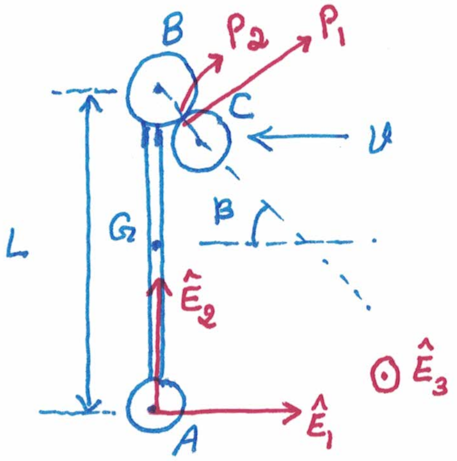
TUTORIAL-12, PROBLEM-4



radius — R

mass — m

coefficient of restitution — ε

Post-collision,

$$\underline{V}^C = v_{c1}\hat{E}_1 + v_{c2}\hat{E}_2$$

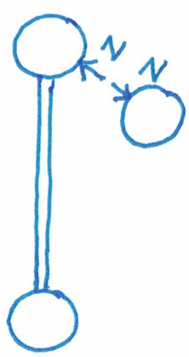
2-D problem

$$\underline{V}^G = v_{G1}\hat{E}_1 + v_{G2}\hat{E}_2$$

$$\underline{\omega}_f = \omega\hat{E}_3 \quad (\text{about } G)$$

- No external impulses
- Weight is bounded, hence not considered

FBD



Pre-collision,

$$\underline{U}^C = -v\hat{E}_1$$

$$\underline{U}^G = 0$$

LMB,

$$\Sigma \underline{I}_f^{ext} = \underline{p}_{sys}(t+\delta t) - \underline{p}_{sys}(t-\delta t) = 0$$

$$m\underline{V}^C + 2m\underline{V}^G = -mv\hat{E}_1 \quad \text{------} \ (1)$$

two eqs.

$AMB_{/G}$ (about G),

$$\Sigma \underline{I}_m^{G,ext} = \underline{h}_{sys}(t+\delta t) - \underline{h}_{sys}(t-\delta t) = 0$$

$$\underbrace{m\,\underline{r}^{C/G} \times (-v\,\hat{E}_1)}_{\{\underline{h}_{sys}(t-\delta t)\}} = \underbrace{m\,\underline{r}^{C/G} \times \underline{V}^{C} + \underline{\underline{I}}^{G} \cdot \underline{w}_f}_{\{\underline{h}_{sys}(t+\delta t)\}}$$

$$\underline{r}^{C/G} = 2R\cos\beta\,\hat{E}_1 + \left(\frac{L}{2} - 2R\sin\beta\right)\hat{E}_2 \quad \text{------- (2)}$$

AMB$_{/G}$,

$$m\,\underline{r}^{C/G} \times (-v\,\hat{E}_1) = m\,\underline{r}^{C/G} \times \underline{V}^{C} + m\left(R^2 + \frac{L^2}{2}\right)^{1} \omega\,\hat{E}_3 \quad \text{------- (3)}$$

one eq.

* 5 unknowns, 3 equations

Now, using the impact model,

(Note, $\hat{n} = -\cos\beta\,\hat{E}_1 + \sin\beta\,\hat{E}_2$)

$$\left(\underline{V}^{P_2} - \underline{V}^{P_1}\right)\cdot\hat{n} = -\varepsilon\left(0 + v\,\hat{E}_1\right)\cdot\hat{n}$$

Note that before collision,

$$\underline{v}^{P_2} = 0 \quad \text{and} \quad \underline{v}^{P_1} = \cancel{\underline{V}^{C}}\ \underline{v}^{C} = -v\,\hat{E}_1$$

After collision,

$$\underline{V}^{P_1} = \underline{V}^{C} \quad \text{and} \quad \underline{V}^{P_2} = \underline{V}^{G} + \underline{w}_f \times \underline{r}^{P_2/G}$$

