MSO 201A : Probability and Statistics

Assignment - V

Solutions

Problem No.1 (a) For $r \in \{1, 2, \ldots\}$

$$\boxed{C_r(\mu,\lambda) = E\left((X-\mu)^r\right) = \int_{\mu}^{\infty} (x-\mu)^r \frac{1}{\lambda} e^{-\frac{(x-\mu)}{\lambda}} dx = \lambda^r \int_0^{\infty} t^r e^{-t} dt = \lambda^r \, |\underline{r}.}$$

$E(X-\mu) = \lambda \Rightarrow E(X) = \mu+\lambda$, i.e. $\boxed{\mu_1'(\mu,\lambda) = \mu+\lambda}$

$E((X-\mu)^2) = 2\lambda^2 \Rightarrow E(X^2) - 2\mu E(X) + \mu^2 = 2\lambda^2 \Rightarrow \mu_2'(\mu,\lambda) = E(X^2) = 2\lambda^2 + 2\mu\lambda + \mu^2$

$\Rightarrow \boxed{\mu_2'(\mu,\lambda) = 2\lambda^2 + 2\mu\lambda + \mu^2}$

(b) $F_{\mu,\lambda}(\xi_p) = p \Rightarrow \int_{\mu}^{\xi_p} \frac{1}{\lambda} e^{-\frac{x-\mu}{\lambda}} dx = p \Rightarrow 1 - e^{-\frac{\xi_p-\mu}{\lambda}} = p \Rightarrow \boxed{\xi_p = \mu - \lambda \ln(1-p)}$

(c) $\boxed{q_1(\mu,\lambda) = \xi_{1/4} = \mu - \lambda \ln \frac{3}{4}}$, $\boxed{m(\mu,\lambda) = \xi_{1/2} = \mu - \lambda \ln(\frac{1}{2})}$

$\boxed{q_3(\mu,\lambda) = \xi_{3/4} = \mu - \lambda \ln \frac{1}{4}}$. (lower quartile, median and upper quartile)

(d) Clearly $f_{\mu,\sigma}(x) \downarrow$ on $[\mu, \infty) \Rightarrow \sup\{f_{\mu,\sigma}(x) : x \in \mathbb{R}\} = f_{\mu,\sigma}(\mu) = \frac{1}{\lambda}$

$\Rightarrow \boxed{mo(\mu,\lambda) = \mu.}$ (Mode)

(e) $\sigma(\mu,\lambda) = \sqrt{\mu_2} = \boxed{\sqrt{2}\lambda}$ (from (a)) (Standard deviation)

For $d>0$ $MD(m(\mu,\lambda)) = E(|X-\mu+\lambda \ln \frac{1}{2}|) = E(|X-\mu-\lambda \ln 2|)$

$E(|X-\mu-d\lambda|) = \int_{\mu}^{\infty} |x-\mu-d\lambda| \frac{1}{\lambda} e^{-\frac{x-\mu}{\lambda}} dx = \lambda \int_0^{\infty} |z-d| e^{-z} dz$

$= \lambda(d-1+2e^{-d}) \Rightarrow \boxed{MD(m(\mu,\lambda)) = \lambda \ln 2}$ (Mean deviation about median)

$IQR(\mu,\lambda) = q_3(\mu,\lambda) - q_1(\mu,\lambda) = \boxed{\lambda \ln 3}$ (Inter quartile range)

$CQD(\mu,\lambda) = \frac{q_3 - q_1}{q_3 + q_1} = \boxed{\frac{\lambda \ln 3}{2\mu - \lambda \ln \frac{3}{16}}}$ (Coefficient of quartile deviation)

$CV(\mu,\lambda) = \frac{\sigma(\mu,\lambda)}{\mu_1'(\mu,\lambda)} = \boxed{\frac{\sqrt{2}\lambda}{\mu+\lambda}}$ (Coefficient of Variation)

(f) $\mu_3 = E((X-\mu)^3) = 6\lambda^3 \Rightarrow \beta_1(\mu,\lambda) = \frac{\mu_3^2}{\mu_2^3} = \frac{36\lambda^6}{8\lambda^6} = \boxed{\frac{9}{2}}$ (Coeff. of Skewness)

$\beta_2(\mu,\lambda) = \frac{q_3 - 2m + q_1}{q_3 - q_1} = \boxed{\frac{\ln(4/3)}{\ln 3}}$ (Yule Coefficient of Skewness)

(g) $\mu_4 = E((X-\mu)^4) = 24\lambda^4$; $\gamma_1(\mu,\lambda) = \frac{\mu_4}{\mu_2^2} = \frac{24\lambda^4}{4\lambda^4} = \boxed{6}$ (Kurtosis)

$\gamma_2(\mu,\lambda) = \gamma_1(\mu,\lambda) - 3 = \boxed{3}$ (Excess Kurtosis)

(h) $\beta_1(\mu,\lambda) > 0$ and $\beta_2(\mu,\lambda) > 0 \Rightarrow$ Distribution of $X_{\mu,\sigma}$ is +vely skewed

$\Rightarrow f_{\mu,\sigma}$ has longer tails on the rhs

$\gamma_2(\mu,\lambda) > 0 \Rightarrow$ Distribution of $X_{\mu,\sigma}$ is leptokurtic

$\Rightarrow f_{\mu,\sigma}$ is more peaked around μ than normal distribution.

Problem No. 2 Note that $f_X(x+\mu) = f_X(\mu - x),\ \forall x \in \mathbb{R} \Rightarrow X - \mu \stackrel{d}{=} \mu - X$

(a) $X-\mu \stackrel{d}{=} \mu - X \Rightarrow P(X-\mu \le 0) = P(\mu - X \le 0) \Rightarrow F_X(\mu) + F_X(\mu-) = 1$

$\Rightarrow F_X(\mu-) \le \frac{1}{2} \le F_X(\mu)$ (since $F_X(\mu-) \le F_X(\mu)$)

$\Rightarrow m_e = \mu$

Also

$P(X < q_3) \le \frac{3}{4} \le P(X \le q_3) \Rightarrow P(X-\mu < q_3 - \mu) \le \frac{3}{4} \le P(X-\mu \le q_3-\mu)$

$\Rightarrow P(\mu - X < q_3 - \mu) \le \frac{3}{4} \le P(\mu - X \le q_3 - \mu)$ (since $X-\mu \stackrel{d}{=} \mu - X$)

$\Rightarrow 1 - F_X(2\mu - q_3) \le \frac{3}{4} \le 1 - F_X((2\mu - q_3)-)$

$\Rightarrow F_X((2\mu - q_3)-) \le \frac{1}{4} \le F_X(2\mu - q_3) \Rightarrow q_1 = 2\mu - q_3$

$\Rightarrow \mu = m_e = \frac{q_1 + q_3}{2}$

(b) $X - \mu \stackrel{d}{=} \mu - X \Rightarrow E(X-\mu) = E(\mu - X) \Rightarrow E(X) = \mu = \frac{q_1+q_3}{2}$

Problem No. 3 Consider getting an upper face with 2 or 3 dots as success (S). Then $P(S) = \frac{1}{3}$

$X =$ # of successes in 6 Bernoulli trials $\sim Bin(6, \frac{1}{3})$

Required probability $= P(X=2) = \binom{6}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^4 = \frac{80}{243}$.

$\boxed{\text{Problem No. 4}}$ Consider a sequence of Bernoulli trials with success probability in each trial as p. Define

X_n = # of successes in first n trials

X_{n-1} = # of successes in first $(n-1)$ trials

Then $X_n \sim Bin(n, p)$ and $X_{n-1} \sim Bin(n-1, p)$. Also

$X_n \overset{d}{=} X_{n-1} + Y_n$, where $Y_n \sim Bin(1, p)$ and X_{n-1} and Y_n are indep t.

