Lecture 4

Inverse; Determinant; Orthogonal tensor; Rotation tensor

11-18 August, 2021

Review of L3

I. **Principal values** and **principal vectors**:

$$\mathsf{A} \cdot \hat{\mathbf{v}} = \lambda \hat{\mathbf{v}}, \qquad \left|\hat{\mathbf{v}}\right| = 1$$

1. Find $\{\lambda, [\hat{\mathbf{v}}]_{\mathscr{E}}\}$ for $[\mathsf{A}]_{\mathscr{E}}$ in *any* $\left\{\mathscr{E}, O, \hat{\mathbf{e}}_i\right\}$
2. A is *diagonal* in *principal* CS $\left\{\mathscr{P}, C, \hat{\mathbf{v}}_i\right\}$.

II. **Symmetric** tensor: $\mathsf{S} = \mathsf{S}^T$.

1. S_i are real and $\hat{\mathbf{v}}_i$ are *orthogonal*.
2. *Principal CS* $\left\{\mathscr{P}, C, \hat{\mathbf{v}}_i\right\}$ is *Cartesian*.
3. $\mathsf{S} = \sum_{i=1}^{3} S_i \hat{\mathbf{v}}_i \otimes \hat{\mathbf{v}}_i$ and $[\mathsf{S}]_{\mathscr{P}} = \begin{bmatrix} S_1 & 0 & 0 \\ 0 & S_2 & 0 \\ 0 & 0 & S_3 \end{bmatrix}$

S is *always diagonalizable; stretch* by S_i along $\hat{\mathbf{v}}_i$

III. **Skew-symmetric** tensor: $\mathsf{W} = -\mathsf{W}^T$

1. *Axial vector*: $\mathrm{ax}(\mathsf{W}) =: \mathbf{w} = -\frac{1}{2}\epsilon_{ijk} W_{jk} \hat{\mathbf{E}}_i$
2. $\mathsf{W} \cdot \mathbf{a} = \mathbf{w} \times \mathbf{a}$ for all $\mathbf{a}$.
3. $\mathrm{asym}(\mathbf{w}) =: \mathsf{W} = -\epsilon_{ijk} w_i \left(\hat{\mathbf{E}}_j \otimes \hat{\mathbf{E}}_k\right)$

Inverse, Determinant

I. **Definition**: A^{-1} is the *inverse* of A if
$\mathsf{A} \cdot \mathsf{A}^{-1} = \mathsf{A}^{-1} \cdot \mathsf{A} = \mathsf{1}$ (identity tensor)

1. *Independent* of CS.
2. To find A^{-1}, pick *any* convenient CS $\left\{\mathscr{P}, P, \hat{\mathbf{e}}_i\right\}$ and use $[\mathsf{A}^{-1}]_{\mathscr{P}} = \left([\mathsf{A}]_{\mathscr{P}}\right)^{-1}$.
3. A has *no* zero principal value $\Longrightarrow \mathsf{A}^{-1}$

II. **Definition**: *Determinant* of A is found by

$$\det(\mathsf{A}) = \det([\mathsf{A}]_{\mathscr{P}})$$

in *any* CS $\left\{\mathscr{P}, P, \hat{\mathbf{e}}_i\right\}$ of our choice.

1. $\det(\mathsf{A})$ is *independent* of CS.

III. **Properties**: λ_i are principal values of A.

1. λ_i^{-1} are principal values of A^{-1}.
2. $(\mathsf{A} \cdot \mathsf{B})^{-1} = \mathsf{B}^{-1} \cdot \mathsf{A}^{-1}$.
3. $(\mathsf{A}^T)^{-1} = (\mathsf{A}^{-1})^T = \mathsf{A}^{-T}$.
4. $\det(\mathsf{A}) = \Pi_{i=1}^{3} \lambda_i$.
5. $\det\left(\mathsf{A}^{-1}\right) = (\det \mathsf{A})^{-1}$.
6. $\det(\mathsf{A} \cdot \mathsf{B}) = (\det \mathsf{A})(\det \mathsf{B})$.

Orthogonal tensor

I. **Definition**: Q is *orthogonal* if for all $\mathbf{a}$

$$\left|\mathrm{Q}\cdot\mathbf{a}\right| = |\mathbf{a}|$$

Q preserves *length* of $\mathbf{a}$, but not direction.

II. **Properties**: λ_i are principal values of Q.

1. $\mathrm{Q}^{-1} = \mathrm{Q}^T$.
2. $|\lambda_i| = 1$, with $\lambda_3 = \pm 1$, $\lambda_{1,2} = a \pm i\,b$.
3. *One real* principal vector $\hat{\mathbf{e}}_3$ for λ_3.
4. *Relative orientation*: For all $\mathbf{a}$ and $\mathbf{b}$:
   i. $(\mathrm{Q}\cdot\mathbf{a})\cdot(\mathrm{Q}\cdot\mathbf{b}) = \mathbf{a}\cdot\mathbf{b}$.
   ii. Plane *normal* to $\hat{\mathbf{e}}_3$ is *invariant* under Q

   Q *either* rotates objects about $\hat{\mathbf{e}}_3$
   or reflects them about invariant plane.

   iii. $(\mathrm{Q}\cdot\mathbf{a})\times(\mathrm{Q}\cdot\mathbf{b}) = \lambda_3\mathrm{Q}\cdot(\mathbf{a}\times\mathbf{b})$.
5. $\det(\mathrm{Q}) = \lambda_3(a^2+b^2) = \lambda_3 = \pm 1$.
   i. $\det(\mathrm{Q}) = -1 \Longrightarrow$ Q *inverts* volumes.

