CHAPTER

7

APPLICATIONS OF RESIDUES

We turn now to some important applications of the theory of residues, which was developed in Chap. 6. The applications include evaluation of certain types of definite and improper integrals occurring in *real* analysis and applied mathematics. Considerable attention is also given to a method, based on residues, for locating zeros of functions and to finding inverse Laplace transforms by summing residues.

78. EVALUATION OF IMPROPER INTEGRALS

In calculus, the improper integral of a continuous function $f(x)$ over the semi-infinite interval $0 \le x < \infty$ is defined by means of the equation

$$\int_0^\infty f(x)\,dx = \lim_{R\to\infty} \int_0^R f(x)\,dx. \tag{1}$$

When the limit on the right exists, the improper integral is said to *converge* to that limit. If $f(x)$ is continuous for *all* x, its improper integral over the infinite interval $-\infty < x < \infty$ is defined by writing

$$\int_{-\infty}^\infty f(x)\,dx = \lim_{R_1\to\infty} \int_{-R_1}^0 f(x)\,dx + \lim_{R_2\to\infty} \int_0^{R_2} f(x)\,dx; \tag{2}$$

and when both of the limits here exist, we say that integral (2) converges to their sum. Another value that is assigned to integral (2) is often useful. Namely, the

Cauchy principal value (P.V.) of integral (2) is the number

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx, \tag{3}$$

provided this single limit exists.

If integral (2) converges, its Cauchy principal value (3) exists; and that value is the number to which integral (2) converges. This is because

$$\begin{aligned}\lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx &= \lim_{R\to\infty} \left[\int_{-R}^{0} f(x)\,dx + \int_{0}^{R} f(x)\,dx\right] \\ &= \lim_{R\to\infty} \int_{-R}^{0} f(x)\,dx + \lim_{R\to\infty} \int_{0}^{R} f(x)\,dx\end{aligned}$$

and these last two limits are the same as the limits on the right in equation (2).

It is *not*, however, always true that integral (2) converges when its Cauchy principal value exists, as the following example shows.

EXAMPLE. Observe that

$$\text{P.V.} \int_{-\infty}^{\infty} x\,dx = \lim_{R\to\infty} \int_{-R}^{R} x\,dx = \lim_{R\to\infty} \left[\frac{x^2}{2}\right]_{-R}^{R} = \lim_{R\to\infty} 0 = 0. \tag{4}$$

On the other hand,

$$\begin{aligned}\int_{-\infty}^{\infty} x\,dx &= \lim_{R_1\to\infty} \int_{-R_1}^{0} x\,dx + \lim_{R_2\to\infty} \int_{0}^{R_2} x\,dx \\ &= \lim_{R_1\to\infty} \left[\frac{x^2}{2}\right]_{-R_1}^{0} + \lim_{R_2\to\infty} \left[\frac{x^2}{2}\right]_{0}^{R_2} \\ &= -\lim_{R_1\to\infty} \frac{R_1^2}{2} + \lim_{R_2\to\infty} \frac{R_2^2}{2};\end{aligned} \tag{5}$$

and since these last two limits do not exist, we find that the improper integral (5) fails to exist.

But suppose that $f(x)$ $(-\infty < x < \infty)$ is an *even* function, one where

$$f(-x) = f(x) \quad \text{for all } x,$$

and *assume that the Cauchy principal value* (3) *exists*. The symmetry of the graph of $y = f(x)$ with respect to the y axis tells us that

$$\int_{-R_1}^{0} f(x)\,dx = \frac{1}{2} \int_{-R_1}^{R_1} f(x)\,dx$$

and

$$\int_0^{R_2} f(x)\,dx = \frac{1}{2}\int_{-R_2}^{R_2} f(x)\,dx.$$

Thus

$$\int_{-R_1}^{0} f(x)\,dx + \int_0^{R_2} f(x)\,dx = \frac{1}{2}\int_{-R_1}^{R_1} f(x)\,dx + \frac{1}{2}\int_{-R_2}^{R_2} f(x)\,dx.$$

If we let R_1 and R_2 tend to ∞ on each side here, the fact that the limits on the right exist means that the limits on the left do too. In fact,

$$\int_{-\infty}^{\infty} f(x)\,dx = \text{P.V.}\int_{-\infty}^{\infty} f(x)\,dx. \tag{6}$$

Moreover, since

$$\int_0^{R} f(x)\,dx = \frac{1}{2}\int_{-R}^{R} f(x)\,dx,$$

it is also true that

$$\int_0^{\infty} f(x)\,dx = \frac{1}{2}\left[\text{P.V.}\int_{-\infty}^{\infty} f(x)\,dx\right]. \tag{7}$$

We now describe a method involving sums of residues, to be illustrated in the next section, that is often used to evaluate improper integrals of *rational functions* $f(x) = p(x)/q(x)$, where $p(x)$ and $q(x)$ are polynomials with real coefficients and no factors in common. We agree that $q(z)$ has no real zeros but has at least one zero *above* the real axis.

The method begins with the identification of all the *distinct* zeros of the polynomial $q(z)$ that lie above the real axis. They are, of course, finite in number (see Sec. 53) and may be labeled $z_1, z_2, \ldots, z_n$, where n is less than or equal to the degree of $q(z)$. We then integrate the quotient

$$f(z) = \frac{p(z)}{q(z)} \tag{8}$$

around the positively oriented boundary of the semicircular region shown in Fig. 93.

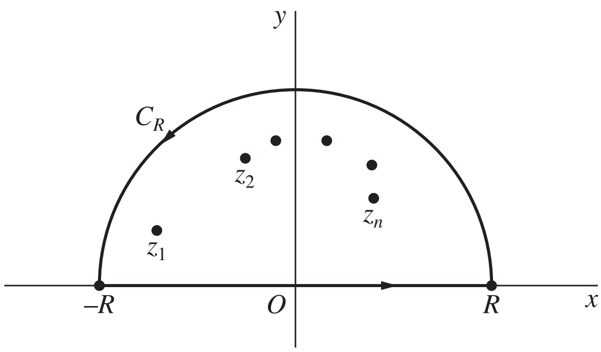

FIGURE 93

That simple closed contour consists of the segment of the real axis from $z = -R$ to $z = R$ and the top half of the circle $|z| = R$, described counterclockwise and denoted by C_R. It is understood that the positive number R is large enough so that the points $z_1, z_2, \ldots, z_n$ all lie inside the closed path.