Thus

$$P(X_n \geq r) = P(X_{n-1}+Y_n \geq r) = P(X_{n-1}+Y_n \geq r, Y_n = 0) + P(X_{n-1}+Y_n \geq r, Y_n = 1)$$
$$= P(X_{n-1} \geq r, Y_n = 0) + P(X_{n-1} \geq r-1, Y_n = 1)$$
$$= P(X_{n-1} \geq r)\, P(Y_n = 0) + P(X_{n-1} \geq r-1)\, P(Y_n = 1)$$
$$= P(X_{n-1} \geq r)(1-p) + P(X_{n-1} \geq r-1)\, p$$
$$= P(X_{n-1} \geq r) + p\,\big(P(X_{n-1} \geq r-1) - P(X_{n-1} \geq r)\big)$$
$$= P(X_{n-1} \geq r) + p\, P(X_{n-1} = r-1)$$
$$= P(X_{n-1} \geq r) + \left\{\binom{n-1}{r-1} p^{r-1}(1-p)^{n-r}\right\} p$$

$\boxed{\text{Problem No. 5}}$ $f_X(x) = P(X=x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, \dots, n$

$$f_X(x+1) > f_X(x) \iff \frac{\lfloor n}{\lfloor x+1 \, \lfloor n-x-1} p^{x+1}(1-p)^{n-x-1} > \frac{\lfloor n}{\lfloor x \, \lfloor n-x} p^x (1-p)^{n-x}$$
$$\iff x < (n+1)p - 1 \quad \dots\dots (A)$$

$$f_X(x+1) < f_X(x) \iff x > (n+1)p - 1 \quad \text{- - - -} (B)$$

Case I $(n+1)p$ is an integer

We have from (A) and (B) (along with the fact that $P(X = (n+1)p-1) = P(X = (n+1)p)$)

$$P(X=0) < P(X=1) < \dots < P(X=(n+1)p-1) = P(X=(n+1)p) > P(X=(n+1)p+1) > \dots > P(X=n)$$

$\Rightarrow$ We have two modes $(n+1)p-1$ and $(n+1)p$.

Case II $(n+1)p$ is not an integer.

Let $M = [(n+1)p]$. Then we have from (A) and (B)

$$P(X=0) < P(X=1) < \dots < P(X=M) > P(X=M+1) > \dots > P(X=n)$$

$\Rightarrow$ $M_0 = [(n+1)p]$ is the mode.

$\boxed{\text{Problem No. 6}}$ Denote the event of a ball going into any of the boxes B_1 B_2 and B_3 as success, so that $P(S) = 3/7$

X = # of balls in boxes B_1, B_2 and B_3 taken together $\sim Bin(18, \frac{3}{7})$.

Required probability $= P(X=6) = \binom{18}{6} \left(\frac{3}{7}\right)^6 \left(\frac{4}{7}\right)^{12}$.

$\boxed{\text{Problem No. 7}}$ Suppose that $T \sim Ge(p)$. Then $P(T \geq j) = (1-p)^j$, $j = 0, 1, 2, \dots$

$$P(T \geq j+k) = p^{j+k} = p^j p^k = P(T \geq j)\, P(T \geq k), \ \forall j, k \in S_T.$$

Conversely suppose that T has LOM property i.e.

$$P(T \geq j+k) = P(T \geq j)\, P(T \geq k) \quad \forall j, k \in S_T$$

$$\Rightarrow P(T \geq j+1) = P(T \geq j)\, P(T \geq 1)$$
$$= P(T \geq j-1)\, (P(T \geq 1))^2$$
$$\vdots$$
$$= P(T \geq 0)\, (P(T \geq 1))^{j+1}$$
$$= (1-p)^{j+1},$$

where $p = P(T=0) \in (0, 1)$.

Thus, for $k \in \{0, 1, 2, \dots\}$

$$P(T=k) = P(T \geq k) - P(T \geq k+1)$$
$$= p^k - p^{k+1} = p(1-p)^k$$

$\Rightarrow T \sim Ge(p)$.

$\boxed{\text{Problem No. 8}}$ In each trial, label the outcome of observing an upper face with two or three dots as success and observing any other outcomes as failure. Then $P(S) = \frac{1}{3}$.

X = # of failures preceding the 2^{nd} success $\sim NB(2, \frac{1}{3})$

Required prob $= P(X+2 = 8)$
$$= P(X=6) = \binom{7}{1}\left(\frac{1}{3}\right)^2\left(1-\frac{1}{3}\right)^6 = \frac{448}{6561}.$$

$\boxed{\text{Problem No. 9}}$ (a) $X = Z - r$ denotes the number of failures preceding the r-th success $\sim NB(r, p)$.

$$f_Z(z) = P(Z = z) = P(X = z - r) = \begin{cases} \binom{z-1}{r-1} p^r (1-p)^{z-r}, & z = r, r+1, \dots \\ 0, & \text{otherwise} \end{cases}$$

For $r = 1$, $X \sim NB(1, p)$. Thus X has LOM property and

$$P(Z > m+n) = P(X > m+n-1) = P(X \geq m+n) = P(X \geq m)\, P(X \geq n)$$
$$= P(Z \geq m+1)\, P(Z \geq n+1)$$
$$= P(Z > m)\, P(Z > n).$$

(b) X_1 = # of games team A will have to play to secure 5^{th} win

X_2 = # of games team B will have to play to secure 5^{th} win

Required prob $= p = P(X_1 = 8) + P(X_2 = 8)$

$$= P(Y_1 = 13) + P(Y_2 = 13),$$

Where $Y_1 = X_1 - 5$ ($Y_2 = X_2 - 5$) is the number of failures preceding the 5^{th} success for Team A (Team B); here success for a team is win. Then $Y_1 \sim NB(5, 0.7)$ and $Y_2 \sim NB(5, 0.3)$.

$$P(X_1 = 8) = P(Y_1 = 3) = \binom{7}{4} (0.7)^5 (1 - 0.7)^3 \approx .1588$$

$$P(X_2 = 8) = P(Y_2 = 3) = \binom{7}{4} (0.3)^5 (1 - 0.3)^3 \approx 0.0292$$

Required prob $= P(X_1 = 8) + P(X_2 = 8) = 0.188$ (approx.).

$\boxed{\text{Problem No. 10}}$ Consider a sequence of independent Bernoulli trials with probability of success in each trial being p. Then

$$\sum_{k=r}^{n} \binom{n}{k} p^k (1-p)^{n-k} = P(\text{at least } r \text{ successes in } n \text{ Bernoulli trials})$$

$$= P\left(\sum_{k=0}^{n-r} \{ r\text{th success in } (r+k)\text{-th trial} \} \right)$$

$$= \sum_{k=0}^{n-r} P(r\text{th success in } (r+k)\text{-th trial})$$

$$= \sum_{k=0}^{n-r} \underbrace{\binom{r+k-1}{r-1} p^{r-1} (1-p)^k}_{\substack{(r-1) \text{ successes} \\ \text{in first } (r+k-1) \text{ trials}}} \; \underbrace{p}_{\substack{\text{success in} \\ (r+k)\text{-th trial}}} = \text{RHS}$$

$\boxed{5/V}$

Direct Method: Let $q = 1-p$. We need to show that

$$\sum_{k=r}^{n}\binom{n}{k}(1-q)^{k-r}q^{n-k} = \sum_{k=0}^{n-r}\binom{r+k-1}{k}q^k$$

LHS = polynomial in q of degree $(n-r)$ = $\sum_{k=0}^{n-r} c_k q^k$

It is enough to show that $c_k = \binom{r+k-1}{k}$, $k = 0, 1, 2, \dots, n-r$.

For $k \in \{0, 1, 2, \dots, n-r\}$

$$c_k = \text{coefficient of } q^k \text{ in } \sum_{j=r}^{n}\binom{n}{j}(1-q)^{j-r}q^{n-j} = \sum_{j=0}^{n-r}\underbrace{\binom{n}{n-j}}_{=\binom{n}{j}}(1-q)^{n-j-r}q^j$$

$$= \text{Coefficient of } q^k \text{ in } \sum_{j=0}^{k}\binom{n}{j}(-1)^{k-j}\binom{n-j-r}{k-j}$$

$$\hookrightarrow = \sum_{j=0}^{k}\binom{n}{k-j}(-1)^j\binom{n-k-r+j}{j}$$

But

$$(-1)^j\binom{n-k-r+j}{j} = (-1)^j\frac{(n-k-r+j)(n-k-r+j-1)\cdots(n-k-r+1)}{j!}$$

$$= \frac{(-n+k+r-j)(-n+k+r-j+1)\cdots(-n+k+r-1)}{j!}$$

$$= \binom{-n+k+r-j}{j}$$

$$\Rightarrow c_k = \sum_{j=0}^{k}\binom{n}{k-j}\binom{-n+k+r-j}{j} = \binom{k+r-1}{k}.$$

Problem No. 11 Let us call the event of choosing Box 1 as success and that of choosing Box 2 as failure. Then we have a sequence of independent Bernoulli trials with probability of success in each trial being p.

