4. Show that

   (a) $\epsilon_{ijk}\epsilon_{jki} = 6$,

   (b) $\epsilon_{ijk}A_jA_k = 0$ for all i, and

   (c) $\epsilon_{ilm}\epsilon_{jlm} = 2\delta_{ij}$.

4(a): $\varepsilon_{ijk}\,\varepsilon_{jki} = \varepsilon_{ijk}\,\varepsilon_{ijk}$

$$= \underline{\underline{\delta_{jj}}}\,\underline{\underline{\delta_{kk}}} - \delta_{jk}\,\delta_{kj}$$

$$= (\delta_{11} + \delta_{22} + \delta_{33})\,\delta_{kk} - \delta_{jj}$$

$$= 3 \times 3 - 3$$

$$= 9 - 3$$

$$= 6$$

4(b): $\varepsilon_{ijk} A_j A_k = 0 \quad \forall\, i$

Long expansion

$$\underbrace{\varepsilon_{i\underline{\underline{11}}}}_{0} A_1 A_1 + \varepsilon_{i12} A_1 A_2 + \varepsilon_{i13} A_1 A_3$$

$$+ \varepsilon_{i21} A_2 A_1 + \underbrace{\varepsilon_{i\underline{\underline{22}}}}_{0} A_2 A_2 + \varepsilon_{i23} A_2 A_3$$

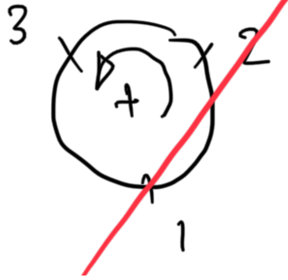
$$+ \varepsilon_{i31} A_3 A_1 + \varepsilon_{i32} A_3 A_2 + \underbrace{\varepsilon_{i\underline{\underline{33}}}}_{0} A_3 A_3$$

$$= \underline{\varepsilon_{312} A_1 A_2} + \underline{\varepsilon_{213} A_1 A_3} + \underline{\varepsilon_{321} A_2 A_1}$$

$$+ \varepsilon_{123} A_2 A_3 + \underline{\varepsilon_{231} A_3 A_1} + \varepsilon_{132} A_3 A_2$$

$$= \cancel{A_1 A_2} - \cancel{A_2 A_1} - \cancel{A_1 A_3} + \cancel{A_3 A_1}$$

$$+ \cancel{A_2 A_3} - \cancel{A_3 A_2}$$



$$= 0$$

The work in red loop is <u>WRONG</u>.
We continue from the previous step.
We write the expression for $i = 1, 2,$ and 3 as follows:

<u>$i = 1$</u>

$$\cancelto{0}{\varepsilon_{112}} A_1 A_2 + \cancelto{0}{\varepsilon_{113}} A_1 A_3 + \cancelto{0}{\varepsilon_{121}} A_2 A_1 + \varepsilon_{123} A_2 A_3 + \cancelto{0}{\varepsilon_{131}} A_3 A_1 + \varepsilon_{132} A_3 A_2$$

$$= \varepsilon_{123} A_2 A_3 + \varepsilon_{132} A_3 A_2$$

$$= A_2 A_3 - A_3 A_2$$

$$= 0$$

<u>$i = 2$</u>

$$\cancelto{0}{\varepsilon_{212}} A_1 A_2 + \varepsilon_{213} A_1 A_3 + \cancelto{0}{\varepsilon_{221}} A_2 A_1 + \cancelto{0}{\varepsilon_{223}} A_2 A_3 + \varepsilon_{231} A_3 A_1 + \cancelto{0}{\varepsilon_{232}} A_3 A_2$$

$$= \varepsilon_{213} A_1 A_3 + \varepsilon_{231} A_3 A_1$$

$$= - A_1 A_3 + A_3 A_1$$

$$= 0$$

$i = 3$

$$\varepsilon_{312} A_1 A_2 + \cancel{\varepsilon_{313} A_1 A_3} + \varepsilon_{321} A_2 A_1 + \cancel{\varepsilon_{323} A_2 A_3}$$
$$+ \underbrace{\cancel{\varepsilon_{331}}}_{0} A_3 A_1 + \underbrace{\cancel{\varepsilon_{332}}}_{0} A_3 A_2$$

$$= \varepsilon_{312} A_1 A_2 + \varepsilon_{321} A_2 A_1$$

$$= A_1 A_2 - A_2 A_1$$

$$= 0$$

Thus we have shown that $\forall i$
$\varepsilon_{ijk} A_j A_k = 0$.

Second method

$$\underline{A} \times \underline{B} = \varepsilon_{ijk} A_j B_k \hat{e}_i$$

$\therefore (\underline{A} \times \underline{B})_i = \varepsilon_{ijk} A_j B_k$ ——— (1)

further, $\underline{A} \times \underline{B} = |A||B| \sin\theta \, \hat{n}$, where $\hat{n}$ is a unit vector $\hat{n} = n_1 \hat{e}_1 + n_2 \hat{e}_2 + n_3 \hat{e}_3$

$\therefore (\underline{A} \times \underline{B})_i = |A||B| \sin\theta \, n_i$ ——— (2)

for $\underline{B} = \underline{A}$, $\theta = 0^\circ$

$\therefore$ from (1) and (2) $\varepsilon_{ijk} A_j A_k = 0 \ \forall i$

4 (c) : $\varepsilon_{ilm} \varepsilon_{jlm} = 2\delta_{ij}$

L.H.S. $\varepsilon_{ilm}\varepsilon_{jlm} = \varepsilon_{mil}\varepsilon_{mjl}$

$$= \delta_{ij}\delta_{ll} - \delta_{il}\delta_{lj}$$

$$= 3\delta_{ij} - \delta_{ij}$$

$$= 2\delta_{ij}$$