Lecture 14

Rigid body kinetics: Inertial CS; Linear momentum balance; Angular momentum balance.

22-28 September, 2021

Inertial CS

I. The **laws of motion** are written in an *Inertial Coordinate System*.

II. **Definition**. An *inertial coordinate system* is one in which the laws of motion hold.

III. **No** inertial CS has yet been found.

1. Ref.: *Science of Mechanics* by *E. Mach*

IV. **However,** laws of mechanics hold excellently in a given CS, provided that the non-inertial nature of the CS is small compared to the motion being studied. For example, when studying a

1. satellite around the Earth ($a_{sat} \sim 1$ m-s^{-2}), we can ignore the Earth's acceleration around the sun (≈ 0.006 m-s^{-2}).
2. car ($a_{car} \sim 1$ m-s^{-2}), the Earth's centripetal acceleration (≈ 0.034 m-s^{-2}) due to its rotation may be ignored.

Balance laws

I. All mechanical system <u>must</u> satisfy
   1. Mass balance.
   2. Linear momentum balance (LMB)
   3. Angular momentum balance (AMB)

II. **Two** ways to study a mechanical system:
   1. *Control mass*: Follow evolution of the <u>same</u> set of bodies and material points.
      i. Good for systems with <u>solids</u>.
   2. *Control volume*: Follow evolution of only those bodies and material points that occupy a given volume.
      i. Good for systems with <u>fluids</u>.

III. Rigid body system has <u>*no*</u> mass exchange
   1. Follow a *control mass* analysis, i.e. focus on the <u>same</u> set of rigid bodies.
   2. Mass balance is trivially true.

LMB for particles

वैशेषिक सूत्र (Vaiśeṣika Sūtra) — कणाद (Kaṇāda) in about 600 BC (300 years *before* Aristotle).

1. संयोगाभावे गुरुत्वात् पतनम् — *In the absence of conjunction, gravity [causes] fall.* [Newton's 1st]
2. नोदनविशेषाभावान्नोर्ध्वं न तिर्य्यग्गमनम् — *In the absence of a force, there is no upward motion, sideward motion or motion in general.*
3. नोदनादाद्यमिषोः कर्म तत्कर्मकारिताच्च संस्कारादुत्तरं तथोत्तरमुत्तरञ्च् — *The initial pressure [on the bow] leads to the arrow's motion; from that motion is momentum, from which is the motion that follows and the next and so on similarly.* [$\leqslant$ Newton's 2nd]
4. कार्य्यविरोधि कर्म — *Action (kārya) is opposed by reaction (karman)* [Newton's 3rd]

Further reading

1. *Kaṇāda, Great Physicist and Sage of Antiquity* — Subhask Kak (Link)
2. *Matter and Mind: The Vaisheshika Sutra of Kanada* — Subhash Kak (Link to Amazon)

LMB

I. (**Newton's 2nd law, 1686 AD**) Rate of change of linear momentum of a particle equals the total applied force.

II. (**Euler's 1st law, ~1730 AD**) Rate of change of linear momentum of a rigid body equals the total applied force.

   1. Can<u>not</u> derive this from Newton's law

III. Linear momentum of a rigid body

$$\mathbf{p} = \int_V \rho(\mathbf{r})\mathbf{v}(\mathbf{r})\mathrm{d}V = m\mathbf{v}^G \, .$$

IV. Then, LMB gives: $\boxed{\sum \mathbf{F}^i = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = m\mathbf{a}^G \, .}$

AMB

I. (**Euler's 2nd law, ~1730 AD**) Rate of change of angular momentum of a rigid body about a non-accelerating point P, i.e. $\mathbf{a}^P = \mathbf{0}$ in an inertial CS, equals the total applied moment about P.

1. Newton had no such law.
2. Cannot derive from LMB.

II. *Angular momentum* about a point P :

$$\mathbf{h}^P = \mathbf{r}^{G/P} \times m\mathbf{v}^G + \underbrace{\mathrm{I}^G \cdot \boldsymbol{\omega}^{\mathscr{B}}}_{\mathbf{h}^G}.$$

III. Total moment about P:

$$\mathbf{M}^P = \sum \mathbf{r}^{i/P} \times \mathbf{F}^i + \sum \mathbf{M}^j$$

IV. AMB about P : $\mathbf{M}^P = \dfrac{\mathrm{d}\mathbf{h}^P}{\mathrm{d}t} \implies$

$$\boxed{\mathbf{M}^P = \mathbf{r}^{G/P} \times m\mathbf{a}^G + \boldsymbol{\omega}^{\mathscr{B}} \times \left(\mathrm{I}^G \cdot \boldsymbol{\omega}^{\mathscr{B}}\right) + \mathrm{I}^G \cdot \boldsymbol{\alpha}^{\mathscr{B}}}$$

