ESO209A: Dynamics
Tutorial 4
(Week 18 - 24 Aug. Based on L6 and L7)

1. For the rotation tensor obtained in Problem 6 of Tutorial 3 determine the rotation axis $\hat{\mathbf{n}}$ and the angle θ of rotation so that the plate rotates from it's orientation A to achieve the orientation B. Make sure that the choice of $\hat{\mathbf{n}}$ is such that the sense of rotation is counter-clockwise. You will find that all your principal values are real. How will you determine θ?

2. Use $\hat{\mathbf{n}}$ and the θ found in Problem 1 to calculate the matrix of the rotation tensor employing the axis-angle formula. Verify the result with the solution of Problem 6 of Tutorial 3.

3. We saw that the 3-1-3 Euler angle sequence is represented by the flowchart
$$\{\mathscr{E}_0, \hat{\mathbf{E}}_i\} \xrightarrow{\mathsf{R}_\varphi} \{\mathscr{E}', \hat{\mathbf{e}}_i'\} \xrightarrow{\mathsf{R}_\theta} \{\mathscr{E}'', \hat{\mathbf{e}}_i''\} \xrightarrow{\mathsf{R}_\psi} \{\mathscr{E}, \hat{\mathbf{e}}_i\},$$
so that the rotation tensor R in $\{\mathscr{E}_0, G, \hat{\mathbf{E}}_i\} \xrightarrow{\mathsf{R}} \{\mathscr{E}, G, \hat{\mathbf{e}}_i\}$ is given by
$$\mathsf{R} = \mathsf{R}_\psi(\hat{\mathbf{e}}_3'', \psi) \cdot \mathsf{R}_\theta(\hat{\mathbf{e}}_1', \theta) \cdot \mathsf{R}_\varphi(\hat{\mathbf{E}}_3, \varphi).$$
Do the following

   i. Show $[\mathsf{R}_\varphi]_{\mathscr{E}_0} = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $[\mathsf{R}_\theta]_{\mathscr{E}'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$, and
   $[\mathsf{R}_\psi]_{\mathscr{E}''} = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$

   ii. Show $[\mathsf{R}]_{\mathscr{E}_0} = [\mathsf{R}_\varphi(\hat{\mathbf{E}}_3)]_{\mathscr{E}_0} [\mathsf{R}_\theta(\hat{\mathbf{e}}_1')]_{\mathscr{E}'} [\mathsf{R}_\psi(\hat{\mathbf{e}}_3'')]_{\mathscr{E}''}$, and compute $[\mathsf{R}]_{\mathscr{E}_0}$.

   iii. Outline a procedure to find φ, θ and ψ given the matrix $[\mathsf{R}]_{\mathscr{E}_0}$.

4. Aeronautical engineers use the 3-2-1 *intrinsic*[1] Euler angle sequence. Express the rotation tensor R in $\{\mathscr{E}_0, G, \hat{\mathbf{E}}_i\} \xrightarrow{\mathsf{R}} \{\mathscr{E}, G, \hat{\mathbf{e}}_i\}$ using this Euler angle sequence. Then find the three rotation angles in terms of the components of $[\mathsf{R}]_{\mathscr{E}_0}$.

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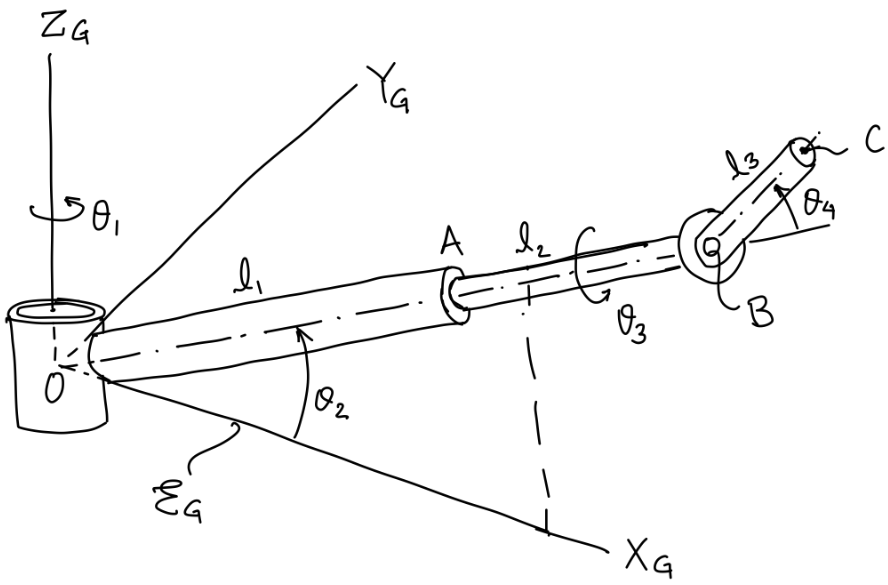
[1] ***Intrinsic*** is used to indicate that every subsequent rotation takes place about an axis of the frame obtained from the previous rotation. Thus, the first rotation is about $\hat{\mathbf{E}}_3$, the second about the $\hat{\mathbf{e}}_2$ axis of the CS obtained from the first rotation, and the third about the $\hat{\mathbf{e}}_1$ axis of the CS obtained after rotating $\mathscr{E}_0$ twice.

5. Axes of all three links of a robot are perfectly aligned to the ground X_G axis when $\theta_1 = \theta_2 = \theta_3 = \theta_4 = 0$. In this position the axis of the pin B is parallel to ground Y_G axis so that BC would be able to rotate in the X_G -Z_G plane. For $\theta_1 = 45^0$, $\theta_2 = \theta_3 = 30^0$ and $\theta_4 = 60^0$ find the coordinates of point C with respect to O.



6. Show that in any CS $\{\mathscr{E}, \hat{\mathbf{e}}_i\}$, the axis of rotation $\hat{\mathbf{n}}$ of a rotation tensor $\mathsf{R}(\hat{\mathbf{n}}, \theta)$ may be obtained from the formula

$$\hat{\mathbf{n}} = -\,\epsilon_{ijk}\frac{R_{jk}}{2\sin\theta}\,\hat{\mathbf{e}}_i.$$

The advantage of this formula is that it fixes the *correct* direction of $\hat{\mathbf{n}}$ for the particular choice of θ.