Required prob. = P(when box 1 is found empty, box 2 has k matches)
+ P(when box 2 is found empty, box 1 has k matches)

= P($(n-k)$ failures precede the $(n+1)$th success)
+ P($(n-k)$ successes precede the $(n+1)$th failure)

$$= \binom{2n-k}{n}\left(\tfrac{1}{2}\right)^n\left(\tfrac{1}{2}\right)^{n-k}\times\tfrac{1}{2} + \binom{2n-k}{n}\left(\tfrac{1}{2}\right)^n\left(\tfrac{1}{2}\right)^{n-k}\times\tfrac{1}{2}$$

$$= \binom{2n-k}{n}\left(\tfrac{1}{2}\right)^{2n-k} \qquad k = 0, 1, 2, \dots, n.$$

Problem No. 12 Clearly $P(A_1) = \frac{a}{N}$ and

$$P(A_2) = P(A_1^c \cap A_2) + P(A_1 \cap A_2)$$
$$= P(A_1^c)P(A_2 | A_1^c) + P(A_1)P(A_2|A_1)$$
$$= \frac{N-a}{N}\cdot\frac{a}{N-1} + \frac{a}{N}\cdot\frac{a-1}{N-1} = \frac{a}{N}$$

Now suppose that $P(A_m) = \frac{a}{N}$, for some $m \in \{1, 2, \dots, N-1\}$

Then

$$P(A_{m+1}) = \sum_{k=0}^{m} P(A_{m+1} \cap \{X_{a,m,N} = k\})$$
$$= \sum_{k=0}^{m} f_{a,m,N}(k)\, P(A_{m+1} | X_{a,m,N} = k)$$
$$= \sum_{k=0}^{m} f_{a,m,N}(k)\, \frac{a-k}{N-m},$$

where $f_{a,m,N}$ is the p.m.f. of $X_{a,m,N}$.

Thus

$$P(A_{m+1}) = \sum_{k=\max\{0,\, m-N+a\}}^{\min\{m,a\}} \frac{\binom{a}{k}\binom{N-a}{m-k}}{\binom{N}{m}}\, \frac{a-k}{N-m}$$
$$= \frac{a}{N-m} - \frac{1}{N-m} E(X_{a,m,N})$$
$$= \frac{a}{N-m} - \frac{1}{N-m}\, m\frac{a}{N} = \frac{a}{N}.$$

Hence the result follows by induction.

Problem No. 13 (a) Let $N = a+b$. Then for $x \in \{0, 1, \dots, n\}$, $n \le a+b$ and $\max\{0, n-b\} \le x \le \min\{n, a\}$

$$f_X(x) = \frac{\binom{a}{x}\binom{b}{n-x}}{\binom{a+b}{n}}$$
$$= \binom{n}{x}\frac{a(a-1)\cdots(a-x+1)\,b(b-1)\cdots(b-n+x+1)}{(a+b)(a+b-1)\cdots(a+b-n+1)}$$
$$= \binom{n}{x}\frac{\frac{a}{a+b}\left(\frac{a}{a+b}-\frac{1}{a+b}\right)\cdots\left(\frac{a}{a+b}-\frac{x-1}{a+b}\right)\frac{b}{a+b}\left(\frac{b}{a+b}-1\right)\cdots\left(\frac{b}{a+b}-\frac{n-x-1}{a+b}\right)}{\left(1-\frac{1}{a+b}\right)\cdots\left(1-\frac{n-1}{a+b}\right)}$$

$$\rightarrow \binom{n}{x} p^x (1-p)^{n-x} \quad \left(\text{since } \frac{a}{a+b} = \frac{a}{N} \rightarrow p,\ \frac{b}{a+b} = 1 - \frac{a}{a+b} \rightarrow 1-p\right)$$

(b) $N = 120$ is large; $a = 80$ is large; $n = 5$; $p = \frac{a}{N} = \frac{2}{3}$

X = # of applicants (out of 5 selected for interview) qualified for job

Using (a)

$$X \overset{\text{approx}}{\sim} Bin(5, p)$$

Required prob $= P(X \geq 2)$

$$= 1 - P(X=0) - P(X=1)$$

$$= 1 - \frac{\binom{40}{5}}{\binom{120}{5}} - \frac{\binom{80}{1}\binom{40}{4}}{\binom{120}{5}}$$

Approx. Prob $= P(X\geq 2) = 1 - P(X=0) - P(X=1)$

$$= 1 - (1-p)^n - \binom{n}{1} p (1-p)^{n-1} \quad (n=5,\ p=\tfrac{2}{3})$$

$$= 1 - \left(\tfrac{1}{3}\right)^5 - \tfrac{10}{3}\left(\tfrac{1}{3}\right)^4.$$

Problem No. 14 Let

$X =$ # of red balls among the n balls drawn from U_1

E: both the balls drawn from urn U_2 are red

Then $X \sim Hyp(r_1, n, N_1)$ and

Required prob. $= P(E)$

$$= \sum_{x=0}^{n} P(E|X=x)\, P(X=x)$$

$$= \sum_{x=\max\{0,\, n-N_1+r_1\}}^{\min\{n, r_1\}} P(E|X=x)\, P(X=x)$$

$$P(E|X=x) = \frac{\binom{r_2+x}{2}}{\binom{N_2+n}{2}} = \frac{r_2(r_2-1) + 2r_2 x + x(x-1)}{(N_2+n)(N_2+n-1)}$$

Thus

$$P(E) = \frac{[r_2(r_2-1) + 2r_2 E(X) + E(X(X-1))]}{(N_2+n)(N_2+n-1)}$$

$$= \frac{1}{(N_2+1)(N_2+n-1)}\left[r_2(r_2-1) + 2r_2 \frac{nr_1}{N_1} + n(n-1)\frac{r_1(r_1-1)}{N_1(N_1-1)}\right].$$

Problem No. 15 $f_X(x) = \dfrac{e^{-\lambda}\lambda^x}{\underline{|x}}, \quad x = 0, 1, 2 \dots$

$$f_X(x+1) > f_X(x) \iff x < \lambda - 1 \quad \dots \quad (A)$$

$$f_X(x+1) < f_X(x) \iff x > \lambda - 1 \quad \dots \quad (B)$$

Case I λ is an integer

Here

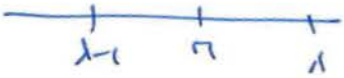
$$f_x(0) < f_x(1) < \dots < f_x(\lambda-2) < f_x(\lambda-1) = f_x(\lambda) > f_x(\lambda+1) > f_x(\lambda+2) > \dots$$

In this case there are two modes $\lambda-1$ and λ (if $\lambda > 1$) and one mode λ if $\lambda \in \{0, 1\}$.

Case II λ is not an integer

Let $m = [\lambda]$

Here



$$f_x(0) < f_x(1) < \dots < f_x(m-1) < f_x(m) > f_x(m+1) > f_x(m+2) > \dots$$

$$\Rightarrow \text{mode} = \dots m = [\lambda].$$

Problem No. 16 (a) We will use the Stirling approximation

$$\lim_{n\to\infty} \frac{\lfloor\underline{n}}{\sqrt{2\pi}\, e^{-n}\, n^{n+\frac{1}{2}}} = 1, \text{ i.e. } \lfloor\underline{n} \simeq \sqrt{2\pi}\, e^{-n}\, n^{n+\frac{1}{2}} \text{ for large } n.$$

Then

$$\text{LHS} = \frac{\lfloor\underline{r+k-1}}{\lfloor\underline{k}\ \lfloor\underline{r-1}} p_r^r (1-p_r)^k$$

$$\overset{r\to\infty}{\simeq} \frac{\sqrt{2\pi}\, e^{-(r+k-1)} (r+k-1)^{r+k-\frac{1}{2}}}{\lfloor\underline{k}\ \sqrt{2\pi}\, e^{-(r-1)} (r-1)^{r-\frac{1}{2}}} \left(1-\frac{\lambda}{r}\right)^r \left(\frac{\lambda}{r}\right)^k$$

$$= \frac{e^{-k}\lambda^k}{\lfloor\underline{k}} \left(1-\frac{\lambda}{r}\right)^r \frac{\left(1+\frac{k-1}{r}\right)^{r+k-\frac{1}{2}}}{\left(1-\frac{1}{r}\right)^{r-\frac{1}{2}}}$$

$$\simeq \frac{e^{-k}\lambda^k}{\lfloor\underline{k}}\, e^{-\lambda}\, \frac{e^{k-1}}{e^{-1}} = \frac{e^{-\lambda}\lambda^k}{\lfloor\underline{k}},\ k = 0, 1, 2, \dots$$

(b) Let us label winning of the game by a person as success and his/her losing the game as failure.

