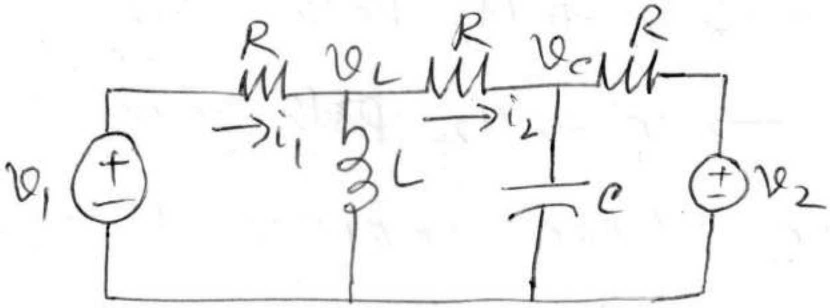
EE250 Mid Sem Solution Sheet

Q1



i. $I_1 = \dfrac{V_1 - V_L}{R}$ — ① (Laplace domain) ($\frac{1}{2}$ marks)

$I_2 = \dfrac{V_L - V_C}{R} = \dfrac{V_L - V_C}{R}$ — ② ($\frac{1}{2}$ marks)

$L\left(\dfrac{di_1}{dt} - \dfrac{di_2}{dt}\right) = v_L$

$V_L = Ls(I_1 - I_2)$ — (3) (1 mark)

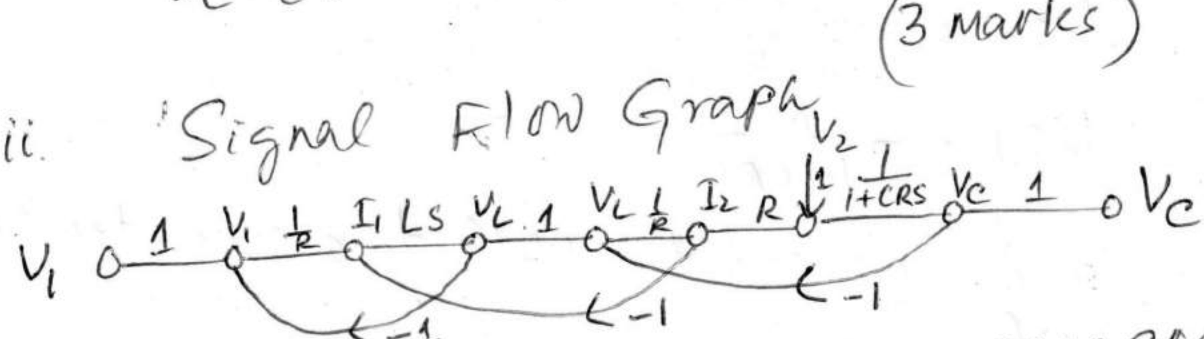
$v_c - \left(i_2 - C\dfrac{dv_c}{dt}\right)R = v_2$

$V_C - (I_2 - CsV_C)R = V_2$ (1 mark)

$V_C(1 + CRs) = I_2R + V_2$ — (4)

(3 marks)

ii. Signal Flow Graph



One can check that all four eqns are accounted in this SFG.

There are two feed forward paths

1. $V_1 \longrightarrow V_C \to M_1$ Path gain
2. $V_2 \longrightarrow V_C \to M_2$ path gain

There are three individual loops

There is one two non touching loops. [5 marks]

iii Forward path gains

$$M_1 = \frac{1}{R} \cdot LS \cdot \frac{1}{R} \cdot R \cdot \frac{1}{1+CRS}$$

$$= \frac{S}{S+1} \quad \text{as } R = L = C = 1 \text{ unit}$$

$$M_2 = \frac{1}{1+CRS} = \frac{1}{1+CRS} = \frac{1}{S+1}$$

Individual loop gain

$$L_{11} = -\frac{LS}{R}$$

$$L_{21} = -\frac{LS}{R}$$

$$L_{31} = -\frac{1}{1+CRS}$$

Gains associated with two non-touching loops

$$L_{12} = L_{11} \times L_{31}$$
$$= \frac{LS}{R(1+CRS)}$$

$$\Delta = 1 + \frac{Ls}{R} + \frac{Ls}{R} + \frac{1}{1+CRs} + \frac{Ls}{R(1+CRs)}$$

$$= 1 + 2s + \frac{1}{s+1} + \frac{s}{s+1}$$

$$= \frac{(1+2s)(s+1) + (s+1)}{(s+1)} = 1 + 2s + 1$$

$$= 2(s+1)$$

$$\left.\frac{V_C(s)}{V_1(s)}\right|_{V_2=0} = \frac{\left(\frac{s}{s+1}\right)\cdot 1}{2(s+1)} = \frac{s}{2(s+1)^2}$$

