Tutorial sheet 6

EE 250, 17 March 2021

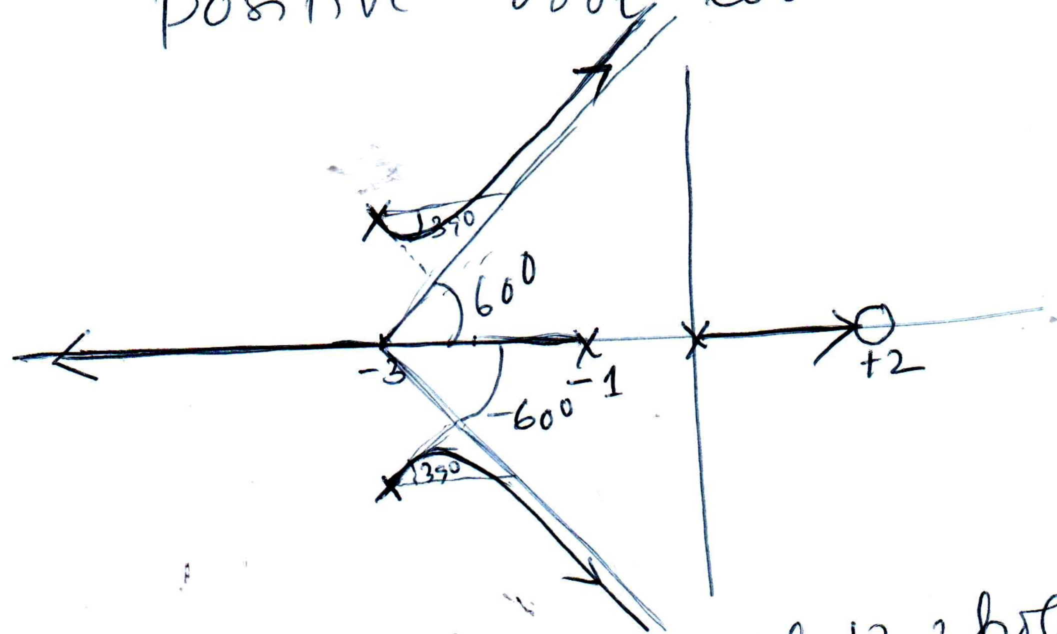
Q1a. Given open loop transfer function

$$G(s) = K \frac{(s-2)}{s(s+1)(s^2+6s+12)},$$

draw the root locus for both positive and negative gain variation. Which one would you use for designing a suitable compensator?

Solution

positive root locus



The system has a zero at +2, 3 poles at $s=0$, $s=-1$, $s=-3 \pm j1.732$

4 branches of root locus, one will converge at open loop zero ($s=2$) and other three will converge at ∞ along three asymptotes.

Asymptote angles are

$+60, +180, +300$

Centroid

$$\sigma = \frac{-0-1-3-3-2}{4-1}$$

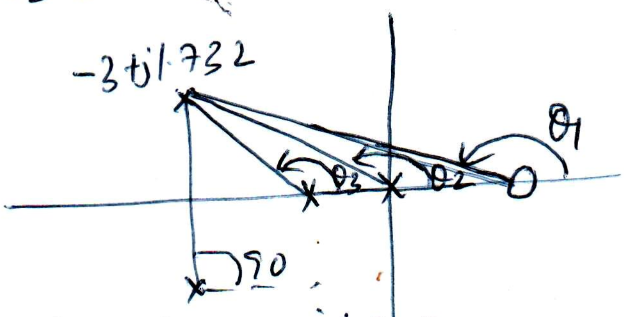
$$= -3$$

Real axis between the pole $s=0$ and zero $s=2$ is part of the root locus. This this pole will converse to zero as shown.

Real axis between the pole at $s=-1$ and to its left is part of the root locus. Thus this pole will converse to ∞ along LHS-plane real axis as shown in the figure.

The poles at complex conjugate $-3 \pm j1.732$ will converse to ∞ along asymptotes at $\pm 60^0$ as shown in the figure.

The angle of departure from $-3+j1.732$ is calculated as follows.