The parametric representation $z = x$ $(-R \le x \le R)$ of the segment of the real axis just mentioned and Cauchy's residue theorem in Sec. 70 can be used to write

$$\int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z),$$

or

$$\int_{-R}^{R} f(x)\,dx = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z) - \int_{C_R} f(z)\,dz. \tag{9}$$

If

$$\lim_{R\to\infty} \int_{C_R} f(z)\,dz = 0,$$

it then follows that

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z); \tag{10}$$

and if $f(x)$ is *even*, equations (6) and (7) tell us that

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z) \tag{11}$$

and

$$\int_{0}^{\infty} f(x)\,dx = \pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z). \tag{12}$$

79. EXAMPLE

We turn now to an illustration of the method in Sec. 78 for evaluating improper integrals.

EXAMPLE. In order to evaluate the integral

$$\int_{0}^{\infty} \frac{x^2}{x^6+1}\,dx,$$

we start with the observation that the function

$$f(z) = \frac{z^2}{z^6 + 1}$$

has isolated singularities at the zeros of $z^6 + 1$, which are the sixth roots of -1, and is analytic everywhere else. The method in Sec. 9 for finding roots of complex numbers reveals that the sixth roots of -1 are

$$c_k = \exp\left[i\left(\frac{\pi}{6} + \frac{2k\pi}{6}\right)\right] \qquad (k = 0, 1, 2, \ldots, 5),$$

and it is clear that none of them lies on the real axis. The first three roots,

$$c_0 = e^{i\pi/6}, \quad c_1 = i, \quad \text{and} \quad c_2 = e^{i5\pi/6},$$

lie in the upper half plane (Fig. 94) and the other three lie in the lower one. When $R > 1$, the points c_k $(k = 0, 1, 2)$ lie in the interior of the semicircular region bounded by the segment $z = x$ $(-R \le x \le R)$ of the real axis and the upper half C_R of the circle $|z| = R$ from $z = R$ to $z = -R$. Integrating $f(z)$ counterclockwise around the boundary of this semicircular region, we see that

$$\int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz = 2\pi i(B_0 + B_1 + B_2), \tag{1}$$

where B_k is the residue of $f(z)$ at c_k $(k = 0, 1, 2)$.

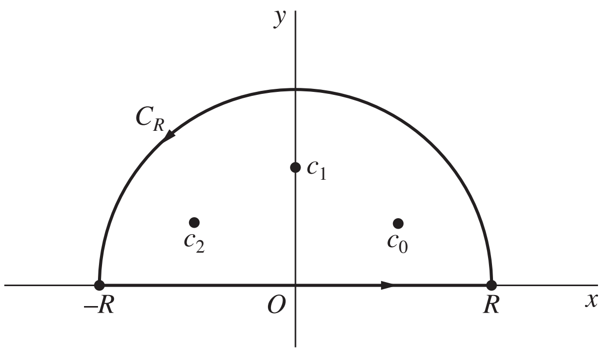

FIGURE 94

With the aid of Theorem 2 in Sec. 76, we find that the points c_k are simple poles of f and that

$$B_k = \operatorname*{Res}_{z=c_k} \frac{z^2}{z^6 + 1} = \frac{c_k^2}{6c_k^5} = \frac{1}{6c_k^3} \qquad (k = 0, 1, 2).$$

Thus

$$2\pi i(B_0 + B_1 + B_2) = 2\pi i\left(\frac{1}{6i} - \frac{1}{6i} + \frac{1}{6i}\right) = \frac{\pi}{3};$$

and equation (1) can be put in the form

$$\int_{-R}^{R} f(x)\,dx = \frac{\pi}{3} - \int_{C_R} f(z)\,dz\,, \tag{2}$$

which is valid for all values of R greater than 1.

Next, we show that the value of the integral on the right in equation (2) tends to 0 as R tends to ∞. To do this, we observe that when $|z| = R$,

$$|z^2| = |z|^2 = R^2$$

and

$$|z^6 + 1| \ge |\,|z|^6 - 1| = R^6 - 1.$$

So, if z is any point on C_R,

$$|f(z)| = \frac{|z^2|}{|z^6 + 1|} \le M_R \quad \text{where} \quad M_R = \frac{R^2}{R^6 - 1};$$

and this means that

$$\left|\int_{C_R} f(z)\,dz\right| \le M_R\,\pi R, \tag{3}$$

πR being the length of the semicircle C_R. (See Sec. 43.) Since the number

$$M_R\,\pi R = \frac{\pi R^3}{R^6 - 1}$$

is a quotient of polynomials in R and since the degree of the numerator is less than the degree of the denominator, that quotient must tend to zero as R tends to ∞. More precisely, if we divide both numerator and denominator by R^6 and write

$$M_R\,\pi R = \frac{\dfrac{\pi}{R^3}}{1 - \dfrac{1}{R^6}},$$

it is evident that $M_R\,\pi R$ tends to zero. Consequently, in view of inequality (3),

$$\lim_{R\to\infty} \int_{C_R} f(z)\,dz = 0.$$

It now follows from equation (2) that

$$\lim_{R\to\infty}\int_{-R}^{R}\frac{x^2}{x^6+1}\,dx=\frac{\pi}{3},$$

or

$$\text{P.V.}\int_{-\infty}^{\infty}\frac{x^2}{x^6+1}\,dx=\frac{\pi}{3}.$$

Since the integrand here is even, we know from equation (7) in Sec. 78 that

$$\int_{0}^{\infty}\frac{x^2}{x^6+1}\,dx=\frac{\pi}{6}. \tag{4}$$

EXERCISES

Use residues to evaluate the improper integrals in Exercises 1 through 5.

1. $\displaystyle\int_{0}^{\infty}\frac{dx}{x^2+1}$.
Ans. $\pi/2$.

2. $\displaystyle\int_{0}^{\infty}\frac{dx}{(x^2+1)^2}$.
Ans. $\pi/4$.

3. $\displaystyle\int_{0}^{\infty}\frac{dx}{x^4+1}$.
Ans. $\pi/(2\sqrt{2})$.

4. $\displaystyle\int_{0}^{\infty}\frac{x^2\,dx}{(x^2+1)(x^2+4)}$.
Ans. $\pi/6$.

