LECTURE-7

Special Power Series
Exponential,
Logarithm &
Trigonometric fns.

Lecture 7 :

§ EXPONENTIAL FUNCTION:

Recall, $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $x \in \mathbb{R}$.

We define the complex exponential analogously: $e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}$

Radius of convergence: $\frac{1}{\lim_{n\to\infty} \frac{|n!|}{|(n+1)!|}} = \frac{1}{0} = \infty$

So, e^z is a genuine analytic fn on $\mathbb{C}$.

Some of the properties of e^x:

1. $e^{x+y} = e^x e^y$
2. $\frac{d(e^x)}{dx} = e^x$
3. $e^0 = 1$

We try to verify if these hold for e^z.

REMARKS:

1) Recall, we already set $e^{i\theta} = \cos\theta + i\sin\theta$

we now verify that there is no clash of notation:

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = \sum_{k=0}^{\infty} \frac{(i\theta)^{2k}}{(2k)!} + \sum_{k=1}^{\infty} \frac{(i\theta)^{2k+1}}{(2k+1)!}$$

E_k = kth partial sum

S_k = kth partial sum

C_k = kth partial sum

then

$E_{2k} = C_k + iS_{k-1}$

$E_{2k+1} = C_k + iS_k$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{\theta^{2k}}{2k!} + i \sum_{k=1}^{\infty} (-1)^k \frac{\theta^{2k+1}}{(2k+1)!}$$

$e^{i\theta}$ $\cos\theta$ $\sin\theta$

$$= \cos\theta + i\sin\theta$$ ∎

2) e^z is analytic on $\mathbb{C}$

and $$(e^z)' = \sum_{n=1}^{\infty} n\frac{(z)^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} = e^z$$

3) $e^0 = 1$

Properties of e^z:

(1) $e^{z_1+z_2} = e^{z_1} \cdot e^{z_2} = e^{z_1} e^{z_2}$

Let $g(z) := e^z e^{z_1+z_2-z}$

Then $g'(z) = (e^z)' e^{z_1+z_2-z} + e^z (e^{z_1+z_2-z})'$

$$= e^z e^{z_1+z_2-z} - e^z e^{z_1+z_2-z} = 0$$

Hence $g(z) =$ constant $= g(z_1+z_2)$

$$= e^{z_1+z_2}$$

ie $e^z e^{z_1+z_2-z} = e^{z_1+z_2}$

take $z = z_2$. Then $e^{z_2} e^{z_1} = e^{z_1+z_2}$

take $z = z_1$. Then $e^{z_1} e^{z_2} = e^{z_1+z_2}$

In particular, $(e^{z_1})^{-1} = e^{-z_1}$

(2) $e^{2n\pi i} = \cos 2n\pi + i \sin 2n\pi = 1$

$\therefore$ $e^{z+2n\pi i} = e^z$ ie e^z has period $2n\pi i$

Conversely,

$$e^z = 1 \Rightarrow e^{x+iy} = 1 \Rightarrow e^x(\cos y + i \sin y) = 1$$

$$\Rightarrow e^x \cos y = 1, \quad e^x \sin y = 0$$

This is possible only if $\sin y = 0$ $(\because e^{x} \neq 0 \ \forall x)$

$$\Rightarrow y = n\pi$$

$$\Rightarrow \cos y = \pm 1 \ ; \quad e^{x} > 0 \ \& \ e^{x}\cos y = 1$$

$$\Rightarrow \cos y = 1 \quad \Rightarrow y = 2n\pi$$

$$\text{and } e^{x} = 1 \quad \Rightarrow x = 0$$

$$\therefore \ e^{z} = 1 \quad \Rightarrow \quad z = 2n\pi i$$

(3) $e^{z_1} = e^{z_2} \Rightarrow z_1 - z_2 = 2n\pi i$

In particular, the complex exponential is not 1-1.

Qn: Is it surjective?

Let $w \in \mathbb{C}$, ie $w = re^{i\theta}$. Does there exist $x+iy \ni e^{x}e^{iy} = re^{i\theta}$

$$\Rightarrow e^{x} = r, \quad e^{iy} = e^{i\theta}$$

$$\Rightarrow y = \theta + 2n\pi.$$

If $w \neq 0$, then $x = \ln(r)$, $y = \theta + 2n\pi$ satisfies $e^{x+iy} = re^{i\theta}$.

Thus, e^z is surjective onto $\mathbb{C}^*$.

