Solution of A8

1.

Let $X = \#$ of heads out of 1000. If the coin is fair, then $X \sim \text{binomial}(1000, 1/2)$. So

$$P(X \geq 560) = \sum_{x=560}^{1000} \binom{1000}{x} \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{n-x} \approx .0000825,$$

where a computer was used to do the calculation. For this binomial, $\mathrm{E}\,X = 1000p = 500$ and $\mathrm{Var}\,X = 1000p(1-p) = 250$. A normal approximation is also very good for this calculation.

$$P\{X \geq 560\} = P\left\{\frac{X-500}{\sqrt{250}} \geq \frac{559.5-500}{\sqrt{250}}\right\} \approx P\{Z \geq 3.763\} \approx .0000839.$$

Thus, if the coin is fair, the probability of observing 560 or more heads out of 1000 is very small. We might tend to believe that the coin is not fair, and $p > 1/2$.

2.

Let $X \sim \text{Poisson}(\lambda)$, and we observed $X = 10$. To assess if the accident rate has dropped, we could calculate

$$P(X \leq 10 | \lambda = 15) = \sum_{i=0}^{10} \frac{e^{-15}15^i}{i!} = e^{-15}\left[1+15+\frac{15^2}{2!}+\cdots+\frac{15^{10}}{10!}\right] \approx .11846.$$

This is a fairly large value, not overwhelming evidence that the accident rate has dropped. (A normal approximation with continuity correction gives a value of .12264.)

3.

The CLT tells us that $Z = (\sum_i X_i - np)/\sqrt{np(1-p)}$ is approximately $\mathrm{n}(0,1)$. For a test that rejects H_0 when $\sum_i X_i > c$, we need to find c and n to satisfy

$$P\left(Z > \frac{c-n(.49)}{\sqrt{n(.49)(.51)}}\right) = .01 \quad \text{and} \quad P\left(Z > \frac{c-n(.51)}{\sqrt{n(.51)(.49)}}\right) = .99.$$

We thus want

$$\frac{c-n(.49)}{\sqrt{n(.49)(.51)}} = 2.33 \quad \text{and} \quad \frac{c-n(.51)}{\sqrt{n(.51)(.49)}} = -2.33.$$

Solving these equations gives $n = 13{,}567$ and $c = 6{,}783.5$.

4.

From the Neyman-Pearson lemma the UMP test rejects H_0 if

$$\frac{f(x \mid \sigma_1)}{f(x \mid \sigma_0)} = \frac{(2\pi\sigma_1^2)^{-n/2} e^{-\Sigma_i x_i^2/(2\sigma_1^2)}}{(2\pi\sigma_0^2)^{-n/2} e^{-\Sigma_i x_i^2/(2\sigma_0^2)}} = \left(\frac{\sigma_0}{\sigma_1}\right)^n \exp\left\{\frac{1}{2}\sum_i x_i^2 \left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right)\right\} > k$$

for some $k \geq 0$. After some algebra, this is equivalent to rejecting if

$$\sum_i x_i^2 > \frac{2\log\left(k\left(\sigma_1/\sigma_0\right)^n\right)}{\left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right)} = c \quad \left(\text{because } \frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2} > 0\right).$$

This is the UMP test of size α, where $\alpha = P_{\sigma_0}(\sum_i X_i^2 > c)$. To determine c to obtain a specified α, use the fact that $\sum_i X_i^2/\sigma_0^2 \sim \chi_n^2$. Thus

$$\alpha = P_{\sigma_0}\left(\sum_i X_i^2/\sigma_0^2 > c/\sigma_0^2\right) = P\left(\chi_n^2 > c/\sigma_0^2\right),$$

so we must have $c/\sigma_0^2 = \chi^2_{n,\alpha}$, which means $c = \sigma_0^2 \chi^2_{n,\alpha}$.

Now, c = 4 χ^2_{15} (0.10) = 4 × 22.307 (see the table) = 89.228.

5.

