Sinusoidal Steady-State Response

- Deals with response of circuits under ***sinusoidal*** (sine or cosine functions) excitations (***voltage*** or ***current***)
- The signals are ***periodic*** function of time
- There is no ***transient*** disturbance (all ***perturbations*** have settled down)
- To discuss about: ***phasors***, ***impedance***, ***admittance***, and ***response*** of RC and RL circuits to ***sinusoidal excitation***

- ***Sinusoidal Signals and Phasors***:

* Sinusoidal signals can be expressed as:

$$v(t) = V_M \sin(\omega t + \phi)$$

V_M: ***Peak Value*** (also known as ***amplitude***)

ω: ***Angular Frequency*** (radians/second)

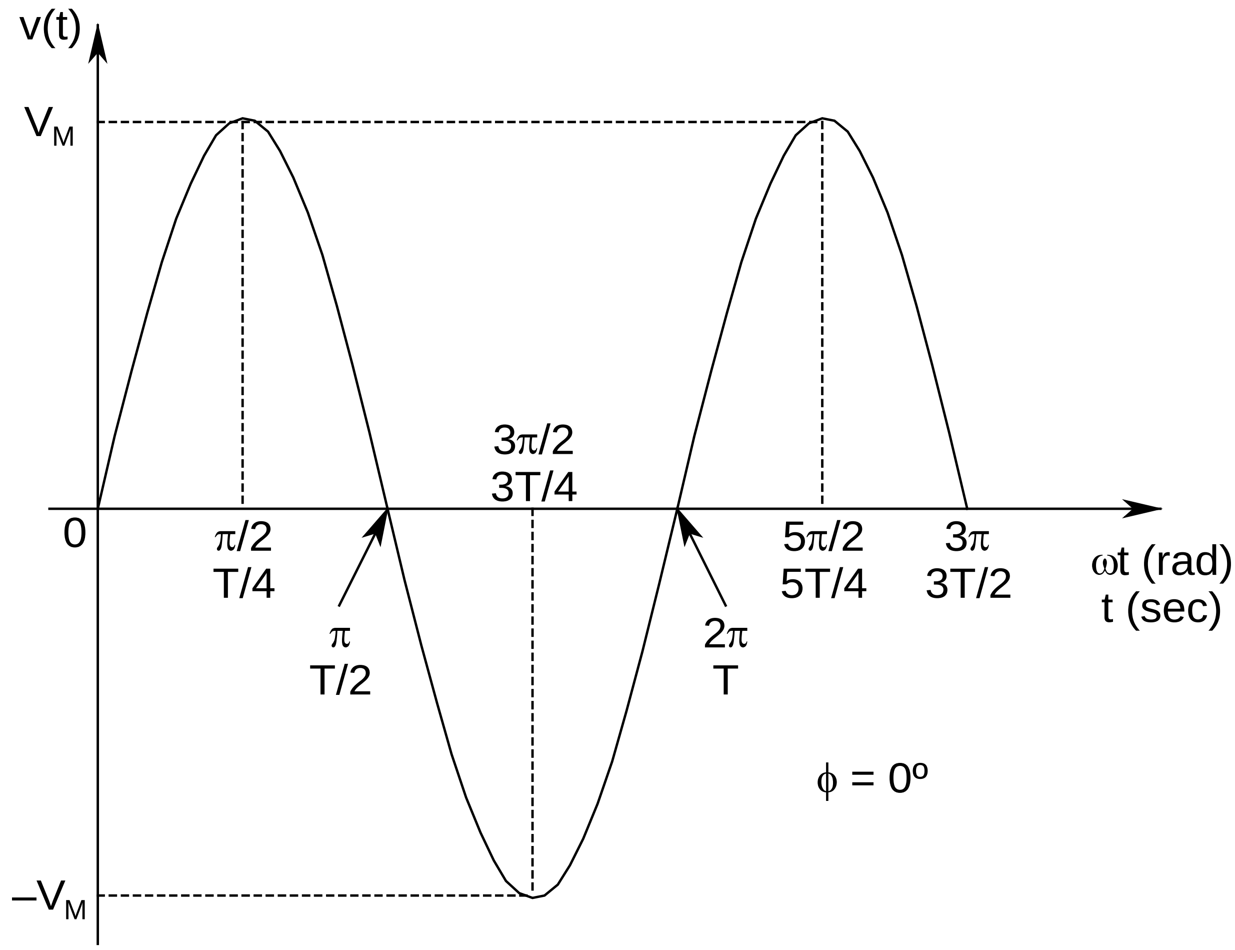
$= 2\pi f$, f: ***Cyclic Frequency*** [per second or Hertz (Hz)]

ϕ: ***Phase Angle*** (radians) (***Note***: π radians = 180°)

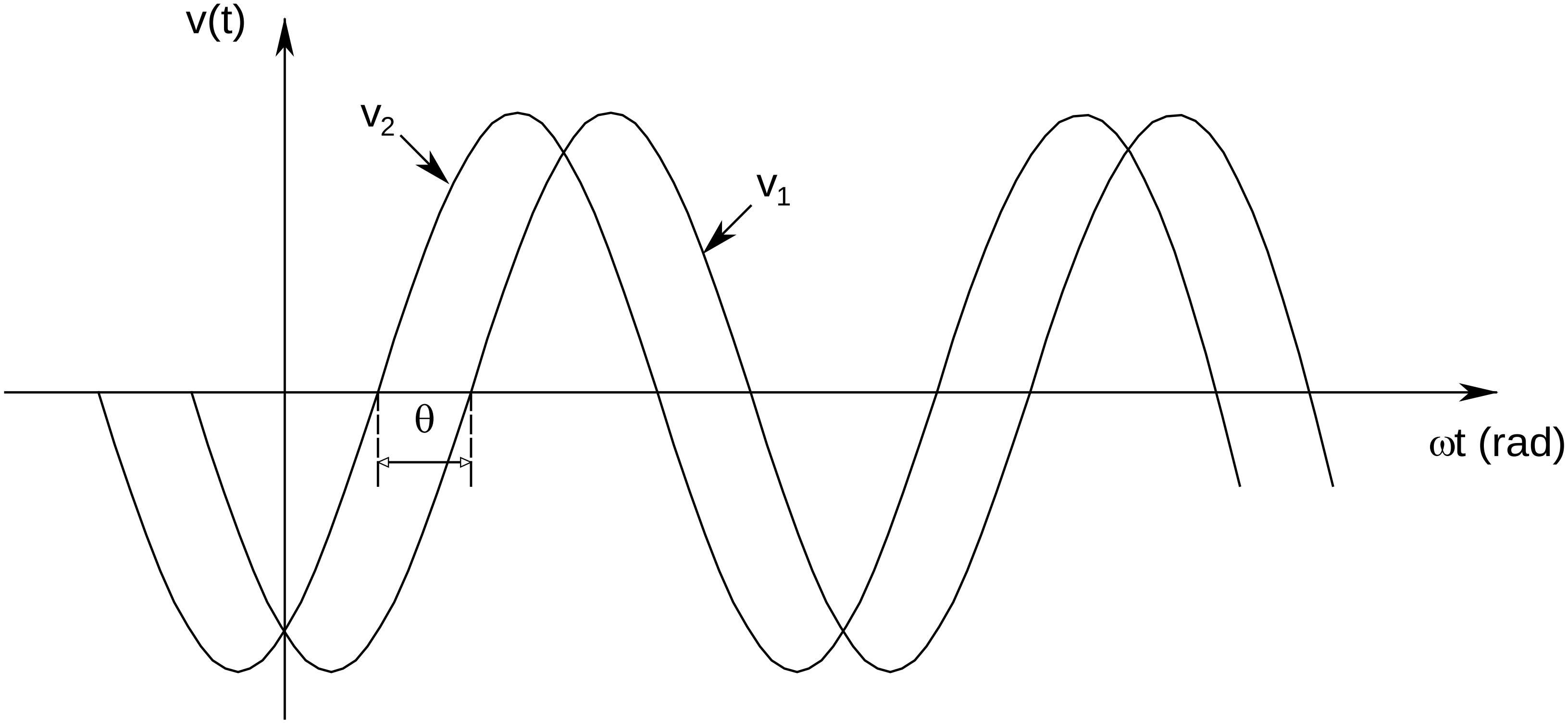
* Another important term: ***Time Period*** (T) (second)

$$T = 1/f \quad = 2\pi/\omega$$

* ***Note***: During one time period, the signal sweeps an angle of 2π


v(t)
V_M
$-V_M$
0
$\pi/2$
$T/4$
π
$T/2$
$3\pi/2$
$3T/4$
2π
T
$5\pi/2$
$5T/4$
3π
$3T/2$
ωt (rad)
t (sec)
$\phi = 0^o$

Concept of Lead/Lag:



v_2 leads v_1, or conversely, v_1 lags v_2

θ: Phase difference between v_1 and v_2

* Going along the time scale, the first signal encountered having a ***positive*** going excursion is ***leading*** the rest of the signals
* ***Ex***: In the figure, v_2 is leading v_1, or conversely, v_1 is lagging v_2
* When $\theta = 0°$, signals v_1 and v_2 overlap, and are known as ***in-phase*** signals
* When $\theta \neq 0°$, these signals are known as ***out-of-phase*** signals
* When $\theta = 180°$, they are ***exactly*** out-of-phase

* ***Note***: The algebraic sum of two exactly equal in-phase signals is ***double*** of either of them, whereas the algebraic sum of two exactly equal and exactly out-of-phase signals is ***zero***
* ***Note***: Leading by θ also implies lagging by $(360° - \theta)$

  $\Rightarrow$ Depends on the ***Frame of Reference***
* ***Useful trigonometric identities***:

$$\pm\sin(\omega t) = \cos\left(\omega t \mp \frac{\pi}{2}\right) \quad \text{and}$$

$$\pm\cos(\omega t) = \sin\left(\omega t \pm \frac{\pi}{2}\right)$$

Concept of Phasors:

* While representing sinusoidal signals either in ***angular sweep domain*** or ***time domain*** (i.e., ωt or t respectively), confusion may arise regarding whether a given waveform is ***leading*** or ***lagging*** some other waveforms

* This confusion can largely be eliminated by representing sinusoidal signals as ***phasors***

* ***Note***: Sinusoidal signals can also be expressed as complex numbers by using ***Euler's identity*** :

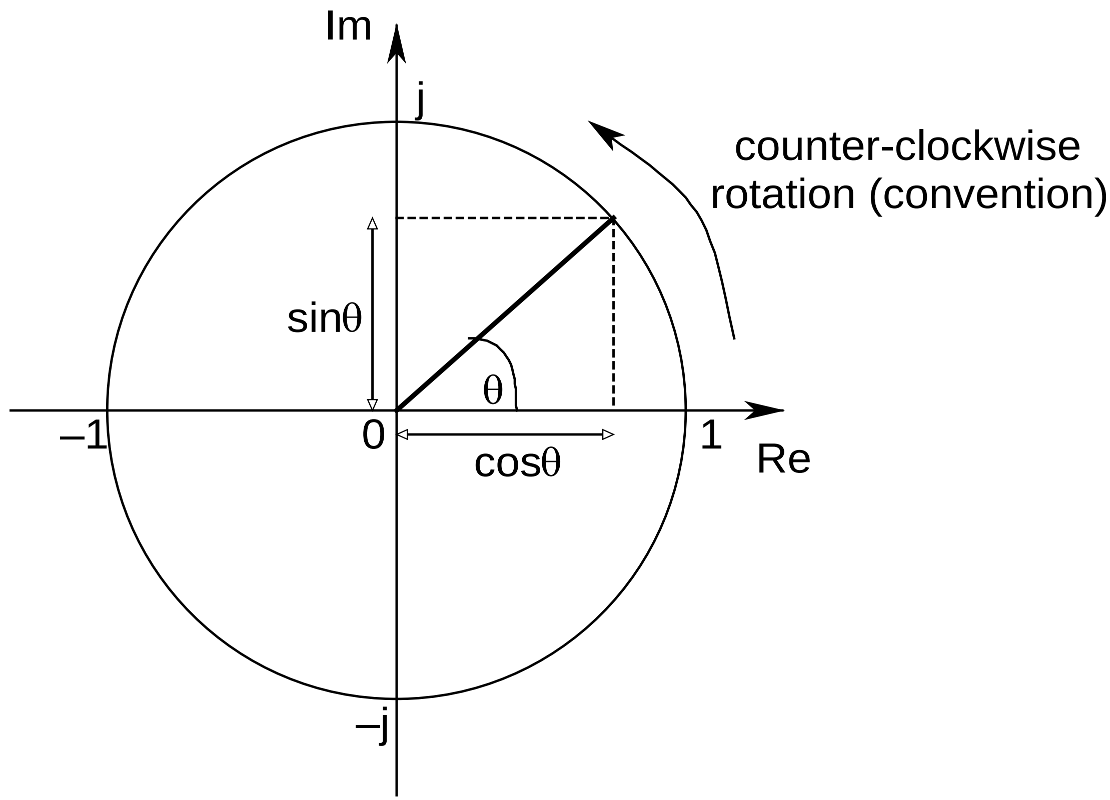
$$\exp(j\theta) = \cos\theta + j\sin\theta$$

* ***Differential equations*** that need to be solved to obtain the response of a circuit become pure ***algebraic equations*** $\Rightarrow$ the advantage is obvious
* ***Euler's identity*** draws out a unit circle in the ***complex plane***, since

$$\left|\exp(j\theta)\right| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

$$\Rightarrow A\exp(j\theta) = A\cos\theta + jA\sin\theta = A\angle\theta$$

$A\angle\theta$: known as a ***phasor***, expressed in ***polar form***
* ***Note***: Any complex number $(a + jb)$ can be expressed as $A\angle\theta$, with $A = \sqrt{a^2 + b^2}$ and $\theta = \tan^{-1}(b/a)$



Rotating Unit Phasor

* A sinusoidal function can be expressed in ***polar form***, and thus can be represented as a ***phasor*** as:

$$A\cos(\omega t+\theta)=\boldsymbol{Re}\left[A\exp\left\{j(\omega t+\theta)\right\}\right]$$

* Thus, the ***time domain*** representation of a sinusoidal function of the form:

$$v(t)=V_M\cos(\omega t+\theta)=\boldsymbol{Re}\left[V_M\exp\left\{j(\omega t+\theta)\right\}\right]$$

$$=\boldsymbol{Re}\left[V_M\exp(j\theta)\exp(j\omega t)\right]$$

can be expressed in ***polar form*** (or in ***phasor representation***) as:

$$\bar{V}(j\omega)=V_M\exp(j\theta)=V_M\angle\theta$$

* Note the ***bar sign*** above V, implying that it is a ***phasor***
* This representation basically implies that the phasor $\bar{V}$ is rotating ***anti-clockwise*** in the complex plane with an angular frequency ω, subtending an angle of θ (***referenced to a pure cosine signal***) at any given time t
* ***Note***: After conversion to the phasor notation, the information regarding the angular frequency ω gets ***suppressed***

* ***Example***: $2\cos(50t) \Leftrightarrow 2\angle 0°$

$$5\cos(100t + 30°) \Leftrightarrow 5\angle 30°$$

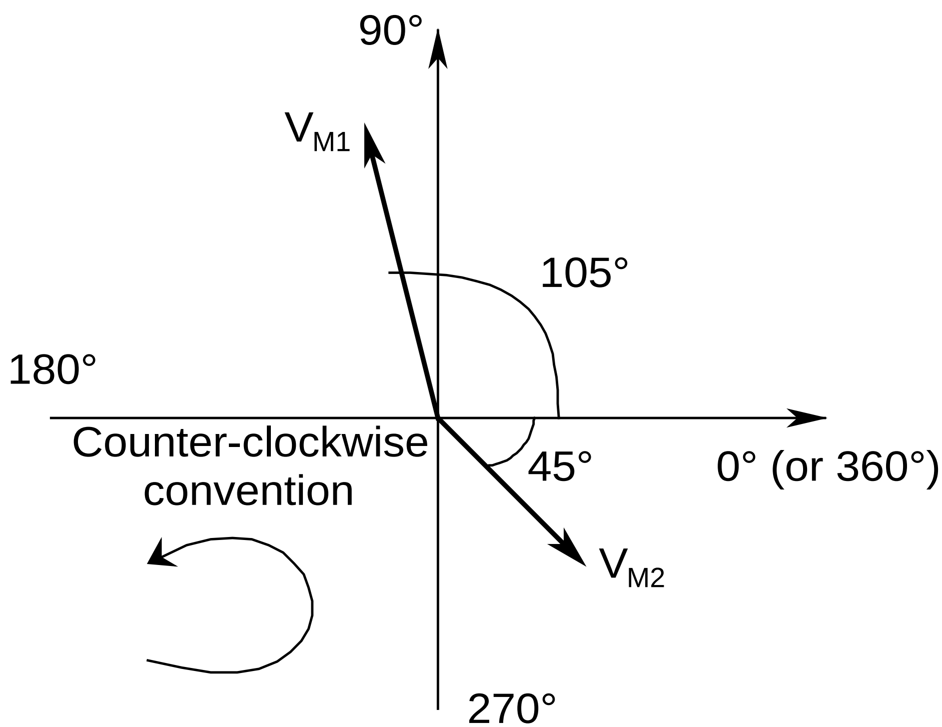
* ***Note***: The two examples are for two different frequencies, however, the phasor representations ***hide*** this information
* That's why the LHS has the term ***(jω)*** associated with $\overline{V}$
* With phasor representations, it becomes easy to visualize the ***leading*** or ***lagging*** phenomenon of various waveforms

Example: Consider two waveforms:

$$v_1 = V_{M1}\cos(\omega t + 105°) \text{ and}$$

$$v_2 = V_{M2}\cos(\omega t - 45°)$$

with $V_{M1} > V_{M2}$



* ***Caution***: For two phasors to be plotted on the same graph, they must have the ***same*** angular frequency ω
* From the plot, it can be observed that v_1 ***leads*** v_2 by 150°, or, equivalently, v_2 ***lags*** v_1 by 150°
* It can also be conversely stated as v_2 ***leads*** v_1 by 210°, or, equivalently, v_1 ***lags*** v_2 by 210°
* Both are technically correct, however, in general, the angle of lead or lag is generally expressed as a number that is ***less*** than 180°

* Therefore, the ***first*** representation is the ***correct*** one
* ***Caution***: To convert a ***sine*** function to a phasor, must transform it to a ***cosine*** function first, using the ***trigonometrical identities*** given earlier
* Phasors can be treated as ***algebraic variables***

  $\Rightarrow$ various ***algebraic functions*** can be performed on them

Phasor Algebra:

* Let $v_1(t) = 5\cos(\omega t + \pi/6)$ and

  $v_2(t) = 10\cos(\omega t - \pi/4)$

* Both signals have ***same frequency***, and both are in ***cosine representation***

  $\Rightarrow$ ***phasor algebra*** can be performed on them

* $\bar{V}_1(j\omega) = 5\angle 30° = 5\cos 30° + j5\sin 30° = 4.33 + j2.5$

  $\bar{V}_2(j\omega) = 10\angle -45° = 10\cos(-45°) + j10\sin(-45°)$

  $= 7.07 - j7.07$

$$v_1(t) + v_2(t) = \bar{V}_1(j\omega) + \bar{V}_2(j\omega)$$

$$= 11.4 - j4.57 = 12.28\angle -21.84°$$

$$= 12.28\cos(\omega t - 21.84°)$$

* Similarly,

$$v_1(t) - v_2(t) = \bar{V}_1(j\omega) - \bar{V}_2(j\omega)$$

$$= -2.74 + j9.57 = 9.95\angle 105.98°$$

$$= 9.95\cos(\omega t + 105.98°)$$

* Be very careful while finding the angle (θ)

