Lecture 8

Rigid body kinematics: Rate of change of orientation in 2*D* and 3*D*; Angular velocity.

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25 - 31 August, 2021

2D versus 3D

I. Rotational motion of a rigid body in 2D?

1. In 2D, *material points* move in *parallel planes* $\mathscr{P}$. (Let $\mathscr{P}$ be normal to $\hat{\mathbf{e}}_3$.)
2. *Orientation*: Angle $\theta(t)$ of a *material line* in $\mathscr{P}$ from a reference axis in $\mathscr{P}$.
3. *Angular velocity*: $\omega\hat{\mathbf{e}}_3 = \mathrm{d}\theta/\mathrm{d}t$.
   i. Same for *any* material line, ref. axis.

II. Does this approach work in 3D? NO!

1. Material line and reference axis do *not* meet at all times $\Longrightarrow \theta(t)$ *not* defined.
2. *Different* material lines and reference axis may give *different* orientations.

III. Why not use rotation tensor $\mathsf{R}(t)$? HARD!

1. Rotation tensors can*not* be added.
2. Rotation tensors do *not* commute.
3. Rate of change $\mathrm{d}\mathsf{R}(t)/\mathrm{d}t$ is a tensor!

Angular velocity

I. **Aim**: Consider a rigid body in motion.
Find its *rate of change* of *orientation.*

II. Rigid body $\Longleftrightarrow$ BFCS (for orientation)

1. BFCS *changes* with time: $\{\mathscr{E}(t), G, \hat{\mathbf{e}}_i(t)\}$.
2. Need to compute $\dot{\hat{\mathbf{e}}}_i(t) := \mathrm{d}\hat{\mathbf{e}}_i(t)\,/\,\mathrm{d}t$.
3. $\dot{\hat{\mathbf{e}}}_i(t) = \lim_{\delta t \to 0} \left\{\hat{\mathbf{e}}_i(t+\delta t) - \hat{\mathbf{e}}_i(t)\right\} \,/\, \delta t$
4. BFCS at $t + \delta t$: $\{\mathscr{E}(t+\delta t), G, \hat{\mathbf{e}}_i(t+\delta t)\}$.
5. Can find rotation tensor $\mathsf{R}\left(\hat{\mathbf{n}}(t), \delta\varphi(t)\right)$:
$$\hat{\mathbf{e}}_i(t+\delta t) = \mathsf{R}\left(\hat{\mathbf{n}}(t), \delta\varphi(t)\right) \cdot \hat{\mathbf{e}}_i(t),$$
where, with $\mathsf{N} = \mathrm{asym}(\hat{\mathbf{n}})$,
$$\mathsf{R}\left(\hat{\mathbf{n}}, \delta\varphi\right) = \mathsf{1} + \sin\delta\varphi\,\mathsf{N} + (1 - \cos\delta\varphi)\,\mathsf{N}^2$$
6. We find $\dot{\hat{\mathbf{e}}}_i(t) = \dot{\varphi}\,\hat{\mathbf{n}}(t) \times \hat{\mathbf{e}}(t)$.

III. Define <u>*angular velocity*</u> $\boldsymbol{\omega}(t) = \dot{\varphi}\hat{\mathbf{n}}(t)$.

1. As in 2D, <u>but</u> rotation axis changes !
2. $\dot{\hat{\mathbf{e}}}_i(t) = \boldsymbol{\omega}(t) \times \hat{\mathbf{e}}(t)$.

Angular velocity

Rigid body's **3D** *angular velocity*: $\boldsymbol{\omega}(t) = \dot{\varphi}\hat{\mathbf{n}}(t)$

I. **Question**: How to compute $\dot{\varphi}$ and $\hat{\mathbf{n}}$?

II. Compute $\dot{\hat{\mathbf{e}}}_i(t)$ for BFCS $\{\mathscr{E}(t), G, \hat{\mathbf{e}}_i(t)\}$

1. Time-varying BFCS: $\mathscr{E}(t) \xrightarrow{\mathsf{R}(t)} \mathscr{E}(t+\delta t)$.
2. Found: $\dot{\hat{\mathbf{e}}}_i(t) = \lim_{\delta t \to 0} \dfrac{1}{\delta t}\left\{ \left(\mathsf{R}(t) - 1\right) \cdot \hat{\mathbf{e}}_i(t) \right\}$.

III. Let motion be <u>*observed*</u> from $\{\mathscr{E}_0, O, \hat{\mathbf{E}}_i\}$:

IV. Get $\mathsf{R}_0(t)$: $\mathscr{E}_0 \xrightarrow{\mathsf{R}_0(t)} \mathscr{E}(t)$; $\mathscr{E}_0 \xrightarrow{\mathsf{R}_0(t+\delta t)} \mathscr{E}(t+\delta t)$

