Lecture 1

Vectors, Coordinate Systems and Index Notation

2 September, 2020

Vectors

I. **Definition**: An object with a magnitude and a direction. Picture

II. **Notation**: $\mathbf{a}, \hat{\mathbf{e}}$

III. Some simple operations Pictures

1. Dot product: $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta$
2. Norm: $|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$
3. Cross product:
   $\mathbf{a} \times \mathbf{b} = -\,\mathbf{b} \times \mathbf{a} = |\mathbf{a}|\,|\mathbf{b}| \sin\theta\,\hat{\mathbf{e}}$

Thumb rule.


ALTERNATIVES TO THE RIGHT-HAND RULE IN VECTOR MULTIPLICATION:


RESULT HOLES
SECOND VECTOR
FIRST VECTOR
HANDGUN RULE:
POINT THE GRIP ALONG THE FIRST VECTOR AND ROTATE IT SO THE SECOND VECTOR IS ON THE SAFETY LATCH SIDE. FIRE. THE RESULT VECTOR IS TOWARD THE BULLET HOLES.

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IV. **Important message**:

"We do **not** require a coordinate system to define vectors."

Coordinate system

I. **Definition**: Any mutually orthogonal triad of unit vectors $\hat{\mathbf{e}}_i$, $(i = 1,2,3)$ defines a <u>Cartesian</u> Coordinate System (CCS).

   1. Right-handed CCS: Satisfies $\hat{\mathbf{e}}_3 = \hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_2$.

II. **Notation**: $\{\mathit{Name, origin, unit\ vectors}\}$

$$\text{e.g. } \left\{ \mathscr{P}, P, \hat{\mathbf{e}}_i \right\}.$$ Picture.

III. **Examples**:

   1. Fixed unit vectors. Picture
   2. Changing unit vectors:
      i. Cylindrical CS. Picture
      ii. Spherical CS. Picture

Index notation

I. Index? Why index notation? Example

II. **Rules**:

1. Any index may appear once or twice in any term in an equation. Examples.
   i. In 3D indices take values 1, 2 and 3.
   ii. In 2D indices take values 1 and 2.
2. **Free index**: An index that appears *just once* in each term. Examples.
   i. Free indices in each term <u>must be same</u>.
   ii. Free indices take values 1, 2 and 3.
3. **Dummy index**: An index that appears twice in a term. Examples
   i. **Summation Convention**: Dummy indices are summed from 1 to 3.
   ii. Name of dummy index is <u>not</u> important.
   iii. Dummy index may <u>not</u> be in all terms

Index notation

IV. **Krönecker delta**: $\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$

V. **Contraction**: $P_{\ldots i \ldots j \ldots}\delta_{ij} = P_{\ldots i \ldots i \ldots}$

Examples

VI. **Alternating tensor**:

$$\epsilon_{ijk} = \begin{cases} +1, & \text{even permutation of } ijk \\ -1, & \text{odd permutation of } ijk \\ 0, & \text{otherwise}\,. \end{cases}$$

Examples

VII. $\delta - \epsilon$ **identity**: $\epsilon_{ijk}\epsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$

Vector operations

I. **Components**: Consider two coordinate systems $\{\mathscr{P}, P, \hat{\mathbf{e}}_i\}$ and $\{\mathscr{S}, S, \hat{\mathbf{e}}_i'\}$.
Picture

1. In $\mathscr{P}$: $\mathbf{a} = a_i \hat{\mathbf{e}}_i$ with $a_i = \mathbf{a} \cdot \hat{\mathbf{e}}_i$
2. In $\mathscr{S}$: $\mathbf{a} = a_i' \hat{\mathbf{e}}_i'$ with $a_i' = \mathbf{a} \cdot \hat{\mathbf{e}}_i'$.
3. *Notation and relation to matrix algebra*. Compile components into a column vector:

$$[\mathbf{a}]_{\mathscr{P}} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \quad \text{and} \quad [\mathbf{a}]_{\mathscr{S}} = \begin{bmatrix} a_1' \\ a_2' \\ a_3' \end{bmatrix}$$

4. In general $a_i \neq a_i'$, i.e.

Vector remains the same, but components change with choice of coordinate system.