Observations:

* If both cos and sin terms are ***positive***

$\Rightarrow$ ***first quadrant*** and θ will be between ***0° and 90°***

* If sin term is ***positive*** but cos term is ***negative***

$\Rightarrow$ ***second quadrant*** and θ will be between ***90° and 180°***

* If both terms are ***negative***

$\Rightarrow$ ***third quadrant*** and θ will be between - ***90° and -180°***

* If cos term is ***positive*** but sin term is ***negative***

$\Rightarrow$ ***fourth quadrant*** and θ will be between ***0° and -90°***

Note:

* Phasor algebra can be done if and only if the sinusoids are of the ***same frequency***

* If the sinusoids have different frequencies, then phasor algebra will ***not*** be applicable

* Under such a condition, the response of the circuit for each individual frequencies should be evaluated, and the net response will be given by a ***superposition*** of all these results

* We will be showing this example later

Example:

Let $v_1(t) = 10\cos(\omega t - 60°)$ and

$v_2(t) = 5\sin(\omega t + 135°)$

Both signals have same frequency, however, while v_1 is in cos form, v_2 is in sin form

In order to apply the phasor algebra, we need to convert v_2 to cos form as well, and then proceed

$$
\begin{aligned}
v_1(t) + v_2(t) &= 10\cos(\omega t - 60°) + 5\sin(\omega t + 135°) \\
&= 10\cos(\omega t - 60°) + 5\cos(\omega t + 45°) \\
&= 10\angle -60° + 5\angle 45° = 8.54 - j5.12 \\
&= 9.96\angle -30.94° = 9.96\cos(\omega t - 30.94°)
\end{aligned}
$$

- ***Impedance***:
  - The value of a pure resistance does not change with frequency (neglecting ***skin effect***)
  - However, for inductors and capacitors under sinusoidal excitation, the ***resistance*** offered by them to the current flow is ***frequency dependent***
  - These are ***complex resistances***, and are known as ***impedance*** (impediment to current flow)
  - Impedances are of two types: ***capacitive*** and ***inductive*** (note that both are ***imaginary***)
  - Both have unit of ***ohm*** (Ω)
  - Resistors have ***real*** impedance, which is equal to their resistance

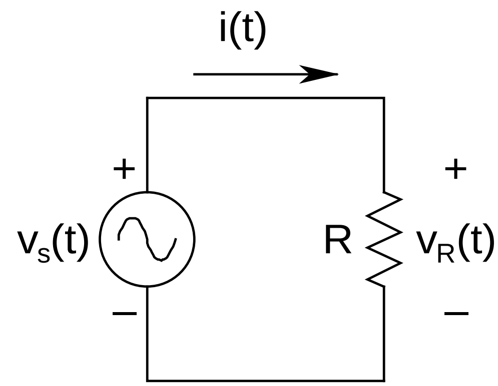
Resistors, Resistance, and Resistive Impedance:

Let $v_s(t) = V_M\cos(\omega t) = v_R(t)$

Phasor: $\bar{V}_R(j\omega) = V_M\angle 0°$

$i(t) = v_R(t)/R = (V_M/R)\cos(\omega t)$

Phasor: $\bar{I}(j\omega) = (V_M/R)\angle 0°$



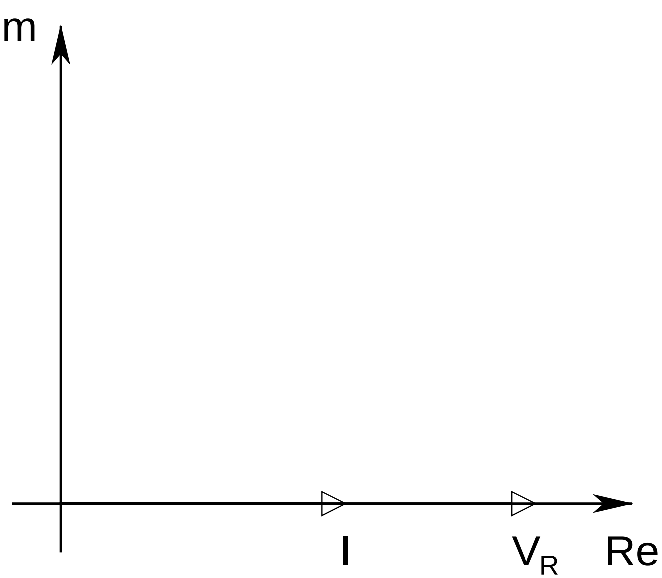
Inference: The current through a resistor and the voltage across it are ***in-phase***

Voltage dropped across R $\left[v_R(t)\right]$ in phasor form:

$\bar{V}_R(j\omega) = \bar{I}(j\omega)R$

Resistive Impedance:

$Z_R = \bar{V}_R(j\omega)/\bar{I}(j\omega) = R$



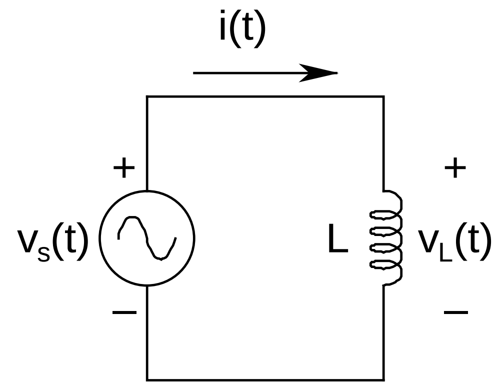
Note: Both the phasors $\bar{I}$ and $\overline{V}_R$ are ***in-phase***

Also, both of them lie on the ***real*** axis on the complex plane

The ratio of $\overline{V}_R$ and $\bar{I}$ is the ***resistive impedance***,

which is ***real*** and is equal to the value of the resistance (R)

Inductors, Inductive Impedance, and Inductive Reactance:



Let $v_s(t) = V_M \cos(\omega t)$

$$v_L(t) = v_s(t) = L\frac{di(t)}{dt}$$

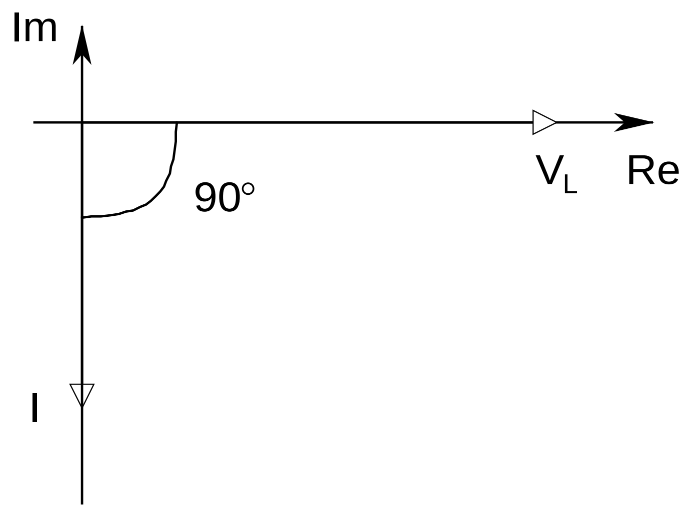
$$\Rightarrow i(t) = \frac{1}{L}\int v_L(t)dt = \frac{V_M}{\omega L}\sin(\omega t)$$

$$= \frac{V_M}{\omega L}\cos\left(\omega t - \frac{\pi}{2}\right)$$

In ***phasor*** notation:

$$\overline{V}_L(j\omega) = V_M\angle 0° \text{ and } \overline{I}(j\omega) = \left[V_M/(\omega L)\right]\angle -90°$$

Thus, the inductor current ***lags*** the inductor voltage by 90°



Note: The phasors $\bar{\text{I}}$ and $\overline{\text{V}}_\text{L}$ are ***out-of -phase***, with $\bar{\text{I}}$ ***lagging*** $\overline{\text{V}}_\text{L}$ by 90°

$$\textbf{\textit{Inductive Impedance}} = \text{Z}_\text{L}\left(\text{j}\omega\right) = \frac{\overline{\text{V}}_\text{L}\left(\text{j}\omega\right)}{\bar{\text{I}}\left(\text{j}\omega\right)} = \omega\text{L}\angle 90° = \text{j}\omega\text{L}$$

where ***j*** implies a rotation of +90°

Important Observations:

* Inductive impedance Z_L is a ***complex*** number

* It is a purely ***imaginary*** number, with no real part

* It is a ***direct*** function of the angular frequency ω

* For very low frequency, as $\omega \to 0$, $Z_L \to 0$

$\Rightarrow$ Thus, for ***dc***, inductors behave as ***short-circuits***

* For very high frequency, as $\omega \to \infty$, $Z_L \to \infty$

$\Rightarrow$ Implies that inductors behave like ***open-circuits*** for ***very high frequencies***

* Z_L is also expressed as: $Z_L = jX_L$

$X_L = \omega L \Rightarrow$ known as ***inductive reactance***

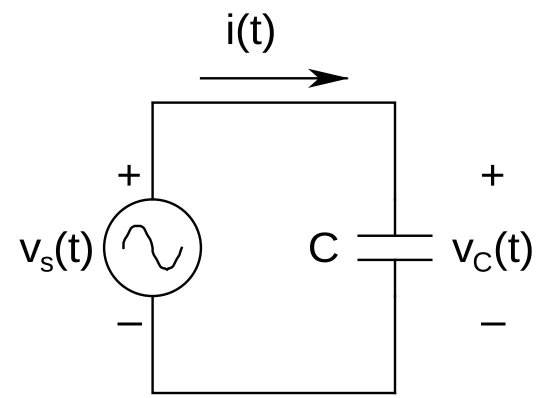
* ***Note***: While Z_L is imaginary, X_L is real

