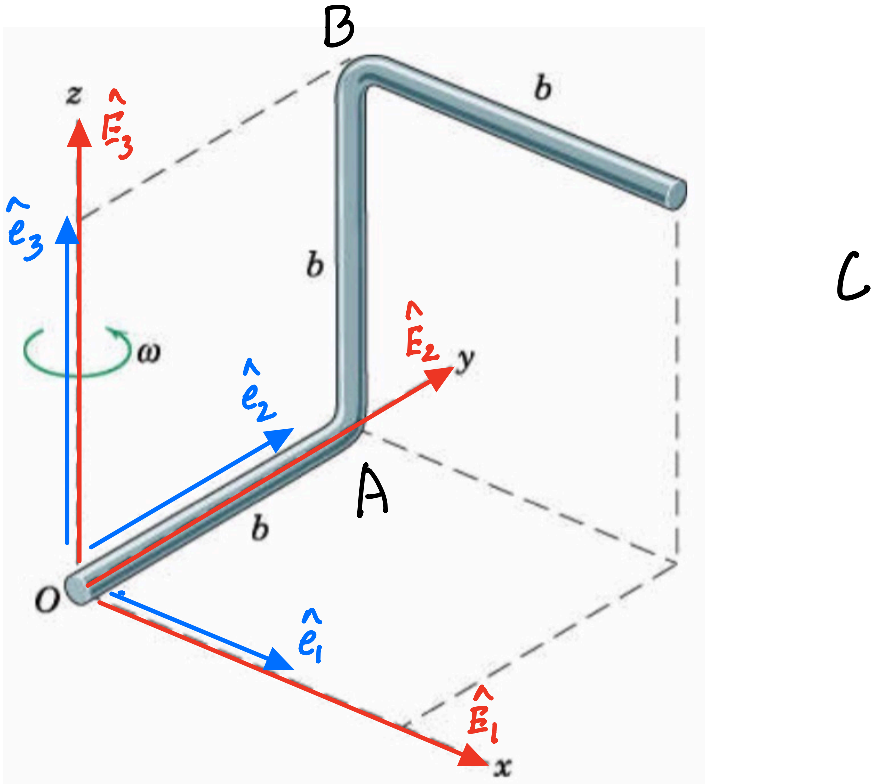
(4) A bent rod as shown in figure has the liner mass density of 1kg/m. The rod is rotating about z-axis at the constant angular velocity of $\omega = 1$ rad/sec. For $b = 1$ m determine the angular momentum about O in the frame O-xyz. Also determine the kinetic energy of the rod.



ANGULAR MOMENTUM ABOUT O

O is a point on the rigid body and $\underline{v}^O = \underline{0}$

$$\Rightarrow \underline{h}^O = \underline{\underline{I}}^O \cdot \underline{\omega}^B \qquad (1)$$

Proof: $\underline{h}^O = \underline{h}^G + \underline{r}^{G/O} \times m\underline{v}^G$ ~ CM of rigid body.

$$= \underline{\underline{I}}^G \cdot \underline{\omega}^B + \underline{r}^{G/O} \times m\underline{v}^G \qquad (2)$$

As O & G are on same rigid body

$$\therefore \quad \underline{v}^G = \cancelto{0}{\underline{v}^O} + \underline{\omega}^B \times \underline{r}^{G/O} \qquad (3)$$

Substituting (3) in (1):

$$\underline{h}^O = \underline{\underline{I}}^G \cdot \underline{\omega}^B + \underline{r}^{G/O} \times m\left(\underline{\omega}^B \times \underline{r}^{G/O}\right) \qquad (4)$$

As always, write $\underline{r}^{G/O} \times m\left(\underline{\omega}^B \times \underline{r}^{G/O}\right)$

$$= m\left(\left|\underline{r}^{G/O}\right|^2 \underline{\underline{1}} - \underline{r}^{G/O} \otimes \underline{r}^{G/O}\right) \cdot \underline{\omega}^B \qquad (5)$$

Equations (4) & (5) $\Rightarrow$

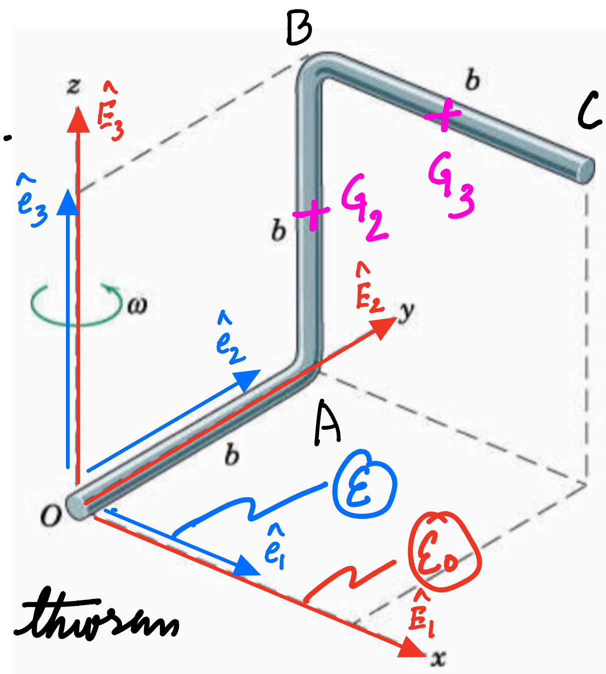
$$\underline{h}^O = \left\{\underline{\underline{I}}^G + m\left(\left|\underline{r}^{G/O}\right|^2 \underline{\underline{1}} - \underline{r}^{G/O} \otimes \underline{r}^{G/O}\right)\right\} \cdot \underline{\omega}^B$$

(by || axis theorem) $= \underline{\underline{I}}^O \cdot \underline{\omega}^B$ ∎

Need to now use (1) for current system.

