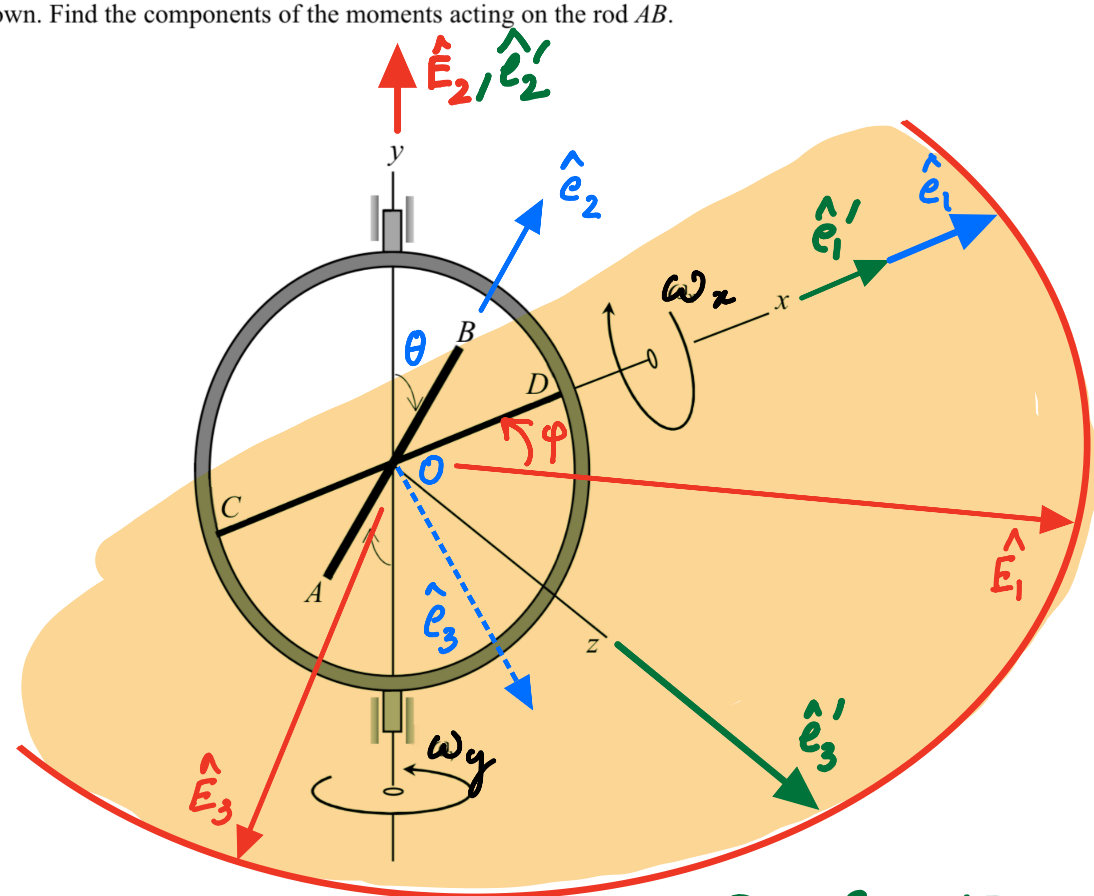
(3) A thin uniform rod AB of mass m and length l is rotating about the shaft CD at a constant angular velocity ω_x as shown. Note that the rod AB is normal to the shaft CD. The shaft CD is attached to a circular gimbal which rotates about the fixed y-axis at a constant angular rate of ω_y, as shown. Find the components of the moments acting on the rod AB.



ORIENTATION:

$$\{\mathcal{E}_0, O, \hat{E}_i\} \xrightarrow[\dot{\varphi}\,\hat{E}_2 = \omega_y \hat{E}_2]{\underset{=}{R}(\hat{E}_2,\, \varphi(t))} \{\mathcal{E}', O, \hat{e}_i'\}$$

