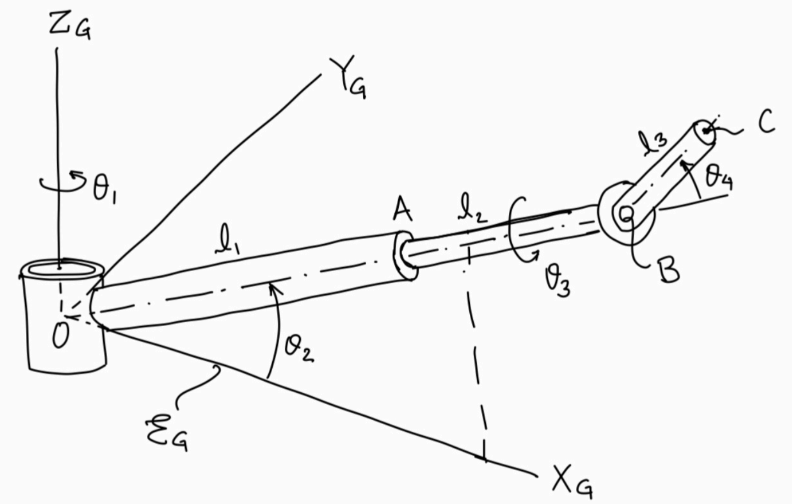
5. Axes of all three links of a robot are perfectly aligned to the ground X_G axis when $\theta_1 = \theta_2 = \theta_3 = \theta_4 = 0$. In this position the axis of the pin B is parallel to ground Y_G axis so that BC would be able to rotate in the X_G -Z_G plane. For $\theta_1 = 45^0$, $\theta_2 = \theta_3 = 30^0$ and $\theta_4 = 60^0$ find the coordinates of point C with respect to O.