Then we have a sequence of $n = 2500$ independent Bernoulli trials with probability of success in each trial as $p = 0.002$.

X = # of successes in n Bernoulli trials $\sim Bin(2500, 0.002)$

Required prob $= P(X \geq 2) = 1 - P(X=0) - P(X=1)$.

$$= 1 - \{(1-0.002)^{2500} + 2500 \times 0.002 \times (1-0.002)^{2499}\}$$

$n = 2500$ is large, $p = 0.002$ is small. $np = 5 = \lambda$ (say).
Thus $X \overset{\text{approx}}{\sim} Po(5)$, and

Required prob $= P(X \geq 2)$

$$\overset{\text{approx.}}{=} P(Y \geq 2) \quad (Y \sim Po(5))$$

$$= 1 - [P(Y=0) + P(Y=1)]$$

$$= 1 - [e^{-\lambda} + \lambda e^{-\lambda}]$$

$$= 1 - 6e^{-5} = 0.9596$$

$\boxed{\text{Problem No. 17}}$ (a) $E\left(\frac{1}{2+X}\right) = \sum_{j=0}^{\infty} \frac{1}{2+j} \frac{e^{-\lambda}\lambda^j}{\lfloor j} = e^{-\lambda} \sum_{j=0}^{\infty} \frac{(j+1)\lambda^j}{\lfloor j+2}$

$$= e^{-\lambda} \sum_{j=2}^{\infty} \frac{(j-1)\lambda^{j-2}}{\lfloor j} = \frac{e^{-\lambda}}{\lambda^2} \left[\sum_{j=0}^{\infty} \frac{(j-1)\lambda^j}{\lfloor j} + 1 \right]$$

$$= \frac{1}{\lambda^2} \left[\sum_{j=0}^{\infty} (j-1) \frac{e^{-\lambda}\lambda^j}{\lfloor j} + e^{-\lambda} \right] = \frac{1}{\lambda^2} [E(X-1) + e^{-\lambda}]$$

$$= \frac{\lambda - 1 + e^{-\lambda}}{\lambda^2}$$

(b) $P(X = n) = p(1-p)^n$, $n = 0, 1, 2, \ldots$

$$E(\min(X, r)) = \sum_{n=0}^{\infty} \min(n, r)\, p(1-p)^n = p\left[\sum_{n=0}^{r} n(1-p)^n + rp \sum_{n=r+1}^{\infty} (1-p)^n\right]$$

$$= \frac{(1-p)[1-(1-p)^r]}{p} - r(1-p)^{r+1} + r(1-p)^{r+1}$$

$$= \frac{(1-p)[1-(1-p)^r]}{p}.$$

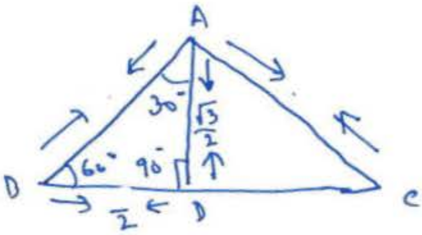
$\boxed{\text{Problem 18}}$ For $r \notin \{1, 2, \ldots, N\}$, $P(Y = r) = 0$. For $r \in \{1, 2, \ldots, N\}$

$$P(Y = r) = \frac{N-1}{N} \cdot \frac{N-2}{N-1} \cdots \frac{N-(r-1)}{N-(r-2)} \cdot \frac{1}{N-(r-1)} = \frac{1}{N}$$

$\Rightarrow Y \sim U(\{1, 2, \ldots, N\})$

$\Rightarrow E(Y) = \frac{N+1}{2}$ and $Var(Y) = \frac{N^2-1}{12}$.

$\boxed{\text{Problem 19}}$ (a) In the equilateral $\triangle ABC$



$AB = BC = X$, $BD = \frac{X}{2}$ and $AD = \frac{\sqrt{3}}{2} X$

$\Rightarrow Y = \frac{1}{2} \cdot X \cdot \frac{\sqrt{3}}{2} X = \frac{\sqrt{3}}{4} X^2$

$E(Y) = \frac{\sqrt{3}}{4} E(X^2) = \frac{\sqrt{3}}{12} a^2$

$$E(Y^2) = \frac{3}{16} E(X^4) = \frac{3}{80} a^4$$

$$\mathrm{Var}(Y) = E(Y^2) - (E(Y))^2 = \frac{a^4}{60}.$$

(b) The required probability is

$$\begin{aligned} p &= P(\max\{X, a-X\} > 2\min\{X, a-X\}) \\ &= P\left(a-X > 2X,\ X \le \frac{a}{2}\right) + P\left(X > 2(a-X),\ X > \frac{a}{2}\right) \\ &= P\left(X \le \frac{a}{3}\right) + P\left(X > \frac{2}{3}a\right) \\ &= \frac{2}{3}. \end{aligned}$$

$\boxed{\text{Problem 20}}$ (a) The d.f. of X is

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{x} \frac{e^{-|t|}}{2}\,dt$$

$$= \begin{cases} \frac{e^{x}}{2}, & \text{if } x < 0 \\ 1 - \frac{e^{-x}}{2}, & \text{if } x \ge 0 \end{cases}$$

Quantile function of X is

$$Q(p) = F^{-1}(p) = \begin{cases} \ln(2p), & 0 < p < \frac{1}{2} \\ -\ln(2(1-p)), & \frac{1}{2} \le p < 1 \end{cases}$$

The desired random observation is

$$X = Q(U) = \begin{cases} \ln(2U), & \text{if } 0 < U < \frac{1}{2} \\ -\ln(2(1-U)), & \text{if } \frac{1}{2} \le U < 1. \end{cases}$$

(b) The d.f. of X is

$$G(x) = \begin{cases} 0, & \text{if } x < 0 \\ \sum_{j=0}^{k} \binom{n}{j} \theta^j (1-\theta)^{n-j}, & \text{if } k \le x < k+1, \ k = 0, 1, \ldots, n-1 \\ 1, & \text{if } x \ge n \end{cases}$$

The quantile function of X is

$$Q(p) = \inf\{x \in \mathbb{R} : G(x) \ge p\}$$

$$= \begin{cases} 0, & \text{if } 0 < p \le (1-\theta)^n \\ k, & \text{if } \sum_{j=0}^{k-1} \binom{n}{j} \theta^j (1-\theta)^{n-j} < p \le \sum_{j=0}^{k} \binom{n}{j} \theta^j (1-\theta)^{n-j}, \ k = 1, 2, \ldots, n-1 \\ n, & \text{if } \sum_{j=0}^{n-1} \binom{n}{j} \theta^j (1-\theta)^{n-j} < U < 1 \end{cases}$$

The derived random observation is

$$X = \begin{cases} 0, & \text{if } 0 < U \le (1-\theta)^n \\ k, & \text{if } \sum_{j=0}^{k-1} \binom{n}{j} \theta^j (1-\theta)^{n-j} < U \le \sum_{j=0}^{k} \binom{n}{j} \theta^j (1-\theta)^{n-j}, \\ & \quad k = 1, 2, \ldots, n-1 \\ n, & \text{if } \sum_{j=0}^{n-1} \binom{n}{j} \theta^j (1-\theta)^{n-j} < U < 1. \end{cases}$$

$\boxed{\text{Problem 21}}$ $F_Y(y) = P(Y \le y) = \sum_{j=1}^{\theta} P(X - [X] \le y,\ j-1 \le X < j) = \sum_{j=1}^{\theta} P(X \le y+j-1,\ j-1 \le X < j)$

$$= \sum_{j=1}^{\theta} P(j-1 \le X \le \min(\theta, y+j-1)), \quad y \in \mathbb{R}.$$

