MSO-203 B ASSIGNMENT 3
IIT, KANPUR

10th October , 2020

Multiple choice questions may have more than one correct answers.

1. Choose the correct answer(s):

   a) $u_x + u_y = u$ is a linear PDE.
   b) $|\nabla u| = 1$ is a non linear PDE.
   c) $|\nabla u| = 1$ is a semilinear PDE.
   d) $uu_x + u_y = \sin(x)$ is a quasilinear PDE.

   Answer: a, b and d.

2. Classify the following PDE and also determine their order.

   a) $\triangle u = \sin^2(x)$ (Poisson Equation)
   b) $u_{tt} - u_{xx} = 0$ (Wave Equation)
   c) $u_t = \triangle u$ (Heat Equation)
   d) $div\left(|\nabla u|^{p-2}\nabla u\right) = 0$ (p-Laplace equation) for $p > 1$.
   e) $det(D^2u) = 1$ (Monge Ampere equation)
   f) $u_t + uu_x = 0$ (Burgers Equation)
   g) $xu_{xxy} + u_{yyy} = 1$.

   Answer:

   a) Linear, second order.
   b) Linear, second order.
   c) Linear, second order.
   d) For $p \neq 2$: fully nonlinear, second order. For $p = 2$: Linear, second order.
   e) Fully nonlinear, first order.
   f) Quasilinear, first order.
   g) Semilinear, third order.

3. Solve the following problem:

$$\begin{cases} u_x - 2u_y = u & \textbf{in } \mathbb{R}^2, \\ u(x,0) = 1. \end{cases}$$

$$\begin{cases} u_x - 2u_y = u & , \\ u(0,y) = y. \end{cases}$$

   and

$$\begin{cases} u_x - 2u_y = u \\ u(x,x) = x. \end{cases}$$

   Solution:

(a)

$$\begin{cases} u_x - 2u_y = u^2 \\ u(x,0) = 1. \end{cases},$$

Characteristic equations are;

$$\dot{x}(t) = 1, x(0) = s$$
$$\dot{y}(t) = -2, y(0) = 0$$
$$\dot{z}(t) = z^2(t), z(0) = 1$$

Solving them we get, $x(t) = t + C_1$ with $x(0) = s$ this gives $x(t) = t + s$.
Also, $y(t) = -2t + C_2$ with $y(0) = 0$ gives $y(t) = -2t$.
Finally, for $z(t)$ we get, $\frac{-1}{z(t)} = t + C_3$ with $z(0) = 1$ this gives $z(t) = \frac{1}{1-t}$.
Eliminating t from expressions of $y(t)$ and $z(t)$ we find $z(t) = \frac{2}{2+y}$.

(b)

$$\begin{cases} u_x - 2u_y = u \\ u(0,y) = y. \end{cases},$$

Step 1: Initial curve in this problem is Y-axis. Let $(0,s)$ be arbitrary point on the initial curve.
Step 2: Write down the characteristic equations;

$$\dot{x}(t) = 1, x(0) = 0$$
$$\dot{y}(t) = -2, y(0) = s$$
$$\dot{z}(t) = z(t), z(0) = u(0,s) = s.$$

Step 3: Solve C.E. to get following;
$x(t) = t$, $y(t) = -2t + s$, $z(t) = e^t s$.
Step 4: Eliminate s and t in $z(t)$ using step 3.

$$u(x,y) = z(t) = e^t s = e^x(y + 2x).$$

(c)

$$\begin{cases} u_x - 2u_y = u \\ u(x,x) = x. \end{cases},$$

Step 1: Initial curve in this case is the line $y = x$. Let (s,s) be arbitrary point on the initial curve.
Step 2: Write down the characteristic equations;

$$\dot{x}(t) = 1, x(0) = 0$$
$$\dot{y}(t) = -2, y(0) = s$$
$$\dot{z}(t) = z(t), z(0) = u(s,s) = s.$$

Step 3: Solve C.E. to get following;

$$x(t) = t + s, y(t) = -2t + s, z(t) = e^t s$$

Step 4: Eliminate s and t in $z(t)$ using step 3.

$$u(x,y) = z(t) = e^t s = e^{\frac{x-y}{3}}(\frac{2x+y}{3})$$

.