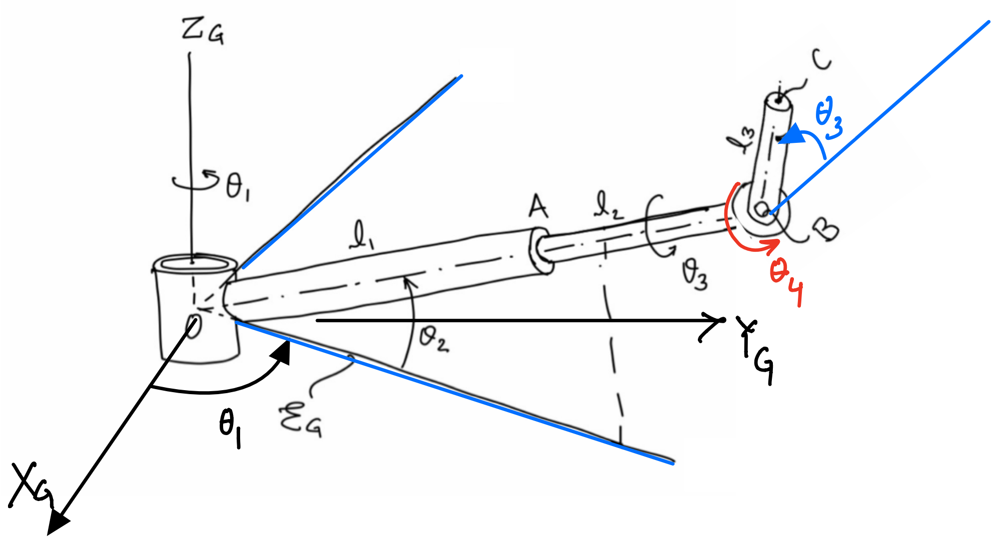


NOTE: This is NOT the most general orientation

In this figure $\theta_1 = \theta_3 = 0^\circ$

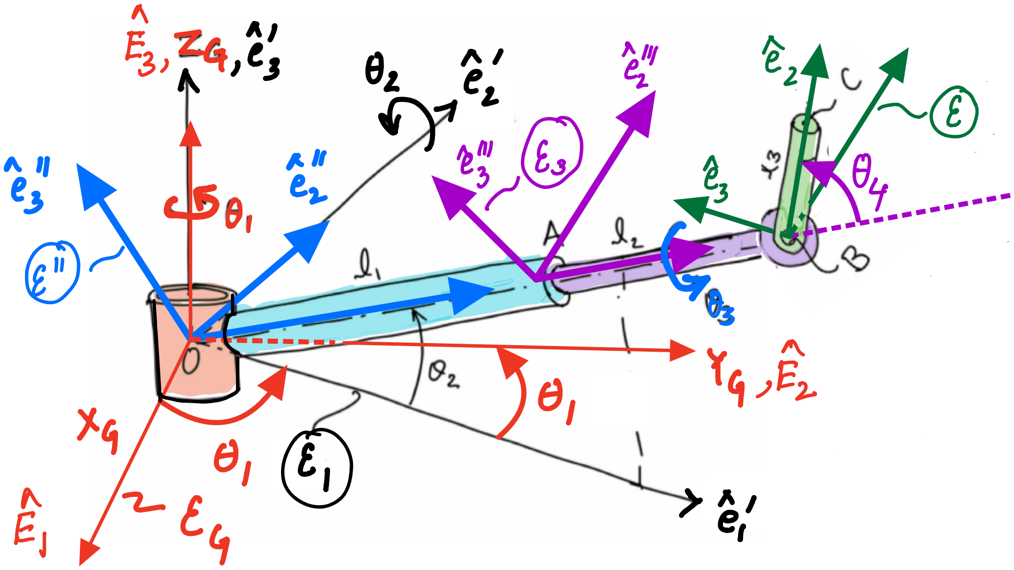
GOOD STRATEGY

- SKETCH the most general configuration.

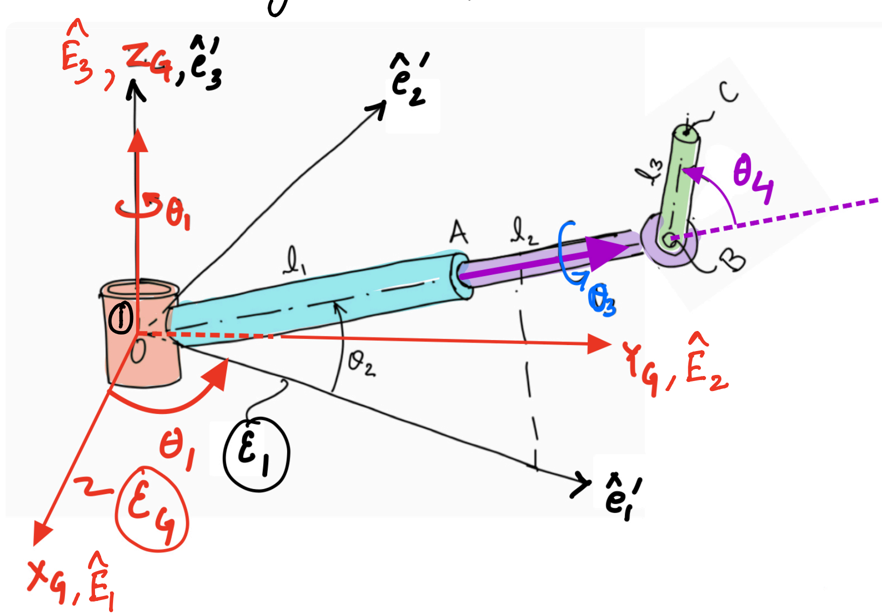


– Identify ALL rigid bodies.

– Attach BFCS to each rigid body with origin at convenient point.



