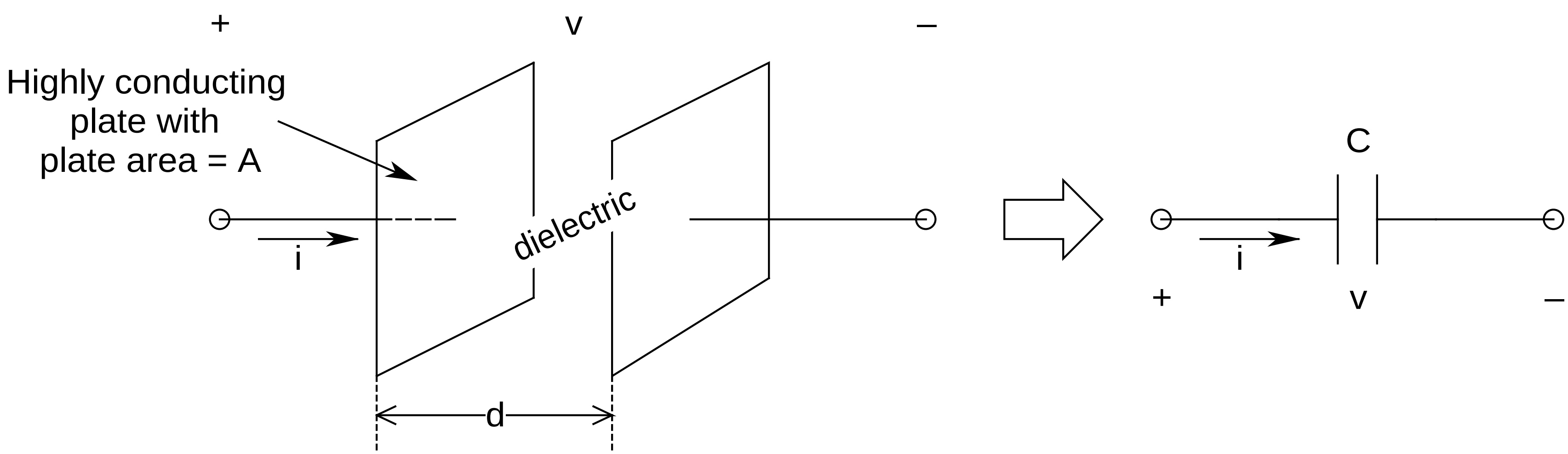
Time Domain (Transient) Response

- ***Inductors*** (L)/***Capacitors*** (C):
  - ***Passive elements***, capable of storing and delivering finite amounts of energy, but average power can't be greater than zero over an infinite time interval
  - I-V relations for these two elements are ***function of time***
- ***Goal***:
  - To investigate the ***time domain*** (transient) response of RL and RC circuits

- ***Capacitors***:



Two highly conducting plates (preferably of metal) separated by a dielectric, e.g., paper, mica, plastic, glass, ceramic, etc.

* The capacitance of the structure: $C = \dfrac{\varepsilon_0 \varepsilon_r A}{d}$

ε_0 = ***permittivity of free space***

$\left(= 8.854 \times 10^{-14}\ \mathrm{F/cm}\right)$

ε_r = ***relative permittivity of the dielectric***

A = ***plate area***, d = ***separation between the plates***

* ε_r for some commonly used dielectric materials:

paper – 3.85, ***teflon*** – 2.1, ***polyethylene*** – 2.25,

polystyrene – 2.4-2.7, ***glass*** – 3.7-10, ***mica*** – 3-6

* Higher values of ε_r are preferred to reduce area

* Unit of C:

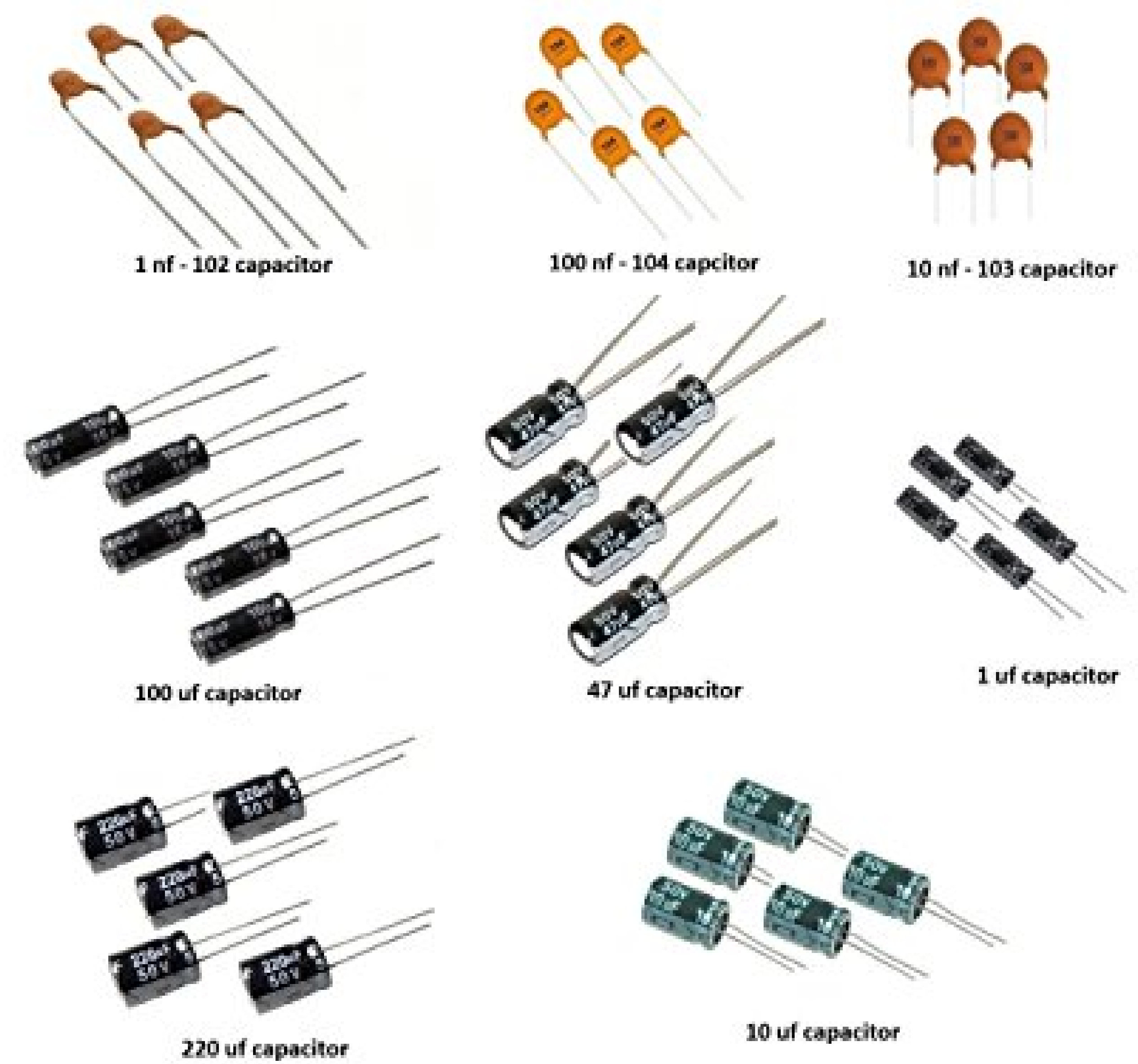
  Farad (F) = Coulomb (C)/Volt (V) = A-s/V

