Lecture 2

Cartesian tensors

Review of L1

I. **Vectors**

1. independent of coordinate system.

II. **Cartesian coordinate sytem** (CCS)

1. Notation: $\left\{\mathscr{P}, P, \hat{\mathbf{e}}_i\right\}$.
2. Fixed or changing unit vectors.

III. **Index notation**

1. Free indices
2. Dummy indices
3. Summation convention:
   $a_i b_i = a_j b_j = a_1 b_1 + a_2 b_2 + a_3 b_3$
4. Krönecker delta $\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$
5. Contraction: $P_{\ldots i \ldots j \ldots} \delta_{ij} = P_{\ldots i \ldots i \ldots}$
6. Alternating tensor

$$\epsilon_{ijk} = \begin{cases} +1, & \text{even permutation of } ijk \\ -1, & \text{odd permutation of } ijk \\ 0, & \text{otherwise}\,. \end{cases}$$

Tensors

I. **Definition**: A *second-order tensor* is a linear transformation that operates on a vector to produce another vector.

II. **Notation**:

1. A when typing; $\underline{\underline{A}}$ when writing.
2. $\mathsf{A} \cdot \mathbf{a} = \mathbf{b}$ OR $\mathbf{a} \cdot \mathsf{A} = \mathbf{b}$.
   "The tensor A *left*/*right* operates ($\cdot$) on the vector $\mathbf{a}$ to produce the vector $\mathbf{b}$."

III. Understood "*physically*" by its action on all vectors *or* on basis. Examples.

IV. Some simple properties

1. **Addition**: Let $\mathsf{C} = (\mathsf{A} + \mathsf{B})$. Then,
   $\mathsf{C} \cdot \mathbf{a} = (\mathsf{A} + \mathsf{B}) \cdot \mathbf{a} = \mathsf{A} \cdot \mathbf{a} + \mathsf{B} \cdot \mathbf{a}$
2. **Multiplication**: Let $\mathsf{D} = \mathsf{A} \cdot \mathsf{B}$. Then,
   $\mathsf{D} \cdot \mathbf{a} = (\mathsf{A} \cdot \mathsf{B}) \cdot \mathbf{a} = \mathsf{A} \cdot (\mathsf{B} \cdot \mathbf{a})$

V. "*Overloaded*" operator: $\mathbf{b} \cdot \left\{ (\mathsf{A} \cdot \mathsf{B}) \cdot \mathbf{a} \right\}$

Tensor product

I. Extend ideas of *unit vector, basis* to tensors.

II. **Definition**: A *tensor product* of two vectors $\mathbf{a}$ and $\mathbf{b}$ is given by $\mathbf{a} \otimes \mathbf{b}$ with properties:

1. $(\mathbf{a} \otimes \mathbf{b}) \cdot \mathbf{c} = (\mathbf{c} \cdot \mathbf{b})\mathbf{a}$
2. $\mathbf{c} \cdot (\mathbf{a} \otimes \mathbf{b}) = (\mathbf{c} \cdot \mathbf{a})\mathbf{b}$

III. **Remarks**

1. Other names: *outer* or *dyadic* product.
2. *Example*: $\mathbf{a} \otimes \mathbf{b} = (a_i b_j)\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j$
3. *Property*: $(\mathbf{a} \otimes \mathbf{b}) \cdot (\mathbf{c} \otimes \mathbf{d}) = (\mathbf{b} \cdot \mathbf{c})\mathbf{a} \otimes \mathbf{d}$

IV. **Definition**: *Tensorial basis* is $\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j$.

1. There are 9 *unit tensors*. Form *basis*.

V. **Application**: Consider a CS $\{\mathscr{P}, P, \hat{\mathbf{e}}_i\}$.

1. Let's write $\mathsf{A} = A_{ij}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j$.
2. $A_{ij} = \hat{\mathbf{e}}_i \cdot \mathsf{A} \cdot \hat{\mathbf{e}}_j = \hat{\mathbf{e}}_i \cdot (\mathsf{A} \cdot \hat{\mathbf{e}}_j) = (\hat{\mathbf{e}}_i \cdot \mathsf{A}) \cdot \hat{\mathbf{e}}_j$
3. A_{ij} are <u>components of A</u> in $\{\mathscr{P}, P, \hat{\mathbf{e}}_i\}$.

More on tensors

I. **Components**: Consider two coordinate systems $\left\{\mathscr{P}, P, \hat{\mathbf{e}}_i\right\}$ and $\left\{\mathscr{S}, S, \hat{\mathbf{e}}_i'\right\}$.

1. In $\mathscr{P}$: $\mathsf{A} = A_{ij}\,\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j\,, A_{ij} = \hat{\mathbf{e}}_i \cdot \mathsf{A} \cdot \hat{\mathbf{e}}_j$
2. In $\mathscr{S}$: $\mathsf{A} = A_{ij}'\hat{\mathbf{e}}_i' \otimes \hat{\mathbf{e}}_j'\,, A_{ij}' = \hat{\mathbf{e}}_i' \cdot \mathsf{A} \cdot \mathbf{e}_j'.$
3. *Relation to matrix algebra*. Compile components into a matrix:

$$[\mathsf{A}]_{\mathscr{P}} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \text{ and } [\mathsf{A}]_{\mathscr{S}} = \begin{bmatrix} A_{11}' & A_{12}' & A_{13}' \\ A_{21}' & A_{22}' & A_{23}' \\ A_{31}' & A_{32}' & A_{33}' \end{bmatrix}$$

We call $[\mathsf{A}]_{\mathscr{P}}$ as the "matrix of A in $\mathscr{P}$".

4. In general $A_{ij} \neq A_{ij}'$, i.e.

Tensor remains the same, but its components change with choice of coordinate system.