Capacitors, Capacitive Impedance, and Capacitive Reactance:

$$\text{Let } v_s(t) = v_C(t) = V_M \cos(\omega t)$$

$$i(t) = C\frac{dv_C(t)}{dt} = -V_M \omega C \sin(\omega t)$$

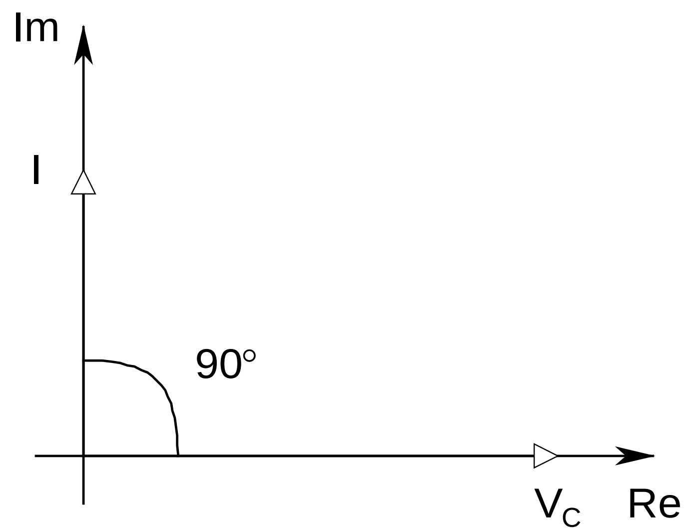
$$= V_M \omega C \cos\left(\omega t + \frac{\pi}{2}\right)$$



In ***phasor*** notation:

$$\bar{V}_C(j\omega) = V_M \angle 0° \text{ and } \bar{I}(j\omega) = V_M \omega C \angle 90°$$

Thus, the capacitor current ***leads*** the capacitor voltage by 90°



Note: The phasors $\bar{I}$ and $\bar{V}_C$ are ***out-of-phase***, with $\bar{I}$ ***leading*** $\bar{V}_C$ by 90°

$$\textbf{\textit{Capacitive Impedance}} = Z_C(j\omega) = \frac{\bar{V}_C(j\omega)}{\bar{I}(j\omega)} = \frac{1}{\omega C}\angle -90^\circ$$

$$= -\frac{j}{\omega C} = \frac{1}{j\omega C}$$

Important Observations:

* Capacitive impedance Z_C also is a ***complex*** number
* It is also a purely ***imaginary*** number, with no real part
* It is an ***inverse*** function of the angular frequency ω
* For very low frequency, as $\omega \to 0$, $Z_C \to \infty$

  $\Rightarrow$ Thus, for ***dc***, capacitors behave as ***open-circuits***

* For very high frequency, as $\omega \to \infty$, $Z_C \to 0$

  $\Rightarrow$ Implies that capacitors behave like ***short-circuits*** for ***very high frequencies***

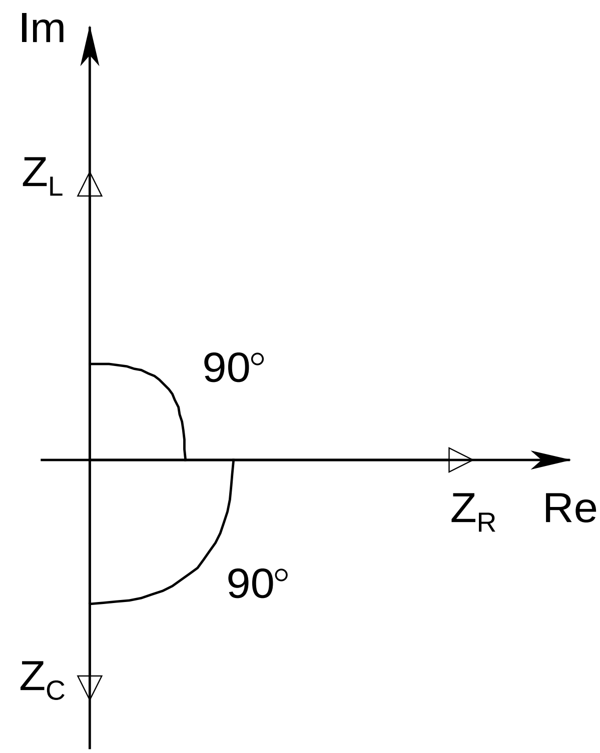
* Z_C is also expressed as: $Z_C = -jX_C$

  $X_C = 1/(\omega C) \Rightarrow$ known as ***capacitive reactance***

* ***Note***: While Z_C is imaginary, X_C is real

Complex Impedance Plane:

* Resistive impedance Z_R is along the ***positive*** real axis
* Inductive and capacitive impedances Z_L and Z_C are along the positive and negative ***imaginary*** axes, respectively
* For a pure resistance, the impedance is ***real***



* For a pure inductor or capacitor, the impedance is totally ***imaginary***

* For circuits having inductors and capacitors along with resistors, the impedance would be ***complex***, having both real and imaginary parts
* If $Z_L > |Z_C|$, then the net impedance of the circuit would be ***inductive***
* On the other hand, if $|Z_C| > Z_L$, then the net impedance of the circuit would be ***capacitive***
* If $Z_L = |Z_C|$, then the net impedance of the circuit would be purely ***resistive***

* The ***general form*** of the impedance Z:

$$Z = Z_R + \left(Z_L - \left|Z_C\right|\right) = R + \left(j\omega L - \frac{j}{\omega C}\right)$$

$$= R + j\left(X_L - X_C\right)$$

* ***Note***: Z is not a phasor, but just a ***complex number***
* Thus, it can be represented in the complex impedance plane by both ***magnitude*** and ***direction***

* In general, impedances are expressed in the ***polar*** form: $|Z|\angle Z$

with $|Z| = \sqrt{R^2 + (X_L - X_C)^2}$

and $\angle Z = \tan^{-1}\left(\dfrac{X_L - X_C}{R}\right)$

* For ***inductive*** circuits, $X_L > X_C$ and $\angle Z$ positive
* For ***capacitive*** circuits, $X_C > X_L$ and $\angle Z$ negative
* For $X_L = X_C$, circuit becomes purely ***resistive*** and $\angle Z = 0°$

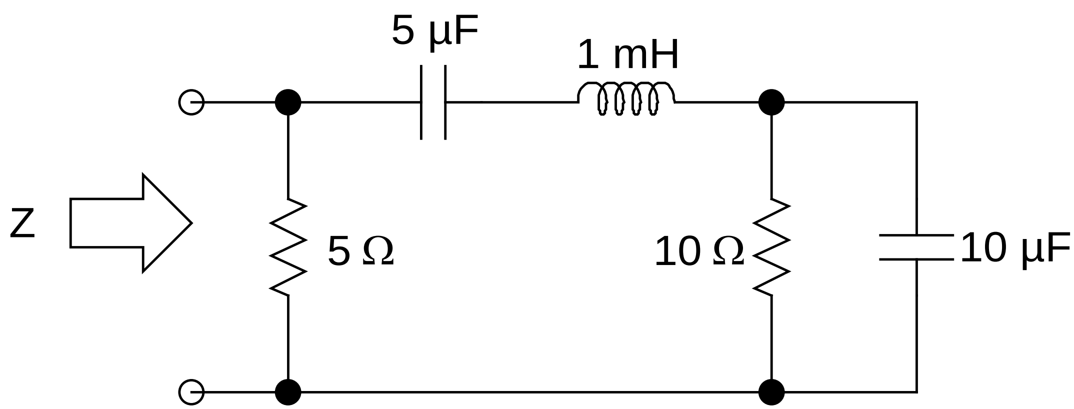
Series - Parallel Combination of Impedances:

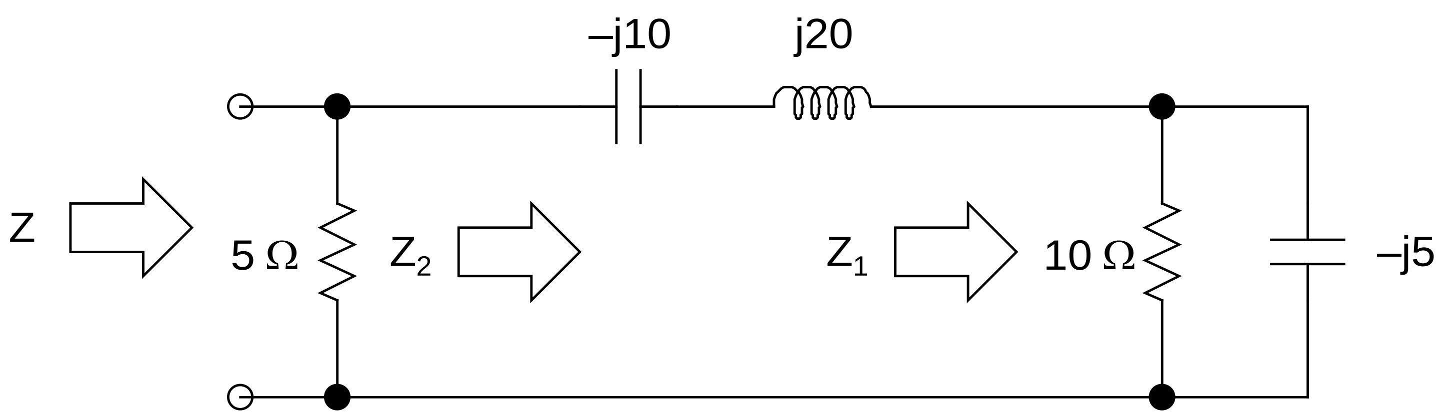
Example: To find Z (ω = 20 krad/sec)

$$5\mu F{:}\ Z_C = -j/(\omega C) = -j10\ \Omega$$

$$1\ mH{:}\ Z_L = j\omega L = j20\ \Omega$$

$$10\mu F{:}\ -j5\ \Omega$$





$$Z_1 = 10 \,||\, \left(-j5\right) = \frac{10 \times \left(-j5\right)}{10 - j5} = \frac{50\angle -90^\circ}{11.18\angle -26.57^\circ}$$

$$= 4.47\angle -63.43^\circ = 2 - j4$$

$$Z_2 = -j10 + j20 + 2 - j4 = 2 + j6 = 6.32\angle 71.57^\circ$$

$$Z = 5 \,||\, \left(6.32\angle 71.57^\circ\right) = \frac{5 \times \left(6.32\angle 71.57^\circ\right)}{5 + 2 + j6} = \frac{31.6\angle 71.57^\circ}{7 + j6}$$

$$= \frac{31.6\angle 71.57^\circ}{9.22\angle 40.6^\circ} = 3.43\angle 30.97^\circ = 2.94 + j1.77\ \Omega$$