$$\left.\frac{V_C(s)}{V_2(s)}\right|_{V_1=0} = \frac{\left(\frac{1}{s+1}\right)\left(1 + \frac{Ls}{R} + \frac{Ls}{R}\right)}{\Delta}$$

$$= \frac{\frac{1}{s+1}\,(1+2s)}{2(s+1)}$$

$$= \frac{2s+1}{2(s+1)^2} = \frac{s+\frac{1}{2}}{(s+1)^2}$$

(7 marks)

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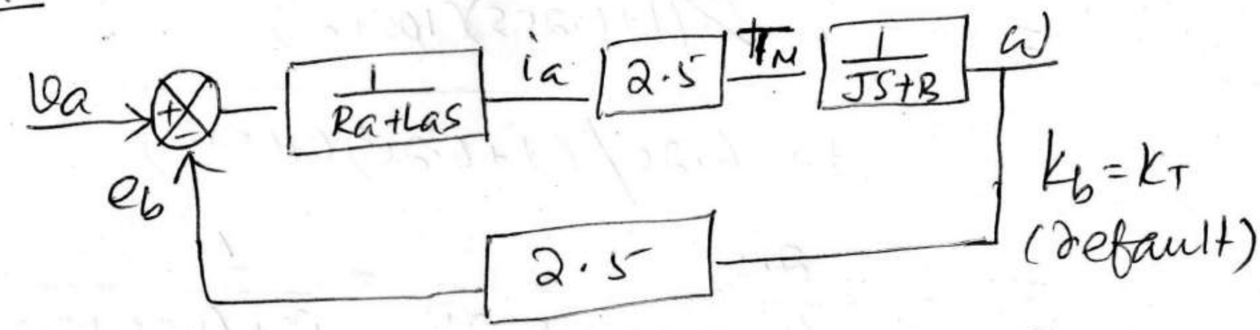
Q2 a.



$$Ra\, i_a + La \frac{di_a}{dt} + e_b = V_a \quad -(1)$$

$$e_b = K_b \omega \quad -(2)$$

$$T_M = K_T i_a = J\frac{d\omega}{dt} + B\omega \quad -(3)$$

The Block diagram is drawn by taking Laplace transform.

$$\boxed{I_a = \frac{V_a(s) - E_b(s)}{Ra + Las}}$$ ← combining eqn (1) & eqn (2)

$$T_M = Js\,\omega(s) + B\,\omega(s)$$
$$= (Js + B)\,\omega(s)$$

$$G(s) = \frac{1}{Ra+Las} \times 2.5 \times \frac{1}{JS+B}$$
$$= \frac{1}{1+0.25s} \times 2.5 \times \frac{1}{10s+2}$$
$$= \frac{2.5}{(1+0.25S)(10S+2)}$$

$$T(s) = \frac{G(s)}{1 + 2.5\,G(s)}$$

$$= \frac{(2.5)/(1+0.25s)(10s+2)}{1 + 6.25/(1+0.25s)(10s+2)}$$

$$= \frac{2.5}{2.5s^2 + 10.5s + 8.25} = \frac{1}{s^2 + 4.2s + 3.3}$$

[4 marks]

<u>part b.</u> $V_a(s) = \frac{100}{s}$

$$\omega(s) = \frac{100}{s(s^2 + 4.2s + 3.3)} = \frac{100}{s(s+3.15)(s+1.05)}$$

$$= \frac{100/3.3}{s} + \frac{100/(-3.15)(-2.1)}{s+3.15}$$

$$+ \frac{100/(-1.05)(2.1)}{(s+1.05)}$$

$$= \frac{30.3}{s} + \frac{15.12}{s+3.15} - \frac{45.35}{s+1.05}$$

$$\omega(t) = 30.3\,u(t) + 15.12\,e^{-3.15t} - 45.35\,e^{-1.05t}$$

$u(t) \triangleq$ unit step

Part b Cont.

$$\omega_n = \sqrt{3.3}$$

$$2\zeta\omega_n = 4.2$$

$$\zeta = 1.156.$$

The system is overdamped.
Hence the overshoot = 0
But system response will be sluggish (slow).

[5 marks.]

