MSO-203 B ASSIGNMENT 2

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10th October , 2020

1. Let P, Q and R are differentiable functions. Consider the following eigenvalue problem for general linear second order equation:

$$P(t)y^{''}(t) + Q(t)y^{'}(t) + R(t)y(t) + \lambda y(t) = 0. \tag{1}$$

Show that the above problem can be reduced to following self adjoint form

$$\big(p(t)y'(t)\big)^{'} + s(t)y(t) + \lambda r(t)y(t) = 0, \tag{2}$$

with the functions

$$p = Pe^{\int \frac{Q-P'}{P}}, s = Re^{\int \frac{Q-P'}{P}}, \text{ and } r = e^{\int \frac{Q-P'}{P}}.$$

HINT: Multiply the first equation by a positive function μ and then equate the two equations to get the expression for the unknown function μ.

Solution. Given that

$$P(t)y^{''}(t) + Q(t)y^{'}(t) + R(t)y(t) + \lambda y(t) = 0. \tag{3}$$

Assume that (3) can be reduced to the form (2). Then from (2) we get

$$p(t)y^{''}(t) + y'(t)p'(t) + s(t)y(t) + \lambda r(t)y(t) = 0. \tag{4}$$

Multiplying (3) with an unknown function "$\mu(t)$" and then equating the coefficients with (4) we get,

[Note: We can not compare directly (3) and (4). Why?]

$$\mu(t)P(t) = p(t) \tag{5}$$
$$\mu(t)Q(t) = p'(t) \tag{6}$$
$$\mu(t)R(t) = s(t) \tag{7}$$
$$\mu(t) = r(t) \tag{8}$$

Differentiating (5) we get

$$\mu'(t)P(t) + \mu(t)P'(t) = p'(t). \tag{9}$$

Comparing (9) with (6), we get

$$\begin{aligned} &\mu'(t)P(t) + \mu(t)P'(t) = \mu(t)Q(t) \\ \iff &P(t)\mu'(t) = \mu(t)(Q(t) - P'(t)) \\ \implies &\frac{d\mu}{\mu} = \frac{Q(t) - P'(t)}{P(t)}dt \\ \implies &\mu(t) = e^{\int \frac{Q-P'}{P}} \end{aligned}$$

Substituting this in (5), (7) and (8) we get the result.

2. There can be more than one correct choices, for multiple type questions. Consider the following eigenvalue problem:

$$\begin{cases} y^{''}(t) + \lambda y(t) = 0 \quad \textbf{on } (0, \pi) \\ y(0) = y(\pi) = 0. \end{cases}$$

Then,

a) $\lambda = 0$ is an eigenvalue of the above problem.

b) Then there exist an eigenvalue which is strictly negative.

c) The function $\sin(2t)$ is an eigenfunction.

d) There are infinitely many eigenvalues.

Solution (a). It is not correct because if $\lambda = 0$ is an eigenvalue, this means, there exists a non-zero eigenfunction (say ψ) such that

$$\psi^{''}(t) = 0, \ \psi(0) = \psi(\pi) = 0.$$

Solving it one obtains

$$\psi \equiv 0,$$

which contradicts that ψ is non-zero function.

Solution (b). It is not true because, if suppose there exists $\lambda_0 < 0$ which is an eigenvalue of the given problem. Also suppose that "ψ_0" is the eigenfunction corresponding to the eigenvalue λ_0. Then from the equation we get that

$$\begin{aligned} & \psi_0^{''} + \lambda_0 \psi_0 = 0, \quad \psi_0(0) = \psi_0(\pi) = 0 \\ \Longrightarrow \ & \psi_0^{''} \psi_0 + \lambda_0 \psi_0^2 = 0 \\ \Longrightarrow \ & \int_0^\pi \psi_0^{''} \psi_0 + \lambda_0 \int_0^\pi \psi_0^2 = 0 \\ \Longrightarrow \ & -\int_0^\pi (\psi_0')^2 + \lambda_0 \int_0^\pi \psi_0^2 = 0. \end{aligned} \tag{10}$$