Clearly, for $y<0$, $F_Y(y) = 0$ and for $y \ge 1$

$$F_Y(y) = \sum_{j=1}^{\theta} P(j-1 \le X \le j) = P(0 \le X \le \theta) = 1.$$

For $0 \le y < 1$, $F_Y(y) = \sum_{j=1}^{\theta} P(j-1 \le X \le y+j-1) = \sum_{j=1}^{\theta} \frac{y}{\theta} = y$

$$\Rightarrow F_Y(y) = \begin{cases} 0, & \text{if } y < 0 \\ y, & \text{if } 0 \le y < 1 \\ 1, & \text{if } y \ge 1 \end{cases} \Rightarrow Y \sim U(0,1)$$

$\boxed{\text{Problem 22}}$ The d.f. corresponding to p.d.f. $f(x)$ is

$$F(x) = \begin{cases} 0, & x < 0 \\ x^2(3-2x), & 0 \le x < 1 \\ 1, & x \ge 1 \end{cases}$$

and the Q.F. corresponding to F is $Q(p) = F^{-1}(p) =$ root of $3t^2 - 2t^3 - p = 0$. We know that $Q(U)$ has p.d.f. f. But

$$Q(U) = \text{root of } 3t^2 - 2t^3 - U = 0.$$

$\boxed{\text{Problem 23}}$ Let $I_n = \frac{1}{\Gamma n\, \theta^n} \int_t^\infty e^{-x/\theta} x^{n-1} dx = \frac{1}{(n-1)!} \int_{t/\theta}^\infty e^{-z} z^{n-1} dz, \quad t > 0$

$$= \frac{e^{-t/\theta} (t/\theta)^{n-1}}{(n-1)!} + \frac{1}{(n-2)!} \int_{t/\theta}^\infty e^{-z} z^{n-2} dz$$

$$= \frac{e^{-t/\theta} (t/\theta)^{n-1}}{(n-1)!} + I_{n-1}$$

$$= \frac{e^{-t/\theta} (t/\theta)^{n-1}}{(n-1)!} + \frac{e^{-t/\theta} (t/\theta)^{n-2}}{(n-2)!} + I_{n-2}$$

$$= \frac{e^{-t/\theta} (t/\theta)^{n-1}}{(n-1)!} + \frac{e^{-t/\theta} (t/\theta)^{n-2}}{(n-2)!} + \cdots + \frac{e^{-t/\theta} (t/\theta)}{1!} + I_1$$

$$= \sum_{j=0}^{n-1} \frac{e^{-t/\theta} (t/\theta)^j}{j!}, \quad t > 0.$$

$\boxed{\text{Problem 24}}$ Suppose that $Y \sim Exp(\theta)$, for some $\theta>0$. Then for $s,t>0$

$$P(Y>s+t) = e^{-\frac{s+t}{\theta}} = e^{-\frac{s}{\theta}} e^{-\frac{t}{\theta}} = P(Y>s)\; P(Y>t).$$

$\Rightarrow$ Y has LOM property.

Conversely suppose that Y has LOM property, i.e.,

$$P(Y>s+t) = P(Y>s)\,P(Y>t) \quad \forall\, s,t>0$$

$$\Rightarrow \bar{F}(s+t) = \bar{F}(s)\bar{F}(t), \ \forall\, s,t>0, \text{ where } \bar{F}(x)=P(Y>x),\ x>0.$$

$$\Rightarrow \bar{F}(s_1+s_2+\dots+s_m) = \bar{F}(s_1)\bar{F}(s_2)\cdots\bar{F}(s_m), \ \forall\, s_i>0,\ i=1,\dots,m$$

$$\Rightarrow \bar{F}\left(\tfrac{m}{n}\right) = \bar{F}(\underbrace{\tfrac{1}{n}+\dots+\tfrac{1}{n}}_{m\text{-times}}) = \left[\bar{F}\left(\tfrac{1}{n}\right)\right]^m, \ m,n\in\mathbb{N} \quad \dots\dots (A)$$

$$\Rightarrow \bar{F}(1) = \left[\bar{F}\left(\tfrac{1}{n}\right)\right]^n, \ \forall n\in\mathbb{N} \quad \dots\dots (B)$$

Using (A) and (B) we get

$$\bar{F}\left(\tfrac{m}{n}\right) = \left[\bar{F}(1)\right]^{\frac{m}{n}}, \ \forall\, m,n\in\mathbb{N} \quad \dots\dots (C)$$

Let $\lambda=\bar{F}(1)$, so that $0\le\lambda\le1$. Clearly, if $\lambda=0$ then using (B)

$$\bar{F}\left(\tfrac{1}{n}\right)=0, \ \forall n\in\mathbb{N}$$

$$\Rightarrow F\left(\tfrac{1}{n}\right)=1 \quad \forall n\in\mathbb{N}$$

$$\Rightarrow \lim_{n\to\infty} F\left(\tfrac{1}{n}\right)=1 \quad \Rightarrow F(0)=1 \ \left(\text{since } F \text{ is continuous}\right),$$

which is not true as $F(0)=0$.

Similarly if $\lambda=1$ then

$$\bar{F}(n) = \bar{F}(\underbrace{1+\dots+1}_{n\text{ times}}) = \left[\bar{F}(1)\right]^n = 1 \quad \forall n\in\mathbb{N}$$

$\Rightarrow F(n)=0, \ \forall n\in\mathbb{N} \Rightarrow \lim_{n\to\infty} F(n)=0$, which is not true. Thus $\lambda\in(0,1)$. Let $\lambda=e^{-\frac{1}{\theta}}$, for some $\theta>0$ ($\theta^{-1}=-\ln\lambda$). Then using (C)

$$\bar{F}(r) = e^{-r/\theta}, \ \forall\, r\in\mathbb{Q}\cap(0,\infty) \text{ and some } \theta>0$$

Now let $x\in\mathbb{Q}^c\cap(0,\infty)$. Then $\exists$ a sequence $\{r_n\}_{n\ge1}$ in $\mathbb{Q}\cap(0,\infty)$ such that $\lim_{n\to\infty} r_n = x$. Therefore

$$\bar{F}(x) = \bar{F}\left(\lim_{n\to\infty} r_n\right)$$
$$= \lim_{n\to\infty} \bar{F}(r_n)$$
$$= \lim_{n\to\infty} e^{-\frac{r_n}{\theta}}$$
$$= e^{-\frac{x}{\theta}}.$$

$$\Rightarrow \bar{F}(x) = P(Y>x) = e^{-x/\theta}, \ \forall x>0 \Rightarrow Y\sim Exp(\theta).$$

Problem 25 We have

$$I_{k,n} = P(Y \le p) = \frac{1}{B(k, n-k+1)} \int_0^p t^{k-1}(1-t)^{n-k}\,dt = \frac{\underline{|n}}{\underline{|k-1}\,\underline{|n-k}} \int_0^p t^{k-1}(1-t)^{n-k}\,dt$$

$$= \frac{\underline{|n}}{\underline{|k-1}\,\underline{|n-k}} \left\{ \frac{p^k(1-p)^{n-k}}{k} + \frac{n-k}{k} \int_0^p t^k (1-t)^{n-k-1}\,dt \right\}$$

$$= \binom{n}{k} p^k (1-p)^{n-k} + I_{k+1,n}$$

$$= \binom{n}{k} p^k (1-p)^{n-k} + \binom{n}{k+1} p^{k+1}(1-p)^{n-k-1} + I_{k+2,n}$$

$$= \binom{n}{k} p^k (1-p)^{n-k} + \binom{n}{k+1} p^{k+1}(1-p)^{n-k-1} + \dots + \binom{n}{n-1} p^{n-1}(1-p) + I_{n,n}$$

$$= \sum_{j=k}^{n} \binom{n}{j} p^j (1-p)^{n-j} = P(X \ge k).$$

Problem 26 (a) $-1 \le \alpha \le 1$, $-1 \le [2\Phi(x)-1] \le 1$

$$\Rightarrow \alpha\,[2\Phi(x_1)-1]\,[2\Phi(x_2)-1] \ge -1$$

$$\Rightarrow f_{\underset{\sim}{X}}(\underset{\sim}{x}) \ge 0, \quad \forall\, \underset{\sim}{x} \in \mathbb{R}^2$$

