# Lecture 13

*Rigid body kinetics*: Rigid body motion; Kinetic quantities; Moment of inertia tensor.

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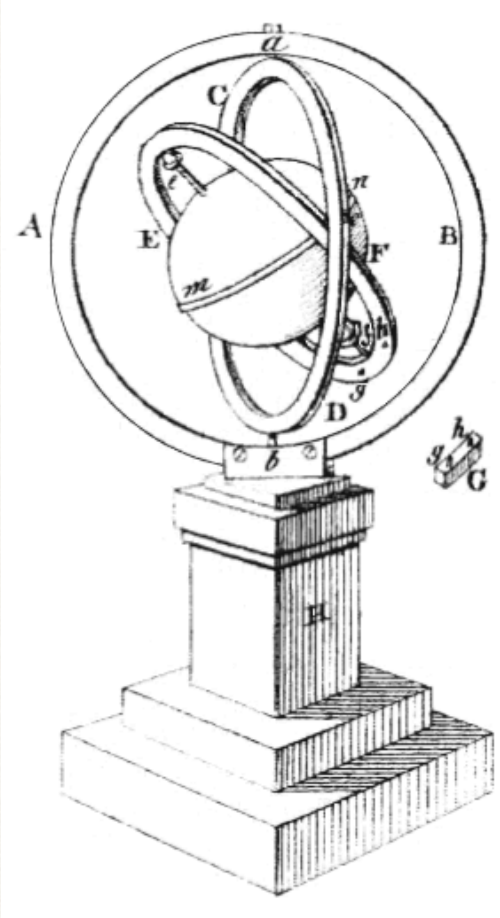
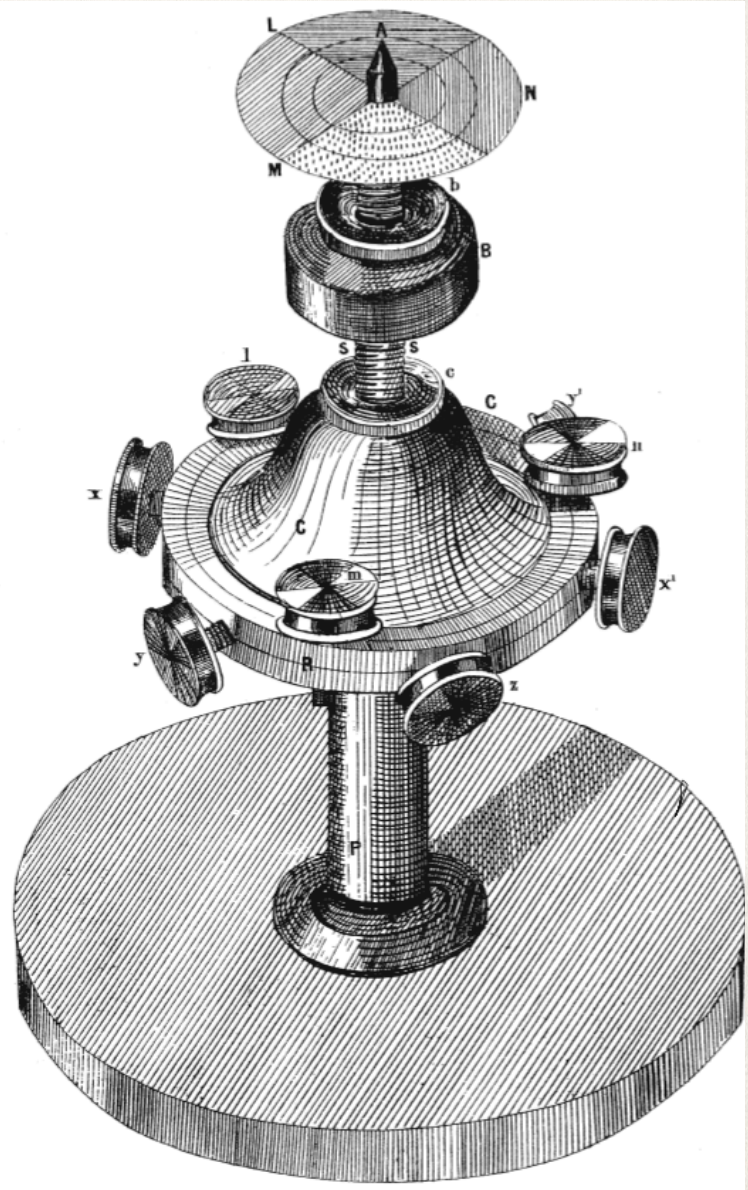
*22-28 September, 2021*

# Rigid body motion




*Bohenenberger's* machine to show Earth's rotation.