In the last step we have applied the boundary conditions and integration by parts formula. Since $\lambda_0 < 0$, assume $\lambda_0 = -a^2$, $a \neq 0$. From (10) we get

$$\int_0^\pi (\psi_0')^2 + a^2 \int_0^\pi \psi_0^2 = 0. \tag{11}$$

Note that since both the integrals are greater than equal to zero (can never be strictly negative). The only possible way for (11) to be true is that both the integral to be zero. In particular

$$\int_0^\pi \psi_0^2 = 0.$$

Since ψ_0 is continuous function, this implies $\psi_0 \equiv 0 \ \text{ on } \ (0, \pi)$. (Why? Exercise) This is a contradiction.

Solution (c). It is true, can be seen directly by putting the function in the equation and boundary conditions. (Or you can solve to get it but that would not be a smart idea.)

Solution (d). It is true by explicit solving or using the theory of SL problems stated in class.

In fact the entire problem could have been answered using the theory stated in lecture notes. I have presented the proof of part (a) and (b) to give an idea, how the proof of the theorem (Theorem for RSLVP problem) would go.

3. Reduce the following eigenvalue problem to SLEVP form:

a) $ty^{''}(t) + 2y^{'}(t) + \lambda y(t) = 0.$
b) $y^{''}(t) + y^{'}(t) + (\lambda + 1)y(t) = 0.$

Solution (a). We want to use our previous problem. Here

$$P(t) = t, \quad Q(t) = 2, \quad R(t) = 0.$$

In the self-adjoint form

$$\begin{aligned} p(t) = P(t)e^{\int \frac{Q(t)-P'(t)}{P(t)}}dt &= te^{\int \frac{(2-1)}{t}dt} \\ &= te^{\log t} \\ &= t^2, \\ s(t) = 0, \quad r(t) = t. \end{aligned}$$

It's self-adjoint form looks like

$$(t^2 y'(t))' + \lambda t y(t) = 0.$$

Solution (b). Here

$$P(t) = 1, \quad Q(t) = 1, \quad R(t) = 1.$$

Therefore using Problem 1 once again, we get

$$\begin{aligned} p(t) &= e^{\int (1-0)dt} = e^t, \\ s(t) &= e^t, \quad r(t) = e^t. \end{aligned}$$

Hence the given equation in self-adjoint form looks like

$$(e^t y'(t))' + e^t y(t) + \lambda e^t y(t) = 0.$$

4. Find the solution of the following eigenvalue problem:

a) $y^{''}(t) + \lambda y(t) = 0$ on $(0, L)$, $y(0) = y(L) = 0$, for fixed $L > 0$.
b) $y^{''}(t) + \lambda y(t) = 0$ on $(0, 1)$, $y(0) = y(1)$, $y'(0) = y'(1)$.
c) $\left(\frac{y'(t)}{t}\right)^{'} + (\lambda + 1)\frac{y(t)}{t^3} = 0$ on $(1, e^\pi)$, $y(1) = 0$, $y(e^\pi) = 0$.
d) $y^{''}(t) + 8y'(t) + (\lambda + 16)y(t) = 0$ on $(0, \pi)$, $y(0) = 0 = y(\pi)$.
e) $y^{''}(t) + \lambda y(t) = 0$ on $(0, 1)$, $y(0) = y'(1) = 0$.