Also

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{\underset{\sim}{X}}(x_1, x_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi(x_1)\phi(x_2)\,dx_1\,dx_2 + \alpha \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [2\Phi(x_1)-1][2\Phi(x_2)-1]\,\phi(x_1)\phi(x_2)\,d\underset{\sim}{x}$$

$$= \left(\int_{-\infty}^{\infty}\phi(x_1)\,dx_1\right)\left(\int_{-\infty}^{\infty}\phi(x_2)\,dx_2\right) + \alpha\left(\underbrace{\int_{-\infty}^{\infty}(2\Phi(x_1)-1)\,\phi(x_1)\,dx_1}_{2\Phi(x_1)-1 = t \Rightarrow 2\phi(x_1)dx_1 = dt}\right) \times \left(\int_{-\infty}^{\infty}(2\Phi(x_2)-1)\,\phi(x_2)\,dx_2\right)$$

$$2\Phi(x_2)-1 = t, \quad 2\phi(x_2)\,dx_2 = dt$$

$$= 1 + \frac{\alpha}{4}\left(\int_{-1}^{1} t\,dt\right)\left(\int_{-1}^{1} t\,dt\right)$$

$$= 1$$

$\Rightarrow f_{\underset{\sim}{X}}$ is a p.d.f.

(b) For fixed $x \in \mathbb{R}$,

$$f_{X_1}(x_1) = \int_{-\infty}^{\infty} f_{\underline{X}}(x_1, x_2)\,dx_2 = \phi(x_1)\int_{-\infty}^{\infty}\phi(x_2)\,dx_2 + \alpha\,\phi(x_1)\,[2\Phi(x_1)-1]\int_{-\infty}^{\infty}[2\Phi(x_2)-1]\phi(x_2)\,dx_2$$

$$= \phi(x_1) \quad \left[\int_{-\infty}^{\infty}\phi(x_2)\,dx_2 = 1,\ \int_{-\infty}^{\infty}[2\Phi(x_2)-1]\,\phi(x_2)\,dx_2 = 0 \text{ as in (a)}\right]$$

By symmetry

$$f_{X_2}(x_2) = \phi(x_2), \quad \forall\, x_2 \in \mathbb{R}$$

(c) For $\alpha=0$, clearly $\underset{\sim}{X}=(X_1, X_2) \sim N_2(0, 0, 1, 1, 0)$. For $\alpha \neq 0$, obviously $\underset{\sim}{X}=(X_1, X_2) \not\sim N_2$.

$\boxed{\text{Problem 27}}$ We know that $(Y_1, Y_2) \sim N_2(\ ) \iff$ every linear combination of Y_1 and Y_2 has univariate normal distribution

(a) $\underbrace{t_1 Y + t_2 Z}_{\text{linear combination of } Y \text{ and } Z} = (t_1 a_1 + t_2 a_3) X_1 + (t_1 a_2 + t_2 a_4) X_2 \sim N_1$ (since $(X_1, X_2) \sim N_2$)

$\Rightarrow (Y, Z) \sim N_2$

$E(Y) = a_1\mu_1 + a_2\mu_2 = \theta_1$ (say); $E(Z) = a_3\mu_1 + a_4\mu_2 = \theta_2$ (say)

$Var(Y) = a_1^2\sigma_1^2 + a_2^2\sigma_2^2 + 2a_1a_2\sigma_1\sigma_2\rho = \tau_1^2$ (say)

$Var(Z) = a_3^2\sigma_1^2 + a_4^2\sigma_2^2 + 2a_3a_4\sigma_1\sigma_2\rho = \tau_2^2$ (say)

$Cov(Y, Z) = a_1a_3\sigma_1^2 + (a_1a_4 + a_2a_3)\sigma_1\sigma_2\rho + a_2a_4\sigma_2^2 = \rho$ (say)

Then $(Y, Z) \sim N_2(\theta_1, \theta_2, \tau_1^2, \tau_2^2, \rho)$

(b) $Y \sim N_1(\theta_1, \tau_1^2)$ and $Z \sim N_1(\theta_2, \tau_2^2)$.

$\boxed{\text{Problem 28}}$ (a) $Y|X=2 \sim N_1\left(8 + \frac{0.6 \times 3}{4}(2-5),\ 9(1-(0.6)^2)\right)$

$= N_1(6.65, 5.76)$

$P(5<Y<11|X=2) = \Phi\left(\frac{11-6.65}{\sqrt{5.76}}\right) - \Phi\left(\frac{5-6.65}{\sqrt{5.76}}\right) = \Phi(1.8125) - \Phi(-0.6875)$

$\simeq \Phi(1.8) + \Phi(.7) - 1 = .722$.

$X \sim N_1(5, 16) \Rightarrow P(4<X<6) = \Phi\left(\frac{6-5}{4}\right) - \Phi\left(\frac{4-5}{4}\right) = \Phi(.25) - \Phi(-.25)$

$= 2\Phi(.25) - 1 = 2 \times .5987 - 1$

$Y \sim N_1(8, 9) \Rightarrow P(7<Y<9) = \Phi\left(\frac{9-8}{3}\right) - \Phi\left(\frac{7-8}{3}\right) = 2\Phi(\frac{1}{3}) - 1$

$\simeq 2 \times 0.6293 - 1$.

(b) $Y|X=5 \sim N_1\left(10 + \frac{\rho \times 5}{1}(5-5),\ 25(1-\rho^2)\right) = N_1(10, 25(1-\rho^2))$

$P(4<Y<16|X=5) = 0.954 \Rightarrow \Phi\left(\frac{16-10}{5\sqrt{1-\rho^2}}\right) - \Phi\left(\frac{4-10}{5\sqrt{1-\rho^2}}\right) = 0.954$

$\Rightarrow \Phi\left(\frac{6}{5\sqrt{1-\rho^2}}\right) = .977 \Rightarrow \frac{6}{5\sqrt{1-\rho^2}} = 2 \Rightarrow \rho = \frac{4}{5} = 0.8$ (as $\rho > 0$).

$\boxed{\text{Problem 29}}$ (a)-(b) $t_1 U + t_2 V = \underbrace{(at_1 + bt_2)X + (bt_1 - at_2)Y}_{\text{linear combination of } (X,Y)} \sim N_1$ (as $(X,Y) \sim N_2$)

$\Rightarrow (U,V) \sim N_2$

$E(U) = E(V) = 0$, $Var(U) = Var(V) = (a^2+b^2)\sigma^2 = \gamma^2$ (say), $Cov(U,V) = 0$

$\Rightarrow (U,V) \sim N_2(0,0,\gamma^2,\gamma^2,0) \Rightarrow$ U and V are i.i.d. $N(0,\gamma^2)$ r.v.s.

(c) Taking $a = b = \frac{1}{\sqrt{2}}$ in (b) the result follows.

$\boxed{\text{Problem 30}}$ (a) $M_Y(t) = E(e^{tX_1X_2}) = E(E(e^{tX_1X_2}|X_1))$

$X_2|X_1 = x_1 \sim N(\rho x_1, 1-\rho^2) \Rightarrow E(e^{tX_1X_2}|X_1) = e^{(tX_1)(\rho X_1) + \frac{t^2X_1^2(1-\rho^2)}{2}}$

$= e^{X_1^2\left(\rho t + \frac{(1-\rho^2)t^2}{2}\right)}$

$X_1^2 \sim \chi_1^2 \Rightarrow M_Y(t) = E\left[e^{X_1^2\left(\rho t + \frac{(1-\rho^2)t^2}{2}\right)}\right]$

$= [1 - 2\rho t - (1-\rho^2)t^2]^{-1/2}, \quad -\frac{1}{1-\rho} < t < \frac{1}{1+\rho}.$

(b) $M_Y^{(1)}(t) = \frac{1}{2}[1-2\rho t - (1-\rho^2)t^2]^{-\frac{3}{2}}[2\rho + 2(1-\rho^2)t]$

$M_Y^{(2)}(t) = \frac{3}{4}[1-2\rho t-(1-\rho^2)t^2]^{-5/2}[2\rho + 2(1-\rho^2)t]^2 +$
$[1-2\rho t - (1-\rho^2)t^2]^{-3/2}(1-\rho^2)$