$\theta_1 = 180 - \tan^{-1}\frac{1.732}{5}$
$= 161^0$

$\theta_2 = 180 - \tan^{-1}\frac{1.732}{3}$
$= 150^0$

$\theta_3 = 180 - \tan^{-1}\frac{1.732}{2}$
$= 140^0$

$\theta_4 = 90^0$

Net angle at $-3+j1.732$

$= \theta_1 - (\theta_2 + \theta_3 + \theta_4)$

$= 161 - 150 - 140 - 90 = -219$

Thus the angle of departure

$\phi_p = 180 + \phi$

$= 180 - 219$

$= -39^0$.

Looking at symmetry along real axis the angle of departure at $s = -3 - 1.732j$ is $\phi_p = 39^0$

Thus the root locus is complete.

We can notice that the closed loop system is always unstable i.e. for $K > 0$

Can we stabilize this system using positive root locus?

The pole at $s=0$ is dragged to zero at $s=2$ as $K>0$. Can we avoid this scenario?

Lets place a zero between poles at $s=0$ & $s=-1$. This will not change the scenario

Lets place a zero to the left of the pole at $s=-1$. This will also not change the scenario.

This will not change even if we place ~~poles so~~ zeros say complex conjugate zeros at $s = -1 \pm j1$. In all situations the zero in RHS plane will pull the pole at $s=0$, hence the system is not stabilizable in the root locus.

Solution — ve root locus or $0°$ root locus

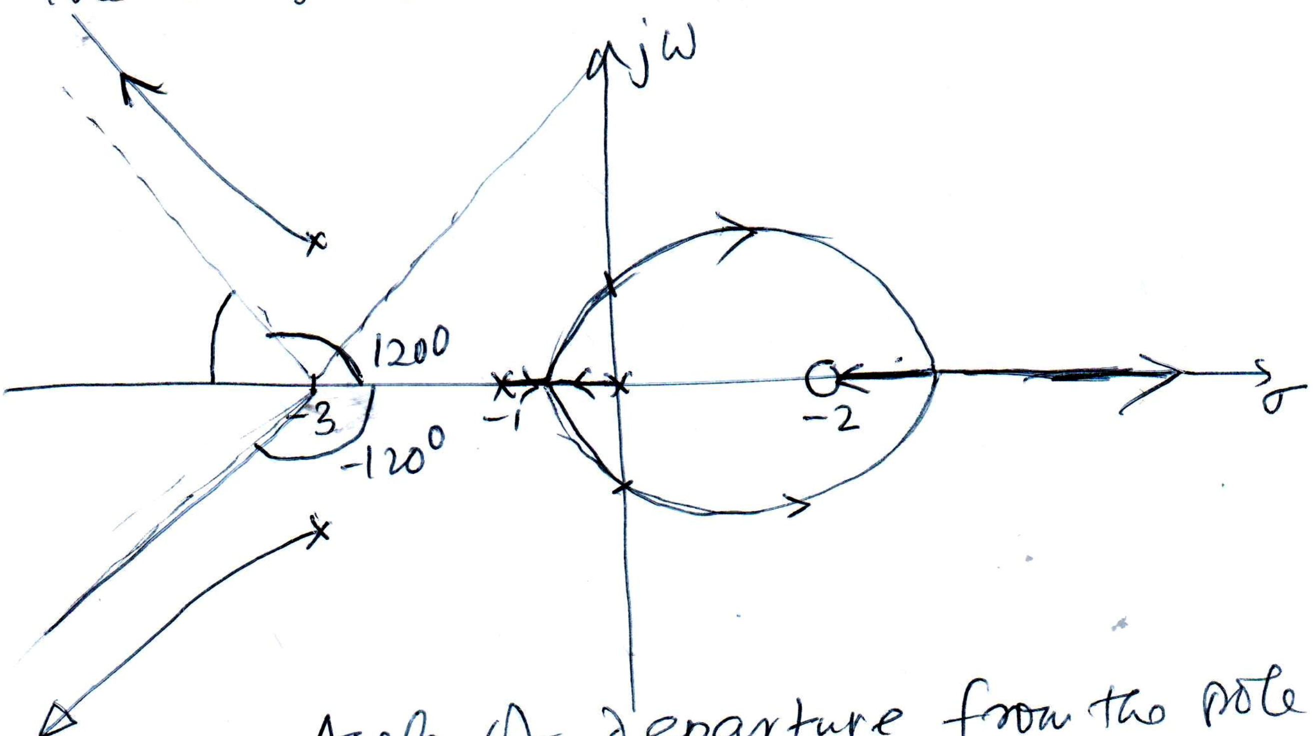
Asymptote angles

$0°, \pm 120°$

Centroid $\sigma = -3$

Real axis between the poles at $s=0$ & $s=-1$ and the real axis to the right of zero at $s=2$ are part of root locus.

