MSO-203 B ASSIGNMENT 1

IIT, KANPUR

18th October , 2020

1. Consider the following functions defined on the interval $[-\pi, \pi]$.

   a) $f(x) = |x|$
   b) $f(x) = |\sin(x)|$
   c) $f(x) = \sin|x|$
   d) $f(x) = x^2$.

   Which one of the above functions admits Fourier series expansion ? (It is understood that the functions are extended 2π periodically to whole of real line). Write down their Fourier series expansion.

Solution (a):

$$f(x) = |x| = \begin{cases} -x, & x \leq 0 \\ x, & x > 0 \end{cases}.$$

Since left hand and right hand derivatives exist at all the points of $(-\pi, \pi)$.

$$\implies |x| = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

where

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |x| dx. \\ a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx. \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx. \end{aligned}$$

Solving

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_{-\pi}^{0} (-x) dx + \frac{1}{2\pi} \int_{0}^{\pi} x dx \\ &= \frac{1}{2\pi} \left(\frac{-x^2}{2} \right) \Big|_{-\pi}^{0} + \frac{1}{2\pi} \left(\frac{x^2}{2} \right) \Big|_{0}^{\pi} \\ &= \frac{1}{2\pi} \frac{\pi^2}{2} + \frac{1}{2\pi} \frac{\pi^2}{2} \\ &= \frac{\pi^2}{2\pi} = \frac{\pi}{2} \end{aligned}$$

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\cos(nx)dx. \\
&= \frac{2}{\pi}\int_{0}^{\pi} x\cos(nx)dx \text{ (why ?)} \\
&= \frac{2}{\pi}\left[\frac{x\sin(nx)}{n}\Big|_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin(nx)dx\right] \\
&= \frac{2}{\pi}\frac{\cos(nx)}{n^2}\Big|_0^{\pi} \\
&= \frac{2}{\pi n^2}((-1)^n - 1)
\end{aligned}
$$

$b_n = 0$, because the integrand $f(x) = |x|\sin(nx)$ is an odd function on $(-\pi, \pi)$.

$$
\implies |x| = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2}((-1)^n - 1)\cos nx, \forall x \in (-\pi, \pi).
$$

Solution (b): $f(x) = |\sin x|$ admits left hand derivative and right hand derivative at all the points. Hence proceed as previous exercise.

Solution (c): Notice

$$
f(x) = \sin|x| = \begin{cases} \sin x, & \text{if } 0 \le x < \pi \\ -\sin x, & \text{if } -\pi \le x < 0 \end{cases} = |\sin x|.
$$

Thus in part (b) and part (c) we have same functions.

Solution (d): Notice the function $f(x) = x^2$ is infinitely differentiable. Proceed as part (a).

2. Find the Fourier even half series of the function $f(x) = x$ on the interval $[0, L]$.

   Solution We have to calculate the Fourier coefficients for the arbitrary length L.

   Since it is even series NO b_n terms will be present or in other other words $b_n = 0$.

   We need to calculate

$$
a_0 = \frac{1}{L}\int_0^L f(x)dx = \frac{1}{L}\int_0^L xdx = \frac{L}{2}.
$$

Fourier cosine coefficients are given by the formula,

$$
a_n = \frac{2}{L}\int_0^L f(x)\cos(\frac{n\pi x}{L})dx = \frac{2}{L}\int_0^L x\cos(\frac{n\pi x}{L})dx.
$$

After calculation $a_n = 0$ if n is even, and $a_n = \frac{-4L}{n^2\pi^2}$, if n is odd.

Therefore, finally since $f(x) = x$ is a differentiable function on $(0, L)$, we have for all $x \in (0, L)$

$$
x = \frac{L}{2} - \frac{4L}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{(2m+1)^2}\cos\left(\frac{(2m+1)\pi x}{L}\right).
$$

3. Consider the function defined by

$$g(x) := \sum_{n=1}^{\infty} \frac{4\cos((2n+1)x)}{\pi(2n+1)^2} + \sum_{m=1}^{\infty} \frac{2\sin((2m+1)x)}{2m+1}, \quad x \in \mathbb{R}.$$

Then which of the following are correct:

a) $g(\frac{\pi}{2}) = \frac{\pi}{2}$
b) $g(0) = 0$
c) $g(0) = \frac{\pi}{2}$
d) $g(1) = 1$.

Hint: Work with the Fourier series of the periodical extension of the following function

$$f(x) = \begin{cases} x, & x \in [-\pi, 0) \\ \pi - x, & x \in (0, \pi]. \end{cases}$$

Solution: Let us work with the Fourier series of the following function

$$f(x) = \begin{cases} x, & x \in [-\pi, 0) \\ \pi - x, & x \in (0, \pi]. \end{cases}$$

Solving

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_{-\pi}^{0} x dx + \frac{1}{2\pi} \int_{0}^{\pi} (\pi - x) dx \\ &= \frac{1}{2\pi} \left(\frac{x^2}{2} \right) \Big|_{-\pi}^{0} + \frac{1}{2\pi} \left(\pi x - \frac{x^2}{2} \right) \Big|_{0}^{\pi} \\ &= -\frac{1}{2\pi} \frac{\pi^2}{2} + \frac{1}{2\pi} \pi^2 - \frac{1}{2\pi} \frac{\pi^2}{2} = 0 \end{aligned}$$

