceleration of A relative to C as measured in the disc fixed frame which is zero since both A and C are points on the same rigid body which is attached to the rotating frame $\mathcal{E}$.

Using $\underline{a}^A_{rel} = 0$ and $\underline{v}^A_{rel} = 0$ and substituting for the angular acceleration $\underline{\alpha}$ and angular velocity $\underline{\omega}$ with $\underline{r}^{A/C} = AC\hat{e}_3$, we get

$$\begin{aligned}
\underline{a}^A &= \underline{a}^C - N\,\omega_0\,\cos\gamma\,\hat{E}_1 \times AC\hat{e}_3 + \left(N\hat{E}_3 + \omega_0\hat{e}_2\right) \times \left(\left(N\hat{E}_3 + \omega_0\hat{e}_2\right) \times AC\hat{e}_3\right)\\
&= \underline{a}^C + N\,\omega_0\,AC\,\cos\gamma\left(\sin\gamma\hat{E}_3 + \cos\gamma\hat{E}_2\right) + \left(N\hat{E}_3 + \omega_0\hat{e}_2\right) \times (N\,AC\,\sin\gamma + \omega_0\,AC)\,\hat{E}_1\\
&= \underline{a}^C + N\,\omega_0\,AC\,\cos\gamma\left(\sin\gamma\hat{E}_3 + \cos\gamma\hat{E}_2\right) + N\,AC\,(N\,\sin\gamma + \omega_0)\,\hat{E}_2\\
&\qquad + \omega_0\,AC\,(N\,\sin\gamma + \omega_0)\left(-\cos\gamma\hat{E}_3 + \sin\gamma\hat{E}_2\right)\\
&= \underline{a}^C + \left[N\,\omega_0\,AC\,\cos^2\gamma + N\,AC\,(N\,\sin\gamma + \omega_0) + \omega_0\,AC\,(N\,\sin\gamma + \omega_0)\sin\gamma\right]\hat{E}_2 - \omega_0^2\,AC\cos\gamma\hat{E}_3\,.
\end{aligned}$$

Finally, we return to the motor fixed frame $\mathcal{E}'$ to evaluate $\underline{a}^C$ as

$$\underline{a}^C = \underline{a}^O + \underline{a}^C_{rel} + 2\,\underline{\omega}_{\mathcal{E}'/\mathcal{E}_0} \times \underline{v}^C_{rel} + \underline{\alpha}_{\mathcal{E}'/\mathcal{E}_0} \times \underline{r}^{C/O} + \underline{\omega}_{\mathcal{E}'/\mathcal{E}_0} \times \left(\underline{\omega}_{\mathcal{E}'/\mathcal{E}_0} \times \underline{r}^{C/O}\right)\,.$$

We note that $\underline{a}^O = \underline{a}^C_{rel} = \underline{v}^C_{rel} = \underline{\alpha}_{\mathcal{E}'/\mathcal{E}_0} = 0$. Hence, the acceleration of the point C becomes

$$\underline{a}^C = N\hat{E}_3 \times \left(N\hat{E}_3 \times OC\hat{e}_2\right) = -N\hat{E}_3 \times N\,OC\,\cos\gamma\,\hat{E}_1 = -N^2\,OC\,\cos\gamma\hat{E}_2\,.$$

The final expression for the acceleration of the point A is

$$\begin{aligned}
\underline{a}^A = &\left[-N^2\,OC\,\cos\gamma \;+\; N\,\omega_0\,AC\,\cos^2\gamma + N\,AC\,(N\,\sin\gamma + \omega_0) + \omega_0\,AC\,(N\,\sin\gamma + \omega_0)\sin\gamma\right]\hat{E}_2\\
&\qquad - \omega_0^2\,AC\cos\gamma\hat{E}_3\,.
\end{aligned}$$

Using the numerical values given in the problem, we get

$$\underline{a}^A = \left(5.5 - 0.5\sqrt{3}\right)\pi^2\,\hat{E}_2 - 2.25\sqrt{3}\pi^2\hat{E}_3 \;= 45.7355\hat{E}_2 - 38.4630\hat{E}_3 \quad m/sec^2\,.$$