Lecture 18

लट्टू — खिलाते भी, संभालते भी

Tops & Gyroscopes

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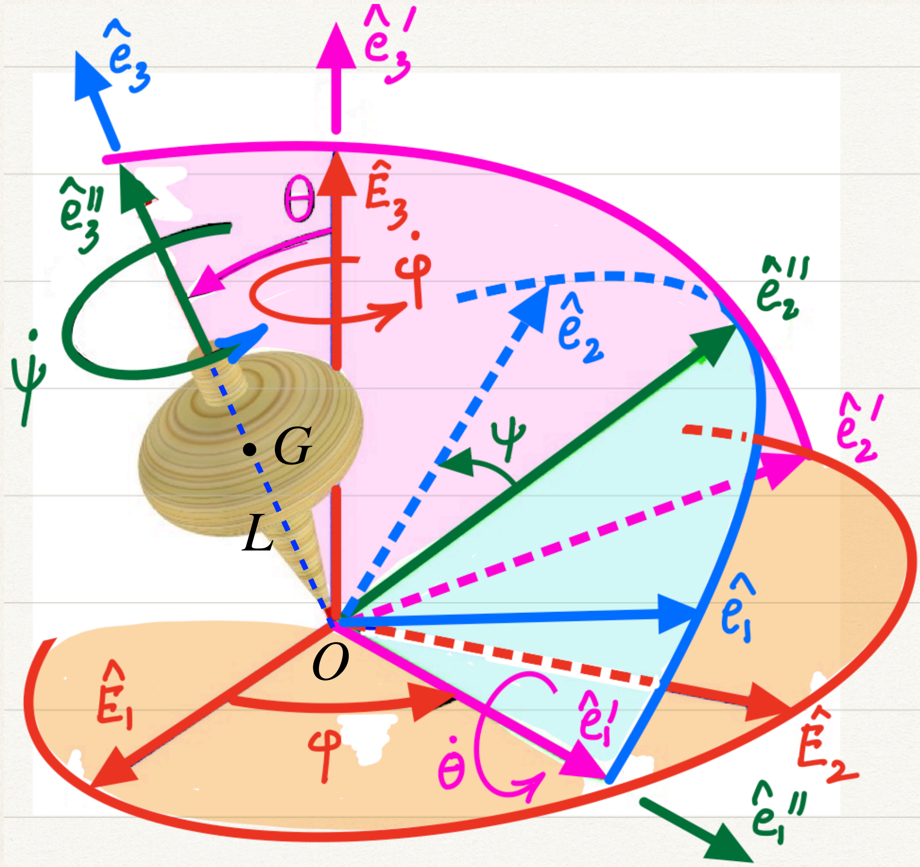
20 - 26 October, 2021

Tops: location

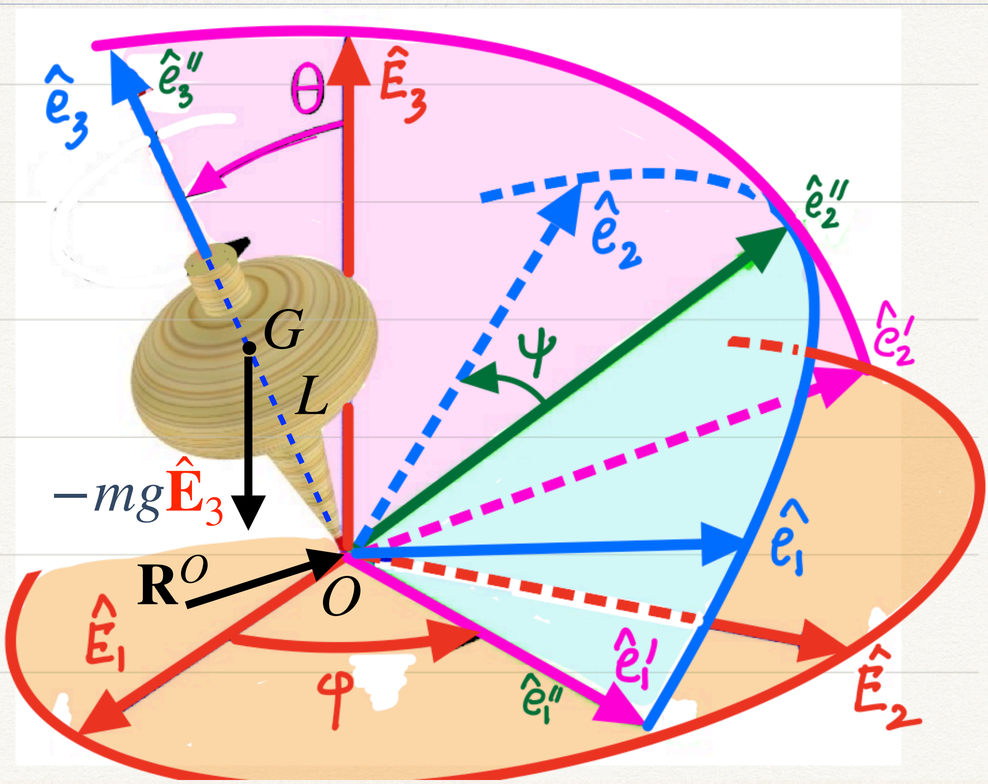
I. **Orientation**: Use *3-1-3* Euler angle sequence.

