Lecture 9

Rigid body kinematics:Angular velocity; Examples; Relation to Euler angle sequence

25 - 31 August, 2021

Angular velocity

I. **Properties**. Rigid body in *3D* motion:
$\boldsymbol{\omega}(t) = \dot{\varphi}\hat{\mathbf{n}}(t) = \mathsf{ax}\left(\dot{\mathsf{R}}_0 \cdot \mathsf{R}_0^T\right) = \mathsf{ax}(\, \boldsymbol{\Omega}(t)\,).$

1. Angular velocity $\boldsymbol{\omega}$ is a vector.
2. Can add / subtract angular velocities.

II. **Example**: Relative angular velocity.

Rigid body $\mathscr{B}_1$ is *observed* by another rotating rigid body $\mathscr{B}_2$ to have angular velocity $\boldsymbol{\omega}_{1/2}$. If $\mathscr{B}_2$ is *measured* to have angular velocity $\boldsymbol{\omega}_{2/0}$ in a CS $\{\mathscr{E}_0, O, \hat{\mathbf{E}}_i\}$, find the angular velocity of $\boldsymbol{\omega}_{1/0}$ of $\mathscr{B}_1$ that would be observed in $\mathscr{E}_0$.

$\Longleftrightarrow$ $\mathscr{B}_1$ is measured by a rotating CS $\{\mathscr{E}'', G_2, \hat{\mathbf{e}}_i''\}$ to have *angular velocity* $\boldsymbol{\omega}_{1/2}$. If $\mathscr{E}''$ *rotates at* $\boldsymbol{\omega}_{1/0}$ with respect to CS $\mathscr{E}_0$, find *rotation rate* of $\mathscr{B}_1$ with respect to $\mathscr{E}_0$.

Angular velocity

Example: *2D rotation from 3D description*. A rigid body $\mathscr{B}$'s motion is described by a <u>moving</u> observer $\{\mathscr{E}_0, O, \hat{\mathbf{E}}_i(t)\}$ to be

$$\{\mathscr{E}_0, O, \hat{\mathbf{E}}_i(t)\} \xrightarrow{\mathsf{R}\left(\hat{\mathbf{E}}_3(t),\, \varphi(t)\right)} \{\mathscr{E}, G, \hat{\mathbf{e}}_i(t)\},$$

where $\mathscr{E}$ is the rigid body's BFCS. Find angular velocity $\boldsymbol{\omega}_{\mathscr{E}/\mathscr{E}_0}$ of the body $\mathscr{B}$ with respect to $\mathscr{E}_0$.

Solution:

1. *Angular velocity tensor*: $\boldsymbol{\Omega}(t) = \dot{\mathsf{R}}(t) \cdot \mathsf{R}^T(t)$.

   i. $\dot{\mathsf{R}}(t)$: Time derivative of $\mathsf{R}(t)$ **<u>w.r.t</u>**. $\mathscr{E}_0(t)$!

$$[\mathsf{R}]_{\mathscr{E}_0} = \begin{pmatrix} \cos\varphi(t) & -\sin\varphi(t) & 0 \\ \sin\varphi(t) & \cos\varphi(t) & 0 \\ 0 & 0 & 1 \end{pmatrix}; \; [\dot{\mathsf{R}}]_{\mathscr{E}_0} = \dot{\varphi} \begin{pmatrix} -\sin\varphi(t) & -\cos\varphi(t) & 0 \\ \cos\varphi(t) & -\sin\varphi(t) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Therefore, $[\,\boldsymbol{\Omega}(t)]_{\mathscr{E}_0} = \dot{\varphi} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

2. Angular velocity $\boldsymbol{\omega}_{\mathscr{E}/\mathscr{E}_0} = \mathrm{ax}(\,\boldsymbol{\Omega}(t)\,) = \dot{\varphi}\hat{\mathbf{E}}_3$.

Remark: Angular velocity of $\mathscr{B}$ *as measured by observer* $\mathscr{E}_0$, ***not*** total (i.e. w.r.t. *non*-rotating CS).