Rewriting the impact model,

$$\left(\underline{V}^{G} + \underline{w}_f \times \underline{r}^{P_2/G} - \underline{V}^{c}\right)\cdot\hat{n} \qquad \text{one eq}$$

$$= -\varepsilon\left(v\hat{E}_1\right)\cdot\hat{n} \qquad \text{--- (4)}$$

Simplifying equations (1) to (4), (scalar form)

$$(1)\left|\begin{aligned} \cancel{m}v_{c_1} + 2\cancel{m}v_{G1} &= -\cancel{m}v \\ \cancel{m}v_{c2} + 2\cancel{m}v_{G2} &= 0 \end{aligned}\right. \qquad (5)$$

$$(3)\left|\begin{aligned} \cancel{m}v\left(\frac{L}{2} - 2R\sin\beta\right) &= \cancel{m}\left(R^2 + \frac{L^2}{2}\right)\omega + \ldots \\ &\ldots + \cancel{m}\left(2R\cos\beta\,.\,v_{c2} - \left(\frac{L}{2} - 2R\sin\beta\right)v_{c1}\right) \end{aligned}\right. \qquad (6)$$

$$(4)\left|\begin{aligned} -v_{G1}\cos\beta + v_{G2}\sin\beta &+ \left(\underline{w}_f \times \underline{r}^{P_2/G}\right)\cdot\hat{n} \\ + v_{c1}\cos\beta - v_{c2}\sin\beta &= \varepsilon v\cos\beta \end{aligned}\right.$$

$$\text{Now,}\quad \underline{r}^{P_2/G} = R\cos\beta\,\hat{E}_1 + \left(\frac{L}{2} - R\sin\beta\right)\hat{E}_2$$

$$\underline{w}_f \times \underline{r}^{P_2/G} = \omega R\cos\beta\,\hat{E}_2 - \omega\left(\frac{L}{2} - R\sin\beta\right)\hat{E}_1$$

$$\left(\underline{w}_f \times \underline{r}^{P_2/G}\right)\cdot\hat{n} = \omega R\cancel{\sin\beta}\cos\beta + \omega\left(\frac{L}{2} - \cancel{R\sin\beta}\right)\cos\beta$$

$$= \frac{1}{2}\omega L\cos\beta$$

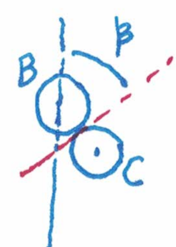
Rewriting Eq. 4,

$$(v_{C_1} - v_{G_1})\cos\beta + (v_{G_2} - v_{C_2})\sin\beta + \frac{\omega L}{2}\cos\beta = \varepsilon u \cos\beta \qquad (7)$$

From equations 5, 6, 7, we have four equations and five unknowns, i.e. $v_{G_1}, v_{G_2}, v_{C_1}, v_{C_2}, \omega$.

we need a fifth equation to solve this problem, and we get that by noting that there are no impulsive forces in a direction $\perp^{ar}$ to BC. We will call this direction, $\hat{t}$, since it is tangent to the disks B and C at the time of impact, at P_2 and P_1, respectively.

$$\hat{t} = \sin\beta\, \hat{E}_1 + \cos\beta\, \hat{E}_2$$



No impulse along $\hat{t}$; component of the velocity of C along $\hat{t}$ will remain the ~~same~~ same before and after the collision,

$$\underline{V}^c \cdot \hat{t} = \underline{U}^c \cdot \hat{t}$$

$$v_{c1} \sin\beta + v_{c2} \cos\beta = -v \sin\beta \quad \text{------} \; (8)$$

Solving the five equations from (5), (6), (7) & (8),

$$v_{c2} = -2 v_{G2}$$
$$v_{c1} = -v - 2 v_{G1} \quad \text{------} \; (9)$$

From above, substituting for v_{G1} & v_{G2} in equation (7),

$$\left[v_{c1} + \frac{1}{2}(v_{c1} + v)\right] \cos\beta - \left(\frac{v_{c2}}{2} + v_{c2}\right) \sin\beta + \frac{WL}{2} \cos\beta = \varepsilon v \cos\beta$$

$$\frac{3 v_{c1}}{2} \cos\beta - \frac{3 v_{c2}}{2} \sin\beta + \frac{WL}{2} \cos\beta = \left(\varepsilon - \frac{1}{2}\right) v \cos\beta \quad \text{------} \; (10)$$

From (8),

$$v_{c1} = -v - \left(\frac{v_{c2} \cos\beta}{\sin\beta}\right)$$

$$v_{c1} = -v - v_{c2} \cot\beta \quad \text{------} \; (11)$$

From (11), substituting for v_{c1} in (6) & (10),

$$v\left(\cancel{\frac{L}{2} - 2R\sin\beta}\right) = \left(R^2 + \frac{L^2}{2}\right)\omega + 2R\cos\beta \cdot v_{c2}$$

$$+ \left(\frac{L}{2} - 2R\sin\beta\right)\left(\cancel{v} + v_{c2}\cot\beta\right)$$

$$\Rightarrow \quad 0 = \left(R^2 + \frac{L^2}{2}\right)\omega + \cancel{2R\cos\beta \cdot v_{c2}}$$