Solution(a). Given eigenvalue problem is

$$\begin{aligned} y^{''}(t) + \lambda y(t) = 0 \quad &\text{on} \quad (0, L), \\ y(0) = y(L) = 0. \end{aligned} \tag{12}$$

The characteristic equation is given by

$$m^2 + \lambda = 0. \tag{13}$$

(Case 1.) Let us first consider the case when $\lambda = 0$. Then

$$m = 0 \implies y(t) = A + Bt$$

for some constant $A, B \in \mathbb{R}$. Applying the boundary condition we get

$$\begin{aligned} & y(0) = 0 \implies A = 0, \\ & y(L) = 0 \implies BL = 0 \implies B = 0 \ \text{(since } L \neq 0), \\ \implies & y(t) \equiv 0. \end{aligned}$$

This contradicts the definition of eigenfunction. Recall that eigenfunction by definition has to be a non-zero function. Therefore $\lambda = 0$ can never be an eigenvalue for the above problem.

(Case 2.) Let $\lambda < 0$. Assume that $\lambda = -k^2$ for some $k \neq 0$. Putting this in (13) we get

$$\begin{aligned} & m = \pm k \\ \implies & y(t) = Ae^{kt} + Be^{-kt} \end{aligned}$$

is the expression for any general solution of (12) in this case. Incorporating the Boundary conditions again, we get

$$\begin{aligned} y(0) = 0 &\implies A + B = 0, \\ y(L) = 0 &\implies Ae^{kL} + Be^{-kL} = 0 \\ &\implies A(e^{kL} - e^{-kL}) = 0. \end{aligned}$$

If $A \neq 0$ then

$$e^{2kL} = 1 = e^0 \implies k = 0 \ \ \text{(since } \ L \neq 0). \tag{14}$$

This is a contradiction to the assumption that $k \neq 0$. Hence $A = 0$ and $B = -A = 0$. Like previous case $y(t) \equiv 0$ becomes the only available choice of eigenfunction, which cannot be the case.

(case 3) Let $\lambda > 0$. Assume that $\lambda = k^2, k \neq 0$. In this case the general solution of (12) is given by

$$y(t) = A\cos(kt) + B\sin(kt). \tag{15}$$

Boundary condition gives

$$\begin{aligned} & y(0) = 0 \implies A + 0 = 0 \implies A = 0, \\ \implies & y(t) = B\sin(kt), \\ & y(L) = 0 \implies B\sin(kL) = 0. \end{aligned}$$

Since we want to avoid $B = 0$, this implies

$$\begin{aligned} & \sin(kL) = 0 \\ \implies & k = \frac{m\pi}{L} \ (m \in \mathbb{Z} \setminus \{0\}), \end{aligned}$$

where $\mathbb{Z}$ is the set of integers. There are infinitely many choices for such k.

The set of eigenvalues are

$$\lambda_m = \frac{m^2\pi^2}{L^2}, \ m \in \mathbb{N} \setminus \{0\}.$$

[Note that in the last line we have changed $m \in \mathbb{N}$ from $m \in \mathbb{Z}$ because same eigenvalue will be counted (eg. λ_1, λ_{-1} are same).]

The eigenfunctions "$y_m(t)$" corresponding to the eigenvalue "λ_m" is given by

$$y_m(t) = B\sin\left(\frac{m\pi}{L}t\right), \ m \in \{1, 2, 3, \ldots\}, \ B \neq 0.$$

Solution (b). Given eigenvalue problem is

$$\begin{aligned} y''(t) + \lambda y(t) = 0 \quad \text{on} \;\; (0,1), \\ y(0) = y(1), y'(0) = y'(1). \end{aligned} \tag{16}$$

Like the previous problem the characteristic equation is given by

$$m^2 + \lambda = 0. \tag{17}$$

(Case 1.) Let us first consider the case when $\lambda = 0$. Then

$$y(t) = A + Bt$$

for some constant $A, B \in \mathbb{R}$. Applying the boundary conditions we get

$$\begin{aligned} y(0) = y(1) \implies A = A + B \implies B = 0, \\ \implies y(t) \equiv A \text{ (constant function)}, \end{aligned}$$

satisfies the second boundary condition. Hence $\lambda = 0$ is an eigenvalue with non-zero constant function as the eigenfunction.

