ESO 209

Tutorial 3, Problem # 5

Problem

The latitude λ and longitude θ of a point on the Earth's surface are shown in the figure (see tutorial sheet).

The Earth-fixed CS is $\{\mathcal{E}_0, O, \hat{E}_i\}$ and the navigation CS is $\{\mathcal{N}, G, \hat{e}_i\}$, where in the latter, $\hat{e}_3$ is downward direction, $\hat{e}_1$ the northerly direction and $\hat{e}_2$ the easterly direction.

a. To find the rotation tensor $\underline{\underline{R}}$ relating $\mathcal{E}_0$ and $\mathcal{N}$ and express its matrix in both CS

b. when $\lambda = 30^\circ$ and $\theta = 60^\circ$ find the single angle of rotation & axis of rotation corresponding to $\underline{\underline{R}}$

Solution

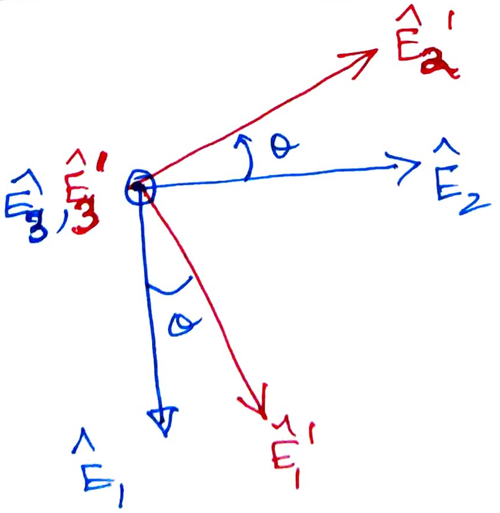
Ignoring the translation of point G from point O, we focus only on the rotations.

The navigation CS can be obtained by a succession of two rotations as will be explained below. Note that the sequence (order) of rotations is important.

The navigation CS (also called as NED, for North-East-Down, system) is obtained ~~E~~ from the Earth-fixed, Earth-Centre system as follows: we call the intermediate CS as $\{\mathcal{E}_\varphi', O, \hat{E}_i'\}$. Thus

$$\{\mathcal{E}_{\varphi_0}, O, \hat{E}_i\} \xrightarrow{\underline{\underline{R}}_1} \{\mathcal{E}_\varphi', O, \hat{E}_i'\} \xrightarrow{\underline{\underline{R}}_2} \{N, G, \hat{e}_i\}$$

The first rotation $\underline{\underline{R}}_1$ is rotating the Earth-fixed CS by the longitude θ about the $\hat{E}_3$ axis. Thus the new CS basis vectors $\hat{E}_1'$, $\hat{E}_2'$, $\hat{E}_3'$ are such that $\hat{E}_3'$ and $\hat{E}_3$ are aligned. The other two axes are oriented as show below (the 1-2 plane is shown for clarity). The view is looking towards the South pole.



The rotation tensor $\underline{\underline{R_1}}$ in the CS $\mathcal{E}_{0}$ can be written as :

$$\left[\underline{\underline{R_1}}\right]_{\mathcal{E}_0} = \begin{array}{c} \\ i=1 \\ i=2 \\ i=3 \end{array} \begin{array}{c} \begin{array}{ccc} j=1 & j=2 & j=3 \end{array} \\ \begin{bmatrix} \hat{E}_1'\cdot\hat{E}_1 & \hat{E}_2'\cdot\hat{E}_1 & \hat{E}_3'\cdot\hat{E}_1 \\ \hat{E}_1'\cdot\hat{E}_2 & \hat{E}_2'\cdot\hat{E}_2 & \hat{E}_3'\cdot\hat{E}_2 \\ \hat{E}_1'\cdot\hat{E}_3 & \hat{E}_2'\cdot\hat{E}_3 & \hat{E}_3'\cdot\hat{E}_3 \end{bmatrix} \end{array}$$