II. **CM position**: Know $\mathbf{r}^{G/O}$ once we know orientation as O is at origin of $\mathscr{E}_0$.



$$\underbrace{\{\mathscr{E}_0, \hat{\mathbf{E}}_i\}}_{\text{Inertial}} \xrightarrow[\dot{\varphi}\hat{\mathbf{E}}_3]{\mathsf{R}_\varphi} \{\mathscr{E}', \hat{\mathbf{e}}_i'\} \xrightarrow[\dot{\theta}\hat{\mathbf{e}}_1']{\mathsf{R}_\theta} \{\mathscr{E}'', \hat{\mathbf{e}}_i''\} \xrightarrow[\dot{\psi}\hat{\mathbf{e}}_3'']{\mathsf{R}_\psi} \underbrace{\{\mathscr{E}, \hat{\mathbf{e}}_i\}}_{\text{BFCS}}$$

Tops: kinetics



I. **FBD**

II. **LMB**: $-mg\hat{\mathbf{E}}_3 + \mathbf{R}^O = m\mathbf{a}^G$

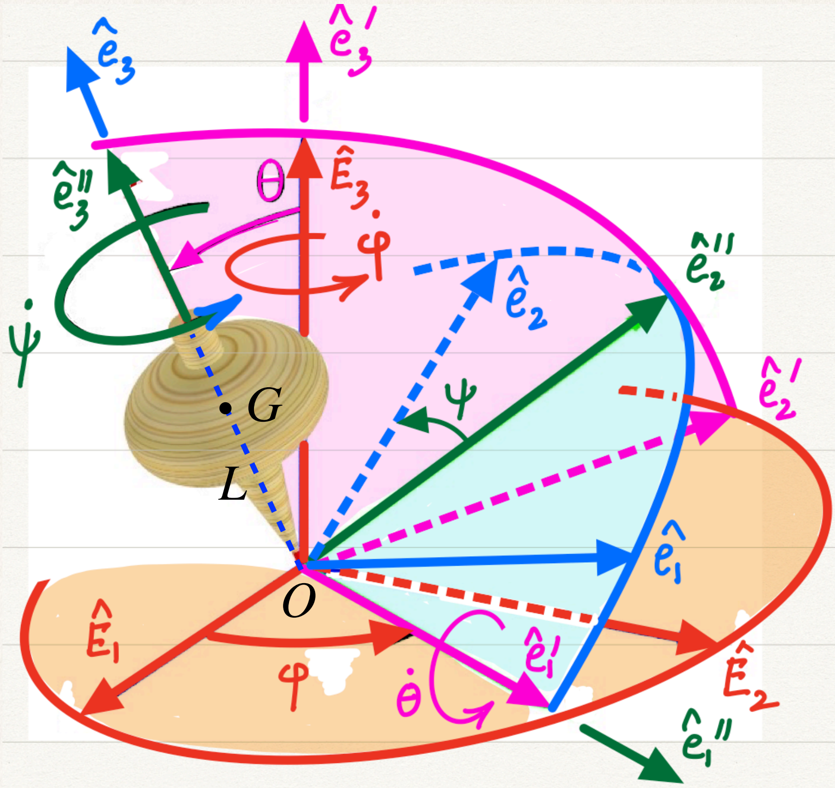
III. $\mathbf{AMB}_{/O}$: $\mathbf{M}^O = \boldsymbol{\omega}^{\mathscr{B}} \times (\mathbb{I}^O \cdot \boldsymbol{\omega}^{\mathscr{B}}) + \mathbb{I}^O \cdot \boldsymbol{\alpha}^{\mathscr{B}}$

