3\. a. $X_1, \ldots, X_n \overset{iid}{\sim} \text{unif}(\theta - a, \theta + a)$

i). $L(\theta, \alpha) = \prod_{i=1}^{n} f(X_i)$

$$= \frac{1}{(2a)^n}, \quad \theta - a \leq X_{(1)} < X_{(n)} \leq \theta + a.$$

free of $\theta$!

Now,

$$X_{(n)} - a \leq \theta \leq X_{(1)} + a.$$

given $a$, $L$ is maximized if

$$\theta \in [X_{(n)} - a, X_{(1)} + a].$$

In particular, let us consider

$$\hat{\theta}_{MLE} = \frac{X_{(n)} - a + X_{(1)} + a}{2}$$

$$= \frac{X_{(1)} + X_{(n)}}{2}. \quad \boxed{1 \text{ mark}}$$

Note that $L$ is $\downarrow$ in $a$.

So, $L$ is maximized if $a$ is minimized. Note that

$$\underline{\theta - X_{(1)} \leq a \leq X_{(n)} - \theta}$$

$$\Rightarrow \hat{a}_{MLE} = \hat{\theta}_{MLE} - X_{(1)}$$

$$= \frac{X_{(n)} - X_{(1)}}{2}. \quad \boxed{1 \text{ mark}}$$

ii) $$E[X_{(1)}] = n\int_{\theta-a}^{\theta+a} x \cdot \frac{1}{2a} \cdot \left[1 - \frac{x-\theta+a}{2a}\right]^{n-1} dx$$

$$z = \frac{x-\theta+a}{2a}, \quad dz = \frac{dx}{2a}$$

$$= n\int_0^1 (2az+\theta-a)\cdot(1-z)^{n-1}\, dz$$

$$= 2an\int_0^1 z(1-z)^{n-1} dz +$$

$$(\theta - a)\, n \int_0^1 (1-z)^{n-1}\, dz$$

$$= 2an\,\mathrm{Beta}(2,n) + \frac{(\theta - a)}{\cancel{n}} \cdot \cancel{n}$$

$$= 2an \cdot \frac{1!\,(n-1)!}{(n+1)!} + (\theta - a)$$

$\boxed{\text{1 mark}}$

$$= \frac{2a}{(n+1)} + (\theta - a).$$

$$E[X_{(n)}] = n \int_{\theta - a}^{\theta + a} x \cdot \frac{1}{2a} \cdot \left(\frac{x - \theta + a}{2a}\right)^{n-1} dx$$

$$z = \frac{x - \theta + a}{2a}, \quad dz = \frac{dx}{2a}$$

$$= n \int_0^1 (2az + \theta - a) \cdot z^{n-1}\, dz$$

$$= 2an \int_0^1 z^n\, dz + (\theta - a)\, n \int_0^1 z^{n-1}\, dz$$

$$= \frac{2an}{(n+1)} + (\theta - a)$$

$$= 2a - \frac{2a}{(n+1)} + (\theta - a)$$

1 mark

$$= (\theta + a) - \frac{2a}{n+1}$$

Now,

$$E[\hat{\theta}_{MLE}] = E\left[\frac{X_{(1)} + X_{(n)}}{2}\right]$$

$$= \frac{1}{2}\left[\frac{2\cancel{a}}{\cancel{n}+1} + \theta - \cancel{a} + \theta + \cancel{a} - \frac{2\cancel{a}}{\cancel{n}+1}\right]$$

$$= \theta \text{ (Yes)}$$

1 mark

$$E[\hat{a}_{MLE}] = E\left[\frac{X_{(n)} - X_{(1)}}{2}\right]$$

$$= \frac{1}{2}\left[(\cancel{\theta} + a) - \frac{2a}{n+1} - \frac{2a}{n+1} - \cancel{\theta} + a\right]$$

$$= \frac{1}{2}\left[2a - \frac{4a}{n+1}\right] = a - \frac{2a}{n+1}$$

$$= \frac{(n-1)}{(n+1)} \cdot a$$

$$\neq a \quad (\text{No})$$

$\boxed{\text{1 mark}}$

3. b. Using the NP lemma, we get if $\prod_{i=1}^{n} \frac{2X_i}{1} > c$, then we reject $H_0$.

Critical region: $\prod_{i=1}^{n} X_i > c/2^n$. $\boxed{1 \text{ mark}}$

$$P_{H_0}\left(\prod_{i=1}^{n} X_i > c_1\right)$$

$$= P_{H_0}\left(-2\sum_{i=1}^{n} \log X_i < -2\log c_1\right)$$

Under $H_0$, $X \sim U(0,1)$ $\boxed{1 \text{ mark}}$

$$-2\log X \sim \chi_2^2$$

$$\Rightarrow -2\sum_{i=1}^{n} \log X_i \sim \chi_{2n}^2 \text{ (Additivity of } \chi^2)$$

Now, $P(\chi_{2n}^2 < K) = \alpha$

$\Rightarrow K = \chi_{2n}^2(1-\alpha)$ $\boxed{1 \text{ mark}}$

For $\alpha = 0.10$ and $n = 10$, we get $k = \chi^2_{20}(0.90) = 12.443$.

1 mark