SOLUTION TO PROBLEM 2 OF TUTORIAL 8

Given, a wheel of radius r connected to an inclined axle rolls without slipping at a constant rate as shown in Fig. 1 such that the point of contact of the disc with the ground moves in a circle with radius R about the hinge point O of the axle. Also given that the time taken for the wheel-axle assembly to complete one round on the horizontal surface is t.

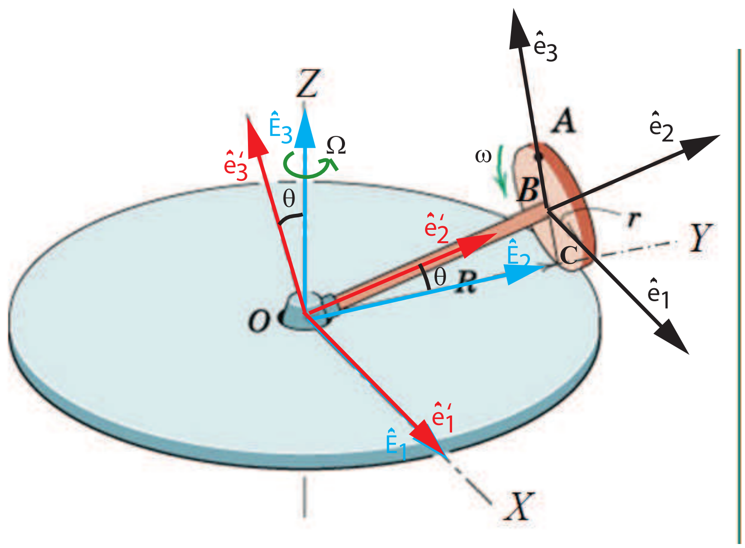

Figure 1: Wheel-axle assembly with different coordinate systems.

We need to find the absolute angular velocity and angular acceleration of the disc/wheel in the ground fixed frame at the instant shown wherein the $\hat{e_1}'$ axis of the axle frame $\mathcal{E}'$ coincides with the $\hat{E}_1$ axis of the ground frame $\mathcal{E}_0$.

In evaluating the solution to this problem, we will consider three reference frames, the ground fixed reference frame $\mathcal{E}_0$, the axle fixed reference frame $\mathcal{E}'$ and the wheel fixed frame $\mathcal{E}$.
Note that this problem is very similar to problem number 3 of tutorial 6 wherein there was an inclined motor with a disc mounted on it rotating on a turntable at a constant rate. As was done in problem 3 of tutorial 6, we could have chosen the axle attached frame to be aligned with the ground frame as well. However, in this solution, I am aligning one axis with the centerline of the axle. I invite the students to do it in the former manner as well.
It has been stated in the problem that we are interested in the instant shown in Fig. 1. At this instant, we have $\hat{E}_1 = \hat{e_1}'$. Furthermore, we have

$$\hat{e_2}' = \cos\theta \hat{E}_2 + \sin\theta \hat{E}_3 \,,$$

$$\hat{e_3}' = -\sin\theta \hat{E}_2 + \cos\theta \hat{E}_3 \,.$$

Also, for convenience, we choose the wheel fixed frame $\mathcal{E}$ to be completely aligned with the axle fixed frame $\mathcal{E}'$ at the given instant. Hence, $\hat{e_1} = \hat{e_1}' = \hat{E}_1$ and

$$\hat{e_2} = \hat{e_2}' = \cos\theta \hat{E}_2 + \sin\theta \hat{E}_3 \,,$$

$$\hat{e_3} = \hat{e_3}' = -\sin\theta \hat{E}_2 + \cos\theta \hat{E}_3 \,.$$

Clearly the axle fixed frame is having a rotating motion about the $\hat{E}_3$. Hence, we can write the angular velocity of the axle as

$$\underline{\omega}_{\mathcal{E}'/\mathcal{E}_0} = \Omega \hat{E}_3 \,.$$

Let, the angular velocity of the wheel about its axis be ω. Accordingly, one can write

$$\underline{\omega}_{\mathcal{E}/\mathcal{E}'} = \omega\hat{e_2}\,.$$

Accordingly, the absolute angular velocity of the wheel or its attached frame $\mathcal{E}$ is

$$\begin{aligned}\underline{\omega}_{\mathcal{E}/\mathcal{E}_0} &= \underline{\omega}_{\mathcal{E}/\mathcal{E}'} + \underline{\omega}_{\mathcal{E}'/\mathcal{E}_0} = \omega\hat{e_2} + \Omega\hat{E}_3 \\ &= \omega\,\cos\theta\hat{E}_2 + (\omega\,\sin\theta + \Omega)\,\hat{E}_3\,. \qquad (1)\end{aligned}$$

The absolute angular acceleration of the disc can be found using

$$\underline{\alpha}_{\mathcal{E}/\mathcal{E}_0} = \underline{\alpha}_{\mathcal{E}/\mathcal{E}'} + \underline{\alpha}_{\mathcal{E}'/\mathcal{E}_0} \;=\; \underline{\alpha}_{\mathcal{E}/\mathcal{E}'}\big|_{\mathcal{E}'} + \underline{\omega}_{\mathcal{E}'/\mathcal{E}_0} \times \underline{\omega}_{\mathcal{E}/\mathcal{E}'} + \underline{\alpha}_{\mathcal{E}'/\mathcal{E}_0}\,. \qquad (2)$$