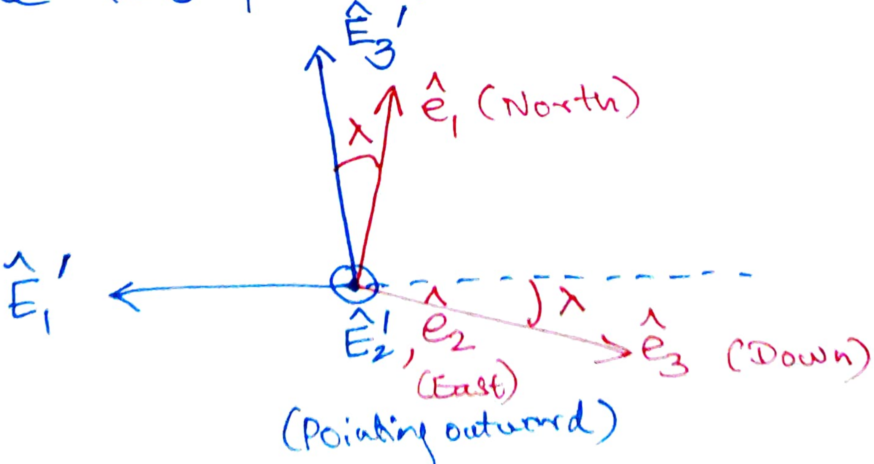
$$\boxed{R_{ij} = \hat{E}_j'\cdot\hat{E}_i}$$

$\cos(90+\theta)$

$$\therefore \left[\underline{\underline{R_1}}\right]_{\mathcal{E}_0} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\cos(90-\theta)$ $\qquad$ $\cos 0^\circ$

Now, the second rotation is by the angle $(-\lambda)$ (latitude) about the axis $\hat{E}_2'$. The navigation CS is oriented w.r.t to the intermediate CS $\mathcal{E}'$ as shown below (again since the axes $\hat{e}_2$ and $\hat{E}_2'$ are aligned, the orientation is shown in the 1-3 plane). The $\hat{e}_2$ & $\hat{E}_2'$ point outward.



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The rotation tensor $\underline{\underline{R_2}}$ in CS $\mathcal{E}_\phi'$ is written as (again using $R_{2ij} = \hat{e}_j' \cdot \hat{E}_i'$)

$$[\underline{\underline{R_2}}]_{\mathcal{E}_\phi'} = \begin{array}{c c} & \begin{array}{ccc} j=1 & j=2 & j=3 \end{array} \\ \begin{array}{c} i=1 \\ i=2 \\ i=3 \end{array} & \begin{bmatrix} \hat{e}_1 \cdot \hat{E}_1' & \hat{e}_2 \cdot \hat{E}_1' & \hat{e}_3 \cdot \hat{E}_1' \\ \hat{e}_1 \cdot \hat{E}_2' & \hat{e}_2 \cdot \hat{E}_2' & \hat{e}_3 \cdot \hat{E}_2' \\ \hat{e}_1 \cdot \hat{E}_3' & \hat{e}_2 \cdot \hat{E}_3' & \hat{e}_3 \cdot \hat{E}_3' \end{bmatrix} \end{array}$$

$$= \begin{bmatrix} \cos(90+\lambda) & 0 & \cos(180-\lambda) \\ 0 & 1 & 0 \\ \cos\lambda & 0 & \cos(90+\lambda) \end{bmatrix}$$

$$= \begin{bmatrix} -\sin\lambda & 0 & -\cos\lambda \\ 0 & 1 & 0 \\ \cos\lambda & 0 & -\sin\lambda \end{bmatrix}$$

Now to find the rotation tensor in both CS.

The Rotation tensor $\underline{\underline{R}} = \underline{\underline{R_2}} \cdot \underline{\underline{R_1}}$

We first find $[\underline{\underline{R}}]_{\mathcal{E}_0}$

$$[\underline{\underline{R}}]_{\xi\phi 0} = [\underline{\underline{R}}_2]_{\xi\phi 0} [\underline{\underline{R}}_1]_{\xi\phi 0} \quad \text{— } (1)$$

We dont have $[\underline{\underline{R}}_2]_{\xi\phi 0}$. So we use the coordinate transformation relation for tensors, viz., $[A]_{\xi\phi 1} = [R]^T_{\xi\phi} [A]_{\xi\phi} [R]_{\xi\phi}$

$$[\underline{\underline{R}}_2]_{\xi\phi'} = [\underline{\underline{R}}_1]^T_{\xi\phi 0} [\underline{\underline{R}}_2]_{\xi\phi 0} [\underline{\underline{R}}_1]_{\xi\phi 0}$$

"Premultiplying" by $[\underline{\underline{R}}_1]_{\xi\phi 0}$ and "post-multiplying" by $[\underline{\underline{R}}_1]^T_{\xi\phi 0}$ and using the orthogonality property,

$$[\underline{\underline{R}}_2]_{\xi\phi 0} = [\underline{\underline{R}}_1]_{\xi\phi 0} [\underline{\underline{R}}_2]_{\xi\phi'} [\underline{\underline{R}}_1]^T_{\xi\phi 0}$$