5. $\displaystyle\int_{0}^{\infty}\frac{x^2\,dx}{(x^2+9)(x^2+4)^2}$.
Ans. $\pi/200$.

Use residues to find the Cauchy principal values of the integrals in Exercises 6 and 7.

6. $\displaystyle\int_{-\infty}^{\infty}\frac{dx}{x^2+2x+2}$.

7. $\displaystyle\int_{-\infty}^{\infty}\frac{x\,dx}{(x^2+1)(x^2+2x+2)}$.
Ans. $-\pi/5$.

8. Use a residue and the contour shown in Fig. 95, where $R>1$, to establish the integration formula

$$\int_{0}^{\infty}\frac{dx}{x^3+1}=\frac{2\pi}{3\sqrt{3}}.$$

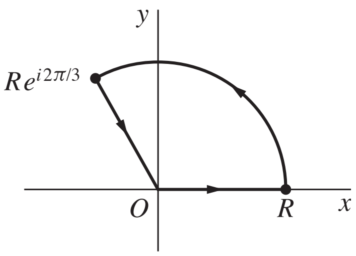

FIGURE 95

9. Let m and n be integers, where $0 \le m < n$. Follow the steps below to derive the integration formula

$$\int_0^\infty \frac{x^{2m}}{x^{2n}+1}\,dx = \frac{\pi}{2n}\csc\left(\frac{2m+1}{2n}\pi\right).$$

(a) Show that the zeros of the polynomial $z^{2n}+1$ lying above the real axis are

$$c_k = \exp\left[i\frac{(2k+1)\pi}{2n}\right] \qquad (k = 0, 1, 2, \ldots, n-1)$$

and that there are none on that axis.

(b) With the aid of Theorem 2 in Sec. 76, show that

$$\operatorname*{Res}_{z=c_k} \frac{z^{2m}}{z^{2n}+1} = -\frac{1}{2n}e^{i(2k+1)\alpha} \qquad (k = 0, 1, 2, \ldots, n-1)$$

where c_k are the zeros found in part *(a)* and

$$\alpha = \frac{2m+1}{2n}\pi.$$

Then use the summation formula

$$\sum_{k=0}^{n-1} z^k = \frac{1-z^n}{1-z} \qquad (z \ne 1)$$

(see Exercise 9, Sec. 8) to obtain the expression

$$2\pi i \sum_{k=0}^{n-1} \operatorname*{Res}_{z=c_k} \frac{z^{2m}}{z^{2n}+1} = \frac{\pi}{n\sin\alpha}.$$

(c) Use the final result in part *(b)* to complete the derivation of the integration formula.

10. The integration formula

$$\int_0^\infty \frac{dx}{[(x^2-a)^2+1]^2} = \frac{\pi}{8\sqrt{2}A^3}[(2a^2+3)\sqrt{A+a} + a\sqrt{A-a}],$$

where a is any real number and $A = \sqrt{a^2+1}$, arises in the theory of case-hardening of steel by means of radio-frequency heating.* Follow the steps below to derive it.

(a) Point out why the four zeros of the polynomial

$$q(z) = (z^2 - a)^2 + 1$$

are the square roots of the numbers $a \pm i$. Then, using the fact that the numbers

$$z_0 = \frac{1}{\sqrt{2}}(\sqrt{A+a} + i\sqrt{A-a})$$

and $-z_0$ are the square roots of $a+i$ (Exercise 5, Sec. 10), verify that $\pm\overline{z_0}$ are the square roots of $a-i$ and hence that z_0 and $-\overline{z_0}$ are the only zeros of $q(z)$ in the upper half plane $\operatorname{Im} z \geq 0$.

(b) Using the method derived in Exercise 7, Sec. 76, and keeping in mind that $z_0^2 = a+i$ for purposes of simplification, show that the point z_0 in part *(a)* is a pole of order 2 of the function $f(z) = 1/[q(z)]^2$ and that the residue B_1 at z_0 can be written

$$B_1 = -\frac{q''(z_0)}{[q'(z_0)]^3} = \frac{a - i(2a^2+3)}{16A^2 z_0}.$$

After observing that $q'(-\overline{z}) = -\overline{q'(z)}$ and $q''(-\overline{z}) = \overline{q''(z)}$, use the same method to show that the point $-\overline{z_0}$ in part *(a)* is also a pole of order 2 of the function $f(z)$, with residue

$$B_2 = \overline{\left\{\frac{q''(z_0)}{[q'(z_0)]^3}\right\}} = -\overline{B_1}.$$

Then obtain the expression

$$B_1 + B_2 = \frac{1}{8A^2 i}\operatorname{Im}\left[\frac{-a + i(2a^2+3)}{z_0}\right]$$

for the sum of these residues.

(c) Refer to part *(a)* and show that $|q(z)| \geq (R - |z_0|)^4$ if $|z| = R$, where $R > |z_0|$. Then, with the aid of the final result in part *(b)*, complete the derivation of the integration formula.

80. IMPROPER INTEGRALS FROM FOURIER ANALYSIS

Residue theory can be useful in evaluating convergent improper integrals of the form

$$(1) \qquad \int_{-\infty}^{\infty} f(x)\sin ax\,dx \quad \text{or} \quad \int_{-\infty}^{\infty} f(x)\cos ax\,dx,$$

*See pp. 359–364 of the book by Brown, Hoyler, and Bierwirth that is listed in Appendix 1.

where a denotes a positive constant. As in Sec. 78, we assume that $f(x) = p(x)/q(x)$ where $p(x)$ and $q(x)$ are polynomials with real coefficients and no factors in common. Also, $q(x)$ has no zeros on the real axis and at least one zero above it. Integrals of type (1) occur in the theory and application of the Fourier integral.*

The method described in Sec. 78 and used in Sec. 79 cannot be applied directly here since (see Sec. 34)

$$|\sin az|^2 = \sin^2 ax + \sinh^2 ay$$

and

$$|\cos az|^2 = \cos^2 ax + \sinh^2 ay.$$

More precisely, since

$$\sinh ay = \frac{e^{ay} - e^{-ay}}{2},$$

the moduli $|\sin az|$ and $|\cos az|$ increase like e^{ay} as y tends to infinity. The modification illustrated in the example below is suggested by the fact that

$$\int_{-R}^{R} f(x)\cos ax\, dx + i\int_{-R}^{R} f(x)\sin ax\, dx = \int_{-R}^{R} f(x)e^{iax}\, dx,$$

together with the fact that the modulus

$$|e^{iaz}| = |e^{ia(x+iy)}| = |e^{-ay}e^{iax}| = e^{-ay}$$

is bounded in the upper half plane $y \geq 0$.