AMB

I. AMB about <u>non</u>-accelerating P $\left(\mathbf{a}^P = \mathbf{0}\right)$:

$$\boxed{\mathbf{M}^P = \mathbf{r}^{G/P} \times m\mathbf{a}^G + \boldsymbol{\omega}^{\mathscr{B}} \times \left(\mathrm{I}^G \cdot \boldsymbol{\omega}^{\mathscr{B}}\right) + \mathrm{I}^G \cdot \boldsymbol{\alpha}^{\mathscr{B}}}$$

II. **Special cases**.

1. $P = G$, and $\mathbf{a}^G$ arbitrary :

$$\boxed{\mathbf{M}^G = \boldsymbol{\omega}^{\mathscr{B}} \times \left(\mathrm{I}^G \cdot \boldsymbol{\omega}^{\mathscr{B}}\right) + \mathrm{I}^G \cdot \boldsymbol{\alpha}^{\mathscr{B}}.}$$

2. P <u>lying</u> on rigid body, and $\mathbf{a}^P$ arbitrary:

$$\boxed{\mathbf{M}^P = \mathbf{r}^{G/P} \times m\mathbf{a}^P + \boldsymbol{\omega}^{\mathscr{B}} \times \left(\mathrm{I}^P \cdot \boldsymbol{\omega}^{\mathscr{B}}\right) + \mathrm{I}^P \cdot \boldsymbol{\alpha}^{\mathscr{B}}}$$

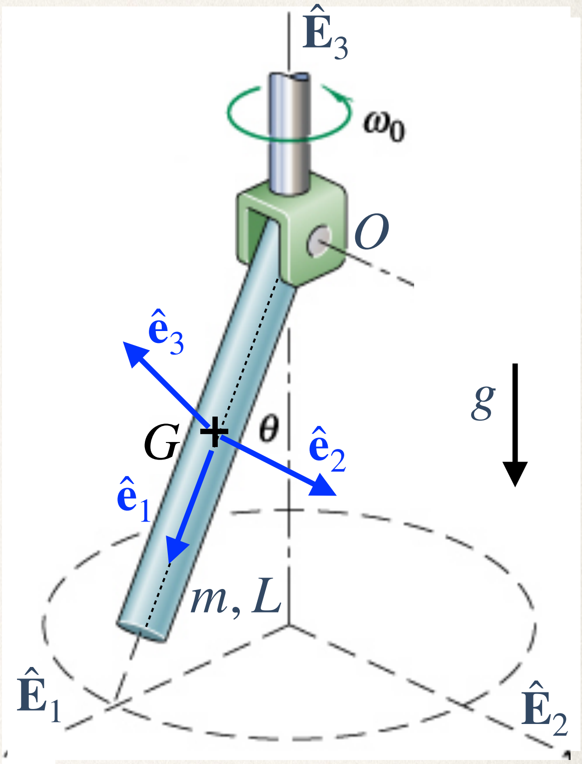
3. *2D rigid body kinetics*. If $\boldsymbol{\omega}^{\mathscr{B}}$ is <u>always</u> along (say) $\hat{\mathbf{e}}_3$ principal axis of I^G:

$$\boxed{\mathbf{M}^P = \mathbf{r}^{G/P} \times m\mathbf{a}^G + I_3^G \boldsymbol{\alpha}^{\mathscr{B}}.}$$

i. 2D kinematics $\supsetneq$ 2D kinetics.

Example 1

Find angle θ given ω_0.



Important. The most crucial thing is to draw neat and correct *Free Body Diagrams* (FBD).

1. *LMB*: $\sum \mathbf{F}^i = m\mathbf{a}^G$
2. *AMB about O*: Point O lies on rod, so use

$$\mathbf{M}^O = \mathbf{r}^{G/O} \times m\mathbf{a}^O + \boldsymbol{\omega}^{\mathscr{B}} \times \left(\mathrm{I}^O \cdot \boldsymbol{\omega}^{\mathscr{B}}\right) + \mathrm{I}^O \cdot \boldsymbol{\alpha}^{\mathscr{B}}$$

3. *Kinematic analysis* $\implies \boldsymbol{\omega}^{\mathscr{B}}, \boldsymbol{\alpha}^{\mathscr{B}}, \mathbf{a}^G$.
4. Use *parallel axis theorem* to get I^O:

$$\mathrm{I}^O = \mathrm{I}^G + m\left(\left|\mathbf{r}^{G/O}\right|^2 1 - \mathbf{r}^{G/O} \otimes \mathbf{r}^{G/O}\right).$$

5. **Answer**: $\cos\theta = 3g/\left(2\omega_0^2 L\right)$.
   i. When $\omega \to \infty, \theta \to 90°$.
   ii. Non-zero θ only if $\omega_0 \geqslant \sqrt{3g/(2L)}$.

Teaching western science to indian brains.

Babu Teacher. "Number One is called a 'right angle,' and you would naturally suppose that Number Two is a 'left angle.' But by order of Government of India Survey Department this is also a right angle."

Punch magazine 1924

Doesn't work well.
Learn science the Indian way.