<u>Part C</u> $x_1 = \omega$, $x_2 = i_a$, $u = v_a$

$$\frac{d\omega}{dt} = -\frac{B}{J}\omega + \frac{K_T}{J} i_a$$

$$\frac{di_a}{dt} = -\frac{K_b}{L_a}\omega - \frac{R_a}{L_a} i_a + \frac{1}{L_a} v_a$$

$$B/J = 2/10 = 0.2, \quad \frac{K_T}{J} = \frac{2.5}{10} = 0.25$$

$$\frac{K_b}{L_a} = \frac{2.5}{0.25} = 10, \quad \frac{R_a}{L_a} = \frac{1}{0.25} = 4$$

State eqns are:

$$\frac{dx_1}{dt} = -0.2x_1 + 0.25x_2$$

$$\frac{dx_2}{dt} = -10x_1 - 4x_2 + 4u$$

$$\dot{x} = Ax + Bu$$

$$A = \begin{bmatrix} -0.2 & 0.25 \\ -10 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

$$y = \omega = x_1$$

$$= C\underline{x} = \begin{bmatrix} 1 & 0 \end{bmatrix} \underline{x}$$

[2 marks]

<u>part d</u> $e^{At} = \mathcal{L}^{-1}(sI - A)^{-1}$

$$(sI - A)^{-1} = \left\{ \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -0.2 & 0.25 \\ -10 & -4 \end{bmatrix} \right\}^{-1}$$

$$= \begin{bmatrix} s+0.2 & -0.25 \\ 10 & s+4 \end{bmatrix}^{-1}$$

$$= \frac{1}{(s+3.15)(s+1.05)} \begin{bmatrix} s+4 & 0.25 \\ -10 & s+0.2 \end{bmatrix}$$

$$\mathcal{L}^{-1}\frac{s+4}{(s+3.15)(s+1.05)} = \mathcal{L}^{-1}\frac{-0.4}{s+3.15}+\frac{1.4}{s+1.05}$$

$$= -0.4\,e^{-3.15t}+1.4\,e^{-1.05t}$$

$$\mathcal{L}^{-1}\frac{0.25}{(s+3.15)(s+1.05)} = \mathcal{L}^{-1} -\frac{0.119}{s+3.15}+\frac{0.119}{s+1.05}$$

$$= -0.119\,e^{-3.15t}+0.119\,e^{-1.05t}$$

$$\mathcal{L}^{-1}\frac{-10}{(s+3.15)(s+1.05)} = 4.76\,e^{-3.15t} - 4.76\,e^{-1.05t}$$

$$\mathcal{L}^{-1}\frac{s+0.2}{(s+3.15)(s+1.05)} = 1.4\,e^{-3.15t}-0.6\,e^{-1.05t}$$

$$e^{At} = \begin{bmatrix} -0.4e^{-3.15t}+1.4e^{-1.05t} & -0.119e^{-3.15t}+0.119e^{-3.15t} \\ 4.76e^{-3.15t}-4.76e^{-1.05t} & 1.4e^{-3.15t}-0.4e^{-1.05t} \end{bmatrix}$$

[4 marks]

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Q3 a. $s^4 + 1 = 0$

$$s^4 + 0s^3 + 0s^2 + 0s + 1 = 0$$

Routh Array

$$\begin{array}{llll}
s^4 & 1 & 0 & 1 \\
s^3 & 0(4) & 0 & \\
s^2 & 0(\epsilon) & 1 & \\
s^1 & -\frac{4}{\epsilon} & 0. & \\
s^0 & 1 & &
\end{array}$$

Aux eqn
$\frac{d}{ds}(s^4 + 1)$
$4s^3$

s^3 row is computed using Aux eqn method
s^2 is obtained using ϵ method.

There are two sign changes in the first column $\rightarrow$ indicating there are two poles in right half of the s-plane

[4 marks]

Q3 b. $G(s) = \frac{1}{(s-2)^2}$

$$C(s) = K_P + \frac{K_I}{s} + K_D s$$

$$C(s)\,G(s) = \frac{K_P s + K_I + K_D s^2}{s\,(s-2)^2}$$

Char poly $1 + C(s)G(s) = 0$

$$d(s) = s(s-2)^2 + K_P s + K_I + K_D s^2$$

$$= s^3 + (K_D - 4)s^2 + (K_P + 4)s + K_I = 0$$

To make the system stable,

$$\left.\begin{aligned} K_D - 4 &> 0 \\ K_P + 4 &> 0 \\ K_I &> 0 \end{aligned}\right\} \quad (1)$$