II. Example: Two rotated CS.

Tensor algebra

I. **Operations**: Consider a CS $\{\mathscr{P}, P, \hat{\mathbf{e}}_i\}$.

1. $\mathbf{b} = \mathsf{A} \cdot \mathbf{a} = A_{ij} a_j \hat{\mathbf{e}}_i \implies b_i = A_{ij} a_j$
2. $\mathbf{c} = \mathbf{a} \cdot \mathsf{A} = a_i A_{ij} \hat{\mathbf{e}}_j \implies c_j = a_i A_{ij}$
3. $\mathsf{C} = \mathsf{A} \cdot \mathsf{B} = A_{in} B_{nj} \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \Rightarrow C_{ij} = A_{in} B_{nj}$

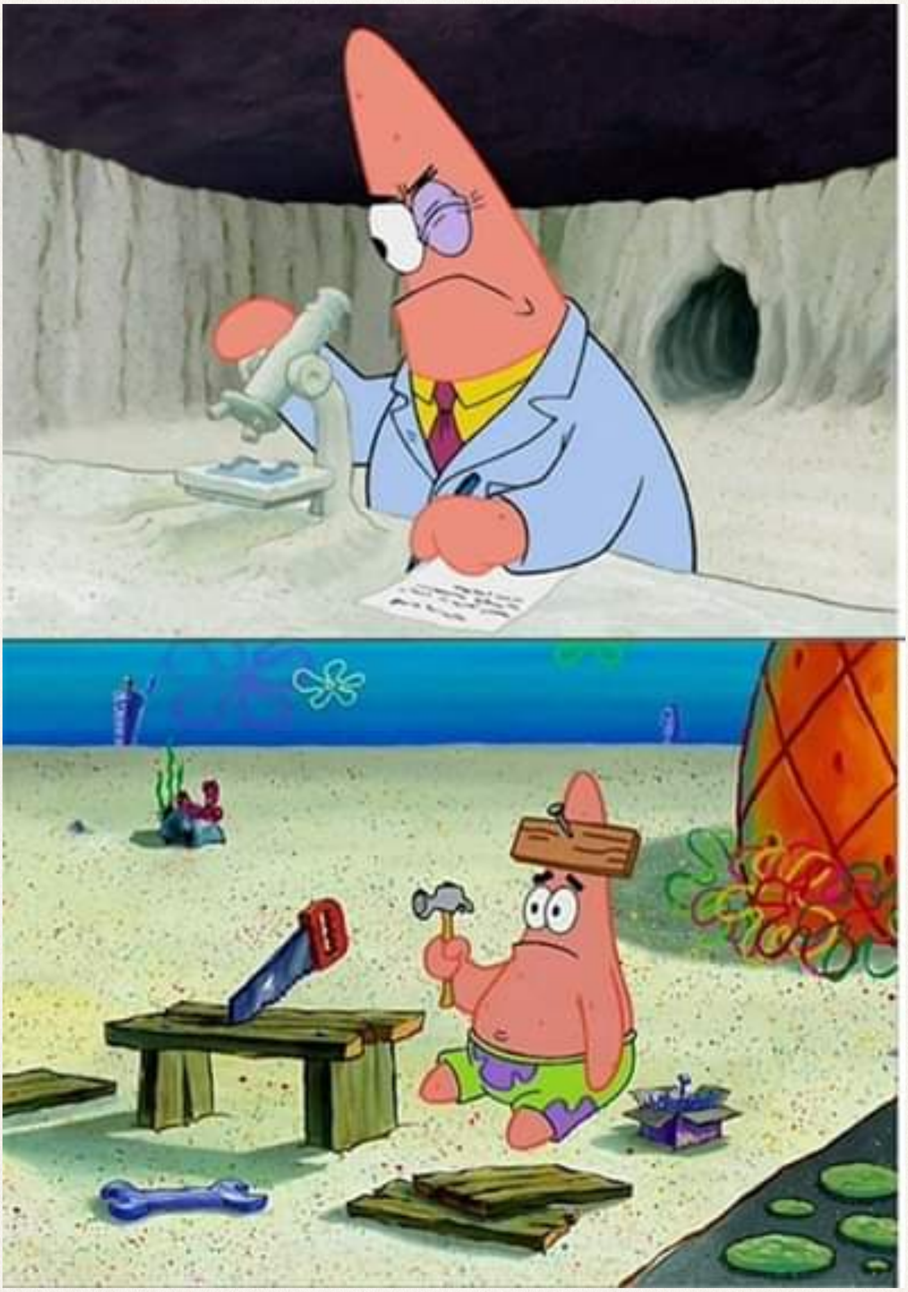
II. **Transpose**: Let $\mathsf{A} = A_{ij}\, \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j$, then

1. $\mathsf{A}^T = A_{ji}\, \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \implies \left(\mathsf{A}^T\right)_{ij} = A_{ji}$.
2. *Properties*
   i. $(\mathsf{A} + \mathsf{B})^T = \mathsf{A}^T + \mathsf{B}^T$
   ii. $(\mathsf{A} \cdot \mathsf{B})^T = \mathsf{B}^T \cdot \mathsf{A}^T$
   iii. $[\mathsf{A}^T] = [\mathsf{A}]^T$
   iv. $(\mathsf{A} \cdot \mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (\mathsf{A}^T \cdot \mathbf{b})$

III. **Trace**: Let $\mathsf{A} = A_{ij}\, \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j = A'_{ij}\, \hat{\mathbf{e}}'_i \otimes \hat{\mathbf{e}}'_j$.

1. $\mathrm{tr}\,(\mathsf{A}) = A_{ii} = A'_{ii} = \mathrm{tr}\,\left(\mathsf{A}^T\right)$.
2. <u>*Independent*</u> of choice of CS.

Mechanics so far …



Now?