EE 200: Solution to Assignment 6

1. Consider the digital system characterised by the input-output relation given by

$$y[n] = x[n] - y[n-1]\,(y[n-1] - 1)\,,$$

where $y[n]$ and $x[n]$ are the output and input sequences. Is the above digital system a linear system? Is it a time invariant system? The algorithm given by the above equation has been proposed to compute the square-root of a positive number α with $0 < \alpha < 1$ by setting $x[n] = \alpha\mu[n]$ with $y[-1] = 1$. Show that $y[n]$ converges to $\sqrt{\alpha}$ as $n \to \infty$.
Solution: The system i/o relation is given by,

$$y[n] = x[n] - y^2[n-1] + y[n-1]$$

For the input $x_i[n]$, i=1,2, the output is,

$$y_i[n] = x_i[n] - y_i^2[n-1] + y_i[n-1]$$

Then, for an input $x[n] = Ax_1[n] + Bx_2[n]$, the output is,

$$y[n] = Ax_1[n] + Bx_2[n] - y^2[n-1] + y[n-1]$$

On the other hand,

$$Ay_1[n]+By_2[n] = Ax_1[n]-Ay_1^2[n-1]+Ay_1[n-1]+Bx_2[n]-By_2^2[n-1]+B_2[n-1] \neq y[n]$$

Hence, the system is non-linear.
If we let $y[n]$ be the output for $x[n]$, then for the input $x[n-n_0]$, we get

$$y[n-n_0] = x[n-n_0] - y^2[n-n_0-1] + y[n-n_0-1]$$

Hence, the system is time-invariant.
For an input $x[n] = \alpha\mu[n]$, the output $y[n]$ converges to some constant K as $n \rightarrow \infty$. The difference equation describing the system $n \rightarrow \infty$ reduces to,

$$K = \alpha - K^2 + K \Rightarrow K^2 = \alpha \Rightarrow K = \sqrt{\alpha}$$

2. Let $\{r[n]\}$ and $\{v[n]\}$ denote two finite length sequences defined for the time index ranges $-K \leq n \leq L$ and $M \leq n \leq N$, respectively, where K, L, M and N are positive integers with $M < N$. Define $x[n] = r[n] \circledast r[n]$, $y[n] = v[n] \circledast v[n]$ and $w[n] = r[n] \circledast v[n]$. Determine the ranges of the time-index n for which $x[n], y[n]$ and $w[n]$ are defined and the duration of the sequences.
Solution: The length for a finite-length sequence defined for the range $M \leq n \leq N$ is given by $N - M + 1$. The convolution of a length $l - L$ sequence with a length R sequence produces a sequence of length $L + R - 1$.
(a) The length of $r[n]$ is $L - (-K) + 1 = L + K + 1$. Hence, the length of $x[n] = r[n] \otimes r[n]$ is $(L + K + 1) + (L + K - 1) - 1 = 2(L + K) + 1$.
The first sample of $x[n]$ is given by $r[-K]r[-K]$, and it occurs at $n = -2K$. The last sample of $x[n]$ is given by $r[L]r[L]$ which occurs at $n = 2L$. The sequence $x[n]$ has the range $-2K \leq n2L$, and its length is $2L - (-2K) + 1 = 2(L + K) + 1$, which is same as derived above.

(b) The length of $v[n]$ is if $N - M + 1$. The length of $y[n] = v[n] \otimes v[n]$ is $2(N - M) + 1$.
The first sample of $y[n]$ is given by $v[M]v[M]$ which occurs at $n = 2M$. The last sample of $y[n]$ is given by $v[N]v[N]$ which occurs at $n = 2N$. The sequence $y[n]$ has the range $2M \leq n \leq 2N$, and its length is $2(N - M) + 1$.

(c) The length of $w[n] = r[n] \otimes v[n]$ is $(L+K-1)+(N-M+1) = (L + K + N - M) + 1$.
The first sample of $w[n]$ is given by $r[-K]v[M]$ which occurs at $n = M - K$. The last sample of $w[n]$ is given by $r[L]v[N]$ which occurs at $n = L + N$. The sequence $w[n]$ is therefore

defined for the range $M - K \leq n \leq L + N$, and its length is $L + N - (M - K) + 1 = (L + K + N - M) + 1$.

