ESO209: Tutorial 2

(Week: 4 Aug 2021. Based on Lecture 2 and 3)

1. Prove the following identities. Note how formulae (a)-(c) “tensorize” vector operations
   (a) $\mathbf{a} \cdot \mathbf{b} = \mathrm{tr}\,(\mathbf{a} \otimes \mathbf{b})$
   (b) $\mathbf{a} \times \mathbf{b} = -\,2\,\mathrm{ax}\{\mathrm{sk}\,(\mathbf{a} \otimes \mathbf{b})\}$; the operation ‘ax’ was defined in lectures. The operation ‘sk’ computes the skew-symmetric tensor associated with a tensor A by the formula $\mathrm{sk}(\mathsf{A}) = (\mathsf{A} - \mathsf{A}^T)/2$; easy to check that $\mathrm{sk}(\mathsf{A})$ is skew-symmetric.
   (c) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathrm{tr}\ (\mathbf{a} \otimes \mathbf{c})\,\mathbf{b} - \mathrm{tr}\ (\mathbf{a} \otimes \mathbf{b})\,\mathbf{c}$
   (d) $\mathbf{a} \cdot (\mathsf{A} \cdot \mathbf{b}) = (\mathsf{A}^T \cdot \mathbf{a}) \cdot \mathbf{b}$; this equality is often used as a definition of A^T.
   (e) $\mathbf{a} \otimes (\mathsf{A} \cdot \mathbf{b}) = (\mathbf{a} \otimes \mathbf{b}) \cdot \mathsf{A}^T$;

2. Find the principal values λ_i and principal vectors $\hat{\mathbf{v}}_i$ of the second order tensor $\mathsf{D} = 6(\hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_1) + \hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_2 + 9(\hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_3) + 2(\hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_2) + 2(\hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_3) + \hat{\mathbf{e}}_3 \otimes \hat{\mathbf{e}}_3$. Confirm that $\hat{\mathbf{v}}_i$ are independent, so that we can define the principal CS $\mathscr{P}$ of D. Is $\mathscr{P}$ Cartesian? Express D in terms of the unit tensorial basis by $\hat{\mathbf{v}}_i \otimes \hat{\mathbf{v}}_j$ in $\mathscr{P}$ and also find $[\mathsf{D}]_{\mathscr{P}}$.

3. Let in some CS $\{\mathscr{B}, G, \hat{\mathbf{e}}_i\}$ there be a vector $\mathbf{r} = x_1\hat{\mathbf{e}}_1 + x_2\hat{\mathbf{e}}_2 + x_3\hat{\mathbf{e}}_3$ and a second order tensor $\mathsf{I} = 6(\hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_1) + 2(\hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_2) + 2(\hat{\mathbf{e}}_3 \otimes \hat{\mathbf{e}}_3)$. Show that $\mathbf{r} \cdot \mathsf{I} \cdot \mathbf{r} = 1$ represents an ellipsoid, and find the lengths of it semi-major axes.

4. Determine the skew-symmetric part of D given in Problem 2 using the formula for $\mathrm{sk}(\mathsf{D})$ given in problem 1(b). Then its axial vector (also called *dual* vector).

5. Find the principal values λ_i and real principal vectors $\hat{\mathbf{v}}_i$ of the second order tensor $\mathsf{W} = 2(\hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_2) - 2(\hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_1) - 4(\hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_3) + 4(\hat{\mathbf{e}}_3 \otimes \hat{\mathbf{e}}_1) - 4(\hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_3) + 4(\hat{\mathbf{e}}_3 \otimes \hat{\mathbf{e}}_2)$. How many principal values are real? Also investigate if all the principal vectors are orthogonal to each other. Find the axial vector $\mathbf{w}$ of W. How does $\mathbf{w}$ relate to W’s principal vector(s)? *Hint:* Try crossing/dotting the principal vectors with $\mathbf{w}$.

6. Find the principal values λ_i and principal vectors $\hat{\mathbf{v}}_i$ of the second order tensor $\mathsf{T} = \hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_1 + 2(\hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_3) + \hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_2 + 2(\hat{\mathbf{e}}_3 \otimes \hat{\mathbf{e}}_1) - 2(\hat{\mathbf{e}}_3 \otimes \hat{\mathbf{e}}_3)$. Comment on the nature of the principal values and vectors. Confirm that $\hat{\mathbf{v}}_i$ are independent, so that we can define a CS $\mathscr{P}$. Is $\mathscr{P}$ Cartesian? Express T in terms of the unit tensorial basis by $\hat{\mathbf{v}}_i \otimes \hat{\mathbf{v}}_j$ in $\mathscr{P}$ and also find $[\mathsf{T}]_{\mathscr{P}}$.