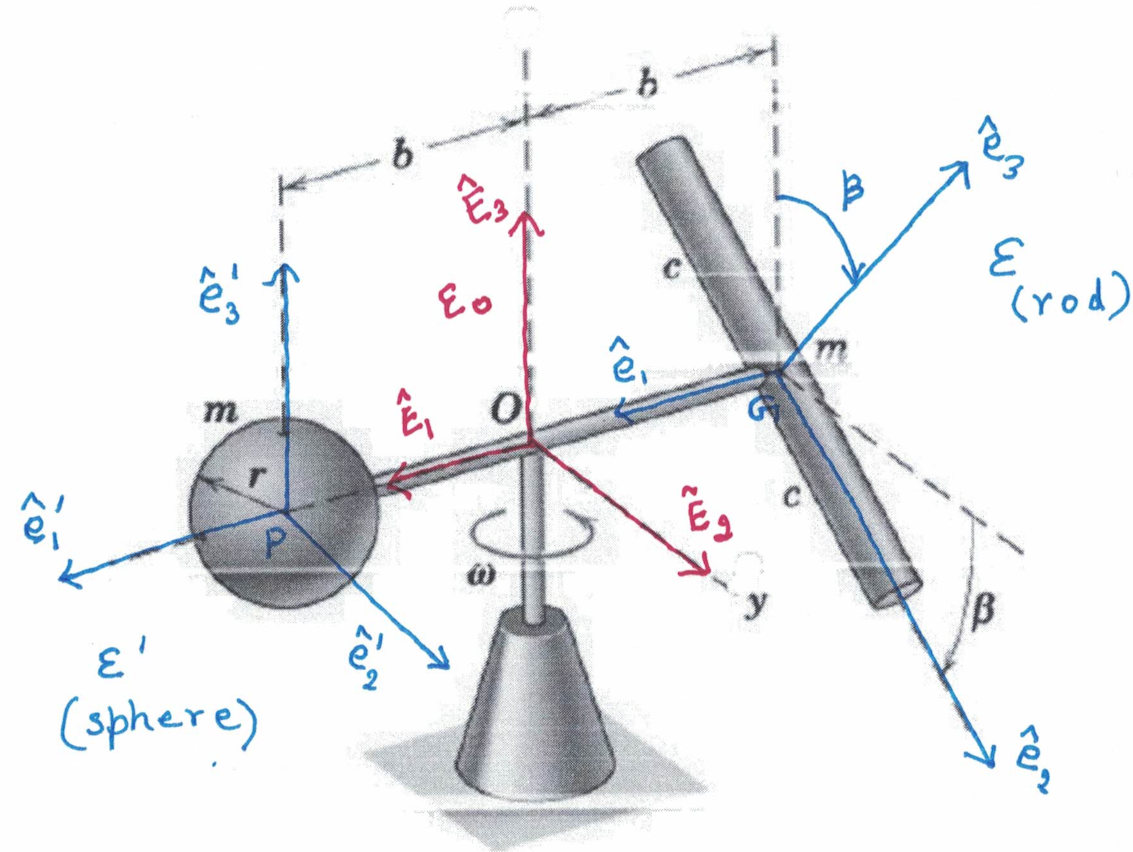
TUTORIAL-8, PROBLEM-4



- The center 'T' arm is massless.
- The sphere is fixed to the arm. That is one rigid body. mass $\rightarrow$ m radius $\rightarrow$ r
- The other rigid body is the uniform rod
  ~~lengt~~ length $\rightarrow$ 2c
  mass $\rightarrow$ m
- P is defined as the center of mass of the sphere, and the point G is the center of mass of the rod.

Defining coordinate systems,

let $\phi = \omega t$

$$\{\mathcal{E}_0, O, \hat{E}_i\} \xrightarrow{\underline{\underline{R}}_1(\hat{E}_3, \phi)} \{\mathcal{E}', P, \hat{e}'_i\} \xrightarrow{\underline{\underline{R}}_2(-\hat{e}'_1, \beta)} \{\mathcal{E}, G, \hat{e}_i\}$$

Inertial ~~frame~~ frame
Fixed CS.

BFCS of the sphere

BFCS of the rod.