~~$+ \frac{L}{2} - 2R\sin\beta$~~

$$+ \left(\frac{L}{2}\cot\beta - 2R\cancel{\cos\beta}\right) v_{c2}$$

$$\Rightarrow \quad v_{c2} = -\left(R^2 + \frac{L^2}{2}\right)\omega \cdot \frac{2}{L}\tan\beta$$

$$\Rightarrow \quad v_{c2} = -\left(2R^2 + L^2\right)\omega L \tan\beta \quad \text{------} \quad (12)$$

$$-\frac{3}{2}\cos\beta\left(v + v_{c2}\cot\beta\right) - \frac{3v_{c2}}{2}\sin\beta$$

$$+ \frac{\omega L}{2}\cos\beta = \left(\varepsilon - \frac{1}{2}\right) v\cos\beta$$

$$\Rightarrow -v\cos\beta - \frac{3v_{c2}}{2}\left(\frac{\cos^2\beta}{\sin\beta} + \sin\beta\right)$$

$$+ \frac{\omega L}{2}\cos\beta = \varepsilon v\cos\beta$$

$\Rightarrow$

$\Rightarrow$

$$+\frac{3 v_{c2}}{2\sin\beta} = \left((\varepsilon+1)v + \frac{\omega L}{2}\right)\cos\beta$$

$$\Rightarrow \quad v_{c2} = \left(\frac{\omega L - 2v(\varepsilon+1)}{3}\right)\sin\beta\cos\beta \quad \text{------- (13)}$$

From equations (12) & (13),

$$-\left(2R^2+L^2\right)\omega L \cancel{\tan\beta} = \left(\frac{\omega L - 2v(\varepsilon+1)}{3}\right)\cancel{\sin\beta}(\cos\beta)^2$$

$$\omega L\left(\frac{1}{3}\cos^2\beta + 2R^2+L^2\right) = \frac{2v}{3}\cos^2\beta(\varepsilon+1)$$

$$\omega = \frac{2v\cos^2\beta.(\varepsilon+1)}{L(\cos^2\beta + 6R^2 + 3L^2)}$$

From (13),

(substituting for ω)

$$v_{c2} = \left(\frac{\frac{2v\cos^2\beta.(\varepsilon+1)}{\cos^2\beta+6R^2+3L^2} - 2v(\varepsilon+1)}{3}\right)\frac{\sin 2\beta}{2}$$

$$\Rightarrow \quad v_{c2} = \frac{-\cancel{2}v(\varepsilon+1)\left(6R^2+3L^2\right)}{\cos^2\beta+6R^2+3L^2}\cdot\frac{\sin 2\beta}{\cancel{6}\,3}$$

$$\Rightarrow \quad v_{c2} = \frac{-v(\varepsilon+1)\left(2R^2+L^2\right)\sin 2\beta}{\cos^2\beta+6R^2+3L^2}$$

$$\Rightarrow \quad v_{c2} = \frac{-v(\varepsilon+1)(2R^2+L^2)\sin 2\beta}{\cos^2\beta + 6R^2 + 3L^2}$$

From 11,

$$v_{c1} = -v + \frac{v(\varepsilon+1)(2R^2+L^2)\,\cancel{\sin 2\beta \cdot \cot\beta}\ 2\cos^2\beta}{\cos^2\beta + 6R^2 + 3L^2}$$

$$\Rightarrow \quad = \frac{-v\cos^2\beta - v(6R^2+3L^2) + 2v(\varepsilon+1)(2R^2+L^2)\cos^2\beta}{\cos^2\beta + 6R^2 + 3L^2}$$

$$\Rightarrow \quad = -v \cdot \frac{\left[\cos^2\beta + 3(2R^2+L^2) - 2\cancel{v}(\varepsilon+1)(2R^2+L^2)\cos^2\beta\right]}{\cos^2\beta + 6R^2 + 3L^2}$$

$$\Rightarrow \quad \cancel{v_{c1} = -v \cdot \left[\frac{3(2R^2+L^2) - 2(\varepsilon+1)(2R^2+L^2)\cos^2\beta + \cos^2\beta}{\cos^2\beta + 6R^2 + 3L^2}\right]}$$

$$\Rightarrow \quad v_{c1} = -v \cdot \left[\frac{(\cos^2\beta + 6R^2 + 3L^2) - 2(\varepsilon+1)(2R^2+L^2)\cos^2\beta}{\cos^2\beta + 6R^2 + 3L^2}\right]$$