(Case 2.) Let $\lambda < 0$. Then $\lambda = -k^2$ for some $k \neq 0$. The general solution is

$$y(t) = Ae^{kt} + Be^{-kt}.$$

The Boundary conditions gives

$$\begin{aligned} y(0) = y(1) &\implies A + B = Ae^k + Be^{-k}, \\ &\implies A(1 - e^k) + B(1 - e^{-k}) = 0. \end{aligned} \tag{18}$$

Second boundary condition gives

$$\begin{aligned} &y'(0) = y'(1) \\ &\implies k(A - B) = k(Ae^k - Be^{-k}) \\ &\implies A - B = Ae^k - Be^{-k} \quad (\text{since } k \neq 0) \\ &\implies A(1 - e^k) + B(e^{-k} - 1) = 0 \end{aligned} \tag{19}$$

Equation (18) and (19) can be written as

$$\begin{pmatrix} 1 - e^k & 1 - e^{-k} \\ 1 - e^k & e^{-k} - 1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{20}$$

Let us calculate,

$$\begin{aligned} \det \begin{pmatrix} 1 - e^k & 1 - e^{-k} \\ 1 - e^k & e^{-k} - 1 \end{pmatrix} &= (1 - e^k)(1 - e^{-k}) \det \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ &= -2(1 - e^k)(1 - e^{-k}) \neq 0 \quad (\text{as } k \neq 0). \end{aligned}$$

Therefore from (20) we have

$$A = B = 0.$$

Therefore $\lambda < 0$ cannot be an eigenvalue.

(case 3) Let $\lambda > 0$. Assume that $\lambda = k^2, k \neq 0$. In this case the general solution is given by

$$y(t) = A\sin(kt) + B\cos(kt). \tag{21}$$

Boundary condition gives

$$y(0) = y(1) \implies B = A\sin k + B\cos k, \tag{22}$$

and, as $k \neq 0$,

$$\begin{aligned} & y'(0) = y'(1) \\ \implies & A\cos(k0) - B\sin(k0) = A\cos k - B\sin k \\ \implies & A = A\cos k - B\sin k. \end{aligned} \tag{23}$$

(22) and (23) together in form of system of equation can be written as

$$\begin{pmatrix} \sin k & \cos k - 1 \\ \cos k - 1 & -\sin k \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{24}$$

$$\det \begin{pmatrix} \sin k & \cos k - 1 \\ \cos k - 1 & -\sin k \end{pmatrix} = -\sin^2 k - (\cos k - 1)^2 = -2 + 2\cos k. \tag{25}$$

We will get non-trivial solutions of A, B only if (25) becomes 0, that is,

$$\begin{aligned} & \cos k = 1 \\ \implies & k = \pm 2n\pi, \;\; n \in \mathbb{N} \setminus \{0\} \\ \implies & \lambda_n = 4n^2\pi^2. \end{aligned}$$

The eigenfunction corresponding to the eigenvalue λ_n is given by

$$\begin{aligned} & y_n(t) = A\cos(2n\pi t), A \neq 0 \\ & \psi_n(t) = B\sin(2n\pi t), B \neq 0, n \in \mathbb{N}. \end{aligned}$$

This means that there are two linearly independent eigenfunctions "y_n" and "ψ_n" corresponding to the eigenvalue λ_n.