*Maxwell's* top: Visualises path of instantaneous rotation axis. Further reading ; Video link .

# Rigid body motion

I. **General rigid body motion**.

1. A rigid body can *translate* and *rotate*.
2. To track *translation*: Follow location of a point on the rigid body.
3. To track *rotation*: Generally use an Euler angle sequence.

II. To find location of a point needs its velocity as function of time. To find rigid body's orientation need its angular velocity as a function of time.

III. To find velocity and angular velocity integrate, respectively, acceleration and angular acceleration.

IV. Accelerations and angular accelerations depend on applied forces and moments through the **laws of motion**.

# Rigid body: Kinetic quantities

**Laws of motion** are in terms of the following :

I. *Mass*: $m = \int_V \rho(\mathbf{r}) \mathrm{d}V.$

II. *Center of mass*: $\mathbf{r}^{G/O} = \frac{1}{m} \int_V \rho(\mathbf{r})\, \mathbf{r}\, \mathrm{d}V\,.$

III. *Linear momentum*: $\mathbf{p} = \int_V \rho(\mathbf{r}) \mathbf{v}(\mathbf{r}) \mathrm{d}V = m\mathbf{v}^G.$

IV. *Angular momentum* about a point $P$ :

$$\mathbf{h}^P = \int_V \mathbf{r}^{/P} \rho(\mathbf{r}) \mathbf{v}(\mathbf{r}) \mathrm{d}V = \mathbf{r}^{G/P} \times m\mathbf{v}^G + \underbrace{\mathbf{I}^G \cdot \boldsymbol{\omega}^{\mathscr{B}}}_{\mathbf{h}^G},$$

where, $\quad \mathbf{I}^G = \int_V \rho(\mathbf{r}) \left( \left| \mathbf{r}^{/G} \right|^2 \mathbb{1} - \mathbf{r}^{/G} \otimes \mathbf{r}^{/G} \right) \mathrm{d}V$

is the *Moment of Inertia* about G of the body.

V. *Kinetic energy*: $E_K = \frac{1}{2} \int_V \rho(\mathbf{r}) \left| \mathbf{v}(\mathbf{r}) \right|^2 \mathrm{d}V$

$$= \frac{1}{2} m \left| \mathbf{v}^G \right|^2 + \frac{1}{2} \omega^{\mathscr{B}} \cdot \mathbf{I}^G \cdot \boldsymbol{\omega}^{\mathscr{B}}$$

# Moment of Inertia

I. **Definition**. *Moment of Inertia Tensor* <u>about</u> a point $P$ of a rigid body is given by

$$\mathbf{I}^P = \int_V \rho(\mathbf{r})\left(\left|\mathbf{r}^{/P}\right|^2 \mathbb{1} - \mathbf{r}^{/P} \otimes \mathbf{r}^{/P}\right) \mathrm{d}V .$$

1. *Symmetric, positive definite* tensor.
2. Three *real, positive* principal values.
3. Three *orthogonal* principal axes which define the rigid body's ***principal CS***.
4. Moment of inertia changes with $P$.
5. Estimates mass distribution *w.r.t.* $P$ .

II. In CS $\{\mathscr{E}, P, \hat{\mathbf{E}}_i\}$, let $\mathbf{I}^P = I^P_{ij}\hat{\mathbf{E}}_i \otimes \hat{\mathbf{E}}_j$ , then

$$I^P_{11} = \int_V \rho(\mathbf{r})\left(X_2^2 + X_3^2\right)\mathrm{d}V,\ I^P_{22} = \int_V \rho(\mathbf{r})\left(X_3^2 + X_1^2\right)\mathrm{d}V$$

$$I^P_{33} = \int_V \rho(\mathbf{r})\left(X_2^2 + X_3^2\right)\mathrm{d}V,\ I^P_{12} = I^P_{21} = -\int_V \rho(\mathbf{r})X_1X_2\mathrm{d}V,$$

$$I^P_{23} = I^P_{32} = -\int_V \rho(\mathbf{r})X_2X_3\mathrm{d}V,\ I^P_{31} = I^P_{13} = -\int_V \rho(\mathbf{r})X_3X_1\mathrm{d}V .$$

# Moment of Inertia

I. **Parallel axes theorem**. Let $CM$ be at $G$:

$$\mathrm{I}^P = \mathrm{I}^G + m\left(\left|\mathbf{r}^{G/P}\right|^2 1 - \mathbf{r}^{G/P} \otimes \mathbf{r}^{G/P}\right).$$