The rough sketch would thus look like:



Angle of departure from the pole at $s = -3 \pm j1.732$

$\phi_p = 360 - \phi$

$= 360 - 219 = 141°$

Break away points $\frac{dK}{ds} = 0$

$$K = \frac{(s^2+s)(s^2+6s+12)}{s-2} = \frac{s^4+7s^3+18s^2+12s}{s+2}$$

$$\frac{dk}{ds} = \frac{(s-2)(4s^3+21s^2+36s+12) - (s^4+7s^3+18s^2+12s)}{(s-2)^2} = 0$$

$\Rightarrow \quad 3s^4+6s^3-24s^2-72s-24=0$

$\Rightarrow \quad s^4+2s^3-8s^2-24s-8=0$

roots are (using MATLAB)

$+3.16, \ -2.39 \pm j0.898, \ -0.3872.$

Looking at the rough sketch we can say that break away points are -0.3872 and 3.16.

Crossing of $J\omega$ axis

$$1 + \frac{K(2-s)}{s(s+1)(s^2+6s+12)} = 0 \text{ char poly}$$

(Note negative sign of K is multiplied with $(s-2)$, so $K>0$)

$$d(s) = s^4+6s^3+12s^2+s^3+6s^2+12s+K(2-s)$$
$$= s^4+7s^3+18s^2+12s+K(2-s)$$
$$= s^4+7s^3+18s^2+(12-K)s+2K = 0$$

Routh Array

s^4	1	18	$2K$
s^3	7	$12-K$	
s^2	$\frac{(114+K)}{7}$	$2K$	
s^1	$\frac{(114+K)(12-K)}{7} - 14K$	0	
s^0	$2K$		

s^1 row needs to zero

$$\frac{(114+k)(12-k)}{7} - 14k = 0$$

$$k^2 + 200k - 1368 = 0$$

$$k = 6.62, -206.62$$

$k = 6.62$ is the valid answer.

Looking at the root locus, we can notice that all poles are in LHS-plane for $0 < K < 6.62$ using negative root locus.

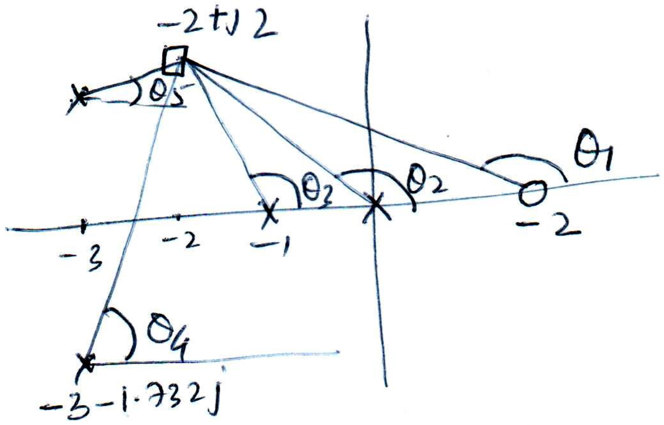
By placing a zero to the left of the pole at $s=0$, two branches of root locus moving towards RHS of root locus can be dragged towards LHS plane can be such that the range of K can be increased.

~~[illegible]~~

A suitable PD compensator can be designed using -ve root locus.

Q1b. Design a stable Controller that would ensure $-2 \pm j2$ are poles of Compensated system.

Soln



Since we are designing the Compensator using 0° root locus, the angle criteria

$$\sum_i \psi_i - \sum_i \theta_i = 0^\circ$$

Contributed by zero

contributed by poles.

$$\theta_1 = 180^\circ - \tan^{-1}\frac{2}{4} = 153.4^\circ$$

$$\theta_2 = 180^\circ - \tan^{-1}\frac{2}{2} = 135^\circ$$

$$\theta_3 = 180^\circ - \tan^{-1}\frac{2}{1} = 116.5^\circ$$

$$\theta_4 = \tan^{-1}\frac{3.732}{1} = 75^\circ$$

$$\theta_5 = \tan^{-1}\frac{2-1.732}{1} = 15^\circ$$

Net angle Contribution

$$= 153.4 - (+135 + 116.5 + 75 + 15^\circ)$$

$$= -188.5$$

To make the net angle contribution to be zero, the compensator must contribute $+188.5^0$

Obviously we have to go for PD or lead compensator.