EXAMPLE. Let us show that

$$\int_{-\infty}^{\infty} \frac{\cos 3x}{(x^2+1)^2}\, dx = \frac{2\pi}{e^3}. \tag{2}$$

Because the integrand is even, it is sufficient to show that the Cauchy principal value of the integral exists and to find that value.

We introduce the function

$$f(z) = \frac{1}{(z^2+1)^2} \tag{3}$$

and observe that the product $f(z)e^{i3z}$ is analytic everywhere on and above the real axis except at the point $z = i$. The singularity $z = i$ lies in the interior of the semicircular region whose boundary consists of the segment $-R \leq x \leq R$ of the real

*See the authors' "Fourier Series and Boundary Value Problems," 7th ed., Chap. 6, 2008.

axis and the upper half C_R of the circle $|z| = R$ $(R > 1)$ from $z = R$ to $z = -R$ (Fig. 96). Integration of $f(z)e^{i3z}$ around that boundary yields the equation

$$\int_{-R}^{R} \frac{e^{i3x}}{(x^2+1)^2}\,dx = 2\pi i B_1 - \int_{C_R} f(z)e^{i3z}\,dz, \tag{4}$$

where

$$B_1 = \operatorname*{Res}_{z=i}[f(z)e^{i3z}].$$

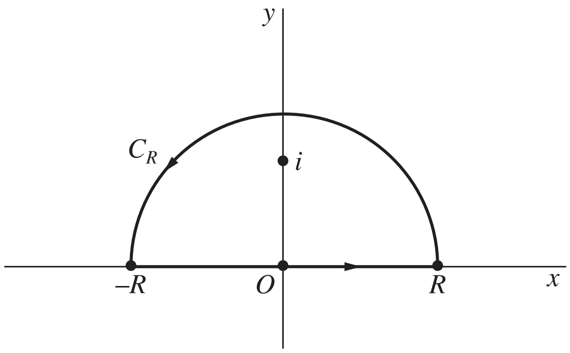

FIGURE 96

Since

$$f(z)e^{i3z} = \frac{\phi(z)}{(z-i)^2} \quad \text{where} \quad \phi(z) = \frac{e^{i3z}}{(z+i)^2},$$

the point $z = i$ is evidently a pole of order $m = 2$ of $f(z)e^{i3z}$; and

$$B_1 = \phi'(i) = \frac{1}{ie^3}.$$

By equating the real parts on each side of equation (4), then, we find that

$$\int_{-R}^{R} \frac{\cos 3x}{(x^2+1)^2}\,dx = \frac{2\pi}{e^3} - \operatorname{Re}\int_{C_R} f(z)e^{i3z}\,dz. \tag{5}$$

Finally, we observe that when z is a point on C_R,

$$|f(z)| \le M_R \quad \text{where} \quad M_R = \frac{1}{(R^2-1)^2}$$

and that $|e^{i3z}| = e^{-3y} \le 1$ for such a point. Consequently,

$$\left|\operatorname{Re}\int_{C_R} f(z)e^{i3z}\,dz\right| \le \left|\int_{C_R} f(z)e^{i3z}\,dz\right| \le M_R\,\pi R. \tag{6}$$

Since

$$\left|f(Re^{i\theta})\right| \le M_R \quad \text{and} \quad \left|\exp(iaRe^{i\theta})\right| \le e^{-aR\sin\theta}$$

and in view of Jordan's inequality (1), it follows that

$$\left|\int_{C_R} f(z)e^{iaz}\,dz\right| \le M_R R \int_0^\pi e^{-aR\sin\theta}\,d\theta < \frac{M_R\pi}{a}.$$

The final limit in the theorem is now evident since $M_R \to 0$ as $R \to \infty$.

EXAMPLE. Let us find the Cauchy principal value of the integral

$$\int_{-\infty}^{\infty} \frac{x\sin x\,dx}{x^2+2x+2}.$$

As usual, the existence of the value in question will be established by our actually finding it.

We write

$$f(z) = \frac{z}{z^2+2z+2} = \frac{z}{(z-z_1)(z-\overline{z_1})},$$

where $z_1 = -1+i$. The point z_1, which lies above the x axis, is a simple pole of the function $f(z)e^{iz}$, with residue

$$B_1 = \frac{z_1\,e^{iz_1}}{z_1 - \overline{z_1}}. \tag{3}$$

Hence, when $R > \sqrt{2}$ and C_R denotes the upper half of the positively oriented circle $|z| = R$,

$$\int_{-R}^{R} \frac{xe^{ix}\,dx}{x^2+2x+2} = 2\pi i B_1 - \int_{C_R} f(z)e^{iz}\,dz;$$

and this means that

$$\int_{-R}^{R} \frac{x\sin x\,dx}{x^2+2x+2} = \operatorname{Im}(2\pi i B_1) - \operatorname{Im}\int_{C_R} f(z)e^{iz}\,dz. \tag{4}$$

Now

$$\left|\operatorname{Im}\int_{C_R} f(z)e^{iz}\,dz\right| \le \left|\int_{C_R} f(z)e^{iz}\,dz\right|; \tag{5}$$

and we note that when z is a point on C_R,

$$|f(z)| \le M_R \quad \text{where} \quad M_R = \frac{R}{(R-\sqrt{2})^2}$$

and that $|e^{iz}| = e^{-y} \le 1$ for such a point. By proceeding as we did in the examples in Secs. 79 and 80, we *cannot* conclude that the right-hand side of inequality

(5), and hence its left-hand side, tends to zero as R tends to infinity. For the quantity

$$M_R \pi R = \frac{\pi R^2}{(R-\sqrt{2})^2} = \frac{\pi}{\left(1 - \dfrac{\sqrt{2}}{R}\right)^2}$$

does not tend to zero. The above theorem does, however, provide the desired limit, namely

$$\lim_{R\to\infty} \int_{C_R} f(z)e^{iz}\,dz = 0,$$

since

$$M_R = \frac{\dfrac{1}{R}}{\left(1 - \dfrac{\sqrt{2}}{R}\right)^2} \to 0 \quad \text{as} \quad R \to \infty.$$

So it does, indeed, follow from inequality (5) that the left-hand side there tends to zero as R tends to infinity. Consequently, equation (4), together with expression (3) for the residue B_1, tells us that

$$\text{(6)} \qquad \text{P.V.} \int_{-\infty}^{\infty} \frac{x \sin x\,dx}{x^2+2x+2} = \operatorname{Im}(2\pi i B_1) = \frac{\pi}{e}(\sin 1 + \cos 1).$$

EXERCISES

Use residues to evaluate the improper integrals in Exercises 1 through 8.