$E(Y^2) = M_Y^{(2)}(0) = 1 + 2\rho^2.$

(c) $E(X_1^2X_2^2) = E(X_1^2 E(X_2^2|X_1)) = E[X_1^2(1-\rho^2+\rho^2X_1^2)]$

$= 1-\rho^2 + \rho^2E(X^4) = 1-\rho^2+3\rho^2 \quad (X_1 \sim N(0,1))$

$= 1+2\sigma^2.$

$\boxed{\text{Problem 31}}$ $f(x,y) = \begin{cases} \frac{1}{\pi}e^{-\frac{1}{2}(x^2+y^2)}, & \text{if } \{x<0 \text{ and } y<0\} \text{ or } \{x>0 \text{ and } y>0\} \\ 0 & \text{otherwise} \end{cases}$

$f_{X_1}(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \begin{cases} \int_{-\infty}^{0} \frac{1}{\pi}e^{-\frac{x^2+y^2}{2}}dy, & \text{if } x<0 \\ \int_{0}^{\infty}\frac{1}{\pi}e^{-\frac{x^2+y^2}{2}}dy, & \text{if } x\geq 0 \end{cases}$

$= \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}, \quad -\infty < x < \infty \Rightarrow X_1 \sim N(0,1)$

By symmetry $X_2 \sim N(0,1)$

Obviously $f(x,y)$ is not a p.d.f. of N_2

Problem 32

$$f_X(x) = \int_{-\infty}^{\infty} g_\rho(x,y)\,dy = \frac{1}{2}\Big[\underbrace{\int_{-\infty}^{\infty} f_\rho(x,y)\,dy}_{\substack{\text{Marginal of } X_1 \sim N(0,1) \\ \text{in } (X_1,X_2) \sim N_2(0,0,1,1,\rho)}} + \underbrace{\int_{-\infty}^{\infty} f_{-\rho}(x,y)\,dy}_{\substack{\text{Marginal of } Y_1 \sim N(0,1) \\ \text{in } (Y_1,Y_2) \sim N_2(0,0,1,1,-\rho)}} \Big]$$

$$= \frac{1}{2}\left[\phi(x) + \phi(x)\right] = \phi(x), \quad -\infty < x < \infty$$

$$\Rightarrow X \sim N(0,1).$$

By symmetry $Y \sim N(0,1)$

Obviously $(X,Y) \not\sim N_2$ unless $\rho = 0$.

Problem 33 (a) Since $X, Y \sim N(0,1)$, $E(X) = E(Y) = 0$, $Var(X) = Var(Y) = 1$

$$E(XY) = \frac{1}{2}\Big[\underbrace{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_\rho(x,y)\,dx\,dy}_{\substack{E(X_1X_2) \\ \text{where } (X_1,X_2) \sim N_2(0,0,1,1,\rho)}} + \underbrace{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{-\rho}(x,y)\,dx\,dy}_{\substack{E(Y_1Y_2) \\ \text{where } (Y_1,Y_2) \sim N_2(0,0,1,1,-\rho)}} \Big]$$

$$= \frac{1}{2}\left[\rho + (-\rho)\right] = 0$$

$$\Rightarrow Cov(X,Y) = E(XY) - E(X)E(Y) = 0 \Rightarrow Cov(X,Y) = 0$$

(b) Obviously X and Y are not independent unless $\rho = 0$.

Problem 34 (a) For $\sum_{i=1}^{4} x_i = 28$, $0 \le x_i \le 28$, $x_i \in \mathbb{N}$, $i = 1(1)4$

$$P(X_1 = x_1, \ldots, X_4 = x_4 \mid X_5 = 2) = \frac{P(X_1 = x_1, \ldots, X_4 = x_4, X_5 = 2)}{P(X_5 = 2)}$$

$$= \frac{\dfrac{\lfloor 30}{\lfloor x_1 \lfloor x_2 \lfloor x_3 \lfloor x_4 \lfloor 2}\, \theta_1^{x_1} \cdots \theta_4^{x_4}\, \theta_5^{2}}{\binom{30}{2}\, \theta_5^2 (1-\theta_5)^{28}} \qquad (X_5 \sim Bin(30, \theta_5))$$

$$= \frac{\lfloor 28}{\lfloor x_1 \cdots \lfloor x_4} \left(\frac{\theta_1}{1-\theta_5}\right)^{x_1} \cdots \left(\frac{\theta_4}{1-\theta_5}\right)^{x_4}$$

$$\Rightarrow (X_1, X_2, X_3) \sim Mult(30, \theta_1, \theta_2, \theta_3) \text{ and } X_4 = 30 - \sum_{i=1}^{3} X_i$$

(b) Let $X_1 =$ # of times we get a sum of 12 in 7 independent casts of a pair of dice

$X_2 =$ " " . . . . sum of 8 " . . .

$\theta_1 = P(\text{getting a sum of 12 in a cast}) = \frac{1}{36}$

$\theta_2 =$ P(getting a sum of 8 in a cast) $= \frac{5}{36}$

Required prob $= P(X_1 = 1, X_2 = 2)$

$$= \frac{\lfloor 7}{\lfloor 1 \lfloor 2 \lfloor 4} \left(\frac{1}{36}\right)\left(\frac{5}{36}\right)^2 \left(1-\frac{6}{36}\right)^4.$$

$\boxed{\text{Problem 35}}$ (a) Let $Z_1 \sim \chi^2_{n_1}$ and $Z_2 \sim \chi^2_{n_2}$ be independent. Then $\frac{Z_1}{2} \sim GAM(\frac{n_1}{2}, 1)$ and $\frac{Z_2}{2} \sim GAM(\frac{n_2}{2}, 1)$ are independent and

$$\Rightarrow X \overset{d}{=} \frac{Z_1/n_1}{Z_2/n_2} = \frac{n_2}{n_1}\frac{Z_1}{Z_2}$$

$$\Rightarrow Y_1 \overset{d}{=} \frac{n_2}{n_2 + n_2 \frac{Z_1}{Z_2}} = \frac{Z_2}{Z_1+Z_2} = \frac{Z_2/2}{\frac{Z_1}{2}+\frac{Z_2}{2}} \sim Be\left(\frac{n_1}{2}, \frac{n_2}{2}\right)$$

(b) $Z_1 \sim N(0,1)$ and $Z_2^2 \sim \chi^2_1$ are independent

$$\Rightarrow \frac{Z_1}{\sqrt{Z_2^2/1}} \sim \chi^2_1, \quad \text{i.e.} \quad \frac{Z_1}{|Z_2|} \sim \chi^2_1.$$

Let $Y = \frac{Z_1}{Z_2}$. Then, for $y \in \mathbb{R}$,

$$F_Y(y) = P\left(\frac{Z_1}{Z_2} \le y\right)$$

$$= P\left(\frac{Z_1}{Z_2} \le y, Z_2 > 0\right) + P\left(\frac{Z_1}{Z_2} \le y, Z_2 < 0\right)$$

$$= P\left(\frac{Z_1}{|Z_2|} \le y, Z_2 > 0\right) + P\left(-\frac{Z_1}{|Z_2|} \le y, Z_2 < 0\right).$$