* Capacitance is defined as ***charge per potential***
* Dielectric should ideally have ***zero conductance***

  $\Rightarrow$ ***infinite resistance*** $\Rightarrow$ won't allow any current

* Real dielectrics are ***leaky***, i.e., they have resistance, however, it is ***very large*** (> MΩ)
* If charges are placed on one plate of a capacitor, then it will immediately induce charges of opposite polarity on the other plate

- **Classification of Capacitors**:
  - ***Mica*** - 1 pF to 0.1 μF
  - ***Ceramic*** - 10 pF to 1 μF
  - ***Mylar*** (***Polyester***) - 1 nF to 10 μF
  - ***Paper*** - 1 nF to 50 μF
  - ***Electrolytic*** - 0.1 μF to 0.2 F
- **Shapes**:
  - ***Discs*** or ***Cylinders***
- Be very careful with ***electrolytic capacitors***
  - Always make sure that you discharge them before using by touching them to a metal plate



40 Capacitors

* ***More rigorous definition of capacitance***:
  Incremental change in charge dq caused by an incremental change in voltage dv across it
  i.e., $C = dq/dV$
* Under steady-state, i.e., when there is no time variation, $C = Q/V$
* Implies that if a charge $+Q$ is placed on one plate of the capacitor, it will induce charge $-Q$ on the other plate
* The plate having positive charges will be at a higher potential than the plate having the negative charges

* Note: Putting charges on one plate of a capacitor needs ***work*** to be done
* With external stimulus removed, these charges are stored in the capacitor, with the appearance of a potential difference (V), and consequently, an electric field $(\mathscr{E})$ across the dielectric of the capacitor (note that it is caused due to the charge separation)

* The resulting energy storage, which is actually the ***work done*** (W) in building up this potential difference, is given by:

$$W = -q\int_0^d \mathscr{E}\,dx = -q\int_0^V (-dv) = \int_0^Q \frac{q\,dq}{C} = \frac{Q^2}{2C} = \frac{1}{2}CV^2$$

* Thus, the energy stored in a capacitor is directly proportional to the value of the capacitance and to the square of the voltage

* Note: Each dielectric has a ***breakdown strength***, i.e., the ***maximum electric field*** that can be applied across it before it electrically breaks down

* ***Electrical breakdown***: high electric field tears off electrons from atoms, which gets accelerated, and knocks off electrons from other atoms

  $\Rightarrow$ known as ***avalanche breakdown***

* Typical breakdown strength of dielectrics in kV/cm:

  Air – 30, ***Polysterene***, ***Teflon*** – 200, ***Mica*** – 1200

* ***Breakdown voltage*** is simply the product of the breakdown strength and the plate separation, since the electric field inside a capacitor is ***constant***

* ***I-V Relation*** (***Derivative Form***):

$$i = \frac{dq}{dt} = \frac{dq}{dv}\frac{dv}{dt} = C\frac{dv}{dt}$$

* Note that for ***positive dv/dt***, current flows from the positive terminal to the negative terminal, implying ***charging*** of a capacitor

* Similarly, ***negative*** $\boldsymbol{dv/dt}$ implies that the current direction is reversed, and the capacitor is ***discharging***

* Note: Under steady-state, i = 0 (***no dc current can pass through a capacitor***)

$\Rightarrow$ Under ***dc*** conditions, capacitors behave as ***open-circuits***

$\Rightarrow$ extremely important observation

* ***I-V Relation*** (***Integral Form***):

$$dv = \frac{i}{C} dt \qquad \text{or} \qquad v = \frac{1}{C} \int i dt + K_1$$

K_1 : constant of integration (***initial voltage across the capacitor***)

* Note: Once charges are stored in a capacitor, thus developing a potential across it, and then the external stimulus is removed, then these charges would remain within the capacitor for an ***infinite time*** (ideally)

* However, recall that actual capacitors have ***leakage resistance*** (very high though), and thus, the charges stored in the capacitor actually keep on ***discharging*** with time, albeit at a very slow rate
* Also, capacitors can ***acquire*** charge from the ambient and start developing a ***potential difference*** between its terminals, which may reach dangerous proportions
* To ensure that such a situation does not arise, it is advisable to store capacitors with their two terminals ***shorted***

* ***Instantaneous power*** delivered to a capacitor:

$$p = vi = Cv\frac{dv}{dt}$$

* If the capacitor was initially discharged, i.e., $v(0) = 0$, and after a finite time t_1, acquires voltage V, then the total ***energy stored*** in the capacitor at time t_1 :

$$\text{Energy stored} = \int_0^{t_1} p\,dt = C\int_0^{V} v\,dv = \frac{1}{2}CV^2$$

Some Observations:

* If the voltage across a capacitor changes ***linearly*** with time, then the ***current*** through the capacitor becomes a ***constant***

  $\Rightarrow$ Known as ***constant current charging***/ ***discharging*** of a capacitor

* In case the voltage across a capacitor changes ***instantly*** (e.g., a step function), then from the I-V relation, it becomes apparent that the current would be ***infinite***

* In reality, that's not the case, since the current would be controlled by the ***internal resistance*** of the capacitor
* $\because Q = CV,$ a change in V (potential) implies a change in Q (charge)
* A change in charge across a capacitor involves a change in ***dielectric polarization***, which takes time
* Governed by ***dielectric relaxation time***, i.e., the time needed for the ***dipole moments*** to reorient

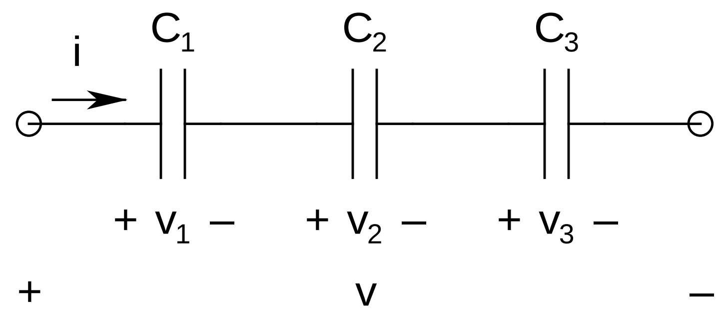
* This process can be hastened/slowed down by increasing/decreasing the ***driving force***
* Thus, an ***instantaneous change in charge across a capacitor is impossible***
* **Key Inference**:

  The voltage across a capacitor cannot change instantly, however, the current through it can

  $\Rightarrow$ This is an extremely important point, and should be noted carefully

- ***Series Combination of Capacitors***:

Note: Same current (i) flows through all the capacitors



$$v = v_1 + v_2 + v_3 \quad \Rightarrow \quad \frac{1}{C_{net}}\int idt = \frac{1}{C_1}\int idt + \frac{1}{C_2}\int idt + \frac{1}{C_3}\int idt$$

$$\Rightarrow C_{net} = \left(C_1^{-1} + C_2^{-1} + C_3^{-1}\right)^{-1} \Rightarrow$$ ***smallest one dominates***

