EE250 Control Systems

Understanding the linearization of non-linear models

1.8.3 Inverted Pendulum Mounted on A Cart

An inverted pendulum mounted on a motor-driven cart [7] is shown in Figure 1.4. This is a model of the attitude control of a space booster on takeoff. The inverted pendulum is unstable in the sense that it may fall over any time in any direction unless a suitable control force is applied. Here we consider only a two-dimensional problem in which the pendulum moves only in the plane of page. The control force is applied to the cart. Assume that the center of gravity of the pendulum rod is at its geometric center.

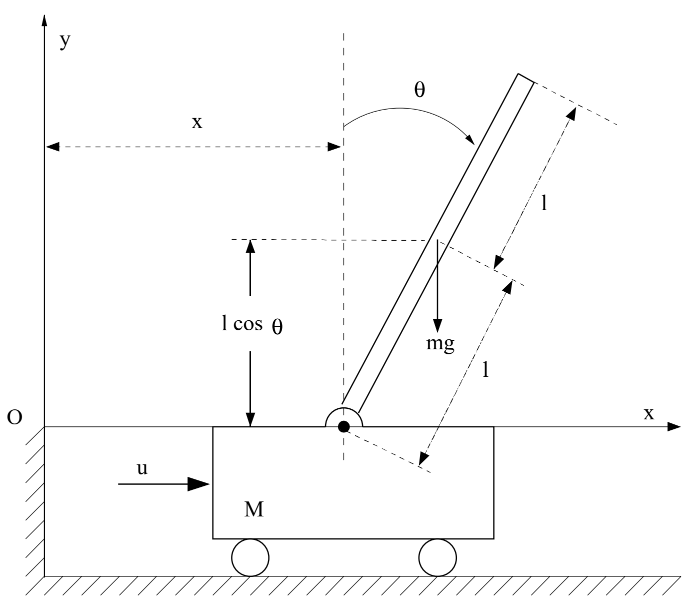

Figure 1.4: Inverted Pendulum Mounted on A Cart

Mathematical Model of the system

De ne the angle of the rod from the vertical line as θ. De ne also the $(x,\ y)$ coordinates of the center of gravity of the pendulum rod as $(x_G,\ y_G)$. Then

$$\begin{aligned} x_G &= x + l\sin\theta \\ y_G &= l\cos\theta \end{aligned}$$

The rotational motion of the pendulum rod about its center of gravity can be described by

$$I\ddot{\theta} = Vl\sin\theta - Hl\cos\theta \tag{1.84}$$

where I is the moment of inertia of the rod about its center of gravity and is given by the following expression.

$$I = \int_{-l}^{l} r^2 dm = \frac{ml^2}{3} \tag{1.85}$$

The horizontal motion of center of gravity of the pendulum rod is given by

$$m\frac{d^2}{dt^2}(x + l\sin\theta) = H \tag{1.86}$$

The vertical motion of center of gravity of the pendulum rod is

$$m\frac{d^2}{dt^2}(l\cos\theta) = V - mg \tag{1.87}$$

The horizontal motion of the cart is described by

$$M\frac{d^2x}{dt^2} = u - H \tag{1.88}$$

Equations (1.84) - (1.88) describe the motion of inverted pendulum-on-the cart system. Differentiating (1.86) and combining with (1.88), we get

$$m\ddot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta = u - M\ddot{x} \tag{1.89}$$

Differentiating (1.87) and combining with (1.88) and (1.84), we get

$$I\ddot{\theta} = (mg - ml\dot{\theta}^2\cos\theta - ml\ddot{\theta}\sin\theta)l\sin\theta - (u - M\ddot{x})l\cos\theta \tag{1.90}$$

Substituting $u - M\ddot{x}$ from equation (1.88) into (1.89) and doing manipulations, we get

$$I\ddot{\theta} = mgl\sin\theta - ml^2\ddot{\theta} - m\ddot{x}l\cos\theta \tag{1.91}$$

Denoting $\frac{1}{m+M}$ as a, we can represent equation (1.89) as

$$\ddot{x} = -mal\ddot{\theta}\cos\theta + mal\dot{\theta}^2\sin\theta + au \tag{1.92}$$

We substitute (1.92) into (1.91) to get

$$\ddot{\theta} = \frac{mgl\sin\theta - m^2l^2a\dot{\theta}^2\sin\theta\cos\theta - mal\cos\theta u}{I - m^2l^2a\cos^2\theta + ml^2} \tag{1.93}$$

Equations (1.92) and (1.93) represent the nonlinear dynamics of the inverted pendulum mounted on a cart.

