SOLUTION TO PROBLEM 3 OF TUTORIAL 4

Given, the $3-1-3$ Euler angle sequence represented by the flowchart

$$\{\mathcal{E}_0, \hat{\mathbf{E}}_i\} \xrightarrow{\underline{\underline{R}}_\varphi} \{\mathcal{E}', \hat{\mathbf{e}}'_i\} \xrightarrow{\underline{\underline{R}}_\theta} \{\mathcal{E}'', \hat{\mathbf{e}}''_i\} \xrightarrow{\underline{\underline{R}}_\psi} \{\mathcal{E}, \hat{\mathbf{e}}_i\}\,.$$

The final rotation tensor $\underline{\underline{R}}$ for the transformation $\{\mathcal{E}_0, G, \hat{\mathbf{E}}_i\} \xRightarrow{\underline{\underline{R}}} \{\mathcal{E}, G, \hat{\mathbf{e}}_i\}$ is given by

$$\underline{\underline{R}} = \underline{\underline{R}}_\psi(\hat{\mathbf{e}}''_3, \psi) \cdot \underline{\underline{R}}_\theta(\hat{\mathbf{e}}'_1, \theta) \cdot \underline{\underline{R}}_\varphi(\hat{\mathbf{E}}_3, \varphi)\,.$$

1. We first need to show that

$$[\underline{\underline{R}}_\varphi]_{\mathcal{E}_0} = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$[\underline{\underline{R}}_\theta]_{\mathcal{E}'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix},$$

$$[\underline{\underline{R}}_\psi]_{\mathcal{E}''} = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

2. Further we need to show that $[\underline{\underline{R}}]_{\mathcal{E}_0} = [\underline{\underline{R}}_\varphi(\hat{\mathbf{E}}_3)]_{\mathcal{E}_0}\,[\underline{\underline{R}}_\theta(\hat{\mathbf{e}}'_1)]_{\mathcal{E}'}\,[\underline{\underline{R}}_\psi(\hat{\mathbf{e}}''_3)]_{\mathcal{E}''}$, and compute $[\underline{\underline{R}}]_{\mathcal{E}_0}$.

3. We finally need to find a systematic procedure to get φ, θ and ψ for a given $[\underline{\underline{R}}]_{\mathcal{E}_0}$.

In evaluating the first part of the solution, we shall use the axis-angle formula for the rotation tensor given by

$$\underline{\underline{R}}(\hat{n}, \theta) = \underline{\underline{1}} + \sin\theta\,\underline{\underline{N}} + (1-\cos\theta)\,\underline{\underline{N}}^2\,,$$

where $\underline{\underline{N}} = asym(\hat{n})$ is the skew-symmetric tensor corresponding to the axis of rotation given by the unit vector $\hat{n}$. Accordingly, in any given coordinate system $\mathcal{C}$, the rotation tensor is

$$\left[\underline{\underline{R}}(\hat{n}, \theta)\right]_{\mathcal{C}} = \underline{\underline{1}} + \sin\theta\,\left[\underline{\underline{N}}\right]_{\mathcal{C}} + (1-\cos\theta)\,\left[\underline{\underline{N}}\right]^2_{\mathcal{C}}\,.$$

Now, $[\underline{\underline{R}}_\varphi]_{\mathcal{E}_0} = \left[\underline{\underline{R}}(\hat{E}_3, \varphi)\right]_{\mathcal{E}_0}$ and $\left[\hat{E}_3\right]_{\mathcal{E}_0} = [0, 0, 1]^T$ which results in

$$[\underline{\underline{N}}]_{\mathcal{E}_0} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad [\underline{\underline{N}}]^2_{\mathcal{E}_0} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$\text{Therefore, } [\underline{\underline{R}}_\varphi]_{\mathcal{E}_0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \sin\varphi \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + (1-\cos\varphi) \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Similarly, $[\underline{\underline{R}}_\theta]_{\mathcal{E}'} = \left[\underline{\underline{R}}(\hat{e_1}', \theta)\right]_{\mathcal{E}'}$ and $[\hat{e_1}']_{\mathcal{E}'} = [1,0,0]^T$ which results in

$$[\underline{\underline{N}}]_{\mathcal{E}'} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad [\underline{\underline{N}}]^2_{\mathcal{E}'} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

$$\text{Therefore, } [\underline{\underline{R}}_\theta]_{\mathcal{E}'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \sin\theta \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} + (1-\cos\theta) \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}.$$

Proceeding along the same line, one can show that

$$[\underline{\underline{R}}_{\psi}]_{\mathcal{E}''} = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

since $[\underline{\underline{R}}_\psi]_{\mathcal{E}''} = \left[\underline{\underline{R}}(\hat{e_3}'', \psi)\right]_{\mathcal{E}''}$ and $[\hat{e_3}'']_{\mathcal{E}''} = [0,0,1]^T$.

