SOLUTION TO PROBLEM 4 OF TUTORIAL 4

We need to find the final rotation tensor $\underline{\underline{R}}$ evaluated in $\left\{\mathcal{E}_0, G, \hat{E}_i\right\}$ for the transformation $\{\mathcal{E}_0, G, \hat{\mathbf{E}}_i\} \xrightarrow{\underline{\underline{R}}} \{\mathcal{E}, G, \hat{\mathbf{e}}_i\}$ using the $3-2-1$ Euler angle sequence. We also need to find the three rotation angles in terms of the components of $\left[\underline{\underline{R}}\right]_{\mathcal{E}_0}$.

We first note the sequence of coordinate systems obtained under the $3-2-1$ Euler angle sequence as shown in Fig. 1.

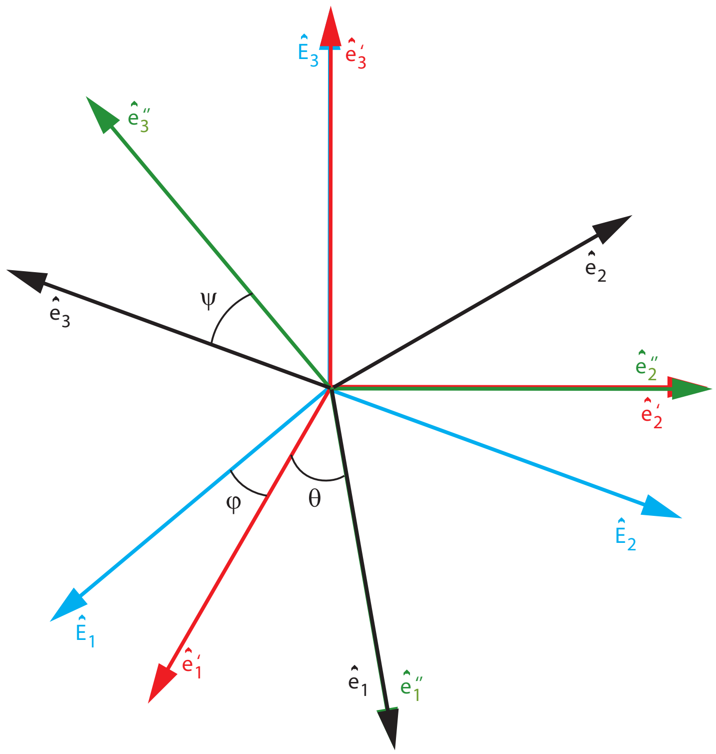

Figure 1: Various coordinate systems involved in $3-2-1$ Euler angle sequence for representing the rotation tensor.

The flowchart for the various transformation under this sequence is

$$\{\mathcal{E}_0, \hat{\mathbf{E}}_i\} \xrightarrow{\underline{\underline{R}}_{\varphi}} \{\mathcal{E}', \hat{\mathbf{e}}'_i\} \xrightarrow{\underline{\underline{R}}_{\theta}} \{\mathcal{E}'', \hat{\mathbf{e}}''_i\} \xrightarrow{\underline{\underline{R}}_{\psi}} \{\mathcal{E}, \hat{\mathbf{e}}_i\}\,.$$

The final rotation tensor $\underline{\underline{R}}$ for the transformation $\{\mathcal{E}_0, G, \hat{\mathbf{E}}_i\} \xrightarrow{\underline{\underline{R}}} \{\mathcal{E}, G, \hat{\mathbf{e}}_i\}$ is given by

$$\underline{\underline{R}} = \underline{\underline{R}}_{\psi}(\hat{\mathbf{e}}''_1, \psi) \cdot \underline{\underline{R}}_{\theta}(\hat{\mathbf{e}}'_2, \theta) \cdot \underline{\underline{R}}_{\varphi}(\hat{\mathbf{E}}_3, \varphi)\,.$$

Using the same line of argument as in the solution for problem 3 and using the various coordinate transformation laws appropriately, we get the final form as

$$[\underline{\underline{R}}]_{\mathcal{E}_0} = [\underline{\underline{R}}_{\varphi}(\hat{\mathbf{E}}_3)]_{\mathcal{E}_0}\,[\underline{\underline{R}}_{\theta}(\hat{\mathbf{e}}'_2)]_{\mathcal{E}'}\,[\underline{\underline{R}}_{\psi}(\hat{\mathbf{e}}''_1)]_{\mathcal{E}''}\,.$$

Also, using similar concepts as in solution for problem 3, one can easily obtain

$$[\underline{\underline{R}}_{\varphi}]_{\mathcal{E}_0} = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$[\underline{\underline{R}}_\theta]_{\mathcal{E}'} = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix},$$

$$[\underline{\underline{R}}_\psi]_{\mathcal{E}''} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{pmatrix}.$$

Putting everything together, we have

$$\begin{aligned}
[\underline{\underline{R}}]_{\mathcal{E}_0} &= \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{pmatrix} \\
&= \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta\,\sin\psi & \sin\theta\,\cos\psi \\ 0 & \cos\psi & -\sin\psi \\ -\sin\theta & \cos\theta\,\sin\psi & \cos\theta\,\cos\psi \end{pmatrix} \\
&= \begin{pmatrix} \cos\varphi\,\cos\theta & \cos\varphi\,\sin\theta\,\sin\psi - \sin\varphi\,\cos\psi & \cos\varphi\,\sin\theta\,\cos\psi + \sin\varphi\,\sin\psi \\ \sin\varphi\,\cos\theta & \sin\varphi\,\sin\theta\,\sin\psi + \cos\varphi\,\cos\psi & \sin\varphi\,\sin\theta\,\cos\psi - \cos\varphi\,\sin\psi \\ -\sin\theta & \cos\theta\,\sin\psi & \cos\theta\,\cos\psi \end{pmatrix}.
\end{aligned}$$

Finally, comparing the various entries in the matrix representation of the final rotation tensor $\underline{\underline{R}}$, we can get the various Euler angles as

$$\varphi = \tan^{-1}\left(\frac{R_{12}}{R_{11}}\right) \qquad \theta = \sin^{-1}(-R_{31}), \qquad \psi = \tan^{-1}\left(\frac{R_{32}}{R_{33}}\right).$$