Note that the fixed CS $\mathcal{E}_0$ is not rotating with the arm. This fixed frame has been chosen such that it coincides with the $\mathcal{E}'$ frame of the sphere at the given time instant. (i.e. $\phi = 0$)

Thus, at the given time instant, $\mathcal{E}_0$ and $\mathcal{E}'$ are parallel, and $\hat{e}'_1 = \hat{E}_1 = \hat{e}_1$

$$\hat{e}_2 = \underline{\underline{R}}_2 \cdot \hat{e}'_2 \qquad \left[\underline{\underline{R}}_2\right]_{\mathcal{E}'} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta \\ 0 & \sin\beta & \cos\beta \end{bmatrix}$$

$$\hat{e}_3 = \underline{\underline{R}}_2 \cdot \hat{e}'_3$$

The total angular momentum of the system, (about O)

$$\underline{h}^O_{sys} = \underline{h}^O_{sphere} + \underbrace{\cancel{\underline{h}^O_{arm}}}_{0} + \underline{h}^O_{rod}$$

$$\underline{h}^O_{sys} = \underline{h}^O_{sphere} + \underline{h}^O_{rod}$$

Note $\quad \underline{\omega}_{\mathcal{E}'/\mathcal{E}_0} \ \cancel{\underline{\omega}^{B}_{sphere}} = \underline{\omega}_{sys} = \omega \hat{E}_3$

$$\underline{\omega}_{\mathcal{E}/\mathcal{E}'} = \underline{\omega}^{B}_{rod} = -\dot{\beta}\,\hat{e}_1$$

$$\underline{\omega}^{B}_{sphere} = 0$$

$$\underline{\omega}_{rod} = \underline{\omega}_{\mathcal{E}/\mathcal{E}_0} = \underline{\omega}_{\mathcal{E}/\mathcal{E}'} + \underline{\omega}_{\mathcal{E}'/\mathcal{E}_0}$$

$$= -\dot{\beta}\,\hat{e}_1 + \omega \hat{E}_3$$

Now,

$$\underline{h}^{O}_{sphere} = \underline{\underline{I}}^{O}_{sphere} \cdot \underline{\omega}_{sys}$$

where $\quad \underline{\underline{I}}^{O}_{sphere} = \underline{\underline{I}}^{P}_{sphere}$

$$+ \, m\left(|\underline{r}^{P/O}|^2 \mathbb{1} - \underline{r}^{P/O} \otimes \underline{r}^{P/O}\right)$$

Using $\quad \underline{r}^{P/O} = b\,\hat{e}_1'$

$$\left[\underline{r}^{P/O} \otimes \underline{r}^{P/O}\right]_{\mathcal{E}'} = \begin{bmatrix} b^2 & & \\ & 0 & \\ & & 0 \end{bmatrix}$$

$$\left[\underline{\underline{I}}^{P}_{sphere}\right]_{\mathcal{E}'} = \frac{2}{5} m r^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus, $\quad \left[\underline{\underline{I}}^{O}_{sphere}\right]_{\mathcal{E}'} = \begin{bmatrix} \frac{2}{5} m r^2 & & \\ & \frac{2}{5} m r^2 + m b^2 & \\ & & \frac{2}{5} m r^2 + m b^2 \end{bmatrix}$

Thus, $\underline{h}^{o}_{sphere} = m\omega\left(\frac{2}{5}r^2 + b^2\right)\hat{E}_3$

(using $\mathcal{E}' \equiv \mathcal{E}_o$) ------- ① eq

Now,

$$\underline{h}^{o}_{rod} = \underline{r}^{G/o} \times m\underline{V}^{G} + \underline{h}^{G}_{rod}$$

$$= \underline{r}^{G/o} \times m\underline{V}^{G} + \underline{\underline{I}}^{G}_{rod} \cdot \underline{\omega}_{rod}$$

------ ② eq

$$\underline{r}^{G/o} = -b\hat{e}_1 = -b\hat{E}_1$$

$$\underline{V}^{G} = \underline{V}^{o} + \underline{\omega}_{sys} \times \underline{r}^{G/o}$$

$$= \omega\hat{E}_3 \times -b\hat{e}_1 \qquad (\text{using } \hat{e}_1 = \hat{E}_1)$$

$$= -b\omega\hat{E}_2$$

$$\underline{\omega}_{rod} = -\dot{\beta}\hat{e}_1 + \omega\hat{E}_3$$

$$= -\dot{\beta}\hat{E}_1 + \omega\hat{E}_3$$

The rod, with mass m, is a thin cylinder with negligible thickness (radius) as compared to its length (2c). Using the expression from the lectures for a cylinder, with $r \to 0$,

$$\left[\underline{\underline{I}}^{G}_{rod}\right]_{\mathcal{E}} = \frac{m(2c)^2}{3} \begin{bmatrix} 1 & & 0 \\ & 0 & \\ 0 & & 1 \end{bmatrix}$$