KINEMATIC ANALYSIS $\Rightarrow$ $\underline{\omega}^B = \omega \hat{E}_3$ (6)

Remains to compute $\underline{\underline{I}}^O$.



Clearly,

$$\underline{\underline{I}}^O = \underline{\underline{I}}^O_{OA} + \underline{\underline{I}}^O_{AB} + \underline{\underline{I}}^O_{BC} \qquad \text{— (7)}$$

Employing the ∥ axis theorem

$$\left[\underline{\underline{I}}^O_{OA}\right]_{\mathcal{E}} = \frac{mb^2}{3}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (8)$$

$$\underline{\underline{I}}^O_{AB} = \underline{\underline{I}}^{G_2}_{AB} + m\left(\left|\underline{r}^{G_2/O}\right|^2 \underline{\underline{1}} - \underline{r}^{G_2/O} \otimes \underline{r}^{G_2/O}\right)$$

$$\underline{r}^{G_2/O} = b\,\hat{e}_2 + \frac{b}{2}\hat{e}_3$$

$$\Rightarrow \left[\underline{\underline{I}}^O_{AB}\right]_{\mathcal{E}} = \frac{mb^2}{12}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{mb^2}{4}\begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & -2 & 4 \end{bmatrix}$$

$$= \frac{mb^2}{12}\begin{bmatrix} 16 & 0 & 0 \\ 0 & 4 & -6 \\ 0 & -6 & 12 \end{bmatrix} \qquad (9)$$

Similarly $\frac{9}{4}\begin{bmatrix}1&&\\&1&\\&&1\end{bmatrix} - \begin{bmatrix}1/4 & 1/2 & 1/2\\ 1/2 & 1 & 1\\ 1/2 & 1 & 1\end{bmatrix}$

$$\underline{\underline{I}}^{O}_{BC} = \underline{\underline{I}}^{G_3}_{BC} + m\left(|\underline{r}^{G_3/O}|^2 \underline{\underline{1}} - \underline{r}^{G_3/O} \otimes \underline{r}^{G_3/O}\right)$$

with $\underline{r}^{G_3/O} = b\hat{e}_2 + b\hat{e}_3 + \frac{b}{2}\hat{e}_1$.

We obtain :

$$[\underline{\underline{I}}^{O}_{BC}]_{\mathcal{E}} = \frac{mb^2}{12}\begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix} + \frac{mb^2}{4}\begin{bmatrix}8&-2&-2\\-2&5&-4\\-2&-4&5\end{bmatrix}$$

$$= \frac{mb^2}{12}\begin{bmatrix}24&-6&-6\\-6&16&-12\\-6&-12&16\end{bmatrix} \qquad (10)$$

Combining (7) – (10)

$$[\underline{\underline{I}}^{O}]_{\mathcal{E}} = \frac{mb^2}{12}\begin{bmatrix}44&-6&-6\\-6&20&-18\\-6&-18&32\end{bmatrix} \qquad (11)$$

We can now combine (1), (6) & (11) to obtain $\underline{h}^O$.

(Note that at the instant shown $\hat{E}_i = \hat{e}_i$).

$$[\underline{h}^O]_{\mathcal{E}_0} = [\underline{\underline{I}}^O]_{\mathcal{E}_0} [\underline{\omega}]_{\mathcal{E}_0} = \frac{mb^2}{12}\begin{bmatrix} 44 & -6 & -6 \\ -6 & 20 & -18 \\ -6 & -18 & 32 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix}$$

$$= \frac{mb^2\omega}{12}\begin{bmatrix} -6 \\ -18 \\ 32 \end{bmatrix}$$

$$\Rightarrow \boxed{\underline{h}^O = mb^2\left(-\frac{1}{2}\hat{E}_1 - \frac{3}{2}\hat{E}_2 + \frac{8}{3}\hat{E}_3\right)}$$

KINETIC ENERGY

Claim: $E_K = \frac{1}{2}\underline{\omega}^B \cdot \underline{\underline{I}}^O \cdot \underline{\omega}^B$ ——— (12)

as O is a fixed point ($\underline{v}^O = 0$) on the rigid body

Proof: $E_K = \frac{1}{2}m|\underline{v}^G|^2 + \frac{1}{2}\underline{\omega}^B \cdot \underline{\underline{I}}^G \cdot \underline{\omega}^B$ ——— (13)

As in the proof above for angular momentum, set $\underline{v}^G = \underbrace{\underline{v}^O}_{0} + \underline{\omega}^B \times \underline{r}^{G/O}$ in (13) and follow similar steps to obtain (12).

Finally to compute E_K, use (6) & (11) in (12):

$$\boxed{E_K = \frac{4}{3}mb^2\omega^2.}$$