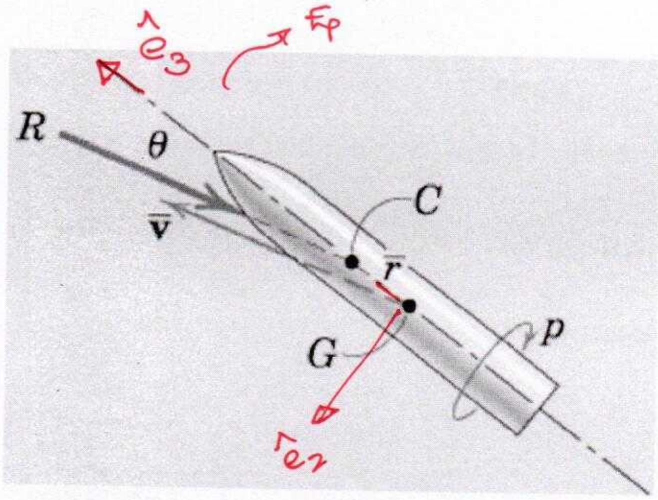
ES0209 - Tutorial 12

Problem 2

To find spin 'p' such that $\dot{\theta} = 0$.



Defining a BFCS $\{F_p, G, \hat{e}_i\}$ such that $\hat{e}_3$ is along symmetry axis. Assuming that at this instant F_p is aligned with F_{p_0}, the observer CS $\{F_p, G, \hat{E}_i\}$

KINEMATICS

Choosing the 3-1-3 Euler angle sequence, the Kinematics of the motion of the rocket can be defined as per equations developed in Lecture 18, viz.

if $\underline{\omega}^B = \dot{\varphi}\hat{E}_3 + \dot{\theta}\hat{e}_1' + \dot{\psi}\hat{e}_3$ which can then be written as

$$\omega_1 = \dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi$$

$$\omega_2 = \dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi$$

$$\omega_3 = \dot{\varphi}\cos\theta + \dot{\psi}$$

and the angular accelerations are

$$\alpha_1 = \frac{d}{dt}(\dot{\omega}_1) = \frac{d}{dt}(\dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi)$$

$$\alpha_1 = \ddot{\varphi}\sin\theta\sin\psi + \dot{\varphi}\dot{\theta}\cos\theta\sin\psi + \dot{\varphi}\sin\theta\dot{\psi}\cos\psi + \ddot{\theta}\cos\psi + \dot{\theta}\dot{\psi}(-\sin\psi)$$

$|||^{ly}$

$$\alpha_2 = \ddot{\varphi}\sin\theta\cos\psi + \dot{\varphi}\dot{\theta}\cos\theta\cos\psi - \dot{\varphi}\dot{\psi}\sin\theta\sin\psi - \ddot{\theta}\sin\psi - \dot{\theta}\dot{\psi}\cos\psi$$

$$\alpha_3 = \ddot{\varphi}\cos\theta - \dot{\varphi}\dot{\theta}\sin\theta + \ddot{\psi}$$

Since we wish to find the stabilized solution,
$\ddot{\varphi} = 0$, $\ddot{\psi} = 0$, $\ddot{\theta} = 0$, given that $\dot{\theta} = 0$.
Moreover since ψ is the spin angle (about $\hat{e}_3$) the symmetry axis, $\psi = 0$ when BFCS and observer CS are aligned (or ψ does not affect the solution).

Applying the above values, we get,

$$\omega_1 = 0 \quad (\because \sin\psi = 0 \ , \ \dot{\theta} = 0)$$

$$\omega_2 = \dot{\varphi}\sin\theta \quad (\because \cos\psi = 1 \ , \ \dot{\theta} = 0)$$

$$\omega_3 = \dot{\varphi}\cos\theta + \dot{\psi}$$

and

$$\dot{\omega}_1 = \alpha_1 = \dot{\varphi}\dot{\psi}\sin\theta$$

$$\dot{\omega}_2 = \alpha_2 = 0$$

$$\dot{\omega}_3 = \alpha_3 = 0$$

Now lets look at the kinetics.

Using the Euler's equations

$$M_1 = I_1\dot{\omega}_1 - (I_2 - I_3)\,\omega_2\omega_3$$

$$M_2 = I_2\dot{\omega}_2 - (I_3 - I_1)\,\omega_3\omega_1$$

$$M_3 = I_3\dot{\omega}_3 - (I_1 - I_2)\,\omega_1\omega_2$$

The applied moments are

$$M_1 = R\bar{r}\sin\theta$$

$$M_2 = 0$$

$$M_3 = 0$$

Thus the Euler eqns give (recognizing $I_1 = I_2 = I_0$ & $I_3 = I$)

$\underline{M_1 \text{ eqn}}$ $\quad R\bar{r}\sin\theta = I_0\dot{\varphi}\dot{\psi}\sin\theta - (I_0 - I)\dot{\varphi}\sin\theta(\dot{\varphi}\cos\theta + \dot{\psi})$

$\underline{M_2 \text{ Eqn}}$ $\quad 0 = I_0(0) - (I - I_0)(\dot{\varphi}\cos\theta + \dot{\psi})(0) = 0$

$\Rightarrow$ identically satisfied ✓

$\underline{M_3 \text{ eqn}}$ $\quad 0 = I(0) - (I_0 - I_0)(0)(\dot{\varphi}\sin\theta) = 0$

$\Rightarrow$ identically satisfied. ✓

The M_1 eqn gives.

$$R\bar{r}\cancel{\sin\theta} = I_0\dot{\varphi}\dot{\psi}\cancel{\sin\theta} - (I_0 - I)(\dot{\varphi}^2\cancel{\sin\theta}\cos\theta + \dot{\varphi}\dot{\psi}\cancel{\sin\theta})$$

$$\Rightarrow \quad R\bar{r} = I_0\dot{\varphi}\dot{\psi} + (I - I_0)\dot{\varphi}^2\cos\theta + (I - I_0)\dot{\varphi}\dot{\psi}$$

$$= I\dot{\varphi}\dot{\psi} + (I - I_0)\dot{\varphi}^2\cos\theta$$

$$\Rightarrow \quad [(I - I_0)\cos\theta]\dot{\varphi}^2 + [I\dot{\psi}]\dot{\varphi} - R\bar{r} = 0$$

Note. $\dot{\psi}$ is the spin rate 'p' & $\dot{\varphi}$ = precession rate = s

$$\Rightarrow \quad [(I - I_0)\cos\theta]s^2 + [Ip]s - R\bar{r} = 0$$

We thus have a quadratic equation for the precession rate in terms of the spin rate, 'p'

$$s = \dot{\varphi} = \frac{-Ip \pm \sqrt{(Ip)^2 - 4[(I-I_0)\cos\theta][-R\bar{r}]}}{2(I-I_0)\cos\theta}$$

For a feasible solution, term within the square root should be ≥ 0, ie,

$$(Ip)^2 - 4\left[(I-I_0)\cos\theta\right]\left[-R\bar{r}\right] \geq 0$$

Noting $I_0 > I$ (for the rocket), the condition for 'p' is

$$\boxed{p \geq \frac{2}{I}\sqrt{(I_0-I)R\bar{r}\cos\theta}}$$