Solution (c). For this problem we want to change the independent variable first to reduce the problem into a constant coefficient problem (linear) for which we know how to find the general solution.

$$\begin{aligned} & \text{Put} \;\; \log t = z, \;\; \text{and} \\ & \text{define} \;\; \psi(z) = y(t), \end{aligned}$$

where $y(t)$ satisfies $3c$ (the given problem).

$$\begin{aligned} \frac{dy}{dt} &= \frac{d\psi}{dz}\frac{dz}{dt} = \frac{1}{t}\frac{d\psi}{dz} \\ \frac{d^2y}{dt^2} &= -\frac{1}{t^2}\frac{d\psi}{dz} + \frac{1}{t^2}\frac{d^2\psi}{dz^2}. \end{aligned}$$

Using this we get

$$\begin{aligned} & \left(\frac{y'(t)}{t}\right)' + \left(\frac{\lambda+1}{t^3}\right)y(t) = 0 \\ \implies & -\frac{y'(t)}{t^2} + \frac{y''(t)}{t} + (\lambda+1)\frac{y(t)}{t^3} = 0 \\ \implies & -2\frac{d\psi}{dz}\frac{1}{t^3} + \frac{1}{t^3}\frac{d^2\psi}{dz^2} + (\lambda+1)\frac{\psi(z)}{t^3} = 0 \\ \implies & \frac{d^2\psi}{dz^2} - 2\frac{d\psi}{dz} + (\lambda+1)\psi(z) = 0 \;\; (\text{since} \;\; t > 0), \end{aligned} \tag{26}$$

and the initial condition changes as

$$\begin{aligned} \text{at} \quad t = 1, &\quad z = 0 \\ t = e^{\pi}, &\quad z = \pi \\ \implies \psi(0) = \psi(\pi) = 0. \end{aligned}$$

Look that we know the general solution of (26) . Characteristic equation is given by

$$\begin{aligned} (m-1)^2 + \lambda = 0 &\iff m = 1 \pm i\sqrt{\lambda} \\ \implies \text{“}e^z \cos\sqrt{\lambda}z\text{”} &\quad \text{and} \quad \text{“}e^z \sin\sqrt{\lambda}z\text{”} \end{aligned}$$

are two linearly independent solution of (26). Thus the general solution is **for the case $\lambda > 0$,**

$$\psi(z) = Ae^z \cos(\sqrt{\lambda}z) + Be^z \sin(\sqrt{\lambda}z).$$

The initial conditions are

$$\psi(0) = \psi(\pi) = 0.$$

Now

$$\begin{aligned} \psi(0) = 0 &\implies A = 0 \\ \psi(\pi) = 0 &\implies Be^{\pi} \sin(\sqrt{\lambda}\pi) = 0. \end{aligned}$$

Since $e^{\pi} \neq 0$ and we want $B \neq 0$,

$$\begin{aligned} &\implies \sin(\sqrt{\lambda}\pi) = 0 = \sin(n\pi), \ n \in \mathbb{Z} \\ &\implies \lambda = n^2, \ n \in \mathbb{N} \setminus \{0\}. \end{aligned}$$

Since for distinct values of natural numbers one obtains different eigenvalue, so it is more appropriate to index the set of eigenfunctions as ψ_n for each $n \in \mathbb{N}$,

$$\psi_n(z) = Be^z \sin(nz) \quad \text{for some constant} \quad B \in \mathbb{R} \setminus \{0\}.$$

Now going back to our original coordinate system “t” by using the relation $\log t = z$, we get

$$\begin{aligned} y_n(t) = \psi_n(z) = Be^z \sin(nz) = Bt \sin(n \log t) \\ \implies y_n(t) = Bt \sin(n \log t), \end{aligned}$$

are the countable sequence of eigenfunctions for the above problem.

What happens in the case $\lambda \leq 0$? Do it yourself.

SOLUTION OF PART 3(d) and 3(e) are kept as exercise for you.

5. It is given to you that the following is an orthogonal family of functions on the interval $(-\pi, \pi)$:

$$\Big\{ \sin(nx), \cos(nx) \Big\}_{n \in \mathbb{N}}.$$

Using this fact prove that the following family of functions are also orthogonal on the interval $(-1, 1)$:

$$\Big\{ \sin(n\pi x), \cos(n\pi x) \Big\}_{n \in \mathbb{N}}.$$

Solution: Do it yourself by appropriately changing the variable and the using the given information in the question.

