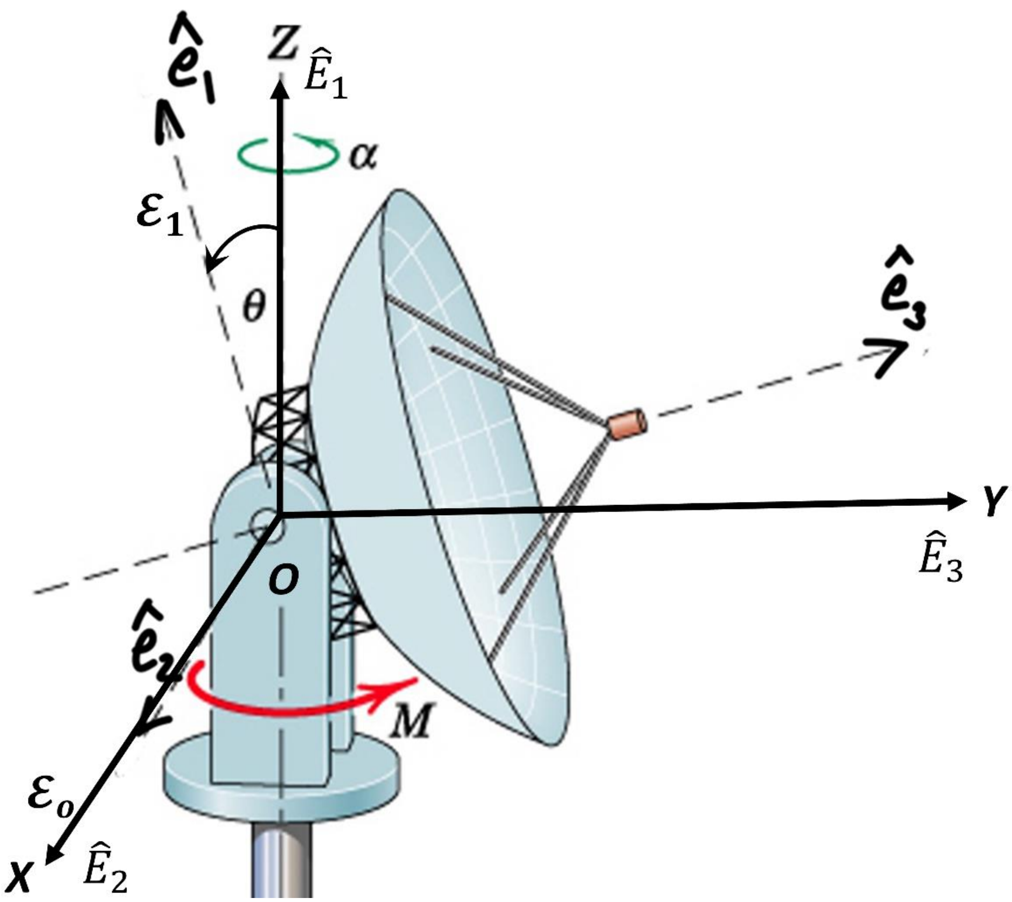
Problem 5:



Solution:

$$\{\mathcal{E}_o, O, \hat{E}_i\} \xrightarrow{R\{\hat{e}_2, \theta\}} \{\mathcal{E}_1, O, \hat{e}_i\}$$

Rotation matrix R is given as

$$[R]_{\mathcal{E}_o} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \dots\dots (1)$$

We know that $[R]_{\mathcal{E}_o} = [R]_{\mathcal{E}_1}$

Given, antenna has Moment of inertia I about its $\hat{e}_3$ axis of symmetry and Moment of inertia I_o about each $\hat{e}_1$ and $\hat{e}_2$ axis.

Clearly cross product of inertia must be zero because $\hat{e}_3$ is the axis of symmetry. Therefore

$$I_{xy} = I_{yz} = I_{xz} = 0 \text{ and } I_{xx} = I_{yy} = I_o, I_{zz} = I$$

Then moment of inertia tensor is given as

$$[I]_{\mathcal{E}_1} = \begin{bmatrix} I_o & 0 & 0 \\ 0 & I_o & 0 \\ 0 & 0 & I \end{bmatrix} \ldots\ldots(2)$$

Now find $[I]_{\mathcal{E}_o}$ which will be

$$[I]_{\mathcal{E}_o} = [R]_{\mathcal{E}_1}^T [I]_{\mathcal{E}_1} [R]_{\mathcal{E}_1}$$

$$[I]_{\mathcal{E}_o} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} I_o & 0 & 0 \\ 0 & I_o & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$[I]_{\mathcal{E}_o} = \begin{bmatrix} I_o(\cos\theta)^2 + I(\sin\theta)^2 & 0 & I_o \sin\theta\cos\theta - I\sin\theta\cos\theta \\ 0 & I_o & 0 \\ I_o \sin\theta\cos\theta - I\sin\theta\cos\theta & 0 & I_o(\cos\theta)^2 + I(\sin\theta)^2 \end{bmatrix} \ldots\ldots(3)$$

We know Total moment about non-accelerating point P (see lecture notes):

$$\underline{M}^P = \underline{r}^{G/P} \times m\underline{a}^G + \underline{\omega}^B \times \left(I^G \cdot \underline{\omega}^B\right) + I^G \cdot \underline{\alpha}^B \ldots\ldots\ldots.(4)$$

Since when Torque M is applied, complete system is at rest therefore $\underline{\omega}^B = 0$ and eq. (4) becomes

$$\underline{M}^P = \underline{r}^{G/P} \times m\underline{a}^G + I^G \cdot \underline{\alpha}^B \ldots\ldots\ldots.(5)$$

Taking Total moment about G of antenna and assuming that G coincides with O (see figure)

$$\underline{M}^G = \underline{r}^{G/G} \times m\underline{a}^G + I^G \cdot \underline{\alpha}^B \ldots\ldots\ldots.(6)$$

Since $\underline{r}^{G/G} = 0$

$$\underline{M}^G = I^G \cdot \underline{\alpha}^B \ldots\ldots\ldots.(7)$$

$$\underline{\alpha}^B = \alpha\hat{E}_1 \ldots\ldots(8)$$

Calculating RHS of eq. (7), use eq. (3) and eq. (8)

$$I^G \cdot \underline{\alpha}^B = \{I_o(\cos\theta)^2 + I(\sin\theta)^2\}\alpha\hat{E}_1 + (I_o\sin\theta\cos\theta - I\sin\theta\cos\theta)\alpha\hat{E}_3 \ldots\ldots(9)$$

$$\underline{M}^G = M\hat{E}_1 \ldots\ldots(10)$$

Take dot product of eq. (9) and eq. (10) with $\hat{E}_1$

$$\{I_o(\cos\theta)^2 + I(\sin\theta)^2\}\alpha = M$$

$$\alpha = \frac{M}{I_o(\cos\theta)^2 + I(\sin\theta)^2}$$

Here if we take dot product of eq. (9) and eq. (10) with $\hat{E}_3$, we also get an eq. in α,

$$(I_o\sin\theta\cos\theta - I\sin\theta\cos\theta)\alpha = 0$$

From here we can find a relation for θ.