Tutorial 4, Problem 2

From Problem 1, we have

$$\hat{n} = \frac{1}{\sqrt{2}}\hat{e}_1 + \frac{1}{\sqrt{2}}\hat{e}_3$$

$$\theta = 180^\circ$$

Finding the matrix of rotation tensor $\underline{\underline{R}}$ using axis-angle formula,

$$\underline{\underline{R}} = \underline{\underline{R}}(\hat{n},\theta) = 1 + \sin\theta\,\underline{\underline{N}} + (1-\cos\theta)\,\underline{\underline{N}}^2$$

where $\underline{\underline{N}} = \text{asym}(\hat{n})$

where $\hat{n} = n_i\,\hat{e}_i$

or $\hat{n} = n_i\,\hat{E}_i$

~~In matrix form~~

Now $\hat{n} = \frac{1}{\sqrt{2}}\hat{e}_1 + \frac{1}{\sqrt{2}}\hat{e}_3$ has been expressed in the initial body-fixed cs $\{\mathcal{B}, 0, \hat{e}_i\}$ which is aligned with the CS $\{\mathcal{E}_1, 0, \hat{E}_i\}$.
(TUTORIAL-3, PROBLEM-6)

So we can also write,

$$\hat{n} = \frac{1}{\sqrt{2}}\hat{E}_1 + \frac{1}{\sqrt{2}}\hat{E}_3 \quad \left(\text{in CS } \{\mathcal{E}_1, 0, \hat{E}_i\}\right)$$

Now expressing the tensor $\underline{\underline{N}}$, in the same $\mathcal{E}_1$ CS, in matrix form,

$$[\underline{\underline{N}}]_{\mathcal{E}_1} = \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 0 \end{bmatrix}$$

$$[\underline{\underline{N}}^2]_{\mathcal{E}_1} = \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -n_3^2 - n_2^2 & n_1 n_2 & n_3 n_1 \\ n_1 n_2 & -n_1^2 - n_3^2 & n_2 n_3 \\ n_3 n_1 & n_2 n_3 & -n_2^2 - n_1^2 \end{bmatrix}$$

$$= \begin{bmatrix} -1/2 & 0 & 1/2 \\ 0 & -1 & 0 \\ 1/2 & 0 & -1/2 \end{bmatrix}$$

Now, using the axis-angle formula,

$$[\underline{\underline{R}}]_{\mathcal{E}_1} = [\underline{\underline{I}}]_{\mathcal{E}_1} + \underbrace{\sin 180^\circ}_{0} \cdot [\underline{\underline{N}}]_{\mathcal{E}_1} + (1 - \cos 180^\circ) [\underline{\underline{N}}^2]_{\mathcal{E}_1}$$

Thus, $$[\underline{\underline{R}}]_{\mathcal{E}_1} = [\underline{\underline{I}}]_{\mathcal{E}_1} + 2.\begin{bmatrix} -1/2 & 0 & 1/2 \\ 0 & -1 & 0 \\ 1/2 & 0 & -1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

$$= \underline{\underline{\begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}}}$$

This is the same rotation tensor $\underline{\underline{R}}$ that we found out in Problem 6 of Tutorial 3 (as expected). Hence verified.