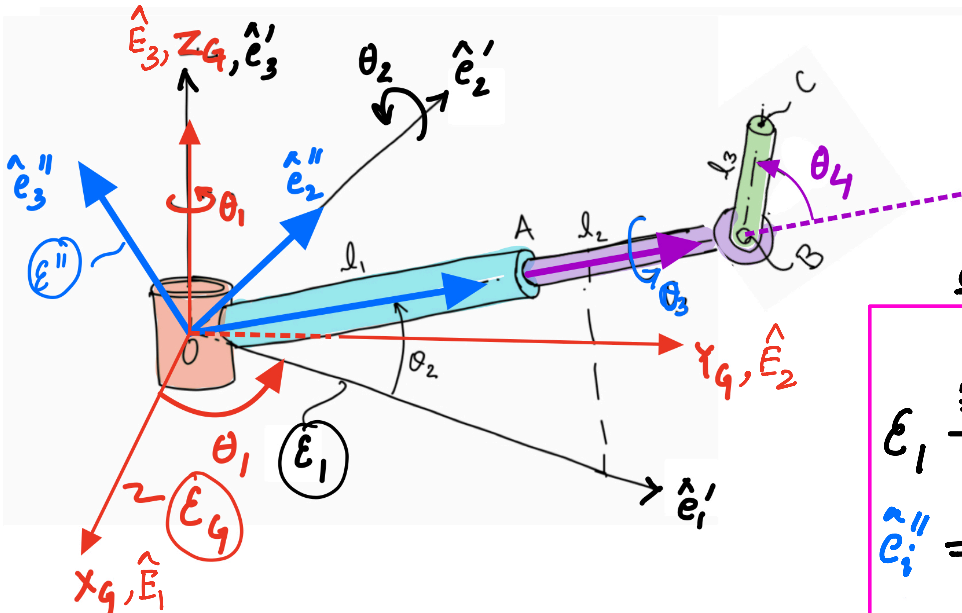
– Identify the sequence of rotations systematically.



STEP I

$$\mathcal{E}_G \xrightarrow{\underline{\underline{R}}_1(\hat{E}_3,\,\theta_1)} \mathcal{E}_1$$

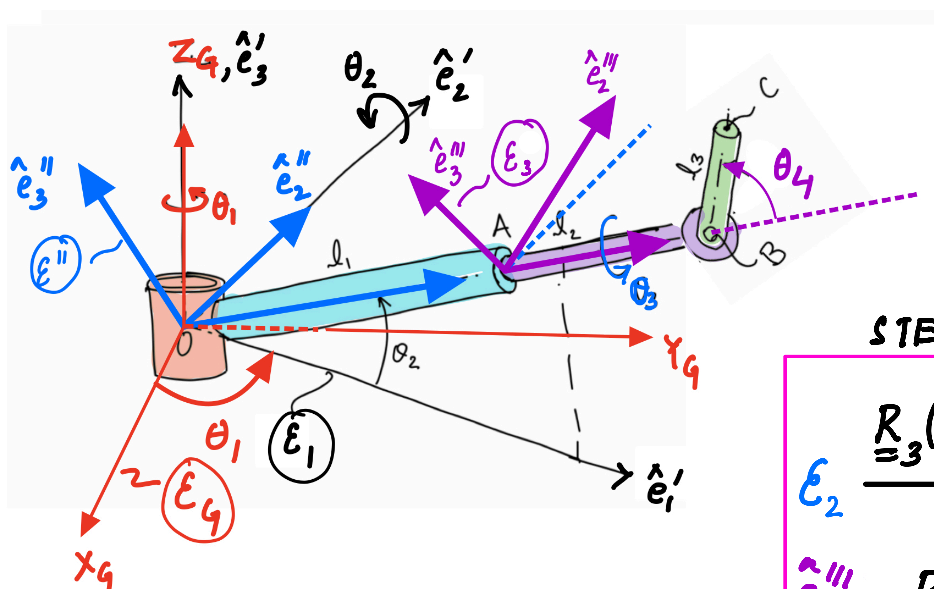
$$\hat{e}_i' = \underline{\underline{R}}_1 \cdot \hat{E}_i$$



STEP II

$$\mathcal{E}_1 \xrightarrow{\underline{\underline{R}}_2(-\hat{e}_2', \theta_2)} \mathcal{E}_2$$

