Problem 6:

6. Show that in any CS $\{\mathscr{E}, \hat{\mathbf{e}}_i\}$, the axis of rotation $\hat{\mathbf{n}}$ of a rotation tensor $\mathsf{R}(\hat{\mathbf{n}}, \theta)$ may be obtained from the formula

$$\hat{\mathbf{n}} = -\epsilon_{ijk}\frac{R_{jk}}{2\sin\theta}\hat{\mathbf{e}}_i.$$

The advantage of this formula is that it fixes the *correct* direction of $\hat{\mathbf{n}}$ for the particular choice of θ.

Solution:

Axis-angle formula (see lecture 6 for proof),

$$\mathrm{R} = \mathrm{R}(\widehat{\mathbf{n}}, \theta) = 1 + \sin\theta\mathrm{N} + (1-\cos\theta)\mathrm{N}^2$$

where

$\widehat{\mathbf{n}}$ = real principal vector of R

N = asym ($\widehat{\mathbf{n}}$)$= -\epsilon_{ijk}\widehat{\mathbf{n}}_{\mathbf{i}}(\hat{\mathbf{e}}_{\mathbf{j}} \otimes \hat{\mathbf{e}}_{\mathbf{k}})$, $(\widehat{\mathbf{n}} \times \mathbf{r_0}) = \mathrm{N}.\mathbf{r_0}$

$\theta = \cos^{-1}\left(\dfrac{\mathrm{tr}(\mathrm{R})-1}{2}\right)$

we know (from lecture 6)

$\mathbf{r} = \mathbf{r_0} + (\cos\theta - 1)\mathbf{r_0}_{\perp} + \sin\theta(\widehat{\mathbf{n}} \times \mathbf{r_0})$

since $\mathbf{r_0}_{\perp} = -\widehat{\mathbf{n}} \times (\widehat{\mathbf{n}} \times \mathbf{r_0})$

therefore

$\mathbf{r} = \mathbf{r_0} - (\cos\theta - 1)\big(\widehat{\mathbf{n}} \times (\widehat{\mathbf{n}} \times \mathbf{r_0})\big) + \sin\theta(\widehat{\mathbf{n}} \times \mathbf{r_0})$ .... (1)

use two identities: $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a}.\mathbf{c})\mathbf{b} - (\mathbf{a}.\mathbf{b})\mathbf{c}$ and $(\mathbf{a} \otimes \mathbf{b})\mathbf{c} = \mathbf{a}(\mathbf{b}.\mathbf{c})$

now

$$\big(\widehat{\mathbf{n}} \times (\widehat{\mathbf{n}} \times \mathbf{r_0})\big) = (\widehat{\mathbf{n}} \otimes \widehat{\mathbf{n}} - \mathrm{I})\mathbf{r_0}$$

then

$\mathbf{r} = \mathbf{r_0} - (\cos\theta - 1)(\widehat{\mathbf{n}} \otimes \widehat{\mathbf{n}} - \mathrm{I})\mathbf{r_0} + \sin\theta(\widehat{\mathbf{n}} \times \mathbf{r_0})$

$\mathbf{r} = [\mathrm{I} + (1-\cos\theta)(\widehat{\mathbf{n}} \otimes \widehat{\mathbf{n}} - \mathrm{I}) + \sin\theta\mathrm{N}]\mathbf{r_0}$

$\mathbf{r} = \mathrm{R}.\mathbf{r_0}$

$\mathrm{R} = \mathrm{I} + (1-\cos\theta)(\widehat{\mathbf{n}} \otimes \widehat{\mathbf{n}} - \mathrm{I}) + \sin\theta\mathrm{N}$

writing R_{jk}

$$\mathrm{R}_{jk}\left(\hat{\mathbf{e}}_{\mathbf{j}}\otimes\hat{\mathbf{e}}_{\mathbf{k}}\right)=\delta_{jk}\left(\hat{\mathbf{e}}_{\mathbf{j}}\otimes\hat{\mathbf{e}}_{\mathbf{k}}\right)+(1-\cos\theta)\left(\mathrm{n}_j n_k-\delta_{jk}\right)\left(\hat{\mathbf{e}}_{\mathbf{j}}\otimes\hat{\mathbf{e}}_{\mathbf{k}}\right)$$
$$-\sin\theta\epsilon_{ijk}n_i\left(\hat{\mathbf{e}}_{\mathbf{j}}\otimes\hat{\mathbf{e}}_{\mathbf{k}}\right) \qquad \ldots\ldots(2)$$

consider $\left(\hat{\mathbf{e}}_{\mathbf{j}}\otimes\hat{\mathbf{e}}_{\mathbf{k}}\right)$ component and multiply eq.2 by ϵ_{jkm}

$$\mathrm{R}_{jk}\epsilon_{jkm}=\delta_{jk}\epsilon_{jkm}+(1-\cos\theta)\left(\mathrm{n}_j n_k-\delta_{jk}\right)\epsilon_{jkm}-\sin\theta\epsilon_{ijk}\epsilon_{jkm}n_i$$

now

$\delta_{jk}\epsilon_{jkm}=0$; $\mathrm{n}_j n_k\epsilon_{jkm}=0$; for non-zero $j\neq k$

$\epsilon_{ijk}\epsilon_{jkm}=-\epsilon_{jik}\epsilon_{jkm}=-\delta_{ik}\delta_{km}+\delta_{im}\delta_{kk}=-\delta_{im}+3\delta_{im}=2\delta_{im}$

again

$$\mathrm{R}_{jk}\epsilon_{jkm}=-2\sin\theta\delta_{im}n_i=-2\sin\theta n_m$$

$$n_m=-\frac{\mathrm{R}_{jk}\epsilon_{jkm}}{2\sin\theta}$$

change $m\to i$

$$n_i=-\frac{\mathrm{R}_{jk}\epsilon_{jki}}{2\sin\theta}=-\frac{\epsilon_{ijk}\mathrm{R}_{jk}}{2\sin\theta}$$