Substituting in Eqn (1) above,

$$[\underline{\underline{R}}]_{\xi\phi 0} = [\underline{\underline{R}}_1]_{\xi\phi 0} [\underline{\underline{R}}_2]_{\xi\phi'} \underbrace{[\underline{\underline{R}}_1]^T_{\xi\phi 0} [\underline{\underline{R}}_1]_{\xi\phi 0}}_{\Downarrow \; \underline{\underline{1}}}$$

$$= [\underline{\underline{R}}_1]_{\xi\phi 0} [\underline{\underline{R}}_2]_{\xi\phi'}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\sin\lambda & 0 & -\cos\lambda \\ 0 & 1 & 0 \\ \cos\lambda & 0 & -\sin\lambda \end{bmatrix}$$

$$[\underline{\underline{R}}]_{\xi\phi 0} = \begin{bmatrix} -\sin\lambda\cos\theta & -\sin\theta & -\cos\lambda\cos\theta \\ -\sin\lambda\sin\theta & \cos\theta & -\cos\lambda\sin\theta \\ \cos\lambda & 0 & -\sin\lambda \end{bmatrix}$$

Now to find $[\underline{\underline{R}}]_{\mathcal{N}}$

$$[\underline{\underline{R}}]_{\mathcal{N}} = [\underline{\underline{R_2}}]_{\mathcal{N}} [\underline{\underline{R_1}}]_{\mathcal{N}}$$

2 $\leftarrow$ $[\underline{\underline{R_2}}]_{\mathcal{N}} = [\underline{\underline{R_2}}]_{\xi'} = \begin{bmatrix} -\sin\lambda & 0 & -\cos\lambda \\ 0 & 1 & 0 \\ \cos\lambda & 0 & -\sin\lambda \end{bmatrix}$.

3 $\leftarrow$ $[\underline{\underline{R_1}}]_{\mathcal{N}} = [\underline{\underline{R_2}}]^T_{\xi'} [\underline{\underline{R_1}}]_{\xi'} [\underline{\underline{R_2}}]_{\xi'}$

$\downarrow$

since this is nothing but $[\underline{\underline{R_1}}]_{\xi_0}$

and $\underline{\underline{R_2}}$ is the tensor transformation from CS ξ_1' to CS $\mathcal{N}$

we treat ~~$[\underline{\underline{R_1}}]$~~ $\underline{\underline{R_1}}$ as any other tensor.

~~$\therefore [\underline{\underline{R}}]_{\mathcal{N}} =$~~ Substituting for $[\underline{\underline{R_2}}]_{\mathcal{N}}$ $[\underline{\underline{R_1}}]_{\mathcal{N}}$ from 2 & 3

$$[\underline{\underline{R}}]_{\mathcal{N}} = \underbrace{[\underline{\underline{R_2}}]_{\xi'} [\underline{\underline{R_2}}]^T_{\xi'}}_{\Downarrow \; \underline{\underline{1}}} [\underline{\underline{R_1}}]_{\xi'} [\underline{\underline{R_2}}]_{\xi'}$$

$\therefore$

$$= [\underline{\underline{R_1}}]_{\xi'} [\underline{\underline{R_2}}]_{\xi'}$$

$$= [\underline{\underline{R_1}}]_{\xi_0} [\underline{\underline{R_2}}]_{\xi'} = [\underline{\underline{R}}]_{\xi_0}$$

which is as expected!

Problem 5(b)

When $\lambda = 30^\circ$, $\theta = 60^\circ$

$$[\underline{R}]_{\varepsilon_0} = \begin{bmatrix} -\sin 30 \cos 60 & -\sin 60 & -\cos 30 \cos 60 \\ -\sin 30 \sin 60 & \cos 60 & -\cos 30 \sin 60 \\ \cos 30 & 0 & -\sin 30 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{4} & -\frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & 1/2 & -\frac{3}{4} \\ \sqrt{3}/2 & 0 & -1/2 \end{bmatrix}$$

~~The~~ It can be checked that determinant $= 1$

The principal values and principal vectors are (obtained from Matlab).

$\lambda_1 = 1$ $\qquad \lambda_{2,3} = -0.625 \pm 0.7806 i$

$$[\hat{\nu}_1]_{\varepsilon_0} = \begin{bmatrix} -0.4804 \\ 0.8321 \\ -0.2773 \end{bmatrix}$$

Rotation axis is $-0.48 \hat{E}_1 + 0.8321 \hat{E}_2 - 0.2773 \hat{E}_3$

Angle of Rotation $= \tan^{-1}\left(\frac{0.7806}{-0.625}\right)$

$= -51.32^\circ$