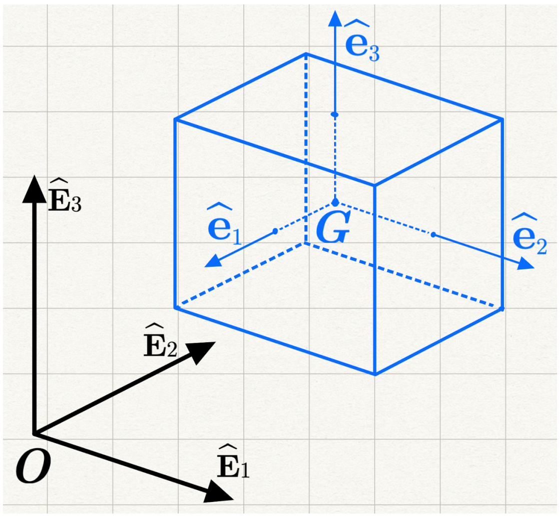
ESO209A: Tutorial 3
(Week 3: 11-18 Aug 2021. Based on Lectures 4 and 5)

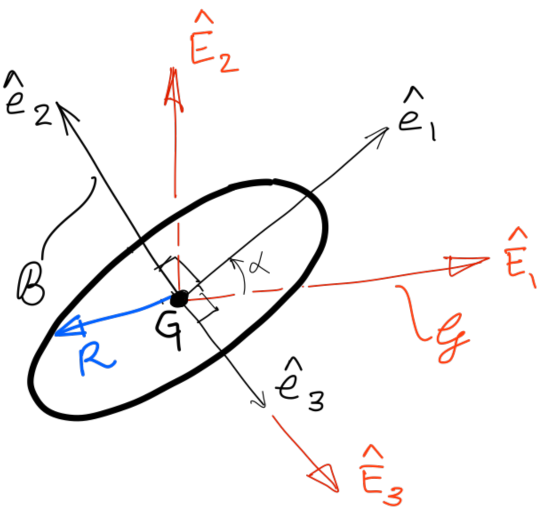
1. For any two vectors $\mathbf{a}$ and $\mathbf{b}$, prove the following properties for an orthogonal tensor Q:
   a. $(\mathsf{Q} \cdot \mathbf{a}) \cdot (\mathsf{Q} \cdot \mathbf{b}) = \mathbf{a} \cdot \mathbf{b}$.
   b. $(\mathsf{Q} \cdot \mathbf{a}) \times (\mathsf{Q} \cdot \mathbf{b}) = \lambda \mathsf{Q} \cdot (\mathbf{a} \times \mathbf{b})$ with λ being the real principal value of Q.

2. With $\hat{\mathbf{n}}$ a unit vector and 1 the identity tensor, let $\mathsf{Q} = \mathsf{1} - 2\hat{\mathbf{n}} \otimes \hat{\mathbf{n}}$.
   a. Show that Q is an orthogonal tensor.
   b. Calculate the principal values and vectors of Q.
   c. Take a vector $\mathbf{a} = \hat{\mathbf{e}}_1 + 2\hat{\mathbf{e}}_2 + 3\hat{\mathbf{e}}_3$, evaluate $\mathsf{Q} \cdot \mathbf{a}$ and describe the action of Q on $\mathbf{a}$ by drawing sketches of $\mathbf{a}$ and $\mathsf{Q} \cdot \mathbf{a}$. Let $\hat{\mathbf{n}} = \dfrac{1}{\sqrt{3}}(\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3)$.