3. The convolution sum of the sequence

$$r[n] = \{2, -4, 0, 7, 3\},\ -2 \leq n \leq 2,$$

and the sequence

$$v[n] = \{-3, -2, 4, 1, 6\},\ -1 \leq n \leq 3$$

is

$$y[n] = \{-6, 8, 16, -35, -15, -2, 19, 45, 18\}\ -3 \leq n \leq 5.$$

Verify the above result without computing the convolution sum.
Solution: We note

$$\sum_{n=-2}^{2} r[n] = 2 - 4 + 0 + 7 + 3 = 8,$$

$$\sum_{n=-1}^{3} v[n] = -3 - 2 + 4 + 1 + 6 = 6, \text{ and}$$

$$\sum_{n=-3}^{5} y[n] = -6 + 8 + 16 - 35 - 15 - 2 + 19 + 45 + 18 = 48$$

Therefore, the first test

$$\left(\sum r[n]\right)\left(\sum v[n]\right) = \left(\sum y[n]\right)$$

is satisfied. Next, we note

$$\sum_{n=-2}^{2} (-1)^n r[n] = 2 + 4 + 0 - 7 + 3 = 2,$$

$$\sum_{n=-1}^{3} (-1)^n v[n] = 3 - 2 - 4 + 1 - 6 = -8, \text{ and}$$

$$\sum_{n=-3}^{5} (-1)^n y[n] = 6 + 8 - 16 - 35 + 15 - 2 - 19 + 45 - 18 = -16$$

Therefore, the second test

$$\left(\sum(-1)^n r[n]\right)\left(\sum(-1)^n v[n]\right) = \left(\sum(-1)^n y[n]\right)$$

is also satisfied.

4. The cross-correlation sequence $r_{xy}[k]$ of two real sequences $x[n]$ and $y[n]$ is defined by

$$r_{xy}[n] = \sum_{l=-\infty}^{\infty} x[l]y[l-n].$$

Show that the cross-correlation sequence can be implemented as a convolution sum given by

$$r_{xy}[n] = x[n] \circledast y[-n].$$

Prove that

$$r_{xy}[n] = r_{yx}[-n].$$

Solution: The cross-correlation sequence $r_{xy}[n]$ is defined as

$$\begin{aligned} r_{xy}[n] &= \sum_{l=-\infty}^{\infty} x[l]y[l-n] \\ &= \sum_{l=-\infty}^{\infty} x[l]y[-n+l] \\ &= \sum_{l=-\infty}^{\infty} x[l]y[-(n-l)] \\ &= x[n] \circledast y[-n] \end{aligned}$$

Now,

$$\begin{aligned} r_{yx}[n] &= \sum_{l=-\infty}^{\infty} y[l]x[l-n] \\ &= \sum_{m=-\infty}^{\infty} y[m+n]x[m] \\ &= \sum_{m=-\infty}^{\infty} x[m]y[m-(-n)] \\ &= r_{xy}[-n] \end{aligned}$$

Hence, the result is obtained.

5. The auto correlation sequence $r_{xx}[n]$ of a real sequence $x[n]$ is defined by

$$r_{xx}[n] = \sum_{l=-\infty}^{\infty} x[l]x[l-n].$$

Show that $r_{xx}[n]$ is an even sequence, that is, $r_{xx}[n] = r_{xx}[-n]$. Determine the correlation sequences of

$$\{x[n]\} = \{-3, 4, 0, -2, 5, 4\},\ -1 \leq n \leq 4$$

and

$$\{y[n]\} = \{1, 3, -2, 0, 6, -7\},\ -3 \leq n \leq 2.$$

Verify your results using MATLAB. Use the codes:

$$\begin{aligned}
&x = [-3\ 4\ 0\ \ -2\ 5\ 4];\\
&r = \text{conv}(x, \text{fliplr}(x))\\
&\text{and}\\
&y = [1\ 3\ \ -2\ 0\ 6\ \ -7];\\
&r = \text{conv}(y, \text{fliplr(y)})
\end{aligned}$$

Solution: The auto-correlation sequence $r_{xx}[n]$ is defined as

$$\begin{aligned}
r_{xx}[n] &= \sum_{l=-\infty}^{\infty} x[l]x[l-n]\\
&= \sum_{m=-\infty}^{\infty} x[m+n]x[m]\\
&= \sum_{m=-\infty}^{\infty} x[m]x[m-(-n)]\\
&= r_{xx}[-n]
\end{aligned}$$

Hence, the sequence is an even sequence.

$$r_{xx}[n] = \{-12, 1, 26, -16, -2, 70, -2, -16, 26, 1, -12\}$$

$$r_{yy}[n] = \{-7, -15, 32, -14, -45, 99, -45, -14, 32, -15, -7\}.$$