As it will be tough to find a single lead compensator that can do the job, lets go for double lead compensator.

$$C(s) = K\left(\frac{s+z}{s+p}\right)^2$$

Let the zero be at $s=-1$

Thus double zeros at $s=-1$ would contribute $116.5 \times 2 = 233^0$

This allows double poles to contribute $233 - 188.5 = 44.5^0$

Each pole would contribute $\frac{44.5^0}{2} = 22.25^0$

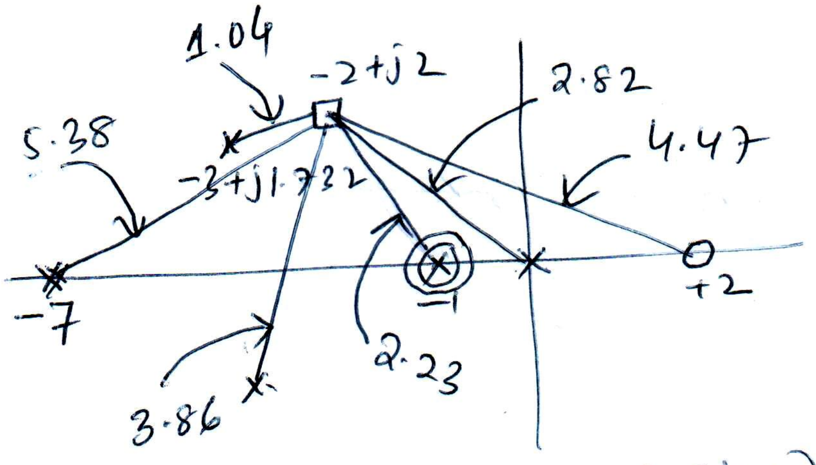
$$\tan 22.25 = \frac{2}{x} = 0.409$$

$$x = \frac{2}{0.4} = 5$$

pole is located at $s=-7$

$$C(s) = K\left(\frac{s+1}{s+7}\right)^2$$

Computation of K



The distance of the desired pole and the pole at $-7 = \sqrt{5^2+2^2}$
$= \sqrt{29} = 5.38$

$$K = \frac{\prod (\text{distance from poles})}{\prod (\text{distance from zeros})}$$

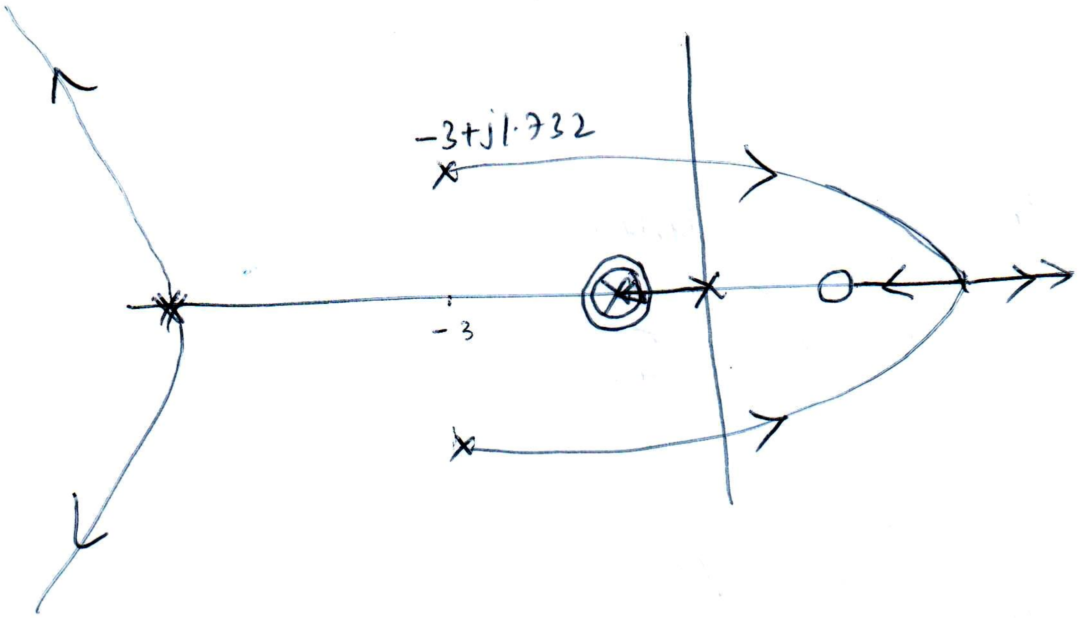
$$= \frac{5.38^2 \times 2.23 \times 2.82 \times 3.86 \times 1.04}{4.47 \times 2.23^2}$$