Infact, if we restrict e^z to

$\mathbb{H} = \{x+iy \,/\, -\pi < y \leq \pi\}$ then

$e^z : \mathbb{H} \longrightarrow \mathbb{C}^*$ is bijective.

$$\left(\because \quad e^{z_1} = e^{z_2} \Rightarrow z_1 - z_2 = 2n\pi i \quad \text{but } z_1, z_2 \in \mathbb{H} \Rightarrow \operatorname{Im} z_1 - \operatorname{Im} z_2 < 2\pi \quad \therefore z_1 - z_2 = 0 \right)$$

$$\Rightarrow z_1 = z_2$$

$$\& \quad e^{\ln|w| + i\operatorname{Arg} w} = w \Big)$$

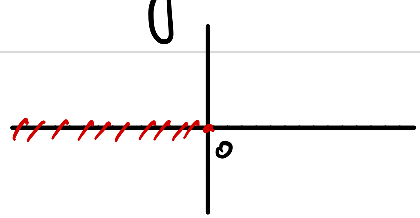
Define the inverse map as Logarithm (Principal Logarithm)

$$\operatorname{Log} : \mathbb{C}^* \longrightarrow \mathbb{C}$$
$$w \longmapsto \ln|w| + i\operatorname{Arg} w.$$

$$\left(\begin{array}{l} \log : \mathbb{C}^* \longrightarrow \mathbb{C} \\ \quad w \longmapsto \ln|w| + i\arg w \end{array} \right.$$

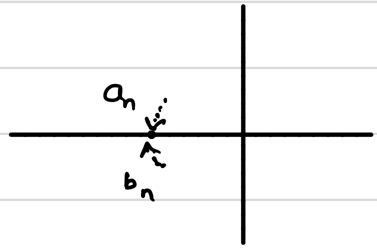
is multivalued !! $\Big)$

REMARK: 1) Log is not continuous on the negative real axis ie $\{x+iy \,/\, x \leq 0, y = 0\}$



To see this, consider $\alpha \in \mathbb{R}^-$

Let $a_n = |\alpha| e^{i(\pi - 1/n)}$ & $b_n = |\alpha| e^{i(-\pi + 1/n)}$



$$\mathrm{Log}(a_n) = \ln|\alpha| + i(\pi - 1/n)$$

$$\mathrm{Log}(b_n) = \ln|\alpha| + i(-\pi + 1/n)$$

$$\lim_{n \to \infty} \mathrm{Log}(a_n) = \ln|\alpha| + i\pi$$

$$\#$$

$$\lim_{n \to \infty} \mathrm{Log}(b_n) = \ln|\alpha| - i\pi$$

2) However, on the rest of the complex plane ie $\mathbb{C} \setminus (\mathbb{R}^- \cup \{0\})$ $\mathrm{Log}\, z$ is not just continuous but also analytic.

Continuity of $\operatorname{Log} z$ on $\mathbb{C}^* \setminus \mathbb{R}^-$

$$\operatorname{Arg} z : \mathbb{C}^* \setminus \mathbb{R}^- \to (-\pi, \pi)$$

is continuous.

hence $\underbrace{\ln|z| + i\operatorname{Arg} z}_{= \operatorname{Log} z}$ is continuous on $\mathbb{C}^* \setminus \mathbb{R}^-$.

Analyticity of $\operatorname{Log} z$ on $\mathbb{C}^* - \mathbb{R}^-$

Let $z = re^{i\theta}$ $\quad r \in \mathbb{R}^{>0}$ & $\theta \in (-\pi, \pi)$

then $\operatorname{Log} z = \ln r + i\theta$

$\uparrow \operatorname{Arg} z$

$\therefore \quad u(r, \theta) = \ln r$

$v(r, \theta) = \theta$

$u_r = \frac{1}{r}, \quad u_\theta = 0$

$v_r = 0 \quad v_\theta = 1$

CR eqns in polar form: $\quad u_r = \frac{1}{r} v_\theta$

& $\quad v_r = \frac{-1}{r} u_\theta$

are satisfied.

$$\begin{pmatrix} u_r \\ u_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix} \begin{pmatrix} u_x \\ u_y \end{pmatrix} \Rightarrow \begin{array}{l} u_r, u_\theta, v_r, v_\theta \text{ cont} \\ \Updownarrow \\ u_x, u_y, v_x, v_y \text{ cont.} \end{array}$$

$u_r, u_\theta, v_r, v_\theta$ are all continuous

$\Rightarrow$ Log z is analytic

Let's find the derivative of Log z for $z \in \mathbb{C}^* \setminus \mathbb{R}^-$

$$(\operatorname{Log} z)' = \lim_{w \to z} \frac{\operatorname{Log} w - \operatorname{Log} z}{w - z}$$

Since, $\operatorname{Log} : \mathbb{C}^* \setminus \mathbb{R}^- \longrightarrow \underbrace{\{x + iy \mid x \in \mathbb{R},\ y \in (-\pi, \pi)\}}_{= S}$

is continuous and bijective with

inverse $\exp : S \longrightarrow \mathbb{C}^* \setminus \mathbb{R}^-$,

so $\left.\begin{array}{l} w = e^{\zeta},\ z = e^{\zeta_0} \\ \text{for } \zeta, \zeta_0 \in S \end{array}\right\} \Rightarrow \left.\begin{array}{l} w \to z \\ \text{as } \zeta \to \zeta_0 \end{array}\right\} \Rightarrow \operatorname{Log} w \to \operatorname{Log} z$