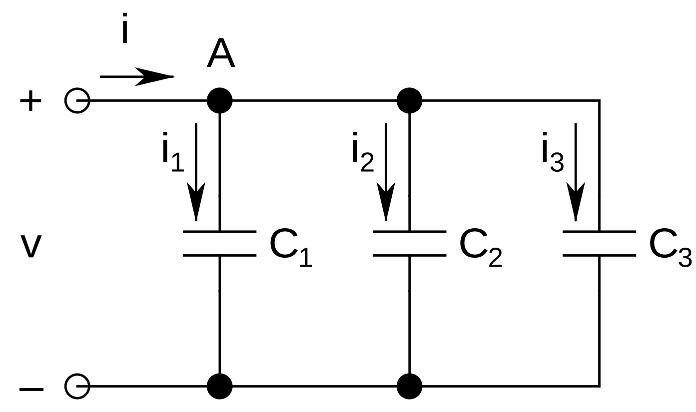
Note: This rule follows that of parallel combination of resistors!

Note: ***All capacitors store the same amount of charge Q***

$\because\ V = Q/C,\ \therefore$ ***the smallest capacitance would have the largest voltage across it***

- ***Parallel Combination of Capacitors***:

Note: All the capacitors have the same voltage v across them



$$i = i_1 + i_2 + i_3 \quad \Rightarrow \quad C_{net}\frac{dv}{dt} = C_1\frac{dv}{dt} + C_2\frac{dv}{dt} + C_3\frac{dv}{dt}$$

$\Rightarrow C_{net} = C_1 + C_2 + C_3 \ \Rightarrow$ ***largest one dominates***

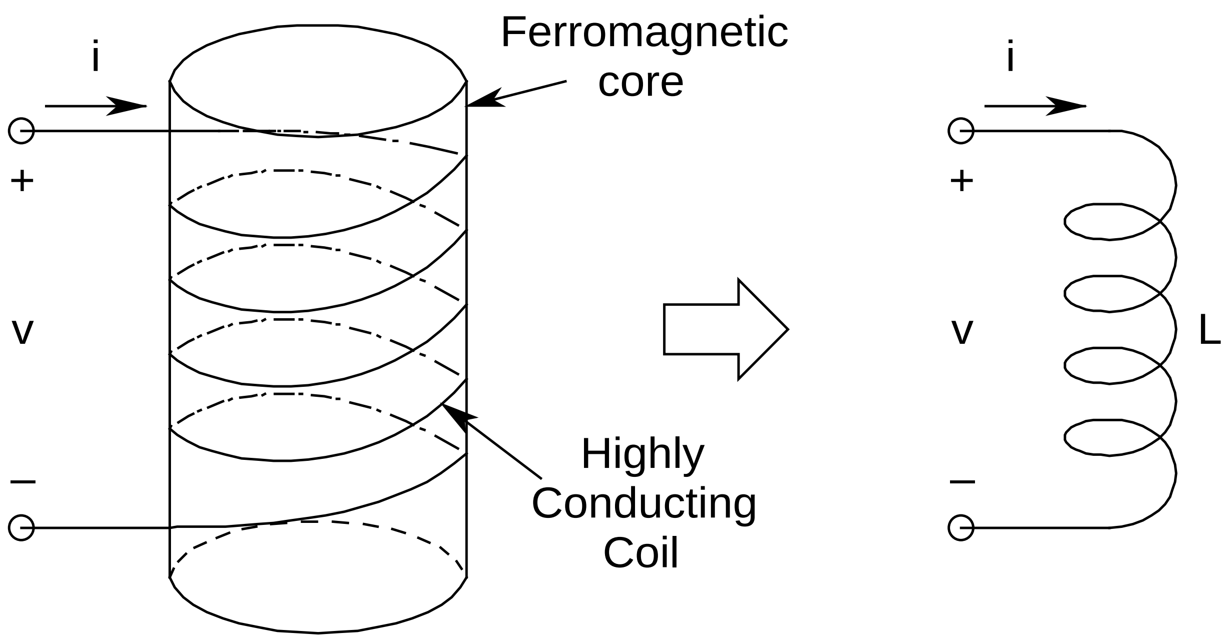
Note: This rule follows that of series combination of resistors!

Charge analogue: $Q_1 = C_1 v,\ Q_2 = C_2 v,\ Q_3 = C_3 v$

$$\Rightarrow Q_{net} = C_{net} v = Q_1 + Q_2 + Q_3 = \left(C_1 + C_2 + C_3\right) v$$

$$\Rightarrow C_{net} = C_1 + C_2 + C_3$$

- ***Inductors***:
  - Highly conducting coil [typically made of ***copper*** (Cu)] wound around a ferromagnetic [e.g., ***cobalt*** (Co), ***nickel*** (Ni), ***iron*** (Fe), or their compounds] core



- Passive element that tries to ***oppose*** any change in the ***current*** flowing through it
- Current flowing through the coil builds up a ***magnetic flux*** within the ferromagnetic material
- Now, if for any reason, the current changes, then it would produce a change in this flux
- By ***Lenz' law***, it would produce an ***electro-motive force*** (emf) that would try to ***oppose*** the very cause that is producing it
  - ***Current through an inductor cannot change instantly, however, voltage across it can***
- Unit of inductance: ***Henry*** (H) = V-s/A = Ω-s


12A2N
12A2N

– For a cylindrical core wire-wound inductor, the inductance L can be expressed as:

$$L = \frac{\mu_0 \mu_r N^2 A}{l}$$

- μ_0 ($= 4\pi \times 10^{-7}$ H/m) = ***permeability of free space***
- μ_r = ***relative permeability of the ferromagnetic core***
- N = ***number of turns of the coil***
- A = ***cross-sectional area of the coil in m²***
- l = ***length of the coil in m***

– μ_r of some ferromagnetic materials:

- ***mu-metal*** − 20,000, ***permalloy*** − 8,000, ***electrical steel*** − 4,000, ***steel*** − 700, ***nickel*** − 100, etc.

* ***I-V Relation*** (***Derivative Form***): $v = L\frac{di}{dt}$

* ***Note*** : Under steady-state, v = 0

$\Rightarrow$ Under ***dc*** conditions, inductors behave as ***short-circuits***

$\Rightarrow$ extremely important observation

* ***Note***: This observation is ***dual*** to that of a capacitor

* ***I-V Relation*** (***Integral Form***):

$$di = \frac{v}{L}dt \qquad \text{or} \qquad i = \frac{1}{L}\int v\,dt + K_2$$

K_2 : constant of integration (initial current flowing through the inductor)

Why current through inductors cannot change instantly?

* From the I-V relation, it will entail an ***infinite*** voltage change, which is impossible
* In reality, the coil has ***resistance***, and that won't let the voltage change infinitely
* Whenever there is a change in current flowing through an inductor, it produces a change in flux linkage of the ferromagnetic core, which produces a ***back emf*** that opposes the very cause producing it (***Lenz's law***)

Note: ***Whereas in a capacitor, the voltage across it cannot change instantly, for the case of an inductor, the current flowing through it cannot change instantly***

Keep these two observations in mind!