BFCS of outer ring

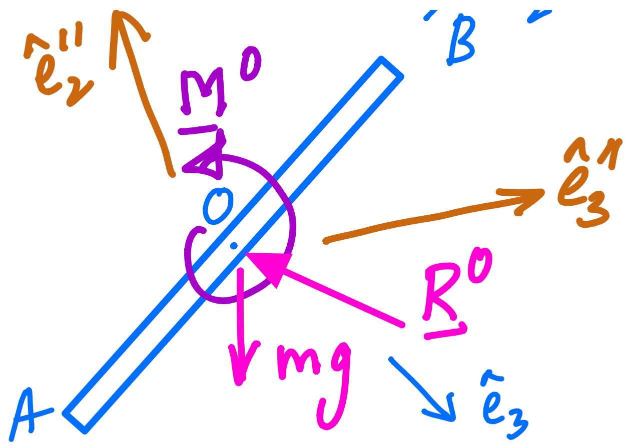
Inertial CS

$$\dot{\theta}\,\hat{e}_1' = \omega_x \hat{e}_1' \quad \Big\downarrow \quad \underset{=}{R}(\hat{e}_1', \theta(t))$$

BFCS of the rod AB $\quad \{\mathcal{E}, O, \hat{e}_i\}$

$\hat{e}_2$

FBD



$$\text{AMB}/_O: \quad \underline{M}^O = \underline{\omega}^B \times \left(\underline{\underline{I}}^O \cdot \underline{\omega}^B\right) + \underline{\underline{I}}^O \cdot \underline{\alpha}^B \qquad (1)$$

Express (1) in BFCS $\mathcal{E}$ (Euler's equations):

$$(1') \qquad \begin{aligned} \hat{e}_1: &\quad M_1^O = I_1\alpha_1 - (I_2 - I_3)\,\omega_2\omega_3 \\ \hat{e}_2: &\quad M_2^O = I_2\alpha_2 - (I_3 - I_1)\,\omega_3\omega_1 \\ \hat{e}_3: &\quad M_3^O = I_3\alpha_3 - (I_1 - I_2)\,\omega_1\omega_2 \end{aligned}$$

where

$$\underline{\underline{I}}^O = \sum_{i=1}^{3} I_i \, \hat{e}_i \otimes \hat{e}_i \, ,$$

This is because the BFCS $\mathcal{E}$ is also the principal CS of the rod

$$\underline{\omega}^B = \omega_i \hat{e}_i \quad \text{and} \quad \underline{\alpha}^B = \alpha_i \hat{e}_i \, .$$

Clearly $\quad I_1 = I_3 = \dfrac{m l^2}{12}, \quad I_2 = 0. \qquad (2)$

Need to find $\underline{\omega}^B$ & $\underline{\alpha}^B$ and express them in $\mathcal{E}$.

KINEMATICS: $\quad \underline{\omega}^B = \underline{\omega}_{\mathcal{E}/\mathcal{E}'} + \omega_{\mathcal{E}'/\mathcal{E}_0}$

$$\Rightarrow \underline{\omega}^B = \omega_x \hat{e}_1' + \omega_y \hat{E}_2$$
$$= \omega_x \hat{e}_1 + \omega_y (\cos\theta\, \hat{e}_2 - \sin\theta\, \hat{e}_3)$$

$$\therefore \omega_1 = \omega_x, \; \omega_2 = \omega_y \cos\theta, \; \omega_3 = -\omega_y \sin\theta \qquad \text{——— (3)}$$

Then, $\underline{\alpha}^B = \underline{\dot{\omega}}^B = \underline{\overset{\circ}{\omega}}^B$

$$\Rightarrow \underline{\alpha}^B = \dot{\omega}_i \hat{e}_i$$
$$= -\omega_y \sin\theta\, \dot{\theta}\, \hat{e}_2 - \omega_y \cos\theta\, \dot{\theta}\, \hat{e}_3$$

But $\dot{\theta} = \omega_x$.

$$\therefore \alpha_1 = 0, \; \alpha_2 = -\omega_x \omega_y \sin\theta, \; \alpha_3 = -\omega_x \omega_y \cos\theta \qquad \text{——— (4)}$$

Employing (3) – (4) in (1'):

$$\underline{M}^O = -\frac{ml^2}{12} \omega_y^2 \cos\theta \sin\theta\, \hat{e}_1 - \frac{ml^2}{6} \omega_x \omega_y \cos\theta\, \hat{e}_3$$

$\dot{\theta} = \omega_x \Rightarrow \theta(t) = \omega_x t.$