1. $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x\,dx}{(x^2+a^2)(x^2+b^2)} \quad (a > b > 0).$

Ans. $\displaystyle\frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right).$

2. $\displaystyle\int_0^{\infty} \frac{\cos ax}{x^2+1}\,dx \quad (a > 0).$

Ans. $\displaystyle\frac{\pi}{2}e^{-a}.$

3. $\displaystyle\int_0^{\infty} \frac{\cos ax}{(x^2+b^2)^2}\,dx \quad (a > 0, b > 0).$

Ans. $\displaystyle\frac{\pi}{4b^3}(1+ab)e^{-ab}.$

4. $\displaystyle\int_0^{\infty} \frac{x \sin 2x}{x^2+3}\,dx.$

Ans. $\displaystyle\frac{\pi}{2}\exp(-2\sqrt{3}).$

5. $\displaystyle\int_{-\infty}^{\infty} \frac{x \sin ax}{x^4+4}\, dx \quad (a > 0).$

Ans. $\dfrac{\pi}{2} e^{-a} \sin a.$

6. $\displaystyle\int_{-\infty}^{\infty} \frac{x^3 \sin ax}{x^4+4}\, dx \quad (a > 0).$

Ans. $\pi e^{-a} \cos a.$

7. $\displaystyle\int_{-\infty}^{\infty} \frac{x \sin x\, dx}{(x^2+1)(x^2+4)}.$

8. $\displaystyle\int_{0}^{\infty} \frac{x^3 \sin x\, dx}{(x^2+1)(x^2+9)}.$

Use residues to find the Cauchy principal values of the improper integrals in Exercises 9 through 11.

9. $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x\, dx}{x^2+4x+5}.$

Ans. $-\dfrac{\pi}{e} \sin 2.$

10. $\displaystyle\int_{-\infty}^{\infty} \frac{(x+1)\cos x}{x^2+4x+5}\, dx.$

Ans. $\dfrac{\pi}{e}(\sin 2 - \cos 2).$

11. $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x\, dx}{(x+a)^2+b^2} \quad (b > 0).$

12. Follow the steps below to evaluate the *Fresnel integrals*, which are important in diffraction theory:

$$\int_0^{\infty} \cos(x^2)\, dx = \int_0^{\infty} \sin(x^2)\, dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

(a) By integrating the function $\exp(iz^2)$ around the positively oriented boundary of the sector $0 \le r \le R, 0 \le \theta \le \pi/4$ (Fig. 99) and appealing to the Cauchy–Goursat theorem, show that

$$\int_0^R \cos(x^2)\, dx = \frac{1}{\sqrt{2}} \int_0^R e^{-r^2}\, dr - \text{Re} \int_{C_R} e^{iz^2}\, dz$$

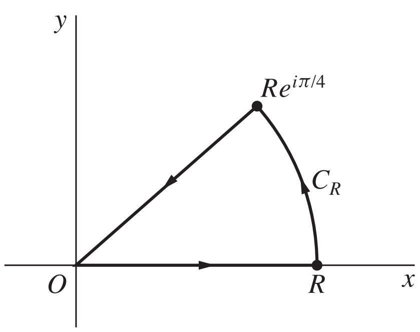

FIGURE 99

EXAMPLE. Modifying the method used in Secs. 80 and 81, we derive here the integration formula*

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2} \tag{4}$$

by integrating e^{iz}/z around the simple closed contour shown in Fig. 101. In that figure, ρ and R denote positive real numbers, where $\rho < R$; and L_1 and L_2 represent the intervals

$$\rho \le x \le R \quad \text{and} \quad -R \le x \le -\rho,$$

respectively, on the real axis. While the semicircle C_R is as in Secs. 80 and 81, the semicircle C_ρ is introduced here in order to avoid passing through the singularity $z = 0$ of the quotient e^{iz}/z.

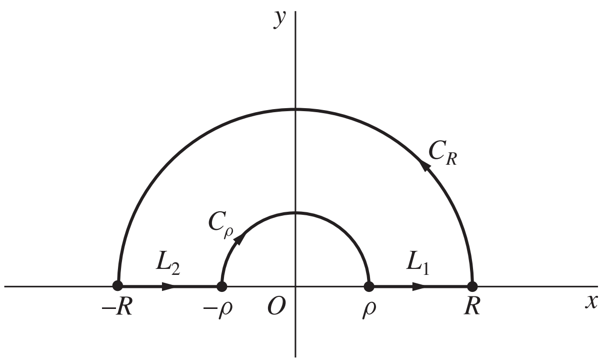

FIGURE 101

The Cauchy–Goursat theorem tells us that

$$\int_{L_1} \frac{e^{iz}}{z}\,dz + \int_{C_R} \frac{e^{iz}}{z}\,dz + \int_{L_2} \frac{e^{iz}}{z}\,dz + \int_{C_\rho} \frac{e^{iz}}{z}\,dz = 0,$$

or

$$\int_{L_1} \frac{e^{iz}}{z}\,dz + \int_{L_2} \frac{e^{iz}}{z}\,dz = -\int_{C_\rho} \frac{e^{iz}}{z}\,dz - \int_{C_R} \frac{e^{iz}}{z}\,dz. \tag{5}$$

Moreover, since the legs L_1 and $-L_2$ have parametric representations

$$z = re^{i0} = r \ (\rho \le r \le R) \quad \text{and} \quad z = re^{i\pi} = -r \ (\rho \le r \le R), \tag{6}$$

*This formula arises in the theory of the *Fourier integral*. See the authors' "Fourier Series and Boundary Value Problems," 7th ed., pp. 150–152, 2008, where it is derived in a completely different way.