* ***Instantaneous power*** delivered to an inductor:

$$p = vi = Li\frac{di}{dt}$$

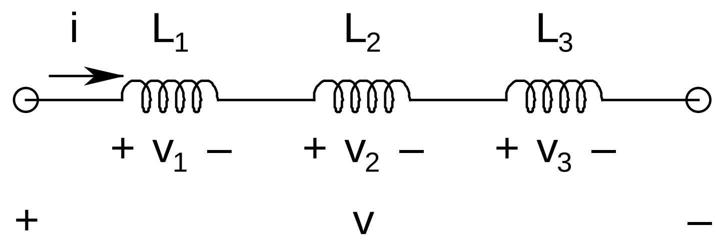
* If the initial current through the inductor was zero, i.e., $i(0) = 0$, and after a finite time t_1, a current I flows through it, then the total ***energy stored*** in the inductor at time t_1:

$$\text{Energy stored} = \int_0^{t_1} p\,dt = L\int_0^{I} i\,di = \frac{1}{2}LI^2$$

* Practical inductors have ***series resistance*** associated with them, due to the finite resistance of the coil wire, however, it is ***very small*** (typically a few Ωs at the most)

- ***Series Combination of Inductors***:

Note: Same current (i) flows through all the inductors



$$v = v_1 + v_2 + v_3 \quad \Rightarrow \quad L_{net}\frac{di}{dt} = L_1\frac{di}{dt} + L_2\frac{di}{dt} + L_3\frac{di}{dt}$$

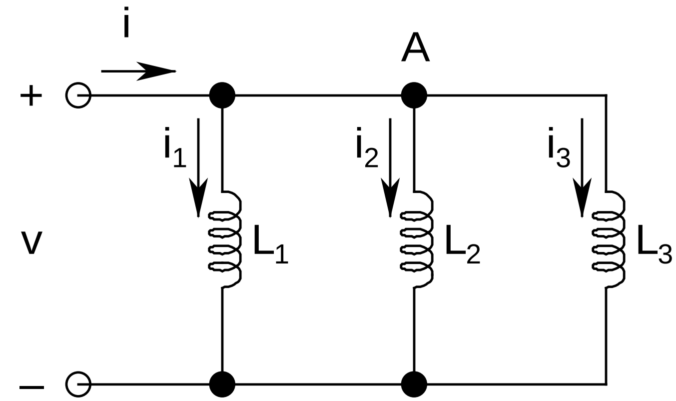
$\Rightarrow L_{net} = L_1 + L_2 + L_3 \Rightarrow$ ***largest one dominates***

Note: This rule follows that of series combination of resistors!

Contrast this result with the series combination of capacitors, where the smallest one dominates

- ***Parallel Combination of Inductors***:

Note: All the inductors have the same voltage v across them



$$i = i_1 + i_2 + i_3 \qquad \Rightarrow \qquad \frac{1}{L_{net}}\int v dt = \frac{1}{L_1}\int v dt + \frac{1}{L_2}\int v dt + \frac{1}{L_3}\int v dt$$

$\Rightarrow L_{net} = \left(L_1^{-1} + L_2^{-1} + L_3^{-1}\right)^{-1} \Rightarrow$ ***smallest one dominates***

Note: This rule follows that of parallel combination of resistors!

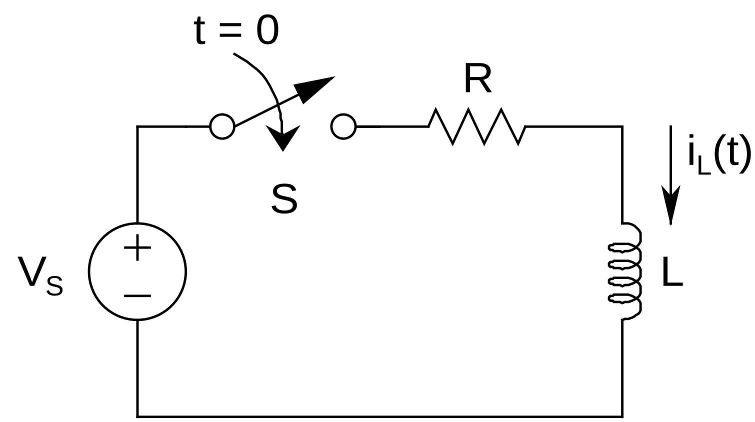
Contrast this result with the parallel combination of capacitor, where the largest one dominates

- ***Transient Response***:
  - Time response of RC and RL circuits
  - These are known as ***first-order circuits***
  - There is another type of circuit, having R, L, and C: these are known as ***second-order circuits***
  - We will consider only first-order circuits
  - The input is assumed to be a ***step function***, either going from zero to maximum or from maximum to zero within an infinitesimally small time
  - The behavior of the circuit, which is a function of time, is known as the ***transient response***

- ***RL Circuit***:

V_S : DC Voltage Source of magnitude V_1

Switch S was open for a long time and is closed at t = 0



Note: $i_L(t)$ for both $t = 0^-$ and 0^+ are zero, since no current was flowing in the circuit at $t = 0^-$, and that the current through an inductor cannot change instantly

* For $t > 0^+$: $V_1 = i_L(t)R + L\dfrac{di_L(t)}{dt}$

* Simple first-order differential equation with constant coefficients

Solution: $i_L(t) = \dfrac{V_1}{R}\left[1 - \exp\left(-\dfrac{Rt}{L}\right)\right]$