$\Leftarrow \left\{\begin{array}{l} \operatorname{Log}(e^{\zeta}) \to \operatorname{Log}(e^{\zeta_0}) \\ \text{ie } \zeta \longrightarrow \zeta_0 \end{array}\right.$

Thus, $$(\operatorname{Log} z)' = \lim_{\zeta \to \zeta_0} \frac{\zeta - \zeta_0}{e^{\zeta} - e^{\zeta_0}}$$

$$= \frac{1}{\lim_{\zeta \to \zeta_0} \frac{e^{\zeta} - e^{\zeta_0}}{\zeta - \zeta_0}} = \frac{1}{e^{\zeta_0}} = \frac{1}{z}.$$

$\blacksquare$

REMARK 1: For every $\alpha \in \mathbb{R}$ fixed, the function

$$\log z = \ln|z| + i\arg z \qquad \alpha < \arg z \leq \alpha + 2\pi$$

is single-valued, called a branch of the Logarithm

Note that the above branch of the logarithm is analytic in $\mathbb{C}^* \setminus \{re^{i\alpha} / r \in \mathbb{R}\setminus\{0\}\}$, by a similar reasoning as for Log.

CAUTION: Domain of analyticity may be different for different branches of the logarithm.

REMARK 2: The property of Log that $e^{\operatorname{Log} z} = z$ $\forall z \in \mathbb{C}^*$

leads us to a meaningful definition of

z^α where $z \in \mathbb{C}^*$, $\alpha \in \mathbb{C}$

P.V of $z^\alpha := e^{\alpha(\ln|z| + i\operatorname{Arg} z)} \longrightarrow$ Principal value of z^α

z^α is a multivalued fn owing to "Arg z" replaced by arg z.

ie the different values of z^α are

$$e^{\alpha(\ln|z| + i\arg z)}.$$

Remark 3: Each branch, $\log z$, of the logarithm leads to a branch of z^α.

More Power Series . . .

Trigonometric functions

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^k \frac{x^{2k+1}}{(2k+1)!} + \cdots$$

$$\boxed{\sin z := z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots + (-1)^k \frac{z^{2k+1}}{(2k+1)!} + \cdots}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^k \frac{x^{2k}}{(2k)!} + \cdots$$

$$\boxed{\cos z := 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots + (-1)^k \frac{z^{2k}}{(2k)!} + \cdots}$$

(*) Each of the above series is absolutely convergent $\forall z \in \mathbb{C}$, since the corresponding real series is convergent $\forall x \in \mathbb{R}$.

Properties: ① $\cos(-z) = \cos z$; $\sin(-z) = -\sin z$

② $e^{iz} = \cos z + i \sin z$; $e^{-iz} = \cos(-z) + i\sin(-z)$
$= \cos z - i \sin z$.

& $(e^{iz})^n = \cos nz + i \sin nz$.

③ $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$; $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$ (By ②)

④ $\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$

$\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$ ▨

Appendix

(*) $$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots$$

$$e^{|z|} = 1 + |z| + \frac{(|z|)^2}{2!} + \frac{(|z|)^3}{3!} + \frac{(|z|)^4}{4!} + \cdots$$

k-th partial sum of $\cos z = S_k = \sum_{n=0}^{k} (-1)^n \frac{z^{2n}}{(2n)!}$

. . . . $e^{|z|} = t_k$.

$$|S_k - S_{k'}| = \left| \sum_{n=k}^{k'} (-1)^n \frac{z^{2n}}{(2n)!} \right| \leq \sum_{n=k}^{k'} \frac{|z|^{2n}}{(2n)!}$$

$$\leq \sum_{n=2k}^{2k'} \frac{|z|^n}{n!}$$

$$= |t_k - t_{k'}| < \varepsilon$$

$\forall\, k, k' \geq N$ (by abs cgs of $e^{|z|}$)

Hyperbolic functions

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^{2k+1}}{(2k+1)!} + \cdots$$

$$\boxed{\sinh z := z + \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots + \frac{z^{2k+1}}{(2k+1)!} + \cdots}$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2k}}{(2k)!} + \cdots$$

$$\boxed{\cosh z := 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots + \frac{z^{2k}}{(2k)!} + \cdots}$$

Each of the above series is absolutely
(*) convergent $\forall z \in \mathbb{C}$, since the corresponding real series is convergent $\forall x \in \mathbb{R}$.

<u>Properties</u>: ① $\cos(iz) = \cosh z$; $\sin(iz) = \sinh z$

② $\cosh z = \dfrac{e^z + e^{-z}}{2}$; $\sinh z = \dfrac{e^z - e^{-z}}{2}$ (By defn.)