Lecture 16

Rigid body kinetics: Problem formulation; Rigid bodies in space.

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6 - 19 October, 2021

Problem formulation

Aim: Find *location* and *orientation* of a rigid body given forces and moments acting.

I. Draw a neat and correct *FBD*.

II. *Laws of motion*: $\sum \mathbf{F} = m\mathbf{a}^G$ (LMB)

$$\mathbf{M}^G = \boldsymbol{\omega}^{\mathscr{B}} \times \left(\mathrm{I}^G \cdot \boldsymbol{\omega}^{\mathscr{B}}\right) + \mathrm{I}^G \cdot \boldsymbol{\alpha}^{\mathscr{B}} \quad \text{(AMB)}$$

III. *Kinematic analysis* to include constraints.

IV. Relate kinematic variables $\mathbf{a}^G, \boldsymbol{\omega}^{\mathscr{B}}, \boldsymbol{\alpha}^{\mathscr{B}}$ to *location* and *orientation* of the rigid body:

Location: $\mathbf{a}^G = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}^G,\ \mathbf{v}^G = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r}^{G/O}$.

Orientation: Use Euler angle seq., e.g. *3-1-3*.

$$\boldsymbol{\alpha}^{\mathscr{B}} = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\omega}^{\mathscr{B}}, \quad \begin{cases} \dot{\varphi} = (\omega_1 \sin\psi + \omega_2 \cos\psi)\csc\theta \\ \dot{\theta} = \omega_1 \cos\psi - \omega_2 \sin\psi \\ \dot{\psi} = -(\omega_1 \sin\psi + \omega_2 \cos\psi)\cot\theta + \omega_3 \end{cases}$$

V. **Integrate** ODEs to find $\mathbf{r}^{G/O}(t), \{\phi, \theta, \psi\}(t)$

1. *Initial conditions*: $12 = 3 \times 2 + 3 \times 2$.

Arjun & Karṇ
Hoysaleswara Temple, Halebidu, KA

Spacecrafts & Missiles

The Mysore Rocket Man

Rigid body in space

I. LMB: $\sum \mathbf{F} = m\mathbf{a}^G = \mathbf{0} \implies G$ is <u>fixed</u>.

II. AMB$_{/G}$: $\mathbf{M}^G = \mathbf{0} = \dfrac{\mathrm{d}\mathbf{h}^G}{\mathrm{d}t} \implies \mathbf{h}^G =$ <u>const</u>.

$$\implies \mathbf{M}^G = \mathbf{0} = \boldsymbol{\omega}^{\mathscr{B}} \times \left(\mathbf{I}^G \cdot \boldsymbol{\omega}^{\mathscr{B}}\right) + \mathbf{I}^G \cdot \boldsymbol{\alpha}^{\mathscr{B}}$$

III. How to define inertial CS $\mathscr{E}_0$?

1. Locate *origin* at G, as it is fixed.
2. Align $\hat{\mathbf{E}}_3$ with $\mathbf{h}^G$; others normal to $\hat{\mathbf{E}}_3$.

IV. *Principal* CS of $\mathbf{I}^G$ set as *BFCS* $\{\mathscr{E}, G, \hat{\mathbf{e}}_i\}$:

1. $\mathbf{I}^G = I_1\,\hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_2 + I_2\,\hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_2 + I_3\,\hat{\mathbf{e}}_3 \otimes \hat{\mathbf{e}}_3$
2. Let $\boldsymbol{\omega}^{\mathscr{B}} = \omega_i \hat{\mathbf{e}}_i$ and $\boldsymbol{\alpha}^{\mathscr{B}} = \alpha_i \hat{\mathbf{e}}_i = \dot{\omega}_i \hat{\mathbf{e}}_i$.