From equation 9,

$$v_{G2} = -\frac{v_{c2}}{2}$$

$$v_{G1} = -\left(\frac{v_{c1} + v}{2}\right)$$

The quantities that are to be collected in the question are,

$$\underline{V}^{C} = v_{c1}\hat{E}_1 + v_{c2}\hat{E}_2 \checkmark$$

(we have this)

$$\underline{V}^{B} = \underline{V}^{G} + \underline{\omega}_f \times \underline{r}^{B/G}$$

$$= v_{G1}\hat{E}_1 + v_{G2}\hat{E}_2 + \omega\hat{E}_3 \times \frac{L}{2}\hat{E}_2$$

$$= \left(v_{G1} - \frac{\omega L}{2}\right)\hat{E}_1 + v_{G2}\hat{E}_2$$

$$= \left(-v\left(\frac{v_{c1}}{v} + 1\right)\frac{1}{2} - \frac{\omega L}{2}\right)\hat{E}_1$$

$$- \frac{v_{c2}}{2}\hat{E}_2$$

(Let us write $\cos^2\beta + 6R^2 + 3L^2 \equiv$ "gamma"

and $(\varepsilon + 1)(2R^2 + L^2) \equiv$ "alpha")

So,

$$\underline{V}^{B} = \frac{v}{\cancel{2}}\left(\cancel{1} - \frac{\cancel{2} * alpha * \cos^2\beta}{gamma} \cancel{-1}\right)\hat{E}_1$$

$$- \frac{v \cos^2\beta\,(\varepsilon+1)}{gamma}\hat{E}_1$$

$$+ \frac{v}{2}\cdot\frac{alpha * \sin 2\beta}{gamma}\hat{E}_2$$

$$\Rightarrow \underline{V}^{B} = \frac{-v(\varepsilon+1)(2R^2+L^2)\cos^2\beta - v\cos^2\beta(\varepsilon+1)}{gamma}\hat{E}_1$$

$$+ \frac{v}{2}\,\frac{alpha * \sin 2\beta}{gamma}\hat{E}_2$$

$$\Rightarrow \underline{V}^{B} = \left[\frac{-v(\varepsilon+1)\cos^2\beta\,(2R^2+L^2+1)}{\cos^2\beta + 6R^2 + 3L^2}\right]\hat{E}_1$$

$$+ \left[\frac{v(\varepsilon+1)(2R^2+L^2)\sin 2\beta}{2(\cos^2\beta + 6R^2 + 3L^2)}\right]\hat{E}_2$$

Similarly, finding $\underline{V}^{A}$,

$$\underline{V}^A = \underline{V}^G + \underline{\omega}_f \times \underline{r}^{A/G}$$

$$= v_{G1}\hat{E}_1 + v_{G2}\hat{E}_2 + \omega \hat{E}_3 \times \left(-\frac{L}{2}\hat{E}_2\right)$$

$$= \left(v_{G1} + \frac{\omega L}{2}\right)\hat{E}_1 + v_{G2}\hat{E}_2$$

$\Rightarrow$

$$\underline{V}^A = \left[\frac{-v(\varepsilon+1)\cos^2\beta\,(2R^2+L^2-1)}{\cos^2\beta + 6R^2 + 3L^2}\right]\hat{E}_1$$

$$+ \frac{v(\varepsilon+1)(2R^2+L^2)\sin 2\beta}{2(\cos^2\beta + 6R^2 + 3L^2)}\hat{E}_2$$

Looking at the special case of $\cancel{\beta=0}$ $\beta = 0$ and $R \to 0$,

$$\underline{\omega}_f = \frac{2v(\varepsilon+1)}{L(1+3L^2)}\hat{E}_3$$

$$\underline{V}^C = -v\left(\frac{(1+3L^2) - 2(\varepsilon+1)L^2}{1+3L^2}\right)\hat{E}_1$$

$$+ 0\hat{E}_2$$

$\Rightarrow$

$$\underline{V}^{C} = -v\left[1 - \frac{2(\varepsilon+1)L^2}{1+3L^2}\right]\hat{E}_1$$

$$\underline{V}^{G} = -\frac{1}{2}\dot{v}\left[\cancel{-1} + \frac{2(\varepsilon+1)L^2}{1+3L^2}\cancel{+1}\right]\hat{E}_1$$

$$= -\frac{v(\varepsilon+1)L^2}{1+3L^2}\cdot\hat{E}_1$$

$$\underline{V}^{B} = \frac{-v(\varepsilon+1)(1+L^2)}{1+3L^2}\hat{E}_1$$

$$\underline{V}^{A} = \frac{-v(\varepsilon+1)(L^2-1)}{1+3L^2}\hat{E}_1$$