Since the imaginary component of Z is ***positive***, hence, the net impedance of the circuit is ***inductive*** in nature

Admittance (Y):

* Y is the ***dual*** (or ***inverse***) of Z:

$$Y = \frac{1}{Z} = \frac{\overline{I}}{\overline{V}} = G + jB$$

* Has unit of inverse ohm, known as ***mho*** (℧)

* G: ***Conductance*** (inverse of resistance)

* B: ***Susceptance*** (inverse of reactance)

* **Inductor**: ***Inductive Admittance***:

$$jB_L = 1/Z_L = 1/(j\omega L) = -j/(\omega L)$$

$\Rightarrow$ ***Inductive Susceptance*** $B_L = -1/(\omega L)$

* ***Note***: While the inductive reactance (X_L) is ***positive***, the inductive susceptance (B_L) is ***negative***
* **Capacitor**: ***Capacitive Admittance***:

  $jB_C = 1/Z_C = j\omega C$

  $\Rightarrow$ ***Capacitive Susceptance*** $B_C = \omega C$
* Again note that while the capacitive impedance (Z_C) is ***negative***, the capacitive susceptance (B_C) is ***positive***

* The ***general form*** of admittance:

$$Y = G + j\left(B_C - \left|B_L\right|\right) = G + j\left(\omega C - \frac{1}{\omega L}\right)$$

* ***Caution***: If some impedance is expressed as $Z = \left(R + jX\right)$, and its equivalent admittance representation is $Y = \left(G + jB\right)$, then $G \neq 1/R$, and $B \neq 1/X$

* ***Example***: Let $Z = \left(8 - j6\right)$

  Then, $Y = 1/\left(8 - j6\right) = \left(8 + j6\right)/10 = 0.8 + j0.6$

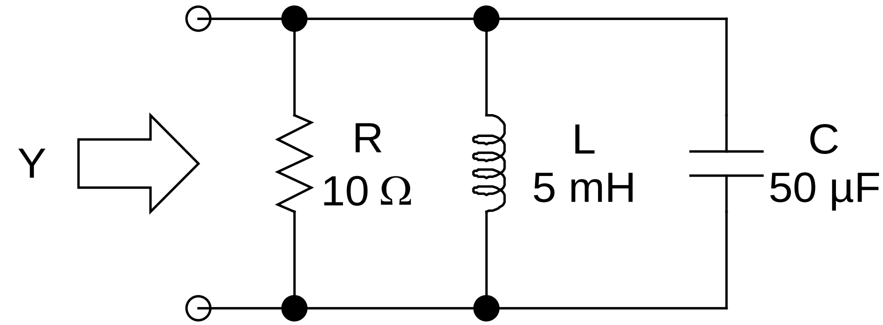
Advantage of Admittance Representation:

* For impedances in ***parallel***, their respective admittances add in ***series***
* This makes the analysis of parallel circuits using the admittance technique quite easy and straightforward

Example: Find Y (ω = 1 krad/sec)

$$G = 1/R = 1/10 = 0.1\ \mho$$

$$\omega L = 5\ \Omega,\ \omega C = 0.05\ \mho$$

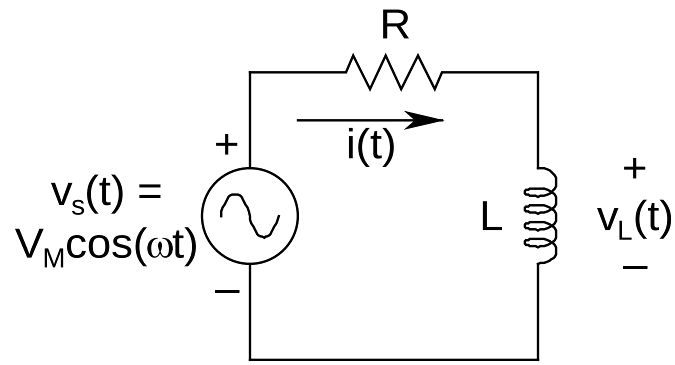


$$Y = G + j\left(\omega C - \frac{1}{\omega L}\right)$$

$$= 0.1 + j\left(0.05 - \frac{1}{5}\right) = 0.1 - j0.15\ \mho$$

$$Z = \frac{1}{Y} = \frac{1}{0.1 - j0.15} = \frac{0.1 + j0.15}{0.18} = 0.56 + j0.83\ \Omega$$

RL Circuit:



$$\bar{V}_S = V_M\angle 0°,\ Z_L = jX_L = j\omega L$$

$$Z = R + jX_L = |Z|\angle Z$$

$$|Z| = \sqrt{R^2 + X_L^2},\ \angle Z = \tan^{-1}\left(X_L / R\right)$$

$$\bar{I} = \frac{\bar{V}_S}{Z} = \frac{V_M\angle 0°}{|Z|\angle Z} = I_M\angle -\tan^{-1}\left(X_L/R\right)\ \text{(phasor)}$$

$$I_M = V_M / |Z|$$

$$i(t) = I_M\cos\left[\omega t - \tan^{-1}\left(\frac{\omega L}{R}\right)\right]\ \text{(sinusoidal)}$$

* ***Caution***: Sinusoidal and phasor representations should ***never*** be mixed up

[Ex.: $10\sin(50t)/3\angle 30° \rightarrow$ ***Invalid***]

<u>Observations</u>:

* For an inductive circuit, the current ***lags*** the excitation voltage
* If $L = 0$, then the circuit becomes purely ***resistive***, the phase angle of the current becomes ***zero***, which makes $\overline{V}_S$ and $\overline{I}$ ***in-phase***

* If R = 0, then the circuit turns purely ***inductive***, and the phase angle of the current becomes exactly equal to ***- 90°***, which implies that the current ***lags*** $\overline{V}_S$ exactly by 90°
* For other values of R, the phase angle ranges between 0° and $-90°$

Another Interesting Observation:

* The voltage appearing across the inductor :

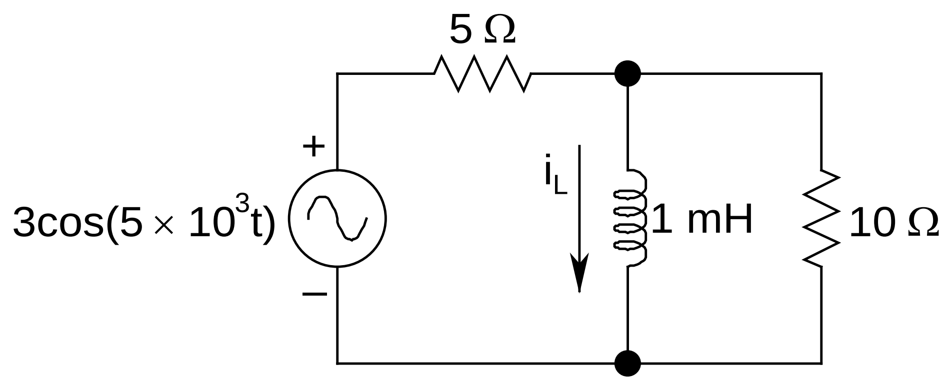
$$\overline{V}_L = \overline{I}Z_L = \left[I_M \angle -\tan^{-1}\left(\frac{\omega L}{R}\right)\right](\omega L)\angle 90°$$

$$= \omega L I_M \angle \left[90° - \tan^{-1}\left(\frac{\omega L}{R}\right)\right]$$

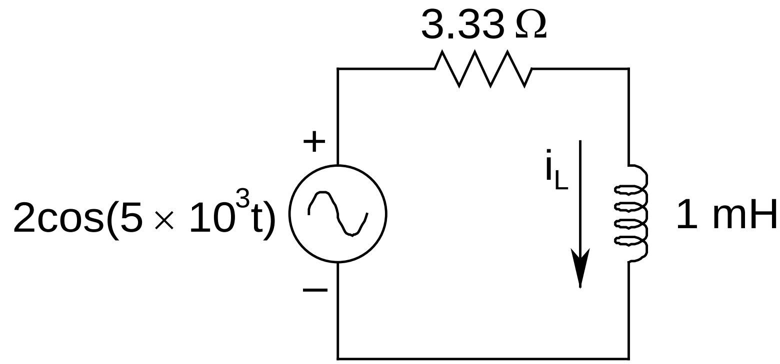
* Thus, the inductor voltage always ***leads*** the inductor current by ***90°***

Example: Find i_L, and express it in both ***phasor*** and ***sinuosidal*** representations:

First, apply Thevenin's technique by opening the inductor from the circuit and finding the open-circuit (or Thevenin) voltage V_T and the Thevenin resistance R_T appearing between these two terminals