V. AMB$_{/G}$ $\implies$
(*Euler's equations*)

$$\underbrace{\left\{\boxed{\begin{aligned} 0 &= I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 \\ 0 &= I_2\dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1 \\ 0 &= I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2 \end{aligned}}\right.}_{\text{In BFCS}}$$

Rigid body in space

I. **Special case**. Symmetric body: $I_1 = I_2 = I_o$

$$\left.\begin{aligned} 0 &= I_o\dot{\omega}_1 - (I_o - I_3)\omega_2\omega_3 \\ 0 &= I_o\dot{\omega}_2 - (I_3 - I_o)\omega_3\omega_1 \end{aligned}\right\} \text{ and } 0 = I_3\dot{\omega}_3$$

II. **Solve**: $\omega_3 = \Omega_0$. With $\nu = \Omega_0(I_3 - I_o)/I_o$,

$$\omega_1 = \omega_0 \cos(\nu t + \phi); \;\; \omega_2 = \omega_0 \sin(\nu t + \phi).$$

III. Use *3-1-3* Euler angle sequence:

$$\dot{\varphi} = \omega_0 \sin\{\nu t + \psi(t) + \phi\}\csc\theta(t)$$

$$\dot{\theta} = \omega_0 \cos\{\nu t + \psi(t) + \phi\}$$

$$\dot{\psi} = \Omega_0 - \omega_0 \sin\{\nu t + \psi(t) + \phi\}\cot\theta(t)$$

IV. **Sol.**
$$\begin{cases} \dot{\theta} = 0 \implies \theta(t) = \theta_0, \; (\because \mathbf{h}^G \cdot \hat{\mathbf{e}}_3 = \text{const.}) \\ \dot{\varphi} = p = \omega_0 \csc\theta_0 \implies \varphi(t) = pt + \varphi_0 \\ \dot{\psi} = s = \Omega_0 - \omega_0 \cot\theta_0 \implies \psi(t) = st + \psi_0 \end{cases}$$

V. **Given**: $\mathbf{h}^G = H\hat{\mathbf{E}}_3$, $E_k = T \implies p = H/I_o$;

$$s = H\cos\theta_0 \frac{I_o - I_3}{I_o I_3}; \; \cos\theta_0 = \left| \frac{I_o I_3}{I_o - I_3}\left(\frac{2T}{H^2} - \frac{1}{I_o}\right)\right|^{1/2}$$

We call p: *Precession*, s: *Spin*

Rigid body in space

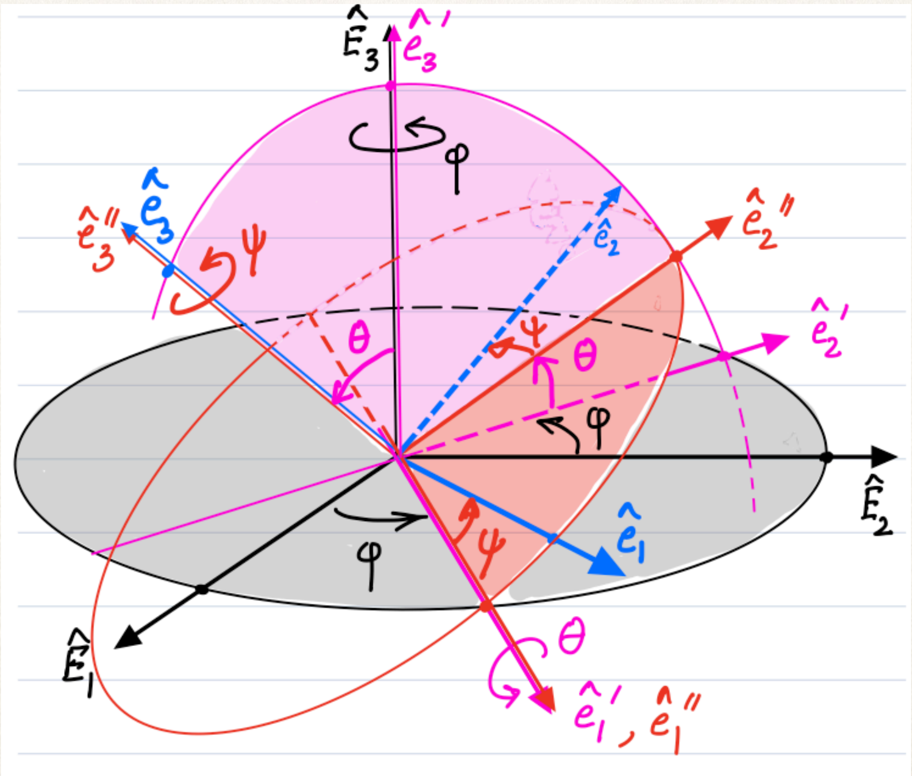
I. Symmetric body with $\mathbf{h}^G = H\hat{\mathbf{E}}_3$, $E_k = T$.