1. In $\mathscr{E}$:

Euler's eqns.
$$\begin{cases} M_1 = I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 \\ M_2 = I_2\dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1 \\ M_3 = I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2 \end{cases}$$

2. $$\left.\begin{matrix} M_1 = mgL\sin\theta\cos\psi \\ M_2 = -mgL\sin\theta\sin\psi \end{matrix}\right\}, \; M_3 = 0$$

Tops: kinematics



$$\underbrace{\{\mathscr{E}_0, \hat{\mathbf{E}}_i\}}_{\text{Inertial}} \xrightarrow[\dot{\varphi}\hat{\mathbf{E}}_3]{\mathsf{R}_\varphi} \{\mathscr{E}', \hat{\mathbf{e}}_i'\} \xrightarrow[\dot{\theta}\hat{\mathbf{e}}_1']{\mathsf{R}_\theta} \{\mathscr{E}'', \hat{\mathbf{e}}_i''\} \xrightarrow[\dot{\psi}\hat{\mathbf{e}}_3'']{\mathsf{R}_\psi} \underbrace{\{\mathscr{E}, \hat{\mathbf{e}}_i\}}_{\text{BFCS}}$$

Kinematics: $\boldsymbol{\omega}^{\mathscr{B}} = \dot{\varphi}\hat{\mathbf{E}}_3 + \dot{\theta}\hat{\mathbf{e}}_1' + \dot{\psi}\hat{\mathbf{e}}_3$, $\boldsymbol{\alpha}^{\mathscr{B}} = \dot{\boldsymbol{\omega}}^{\mathscr{B}}$

$$\left.\begin{aligned} \omega_1 &= \dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \\ \omega_2 &= \dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \\ \omega_3 &= \dot{\varphi}\cos\theta + \dot{\psi} \end{aligned}\right| \quad \begin{aligned} \alpha_1 &= \dot{\omega}_1 \\ \alpha_2 &= \dot{\omega}_2 \\ \alpha_3 &= \dot{\omega}_3 \end{aligned}$$

Symmetric tops

I. **Governing equations** for $I_1 = I_2 \neq I_3$.

1. In the $\hat{\mathbf{e}}_3$ direction:

$$M_3 = 0 \implies \omega_3 = \dot{\varphi}\cos\theta + \dot{\psi} = \text{const.} =: \omega_3^0$$

2. $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ direction equations combine:

$$I_1(\ddot{\varphi}\sin\theta + 2\dot{\varphi}\dot{\theta}\cos\theta) - I_3\omega_3^0\dot{\theta} = 0\,;$$

$$I_1(\ddot{\theta} - \dot{\varphi}^2\sin\theta\cos\theta) + I_3\omega_3^0\dot{\varphi}\sin\theta = mgL\sin\theta$$

3. Note ψ drops out completely. Why?

II. **Special case**: *Steady precession*

1. $\theta \approx \text{const.} \neq 0^\circ \implies \ddot{\theta} = \dot{\theta} = 0$

$$\implies -I_1\dot{\varphi}^2\cos\theta + I_3\dot{\varphi}\omega_3^0 = mgL$$

2. If $|\dot{\varphi}| \ll |\dot{\psi}|$, then

   i. $\dot{\psi} \approx \omega_3^0 = \text{const.}$

   ii. $\dot{\varphi}^2 \ll |\omega_3^0\dot{\varphi}| \implies \dot{\varphi} \approx \dfrac{mgL}{I_3\dot{\psi}} = \text{const.}$

