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Q5)

a) $\sin^3 t = \frac{3}{4}\sin t - \frac{1}{4}\sin 3t$

We know that, $\sin(t+2\pi) = \sin t \Rightarrow T_1 = 2\pi$

and, $\sin 3(t+\frac{2\pi}{3}) = \sin 3t \Rightarrow T_2 = \frac{2\pi}{3}$

So, since $f(t) = \sin^3 t$ is a linear combination of two periodic functions $f_1(t) = \sin t$ and $f_2(t) = \sin 3t$,

therefore, $f(t) = \sin^3 t$ is a periodic function □

b) Now, fundamental period of $f_1(t)$, $T_1 = 2\pi$

fundamental period of $f_2(t)$, $T_2 = 2\pi/3$

So, fundamental period of $f(t)$,

T = LCM of (T_1, T_2)

= LCM of $(2\pi, \frac{2\pi}{3})$

$\Rightarrow \boxed{T = 2\pi}$

c) As we saw, $\sin^3 t = \frac{3}{4}\sin t - \frac{1}{4}\sin 3t \quad (t \in \mathbb{R})$

$\sin^3 t = a_0 + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt \quad (t \in \mathbb{R})$

where, $a_n = a_0 = 0 \quad \forall\, n \in \mathbb{N}$

and $b_n = 0 \quad \forall\, n \in \mathbb{N} \setminus \{1, 3\}$

$b_1 = \frac{3}{4}$, $b_3 = -\frac{1}{4}$

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d) Using Parseval's identity,

$$\frac{1}{\pi}\int_{-\pi}^{\pi}(\sin^3 t)^2\,dt = 2a_0^2 + \sum_{n=1}^{\infty}(a_n^2+b_n^2)$$

$$\Rightarrow \frac{1}{\pi}\int_{-\pi}^{\pi}\sin^6 t\,dt = b_1^2 + b_3^2 = \frac{9}{16}+\frac{1}{16} = \frac{10}{16}$$

$$\Rightarrow \boxed{\int_{-\pi}^{\pi}\sin^6 t\,dt = \frac{5\pi}{8}}$$

Q6) $\begin{cases} 4ty'' + 2y' + \lambda y = 0 \\ y(0) = y(1) = 0 \end{cases}$

a) let $x^2 = t \Rightarrow 2x\,dx = dt$

also, let $y(t) = u(x)$

$$\therefore \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{du}{dx} \Rightarrow u'(x) = y'\cdot 2\sqrt{t} \quad \text{--- (1)}$$

$$\therefore \frac{d}{dt}(y'\cdot 2\sqrt{t})\frac{dt}{dx} = \frac{d^2u}{dx^2} \Rightarrow u''(x) = 2\sqrt{t}\left\{y''2\sqrt{t} + \frac{y'}{\sqrt{t}}\right\}$$

$$= 4y''\cdot t + 2y' \quad \text{--- (2)}$$

So, substituting (1) & (2) in main equation, we get —

$$\begin{cases} u'' + \lambda u = 0 \\ u(0) = u(-1) = u(1) = 0 \end{cases}$$

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b) Now, this is a SLEVP, whose solution has been discussed in the lectures,

$$\boxed{\lambda_n = -n^2\pi^2}$$

$u_n(x) = \sin(n\pi x)$

$\therefore\ y_n(t) = \sin(n\pi\sqrt{t})$

Q7) $\begin{cases} (u-1)^2 u_x + u_y = 1 \\ u(x,0) = 1 \end{cases}$

a) here, let us write the characteristic equations,

$x'(t) = (z-1)^2$; $x(0) = s$

$y'(t) = 1$; $y(0) = 0$

$z'(t) = 1$; $z(0) = 1$

Solving $y'(t) = 1$, we get

$y(t) = t$ ——— (1) $(\because y(0) = 0)$

Solving $z'(t) = 1$, we get

$z(t) = t + 1$ ——— (2) $(\because z(0) = 1)$

$\therefore\ z - 1 = t$

$\therefore\ x'(t) = t^2$

$\Rightarrow x(t) = \frac{t^3}{3} + s$ ——— (3)

from (1), (2) & (3), $z(x,y) = y + 1$

$$\Rightarrow \boxed{u(x,y) = y + 1}$$

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b) The solution $u(x,y) = y+1$ exists in the entire $\mathbb{R}^2$ region.

c) Now, from ① & ③, the projected characteristics that we get is

$$x = \frac{y^3}{3} + s$$

Let us assume that there exists two different projected characteristics which intersect.
Let (x_0, y_0) be such an intersection point of two projected characteristics,

$$x = y^3/3 + s_1 \quad , \quad x = y^3/3 + s_2$$

where $s_1 \neq s_2$

So, Solving them with the point of intersection,

$$x_0 - \frac{y_0^3}{3} = s_1 \quad , \quad x_0 - \frac{y_0^3}{3} = s_2$$

So, $s_1 = s_2$; which is a contradiction

So, there DOES NOT exist two different projected characteristics which intersect.

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d) let the arbitrary point in $\mathbb{R}^2$ be (p, q)

then let us take $s_0 = p - \frac{q^3}{3}$

then the projected characteristics

$$x - \frac{y^3}{3} = p - \frac{q^3}{3}$$

passes through the point (p, q) for any arbitrary p and q.

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Q2) Let us assume that there exists a harmonic function f on $\mathbb{R}^2$ such that,

$f(0,0)=0$ and $f(x,y)>0 \quad \forall (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$

~~Now, let us take a region in $\mathbb{R}^2$ with~~

Now, let us consider a region Ω in $\mathbb{R}^2$ s.t. $(0,0) \in \Omega$ and $(0,0) \notin \partial\Omega$

Now, applying the Maximum Principle

$$\min_{\bar{\Omega}} f(x,y) = \min_{\partial\Omega} f(x,y)$$

where $\bar{\Omega} = \Omega \cup \partial\Omega$

Now, we know that

$$\min_{\bar{\Omega}} f(x,y) = 0 \qquad (\because f(0,0)=0)$$

$$\therefore \min_{\partial\Omega} f(x,y) = 0$$

but $(0,0) \notin \partial\Omega$

hence $f(x,y) > 0 \quad \forall (x,y) \in \partial\Omega$

which is a contradiction.

So, there CANNOT exist any harmonic function f on $\mathbb{R}^2$ such that $f(0,0)=0$ and $f(x,y)>0 \quad \forall (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$