respectively, the left-hand side of equation (5) can be written

$$\int_{L_1} \frac{e^{iz}}{z}\,dz - \int_{-L_2} \frac{e^{iz}}{z}\,dz = \int_\rho^R \frac{e^{ir}}{r}\,dr - \int_\rho^R \frac{e^{-ir}}{r}\,dr = 2i\int_\rho^R \frac{\sin r}{r}\,dr.$$

Consequently,

$$2i\int_\rho^R \frac{\sin r}{r}\,dr = -\int_{C_\rho} \frac{e^{iz}}{z}\,dz - \int_{C_R} \frac{e^{iz}}{z}\,dz\,. \tag{7}$$

Now, from the Laurent series representation

$$\frac{e^{iz}}{z} = \frac{1}{z}\left[1 + \frac{(iz)}{1!} + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \cdots\right] = \frac{1}{z} + \frac{i}{1!} + \frac{i^2}{2!}z + \frac{i^3}{3!}z^2 + \cdots \qquad (0 < |z| < \infty),$$

it is clear that e^{iz}/z has a simple pole at the origin, with residue unity. So, according to the theorem at the beginning of this section,

$$\lim_{\rho\to 0}\int_{C_\rho} \frac{e^{iz}}{z}\,dz = -\pi i.$$

Also, since

$$\left|\frac{1}{z}\right| = \frac{1}{|z|} = \frac{1}{R}$$

when z is a point on C_R, we know from Jordan's lemma in Sec. 81 that

$$\lim_{R\to\infty}\int_{C_R} \frac{e^{iz}}{z}\,dz = 0.$$

Thus, by letting ρ tend to 0 in equation (7) and then letting R tend to ∞, we arrive at the result

$$2i\int_0^\infty \frac{\sin r}{r}\,dr = \pi i,$$

which is, in fact, formula (4).

83. AN INDENTATION AROUND A BRANCH POINT

The example here involves the same indented path that was used in the example in Sec. 82. The indentation is, however, due to a branch point, rather than an isolated singularity.

EXAMPLE. The integration formula

$$\int_0^\infty \frac{\ln x}{(x^2+4)^2}\,dx = \frac{\pi}{32}(\ln 2 - 1) \tag{1}$$

can be derived by considering the branch

$$f(z) = \frac{\log z}{(z^2+4)^2} \qquad \left(|z| > 0, -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}\right)$$

of the multiple-valued function $(\log z)/(z^2+4)^2$. This branch, whose branch cut consists of the origin and the negative imaginary axis, is analytic everywhere in the stated domain except at the point $z = 2i$. See Fig. 102, where the same indented path and the same labels L_1, L_2, C_ρ, and C_R as in Fig. 101 are used. In order that the isolated singularity $z = 2i$ be inside the closed path, we require that $\rho < 2 < R$.

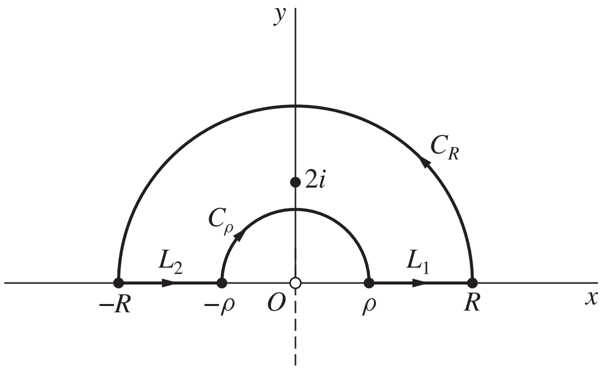

FIGURE 102

According to Cauchy's residue theorem,

$$\int_{L_1} f(z)\,dz + \int_{C_R} f(z)\,dz + \int_{L_2} f(z)\,dz + \int_{C_\rho} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=2i} f(z).$$

That is,

$$\int_{L_1} f(z)\,dz + \int_{L_2} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=2i} f(z) - \int_{C_\rho} f(z)\,dz - \int_{C_R} f(z)\,dz. \tag{2}$$

Since

$$f(z) = \frac{\ln r + i\theta}{(r^2 e^{i2\theta} + 4)^2} \qquad (z = re^{i\theta}),$$

the parametric representations

$$z = re^{i0} = r \ (\rho \le r \le R) \quad \text{and} \quad z = re^{i\pi} = -r \ (\rho \le r \le R) \tag{3}$$

for the legs L_1 and $-L_2$, respectively, can be used to write the left-hand side of equation (2) as

$$\int_{L_1} f(z)\,dz - \int_{-L_2} f(z)\,dz = \int_\rho^R \frac{\ln r}{(r^2+4)^2}\,dr + \int_\rho^R \frac{\ln r + i\pi}{(r^2+4)^2}\,dr.$$

Also, since

$$f(z) = \frac{\phi(z)}{(z-2i)^2} \quad \text{where} \quad \phi(z) = \frac{\log z}{(z+2i)^2},$$

the singularity $z = 2i$ of $f(z)$ is a pole of order 2, with residue

$$\phi'(2i) = \frac{\pi}{64} + i\frac{1-\ln 2}{32}.$$

Equation (2) thus becomes

$$(4) \quad 2\int_\rho^R \frac{\ln r}{(r^2+4)^2}\,dr + i\pi\int_\rho^R \frac{dr}{(r^2+4)^2} = \frac{\pi}{16}(\ln 2 - 1) + i\frac{\pi^2}{32} - \int_{C_\rho} f(z)\,dz - \int_{C_R} f(z)\,dz;$$

and, by equating the real parts on each side here, we find that

$$(5) \quad 2\int_\rho^R \frac{\ln r}{(r^2+4)^2}\,dr = \frac{\pi}{16}(\ln 2 - 1) - \operatorname{Re}\int_{C_\rho} f(z)\,dz - \operatorname{Re}\int_{C_R} f(z)\,dz.$$

It remains only to show that

$$(6) \quad \lim_{\rho\to 0} \operatorname{Re}\int_{C_\rho} f(z)\,dz = 0 \quad \text{and} \quad \lim_{R\to\infty} \operatorname{Re}\int_{C_R} f(z)\,dz = 0.$$

For, by letting ρ and R tend to 0 and ∞, respectively, in equation (5), we then arrive at

$$2\int_0^\infty \frac{\ln r}{(r^2+4)^2}\,dr = \frac{\pi}{16}(\ln 2 - 1),$$

which is the same as equation (1).