$= 32.8$

Thus the Compensator form is

$$C(s) = 32.8 \frac{(s+1)^2}{(s+7)^2}$$

Let's see how this compensator reshapes the root locus.



1. There are 6 poles and 3 zeros ⇒ 3 root locus branches conversing to ∞ along asymptotes $0^0, \pm 120^0$

2. The zero at -1 will lock the pole at -1 permanently.

3. pole at $s=0$ would converse to zero at $s=-1$.

4. Two poles at -7 would break away from each other and converse along asymptotes $\pm 120^0$ as the centroid is at $s=-7$ $\left(= \frac{-14-6+1-2}{3}\right)$

5. Two complex conjugate poles would branch towards RHS plane — one branch would converse to zero at $s=2$ & the other conversing to ∞ at along 0^0 asymptote

6. Although $-2 \pm j2$ are closed loop poles of the compensated system, they are not dominant poles.

$$T(s) = \frac{-G(s)C(s)}{1 + C(s)G(s)}$$

$$= \frac{-32.80s^3 + 98.4s + 65.6}{s^6 + 21s^5 + 165s^4 + 574.2s^3 + 1050s^2 + 686.4s + 65.6}$$

Closed loop poles are

$-7.84 \pm j2.82$
$-2 \pm j2$
-1.0
-0.1144

Since the pole -0.1144 is dominant, the system will be very sluggish.

The system gain is fixed at 32.8. The root locus would cross the $j\omega$ axis at gain $K \approx 270$. Compare this to the uncompensated system where $K = 6.62$. Thus the range of K has increased by 45 times.

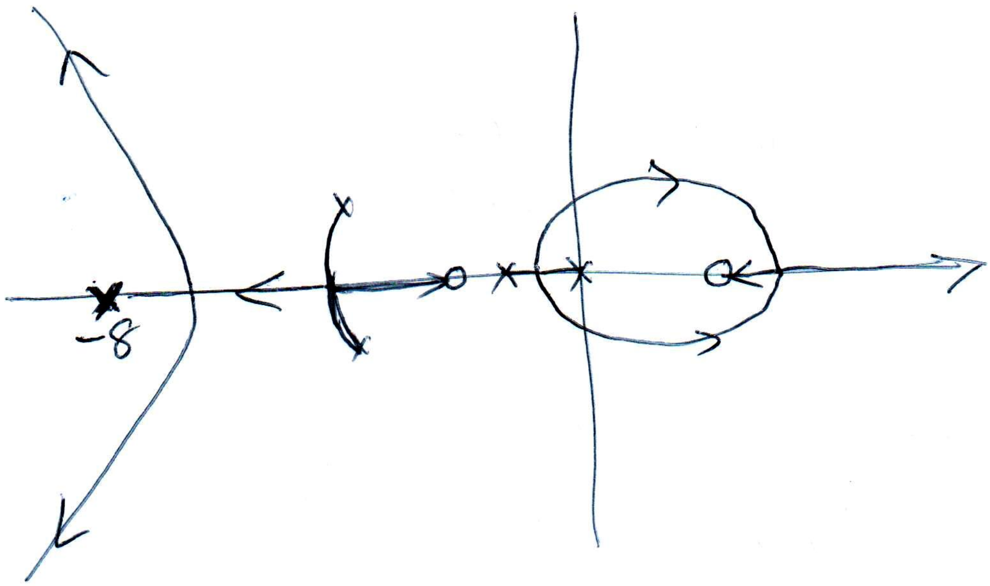
If we want to keep shape of the root locus similar to uncompensated one, one possible design is

$$C(s) = 13.5 \frac{(s+1.5)}{s+8}$$

Closed loop poles are:

$-0.44 \pm j0.548$ ← dominant poles
-2.6697
-4.19
-7.246

The root locus would look as



Home assignment:

Can you design a compensator where the dominant poles are $-0.8 \pm j0.8$?