II. Example: Two rotated CS.

Vector operations

III. Simple operations revisited with index notation

1. Dot product: $\mathbf{a} \cdot \mathbf{b} = a_i b_i = a_i' b_i'$
2. Norm: $|\mathbf{a}| = \left(a_i a_i\right)^{1/2} = \left(a_i' a_i'\right)^{1/2}$
3. Cross product: $\mathbf{a} \times \mathbf{b} = \epsilon_{ijk} a_i b_j \hat{\mathbf{e}}_k$

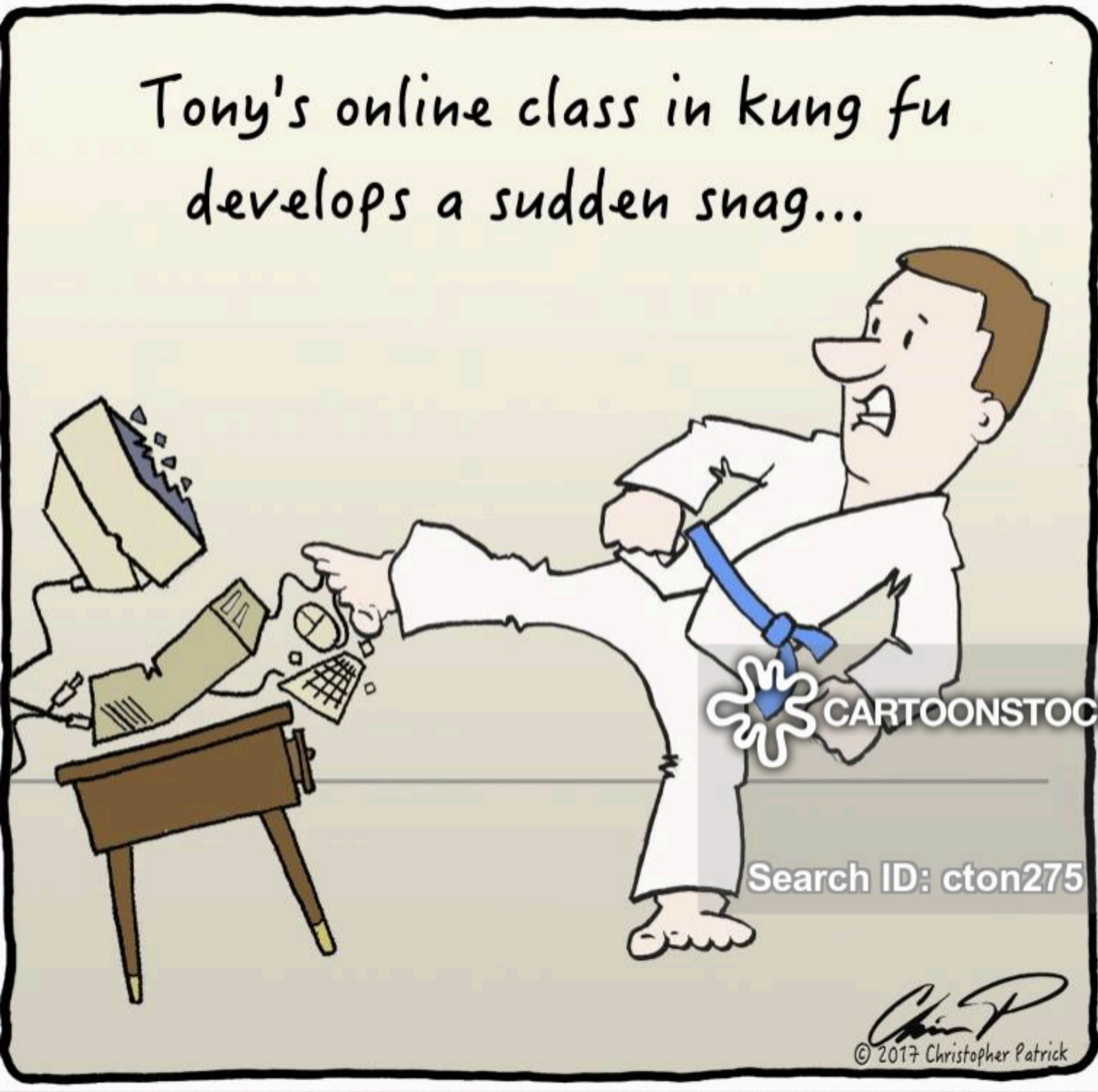
IV. Some vector identities

1. Box product:

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a} = (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b}$$
$$= \epsilon_{ijk} a_i b_j c_k$$

2. Triple product:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\, \mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\, \mathbf{c}$$
$$= \epsilon_{ijk} \epsilon_{lmk} a_j b_l c_m \hat{\mathbf{e}}_i$$


Tony's online class in kung fu develops a sudden snag...
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