Tops: Simple analysis

I. For *symmetric* top $(I_1 = I_2 \neq I_3)$, **assume**:

1. *Steady precession* $(\ddot{\theta} = \dot{\theta} = 0)$;
2. *High spin* $(|\dot{\varphi}| \ll |\dot{\psi}|)$.
3. *No moment* about *spin* axis: $\mathbf{M}^O \cdot \hat{\mathbf{e}}_3 = 0$

$$\Longrightarrow \begin{cases} \omega_1 = \dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi = \dot{\varphi}\sin\theta\sin\psi \\ \omega_2 = \dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi = \dot{\varphi}\sin\theta\cos\psi \\ \omega_3 = \dot{\varphi}\cos\theta + \dot{\psi} \approx \dot{\psi} \end{cases}$$

$$\Longrightarrow \ |\omega_3| \gg |\omega_1|, |\omega_2|$$

II. $\mathbf{h}^O = \sum I_i \omega_i \hat{\mathbf{e}}_i \approx I_3 \dot{\psi} \hat{\mathbf{e}}_3 \Longrightarrow \mathring{\mathbf{h}}^O = I_3 \ddot{\psi} \hat{\mathbf{e}}_3$

III. $\mathbf{AMB}_{/O} : \mathbf{M}^O = \dot{\mathbf{h}}^O = \boldsymbol{\omega}^{\mathscr{B}} \times \mathbf{h}^O + \mathring{\mathbf{h}}^O$

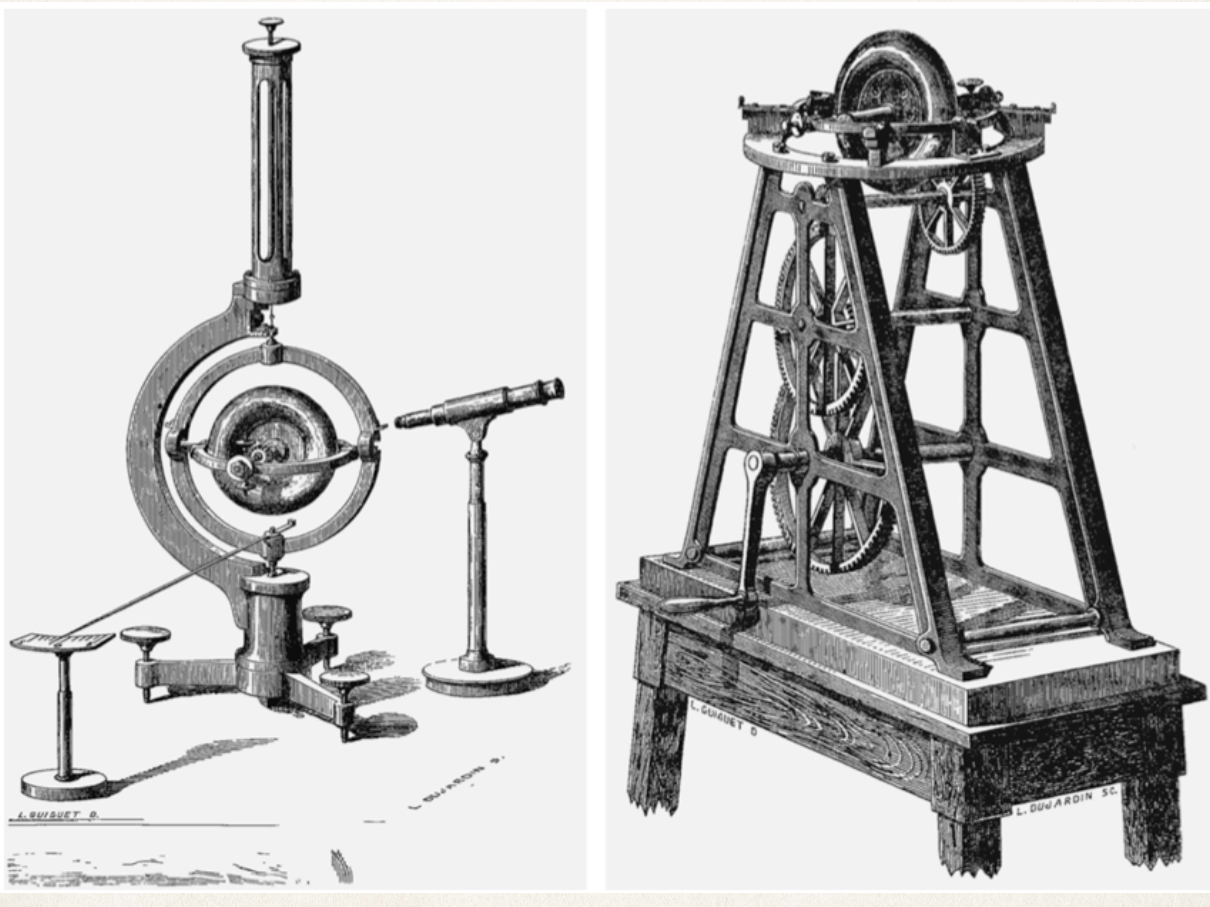
1. $$\Longrightarrow \ \mathbf{M}^O \approx I_3 \dot{\varphi} \hat{\mathbf{E}}_3 \times \dot{\psi} \hat{\mathbf{e}}_3 + \cancelto{0}{I_3 \ddot{\psi} \hat{\mathbf{e}}_3}$$