4. Consider the following semilinear equation:

$$\begin{cases} u_x + u_y = u^{\frac{1}{2}}, \\ u(x,0) = 0. \end{cases} \tag{1}$$

Show that solution of this problem is not unique. Try to explain a possible reason for this. Does it contradict the Existence uniqueness theorem provided in lectures?

Solution : Arbitrary point on IC is given by $(s, 0)$. C.E. are following;

$$\begin{aligned} \dot{x}(t) &= 1, x(0) = s \\ \dot{y}(t) &= 1, y(0) = 0 \\ \dot{z}(t) &= \sqrt{z(t)}, z(0) = 0. \end{aligned}$$

Solving them we get $x(t) = t + s$, $y(t) = t$, $z(t) = \frac{t^2}{4}$. Eliminating t for $z(t)$ we find $u(x, y) = z(t) = \frac{t^2}{4} = \frac{y^2}{4}$, for $y \geq 0$. $u(x, y) \equiv 0$ is also a solution, therefore solution is not unique.

The non uniqueness appears because the C.E for z has non unique solution.

The condition of the main theorem is not contradicted as the assumption on the regularity of the coefficient function is not satisfied.

5. Consider the following semilinear equation:

$$\begin{cases} uu_x + u_y = u^2 \quad, \\ u(x,0) = 1. \end{cases} \tag{2}$$

Solution: Let $(s, 0)$ is arbitrary point to the initial curve. C.E. are;

$$\begin{aligned} \dot{x}(t) &= z(t), x(0) = s \\ \dot{y}(t) &= 1, y(0) = 0 \\ \dot{z}(t) &= z^2(t), z(0) = 1 \end{aligned}$$

Above gives $y(t) = t$ and $z(t) = \frac{1}{1-t}$.

Notice here that you can act and do not need to solve for $x(t)$, because finally we need to eliminate t from $z(t)$ which is clearly possible since we already know $y(t) = t$. Therefore $u(x, y) = z(t) = \frac{1}{1-t} = \frac{1}{1-y}$.

6. Consider the following problem:

$$\begin{cases} u_x + u_y = 0 \quad, \\ u(x,x) = 1. \end{cases} \tag{3}$$

Then the above problem has

a) infinitely many solutions

b) no solution

c) atmost finitely many solutions

d) unique solution.

Solution: let f be any arbitrary differentiable function defined on real line with $f(0) = 1$. Define the function,

$$u(x, y) = f(x - y).$$

It is easy to see, that such a u solves the given problem and since choice of such f are infinite in number, (a) is the only correct option.

7. Does the projected characteristics in Problem 4 and 5 intersects?

Solution: Part 1: In Problem 4, recall that $x(t) = t + s$ and $y(t) = t$ is the PC starting from $(s, 0)$. Eliminating t, we see that $x(t) = y(t) + s$ that is PC starting from $(s, 0)$ is straight line with gradient 1. Therefore any two PC are parallel lines hence do not intersect.

Part 2: Now to do this part we have to solve for $x(t)$(recall while solving the problem we skipped that).

$$x'(t) = \frac{1}{1-t} \Rightarrow x(t) = -log(1-t) + C$$

At $t = 0$, $x(0) = s \Rightarrow c = s \Rightarrow x(t) = s - log(1-t)$.

Eliminating t we get $x(t) = s - log(1 - y(t))$ is the equation of the curve through $(s, 0)$.

If possible, $\exists\ s_1 \neq s_2$ s.t. the curves $x = s_1 - log(1-y)$ and $x = s_2 - log(1-y)$ intersect. Substracting implies $s_1 = s_2$, contradiction. Therefore PC do not intersect.