II. *3-1-3* Euler angle sequence:

$$\begin{cases} \theta(t) = \theta_0 \\ \dot{\varphi} = p \\ \dot{\psi} = s \end{cases} \quad \begin{cases} \omega_1 = p \sin\theta_0 \sin\psi(t); \\ \omega_2 = p \sin\theta_0 \cos\psi(t) \\ \omega_3 = s + p\cos\theta_0 \end{cases}$$

Precession: $p = H/I_o$, Spin: $s = H\cos\theta_0 \dfrac{I_o - I_3}{I_o I_3}$

$$\text{Tilt: } \cos\theta_0 = \left| \frac{I_o I_3}{I_o - I_3}\left(\frac{2T}{H^2} - \frac{1}{I_o}\right)\right|^{1/2}$$

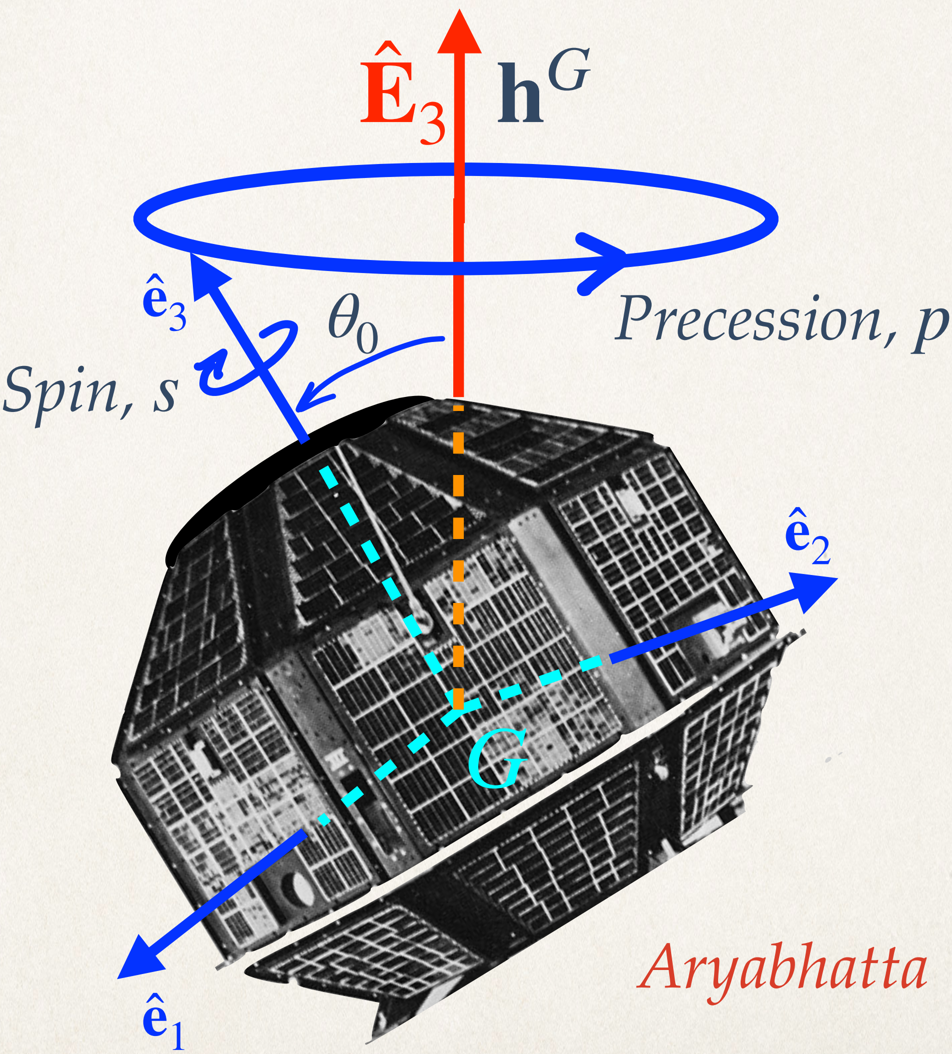


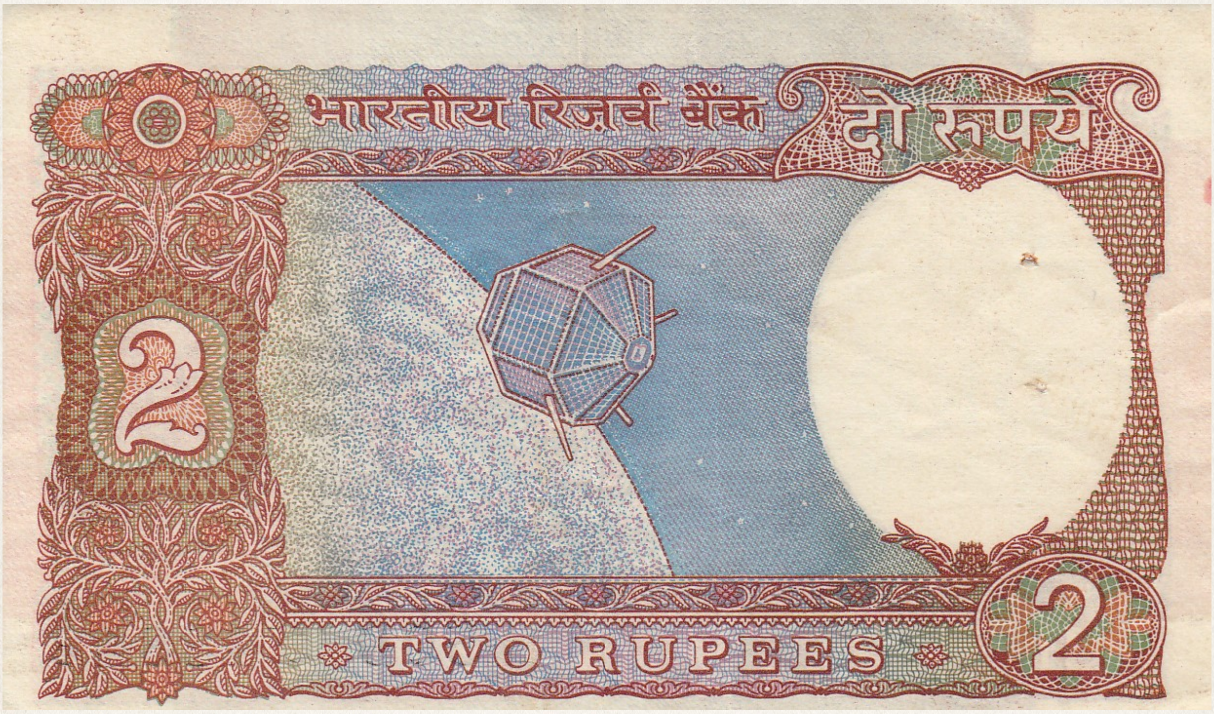
Case I: Spacecrafts

I. Generally, $I_1 = I_2 = I_o \leqslant I_3$.

II. We have $s = p\cos\theta_0\,(I_o - I_3)/I_3 \Longrightarrow$

1. *precession* p and *spin* s are in opposite direction $\Longrightarrow$ *Retrograde* motion.