The pdf of Y is

$$f(y|\theta) = \frac{1}{\theta} y^{(1/\theta)-1} e^{-y^{1/\theta}}, \quad y > 0.$$

By the Neyman-Pearson Lemma, the UMP test will reject if

$$\frac{1}{2} y^{-1/2} e^{y - y^{1/2}} = \frac{f(y|2)}{f(y|1)} > k.$$

To see the form of this rejection region, we compute

$$\frac{d}{dy}\left(\frac{1}{2} y^{-1/2} e^{y-y^{1/2}}\right) = \frac{1}{2} y^{-3/2} e^{y-y^{1/2}} \left(y - \frac{y^{1/2}}{2} - \frac{1}{2}\right)$$

which is negative for $y < 1$ and positive for $y > 1$. Thus $f(y|2)/f(y|1)$ is decreasing for $y \leq 1$ and increasing for $y \geq 1$. Hence, rejecting for $f(y|2)/f(y|1) > k$ is equivalent to rejecting for $y \leq c_0$ or $y \geq c_1$. To obtain a size α test, the constants c_0 and c_1 must satisfy

$$\alpha = P(Y \leq c_0|\theta = 1) + P(Y \geq c_1|\theta = 1) = 1 - e^{-c_0} + e^{-c_1} \quad \text{and} \quad \frac{f(c_0|2)}{f(c_0|1)} = \frac{f(c_1|2)}{f(c_1|1)}.$$

Solving these two equations numerically, for $\alpha = .10$, yields $c_0 = .076546$ and $c_1 = 3.637798$. The Type II error probability is

$$P(c_0 < Y < c_1|\theta = 2) = \int_{c_0}^{c_1} \frac{1}{2} y^{-1/2} e^{-y^{1/2}}\, dy = -e^{-y^{1/2}}\Big|_{c_0}^{c_1} = .609824.$$

6.

By the Neyman-Pearson Lemma, the UMP test rejects for large values of $f(x|H_1)/f(x|H_0)$. Computing this ratio we obtain

x	1	2	3	4	5	6	7
$\frac{f(x\|H_1)}{f(x\|H_0)}$	6	5	4	3	2	1	.84

The ratio is decreasing in x. So rejecting for large values of $f(x|H_1)/f(x|H_0)$ corresponds to rejecting for small values of x. To get a size α test, we need to choose c so that $P(X \leq c|H_0) = \alpha$. The value $c = 4$ gives the UMP size $\alpha = .04$ test. The Type II error probability is $P(X = 5, 6, 7|H_1) = .82$.

7.

From Corollary 8.3.13 we can base the test on $\sum_i X_i$, the sufficient statistic. Let $Y = \sum_i X_i \sim \text{binomial}(10, p)$ and let $f(y|p)$ denote the pmf of Y. By Corollary 8.3.13, a test that rejects if $f(y|1/4)/f(y|1/2) > k$ is UMP of its size. By Exercise 8.25c, the ratio $f(y|1/2)/f(y|1/4)$ is increasing in y. So the ratio $f(y|1/4)/f(y|1/2)$ is decreasing in y, and rejecting for large value of the ratio is equivalent to rejecting for small values of y. To get $\alpha = .0547$, we must find c such that $P(Y \leq c|p = 1/2) = .0547$. Trying values $c = 0, 1, \ldots,$ we find that for $c = 2$, $P(Y \leq 2|p = 1/2) = .0547$. So the test that rejects if $Y \leq 2$ is the UMP size $\alpha = .0547$ test. The power of the test is $P(Y \leq 2|p = 1/4) \approx .526$.

8.

By the Neyman-Pearson Lemma, the most powerful test of $H_0\colon \theta = 1$ vs. $H_1\colon \theta = 2$ is given by Reject H_0 if $f(x \mid 2)/f(x \mid 1) > k$ for some $k \geq 0$. Substituting the beta pdf gives

$$\frac{f(x|2)}{f(x|1)} = \frac{\frac{1}{\beta(2,1)} x^{2-1}(1-x)^{1-1}}{\frac{1}{\beta(1,1)} x^{1-1}(1-x)^{1-1}} = \frac{\Gamma(3)}{\Gamma(2)\Gamma(1)} x = 2x.$$

Thus, the MP test is Reject H_0 if $X > k/2$. We now use the α level to determine k. We have

$$\alpha = \sup_{\theta \in \Theta_0} \beta(\theta) = \beta(1) = \int_{k/2}^{1} f_X(x|1)\, dx = \int_{k/2}^{1} \frac{1}{\beta(1,1)} x^{1-1}(1-x)^{1-1}\, dx = 1 - \frac{k}{2}.$$

Thus $1 - k/2 = \alpha$, so the most powerful α level test is reject H_0 if $X > 1 - \alpha$.