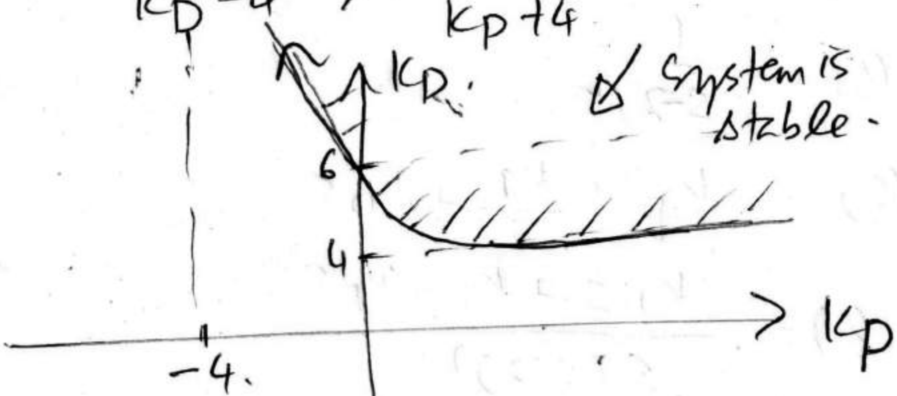
Routh array

s^3	1	$K_P + 4$
s^2	$K_D - 4$	K_I
s^1	$\dfrac{(K_D-4)(K_P+4) - K_I}{K_D - 4}$	0
s^0	K_I	

$(K_I = 8 \text{ Given})$

$$(K_D - 4)(K_P + 4) - K_I > 0$$

$$K_D - 4 > \frac{8}{K_P + 4} \Rightarrow K_D > 4 + \frac{8}{K_P + 4}$$



$K_D = 6,\ K_P = 0$ (At the boundary)

$K_D = 4,\ K_P = \infty$ (At the boundary)

Among all gains K_D is vital here to give stabilise as P or PI can't stabilize.

Q3 C. $d(s) = s^7 + 2s^6 - 4s^5 - 8s^4 - 25s^3 - 50s^2 + 100s + 200$

$$\begin{array}{lcccc}
s^7 & 1 & -4 & -25 & 100 \\
s^6 & 2 & -8 & -50 & 200 \\
s^5 & 0 & 0 & 0 &
\end{array}$$

Aux eqn $2s^6 - 8s^4 - 50s^2 + 200 = 0$
is a factor of $d(s)$

$$s^6 - 4s^4 - 25s^2 + 100 = 0$$

$$(s^4 - a^2)(s^2 - b^2) = 0$$

Comparing coeff $\Rightarrow$ $a^2 b^2 = 100$
$b^2 = 4$
$a^2 = 25$

$(s^4 - 25)(s^2 - 4)$ are factors.

$$\begin{array}{r|l}
 & s+2 \\
\hline
(s^6 - 4s^4 - 25s^2 + 100) & s^7 + 2s^6 - 4s^5 - 8s^4 - 25s^3 - 50s^2 + 100s + 200 \\
 & s^7 \quad - 4s^5 \quad - 25s^3 \quad + 100s \\
\hline
 & 2s^6 - 8s^4 - 50s^2 + 200 \\
 & 2s^6 - 8s^4 - 50s^2 + 200 \\
\hline
 & 0
\end{array}$$

$$d(s) = (s^4 - 25)(s^2 - 4)(s+2) = 0$$

poles are $s_{1,2} = \pm\sqrt{5}$; $s_{3,4} = \pm j\sqrt{5}$, $s_{5,6} = \pm 2$

$s_7 = -2$

Routh Array

s^7	1	-4	-25	100
s^6	2	-8	-50	200
s^5	0 (12)	0 (−32)	0 (−100)	0
s^4	$-32/12$	$-\frac{400}{12}$	200	
s^3	-182	800	0	
s^2	-45	200		
s^1	36	0		
s^0	200			

$$\frac{d}{ds}(2s^6 - 8s^4 - 50s^2 + 200) = 12s^5 - 32s^3 - 100s$$

There are two sign changes.

There are two poles $s = +\sqrt{5}$ & $s = +2$ in RHS plane – Validated.

Q4 solution

part(a)

$$G(s) = \frac{K}{s(s+2)(s^2+2s+2)}$$

poles are $s_1=0$, $s_2=-2$, $s_{3,4} = -1 \pm j1$

$n=4$, $m=0$, 4 root locus branches converging at infinity.