By inspection:



$$V_T = (10/15)\times 3\cos\left(5\times 10^3 t\right)$$

$$= 2\cos\left(5\times 10^3 t\right)$$

$$R_T = 5 \| 10 = 3.33\ \Omega$$

$$X_L = \omega L = 5\times 10^3 \times 10^{-3} = 5\ \Omega$$

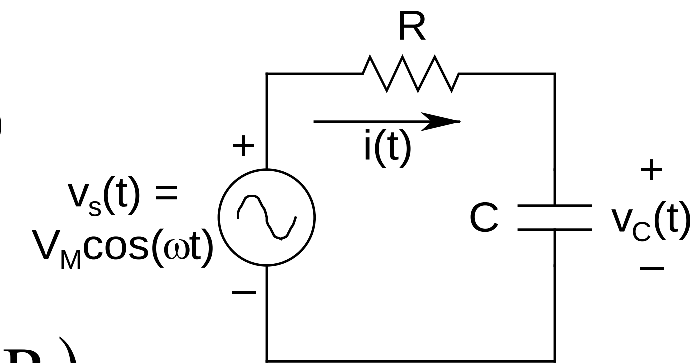
$$|Z| = \sqrt{R^2 + X_L^2} = \sqrt{3.33^2 + 5^2} = 6\ \Omega$$

$$\angle Z = \tan^{-1}\left(X_L / R\right) = \tan^{-1}\left(5/3.33\right) = 56.34°$$

$$\Rightarrow \bar{I}_L = \left(2/6\right)\angle -56.34° = 0.33\angle -56.34°\ \text{A}\ (\textit{\textbf{phasor}})$$

$$i_L(t) = 0.33\cos\left(5\times 10^3 t - 56.34°\right)\ \text{A}\ (\textit{\textbf{sinusoidal}})$$

RC Circuit:



$$\bar{V}_S = V_M\angle 0°,\ Z_C = -jX_C = -j/(\omega C)$$

$$Z = R - jX_C = |Z|\angle Z$$

$$|Z| = \sqrt{R^2 + X_C^2},\ \angle Z = -\tan^{-1}\left(X_C / R\right)$$

$$\bar{I} = \frac{\bar{V}_S}{Z} = \frac{V_M\angle 0°}{|Z|\angle Z} = I_M\angle\tan^{-1}\left(X_C/R\right)\ (\textbf{\textit{phasor}})$$

$$I_M = V_M / |Z|$$

$$i(t) = I_M\cos\left[\omega t + \tan^{-1}\left(\frac{1}{\omega RC}\right)\right]\ (\textbf{\textit{sinusoidal}})$$

Observations:

* For a ***capacitive*** circuit, the current ***leads*** the excitation voltage

* If $C \rightarrow \infty$, consequently $Z_C \rightarrow 0$, then the circuit becomes purely ***resistive***, the phase angle of the current becomes ***zero***, which makes $\overline{V}_S$ and $\overline{I}$ ***in-phase***

* If $R = 0$, then the circuit turns purely ***capacitive***, and the phase angle of the current becomes ***exactly*** equal to $90°$, which implies that the current ***leads*** $\overline{V}_S$ exactly by $90°$

* For other values of R, the phase angle ranges between 0° and 90°
* The voltage appearing across the capacitor:

$$\overline{V}_C = \overline{I}Z_C = \left[I_M \angle \tan^{-1}\left(\frac{1}{\omega RC}\right)\right]\left(\frac{1}{\omega C}\right)\angle -90°$$

$$= \left(\frac{I_M}{\omega C}\right)\angle\left[\tan^{-1}\left(\frac{1}{\omega RC}\right) - 90°\right]$$

* Thus, the capacitor voltage always ***lags*** the capacitor current by 90°

Phasor Analysis of AC Circuits:

All the circuit analysis techniques discussed earlier, viz. the ***node voltage method***, the ***mesh current method***, the ***superposition principle***, and the ***Thevenin and Norton techniques***, are applicable for ac circuits as well

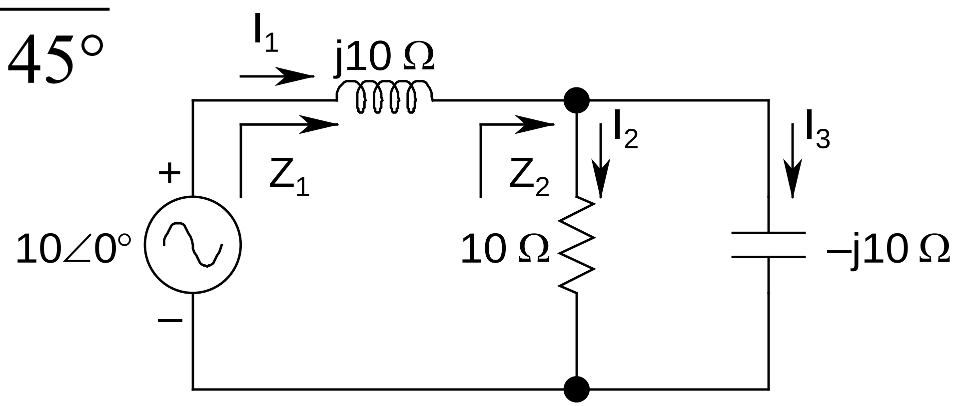
Example: Using phasor analysis, find $\overline{I}_1$, $\overline{I}_2$, and $\overline{I}_3$, and plot them.

$$Z_2 = \frac{10\times(-j10)}{10-j10} = \frac{100\angle-90°}{14.14\angle-45°}$$

$$= 7.07\angle-45° = 5 - j5\ \Omega$$

$$Z_1 = j10 + Z_2 = 5 + j5$$

$$= 7.07\angle 45°\ \Omega$$



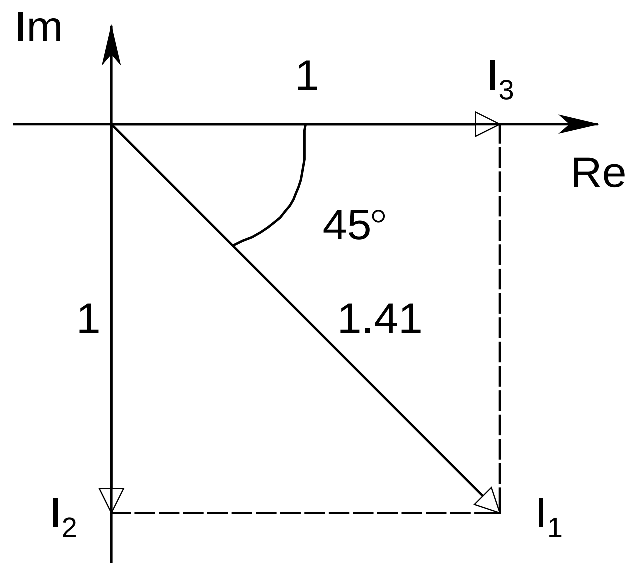
$$\overline{I}_1 = \frac{10\angle 0°}{7.07\angle 45°} = 1.41\angle-45°\ \text{A}$$

Note: $\overline{I}_1$ ***lags*** the excitation voltage $\Rightarrow$ net impedance is ***inductive*** $\Rightarrow$ corroborated by the expression of Z_1

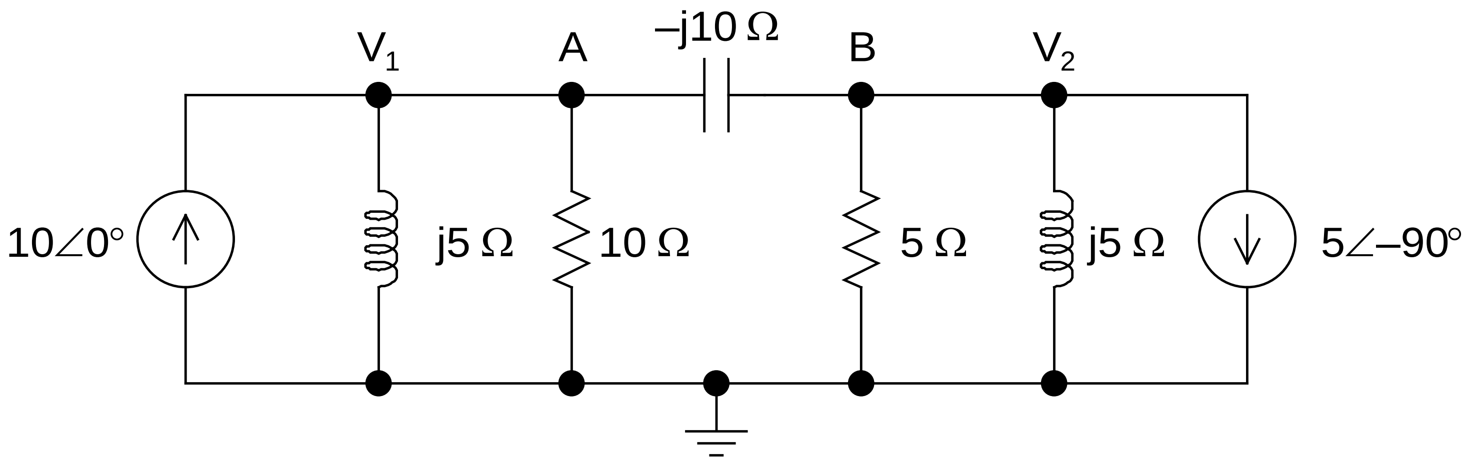
$$\overline{I}_2 = \frac{-j10}{10 - j10} \times \overline{I}_1 = \frac{10\angle -90^\circ}{14.14\angle -45^\circ} \times 1.41\angle -45^\circ = 1\angle -90^\circ \text{ A}$$

$$\overline{I}_3 = \frac{10}{10 - j10} \times \overline{I}_1 = \frac{10\angle 0^\circ}{14.14\angle -45^\circ} \times 1.41\angle -45^\circ = 1\angle 0^\circ \text{ A}$$

Note: $\overline{I}_1$ is somewhat like a vector sum of $\overline{I}_2$ and $\overline{I}_3$



Example: Using node voltage method, find $\overline{V}_1$ and $\overline{V}_2$.