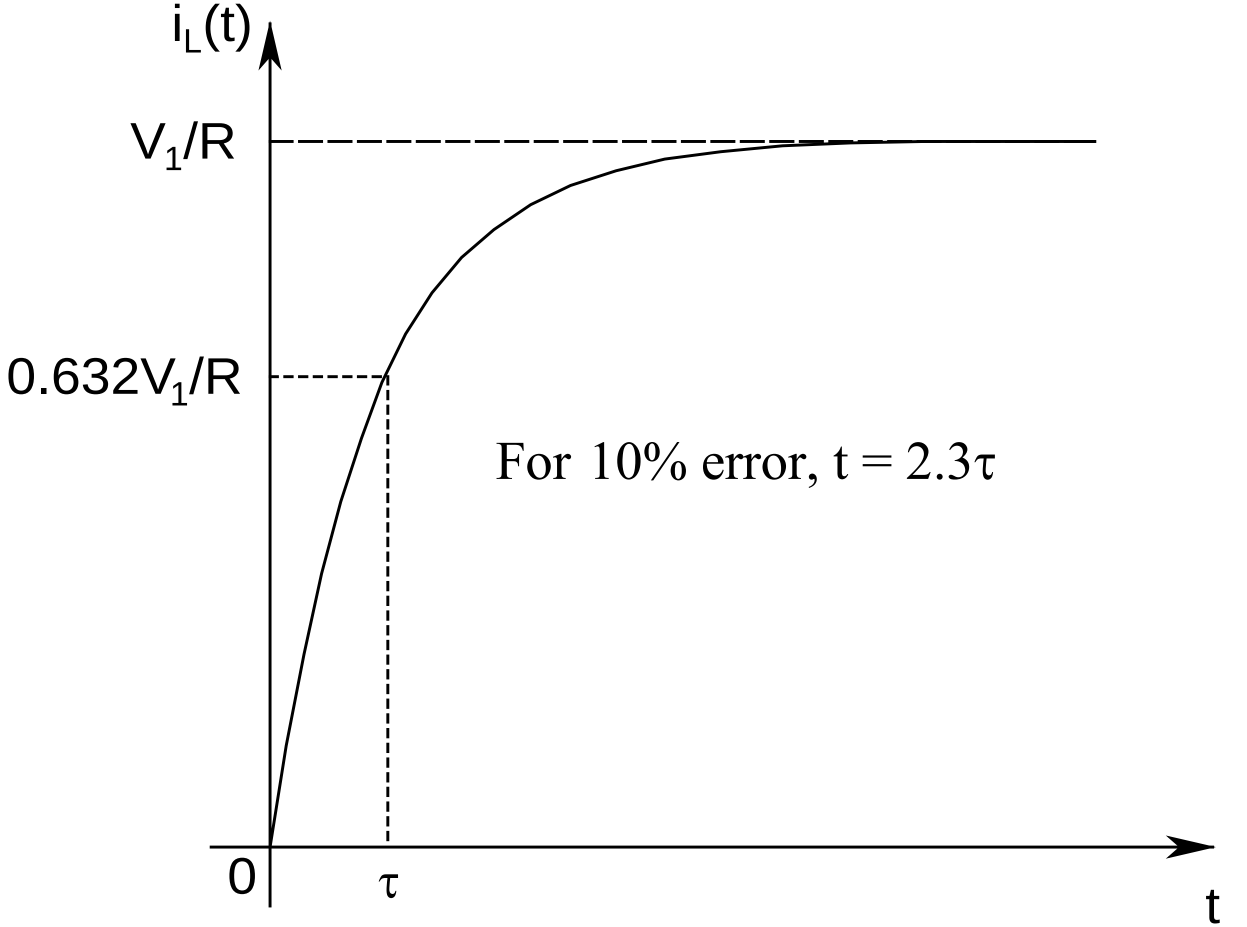
$$= \frac{V_1}{R}\left[1 - \exp\left(-\frac{t}{\tau}\right)\right]$$

* $\tau = L/R$ is known as the ***time constant*** of the circuit (in sec)

* Recall: the unit of L is Ω-sec

* At $t = 0$, $i_L(0) = 0$, as expected
* As $t \rightarrow \infty$, $i_L(\infty) = V_1/R$, since the inductor becomes a short-circuit, and the current gets controlled only by R
* Note: Since the current varies exponentially with time, hence, it would take ***infinite*** amount of time for the current to become exactly equal to V_1/R

* Hence, need to make some ***engineering approximations***
* Note: For $t = 4\tau$, $i_L(4\tau) = 0.98V_1 / R$
  (within 2% of the final steady-state value)
* For $t = 5\tau$, $i_L(5\tau) = 0.99V_1 / R$
  (within 1% of the final steady-state value)
* Thus, it is frequently assumed that within a time of $(4\text{-}5)\tau$, the transient gets more or less completed (1-2% error)
* This assumption is widely used in all branches of science
* Also, note that at $t = \tau$, the current reaches 63.2% of its maximum value



$i_L(t)$
V_1/R
$0.632V_1/R$
For 10% error, t = 2.3τ
0
τ
t

- ***Trick to obtain transient response by inspection***:

  * For both rising and falling exponents, the following relation holds:

    $$i_L(t) = i_L(\infty) + \left[i_L(0) - i_L(\infty)\right]\exp(-t/\tau)$$

    where

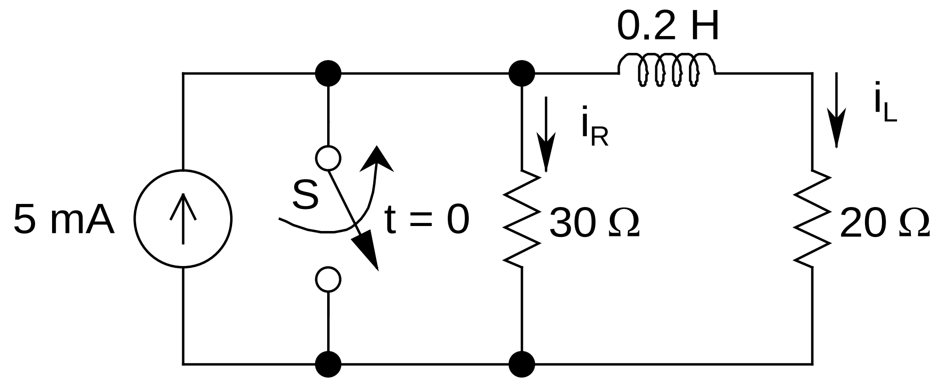
    $i_L(0)$ = current at the start of the transient,

    and

    $i_L(\infty)$ = current after all the transients have died down and the system has reached its final steady state

For the example considered, $i_L(0) = 0$, and $i_L(\infty) = V_1 / R,$ since after infinite time, all transients would have died down, and the inductor would have behaved like a short-circuit, with the same result as before

- ***Example***: S was closed for a long time, and opened at t = 0. To find i_R at t = 0^-, 0^+, ∞, and 6 ms:



With S closed, the entire current of 5 mA would flow through S, thus, at $t = 0^-,\ i_R = 0 = i_L$

After S is opened at t = 0,

i_L will remain 0 at $t = 0^+$, since the inductor would oppose any instantaneous change in current through it

$\therefore i_R(t = 0^+) = 5\ \text{mA}$

As $t \to \infty$, all transients die down, and the inductor behaves like a short-circuit

The 5 mA source current gets divided between the two branches depending on the values of the resistors present in these two branches:

$$\Rightarrow i_R\left(t \to \infty\right) = \frac{20}{20+30} \times 5 = 2 \text{ mA}$$

The real transient problem is the evaluation of i_R at t = 6 ms

Can find the time constant τ of the circuit by applying the Thevenin's technique:

Open the independent current source, remove the inductor, and look across its two terminals to find the total resistance between them

$$\Rightarrow R = 30 + 20 = 50\ \Omega, \text{ and } \tau = L/R = 0.2 \text{ H}/50\ \Omega = 4 \text{ ms}$$

$$\textbf{\textit{Note}}: \; i_L(0) = 0, \text{ and } i_L(t \to \infty) = (5 \text{ mA}) - i_R(t \to \infty)$$
$$= (5 - 2) \text{ mA} = 3 \text{ mA}$$

$$\Rightarrow i_L(6 \text{ ms}) = i_L(\infty) + [i_L(0) - i_L(\infty)] \exp(-t/\tau)$$
$$= 3 + (0 - 3)\exp(-6/4) = 2.33 \text{ mA}$$

$$\Rightarrow i_R(6 \text{ ms}) = (5 \text{ mA}) - i_L(6 \text{ ms}) = (5 - 2.33) \text{ mA}$$
$$= 2.67 \text{ mA}$$