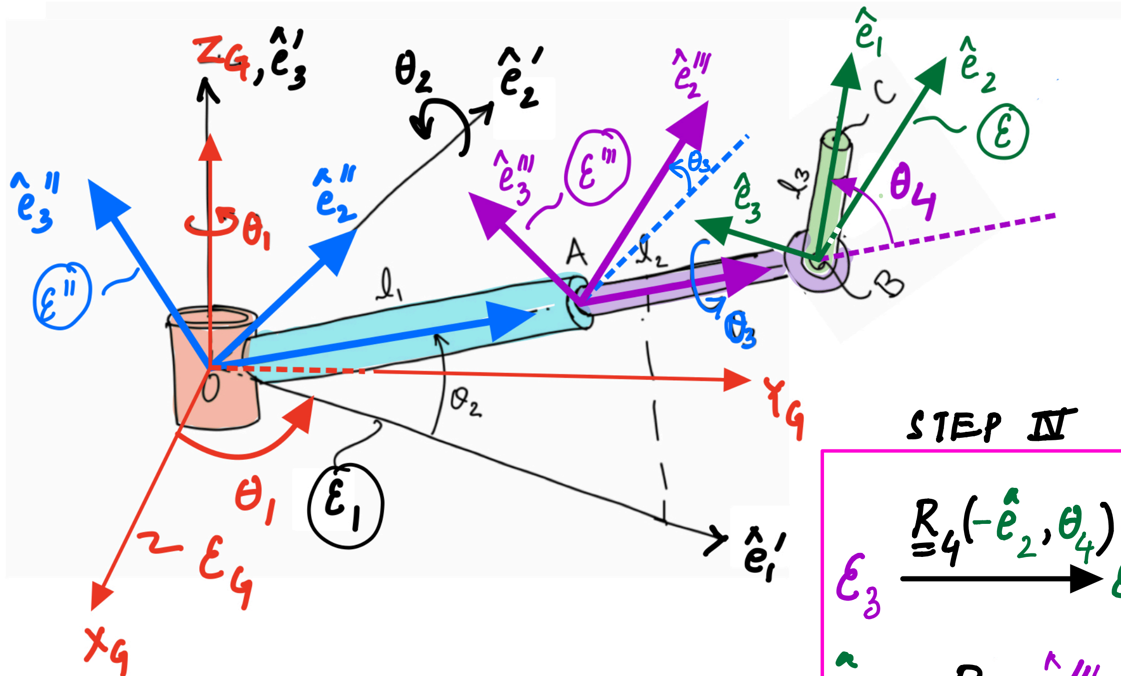
$$\hat{e}_i'' = \underline{\underline{R}}_2 \cdot e_i'$$



STEP III

$$\mathcal{E}_2 \xrightarrow{\underline{\underline{R}}_3(\hat{e}_1'', \theta_3)} \mathcal{E}_3$$

$$\hat{e}_i''' = \underline{\underline{R}}_3 \cdot e_i''$$



STEP IV

$$\mathcal{E}_3 \xrightarrow{\underline{\underline{R}}_4(-\hat{e}_2,\theta_4)} \mathcal{E}$$

$$\hat{e}_i = \underline{\underline{R}}_4 \cdot \hat{e}_i'''$$

Putting it all together:

$$\mathcal{E}_G \xrightarrow{\underline{\underline{R}}_1(\hat{E}_3,\theta_1)} \underset{\text{COLLAR}}{\mathcal{E}_1} \xrightarrow{\underline{\underline{R}}_2(-\hat{e}_2',\theta_2)} \underset{\text{LINK OA}}{\mathcal{E}_2} \xrightarrow{\underline{\underline{R}}_3(\hat{e}_1'',\theta_3)} \underset{\text{LINK AB}}{\mathcal{E}_3} \xrightarrow{\underline{\underline{R}}_4(-\hat{e}_2,\theta_4)} \underset{\text{LINK BC}}{\mathcal{E}} ;$$