KCL at node A:

$$10\angle 0^\circ = \frac{\overline{V}_1}{j5} + \frac{\overline{V}_1}{10} + \frac{\overline{V}_1 - \overline{V}_2}{-j10}$$

$$\Rightarrow \left(0.1 - j0.1\right)\overline{V}_1 - j0.1\overline{V}_2 = 10 \qquad (1)$$

KCL at node B:

$$\frac{\bar{V}_1 - \bar{V}_2}{-j10} = \frac{\bar{V}_2}{5} + \frac{\bar{V}_2}{j5} + 5\angle -90°$$

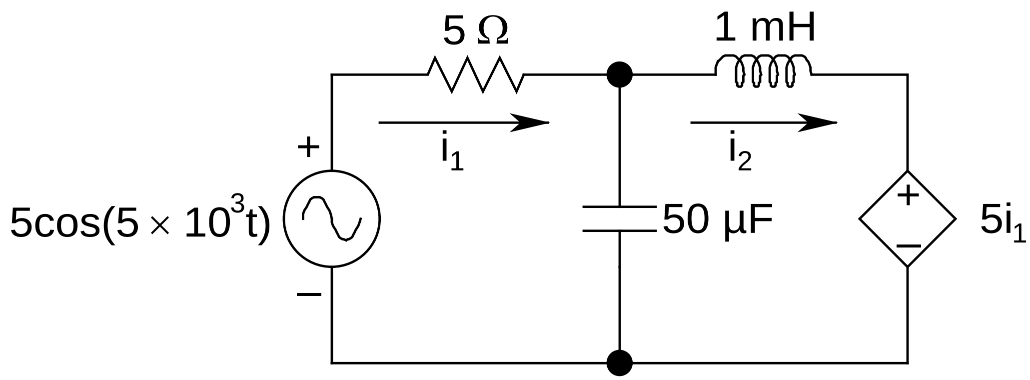
$$\Rightarrow j0.1\bar{V}_1 - \left(0.2 - j0.1\right)\bar{V}_2 = -j5 \qquad (2)$$

Simultaneous solution of (1) and (2) gives:

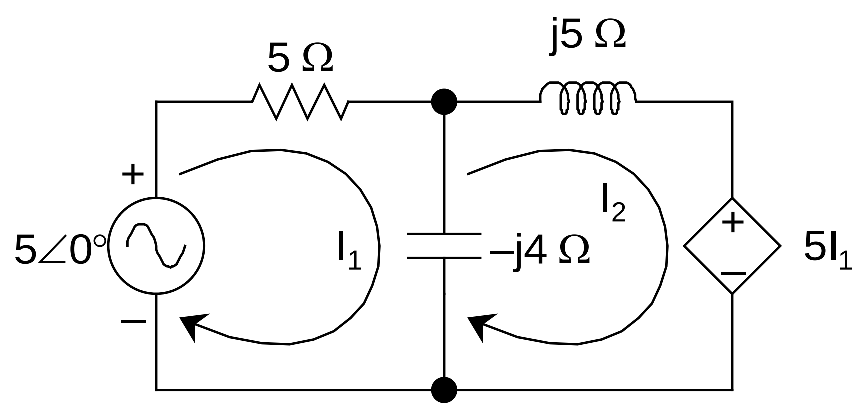
$$\bar{V}_1 = 50.3\angle 22.58° = \left(46.44 + j19.31\right)\ \text{V}$$

$$\bar{V}_2 = 43.9\angle 127.87° = \left(-26.94 + j34.65\right)\ \text{V}$$

Example: Using mesh current method, find the phasor representations of i_1 and i_2 ($\omega = 5$ krad/sec).



$\Rightarrow$



KVL in loop 1:

$$5\angle 0 = 5\overline{I}_1 - j4\left(\overline{I}_1 - \overline{I}_2\right) = \left(5 - j4\right)\overline{I}_1 + j4\overline{I}_2 \qquad (1)$$

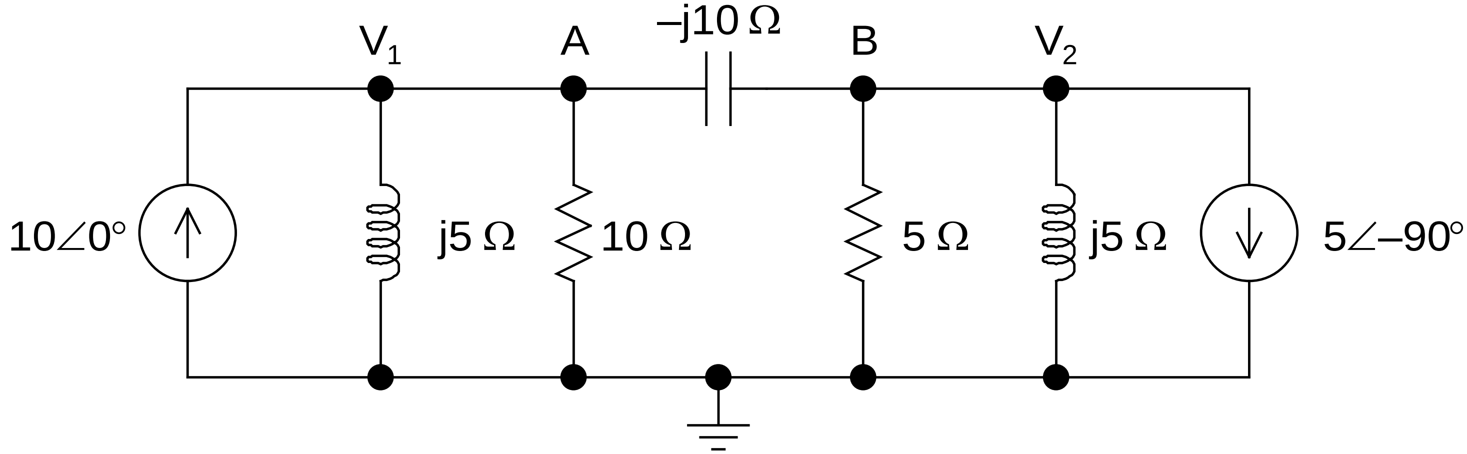
KVL in loop 2:

$$0 = -j4\left(\overline{I}_2 - \overline{I}_1\right) + j5\overline{I}_2 + 5\overline{I}_1 = \left(5 + j4\right)\overline{I}_1 + j\overline{I}_2 \qquad (2)$$

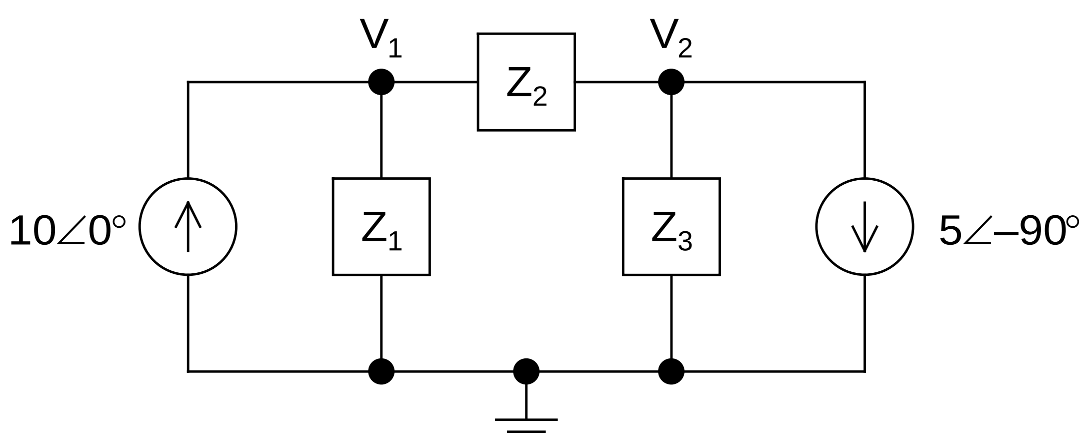
Simultaneous solution of (1) and (2) gives:

$$\overline{I}_1 = 0.2\angle -143.13^\circ \text{ A} \quad \text{and} \quad \overline{I}_2 = 1.28\angle -104.47^\circ \text{ A}$$

Example: Using superposition principle, find $\overline{V}_1$ and $\overline{V}_2$.



$\Rightarrow$



$$Z_1 = \frac{10 \times (j5)}{10 + j5} = \frac{50\angle 90^\circ}{11.18\angle 26.57^\circ} = 4.47\angle 63.43^\circ\ \Omega$$

$$= 2 + j4\ \Omega$$

$$Z_2 = -j10\ \Omega$$

$$Z_3 = \frac{5 \times (j5)}{5 + j5} = \frac{25\angle 90^\circ}{7.07\angle 45^\circ} = 3.54\angle 45^\circ\ \Omega = 2.5 + j2.5\ \Omega$$

First, consider the $10\angle 0^\circ$ source and null (open) the $5\angle -90^\circ$ source:

$$\overline{V}_1 = \frac{Z_2 + Z_3}{Z_1 + Z_2 + Z_3} \times 10\angle 0^\circ \times Z_1 \quad = 62.03\angle 29.73^\circ$$

$$= 53.87 + j30.76\ V$$

$$\bar{V}_2 = \frac{Z_1}{Z_1 + Z_2 + Z_3} \times 10\angle 0° \times Z_3 = 27.76\angle 146.3°$$

$$= -23.1 + j15.4 \text{ V}$$

Next, consider the $5\angle -90°$ source and null (open) the $10\angle 0°$ source:

$$\bar{V}_1 = \frac{Z_3}{Z_1 + Z_2 + Z_3} \times \left(-5\angle -90°\right) \times Z_1 = -13.88\angle 56.3°$$

$$= -7.7 - j11.55 \text{ V}$$

$$\bar{V}_2 = \frac{Z_1 + Z_2}{Z_1 + Z_2 + Z_3} \times \left(-5\angle -90°\right) \times Z_3$$

$$= -19.63\angle -78.7° = -3.85 + j19.25 \text{ V}$$

Hence, by superposition:

$$\overline{V}_1 = 53.87 + j30.76 - 7.7 - j11.55 = 46.17 + j19.21$$

$$= 50\angle 22.6^\circ \text{ V}$$

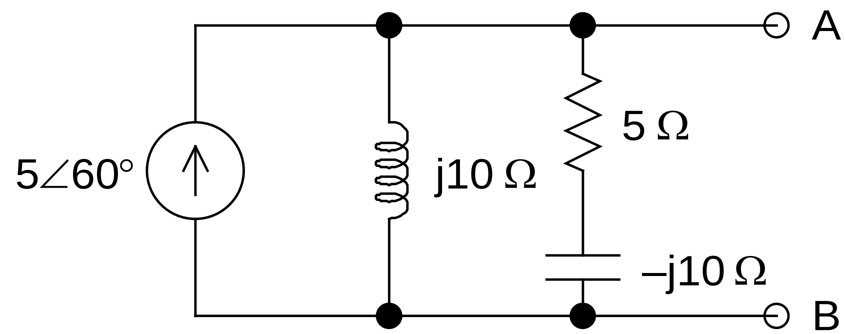
$$\overline{V}_2 = -23.1 + j15.4 - 3.85 + j19.25 = -26.95 + j34.65$$

$$= 43.9\angle 127.88^\circ \text{ V}$$

Note the exact match of the results with the previous example

Expected, since the outcome is independent of the method used

Example: Construct Thevenin and Norton Equivalents, as seen from terminals A and B:



Open the independent current source $5\angle 60°$ A:

Thevenin Impedance:

$$Z_T = (j10) \,||\, (5 - j10)$$

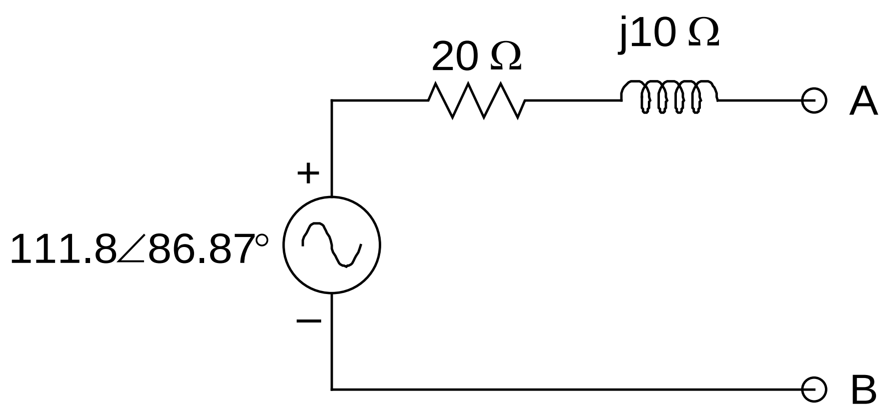
$$= 22.36\angle 26.57° = 20 + j10\ \Omega$$

Thus, the Thevenin impedance is ***inductive***

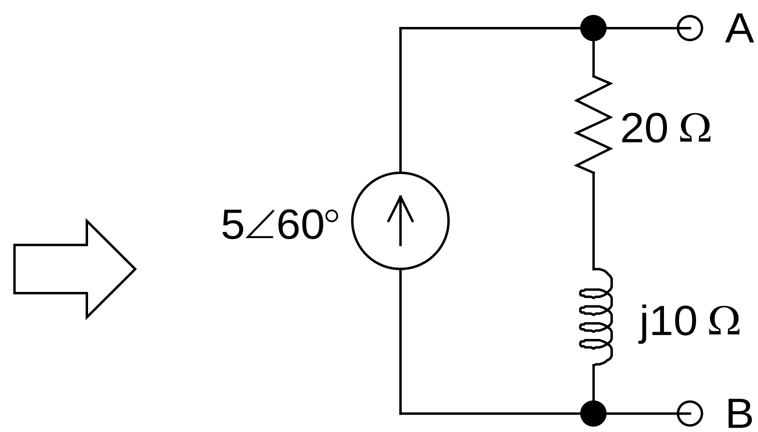
Thevenin Voltage:

$$V_T = (5\angle 60°) \times Z_T = (5\angle 60°) \times (22.36\angle 26.57°)$$

$$= 111.8\angle 86.87°\ \text{V}$$



Thevenin Equivalent



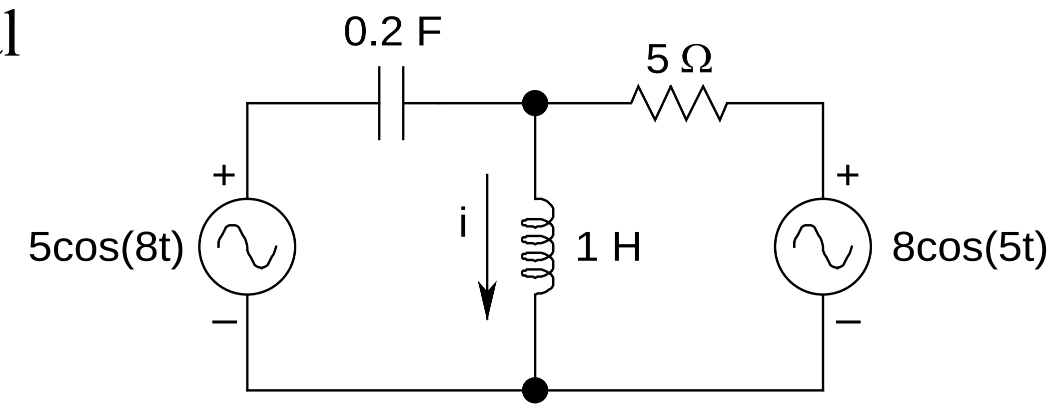
Norton Equivalent

$$\textbf{\textit{Note}}\text{: Norton Current } I_N = \frac{V_T}{Z_T} = 5\angle 60^\circ$$

Example: Sources having different frequencies.

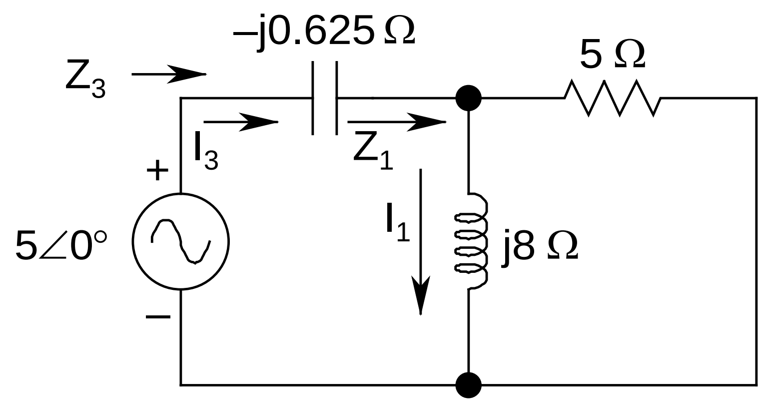
Determine the expression for the current i.

Caution: Phasor analysis can be done for individual frequencies, but the final result can ***never*** be expressed in phasor form



Perform phasor analysis for each frequency, and then apply ***superposition*** to get the net result, which can only be expressed in ***time domain***

First consider the $5\cos(8t)$ source, with $\omega = 8$ rad / sec, and short-circuit the other independent voltage source



$$Z_1 = \frac{5\times(j8)}{5+j8} = \frac{40\angle 90°}{9.43\angle 58°}$$

$$= 4.24\angle 32° = 3.6 + j2.25\ \Omega$$

$$Z_3 = Z_1 - j0.625 = 3.6 + j1.625$$

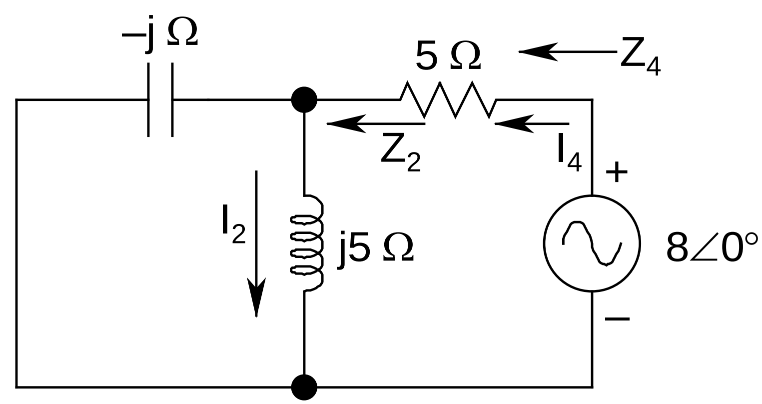
$$= 3.95\angle 24.3°\ \Omega$$

$$\Rightarrow \overline{I}_3 = (5\angle 0°)/(3.95\angle 24.3°) = 1.27\angle -24.3°\ \text{A}$$

$$\Rightarrow \overline{I}_1 = \frac{5}{5+j8}\times\overline{I}_3 = \frac{5}{9.43\angle 58°}\times 1.27\angle -24.3° = 0.67\angle -82.3°\ \text{A}$$

In ***time domain*** representation, $i_1 = 0.67\cos(8t - 82.3°)$ A

Next, consider the $8\cos(5t)$ source, with $\omega = 5$ rad / sec, and short-circuit the other independent voltage source



$$Z_2 = \frac{-j \times (j5)}{-j + j5} = \frac{5}{4\angle 90^\circ}$$

$$= 1.25\angle -90^\circ = -j1.25\ \Omega$$

$$\Rightarrow Z_4 = 5 + Z_2 = 5 - j1.25$$

$$= 5.15\angle -14.04^\circ\ \Omega$$

$$\Rightarrow \overline{I}_4 = (8\angle 0^\circ)/(5.15\angle -14.04^\circ) = 1.55\angle 14.04^\circ\ \text{A}$$

$$\Rightarrow \overline{I}_2 = \frac{-j}{-j + j5} \times \overline{I}_4 = \frac{1\angle -90^\circ}{4\angle 90^\circ} \times 1.55\angle 14.04^\circ$$

$$= 0.39\angle -165.96^\circ \text{ A}$$

In ***time domain***, $i_2 = 0.39\cos(5t - 165.96^\circ)$ A

Hence, the net current i flowing through the 1 H inductance (by ***superposition***):

$$i = i_1 + i_2 = 0.67\cos(8t - 82.3^\circ) + 0.39\cos(5t - 165.96^\circ) \text{ A}$$