Solution of Quiz Q5

1.a. Let $\sigma^2 = k\tau^2$. Here, $\mathbb{E}(X_1^2) = \sigma^2$, $\mathbb{E}(X_1^4) = 3\sigma^4$ and $Var(X_1^2) = 2\sigma^4$. $\boxed{\text{0.5 mark}}$

Observe that $X_1^2, X_2^2, \ldots, X_n^2$ are i.i.d. random variables. So, using CLT we have

$$\sqrt{n}\left[\frac{1}{n}\sum_{i=1}^{n} X_i^2 - \sigma^2\right] \xrightarrow[n\to\infty]{D} N(0, 2\sigma^4 = 2k^2\tau^4). \qquad \boxed{\text{1 mark}}$$

Using delta method, we have

$$\sqrt{n}\left[g(T_n) - g(\sigma^2)\right] \xrightarrow[n\to\infty]{D} N\big(0, \{g'(\sigma^2)\}^2 2\sigma^4\big). \qquad \boxed{\text{0.5 mark}}$$

We require $\{g'(\sigma^2)\}^2 2\sigma^4$ be independent of σ^2, i.e.,

$$\begin{aligned} &\{g'(\sigma^2)\}^2 2\sigma^4 = c, \text{ a constant} \\ \Rightarrow &g'(\sigma^2) \propto \frac{1}{\sigma^2} \\ \Rightarrow &g(t) = \log(t) \text{ is such a function.} \end{aligned} \qquad \boxed{\text{1 mark}}$$

Taking $g(t) = \log(t)$, we have

$$\sqrt{n}\left[\log(T_n) - \log(\sigma^2)\right] \xrightarrow[n\to\infty]{D} N\big(0, 2\big).$$

1.b. The likelihood is given by:

$$L\Big(\theta|x_1,x_2,\ldots,x_n\Big) = \prod_{i=1}^{n} f(x_i|\theta) = \exp\Big(-\frac{\sum_{i=1}^n x_i^k}{k\theta^2}\Big)\prod_{i=1}^{n}\frac{x_i^{k-1}}{\theta^{2n}}$$

$$\log L\Big(\theta|x_1,x_2,\ldots,x_n\Big) = -2n\log\theta + (k-1)\sum_{i=1}^{n}\log x_i - \frac{\sum_{i=1}^n x_i^k}{k\theta^2}. \qquad \boxed{0.5 \text{ mark}}$$

Hence, $\dfrac{\partial \log L\Big(\theta|x_1,x_2,\ldots,x_n\Big)}{\partial\theta} = 0$ gives

$$\frac{-2n}{\theta} + \frac{2\sum_{i=1}^n x_i^k}{k\theta^3} = 0 \implies \widehat{\theta}_{MLE} = \sqrt{\frac{\sum_{i=1}^n x_i^k}{kn}}. \qquad \boxed{0.5 \text{ mark}}$$

$$\frac{\partial^2 \log L\Big(\theta|x_1,x_2,\ldots,x_n\Big)}{\partial\theta^2} = \frac{2n}{\theta^2} - \frac{6\sum_{i=1}^n x_i^k}{k\theta^4} = \frac{2}{\theta^2}\Big(n - \frac{3\sum_{i=1}^n x_i^k}{k\theta^2}\Big)$$

$$\implies \frac{\partial^2 \log L\Big(\theta|x_1,x_2,\ldots,x_n\Big)}{\partial\theta^2}\Bigg|_{\theta=\widehat{\theta}_{MLE}} = \frac{2}{\widehat{\theta}^2_{MLE}}(n-3n) = \frac{-4n}{\widehat{\theta}^2_{MLE}} < 0. \qquad \boxed{1 \text{ mark}}$$

$$E(X) = \frac{1}{\theta^2}\int_0^\infty x^k \exp\Big(-\frac{x^k}{k\theta^2}\Big)dx$$

$$\text{Let } x^k = m \implies x = m^{1/k}, dx = \frac{1}{k}m^{\frac{1}{k}-1}dm$$

$$E(X) = \frac{1}{k\theta^2}\int_0^\infty m^{1+\frac{1}{k}-1}\exp\Big(-\frac{m}{k\theta^2}\Big)dx$$

$$\implies E(X) = \theta^{2/k}c_k, \text{ where } c_k = k^{1/k}\Gamma\Big(1+\frac{1}{k}\Big).$$

So, the method of moments estimate $\widehat{\theta}_{MME}$ of θ is given by:

$$\widehat{\theta}_{MME} = \Big(\frac{\bar{X}_n}{c_k}\Big)^{k/2}, \text{ where } \bar{X}_n \text{ is the sample mean.} \qquad \boxed{1 \text{ mark}}$$