Angular velocity

Example. *Angular velocity using the* ***z-x-z*** *or* ***3-1-3 Euler angle sequence***. Consider a rigid body with BFCS located at any time t by :

$$\{\mathscr{E}_0, \hat{\mathbf{E}}_i\} \xrightarrow{\mathsf{R}_\varphi(t)} \{\mathscr{E}', \hat{\mathbf{e}}_i'(t)\} \xrightarrow{\mathsf{R}_\theta(t)} \{\mathscr{E}'', \hat{\mathbf{e}}_i''(t)\} \xrightarrow{\mathsf{R}_\psi(t)} \{\mathscr{E}, \hat{\mathbf{e}}_i(t)\}$$

$$\mathsf{R}_\varphi(t) = \mathsf{R}\big(\hat{\mathbf{E}}_3, \varphi(t)\big);\ \mathsf{R}_\theta(t) = \mathsf{R}\left(\hat{\mathbf{e}}_1'(t), \theta(t)\right);\ \mathsf{R}_\psi(t) = \mathsf{R}\left(\hat{\mathbf{e}}_3''(t), \psi(t)\right)$$

Find the rigid body's angular velocity $\boldsymbol{\omega}_{\mathscr{E}/\mathscr{E}_0}$

Solution I. $\{\mathscr{E}_0, O, \hat{\mathbf{E}}_i(t)\} \xrightarrow{\mathsf{R}(t)} \{\mathscr{E}, \hat{\mathbf{e}}_i(t)\}$ with $\mathsf{R}(t) = \mathsf{R}_\psi \cdot \mathsf{R}_\theta \cdot \mathsf{R}_\varphi$. Find $\boldsymbol{\Omega}(t) = \dot{\mathsf{R}}(t) \cdot \mathsf{R}^T(t)$.

Solution II. Use *relative* angular velocity.

1. $\boldsymbol{\omega}_{\mathscr{E}/\mathscr{E}_0} = \boldsymbol{\omega}_{\mathscr{E}/\mathscr{E}''} + \boldsymbol{\omega}_{\mathscr{E}''/\mathscr{E}'} + \boldsymbol{\omega}_{\mathscr{E}'/\mathscr{E}_0} = \dot{\psi}\hat{\mathbf{e}}_3'' + \dot{\theta}\hat{\mathbf{e}}_1' + \dot{\varphi}\hat{\mathbf{E}}_3$

2. Need $[\boldsymbol{\omega}_{\mathscr{E}/\mathscr{E}_0}]_{\mathscr{E}_0} = \dot{\psi}[\hat{\mathbf{e}}_3'']_{\mathscr{E}_0} + \dot{\theta}[\hat{\mathbf{e}}_1']_{\mathscr{E}_0} + \dot{\varphi}[\hat{\mathbf{E}}_3]_{\mathscr{E}_0}$