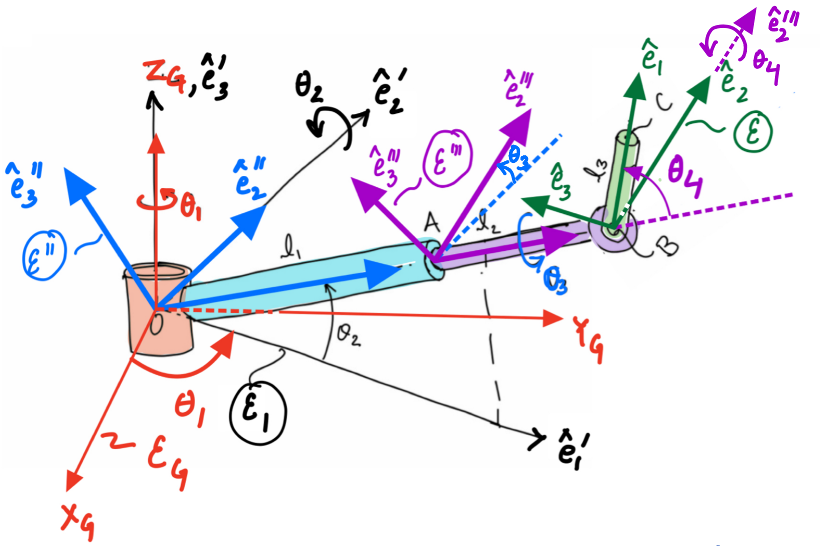
and $\hat{e}_i' = \underline{\underline{R}}_1 \cdot \hat{E}_i$; $\hat{e}_i'' = \underline{\underline{R}}_2 \cdot \underline{\underline{R}}_1 \cdot \hat{E}_i$; $\hat{e}_i''' = \underline{\underline{R}}_3 \cdot \underline{\underline{R}}_2 \cdot \underline{\underline{R}}_1 \cdot \hat{E}_i$

(1) and $\hat{e}_i = \underline{\underline{R}}_4 \cdot \underline{\underline{R}}_3 \cdot \underline{\underline{R}}_2 \cdot \underline{\underline{R}}_1 \cdot \hat{E}_i$

Now need to find the coordinates of C with respect to O, i.e. find $\underline{r}_{C/O}$.

Clearly $\underline{r}_{C/O} = \underline{r}_{C/B} + \underline{r}_{B/A} + \underline{r}_{A/O}$

Further, $\underline{r}_{A/O} = l_1 \hat{e}_1''$; $\underline{r}_{B/A} = l_2 \hat{e}_1'''$; $\underline{r}_{C/B} = l_3 \hat{e}_1$.



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$$\therefore \quad \underline{r}_{C/O} = l_1 \hat{e}_1'' + l_2 \hat{e}_1''' + l_3 \hat{e}_1 \qquad (*) .$$

But this is a MIXED vector. To find coordinates of C we need to express $\underline{r}_{C/O}$ in $\mathcal{E}_G$.

For this we need $\hat{e}_1''$, $\hat{e}_1'''$ and $\hat{e}_1$ in $\mathcal{E}_G$,
i.e. need $[\hat{e}_1'']_{\mathcal{E}_G}$, $[\hat{e}_1''']_{\mathcal{E}_G}$ & $[\hat{e}_1]_{\mathcal{E}_G}$.

To compute them we use relations (1) above:

$$\hat{e}_1'' = \underline{\underline{R}}_2 \cdot \underline{\underline{R}}_1 \cdot \hat{E}_1 \; ; \quad \hat{e}_1''' = \underline{\underline{R}}_3 \cdot \underline{\underline{R}}_2 \cdot \underline{\underline{R}}_1 \cdot \hat{E}_1 \; ;$$
$$\hat{e}_1 = \underline{\underline{R}}_4 \cdot \underline{\underline{R}}_3 \cdot \underline{\underline{R}}_2 \cdot \underline{\underline{R}}_1 \cdot \hat{E}_1 \qquad (2)$$

$$\therefore \ [\hat{e}_1'']_{\mathcal{E}_G} = [\underline{\underline{R}}_2]_{\mathcal{E}_G} [\underline{\underline{R}}_1]_{\mathcal{E}_G} [\hat{E}_1]_{\mathcal{E}_G} \qquad (3)$$