Limits (6) are established as follows. First, we note that if $\rho < 1$ and $z = \rho e^{i\theta}$ is a point on C_ρ, then

$$|\log z| = |\ln \rho + i\theta| \le |\ln \rho| + |i\theta| \le -\ln \rho + \pi$$

and

$$|z^2 + 4| \ge \big|\,|z|^2 - 4\big| = 4 - \rho^2.$$

As a consequence,

$$\left|\operatorname{Re}\int_{C_\rho} f(z)\,dz\right| \le \left|\int_{C_\rho} f(z)\,dz\right| \le \frac{-\ln\rho + \pi}{(4-\rho^2)^2}\pi\rho = \pi\frac{\pi\rho - \rho\ln\rho}{(4-\rho^2)^2};$$

and, by l'Hospital's rule, the product $\rho \ln \rho$ in the numerator on the far right here tends to 0 as ρ tends to 0. So the first of limits (6) clearly holds. Likewise, by writing

$$\left|\operatorname{Re}\int_{C_R} f(z)\,dz\right| \le \left|\int_{C_R} f(z)\,dz\right| \le \frac{\ln R + \pi}{(R^2-4)^2}\pi R = \pi \frac{\dfrac{\pi}{R} + \dfrac{\ln R}{R}}{\left(R - \dfrac{4}{R}\right)^2}$$

and using l'Hospital's rule to show that the quotient $(\ln R)/R$ tends to 0 as R tends to ∞, we obtain the second of limits (6).

Note how another integration formula, namely

$$\int_0^\infty \frac{dx}{(x^2+4)^2} = \frac{\pi}{32}, \tag{7}$$

follows by equating imaginary, rather than real, parts on each side of equation (4):

$$\pi \int_\rho^R \frac{dr}{(r^2+4)^2} = \frac{\pi^2}{32} - \operatorname{Im}\int_{C_\rho} f(z)\,dz - \operatorname{Im}\int_{C_R} f(z)\,dz. \tag{8}$$

Formula (7) is then obtained by letting ρ and R tend to 0 and ∞, respectively, since

$$\left|\operatorname{Im}\int_{C_\rho} f(z)\,dz\right| \le \left|\int_{C_\rho} f(z)\,dz\right| \quad \text{and} \quad \left|\operatorname{Im}\int_{C_R} f(z)\,dz\right| \le \left|\int_{C_R} f(z)\,dz\right|.$$

84. INTEGRATION ALONG A BRANCH CUT

Cauchy's residue theorem can be useful in evaluating a real integral when part of the path of integration of the function $f(z)$ to which the theorem is applied lies along a branch cut of that function.

EXAMPLE. Let x^{-a}, where $x > 0$ and $0 < a < 1$, denote the principal value of the indicated power of x; that is, x^{-a} is the positive real number $\exp(-a \ln x)$. We shall evaluate here the improper real integral

$$\int_0^\infty \frac{x^{-a}}{x+1}\,dx \qquad (0 < a < 1), \tag{1}$$

which is important in the study of the *gamma function*.* Note that integral (1) is improper not only because of its upper limit of integration but also because its integrand has an infinite discontinuity at $x = 0$. The integral converges when $0 < a < 1$ since the integrand behaves like x^{-a} near $x = 0$ and like x^{-a-1} as x

*See, for example, p. 4 of the book by Lebedev cited in Appendix 1.

EXERCISES

In Exercises 1 through 4, take the indented contour in Fig. 101 (Sec. 82).

1. Derive the integration formula

$$\int_0^\infty \frac{\cos(ax)-\cos(bx)}{x^2}\,dx = \frac{\pi}{2}(b-a) \qquad (a \geq 0, b \geq 0).$$

Then, with the aid of the trigonometric identity $1-\cos(2x) = 2\sin^2 x$, point out how it follows that

$$\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}.$$

2. Evaluate the improper integral

$$\int_0^\infty \frac{x^a}{(x^2+1)^2}\,dx, \quad \text{where} \quad -1 < a < 3 \text{ and } x^a = \exp(a \ln x).$$

Ans. $\dfrac{(1-a)\pi}{4\cos(a\pi/2)}$.

3. Use the function

$$f(z) = \frac{z^{1/3}\log z}{z^2+1} = \frac{e^{(1/3)\log z}\log z}{z^2+1} \qquad \left(|z| > 0, -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}\right)$$

to derive this pair of integration formulas:

$$\int_0^\infty \frac{\sqrt[3]{x}\ln x}{x^2+1}\,dx = \frac{\pi^2}{6}, \qquad \int_0^\infty \frac{\sqrt[3]{x}}{x^2+1}\,dx = \frac{\pi}{\sqrt{3}}.$$

4. Use the function

$$f(z) = \frac{(\log z)^2}{z^2+1} \qquad \left(|z| > 0, -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}\right)$$

to show that

$$\int_0^\infty \frac{(\ln x)^2}{x^2+1}\,dx = \frac{\pi^3}{8}, \qquad \int_0^\infty \frac{\ln x}{x^2+1}\,dx = 0.$$

Suggestion: The integration formula obtained in Exercise 1, Sec. 79, is needed here.

5. Use the function

$$f(z) = \frac{z^{1/3}}{(z+a)(z+b)} = \frac{e^{(1/3)\log z}}{(z+a)(z+b)} \qquad (|z| > 0, 0 < \arg z < 2\pi)$$

and a closed contour similar to the one in Fig. 103 (Sec. 84) to show formally that

$$\int_0^\infty \frac{\sqrt[3]{x}}{(x+a)(x+b)}\,dx = \frac{2\pi}{\sqrt{3}} \cdot \frac{\sqrt[3]{a}-\sqrt[3]{b}}{a-b} \qquad (a > b > 0).$$

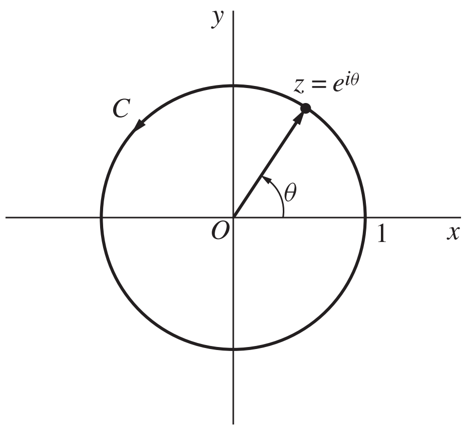

FIGURE 105

which transform integral (1) into the contour integral

$$\int_C F\left(\frac{z - z^{-1}}{2i}, \frac{z + z^{-1}}{2}\right) \frac{dz}{iz} \tag{4}$$

of a function of z around the circle C. The original integral (1) is, of course, simply a parametric form of integral (4), in accordance with expression (2), Sec. 40. When the integrand in integral (4) reduces to a rational function of z, we can evaluate that integral by means of Cauchy's residue theorem once the zeros in the denominator have been located and provided that none lie on C.

