Tutorial - 3

1. For any two vectors **a** and **b**, prove the following properties for an orthogonal tensor Q:
   a. $(Q \cdot \mathbf{a}) \cdot (Q \cdot \mathbf{b}) = \mathbf{a} \cdot \mathbf{b}$.
   b. $(Q \cdot \mathbf{a}) \times (Q \cdot \mathbf{b}) = \lambda Q \cdot (\mathbf{a} \times \mathbf{b})$ with λ being the real principal value of Q.

a. $(\mathrm{Q} \cdot \mathbf{a}) \cdot (\mathrm{Q} \cdot \mathbf{b}) = \mathbf{a} \cdot \mathbf{b}$.

Consider LHS

$$= (\mathrm{Q} \cdot \mathbf{a}) \cdot (\mathrm{Q} \cdot \mathbf{b})$$

using $\mathbf{a} \cdot (\mathrm{A} \cdot \mathbf{b}) = \left(\mathrm{A}^{\mathrm{T}} \cdot \mathbf{a}\right) \cdot \mathbf{b}$

$$= \left(\mathrm{Q} \cdot \mathrm{Q}^{\mathrm{T}}\right) \cdot (\mathbf{a} \cdot \mathbf{b})$$

$$= \mathrm{I} \cdot (\mathbf{a} \cdot \mathbf{b})$$

$$= (\mathbf{a} \cdot \mathbf{b})$$

b. $(\mathrm{Q} \cdot \mathbf{a}) \times (\mathrm{Q} \cdot \mathbf{b}) = \lambda \mathrm{Q} \cdot (\mathbf{a} \times \mathbf{b})$ with λ being the real principal value of Q.

consider $\mathbf{a} \times \mathbf{b} = \mathbf{c}$

using $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

now

$$(\mathrm{Q} \cdot \mathbf{c}) \times \{(\mathrm{Q} \cdot \mathbf{a}) \times (\mathrm{Q} \cdot \mathbf{b})\} = \{(\mathrm{Q} \cdot \mathbf{c}) \cdot (\mathrm{Q} \cdot \mathbf{b})\}(\mathrm{Q} \cdot \mathbf{a}) - \{(\mathrm{Q} \cdot \mathbf{c}) \cdot (\mathrm{Q} \cdot \mathbf{a})\}(\mathrm{Q} \cdot \mathbf{b})$$

$$= (\mathbf{c} \cdot \mathbf{b})(\mathrm{Q} \cdot \mathbf{a}) - (\mathbf{c} \cdot \mathbf{a})(\mathrm{Q} \cdot \mathbf{b})$$

since $\mathbf{c}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ therefore $(\mathbf{c} \cdot \mathbf{b}) = (\mathbf{c} \cdot \mathbf{a}) = 0$

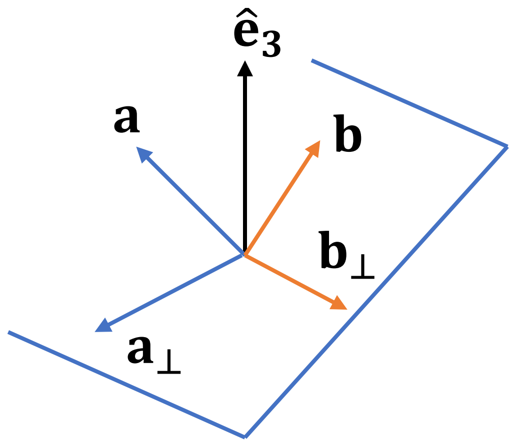
$$(\mathrm{Q} \cdot \mathbf{c}) \times \{(\mathrm{Q} \cdot \mathbf{a}) \times (\mathrm{Q} \cdot \mathbf{b})\} = \mathbf{0}$$

This means $(\mathrm{Q} \cdot \mathbf{c})$ is parallel to $(\mathrm{Q} \cdot \mathbf{a}) \times (\mathrm{Q} \cdot \mathbf{b})$, therefore

Let's assume $(\mathrm{Q} \cdot \mathbf{a}) \times (\mathrm{Q} \cdot \mathbf{b}) = \alpha\, (\mathrm{Q} \cdot \mathbf{c})$ .... (1)

now we must find α

We can write $\mathbf{a} = a_3\hat{\mathbf{e}}_3 + \mathbf{a}_\perp$ and $\mathbf{b} = b_3\hat{\mathbf{e}}_3 + \mathbf{b}_\perp$



we know $\mathrm{Q}\cdot\hat{\mathbf{e}}_3 = \lambda_3\hat{\mathbf{e}}_3$

$$\mathrm{Q}\cdot\mathbf{a} = \lambda_3 a_3\hat{\mathbf{e}}_3 + \mathrm{Q}\cdot\mathbf{a}_\perp$$

$$\mathrm{Q}\cdot\mathbf{b} = \lambda_3 b_3\hat{\mathbf{e}}_3 + \mathrm{Q}\cdot\mathbf{b}_\perp$$

consider LHS

$$\{(\mathrm{Q}\cdot\mathbf{a})\times(\mathrm{Q}\cdot\mathbf{b})\} = (\lambda_3 a_3\hat{\mathbf{e}}_3 + \mathrm{Q}\cdot\mathbf{a}_\perp)\times(\lambda_3 b_3\hat{\mathbf{e}}_3 + \mathrm{Q}\cdot\mathbf{b}_\perp)$$