Clearly $\underline{\underline{R}}_1$ is a 2D rotation linking $\mathcal{E}_G$ & $\mathcal{E}_1$, so that

$$\mathcal{E}_G \xrightarrow{\underline{\underline{R}}_1(\hat{E}_3, \theta_1)} \mathcal{E}_1 \qquad [\underline{\underline{R}}_1]_{\mathcal{E}_G} = [\underline{\underline{R}}_1]_{\mathcal{E}_1} = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 \\ s\theta_1 & c\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \qquad (4)$$

However $\underline{\underline{R}}_2$ links $\mathcal{E}_1$ & $\mathcal{E}_2$ and, so, $[\underline{\underline{R}}_2]_{\mathcal{E}_G}$ is NOT directly known in $\mathcal{E}_G$. We know $\mathcal{E}_1 \xrightarrow{\underline{\underline{R}}_2} \mathcal{E}_2$

$$[\underline{\underline{R}}_2]_{\mathcal{E}_1} = [\underline{\underline{R}}_2]_{\mathcal{E}_2} = \begin{bmatrix} c\theta_2 & 0 & -s\theta_2 \\ 0 & 1 & 0 \\ s\theta_2 & 0 & c\theta_2 \end{bmatrix} \qquad (5)$$

Nb. $\underline{\underline{R}}_2$ rotates about $-\hat{e}_2'$

How to compute $[\underline{\underline{R}}_2]_{\mathcal{E}_G}$? Use the formula for change of coordinates from $\mathcal{E}_1$ to $\mathcal{E}_G$. (See lecture 5)

For any tensor $\underline{\underline{A}}$, as $\mathcal{E}_G \xrightarrow{\underline{\underline{R}}_1} \mathcal{E}_1$: $\quad [\underline{\underline{A}}]_{\mathcal{E}_G} = [\underline{\underline{R}}_1]_{\mathcal{E}_1} [\underline{\underline{A}}]_{\mathcal{E}_1} [\underline{\underline{R}}_1]_{\mathcal{E}_1}^T$

Setting $\underline{\underline{A}} = \underline{\underline{R}}_2 \Rightarrow$

$$[\underline{\underline{R}}_2]_{\mathcal{E}_G} = [\underline{\underline{R}}_1]_{\mathcal{E}_1} [\underline{\underline{R}}_2]_{\mathcal{E}_1} [\underline{\underline{R}}_1]_{\mathcal{E}_1}^T \qquad (6)$$

Substituting (6) in (3):

$$[\hat{e}_1'']_{\mathcal{E}_G} = [\underline{\underline{R}}_1]_{\mathcal{E}_1} [\underline{\underline{R}}_2]_{\mathcal{E}_1} \underbrace{[\underline{\underline{R}}_1]_{\mathcal{E}_1}^T [\underline{\underline{R}}_1]_{\mathcal{E}_G}}_{\underline{\underline{1}} \text{ as } [\underline{\underline{R}}_1]_{\mathcal{E}_1} = [\underline{\underline{R}}_1]_{\mathcal{E}_G}} [\hat{E}_1]_{\mathcal{E}_G}$$

$$= [\underline{\underline{R}}_1]_{\mathcal{E}_G} [\underline{\underline{R}}_2]_{\mathcal{E}_1} [\hat{E}_1]_{\mathcal{E}_G} \qquad (7)$$

As mentioned in lec. 5, the sequence of rotation matrices is reversed when compared to the sequence of rotation tensors in (3)