In the $\mathcal{E}_0$ frame, using $\underline{\underline{R}}_2$

$$\left[\underline{\underline{I}}^{G}_{rod}\right]_{\mathcal{E}_0} = \left[\underline{\underline{R}}_2\right]_{\mathcal{E}_0}\left[\underline{\underline{I}}^{G}_{rod}\right]_{\mathcal{E}}\left[\underline{\underline{R}}_2\right]^{T}_{\mathcal{E}_0}$$

note that $\mathcal{E}' = \mathcal{E}_0$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\beta & -s\beta \\ 0 & s\beta & c\beta \end{bmatrix} \begin{bmatrix} mc^2/3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & mc^2/3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\beta & s\beta \\ 0 & -s\beta & c\beta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\beta & -s\beta \\ 0 & s\beta & c\beta \end{bmatrix} \begin{bmatrix} mc^2/3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -s\beta\frac{mc^2}{3} & c\beta\frac{mc^2}{3} \end{bmatrix}$$

$$= \begin{bmatrix} mc^2/3 & 0 & 0 \\ 0 & (\sin^2\beta)\frac{mc^2}{3} & -(\sin 2\beta)\frac{mc^2}{6} \\ 0 & -(\sin 2\beta)\frac{mc^2}{6} & (\cos^2\beta)\frac{mc^2}{3} \end{bmatrix}$$

------ ... 3 eq

~~Substituting all this in eq. 2~~

$$\cancel{\underline{h}^{o}_{rod} = +b\hat{E}_1 \times m(+b\omega\hat{E}_2) + \left[\underline{\underline{I}}^{G}_{rod}\right]_{\mathcal{E}_0}\begin{bmatrix} -\dot{\beta} \\ 0 \\ \omega \end{bmatrix}}$$

$$\cancel{= m\omega b^2\hat{E}_3 + (-\dot{\beta})\frac{mc^2}{3}\hat{E}_1 - \sin 2\beta\frac{m\omega c^2}{6}\hat{E}_2 + \cos^2\beta\frac{m\omega c^2}{3}\hat{E}_3}$$

~~--- 3 eq~~

Thus calculating the second term in eq. 2, in matrix form,

$$\begin{pmatrix} 2^{nd} \\ term \end{pmatrix} = \left[\underline{\underline{I}}^{G}_{rod}\right]_{\mathcal{E}_0} \begin{bmatrix} -\dot{\beta} \\ 0 \\ \omega \end{bmatrix}_{\mathcal{E}_0}$$

Note that the vector $\underline{\omega}_{rod}$ ~~[illegible]~~ is written as a column vector for this calculation.

$$= \begin{bmatrix} \frac{mc^2}{3}(-\dot{\beta}) \\ -\sin 2\beta \, \frac{m\omega c^2}{6} \\ \cos^2\beta \, \frac{m\omega c^2}{3} \end{bmatrix}_{\mathcal{E}_0}$$

Here, doing the matrix multiplication. Please refer to eq 3 for matrix entries of $\left[\underline{\underline{I}}^{G}_{rod}\right]_{\mathcal{E}_0}$.

This term can also be written in vector format (in $\mathcal{E}_0$ frame) as follows.

$$\begin{pmatrix} 2^{nd} \\ term \end{pmatrix} = \frac{mc^2}{3}(-\dot{\beta})\,\hat{E}_1 - \sin 2\beta \,\frac{m\omega c^2}{6}\,\hat{E}_2 + \cos^2\beta\,\frac{m\omega c^2}{3}\,\hat{E}_3 \quad \text{---- 4 eq.}$$

Including the information from equation 4 in equation 2, we get

$$\underline{h}^{o}_{rod} = m(-b\hat{E}_1) \times (-b\omega \hat{E}_2) + \begin{pmatrix} \text{2nd} \\ \text{term} \end{pmatrix}$$

$$\underline{h}^{o}_{rod} = m\omega b^2 \hat{E}_3 - \frac{mc^2 \dot{\beta}}{3} \hat{E}_1 - \sin 2\beta \, \frac{m\omega c^2}{6} \hat{E}_2 + \cos^2\beta \, \frac{m\omega c^2}{3} \hat{E}_3$$

------------ 5 eq

Putting equations 1 and 5 together, we have the total angular momentum of the system about the point O.

$$\underline{h}^{o}_{sys} = -\frac{mc^2 \dot{\beta}}{3} \hat{E}_1 - (\sin 2\beta) \frac{m\omega c^2}{6} \hat{E}_2 + m\omega \left(\frac{2}{5} r^2 + 2b^2 + \frac{c^2 \cos^2\beta}{3} \right) \hat{E}_3$$

Ans.