   $$\Longrightarrow \ \mathbf{M}^O = I_3 \dot{\boldsymbol{\varphi}} \times \dot{\boldsymbol{\psi}}$$

   $\dot{\boldsymbol{\varphi}}$: *Precessional* rate; $\dot{\boldsymbol{\psi}}$: *Spin* rate

2. $\mathbf{M}^O = mgL\hat{\mathbf{e}}_1'' \ \Longrightarrow \ mgL = I_3 \dot{\varphi} \dot{\psi}$.

Gyroscopes

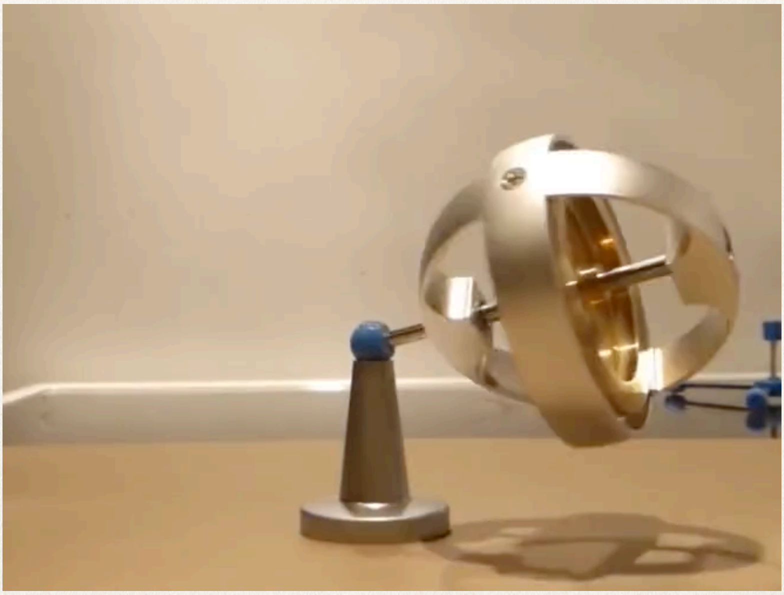


Foucault's gyroscope (left) and its launching device (right) for measuring the Earth's rotation

Gyroscope

***Fast spinning* rotor, generally *heavy*.**

1. *Symmetric rotor*: $\left(I_1 = I_2 \neq I_3\right)$;
2. *Steady precession* $\left(\ddot{\theta} = \dot{\theta} = 0\right)$;
3. *High spin* $\left(\,|\,\dot{\varphi}\,| \ll |\,\dot{\psi}\,|\,\right)$;
4. *No moment* about *spin axis*: $\mathbf{M}^O \cdot \hat{\mathbf{e}}_3 = 0$
5. **Gyroscope equation**: $\mathbf{M}^O = I_3 \dot{\boldsymbol{\varphi}} \times \dot{\boldsymbol{\psi}}$
6. *Spin* stops falling and causes *precession*.
7. *Frictional* losses cause rotor to fall.



