Tutorial 1 #6

Problem: Prove the $\epsilon - \delta$ identity:

$$\epsilon_{ijk}\epsilon_{ipq} = \delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp}$$

Solution: Recall that

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \text{ and } \epsilon_{ijk} = \begin{cases} +1, & \text{even permutation of } ijk \\ -1, & \text{odd permutation of } ijk \\ 0, & \text{otherwise}. \end{cases}$$

We can prove the identity in at least three ways:

1. Brute force: Use the above definitions and check the equality between R.H.S. and L.H.S. This, of course, is extremely tedious and unexciting.

2. Note the following equivalent definition for the alternating symbol (here, an array enclosed by straight lines denotes the determinant)

$$\epsilon_{ijk} = \begin{vmatrix} \delta_{i1} & \delta_{j1} & \delta_{k1} \\ \delta_{i2} & \delta_{j2} & \delta_{k2} \\ \delta_{i3} & \delta_{j3} & \delta_{k3} \end{vmatrix}, \text{ as can be checked directly.}$$

Furthermore, $\begin{bmatrix} \delta_{r1} & \delta_{r2} & \delta_{r3} \\ \delta_{p1} & \delta_{p2} & \delta_{p3} \\ \delta_{q1} & \delta_{q2} & \delta_{q3} \end{bmatrix} \begin{bmatrix} \delta_{i1} & \delta_{j1} & \delta_{k1} \\ \delta_{i2} & \delta_{j2} & \delta_{k2} \\ \delta_{i3} & \delta_{j3} & \delta_{k3} \end{bmatrix} = \begin{bmatrix} \delta_{ir} & \delta_{jr} & \delta_{kr} \\ \delta_{ip} & \delta_{jp} & \delta_{kp} \\ \delta_{iq} & \delta_{jq} & \delta_{kq} \end{bmatrix}$

Taking determinant on both sides we get

$$\epsilon_{ijk}\epsilon_{rpq} = \begin{vmatrix} \delta_{ir} & \delta_{jr} & \delta_{kr} \\ \delta_{ip} & \delta_{jp} & \delta_{kp} \\ \delta_{iq} & \delta_{jq} & \delta_{kq} \end{vmatrix}.$$

Now, it is easy to check that with *i*=*r*,

$$\epsilon_{ijk}\epsilon_{ipq} = \begin{vmatrix} 3 & \delta_{ji} & \delta_{ki} \\ \delta_{ip} & \delta_{jp} & \delta_{kp} \\ \delta_{iq} & \delta_{jq} & \delta_{kq} \end{vmatrix} = \delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp}$$

hence proving our assertion. Moreover,

(ii) With *i*=*r*, *j*=*p*, we get $\epsilon_{ijk}\epsilon_{ijq} = 2\delta_{kq}$, and

(iii) With *i*=*r*, *j*=*p*, *k*=*q*, $\epsilon_{ijk}\epsilon_{ijk} = 6$.

3. This proof does not require determinants. Let $A_{jkpq} = \epsilon_{ijk}\epsilon_{ipq}$. Clearly, A_{jkpq} is non-zero only if $i \neq j \neq k \neq i$ and $i \neq p \neq q \neq i$. That is, either when *j*=*p* and *k*=*q*, or when *j*=*q* and *k*=*p*. As a result, we can write the following general representation for A_{jkpq}:

$$A_{jkpq} = \alpha\delta_{jp}\delta_{kq} + \beta\delta_{jq}\delta_{kp}.$$

Since $A_{jkpq} = -A_{kjpq}$ (due to skew symmetry of ϵ_{ijk}), we get $\alpha = -\beta$. We have then,

$$A_{jkpq} = \alpha(\delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp}).$$

Finally, take (for instance) *j*=*p*=2 and *k*=*q*=3. From the definition of A_{jkpq} we have $A_{2323} = 1$, implying $\alpha = 1$. Our identity is therefore established.