SUBTLETIES OF CONDITIONAL PROBABILITY

Conditional probabilities are quite subtle. Apart from the common mistake of confusing $\mathbf{P}(A \mid B)$ for $\mathbf{P}(B \mid A)$, there are other points one sometimes overlooks. In fact, most of the paradoxical sounding puzzles in probability are based on confusing aspects of conditional probability. Let us see one.

Question 9. A man says "I have two children, and one of them is a boy". What is the chance that the other one is a girl?

There are four possibilities BB, BG, GB, GG, of which GG has been eliminated. Of the remaining three, two are favourable, hence the chance is $2/3$ that the other child is a girl. This is a possible solution. If you accept this as reasonable, here is another question.

Question 10. A man says "I have two children, and one of them is a boy born on a Monday". What is the chance that the other one is a girl?

Does the addition of the information about the boy change the probability? One opinion is that it should not. The other is to follow the same solution pattern as before. Write down all the $2\times2\times7\times7$ possibilities: $BBss$ (boy, boy, Sunday, Sunday), $BBsm$ (boy, boy, Sunday, Monday), etc. The given information that one is a boy who was born on Sunday eliminates many possibilities and what remain are 27 possibilities $BGm*$, $GB*m$, $BBm*$, $BB*m$ where $*$ is any day of the week. Take care to not double count $BBmm$ to see that there are 27 possibilities. Of these, 14 are favourable (i.e., the other child is a girl), hence we conclude that the probability is $14/27$.

Is the correct answer $14/27$ as calculated here or is it $2/3$? Since the information of the day of birth of the boy is irrelevant, why should we change our earlier answer of $2/3$?

Both answers can be correct, depending on the interpretation of what the experiment is. The main reason for all the confusion leading to multiple interpretations is avoided if one realizes this: *To compute conditional probabilities, it is not enough to know what the person said, but also what else he could have said*. Not realizing this point is the main source of confusion in many popular puzzles in probability. We leave you to understand this statement in the context of the problem, but explain it in more general terms.

What are we conditioning on? In talking about conditional probability, we should really think of a *measurement*. To start, we have a discrete probability space (Ω, p) which defines probabilities of various events. A measurement is a function $T : \Omega \mapsto \mathbb{R}$. Let us say that the measurement can take three values, $0, 1, 2$. Let $A_i = \{\omega : T(\omega) = i\}$, for $i = 1, 2, 3$. Then, A_1, A_2, A_3 are pairwise disjoint and their union is Ω. As a result of the measurement, we get to know whether ω belongs to A_1 or to A_2 or to A_3. But, we would not know which of these it belongs to. Based on what the measurement actually shows, we update our probabilities.

The problem with puzzles (like the one above) is that they don't specify what is being measured. Depending on the interpretation, different answers are possible. For example, if a person says "I have two children, one of whom is a girl", it is giving the result of a measurement without saying what was being measured. Did the person check the sex of the eldest child and report "girl" as the measurement? Did he check whether or not he has a girl child and then report "Yes" as the measurement?

One lesson from this is this. We should not think of $\mathbf{P}(\cdot \mid A)$ alone, but of both $\mathbf{P}(\cdot \mid A)$ and $\mathbf{P}(\cdot \mid A^c)$. We make a measurement which corners ω into A or into A^c, and we have to be ready to deploy $\mathbf{P}(\cdot \mid A)$ or $\mathbf{P}(\cdot \mid A^c)$, depending on the outcome of that measurement.