Problem: With $\hat{\mathbf{n}}$ a unit vector and $\mathbf{1}$ the identity tensor Let, $\mathbf{Q} = \mathbf{1} - 2\hat{\mathbf{n}} \otimes \hat{\mathbf{n}}$.

a) Show that $\mathbf{Q}$ is an orthogonal tensor.

b) Calculate the principal values and principal vectors of $\mathbf{Q}$.

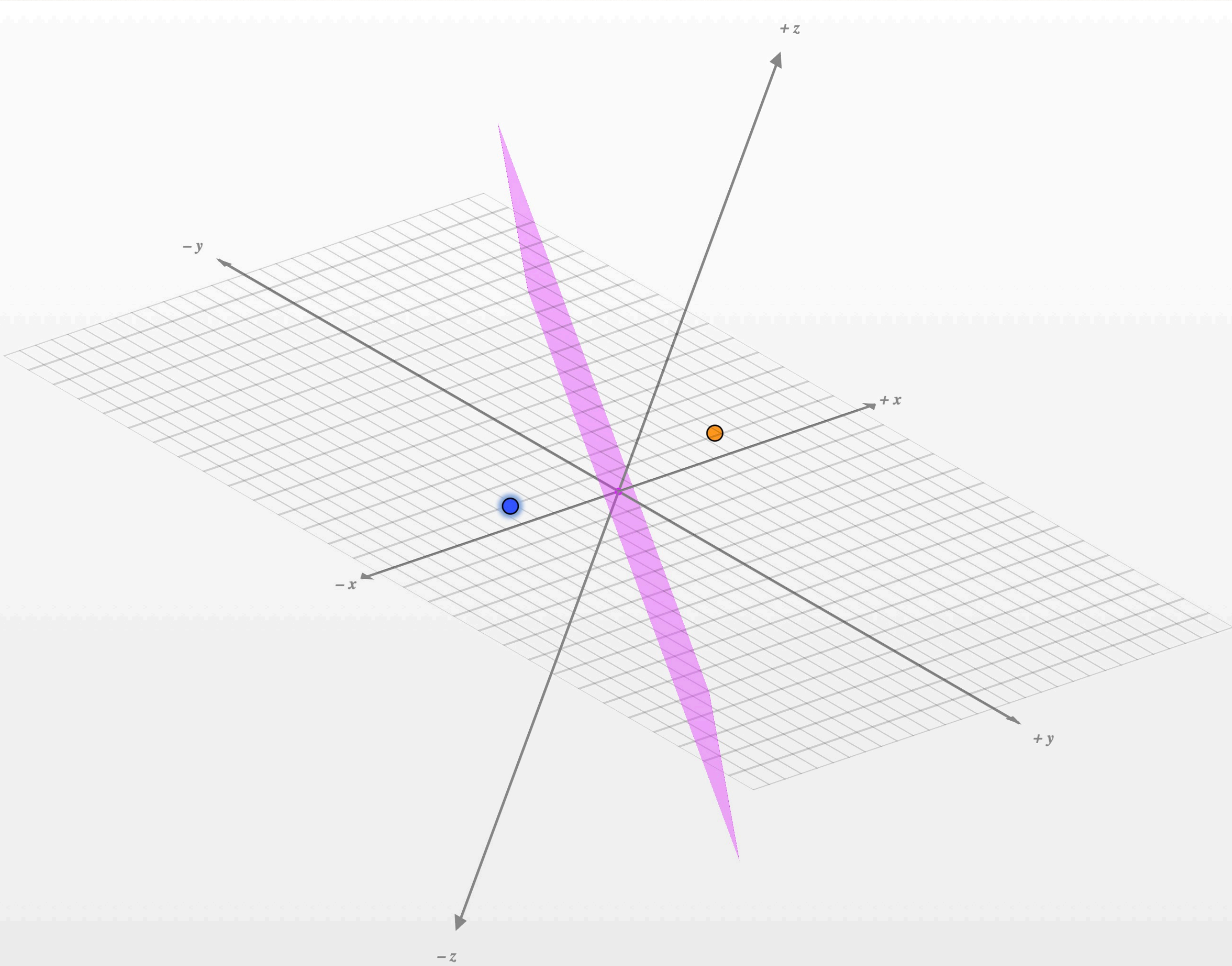
c) Take a vector $\mathbf{a} = \hat{\mathbf{e}}_1 + 2\hat{\mathbf{e}}_2 + 3\hat{\mathbf{e}}_3$, evaluate $\mathbf{Qa}$ and describe the action of $\mathbf{Q}$ on $\mathbf{a}$ by drawing sketches of $\mathbf{a}$ and $\mathbf{Qa}$. Let $\hat{\mathbf{n}} = \dfrac{1}{\sqrt{3}}(\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3)$.

