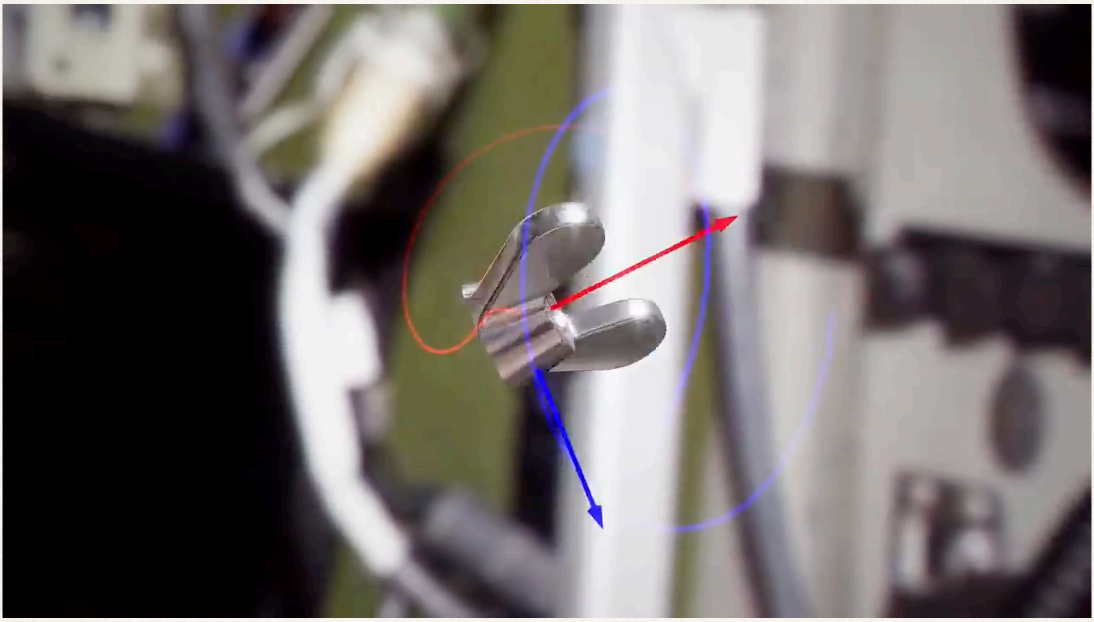
Lecture 6

Rigid body kinematics: Orientation; Axis-angle formula

18-24 August, 2021

Orientation

I. **Example**: Rigid body in space



II. **Question**: How to systematically relate two different orientation of a rigid body?

III. **Euler's theorem** says that *any* two orientation of a rigid body are related by a rotation tensor.

   1. $\Longleftrightarrow$ relating to BFCS' orientation (*Lec. 5*)

IV. **Question** reduces to "How to represent a rotation tensor?"

Axis-angle formula

I. **Lecture 4**: A rotation tensor R rotates objects about its *real* principal axis.

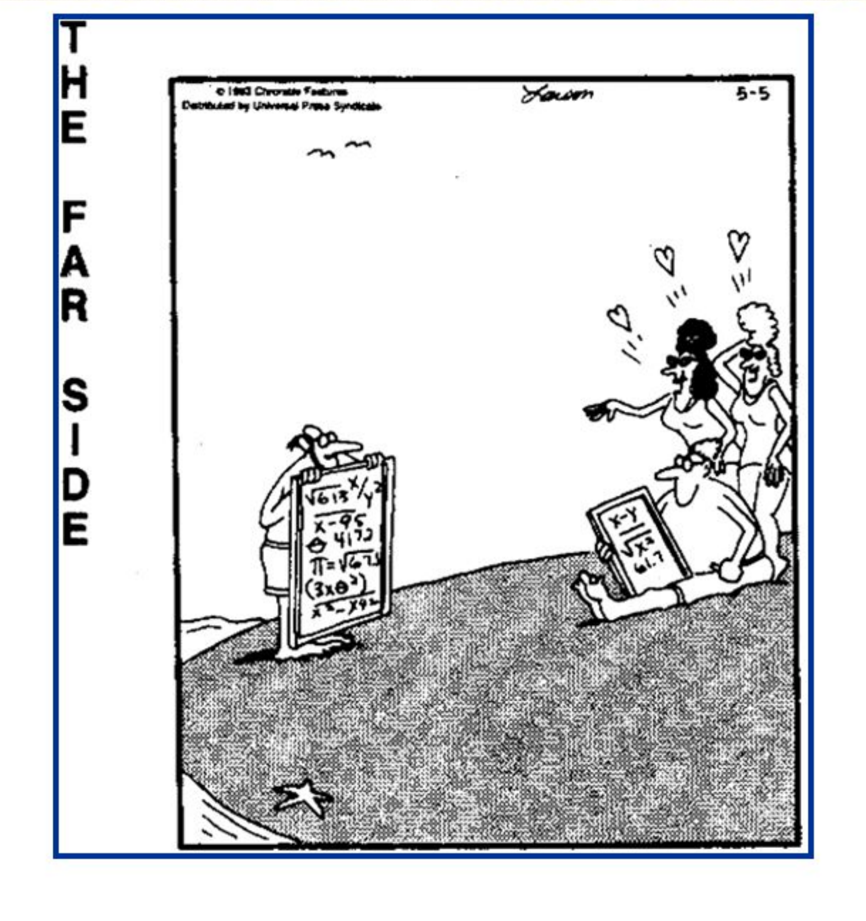
II. **Axis-angle representation**. For *any* R:

$$\mathsf{R} = \mathsf{R}\left(\hat{\mathbf{n}}, \theta\right) = \mathsf{1} + \sin\theta\,\mathsf{N} + (1 - \cos\theta)\,\mathsf{N}^2.$$

1. $\hat{\mathbf{n}}$ is the *real* principal vector of R.
2. $\mathsf{N} = \text{asym}\left(\hat{\mathbf{n}}\right) \Longleftrightarrow \mathsf{N}\cdot\mathbf{b} = \hat{\mathbf{n}}\times\mathbf{b}$ for all $\mathbf{b}$.
3. $\theta = \cos^{-1}[\{\text{tr}\left(\mathsf{R}\right) - 1\}/2]$
   i. $= \arctan\,\{\text{Im}(\lambda)/\text{Re}(\lambda)\}$, λ *complex* pr. val. of R

III. If $\mathbf{r} = \mathsf{R}\left(\hat{\mathbf{n}}, \theta\right)\cdot\mathbf{r}_0$, then

1. $\Delta\mathbf{r} = \mathbf{r} - \mathbf{r}_0 = \mathsf{T}\cdot\mathbf{r}_0$, where
   i. $\mathsf{T} = \sin\theta\,\mathsf{N} + (1 - \cos\theta)\,\mathsf{N}^2$
   ii. T — *rotator* associated with R.
2. $$\left[\mathsf{R}\left(\hat{\mathbf{n}}, \theta\right)\right]_{\hat{\mathbf{i}}-\hat{\mathbf{j}}-\hat{\mathbf{n}}} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$



The axis-angle formula
will make you irresistible.