II. In CS $\{\mathscr{E}, P, \hat{\mathbf{E}}_i\}$, let $\mathbf{r}^{P/G} = d\hat{\mathbf{n}}^{GP} = dn_i^{GP}\hat{\mathbf{E}}_i$ :

$$I_{kk}^P = I_{kk}^G + md^2\, n_k^{GP}\, n_k^{GP} \quad \text{(no sum on } k\text{)},$$

$$I_{ij}^P = I_{ij}^G - md^2\, n_i^{GP}\, n_j^{GP} \quad (i \neq j)\,.$$

III. **Application**. If a rigid body has a *fixed point* $C$ , *i.e.* $\mathbf{v}^C = \mathbf{0}$, then

$$\mathbf{h}^C = \mathrm{I}^C \cdot \boldsymbol{\omega}^{\mathscr{B}}.$$

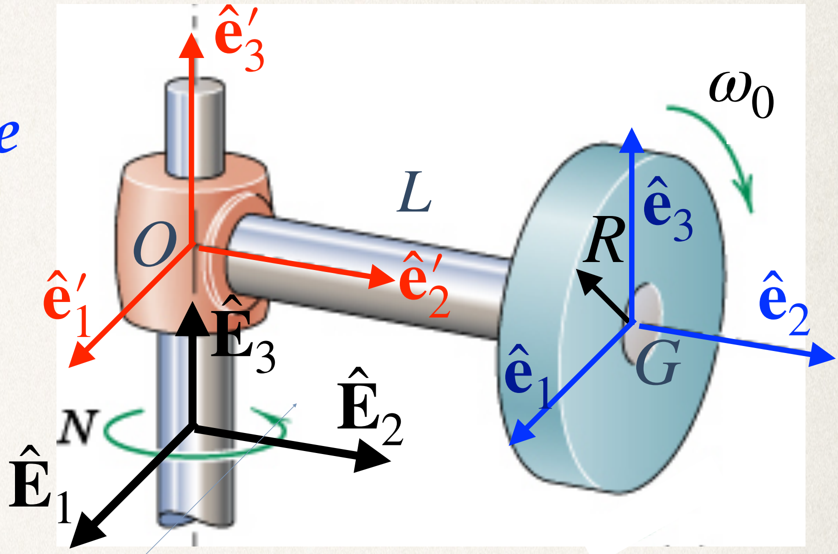
IV. **Perpendicular axes theorem**. If body is ***planar*** with normal $\hat{\mathbf{n}}$, then

$$\mathrm{I}^G = I_n^G\, \hat{\mathbf{n}} \otimes \hat{\mathbf{n}} + I_{a_1}^G \hat{\mathbf{a}}_1 \otimes \hat{\mathbf{a}}_1 + I_{a_1}^G \hat{\mathbf{a}}_2 \otimes \hat{\mathbf{a}}_2\,,$$

with $I_n^G = I_{a_1}^G + I_{a_2}^G$, and $\left\{I_n^G, \hat{\mathbf{n}}\right\}$ and $\left\{I_{a_i}^G, \hat{\mathbf{a}}_i\right\}$ being the *principal pairs* of $\mathrm{I}^G$.

# Example 1

***Find the angular momentum of the system about G and O. The arm OG is massless.***




I. $\{\mathscr{E}_0, \hat{\mathbf{E}}_i\} \xrightarrow{\mathsf{R}(\hat{\mathbf{E}}_3, \varphi_N)} \{\mathscr{E}', \hat{\mathbf{e}}_i'\} \xrightarrow{\mathsf{R}(\hat{\mathbf{e}}_2', \varphi_{\omega_0})} \{\mathscr{E}, \hat{\mathbf{e}}_i\}$

II. $\mathbf{h}^G_{sys} = \mathbf{h}^G_{disk} + \cancelto{0}{\mathbf{h}^G_{arm}} = \mathrm{I}^G_{disk} \cdot \boldsymbol{\omega}^{disk}$

III. $\left[\mathrm{I}^G_{disk}\right]_{\mathscr{E}} = \dfrac{mR^2}{4}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

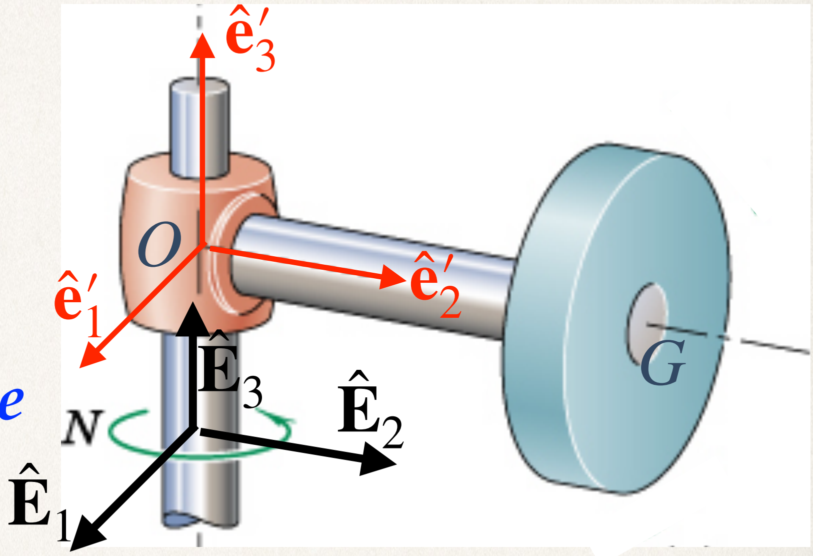
IV. $\mathbf{h}^G_{sys} = mR^2\left(-2\omega_0\hat{\mathbf{E}}_2 + N\hat{\mathbf{E}}_3\right)/4$