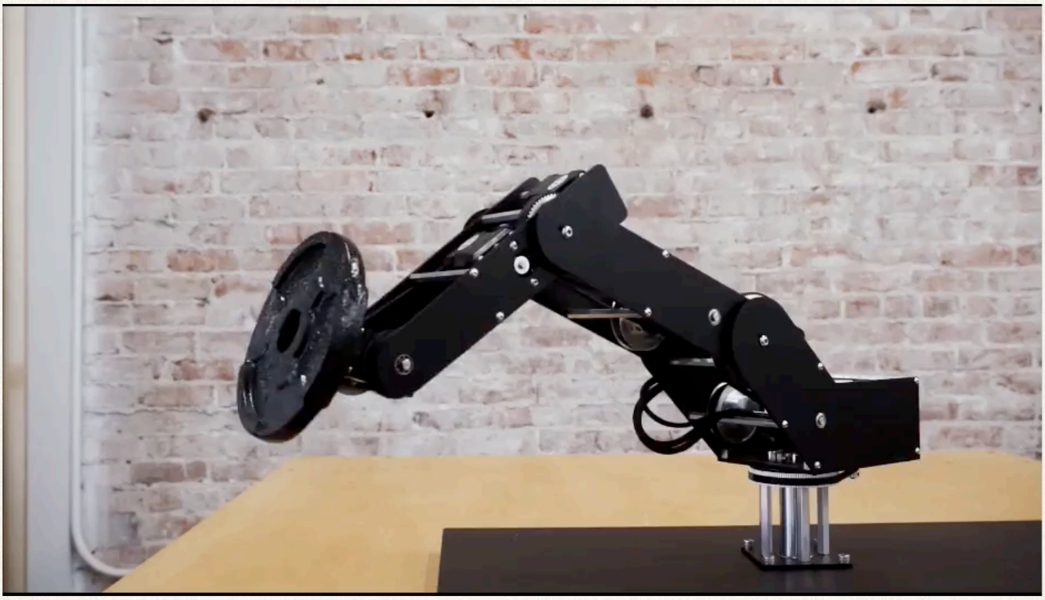
3. $\hat{\mathbf{e}}_3'' = \mathsf{R}_\theta \cdot \mathsf{R}_\varphi \cdot \hat{\mathbf{E}}_3$ and $\hat{\mathbf{e}}_1' = \mathsf{R}_\varphi \cdot \hat{\mathbf{E}}_1 \implies$

$[\hat{\mathbf{e}}_3'']_{\mathscr{E}_0} = [\mathsf{R}_\varphi]_{\mathscr{E}_0}[\mathsf{R}_\theta]_{\mathscr{E}'}[\hat{\mathbf{E}}_3]_{\mathscr{E}_0}$ and $[\hat{\mathbf{e}}_1']_{\mathscr{E}_0} = [\mathsf{R}_\varphi]_{\mathscr{E}_0}[\hat{\mathbf{E}}_1]_{\mathscr{E}_0}$

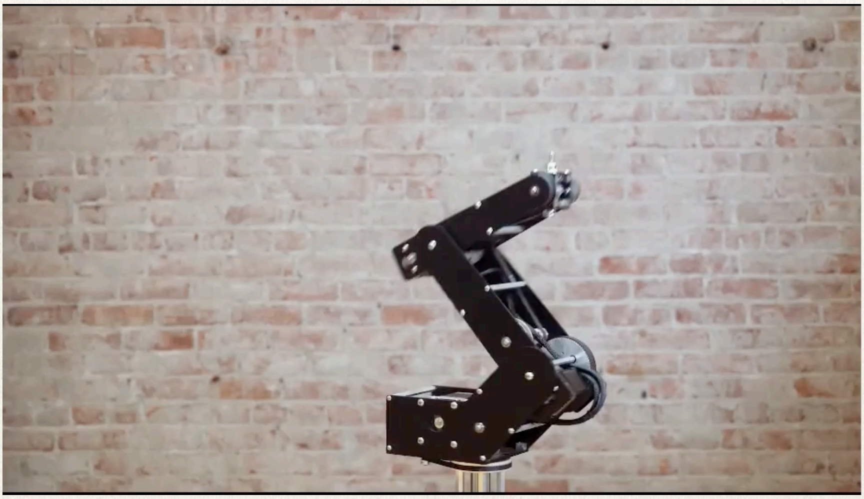
4. <u>Know</u> $[\mathsf{R}_\varphi]_{\mathscr{E}_0}$ and $[\mathsf{R}_\theta]_{\mathscr{E}'}$ — 2D rotations.