Employing (4) & (5) in (7):

$$[\hat{e}_1'']_{\mathcal{E}_G} = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 \\ s\theta_1 & c\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\theta_2 & 0 & -s\theta_2 \\ 0 & 1 & 0 \\ s\theta_2 & 0 & c\theta_2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} c\theta_1 c\theta_2 & -s\theta_1 & -c\theta_1 s\theta_2 \\ s\theta_1 c\theta_2 & c\theta_1 & -s\theta_1 s\theta_2 \\ s\theta_2 & 0 & c\theta_2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} c\theta_1 c\theta_2 \\ s\theta_1 c\theta_2 \\ s\theta_2 \end{bmatrix} \qquad (8)$$

Similarly, we compute

$$[\hat{e}_1''']_{\mathcal{E}_G} = [\underline{\underline{R}}_3]_{\mathcal{E}_G} [\underline{\underline{R}}_2]_{\mathcal{E}_G} [\underline{\underline{R}}_1]_{\mathcal{E}_G} [\hat{E}_1]_{\mathcal{E}_G} = [\underline{\underline{R}}_1]_{\mathcal{E}_G} [\underline{\underline{R}}_2]_{\mathcal{E}_1} [\underline{\underline{R}}_3]_{\mathcal{E}_2} [\hat{E}_1]_{\mathcal{E}_G} \qquad (9)$$

We note: $$[\underline{\underline{R}}_3]_{\mathcal{E}_2} = [\underline{\underline{R}}_3]_{\mathcal{E}_3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta_3 & -s\theta_3 \\ 0 & s\theta_3 & c\theta_3 \end{bmatrix} \qquad (10)$$

Employing (4), (5) & (10) in (9) and multiplying:

$$[\hat{e}_1''']_{\mathcal{E}_G} = \begin{bmatrix} c\theta_1 c\theta_2 \\ s\theta_1 c\theta_2 \\ s\theta_2 \end{bmatrix} \qquad (11)$$

This is the same as in (8), which is expected as $\underline{\underline{R}}_3$ is about $\hat{e}_1''$, so that $\hat{e}_1'' \parallel \hat{e}_1'''$.

Finally, $[\hat{e}_1]_{\mathcal{E}_G} = [\underline{\underline{R}}_4]_{\mathcal{E}_G} [\underline{\underline{R}}_3]_{\mathcal{E}_G} [\underline{\underline{R}}_2]_{\mathcal{E}_G} [\underline{\underline{R}}_1]_{\mathcal{E}_G} [\hat{E}_1]_{\mathcal{E}_G}$

$$= [\underline{\underline{R}}_1]_{\mathcal{E}_G} [\underline{\underline{R}}_2]_{\mathcal{E}_1} [\underline{\underline{R}}_3]_{\mathcal{E}_2} [\underline{\underline{R}}_4]_{\mathcal{E}_3} [\hat{E}_1]_{\mathcal{E}_G} \qquad (12)$$

But, $$[\underline{\underline{R}}_4]_{\mathcal{E}_3} = [\underline{\underline{R}}_4]_{\mathcal{E}_4} = \begin{bmatrix} c\theta_4 & 0 & -s\theta_4 \\ 0 & 1 & 0 \\ s\theta_4 & 0 & c\theta_4 \end{bmatrix} \qquad (13)$$

Nb. $\underline{\underline{R}}_4$ rotates about $-\hat{e}_2'''$

Employing (4), (5), (10) & (13) in (12) and multiplying:

$$[\hat{e}_1]_{\mathcal{E}_G} = \begin{bmatrix} c\theta_1 c\theta_2 c\theta_4 + s\theta_1 s\theta_3 s\theta_4 - c\theta_1 s\theta_2 c\theta_3 s\theta_4 \\ s\theta_1 c\theta_2 c\theta_4 - c\theta_1 s\theta_3 s\theta_4 - s\theta_1 s\theta_2 c\theta_3 s\theta_4 \\ s\theta_2 c\theta_4 + c\theta_2 c\theta_3 s\theta_4 \end{bmatrix} \qquad (14)$$