Solution: (a) First note that, from the given form of $\mathbf{Q}$, $\mathbf{Q} = \mathbf{Q}^T$. Then, $\mathbf{Q}\mathbf{Q}^T = \mathbf{Q}^T\mathbf{Q} = \mathbf{Q}\mathbf{Q} = (\mathbf{1} - 2\hat{\mathbf{n}} \otimes \hat{\mathbf{n}})(\mathbf{1} - 2\hat{\mathbf{n}} \otimes \hat{\mathbf{n}}) = \mathbf{1}$, where we have used the result $(\hat{\mathbf{n}} \otimes \hat{\mathbf{n}})(\hat{\mathbf{n}} \otimes \hat{\mathbf{n}}) = (\hat{\mathbf{n}} \otimes \hat{\mathbf{n}})$, as can be verified directly using the definition of tensor product.

(b) Let $\hat{\mathbf{p}}$ and $\hat{\mathbf{q}}$ be any two unit vectors such that $\hat{\mathbf{p}}$, $\hat{\mathbf{q}}$, and $\hat{\mathbf{n}}$ form an orthonormal triad (i.e., mutually orthogonal unit vectors). We can write the identity tensor as $\mathbf{1} = \hat{\mathbf{p}} \otimes \hat{\mathbf{p}} + \hat{\mathbf{q}} \otimes \hat{\mathbf{q}} + \hat{\mathbf{n}} \otimes \hat{\mathbf{n}}$ and consequently, $\mathbf{Q} = \hat{\mathbf{p}} \otimes \hat{\mathbf{p}} + \hat{\mathbf{q}} \otimes \hat{\mathbf{q}} - \hat{\mathbf{n}} \otimes \hat{\mathbf{n}}$. Hence, $\mathbf{Q}$ has three principal values 1, 1, and -1 corresponding to principal vectors $\hat{\mathbf{p}}$, $\hat{\mathbf{q}}$, and $\hat{\mathbf{n}}$, respectively.

(c) $\mathbf{Qa} = (\mathbf{1} - 2\hat{\mathbf{n}} \otimes \hat{\mathbf{n}})\mathbf{a} = \mathbf{a} - 2(\hat{\mathbf{n}} \cdot \mathbf{a})\hat{\mathbf{n}}$. Substituting the given value of $\mathbf{a}$, and noting that $\hat{\mathbf{n}} \cdot \mathbf{a} = 2\sqrt{3}$, we obtain, $\mathbf{Qa} = -3\hat{\mathbf{e}}_1 - 2\hat{\mathbf{e}}_2 - \hat{\mathbf{e}}_3$. The geometrical illustration is given on the next page. The x, y, and z axes correspond to vectors $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$, and $\hat{\mathbf{e}}_3$, respectively. The pink plane is a plane orthogonal to $\hat{\mathbf{n}}$ passing through the origin. The orange point is obtained using vector $\mathbf{a}$ while the blue point is obtained using vector $\mathbf{Qa}$. Clearly, the blue point is obtained (as if) by reflection, through the mirror represented by the plane, from the orange point. The operation of $\mathbf{Q}$ on any vector $\mathbf{a}$ is therefore to reflect it across a plane whose normal is $\hat{\mathbf{n}}$.

The above can be seen, alternatively, by calculating the principal vectors $\hat{\mathbf{p}} = (1/\sqrt{6})(2\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2 - \hat{\mathbf{e}}_3)$ and $\hat{\mathbf{q}} = (1/\sqrt{2})(\hat{\mathbf{e}}_2 - \hat{\mathbf{e}}_3)$. Any linear combination of these vectors is also a principal vector. Furthermore, we note that $\mathbf{a} = 2\sqrt{3}\hat{\mathbf{n}} - (1/\sqrt{2})\hat{\mathbf{q}} - \sqrt{3}\hat{\mathbf{p}}$ and $\mathbf{Qa} = -2\sqrt{3}\hat{\mathbf{n}} - (1/\sqrt{2})\hat{\mathbf{q}} - \sqrt{3}\hat{\mathbf{p}}$. These expressions clearly confirm with the above stated geometrical interpretation.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