It has been stated in the problem that the wheel rolls without slipping at a *constant rate*. This implies that $\dot{\omega} = \dot{\Omega} = 0$.
Hence,

$$\begin{aligned}\underline{\alpha}_{\mathcal{E}/\mathcal{E}'}\big|_{\mathcal{E}'} &= \dot{\omega}\hat{e_2} = 0\,, \\ \underline{\alpha}_{\mathcal{E}'/\mathcal{E}_0} &= \dot{\Omega}\hat{E}_3 = 0\,.\end{aligned}$$

Substituting the desired values in Eq. (2), we get

$$\underline{\alpha}_{\mathcal{E}/\mathcal{E}_0}\big|_{\mathcal{E}_0} = \Omega\hat{E}_3 \times \omega\hat{e_2} \;= \Omega\,\omega\,\hat{E}_3 \times \left(\cos\theta\hat{E}_2 + \sin\theta\hat{E}_3\right) = -\Omega\,\omega\,\cos\theta\,\hat{E}_1\,. \qquad (3)$$

Now, we make use of the fact that the wheel-axle assembly takes t sec to complete one rotation on the horizontal ground while rotating a constant rate. Hence,

$$\Omega = \frac{2\pi}{t} \quad \text{rad/sec}\,.$$

In order to determine ω, we will make use of the first condition for rolling without slipping according to which the point of contact C of the wheel with the horizontal surface should have zero velocity. Starting with the wheel fixed reference frame $\mathcal{E}$ whose origin is at the point B, the velocity of C can be written as

$$\underline{v}^C = \underline{v}^B + \underline{v}^C_{rel} + \underline{\omega}_{\mathcal{E}/\mathcal{E}_0} \times \underline{r}^{C/B}\,,$$

where $\underline{v}^C_{rel}$ is the velocity of C relative to B as measured in the disc fixed frame and $\underline{v}^B$ is the linear velocity of the point B. Since, both B and C are points on the same rigid body, there is no relative motion between them as measured in a frame attached to that body. Hence, $\underline{v}^C_{rel} = 0$. Now, $\underline{r}^{C/B} = -r\hat{e_3}$. Accordingly, we have

$$\underline{v}^C = \underline{v}^B + \left(\omega\hat{e_2} + \Omega\hat{E}_3\right) \times -r\hat{e_3} = \underline{v}^B - r\,(\omega + \Omega\,\sin\theta)\,\hat{E}_1\,.$$

We now consider the axle fixed frame $\mathcal{E}'$ with its origin at the stationary point O to evaluate $\underline{v}^B$ which gives

$$\underline{v}^B = \underline{v}^B_{rel} + \underline{\omega}_{\mathcal{E}'/\mathcal{E}_0} \times \underline{r}^{B/O}\,.$$

Again, O and B are points on the same rigid body and hence, $\underline{v}^B_{rel} = 0$. Also, $\underline{r}^{B/O} = \sqrt{R^2 - r^2}\hat{e_2}$. With that, the velocity of the point B can be evaluated as

$$\underline{v}^B = \Omega\hat{E}_3 \times \sqrt{R^2 - r^2}\hat{e_2} \;=\; -\Omega\,\sqrt{R^2 - r^2}\,\cos\theta\hat{E}_1\,.$$

Substituting this value of $\underline{v}^B$ in the expression for $\underline{v}^C$, we have

$$\underline{v}^C = -\left[\Omega\sqrt{R^2-r^2}\,\cos\theta + r\,\left(\omega + \Omega\,\sin\theta\right)\right]\hat{E}_1\,.$$

From the geometry of Fig. 1, we have

$$\cos\theta = \frac{\sqrt{R^2-r^2}}{R}\,, \qquad \sin\theta = \frac{r}{R}\,.$$

Substituting these in the above expression for $\underline{v}^C$ and equating to zero gives

$$-\left[\Omega\sqrt{R^2-r^2}\,\frac{\sqrt{R^2-r^2}}{R} + r\,\left(\omega + \Omega\,\frac{r}{R}\right)\right]\hat{E}_1 = -\left[\Omega\,\left(R - \frac{r^2}{R} + \frac{r^2}{R}\right) + r\omega\right] = 0\,.$$

The above can be solved for ω to yield

$$\omega = -\Omega\frac{R}{r} = -\frac{2\pi}{t}\frac{R}{r}\,.$$

Substituting the values of Ω and ω in Eqs. (1) and (3) along with the expressions for $\cos\theta$ and $\sin\theta$, we have the absolute angular velocity and angular acceleration of the wheel as

$$\underline{\omega}_{wheel} = -\frac{2\pi}{t}\frac{R}{r}\frac{\sqrt{R^2-r^2}}{R}\hat{E}_2 + \left(-\frac{2\pi}{t}\frac{R}{r}\frac{r}{R} + \frac{2\pi}{t}\right)\hat{E}_3 = -\frac{2\pi}{t}\frac{\sqrt{R^2-r^2}}{r}\hat{E}_2 \quad \text{rad/sec}$$

and

$$\underline{\alpha}_{wheel} = -\frac{2\pi}{t}\left(-\frac{2\pi}{t}\frac{R}{r}\right)\frac{\sqrt{R^2-r^2}}{R}\,\hat{E}_1 = \left(\frac{2\pi}{t}\right)^2\frac{\sqrt{R^2-r^2}}{r}\,\hat{E}_1 \quad \text{rad/sec}^2\,.$$