Rotation tensor

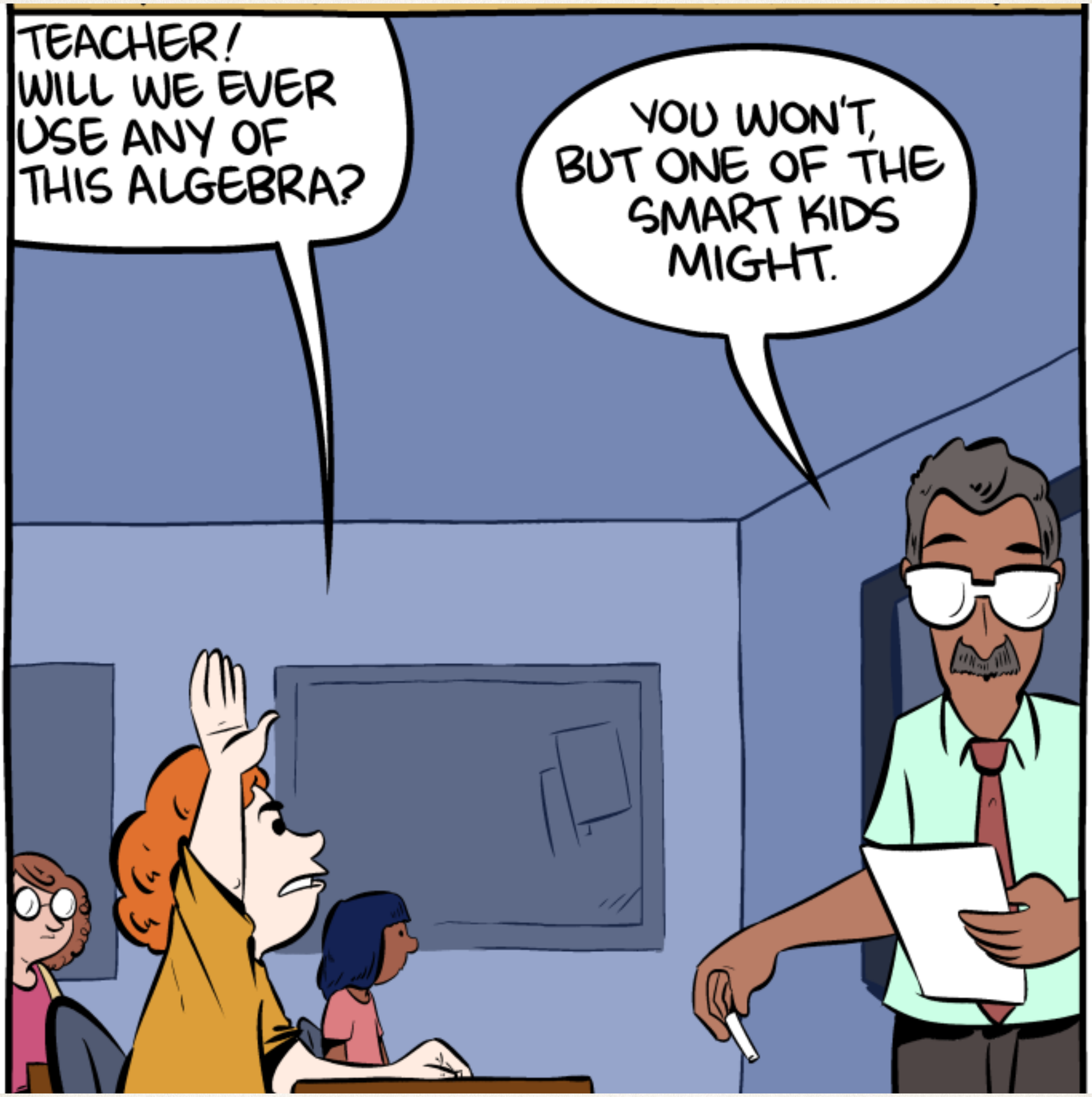
I. **Definition**: An *orthogonal* tensor R with $\det(\mathsf{R}) = +1$ is called a *rotation tensor*.

1. $\det(\mathsf{R}) = +1 \Leftrightarrow$ real principal value $+1$
2. $\det(\mathsf{R}) = +1$: no volume inversion.

II. **Properties**: λ_i are principal values of R.

1. $\mathsf{R}^{-1} = \mathsf{R}^T$, as R is orthogonal.
2. $(\mathsf{R}\cdot\mathbf{a})\cdot(\mathsf{R}\cdot\mathbf{b}) = \mathbf{a}\cdot\mathbf{b}$ for all **a** and **b**.
3. $|\lambda_i| = 1$, with $\lambda_3 = +1$, $\lambda_{1,2} = a \pm i\,b$.
   i. *Rotation angle* $= \arg(\lambda_2) = \arctan(b/a)$
4. *One real* principal vector $\hat{\mathbf{e}}_3$ for $\lambda_3 = +1$:
   i. Plane *normal* to $\hat{\mathbf{e}}_3$ is *invariant* under R
   ii. *Axis of rotation* is along $\hat{\mathbf{e}}_3$.
5. $(\mathsf{R}\cdot\mathbf{a})\times(\mathsf{R}\cdot\mathbf{b}) = +\mathsf{R}\cdot(\mathbf{a}\times\mathbf{b})$ for all **a**, **b**

III. Example



You are <u>all</u> smart.

You need this (tensor) algebra!