$$= \lambda_3 a_3\hat{\mathbf{e}}_3 \times \lambda_3 b_3\hat{\mathbf{e}}_3 + \lambda_3 a_3\hat{\mathbf{e}}_3 \times \mathrm{Q}\cdot\mathbf{b}_\perp + \mathrm{Q}\cdot\mathbf{a}_\perp\times\lambda_3 b_3\hat{\mathbf{e}}_3$$
$$+\mathrm{Q}\cdot\mathbf{a}_\perp\times\mathrm{Q}\cdot\mathbf{b}_\perp$$

projecting LHS along $\hat{\mathbf{e}}_3$

$$\hat{\mathbf{e}}_3\cdot\{(\mathrm{Q}\cdot\mathbf{a})\times(\mathrm{Q}\cdot\mathbf{b})\}$$
$$= \hat{\mathbf{e}}_3\cdot\{\lambda_3 a_3\hat{\mathbf{e}}_3 \times \lambda_3 b_3\hat{\mathbf{e}}_3 + \lambda_3 a_3\hat{\mathbf{e}}_3 \times \mathrm{Q}\cdot\mathbf{b}_\perp + \mathrm{Q}\cdot\mathbf{a}_\perp\times\lambda_3 b_3\hat{\mathbf{e}}_3 + \mathrm{Q}\cdot\mathbf{a}_\perp\times\mathrm{Q}\cdot\mathbf{b}_\perp\}$$

use $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{b} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$ for simplifying last equation (some vector products are zero) and you will be left with only

$$= \hat{\mathbf{e}}_{\mathbf{3}} \cdot (\mathrm{Q} \cdot \mathbf{a}_{\perp} \times \mathrm{Q} \cdot \mathbf{b}_{\perp})$$

$$= \hat{\mathbf{e}}_{\mathbf{3}} \cdot |\mathrm{Q} \cdot \mathbf{a}_{\perp}||\mathrm{Q} \cdot \mathbf{b}_{\perp}| \sin \boldsymbol{\theta}\, \hat{\mathbf{e}}_{\mathbf{3}}$$

$$= |\mathbf{a}_{\perp}||\mathbf{b}_{\perp}| \sin \boldsymbol{\theta} \qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \ (2)$$

now consider RHS

$$\mathrm{Q} \cdot (\mathbf{a} \times \mathbf{b}) = \mathrm{Q} \cdot \{(a_{\mathbf{3}} \hat{\mathbf{e}}_{\mathbf{3}} + \mathbf{a}_{\perp}) \times (b_{\mathbf{3}} \hat{\mathbf{e}}_{\mathbf{3}} + \mathbf{b}_{\perp}\ )\}$$

$$\mathrm{Q} \cdot (\mathbf{a} \times \mathbf{b}) = \mathrm{Q} \cdot \{a_{\mathbf{3}} \hat{\mathbf{e}}_{\mathbf{3}} \times b_{\mathbf{3}} \hat{\mathbf{e}}_{\mathbf{3}} + a_{\mathbf{3}} \hat{\mathbf{e}}_{\mathbf{3}} \times \mathbf{b}_{\perp} + \mathbf{a}_{\perp} \times b_{\mathbf{3}} \hat{\mathbf{e}}_{\mathbf{3}} + \mathbf{a}_{\perp} \times \mathbf{b}_{\perp}\}$$

projecting RHS along $\hat{\mathbf{e}}_{\mathbf{3}}$

$$\hat{\mathbf{e}}_{\mathbf{3}} \cdot \mathrm{Q} \cdot (\mathbf{a} \times \mathbf{b}) = \hat{\mathbf{e}}_{\mathbf{3}} \cdot \mathrm{Q} \cdot \{a_{\mathbf{3}} \hat{\mathbf{e}}_{\mathbf{3}} \times b_{\mathbf{3}} \hat{\mathbf{e}}_{\mathbf{3}} + a_{\mathbf{3}} \hat{\mathbf{e}}_{\mathbf{3}} \times \mathbf{b}_{\perp} + \mathbf{a}_{\perp} \times b_{\mathbf{3}} \hat{\mathbf{e}}_{\mathbf{3}} + \mathbf{a}_{\perp} \times \mathbf{b}_{\perp}\}$$

$$= \hat{\mathbf{e}}_{\mathbf{3}} \cdot \mathrm{Q} \cdot (\mathbf{a}_{\perp} \times \mathbf{b}_{\perp})$$

$$= \hat{\mathbf{e}}_{\mathbf{3}} \cdot \mathrm{Q} \cdot (\mathbf{a}_{\perp} \times \mathbf{b}_{\perp})$$

$$= \lambda_{\mathbf{3}}\ \hat{\mathbf{e}}_{\mathbf{3}} \cdot (\mathbf{a}_{\perp} \times \mathbf{b}_{\perp})$$

$$= \lambda_{\mathbf{3}}\ \hat{\mathbf{e}}_{\mathbf{3}} \cdot |\mathbf{a}_{\perp}||\mathbf{b}_{\perp}| \sin \boldsymbol{\theta} \hat{\mathbf{e}}_{\mathbf{3}}$$

$$= \lambda_{\mathbf{3}}\ \hat{\mathbf{e}}_{\mathbf{3}} \cdot |\mathbf{a}_{\perp}||\mathbf{b}_{\perp}| \sin \boldsymbol{\theta} \quad \cdots\cdots\cdots\cdots\cdots\cdots (3)$$

compare (1), (2) and (3) and hence

$$\alpha = \lambda_{\mathbf{3}}$$

Thus LHS=RHS