6. Consider the following problem

$$y''(t) + (e^{t^2} + 1)y(t) = 0 \quad \text{in } \mathbb{R}.$$

Let y_1 denotes a non-trivial solution (not identically zero function) of the above problem. Then show y_1 has infinitely many zeros, that is, y_1 vanishes at infinitely many points in $\mathbb{R}$.

Solution: Let $p(t) = e^{t^2} + 1$ and $q(t) = 1$ in the following

$$y''(t) + (e^{t^2} + 1)y(t) = 0 \quad \text{in } \mathbb{R} \tag{27}$$

also consider

$$y''(t) + 1y(t) = 0 \quad \text{on } \mathbb{R}. \tag{28}$$

Notice $e^{t^2} + 1 = p(t) > 1 = q(t)$. Now we want to apply Strum Comparison theorem $\widetilde{y(t)} = \sin(t)$ solves (28) which vanishes (zeros) at the point $t = n\pi$, $n \in \mathbb{Z}$.

Between two consecutive zeros of $\widetilde{y}$, lies a zero of the solution of (27) and hence infinitely in number.

7. Show that the following family of functions

$$\Big\{ \sin(n\pi \log(t)) \Big\}_{n\in\mathbb{N}},$$

are orthogonal. The domain of definition for each function in the above family is assumed to be $(1, e)$. Are they orthonormal ? If not, can it be turned in to a orthonormal family ?

Hint: Try finding eigenvalues and eigenfunctions for the following problem:

$$t^2y''(t) + ty'(t) + \lambda y = 0, \quad \text{on } (1, e), \quad y(1) = y(e) = 0.$$

Further Hint: Change the independent variable $t = \log(x)$ *to reduce the above problem to a more known problem.*

Solution: Consider the problem

$$t^2y''(t) + ty'(t) + \lambda y(t) = 0, \quad \text{on } (1, e), \quad y(1) = y(e) = 0 \tag{29}$$

Put $x = \log(t)$ and define $\psi(x) = y(t)$. Then

$$y'(t) = \frac{d\psi}{dx}\frac{dx}{dt} = \frac{1}{t}\frac{d\psi}{dx}$$

$$y''(t) = \frac{1}{t}\frac{d^2\psi}{dx^2}\left(\frac{1}{t}\right) + \frac{d\psi}{dx}\left(\frac{-1}{t^2}\right)$$

Putting the values in (29) we get

$$\begin{aligned} 0 &= t^2y''(t) + ty'(t) + \lambda y(t) \\ &= \psi_{xx} - \psi_x + \psi_x + \lambda\psi(x) \\ &= \psi_{xx} + \lambda\psi(x) \end{aligned}$$

The boundary condition changes as $x = \log(t)$ at $t = 1 \implies x = 0$ and at $t = e \implies x = 1$. Therefore the change problem in x co-ordinate becomes as

$$\psi_{xx} + \lambda\psi(x) = 0 \quad \text{on } (0, 1) \text{ and } \psi(0) = \psi(1) = 0.$$

For this problem it is well known that EV and EF are

$$\psi_n(x) = \sin(n\pi x), \quad n \in \mathbb{N}.$$

Going back to t co-ordinate we get

$$y_n(t) = \psi_n(x) = \sin(n\pi x) = \sin(n\pi \log t), \quad n \in \mathbb{N}.$$

are EF of (29).

Now since (29) RSLVP, we know that the set of EF are orthogonal. This finishes the proof.