We next move on to show that

$$[\underline{\underline{R}}]_{\mathcal{E}_0} = [\underline{\underline{R}}_\varphi(\hat{\mathbf{E}}_3)]_{\mathcal{E}_0}\, [\underline{\underline{R}}_\theta(\hat{\mathbf{e}}_1')]_{\mathcal{E}'}\, [\mathsf{R}_\psi(\hat{\mathbf{e}}_3'')]_{\mathcal{E}''}.$$

We start with the fact that

$$[\underline{\underline{R}}]_{\mathcal{E}_0} = [\underline{\underline{R}}_\psi(\hat{\mathbf{e}}_3'', \psi)]_{\mathcal{E}_0} \cdot [\underline{\underline{R}}_\theta(\hat{\mathbf{e}}_1', \theta)]_{\mathcal{E}_0} \cdot [\underline{\underline{R}}_\varphi(\hat{\mathbf{E}}_3, \varphi)]_{\mathcal{E}_0}.$$

Next, to evaluate $[\underline{\underline{R}}_\theta(\hat{\mathbf{e}}_1', \theta)]_{\mathcal{E}_0}$, we first note from Fig. **??** that the coordinate system $\mathcal{E}'$ is obtained by the action of the rotation tensor $[\underline{\underline{R}}_\varphi(\hat{\mathbf{E}}_3, \varphi)]_{\mathcal{E}_0}$ on the coordinate system $\mathcal{E}_0$. Accordingly, from the coordinate transformation law for second order tensors, we have

$$[\underline{\underline{R}}_\theta(\hat{\mathbf{e}}_1', \theta)]_{\mathcal{E}_0} = [\underline{\underline{R}}_\varphi(\hat{\mathbf{E}}_3, \varphi)]_{\mathcal{E}_0} \cdot [\underline{\underline{R}}_\theta(\hat{\mathbf{e}}_1', \theta)]_{\mathcal{E}'} \cdot [\underline{\underline{R}}_\varphi(\hat{\mathbf{E}}_3, \varphi)]^T_{\mathcal{E}_0}.$$

Substituting in the formula for $[\underline{\underline{R}}]_{\mathcal{E}_0}$ above, we get

$$[\underline{\underline{R}}]_{\mathcal{E}_0} = [\underline{\underline{R}}_\psi(\hat{\mathbf{e}}_3'', \psi)]_{\mathcal{E}_0} \cdot [\underline{\underline{R}}_\varphi(\hat{\mathbf{E}}_3, \varphi)]_{\mathcal{E}_0} \cdot [\underline{\underline{R}}_\theta(\hat{\mathbf{e}}_1', \theta)]_{\mathcal{E}'}.$$

In the above, we have made use of the fact that $[\underline{\underline{R}}_\varphi(\hat{\mathbf{E}}_3, \varphi)]_{\mathcal{E}_0}$ is an orthogonal tensor and hence $[\underline{\underline{R}}_\varphi(\hat{\mathbf{E}}_3, \varphi)]^T_{\mathcal{E}_0} \cdot [\underline{\underline{R}}_\varphi(\hat{\mathbf{E}}_3, \varphi)]_{\mathcal{E}_0} = \underline{\underline{1}}$.
Similarly, we note from Fig. **??** that the coordinate system $\mathcal{E}''$ is obtained by the action of the rotation tensor $[\underline{\underline{R}}(\hat{\mathbf{e}}_1', \theta)]_{\mathcal{E}_0} \cdot [\underline{\underline{R}}_\varphi(\hat{\mathbf{E}}_3, \varphi)]_{\mathcal{E}_0} = [\underline{\underline{R}}_\varphi(\hat{\mathbf{E}}_3, \varphi)]_{\mathcal{E}_0} \cdot [\underline{\underline{R}}_\theta(\hat{\mathbf{e}}_1', \theta)]_{\mathcal{E}'}$ on the coordinate system $\mathcal{E}_0$. Accordingly, again application of the coordinate transformation formula for second order tensor yields

$$[\underline{\underline{R}}_\psi(\hat{\mathbf{e}}_3'', \psi)]_{\mathcal{E}_0} = [\underline{\underline{R}}_\varphi(\hat{\mathbf{E}}_3, \varphi)]_{\mathcal{E}_0} \cdot [\underline{\underline{R}}_\theta(\hat{\mathbf{e}}_1', \theta)]_{\mathcal{E}'} \cdot [\underline{\underline{R}}_\psi(\hat{\mathbf{e}}_3'', \psi)]_{\mathcal{E}''} \cdot [\underline{\underline{R}}_\theta(\hat{\mathbf{e}}_1', \theta)]^T_{\mathcal{E}'} \cdot [\underline{\underline{R}}_\varphi(\hat{\mathbf{E}}_3, \varphi)]^T_{\mathcal{E}_0}.$$