2.a. (i) Observe that

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=1}^{n}\operatorname{Cov}(X_i,X_j) &\le \sum_{i=1}^{n}\zeta + 2\sum_{i=1}^{n}\sum_{j=1}^{i-1} c^{|i-j|} && \boxed{\text{1 mark}}\\
&= n\zeta + 2\sum_{i=1}^{n}\sum_{j=1}^{i-1} c^{|i-j|} = n\zeta + 2\sum_{i=1}^{n}\sum_{j=1}^{i-1} c^{i-j}\\
&= n\zeta + 2\sum_{i=1}^{n} c^{i}\frac{1-c^{-i}}{1-c^{-1}} = n\zeta + 2\sum_{i=1}^{n}\frac{c^{i}+1}{1-c^{-1}}\\
&= n\zeta + \frac{2c}{c-1}\sum_{i=1}^{n} c^{i} - 1 = n\zeta + \frac{2c}{c-1}\Big(\frac{1-c^{n+1}}{1-c} - n\Big)\\
&= n\zeta + \frac{2c}{(c-1)^2}(c^{n+1}+1) + \frac{2c}{c-1}n\\
&\le n\zeta + \frac{4c}{(c-1)^2} + \frac{2c}{c-1}n && \boxed{\text{1 mark}}
\end{aligned}$$

(ii) Fix $\epsilon > 0$ and $n \in \mathbb{N}$, then using Chebyshev's inequality

$$\mathbb{P}\left[\left|\frac{S_n}{n} - \mathbb{E}\left(\frac{S_n}{n}\right)\right| > \epsilon\right] \le \frac{\operatorname{Var}\left(\frac{S_n}{n}\right)}{\epsilon^2}.$$

Observe that $\mathbb{E}(\frac{S_n}{n}) = 0$ and

$$\operatorname{Var}\left(\frac{S_n}{n}\right) = \frac{\operatorname{Var}(S_n)}{n^2} = \frac{\sum_{i=1}^{n}\sum_{j=1}^{n}\operatorname{Cov}(X_i,X_j)}{n^2}. \qquad \boxed{\text{1 mark}}$$

From part (i),

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\operatorname{Cov}(X_i,X_j) \le n\zeta + \frac{2c}{(c-1)^2}(c^{n+1}+1) + \frac{2c}{c-1}n$$

Thus,

$$\lim_{n\to\infty}\mathbb{P}\left[\left|\frac{S_n}{n}\right| > \epsilon\right] = \lim_{n\to\infty}\frac{\operatorname{Var}\left(\frac{S_n}{n}\right)}{\epsilon^2} \le \lim_{n\to\infty}\frac{n\zeta + \frac{2c}{(c-1)^2}(c^{n+1}+1) + \frac{2c}{c-1}n}{n^2\epsilon^2} = 0$$

$\boxed{\text{1 mark}}$

The choice of $\epsilon > 0$ is arbitrary, it follows that $\frac{S_n}{n} \xrightarrow{P} 0$ as $n \to \infty$.

2.b.

$$L(\theta|x_1, x_2, x_3) = P[X_1 = x_1, X_2 = x_2, X_3 = x_3|\theta]$$

using multinomial distribution, we get

$$L(\theta|x_1, x_2, x_3) \propto \theta^{2x_1}\theta^{x_2}(1-\theta)^{x_2}(1-\theta)^{2x_3}. \qquad \boxed{\text{0.5 mark}}$$

To find the maximum likelihood estimate of θ, we maximize the log-likelihood with respect to θ as follows:

$$\log L(\theta|x_1, x_2, x_3) = \text{ constant } + (2x_1 + x_2)\log\theta + (x_2 + 2x_3)\log(1-\theta)$$

$$\implies \frac{\partial \log L(\theta|x_1, x_2, x_3)}{\partial\theta} = 0, \text{ gives}$$

$$\frac{\partial \log L(\theta|x_1, x_2, x_3)}{\partial\theta} = \frac{(2x_1 + x_2)}{\theta} - \frac{(x_2 + 2x_3)}{(1-\theta)} = 0$$

$$\implies (2x_1 + x_2)(1-\theta) - (x_2 + 2x_3)\theta = 0$$

$$\implies \widehat{\theta}_{MLE} = \frac{(2x_1 + x_2)}{2n}. \qquad \boxed{\text{0.5 mark}}$$

$$\frac{\partial^2 \log L(\theta|x_1, x_2, x_3)}{\partial\theta^2} = -\frac{(2x_1 + x_2)}{\theta^2} - \frac{(x_2 + 2x_3)}{(1-\theta)^2}, \text{ which is always negative}$$

$$\implies \left.\frac{\partial^2 \log L(\theta|x_1, x_2, x_3)}{\partial\theta^2}\right|_{\theta=\widehat{\theta}_{MLE}} < 0. \qquad \boxed{\text{1 mark}}$$