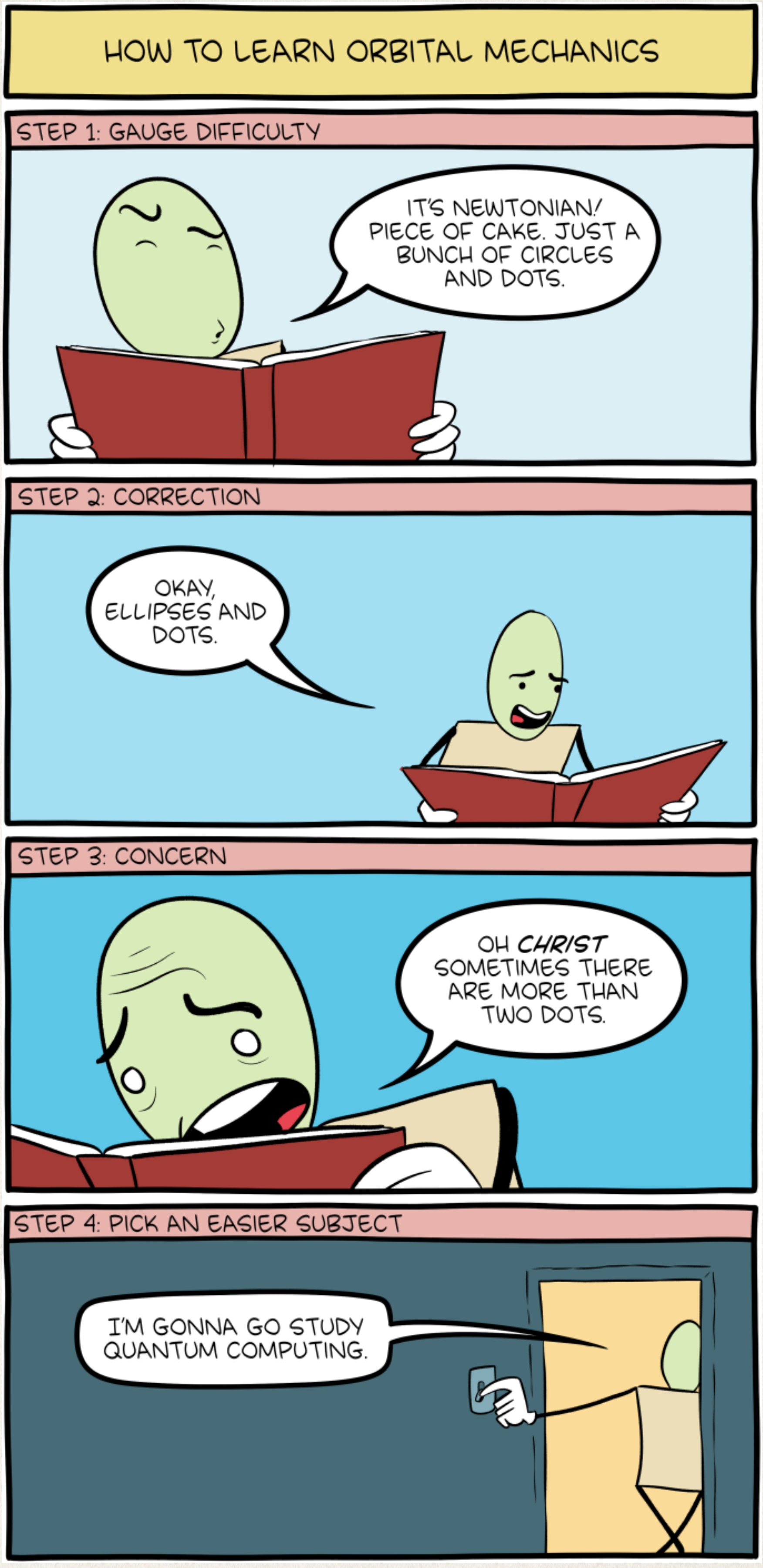
Then, $\hat{\mathbf{e}}_i(t) = \mathsf{R}_0\left(\hat{\mathbf{n}}_0(t), \varphi_0(t)\right) \cdot \hat{\mathbf{E}}_i$,

$\hat{\mathbf{e}}_i(t+\delta t) = \mathsf{R}_0\left(\hat{\mathbf{n}}_0(t+\delta t), \varphi_0(t+\delta t)\right) \cdot \hat{\mathbf{E}}_i$

V. Find: $\mathsf{R}(t) - 1 = \dot{\mathsf{R}}_0(t) \cdot \mathsf{R}_0^T(t) =: \boldsymbol{\Omega}(t)$

1. <u>*Angular velocity tensor*</u>: $\boldsymbol{\Omega}(t) = - \boldsymbol{\Omega}^T(t)$
2. Define <u>*angular velocity*</u> $\boldsymbol{\omega}(t) = \mathrm{ax}(\ \boldsymbol{\Omega}(t)\ )$
3. Can show that this $\boldsymbol{\omega}(t) = \dot{\varphi}\hat{\mathbf{n}}(t)$.

VI. **Ans.** Get $\dot{\varphi}$ and $\hat{\mathbf{n}}$ from $\boldsymbol{\omega}(t) = \mathrm{ax}\left(\dot{\mathsf{R}}_0 \cdot \mathsf{R}_0^T\right)$


HOW TO LEARN ORBITAL MECHANICS
STEP 1: GAUGE DIFFICULTY
IT'S NEWTONIAN! PIECE OF CAKE. JUST A BUNCH OF CIRCLES AND DOTS.
STEP 2: CORRECTION
OKAY, ELLIPSES AND DOTS.
STEP 3: CONCERN
OH *CHRIST* SOMETIMES THERE ARE MORE THAN TWO DOTS.
STEP 4: PICK AN EASIER SUBJECT
I'M GONNA GO STUDY QUANTUM COMPUTING.