V. $\mathbf{h}^O_{disk} = \mathbf{r}^{G/O} \times m\mathbf{v}^G + \mathbf{h}^G_{disk}$

VI. $\mathbf{h}^O_{sys} = -\dfrac{m\omega_0 R^2}{2}\hat{\mathbf{E}}_2 + \dfrac{mN\left(R^2 + 4L^2\right)}{4}\hat{\mathbf{E}}_3$

# Example 2

*The disc is now welded to the massless arm. Find the angular momentum of the system about G and O.*




I. $\{\mathscr{E}_0, \hat{\mathbf{E}}_i\} \xrightarrow{\mathsf{R}(\hat{\mathbf{E}}_3, \varphi_N)} \{\mathscr{E}', \hat{\mathbf{e}}_i'\}$

II. $\mathbf{h}_{sys}^G = \mathrm{I}_{sys}^G \cdot \boldsymbol{\omega}^{sys} = mR^2N\hat{\mathbf{E}}_3/4$

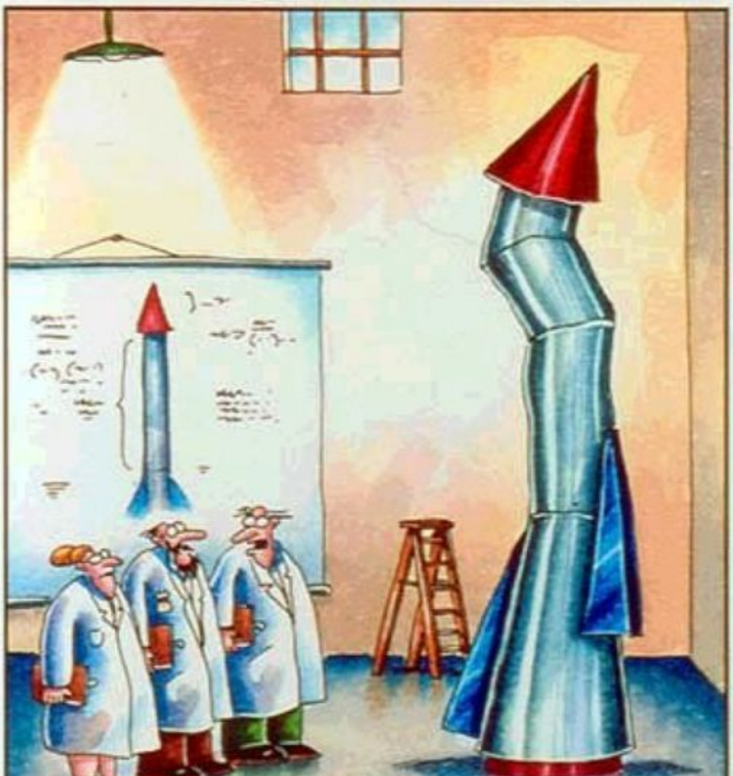
III. $\mathbf{h}_{sys}^O = \mathrm{I}_{sys}^O \cdot \boldsymbol{\omega}^{sys}$

IV. $\mathrm{I}^O = \mathrm{I}^G + m\left(\left|\mathbf{r}^{G/O}\right|^2 \mathsf{1} - \mathbf{r}^{G/O} \otimes \mathbf{r}^{G/O}\right)$

$$\left[\mathrm{I}_{disk}^O\right]_{\mathscr{E}'} = \frac{m}{4}\begin{pmatrix} R^2+4L^2 & 0 & 0 \\ 0 & 2R^2 & 0 \\ 0 & 0 & R^2+4L^2 \end{pmatrix}$$

V. $\mathbf{h}_{sys}^O = \dfrac{mN\left(R^2+4L^2\right)}{4}\hat{\mathbf{E}}_3$

THE FAR SIDE® by GARY LARSON



"It's time we face reality, my friends...
We're not exactly rocket scientists."