EXAMPLE. Let us show that

$$\int_0^{2\pi} \frac{d\theta}{1 + a\sin\theta} = \frac{2\pi}{\sqrt{1 - a^2}} \qquad (-1 < a < 1). \tag{5}$$

This integration formula is clearly valid when $a = 0$, and we exclude that case in our derivation. With substitutions (3), the integral takes the form

$$\int_C \frac{2/a}{z^2 + (2i/a)z - 1}\, dz, \tag{6}$$

where C is the positively oriented circle $|z| = 1$. The quadratic formula reveals that the denominator of the integrand here has the pure imaginary zeros

$$z_1 = \left(\frac{-1 + \sqrt{1 - a^2}}{a}\right)i, \quad z_2 = \left(\frac{-1 - \sqrt{1 - a^2}}{a}\right)i.$$

So if $f(z)$ denotes the integrand in integral (6), then

$$f(z) = \frac{2/a}{(z - z_1)(z - z_2)}.$$

Note that because $|a| < 1$,

$$|z_2| = \frac{1 + \sqrt{1 - a^2}}{|a|} > 1.$$

Also, since $|z_1 z_2| = 1$, it follows that $|z_1| < 1$. Hence there are no singular points on C, and the only one interior to it is the point z_1. The corresponding residue B_1 is found by writing

$$f(z) = \frac{\phi(z)}{z - z_1} \quad \text{where} \quad \phi(z) = \frac{2/a}{z - z_2}.$$

This shows that z_1 is a simple pole and that

$$B_1 = \phi(z_1) = \frac{2/a}{z_1 - z_2} = \frac{1}{i\sqrt{1 - a^2}}.$$

Consequently,

$$\int_C \frac{2/a}{z^2 + (2i/a)z - 1}\, dz = 2\pi i B_1 = \frac{2\pi}{\sqrt{1 - a^2}};$$

and integration formula (5) follows.

The method just illustrated applies equally well when the arguments of the sine and cosine are integral multiples of θ. One can use equation (2) to write, for example,

$$\cos 2\theta = \frac{e^{i2\theta} + e^{-i2\theta}}{2} = \frac{(e^{i\theta})^2 + (e^{i\theta})^{-2}}{2} = \frac{z^2 + z^{-2}}{2}.$$

EXERCISES

Use residues to evaluate the definite integrals in Exercises 1 through 7.

1. $\displaystyle\int_0^{2\pi} \frac{d\theta}{5 + 4\sin\theta}.$

Ans. $\dfrac{2\pi}{3}.$

2. $\displaystyle\int_{-\pi}^{\pi} \frac{d\theta}{1 + \sin^2\theta}.$

Ans. $\sqrt{2}\pi.$

3. $\displaystyle\int_0^{2\pi} \frac{\cos^2 3\theta\, d\theta}{5 - 4\cos 2\theta}.$

Ans. $\dfrac{3\pi}{8}.$

4. $\displaystyle\int_0^{2\pi} \frac{d\theta}{1 + a\cos\theta} \quad (-1 < a < 1).$

Ans. $\dfrac{2\pi}{\sqrt{1-a^2}}.$

5. $\displaystyle\int_0^{\pi} \frac{\cos 2\theta\, d\theta}{1 - 2a\cos\theta + a^2} \quad (-1 < a < 1).$

Ans. $\dfrac{a^2\pi}{1-a^2}.$

6. $\displaystyle\int_0^{\pi} \frac{d\theta}{(a + \cos\theta)^2} \quad (a > 1).$

Ans. $\dfrac{a\pi}{\left(\sqrt{a^2-1}\right)^3}.$

7. $\displaystyle\int_0^{\pi} \sin^{2n}\theta\, d\theta \quad (n = 1, 2, \ldots).$

Ans. $\dfrac{(2n)!}{2^{2n}(n!)^2}\pi.$

86. ARGUMENT PRINCIPLE

A function f is said to be *meromorphic* in a domain D if it is analytic throughout D except for poles. Suppose now that f is meromorphic in the domain interior to a positively oriented simple closed contour C and that it is analytic and nonzero on C. The image Γ of C under the transformation $w = f(z)$ is a closed contour, not necessarily simple, in the w plane (Fig. 106). As a point z traverses C in the positive direction, its images w traverses Γ in a particular direction that determines the orientation of Γ. Note that since f has no zeros on C, the contour Γ does not pass through the origin in the w plane.

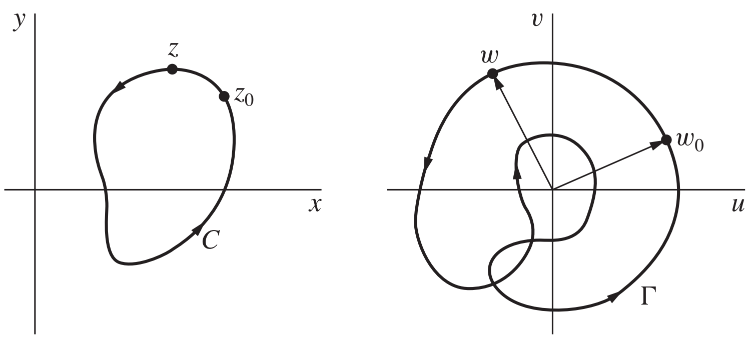

FIGURE 106

Let w_0 and w be points on Γ, where w_0 is fixed and ϕ_0 is a value of $\arg w_0$. Then let $\arg w$ vary continuously, starting with the value ϕ_0, as the point w begins at the point w_0 and traverses Γ once in the direction of orientation assigned to it