Led by UR Rao and IISc. team.

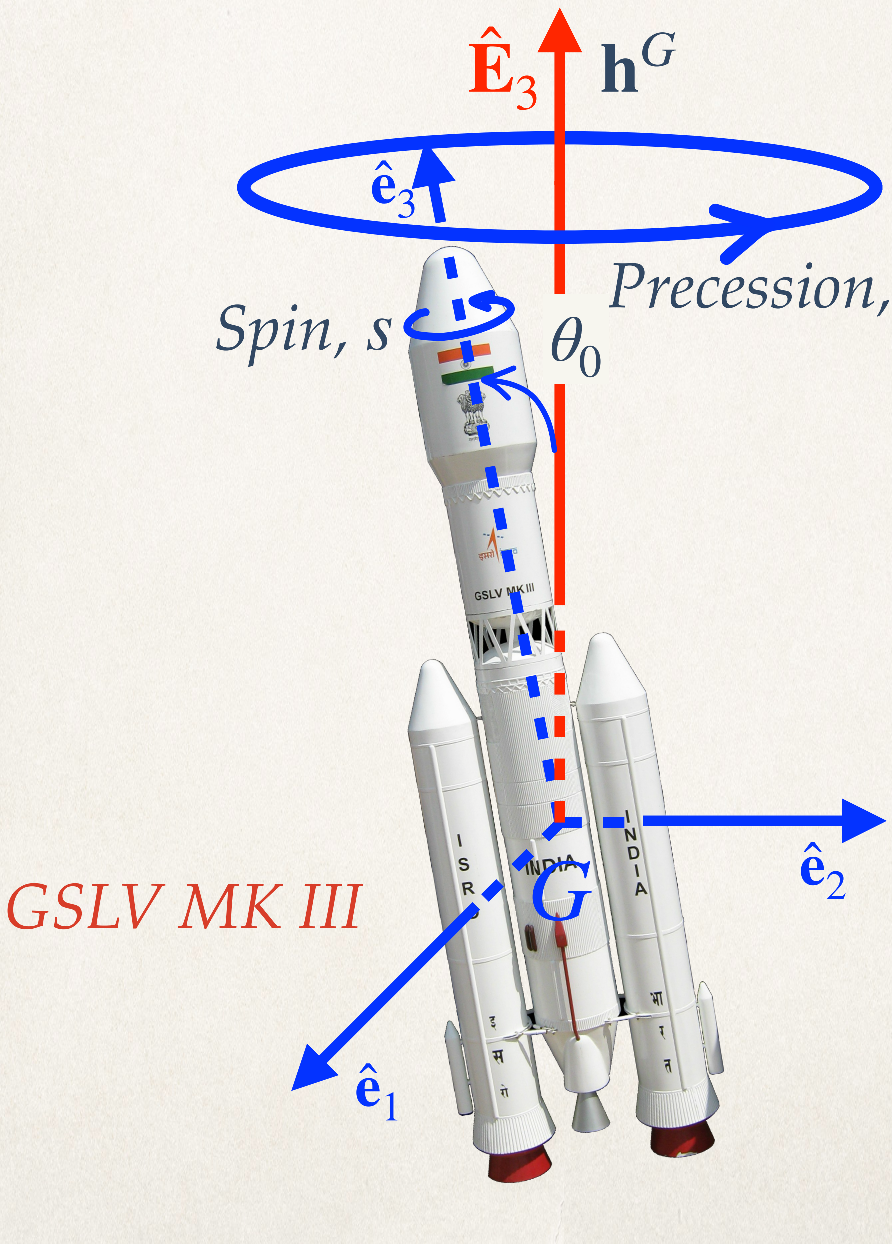
IIT UGs were going into IAS/IT/US.