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{0} x \cos(nx) dx + \frac{1}{\pi} \int_{0}^{\pi} (\pi - x) \cos(nx) dx \\ &= \frac{1}{\pi} \left(\frac{1 - \cos n\pi}{n^2} - \frac{\cos n\pi - 1}{n^2} \right) \\ &= \frac{2}{n^2 \pi} (1 - \cos n\pi) = \begin{cases} 0, & \text{when } n = 2m \\ \frac{4}{n^2 \pi}, & \text{when } n = 2m + 1. \end{cases} \end{aligned}$$

Similarly

$$b_n = \frac{1}{n} (1 - \cos n\pi) = \begin{cases} 0, & \text{when } n = 2m \\ \frac{2}{n}, & \text{when } n = 2m + 1. \end{cases}$$

. Thus

$$f(x) = \sum_{m=1}^{\infty} \frac{4}{(2m+1)^2 \pi} \cos(2m+1)x + \sum_{m=1}^{\infty} \frac{2}{(2m+1)} \sin(2m+1)x.$$

Then we know from our theorem that if $x \in (-\pi, \pi) \backslash \{0\}$, then $f(x)$ is differentiable $\implies f(x) = g(x)$ if $x \in (-\pi, \pi) \backslash \{0\}$ and

$$g(0) = \frac{f(0+) + f(0-)}{2} = \frac{\pi}{2} \implies \quad \text{(c) is true}$$

$$g(\pi/2) = f(\pi/2) = \pi - \pi/2 = \pi/2 \implies \quad \text{(a) is true}$$

Then (b) and (d) will be false.

4. Pick the correct answers from the following:

a) $\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$

b) $\frac{\pi^2}{2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$

c) $0 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos(n\pi)$

d) $0 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{3}{n^2} \cos(n\pi)$.

Hint: Work with the Fourier transform of the periodical extension of the following function $f(x) = x^2$ on the interval $(-\pi, \pi)$

Solution: Let us find the Fourier Series of the function $f(x) = x^2$ on $(-\pi, \pi)$. Fourier coefficients are

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{\pi^2}{3} \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos(nx) dx \\
&= \frac{1}{\pi}\left[\frac{x^2 \sin nx}{n}\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{2x \sin(nx)}{n} dx \right] \\
&= -\frac{2}{\pi n}\left[\frac{x \cos nx}{n}\Big|_{\pi}^{-\pi} + \int_{-\pi}^{\pi} \frac{\cos(nx)}{n} dx \right] \\
&= \frac{4}{n^2}\cos n\pi = \frac{4(-1)^n}{n^2}, \quad n \in \mathbb{N}.
\end{aligned}$$

$b_n = 0$ as $x^2 \sin nx$ is an odd function.

$$\implies x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx. \tag{1}$$

At $x = \pi$, we have from (1)

$$\implies \pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n^2}.$$

$$\implies \frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

(a) is correct. If (a) is correct then definitely (b) is false.

Putting $x = 0$ in (1) and noticing $\cos n\pi = (-1)^n, \forall n$ gives (c) is true. (d) is false if (c) is true.

5. Find the half range series (both even and odd) for the following function:

$$f(x) = \begin{cases} 0, & x \in [0, \frac{\pi}{2}) \\ 1, & x \in [\frac{\pi}{2}, \pi]. \end{cases}$$

Solution: Even half range series is given by

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_0^{\frac{\pi}{2}} 0 dx + \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi} 1 dx = \frac{1}{2} \\
a_n &= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} 0 dx + \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} 1 \cos nx dx = \frac{2}{n\pi}(-\sin\frac{n\pi}{2}) \\
&= \begin{cases} 0, & n = 2m \\ \frac{2}{(2m+1)\pi}(-1)^m, & n = 2m+1 \end{cases}
\end{aligned}$$

$$\Longrightarrow f(x)=\frac{1}{2}+\frac{2}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m}{2m+1}\cos(2m+1)x.$$

Similarly find the odd half range series.

6. Apply Parsevals formula to the function $f(x)=x$ on $[-\pi,\pi]$ to find the sum of the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$.

Solution: Let us calculate the Fourier series of the function "x" on $[-\pi,\pi]$.

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x dx = 0,$$
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx dx = 0$$

as $x\cos nx$ is an odd function.

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx dx \\ &= \frac{1}{\pi}\left[\frac{-2\pi\cos n\pi}{n}\right] = -\frac{2}{n}(-1)^n. \end{aligned}$$

From Parseval's Identity we know that

$$\begin{aligned} \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx &= 2a_0^2+\sum_{n=1}^{\infty} a_n^2+b_n^2 \\ \Longleftrightarrow \frac{2\pi^3}{3.\pi} &= \sum_{n=1}^{\infty}\frac{4}{n^2} \\ \Longleftrightarrow \frac{\pi^2}{6} &= \sum_{n=1}^{\infty}\frac{1}{n^2} \end{aligned}$$