Since $(Z_1, Z_2) \overset{d}{=} (-Z_1, Z_2)$, we have

$$P\left(-\frac{Z_1}{|Z_2|} \le y, Z_2 < 0\right) = P\left(\frac{Z_1}{|Z_2|} \le y, Z_2 < 0\right)$$

$$\Rightarrow F_Y(y) = P\left(\frac{Z_1}{|Z_2|} \le y, Z_2 > 0\right) + P\left(\frac{Z_1}{|Z_2|} \le y, Z_2 < 0\right)$$

$$= P\left(\frac{Z_1}{|Z_2|} \le y\right), \quad \forall y \in \mathbb{R}$$

$$\Rightarrow Y \overset{d}{=} \frac{Z_1}{|Z_2|} \sim t_1$$

(c) $\frac{2X_1}{\theta}$ and $\frac{2X_2}{\theta}$ are i.i.d. χ^2_2

$$\Rightarrow Z = \frac{X_1}{X_2} = \frac{X_1/\theta}{X_2/\theta} = \frac{\chi^2_2/2}{\chi^2_2/2} \Big\} \text{ independent}$$

$$\sim F_{2,2}$$

(d) $$\frac{X_1/n_1}{X_2/n_2} = \frac{\chi^2_{n_1}/n_1}{\chi^2_{n_2}/n_2} \Big\} \text{ independent}$$

$$\sim F_{n_1,n_2}$$

Also $X_3 \sim \chi^2_{n_3}$ and $X_1+X_2 \sim \chi^2_{n_1+n_2}$ are independent

$$\Rightarrow \frac{X_3/n_3}{(X_1+X_2)/(n_1+n_2)} = \frac{\chi^2_{n_3}/n_3}{\chi^2_{n_1+n_2}/n_1+n_2} \Big\} \text{ independent}$$

$$\sim F_{n_3,\, n_1+n_2}$$

It suffices to show that $Y_1 = \frac{X_1}{X_2}$ and $Y_2 = \frac{X_3}{X_1+X_2}$ are independent. The joint p.d.f of (X_1, X_2, X_3) is

$$f_{X_1,X_2,X_3}(x_1,x_2,x_3) = f_{X_1}(x_1) f_{X_2}(x_2) f_{X_3}(x_3)$$

$$= \frac{1}{2^{\frac{n_1+n_2+n_3}{2}} \Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)\Gamma\left(\frac{n_3}{2}\right)} e^{-\frac{x_1+x_2+x_3}{2}} x_1^{\frac{n_1}{2}-1} x_2^{\frac{n_2}{2}-1} x_3^{\frac{n_3}{2}-1}, \quad x_i>0,\ i=1,2,3.$$

Let $Y_1 = \frac{X_1}{X_2}$, $Y_2 = \frac{X_3}{X_1+X_2}$ and $Y_3 = X_2$, i.e., $X_1 = Y_1 Y_3$

$X_2 = Y_3$ and $X_3 = Y_2 Y_3 (1+Y_1)$

$$J = \begin{vmatrix} y_3 & 0 & y_1 \\ 0 & 0 & 1 \\ y_2 y_3 & y_3(1+y_1) & y_2(1+y_1) \end{vmatrix} = -y_3^2(1+y_1)$$

$x_i>0,\ i=1,2,3 \Rightarrow y_1 y_3>0,\ y_3>0,\ y_2 y_3(1+y_1)>0 \Rightarrow y_i>0,\ i=1,2,3$

The joint pdf of (Y_1, Y_2, Y_3) is given by

$$f_{Y_1,Y_2,Y_3}(y_1,y_2,y_3) = \frac{1}{2^{\frac{n_1+n_2+n_3}{2}} \Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)\Gamma\left(\frac{n_3}{2}\right)} e^{-\frac{(y_1 y_3 + y_3 + y_2 y_3 (1+y_1))}{2}} (y_1 y_3)^{\frac{n_1}{2}-1} \times y_3^{\frac{n_2}{2}-1} (y_2 y_3 (1+y_1))^{\frac{n_3}{2}-1} y_3^2 (1+y_1)$$

$$= \frac{1}{2^{\frac{n_1+n_2+n_3}{2}} \Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)\Gamma\left(\frac{n_3}{2}\right)} e^{-\frac{y_3(1+y_1)(1+y_2)}{2}} \times y_1^{\frac{n_1}{2}-1} (1+y_1)^{\frac{n_3}{2}} \times y_2^{\frac{n_3}{2}-1}\, y_3^{\frac{n_1+n_2+n_3}{2}-1},$$

$y_i>0,\ i=1,2,3$

The joint pdf of (Y_1, Y_2) is

$$f_{Y_1,Y_2}(y_1,y_2) = \int_0^\infty f_{Y_1,Y_2,Y_3}(y_1,y_2,y_3)\,dy_3, \qquad y_1>0,\ y_2>0$$

$$= \frac{\Gamma\left(\frac{n_1+n_2+n_3}{2}\right)}{2^{\frac{n_1+n_2+n_3}{2}}\Gamma\frac{n_1}{2}\Gamma\frac{n_2}{2}\Gamma\frac{n_3}{2}}\left(\frac{2}{(1+y_1)(1+y_2)}\right)^{\frac{n_1+n_2+n_3}{2}} y_1^{\frac{n_1}{2}-1}(1+y_1)^{\frac{n_3}{2}} \times y_2^{\frac{n_2}{2}} \cdots$$

$$= h_1(y_1)h_2(y_2) \qquad y_i>0,\ i=1,2$$

$\Rightarrow$ Y_1 and Y_2 are independent.

$\boxed{\text{Problem 36}}$ The joint p.d.f. of $(X_{1:n},\dots,X_{n:n})$ is

$$g(x_1,\dots,x_n) = \underline{|n}\prod_{i=1}^n \frac{1}{\theta}e^{-\frac{x_i}{\theta}}, \qquad 0<x_1<\dots<x_n<\infty$$

$$= \frac{\underline{|n}}{\theta^n}e^{-\frac{\sum_{i=1}^n x_i}{\theta}}, \qquad 0<x_1<\dots<x_n<\infty$$

Let $Z_1 = nX_{1:n}$, $Z_2=(n-1)(X_{2:n}-X_{1:n})$, ..., $Z_n = X_{n:n}-X_{n-1:n}$

$\Rightarrow$ $X_{1:n}=\frac{Z_1}{n}$, $X_{2:n}=\frac{Z_1}{n}+\frac{Z_2}{n-1}$, ..., $X_{n:n}=\frac{Z_1}{n}+\frac{Z_2}{n-1}+\dots+\frac{Z_{n-1}}{2}+Z_n$

$$J = \begin{vmatrix} 1/n & 0 & 0 & \cdots & 0 \\ 1/n & \frac{1}{n-1} & 0 & \cdots & 0 \\ \frac{1}{n} & \frac{1}{n-1} & & \cdots & 1 \end{vmatrix} = \frac{1}{\underline{|n}}$$

$X_{1:n}+X_{2:n}+\dots+X_{n:n} = Z_1+Z_2+\dots+Z_n.$

$0<x_1<\dots<x_n<\infty \Rightarrow 0<\frac{z_1}{n}<\frac{z_1}{n}+\frac{z_2}{n-1}<\dots<\frac{z_1}{n}+\frac{z_2}{n-1}+\dots+z_n<\infty$

$\Rightarrow z_1>0,\ z_2>0,\dots,z_n>0$

Thus the joint pdf of $(Z_1,\dots,Z_n)$ is

$$h(z_1,\dots,z_n) = \frac{\underline{|n}}{\theta^n}e^{-\frac{\sum_{i=1}^n z_i}{\theta}}\times\frac{1}{\underline{|n}}, \quad z_i>0,\quad i=1,\dots,n$$

$$= \prod_{i=1}^n\left(\frac{1}{\theta}e^{-\frac{z_i}{\theta}}\right), \quad z_i>0,\ i=1,\dots,n$$

$\Rightarrow$ $Z_1, Z_2, \dots, Z_n$ are i.i.d. $Exp(\theta)$

$\boxed{20/5}$

We have

$$X_{r:n} = \sum_{i=1}^{r} \frac{Z_i}{n-i+1}$$

$$\Rightarrow E(X_{r:n}) = \sum_{i=1}^{n} \frac{E(Z_i)}{n-i+1} = \theta \sum_{i=1}^{n} \frac{1}{n-i+1}$$

$$Var(X_{r:n}) = \sum_{i=1}^{r} \frac{Var(Z_i)}{(n-i+1)^2} \qquad (Z_i\text{'s are independent})$$

$$= \theta^2 \sum_{i=1}^{r} \frac{1}{(n-i+1)^2}$$

$$Cov(X_{r:n}, X_{s:n}) = Cov\left(\sum_{i=1}^{r} \frac{Z_i}{n-i+1}, \sum_{i=1}^{s} \frac{Z_i}{n-i+1} \right)$$

$$= Var\left(\sum_{i=1}^{r} \frac{Z_i}{n-i+1} \right) \qquad (Z_i\text{'s are independent})$$

$$= \theta^2 \sum_{i=1}^{r} \frac{1}{(n-i+1)^2} = Var(X_{r:n}),$$

$$1 \leq r < s \leq n.$$

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