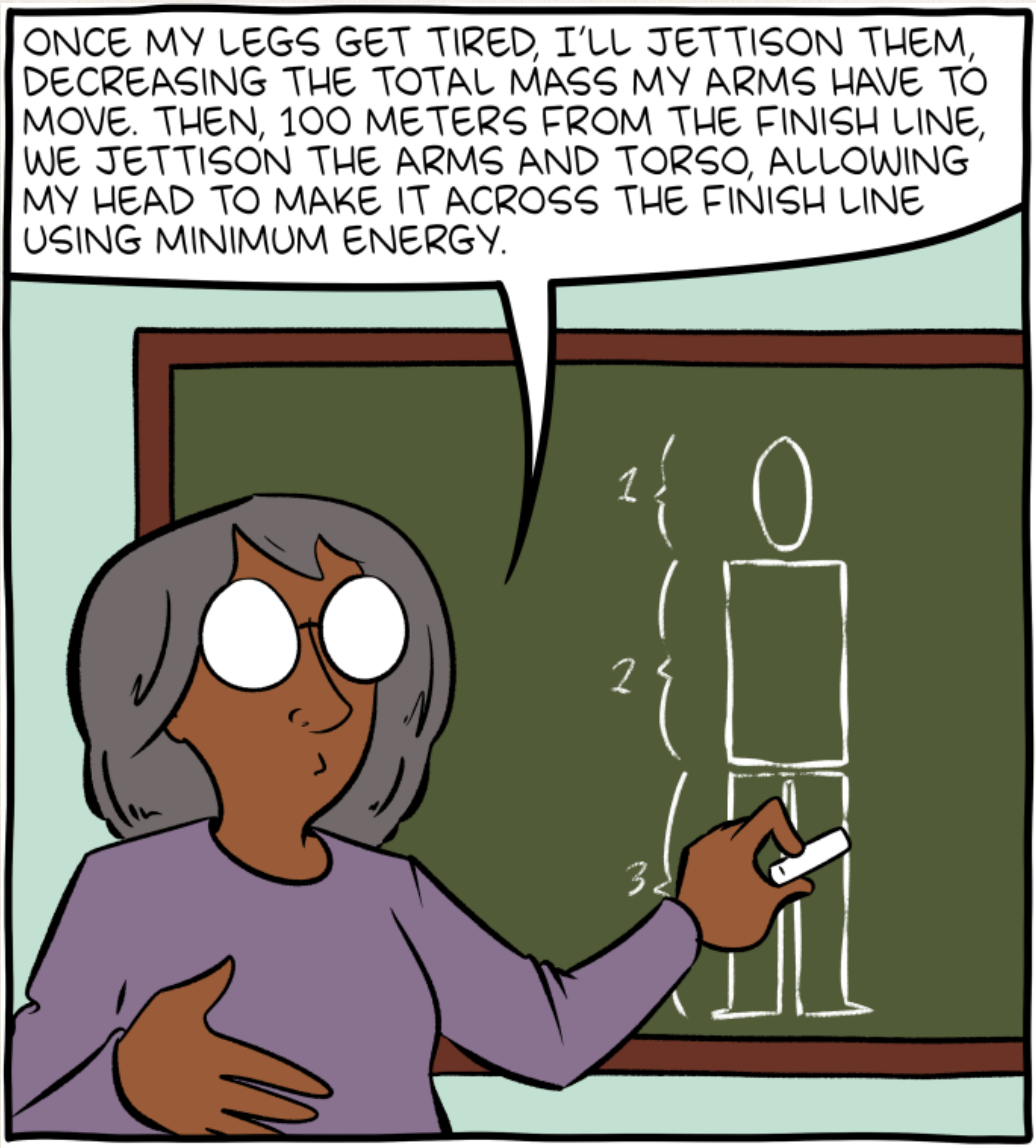
Case II: Rockets

I. In this case, $I_1 = I_2 = I_o \geqslant I_3 \implies$

II. We have $s = p\cos\theta_0 (I_o - I_3)/I_3 \implies$

1. *precession* p and *spin* s are in same direction $\implies$ *Prograde* motion.





Fun fact:
No rocket scientist has ever won a marathon.