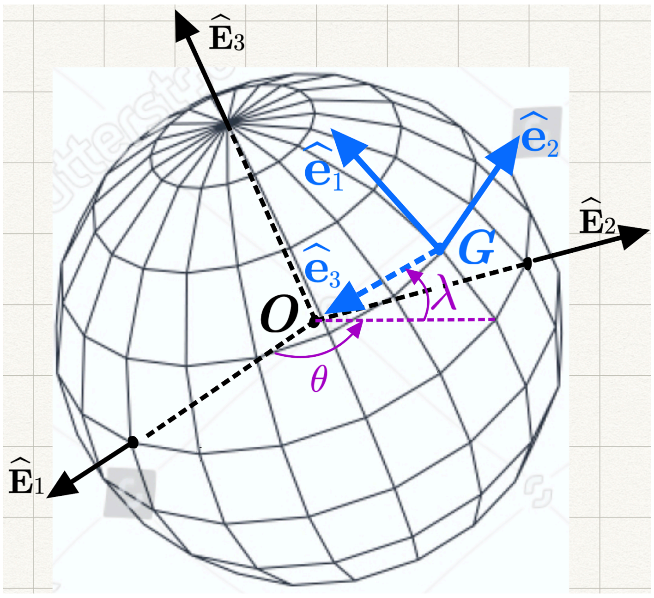
3. Consider the cube in the right figure. Two CS are given: the global CS $\left\{\mathscr{E}_0, O, \hat{\mathbf{E}}_i\right\}$, and the CS $\left\{\mathscr{B}, G, \hat{\mathbf{e}}_i\right\}$ which is fixed to the cube. Initially the cube which is located so that $\mathbf{r}_{G/C} = \hat{\mathbf{E}}_1 + 2\hat{\mathbf{E}}_2 + 3\hat{\mathbf{E}}_3$. The cube is now rotated about the $\hat{\mathbf{e}}_3$ in a counter-clockwise direction by an angle $\theta = 30°$. Do the following:
   a. Find rotation tensor R that relates $\mathscr{E}_0$ and $\mathscr{B}$.
   b. Use R to find the location of the origin O in the CS $\left\{\mathscr{B}, G, \hat{\mathbf{e}}_i\right\}$ *after* the cube's rotation, i.e. $[\mathbf{r}_{O/G}]_{\mathscr{B}}$.



4. The matrix of moment of the *inertia tensor* I of a thin circular disc of mass m and radius R (see figure below) about its center of mass G in the principal (also called *body fixed*) CS $\{\mathscr{B}, G, \hat{\mathbf{e}}_i\}$ is given by $[\mathrm{I}]_{\mathscr{B}} = \dfrac{mR^2}{4}\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 1\end{bmatrix}$. Determine the matrix of I in the <u>global</u> CS $\{\mathscr{G}, G, \hat{\mathbf{E}}_i\}$. *Note*: $\hat{\mathbf{E}}_3$ and $\hat{\mathbf{e}}_3$ are aligned.



5. The latitude λ and longitude θ of a point on the Earth's surface are shown in the figure on the right. These angles are used in navigation systems to define the downward direction $\hat{\mathbf{e}}_3$, the northerly direction $\hat{\mathbf{e}}_1$ and the easterly direction $\hat{\mathbf{e}}_2$. We define the Earth-fixed CS $\left\{\mathscr{E}_0, O, \hat{\mathbf{E}}_i\right\}$ and the navigation CS $\left\{\mathscr{N}, G, \hat{\mathbf{e}}_i\right\}$.
   a. Find the rotation tensor R relating $\mathscr{E}_0$ and $\mathscr{E}$ and express its matrix in both CS
   b. When $\lambda = 30^\circ$ and $\theta = 60^\circ$ find the single angle of rotation and the axis of rotation corresponding to the rotation R.



6. A plate inscribed with letter P is initially positioned in YZ-plane, called *A*, as shown in the figure below. In this position the body-fixed CS $\{\mathscr{B}, O, \hat{\mathbf{e}}_i\}$and the CS $\{\mathscr{E}_1, O, \hat{\mathbf{E}}_i\}$ are aligned. The plate is first rotated about its side *ad*, along $-\hat{\mathbf{E}}_3$ by 180^0 such that the body-fixed CS is now given by $\{\mathscr{B}', O, \hat{\mathbf{e}}_i\}$. Subsequently, the plate is rotated about $\hat{\mathbf{E}}_2$ by 90^0 so as to acquire position *B* as shown. In this position the body-fixed CS is given by $\{\mathscr{B}'', O, \hat{\mathbf{e}}_i\}$. We define this orientation of body-fixed CS as $\{\mathscr{E}_2, O, \hat{\mathbf{E}}_i\}$. Determine the matrix of rotation tensor R in both $\mathscr{E}_1$ and $\mathscr{E}_2$ in going from $\{\mathscr{E}_1, O, \hat{\mathbf{E}}_i\}$ to $\{\mathscr{E}_2, O, \hat{\mathbf{E}}_i\}$. This can be done in two ways: (i) multiplying the rotation tensors R_1 and R_2 in a specific order, or (ii) directly writing R between $\{\mathscr{E}_1, O, \hat{\mathbf{E}}_i\}$ and $\{\mathscr{E}_2, O, \hat{\mathbf{E}}_i\}$ using Euler's theorem. Use both methods to check the correctness of your calculations.