Application I

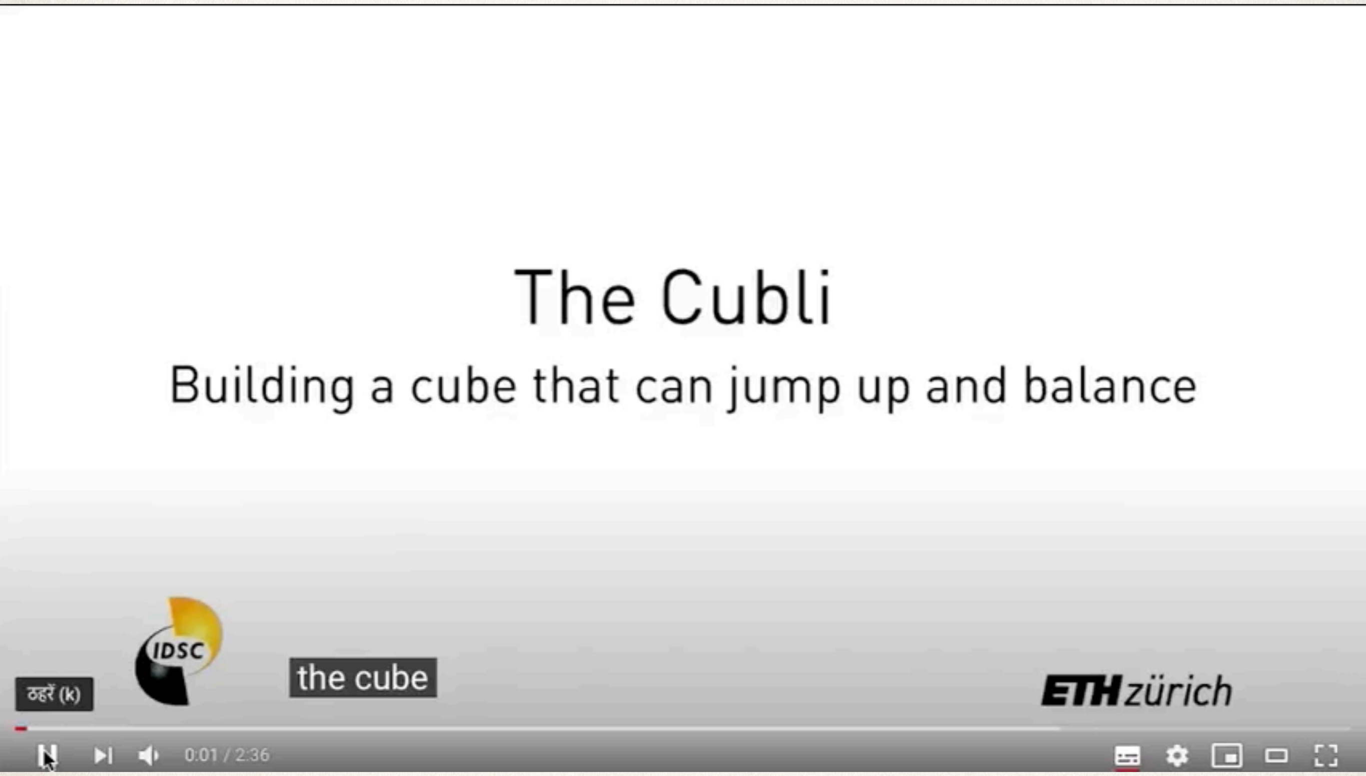
Spin stabilization of projectiles

Application II

Stabilization of ships and spacecrafts

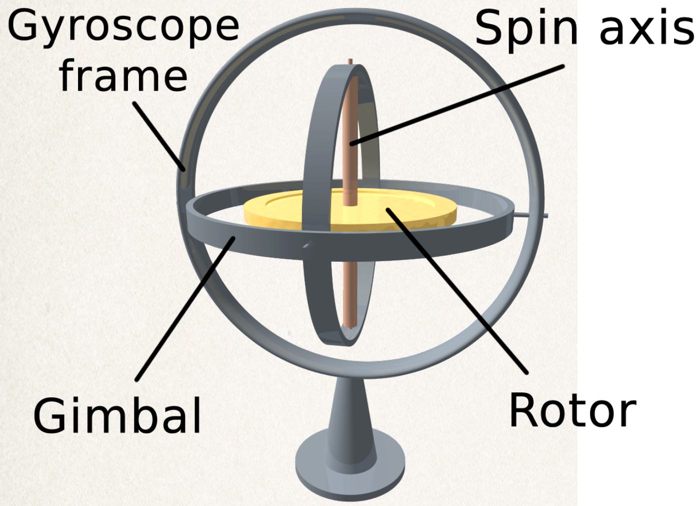
Application III

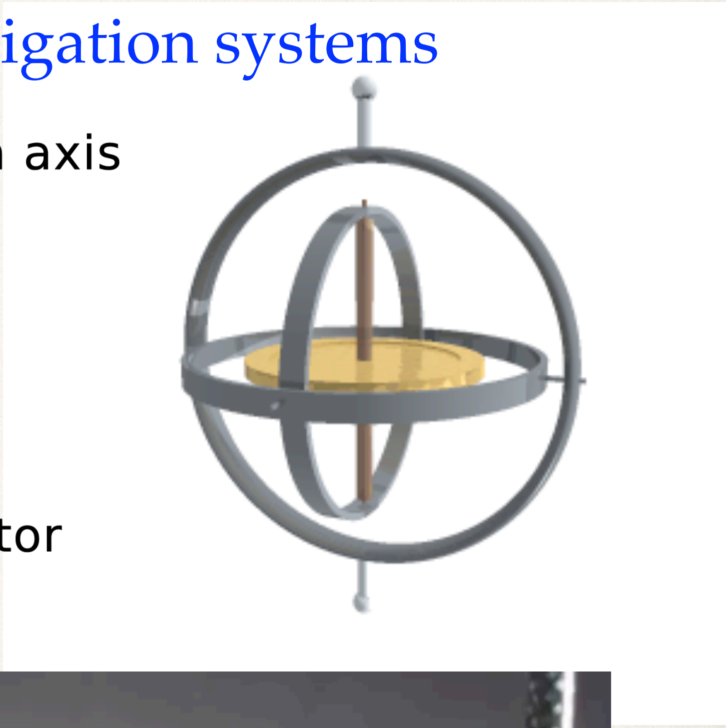