State space model of Inverted Pendulum

Let us consider the state variable as $x_1 = \theta$, $x_2 = \dot{\theta}$, $x_3 = x$ and $x_4 = \dot{x}$. Then using the expression of I, we can rewrite equation (1.92) in state space form as:

$$\begin{aligned} \dot{x}_1 &= x_2 && (1.94) \\ \dot{x}_2 &= \frac{g\sin x_1 - mlax_2^2\sin x_1\cos x_1}{\frac{4l}{3} - mla\cos^2 x_1} - \frac{mla\cos x_1 u}{\frac{4ml^2}{3} - m^2l^2a\cos^2 x_1} && (1.95) \end{aligned}$$

Substituting $\ddot{\theta}$ into the expression $\ddot{x}$ we get

$$\ddot{x} = \frac{-mag\sin x_1\cos x_1 + \frac{4mla}{3}x_2^2\sin x_1 + \frac{4au}{3}}{\frac{4}{3} - ma\cos^2 x_1} \tag{1.96}$$

The above equation can written in the state space form as

$$\begin{aligned} \dot{x}_3 &= x_4 && (1.97) \\ \dot{x}_4 &= \frac{-mag\sin x_1\cos x_1 + \frac{4mla}{3}x_2^2\sin x_1 + \frac{4au}{3}}{\frac{4}{3} - ma\cos^2 x_1} && (1.98) \end{aligned}$$

Thus equations (1.94), (1.95), (1.97) and (1.98) denote the state space model of the inverted pendulum system.

Linearization Using Taylor Series Expansion

The nonlinear dynamics of the inverted pendulum system will be linearized around the vertically upward position using the Taylor's series expansion.

Expanding the system dynamics (1.94), (1.95), (1.97), (1.98) around the equilibrium point $x = [x_1\ x_2\ x_3\ x_4]^T = [0\ 0\ 0\ 0]^T$, we can write the approximated linear model of the system as:

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \frac{\partial}{\partial x_1}\left\{\frac{g\sin x_1 - mlax_2^2\sin x_1\cos x_1}{\frac{4l}{3} - mla\cos^2 x_1}\right\}\Bigg|_{0,0,0,0} + \\
&\quad \frac{\partial}{\partial x_2}\left\{\frac{-mlax_2^2\sin x_1\cos x_1}{\frac{4l}{3} - mla\cos^2 x_1}\right\}\Bigg|_{0,0,0,0} + \\
&\quad \frac{\partial}{\partial u}\left\{-\frac{mla\cos x_1 u}{\frac{4ml^2}{3} - m^2l^2a\cos^2 x_1}\right\}\Bigg|_{0,0,0,0} \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= \frac{\partial}{\partial x_1}\left\{\frac{-mag\sin x_1\cos x_1 + \frac{4mla}{3}x_2^2\sin x_1}{\frac{4}{3} - ma\cos^2 x_1}\right\}\Bigg|_{0,0,0,0} + \\
&\quad \frac{\partial}{\partial x_1}\left\{\frac{\frac{4mla}{3}x_2^2\sin x_1}{\frac{4}{3} - ma\cos^2 x_1}\right\}\Bigg|_{0,0,0,0} + \\
&\quad \frac{\partial}{\partial u}\left\{\frac{\frac{4au}{3}}{\frac{4}{3} - ma\cos^2 x_1}\right\}\Bigg|_{0,0,0,0}
\end{aligned}$$

Simplifying above expression we get the linearized dynamics around the equilibrium point $x = [0\ 0\ 0\ 0]^T$ as

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ \frac{3g}{4l-3mla} & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ \frac{-3mag}{4-3ma} & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} + \begin{bmatrix}0\\ \frac{3a}{3mla-4l}\\ 0\\ \frac{4a}{4-3ma}\end{bmatrix}u \tag{1.99}$$