Substitution of this formula for $[\underline{\underline{R}}_\psi(\hat{\mathbf{e}}_3'', \psi)]_{\mathcal{E}_0}$ in the previous form for $[\underline{\underline{R}}]_{\mathcal{E}_0}$ above results in

$$[\underline{\underline{R}}]_{\mathcal{E}_0} = [\underline{\underline{R}}_\varphi(\hat{\mathbf{E}}_3)]_{\mathcal{E}_0}\, [\underline{\underline{R}}_\theta(\hat{\mathbf{e}}_1')]_{\mathcal{E}'}\, [\underline{\underline{R}}_\psi(\hat{\mathbf{e}}_3'')]_{\mathcal{E}''}.$$

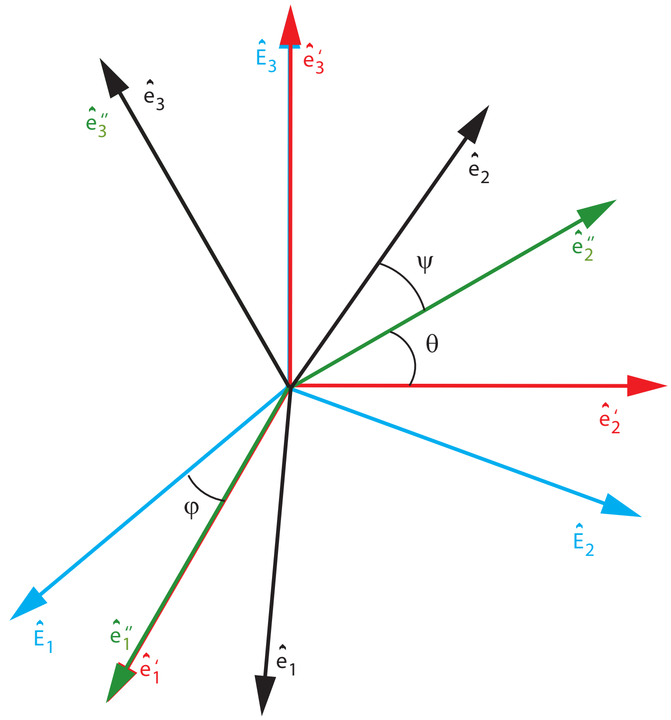

Figure 1: Various coordinate systems involved in $3-1-3$ Euler angle sequence for representing the rotation tensor.

In the above, we have again made use of the fact that $[\underline{\underline{R}}_{\varphi}(\hat{\mathbf{E}}_3, \varphi)]_{\mathcal{E}_0}$ and $[\underline{\underline{R}}_{\theta}(\hat{\mathrm{e}}'_1, \theta)]_{\mathcal{E}'}$ are orthogonal tensors.

Finally, substituting for $[\underline{\underline{R}}_{\varphi}(\hat{\mathbf{E}}_3)]_{\mathcal{E}_0}$, $[\underline{\underline{R}}_{\theta}(\hat{\mathrm{e}}'_1)]_{\mathcal{E}'}$ and $[\underline{\underline{R}}_{\psi}(\hat{\mathbf{e}}''_3)]_{\mathcal{E}''}$ from the solution for the first part, we compute

$$
\begin{aligned}
[\underline{\underline{R}}]_{\mathcal{E}_0} &= \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \cos\varphi\,\cos\psi - \sin\varphi\,\cos\theta\,\sin\psi & -\cos\varphi\,\sin\psi - \sin\varphi\,\cos\theta\,\cos\psi & \sin\varphi\,\sin\theta \\ \sin\varphi\,\cos\psi + \cos\varphi\,\cos\theta\,\sin\psi & -\sin\varphi\,\sin\psi + \cos\varphi\,\cos\theta\,\cos\psi & -\cos\varphi\,\sin\theta \\ \sin\theta\,\sin\psi & \sin\theta\,\cos\psi & \cos\theta \end{pmatrix} .
\end{aligned}
$$

Finally, to get a systematic procedure to extract the three Euler angles φ, θ and ψ from a given rotation tensor $[\underline{\underline{R}}]_{\mathcal{E}_0}$, we compare the various entries in the final matrix representation above to get

$$
\varphi = \tan^{-1}\left(\frac{R_{13}}{-R_{23}}\right) \qquad \theta = \cos^{-1} R_{33}\,, \qquad \psi = \tan^{-1}\left(\frac{R_{31}}{R_{32}}\right) .
$$