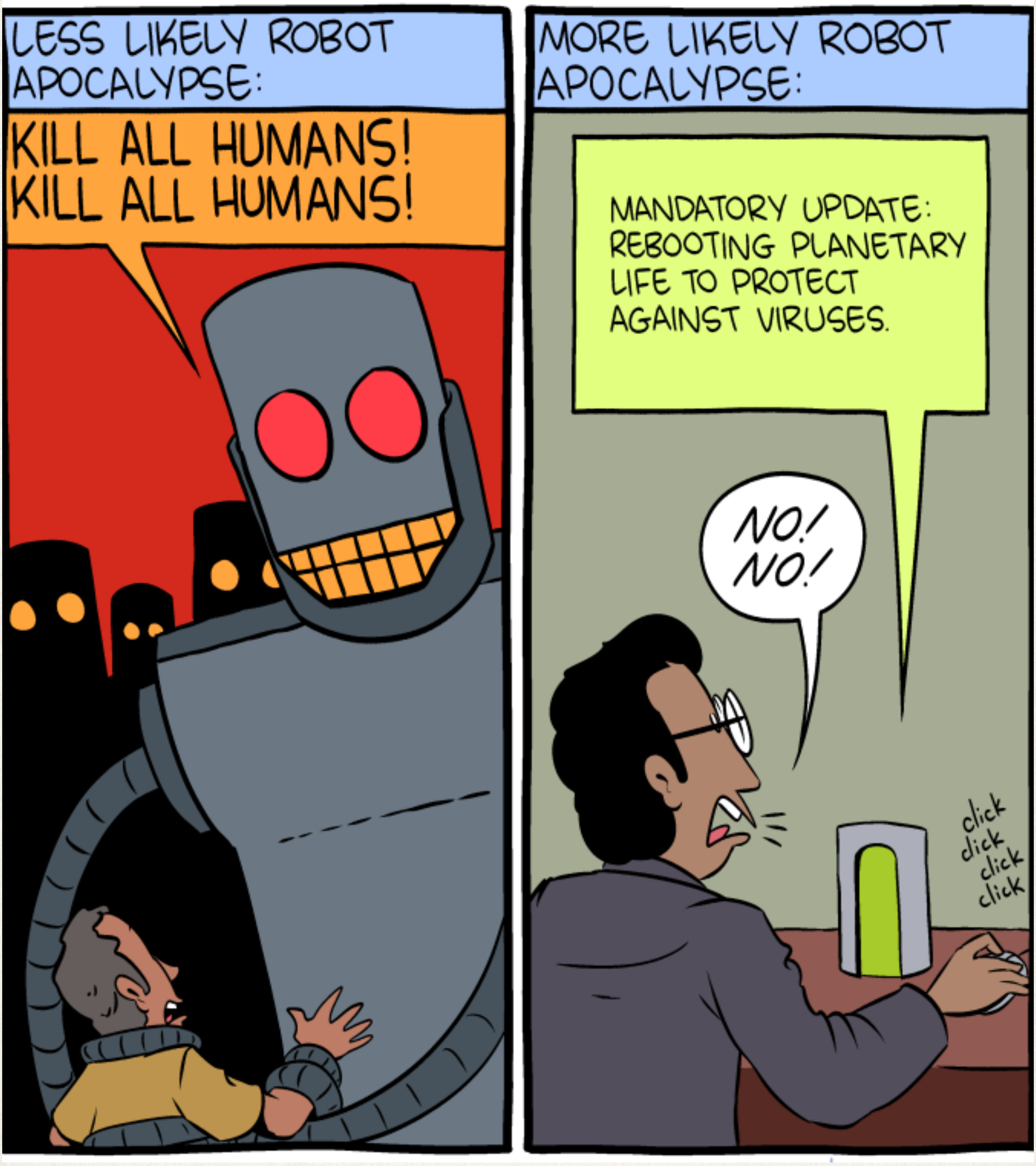
Angular velocity

I. **Example**. Robotic arm carrying a disc $\mathscr{D}$.



Find $\boldsymbol{\omega}_{\mathscr{D}}$ assuming that each joint rotates with respect to the previous joint at the rate $\dot{\theta}$, and the base rotates at the rate $\dot{\varphi}$.




LESS LIKELY ROBOT APOCALYPSE:
KILL ALL HUMANS! KILL ALL HUMANS!
MORE LIKELY ROBOT APOCALYPSE:
MANDATORY UPDATE: REBOOTING PLANETARY LIFE TO PROTECT AGAINST VIRUSES.
NO! NO!
click click click click