For the second part let us calculate

$$\begin{aligned}
\int_1^e y_n^2(t)dt &= \frac{1}{2}\int_1^e 2\sin^2(n\pi \log t)dt \\
&= \frac{1}{2}\int_1^e 1 - \cos(2n\pi \log t)dt
\end{aligned}$$

$$\int_1^e y_n^2(t)dt = \frac{e-1}{2} - \frac{1}{2}\int_1^e \cos(2n\pi \log t)dt \tag{30}$$

Let

$$\begin{aligned}
I_n &= \int_1^e \cos(2n\pi \log t)dt \\
&= \cos(2n\pi \log t)t|_1^e + \int_1^e \frac{2n\pi t}{t}\sin(2n\pi \log t)dt \\
&= (e-1) + 2n\pi\left[t\sin(2n\pi \log t)|_1^e - \int_1^e \frac{2n\pi}{t}\cos(2n\pi \log t)tdt\right] \\
&= (e-1) - 4n^2\pi^2 I_n \\
\implies I_n &= \frac{e-1}{1+4n^2\pi^2}.
\end{aligned}$$

Therefore from (30) we get

$$\begin{aligned}
\int_1^e y_n^2(t)dt &= \frac{e-1}{2} - \frac{1}{2}\frac{e-1}{1+4n^2\pi^2} \\
&= \frac{e-1}{2}\left[1 - \frac{1}{1+4n^2\pi^2}\right] \\
&= \frac{e-1}{2}\frac{4n^2\pi^2}{1+4n^2\pi^2} \neq 1
\end{aligned}$$

Hence they are not orthonormal. Now

$$\int_1^e \left(\sqrt{\frac{2}{e-1}\left(\frac{1+4n^2\pi^2}{4n^2\pi^2}\right)}y_n(t)\right)^2 dt = 1.$$

Thus the functions $\left(\sqrt{\frac{2}{e-1}\left(\frac{1+4n^2\pi^2}{4n^2\pi^2}\right)}y_n(t)\right)_{n\in\mathbb{N}}$ are orthonormal.

8. Consider the following RSLP problem:

$$\big(p(t)y'(t)\big)' + q(t)y(t) + \lambda\sigma(t)y(t) = 0 \quad \text{on } (a,b). \tag{31}$$

Let (μ, ϕ) be any eigenpair for the above problem (Notice that we are not specifying any boundary conditions). Then prove that

$$\mu = \frac{-p\phi\phi'\big|_a^b + \int_a^b \left[p(\phi')^2 - q\phi^2\right]}{\int_a^b \sigma\phi^2}.$$

Now consider $q = 0$ function in (31) , and with the boundary condition

$$y(a) = 0, \; y'(b) = 0.$$

Then prove that any eigenvalue of this problem is strictly positive.

Solution: Consider the problem

$$\big(p(t)y'(t)\big)' + q(t)y(t) + \lambda\sigma(t)y(t) = 0 \quad \text{on} \quad (a, b).$$

Given (μ, ϕ) is the eigenpair of the above problem. This means

$$\big(p(t)\phi'(t)\big)' + q(t)\phi(t) + \mu\sigma(t)\phi(t) = 0 \quad \text{on} \quad (a, b).$$

Multiplying the above equation by $\phi(t)$ and integrating by parts, we get

$$\int_a^b \big(p(t)\phi'(t)\big)' \phi(t)dt + \int_a^b q(t)\phi^2(t)dt + \mu\int_a^b \sigma(t)\phi^2(t)dt = 0.$$

$$\implies -\int_a^b p(t)(\phi'(t))^2 dt + p(t)\phi(t)\phi'(t)\big|_a^b + \int_a^b q(t)\phi^2(t)dt + \mu\int_a^b \sigma(t)\phi^2(t)dt = 0.$$

Changing the sides gives the required answer.

For second part, put $q = 0$ in (31). Let (μ, ϕ) be any arbitrary eigenpair. Then from the above formula after using the boundary condition we get

$$\mu = \frac{\int_a^b p(t)(\phi'(t))^2 dt}{\int_a^b \sigma(t)\phi^2(t)dt}.$$

Now since we are working with RSLVP, we know that $p(t) > 0$, $\sigma(t) > 0$. Therefore the above two integrals are strictly positive and hence $\mu > 0$.