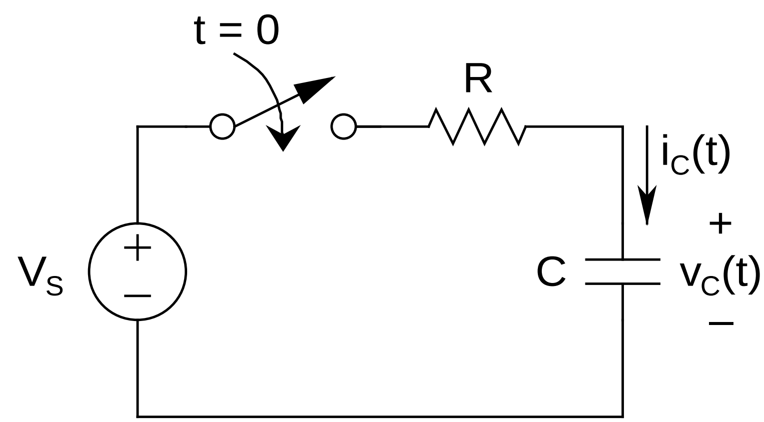
Thus, i_R reduces exponentially from 5 mA to 2 mA, and i_L increases exponentially from 0 to 3 mA, both with a time constant of 4 ms

Note: Once the inductor current (i_L) is known as a function of time, the voltage (v_L) across it can be obtained from the relation $v_L = L(di_L/dt)$ (Caution: take care of the sign)

- ***RC Circuit***:

V_S : DC Voltage Source of magnitude V_1

Switch S was open for a long time and is closed at t = 0



Note: $v_C(t)$ for both $t = 0^-$ and 0^+ are zero, since the capacitor was initially discharged, and that the voltage across a capacitor cannot change instantly

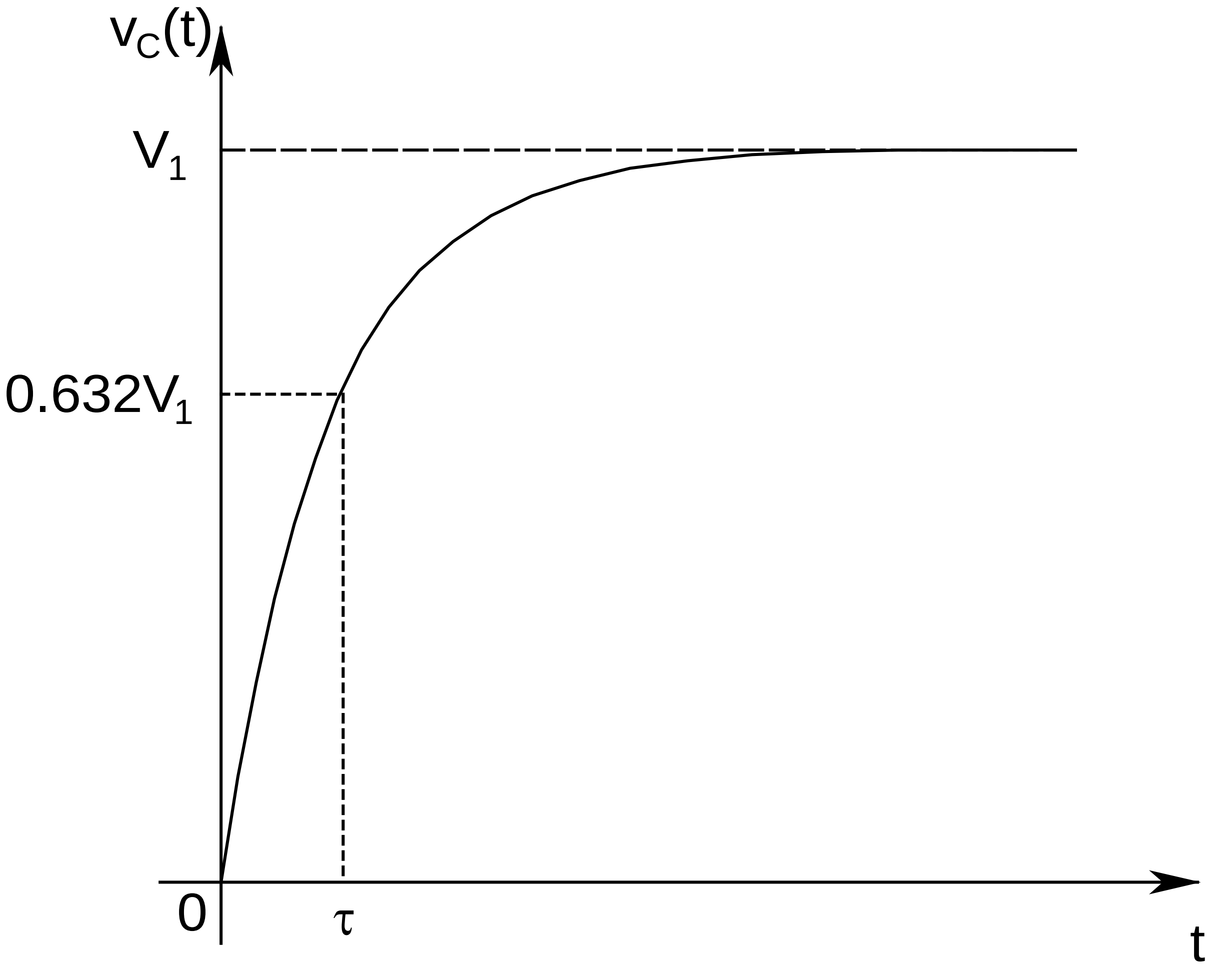
For $t > 0^+$: $V_1 = i_C(t)R + v_C(t) = RC\dfrac{dv_C(t)}{dt} + v_C(t)$

Simple first-order differential equation with constant coefficients

Solution: $v_C(t) = V_1\left[1 - \exp\left(-\dfrac{t}{\tau}\right)\right]$ $(\tau = RC = \text{time constant})$

Observations:

* At $t = 0$, $v_C(0) = 0$, as expected
* As $t \rightarrow \infty$, $v_C(\infty) = V_1$, since the capacitor becomes an open-circuit, and the current through the circuit becomes zero
* ***Note***: Since the capacitor voltage varies exponentially with time, hence, it would take infinite amount of time for this voltage to become exactly equal to V_1
* However, based on the preceding discussion, a time of the order of $\boldsymbol{(4 - 5)\tau}$ is sufficient for the transient to reach close to its final steady-state value
* Also, for 10% error, $t = 2.3\tau$



vC(t)
V1
0.632V1
0
τ
t

- ***Trick to obtain transient response by inspection***:

  * For both rising and falling exponents, the following relation holds:

  $$v_C(t) = v_C(\infty) + \left[v_C(0) - v_C(\infty)\right]\exp(-t/\tau)$$

  where

  $v_C(0)$ = capacitor voltage at the start of the transient, and

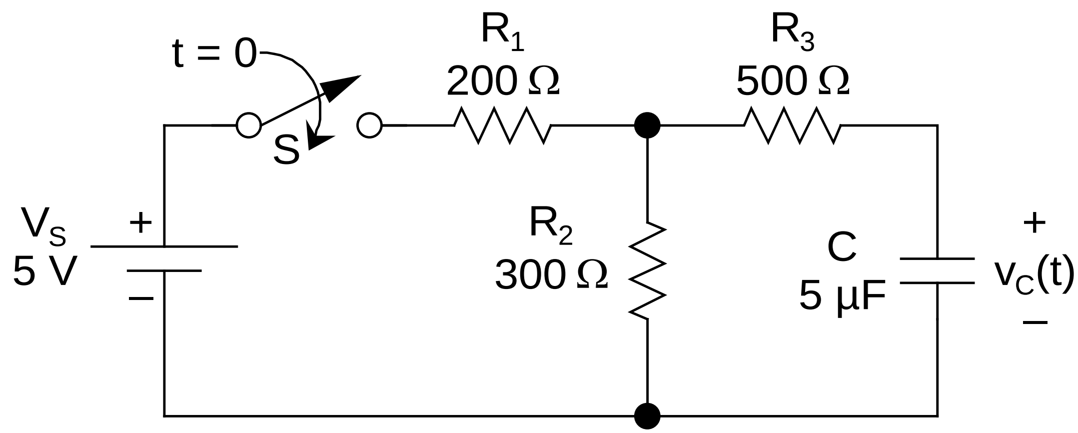
  $v_C(\infty)$ = capacitor voltage after all the transients have died down and the system has reached its final steady state

  * For the example considered, $v_C(0) = 0$, and $v_C(\infty) = V_1$, which would lead to the same result as before

- ***Example***: S was open for a long time, closes at t = 0, and opens again at t = 5 ms. Sketch $v_C(t)$.

S was open for a long time $\Rightarrow$ C will get completely discharged through R_2-R_3

$$\Rightarrow v_C\left(0^-\right) = 0$$



At t = 0, S closes, however, capacitor voltage cannot change instantly $\Rightarrow v_C\left(0^+\right) = 0$

For t > 0, the capacitor voltage would grow exponentially

The problem has ***two part transients***: one between 0 and 5 ms, and the other beyond 5 ms

<u>For t between 0 and 5 ms</u>:

Need to find $v_C(\infty)$ and τ_1 (time constant)

$$v_C(\infty) = \frac{R_2}{R_1 + R_2} V_S = \frac{300}{200+300} \times 5 = 3\ \text{V}$$

since the capacitor would behave like an open-circuit

For τ_1, we need to find R_{eff1}, i.e., the resistance ***seen*** by C

Using Thevenin's approach, by observation:

$$R_{eff1} = R_3 + \left(R_1 \| R_2\right) = 500 + (200 \| 300) = 620\ \Omega$$

$$\Rightarrow \tau_1 = R_{eff1}C = (620\ \Omega) \times (5\ \mu\text{F}) = 3.1\ \text{ms}$$

$$\Rightarrow v_C(t) = v_C(\infty) + \left[v_C(0) - v_C(\infty)\right]\exp\left(-t/\tau\right)$$

$$= 3 + (0-3)\exp(-t/3.1) = 3\left[1 - \exp(-t/3.1)\right]$$

At $t = 5\text{ ms}$, $v_C(5\text{ ms}) = 2.4\text{ V}$

Note: From $t = 0$ to 5 ms, $v_C(t)$ increases exponentially from 0 V to 2.4 V

Note also that if the transient were allowed to proceed all the way (i.e., if S was not reopened at $t = 5\text{ ms}$), then v_C would have reached a final steady state value of 3 V

For t > 5 ms:

S opens again at $t = 5\text{ ms}$, thus removing the source V_S from the circuit

Hence, C would now start to discharge and eventually $v_C(t)$ as $t \to \infty$ would approach zero

For this part of the transient, let us measure time starting from 5 ms, i.e., in our new reference of time,

$v_C(0) = 2.4$ V, and $v_C(\infty) = 0$

The effective resistance ***seen*** by C for this case:

$R_{eff\,2} = R_2 + R_3 = 800\ \Omega$

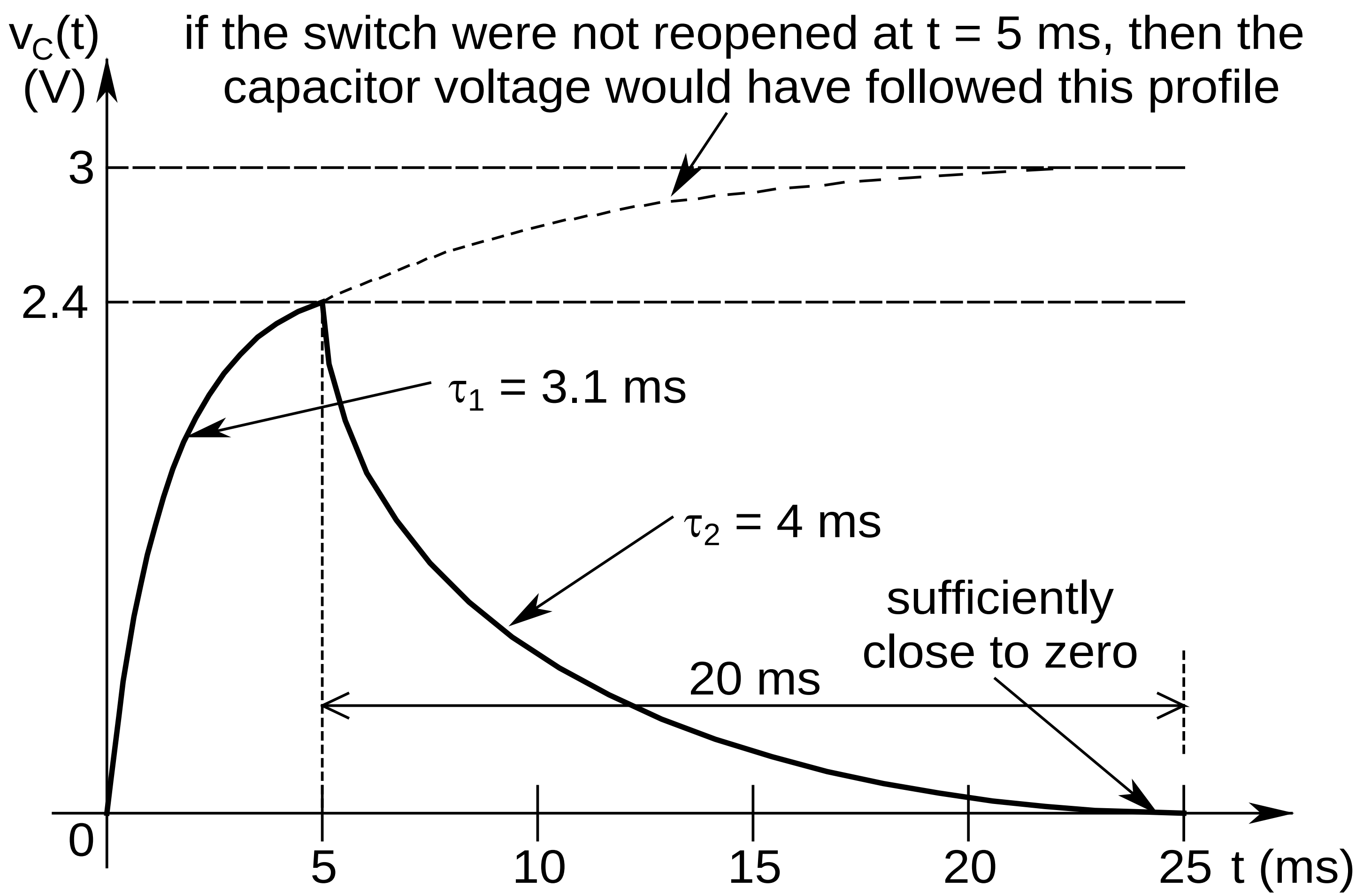
since R_1 gets open-circuited

$\Rightarrow$ time constant $\tau_2 = R_{eff\,2}C = (800\ \Omega) \times (5\ \mu F) = 4$ ms