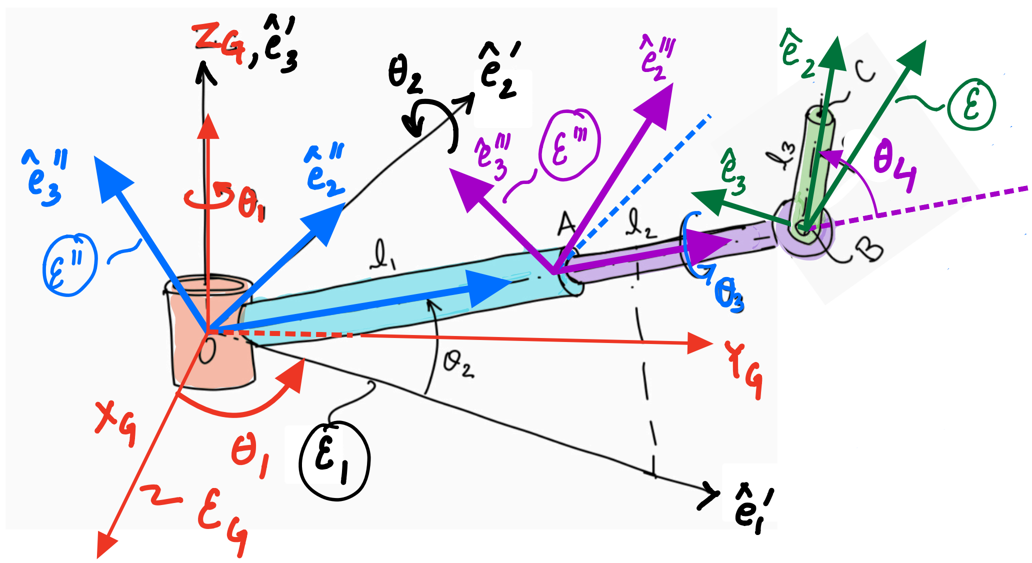
Combining (8), (11) & (14) with (*):

$$[\underline{r}_{c/0}]_{\mathcal{E}_G} = \ell_1 \begin{bmatrix} c\theta_1 c\theta_2 \\ s\theta_1 c\theta_2 \\ s\theta_2 \end{bmatrix} + \ell_2 \begin{bmatrix} c\theta_1 c\theta_2 \\ s\theta_1 c\theta_2 \\ s\theta_2 \end{bmatrix}$$

$$+ \ell_3 \begin{bmatrix} c\theta_1 c\theta_2 c\theta_4 + s\theta_1 s\theta_3 s\theta_4 - c\theta_1 s\theta_2 c\theta_3 s\theta_4 \\ s\theta_1 c\theta_2 c\theta_4 - c\theta_1 s\theta_3 s\theta_4 - s\theta_1 s\theta_2 c\theta_3 s\theta_4 \\ s\theta_2 c\theta_4 + c\theta_2 c\theta_3 s\theta_4 \end{bmatrix}$$

We can now set $\theta_1 = 45°$, $\theta_2 = \theta_3 = 30°$ & $\theta_4 = 60°$:

$$\boxed{\begin{aligned} \underline{r}_{c/0} = {} & \left(\frac{\sqrt{3}}{2\sqrt{2}}\ell_1 + \frac{\sqrt{3}}{2\sqrt{2}}\ell_2 + \frac{3\sqrt{3}}{8\sqrt{2}}\ell_3\right)\hat{E}_1 + \left(\frac{\sqrt{3}}{2\sqrt{2}}\ell_1 + \frac{\sqrt{3}}{2\sqrt{2}}\ell_2 - \frac{3\sqrt{3}}{8\sqrt{2}}\ell_3\right)\hat{E}_2 \\ & + \left(\ell_1/2 + \ell_2/2 + 5\sqrt{3}\,\ell_3/8\right)\hat{E}_3 \end{aligned}}$$



Z_G, ê'_3
θ_2
ê'_2
ê''_3
ε''
θ_1
ê''_2
ê'''_3
ε'''
ê'''_2
A
l_1
l_2
θ_3
ê_2
C
ε
ê_3
l_3
θ_4
B
O
Y_G
θ_2
X_G
θ_1
ε_1
ε_G
ê'_1