Self-assembly robots

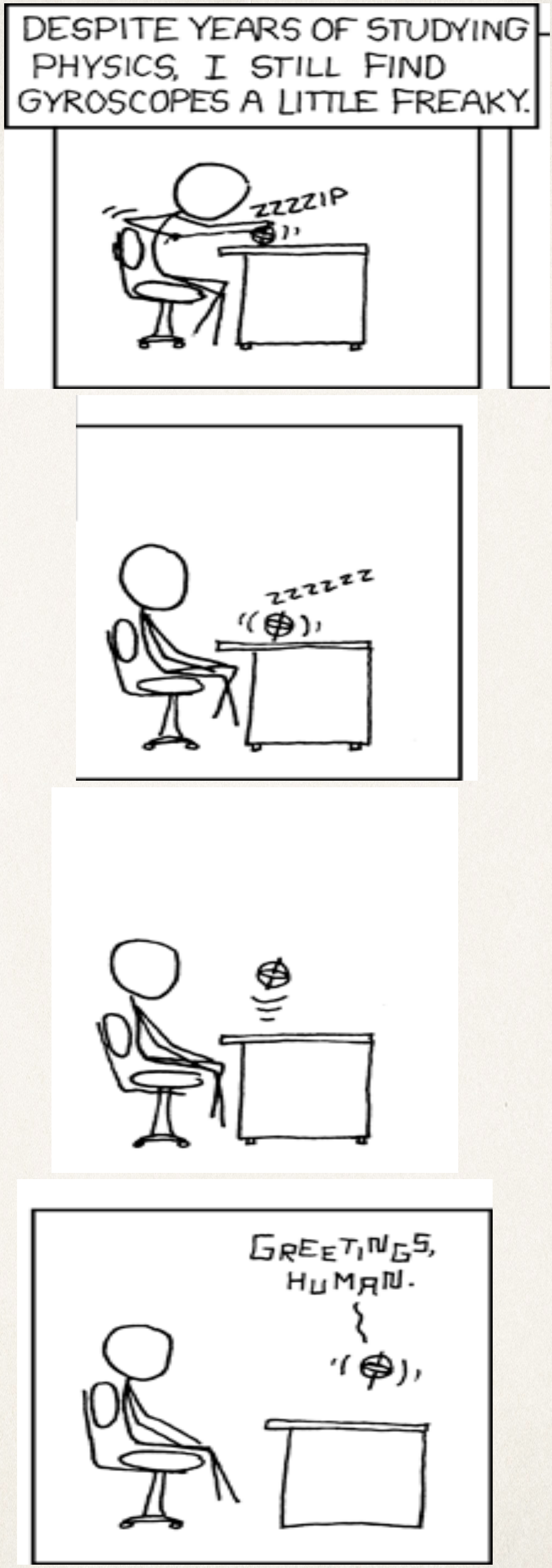


Application IV

Inertial navigation systems






DESPITE YEARS OF STUDYING PHYSICS, I STILL FIND GYROSCOPES A LITTLE FREAKY.
ZZZZIP
ZZZZZZ
GREETINGS, HUMAN.