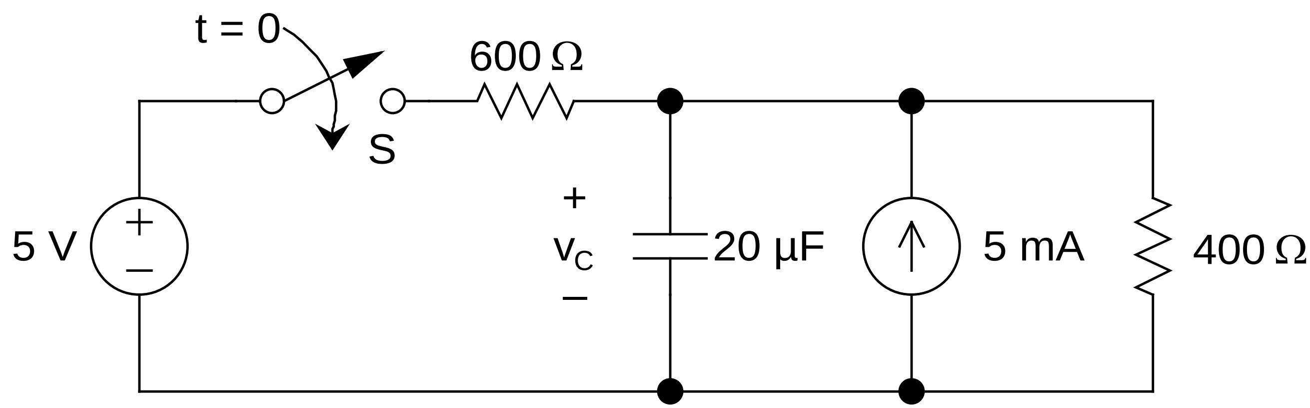
$\Rightarrow v_C(t) = v_C(\infty) + \left[v_C(0) - v_C(\infty)\right]\exp(-t/\tau) = 2.4\exp(-t/4)$

with t = 0 implying an actual time of 5 ms

Allowing a time interval of $5\tau_2$, i.e., after 20 ms (measured in this new time scale), v_C would drop sufficiently close to zero


$v_C(t)$
(V)
if the switch were not reopened at t = 5 ms, then the capacitor voltage would have followed this profile
3
2.4
τ_1 = 3.1 ms
τ_2 = 4 ms
sufficiently
close to zero
20 ms
0
5
10
15
20
25
t (ms)

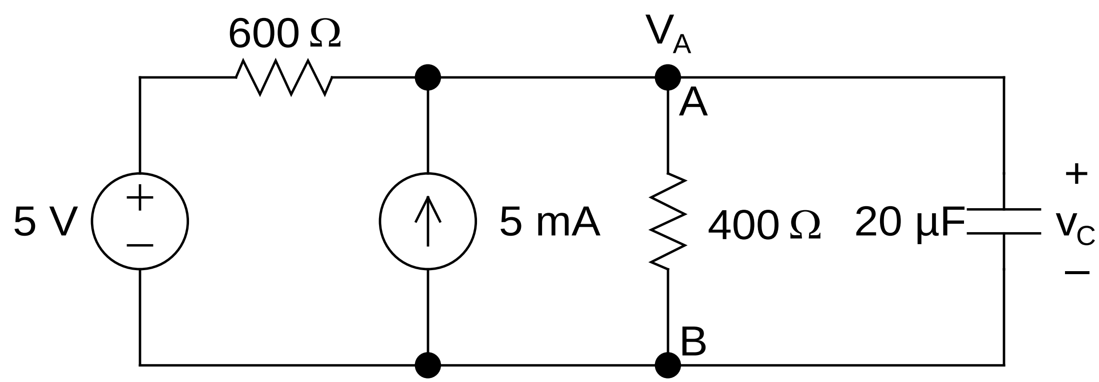
- ***Example***: S was open for a long time, and closes at t = 0. Find v_C for t = 0^-, 0^+, ∞, and 10 ms.



With S open, 5 V source was disconnected from the circuit, and only 5 mA source was active

$$\Rightarrow\ v_C\left(0^-\right) = v_C\left(0^+\right) = \left(5\text{ mA}\right) \times (400\Omega) = 2\text{ V}$$

With S closed at t = 0, we redraw the circuit :



Apply Thevenin's technique: Remove the capacitor

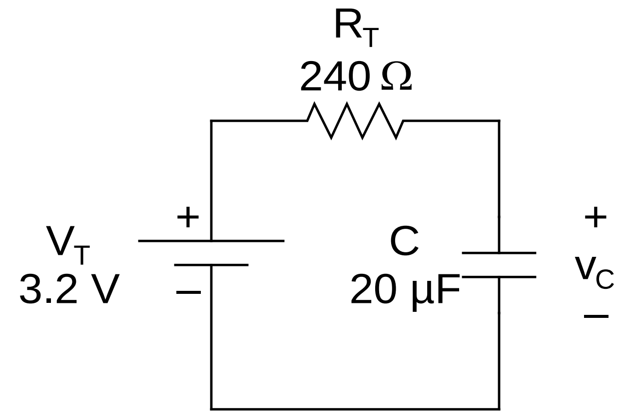
Taking B as the reference potential (ground), KCL at A:

$$\frac{5-V_A}{600}+5\times 10^{-3}=\frac{V_A}{400}$$

$$\Rightarrow V_A = V_{OC} = V_T = 3.2\ \text{V}$$

and Thevenin resistance $R_T = (600)\,||\,(400) = 240\ \Omega$

Thevenin Equivalent:



$$\text{Time constant } \tau = R_T C$$
$$= (240\ \Omega) \times (20\ \mu\text{F})$$
$$= 4.8 \text{ ms}$$

From the figure, it is obvious that $v_C(\infty) = 3.2$ V

Also, $v_C(0) = 2$ V (obtained earlier)

$$\Rightarrow v_C(t) = v_C(\infty) + \left[v_C(0) - v_C(\infty)\right]\exp(-t/\tau)$$
$$= 3.2 + (2 - 3.2)\exp(-t/4.8) = 3.2 - 1.2\exp(-t/4.8)$$
$$\Rightarrow v_C(10 \text{ ms}) = 3.05 \text{ V}$$