MSO201A
Hints to Solution: Quiz-IV

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1. If X and Y are independent, then for a discrete random variable X taking values in $\{x_1, x_2, \ldots, x_n\}$ with positive probability, we have

$$P[X+Y=\alpha] = \sum_{x=x_1}^{x_n} P[X=x]P[Y=\alpha-x] \text{ (due to independence).}$$

So, if Y is a continuous random variable, then $P[X+Y=\alpha]=0$. If Y is a discrete random variable, then $\sum_{x=x_1}^{x_n} P[X=x]P[Y=\alpha-x] < 1$.
Similarly, it can be argued when X is continuous. Under independence, both X and Y are degenerate random variables.

If independence is removed, then take $Y=\alpha-X$, hence both can be non-degenerate random variables.
If X is degenerate at c and Y is non-degenerate random variable, then $P[X+Y=\alpha] = P[Y=\alpha-c] < 1$.

2. Note that $G(Y) \geq c \iff F(c) \leq Y$, hence $P[G(Y) \geq \log_e 5] = \exp(-\log_e 5) = 1/5$.

3. If $h(\cdot)$ and $\phi(\cdot)$ are non-negative, non-decreasing and continuous, then $\phi(h(\cdot))$ is also non-decreasing and continuous. Hence all funcions given are cumulative distribution functions.

4.

$$P[\text{ unit is faulty}|\text{ unit is passed by detector}] = \frac{P[\text{unit is faulty}]P[\text{unit passed}|\text{ unit is faulty}]}{P[\text{ unit is passed}]}$$

$$\implies P[\text{ unit is faulty}|\text{ unit is passed by detector}] = \frac{0.05\times(1-0.5)}{(1-0.05\times 0.5)} = \frac{1}{39}$$

Given $Y=1000$, the number of passed products given they are faulty
$\sim Bin\left(40000-1000, \frac{1}{39}\right)$. So,

$$E(X|Y=1000) = 1000 + 39000 \times \frac{1}{39} = 2000.$$

5. Consider $F(x) = \begin{cases} 0 & x \le 1 \\ 1 - \dfrac{1}{kx} & e^{k-1} < x \le e^k \text{ for } k = 1, 2, \ldots \end{cases}$
   Now, $x(1 - F(x))$ converges to 0 as $x \to \infty$, but $E(X) = \int_0^\infty (1 - F(x))dx = \infty$.

$$P[A \cup B \cup C] = 1 - P[A^{\complement} \cap B^{\complement} \cap C^{\complement}] = 1 - P[A^{\complement} | B^{\complement} \cap C^{\complement}] P[B^{\complement} \cap C^{\complement}]$$

   Let $X \sim U(0,1)$, and define $Y = \begin{cases} X & , \text{ if } X \text{ is irrational} \\ 0 & , \text{ else} \end{cases}$
   then correlation between Y and X is 1.

   Let X and Y be independent and have probability density function (pdf) as follows:

$$f(x) = \begin{cases} \dfrac{1}{x^2} & x \ge 1 \\ 0 & \text{otherwise.} \end{cases}$$

   Then, $\min\{X, Y\}$ has the following pdf:

$$g(x) = 2[1 - F(x)]^{2-1} f(x),$$

   and hence $\min\{X, Y\}$ has finite expectation.

   Consider (X, Y, Z) follow a multivariate normal with following covariance matrix:

$$\Sigma = \begin{bmatrix} 1 & 0.5 & -0.25 \\ 0.5 & 1 & 0.5 \\ -0.25 & 0.5 & 1 \end{bmatrix}.$$

   Then, Y is positively correlated with X and Z, but X and Z are negatively correlated.

6. Observe that $X \sim \mathit{Bernoulli}(\mu)$.

7. A random variable X with mgf $M_X(t) = (1 - \theta t)^{-k}$ with $t < 1/\theta$ is $\mathit{Gamma}(k, \theta)$. So, $\mathbb{E}(X) = k\theta$ and $\mathbb{E}(X^2) = k\theta^2 + k^2\theta^2$.

8. Let $S_n = X_1 + X_2 + \cdots + X_n$, where $X_1, X_2, \ldots, X_n$ are i.i.d. $\mathit{Poisson}(1)$ random variables. Using CLT, we get $\frac{1}{\sqrt{n}}(S_n - n) \xrightarrow{d} N(0,1)$ as $n \to \infty$. Now,

$$e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!} = P\left[S_n \le n\right] = P\left[\frac{1}{\sqrt{n}}(S_n - n) \le 0\right] \xrightarrow{n \to \infty} \Phi(0) = 0.5.$$

9. Observe that $X, -X \sim N(0,1)$ and

$$\begin{aligned}P(Y \leq y) =& P(Y \leq y \mid W=1)P(W=1) + P(Y \leq y \mid W=-1)P(W=-1)\\ =& 0.5P(X \leq y \mid W=1) + 0.5P(-X \leq y \mid W=-1)\\ =& 0.5P(X \leq y) + 0.5P(-X \leq y) \text{ as } X \perp\!\!\!\perp W\\ =& 0.5P(X \leq y) + 0.5P(X \leq y)\\ =& P(X \leq y).\end{aligned}$$

So, $Y \sim N(0,1)$, $\mathbb{E}(Y) = 0$ and $Var(Y) = 1$.
Further, note that

$$Z = \begin{cases} 2X, & \text{with probability } 0.5\\ 0, & \text{with probability } 0.5.\end{cases}$$

So, we get

$$\begin{aligned}P(Z \leq 0) =& 0.5P(2X \leq 0) + 0.5\mathbb{I}(0 \in (-\infty, 0])\\ =& 0.25 + 0.5 = 0.75,\end{aligned}$$

$\mathbb{E}(Z) = 0$ and $Var(Z) = 0.5Var(2X) = 2$.

10. Easy.

11. Observe the following facts:

- If $X \sim Poisson(\lambda_1)$, $Y \sim Poisson(\lambda_2)$ and $X \perp\!\!\!\perp Y$, then $X+Y \sim Poisson(\lambda_1+\lambda_2)$.
- If $X \sim Poisson(\lambda_1)$, $Y \sim Poisson(\lambda_2)$, then $X - Y \nsim Poisson$, because $X - Y$ may be negative.
- If $X \sim Poisson(\lambda_1)$, $Y \sim Poisson(\lambda_2)$ and $X \perp\!\!\!\perp Y$, then $X \mid X+Y = t \sim Binomial(t, \frac{\lambda_1}{\lambda_1+\lambda_2})$.
- If $X \sim Poisson(\lambda_1)$, then $2X \nsim Poisson$ as $2X$ can not be an odd integer.

12. Observe following facts:

- If $\rho = 0$:
  $X \perp\!\!\!\perp Y$.
  $\frac{X}{Y} \sim t_1 \equiv Cauchy$.
  $X+Y \perp\!\!\!\perp X-Y$ and $X+Y, X-Y \sim N(0,2)$.
  $X^2, Y^2 \sim \chi^2$ & $X^2 \perp\!\!\!\perp Y^2 \Rightarrow X^2+Y^2 \perp\!\!\!\perp \dfrac{X^2}{X^2+Y^2}$.
- If $\rho \neq 0$:
  $Cov(X, Y-\rho X) = Cov(X,Y) - \rho Cov(X,X) = \rho - \rho = 0$.
  $Cov(X, Y-2\rho X) = Cov(X,Y) - 2\rho Cov(X,X) = \rho - 2\rho = -\rho$.