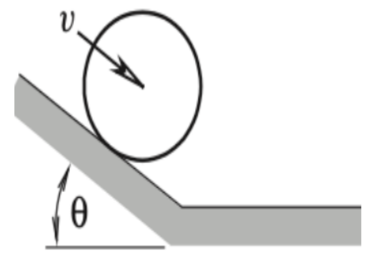
Problem 3:

(3) The sphere rolls without slipping down the incline. At the instant it contacts the ground, its speed was v_0. The coefficient of restitution is ε. Derive expressions for the velocity of its center and its angular velocity at the instant



Solution: In Problem statement it says that we have to obtain velocity of center of sphere and angular velocity of sphere at the instant. Sphere is rolling down the incline and will impact to the base and then it will rebound. We know velocity of center of sphere just before impact and that is v_0. And we will find velocity of center of sphere and angular velocity of sphere just after impact.

Let velocity of center of sphere and angular velocity of sphere as $\underline{v}_1^C$ and $\underline{\omega}_1^C$ just before impact and just after the impact they are $\underline{v}_2^C$ and $\underline{\omega}_2^C$.

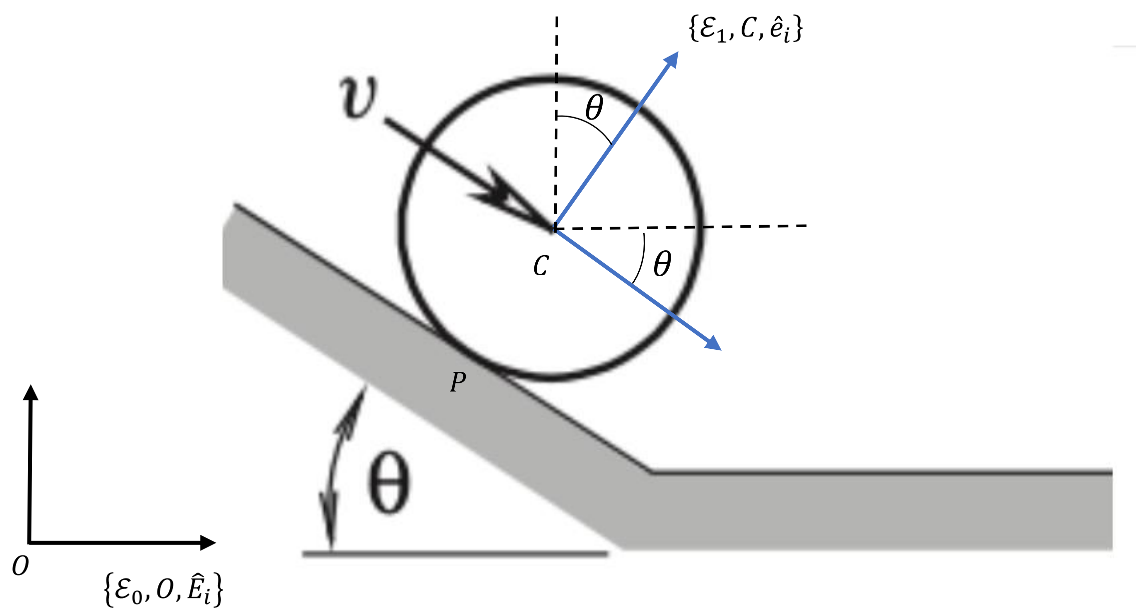


Fig. 1

As we already know $\underline{v}_1^C = v_0\hat{e}_1$. From fig. 1, we can write, $\hat{e}_1 = \cos\theta\hat{E}_1 - \sin\theta\hat{E}_2$ and $\hat{e}_2 = \sin\theta\,\hat{E}_1 + \cos\theta\,\hat{E}_2$.

Therefore, $\underline{v}_1^C = v_0\left(\cos\theta\hat{E}_1 - \sin\theta\hat{E}_2\right)$. To obtain $\underline{\omega}_1^C$, writing velocity of point C, while sitting on sphere

$$\underline{v}_1^C = \underline{v}_1^P + \underline{\omega}_1^C \times \underline{r}_{C/P}, \quad \dots\dots(1)$$

where, point 'P' is the contact point of sphere and base. $\underline{v}_1^{P_1} = \underline{v}_1^{P_2} = 0$ (P_1 on sphere and P_2 on base), $\underline{\omega}_1^C = \omega_1^C \hat{E}_3$ and $\underline{r}_{C/P} = r\hat{e}_2$, where '$r$' is radius of sphere. From Eq. (1)

$$\omega_1^C = -\frac{v_0}{r}.$$

Now we will write the conservation of angular momentum about point C

$$\underline{h}_C^{sys}(t - \delta t) = \underline{h}_C^{sys}(t + \delta t), \quad \dots\dots(2)$$

where

$$\underline{h}_C^{sys}(t - \delta t) = \underline{\underline{I}}_C \cdot \underline{\omega}_1^C + \underline{r}_{C/C} \times m_{sphere}\underline{v}_1^C, \quad \dots\dots(3)$$

and,

$$\underline{h}_C^{sys}(t + \delta t) = \underline{\underline{I}}_C \cdot \underline{\omega}_2^C + \underline{r}_{C/C} \times m_{sphere}\underline{v}_2^C, \quad \dots\dots(4)$$

from Eq. (2), (3) and (4), we will obtain

$$\underline{\omega}_1^C = \underline{\omega}_2^C = -\frac{v_0}{r}\hat{E}_3.$$

FBD of sphere:

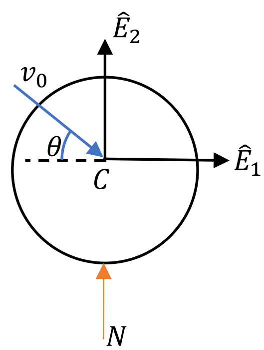


Fig. 2

We begin with the idealization that the contacting surfaces are very smooth, so the impact force on sphere acts in the $\hat{E}_2$ positive direction, whereas it acts on base in the negative $\hat{E}_2$ direction. We are assuming that only a normal impulse at the point of contact with the ground exits therefore from definition of coefficient of restitution, we have

$$\epsilon = -\frac{\underline{v}_2^C \cdot \hat{E}_2}{\underline{v}_1^C \cdot \hat{E}_2} = \frac{\underline{v}_2^C \cdot \hat{E}_2}{v_o \sin\theta},$$

This gives: $\underline{v}_2^C \cdot \hat{E}_2 = \epsilon v_o \sin\theta$. (normal component of velocity)

To obtain the $\underline{v}_2^C \cdot \hat{E}_1$, writing conservation of linear momentum and taking its projection in $\hat{E}_1$ direction and let mass of sphere as m.

$$m\underline{v}_2^C \cdot \hat{E}_1 = mv_o \cos\theta,$$

this gives: $$\underline{v}_2^C \cdot \hat{E}_1 = v_o \cos\theta.$$

Thus, velocity of the center of sphere

$$\underline{v}_2^C = v_o\left(\cos\theta \hat{E}_1 + \epsilon \sin\theta \hat{E}_2\right).$$