Asymptote angles $\phi_\ell = \dfrac{180+360(\ell-1)}{4}$

$\phi_1 = 45^0$, $\phi_2 = 135^0$, $\phi_3 = 225^0$, $\phi_4 = 315^0$

Centroid $= \dfrac{-0-2-1-1}{4} = -1$

Breakaway points

$$K = -\,s(s+2)(s^2+2s+2)$$

$$= -\left(s^4+4s^3+6s^2+4s\right)$$

$$\frac{dK}{ds} = -\left(4s^3+12s^2+12s+4\right) = 0$$

$$\Rightarrow s^3+3s^2+3s+1=0$$

$$(s+1)^3 = 0$$

$$(s+1)^3 = 0$$

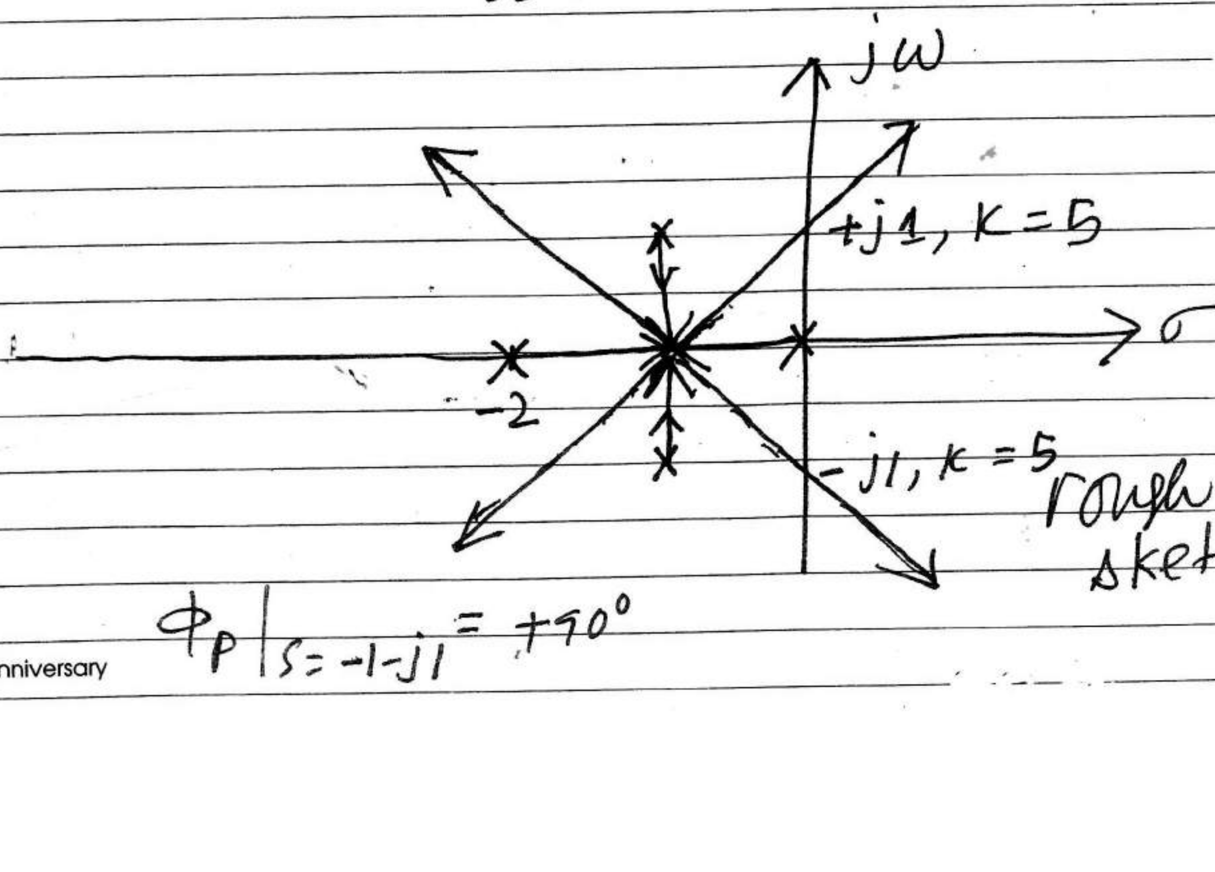
$$s = -1$$

So breakaway point is -1.

Angle of departure at complex conjugate poles: $s = -1 + j1$

$$\phi_p = 180^0 + [-45 - 90^0 - 135^0] = -90^0$$

due to pole at $s=-2$ — due to pole at $s=-1-j1$ — due to pole $s=0$



rough sketch

Notes

$$\phi_p \big|_{s=-1-j1} = +90^0$$

Birthday / Anniversary

part (b)

Char polynomial

$$s^4 + 4s^3 + 6s^2 + 4s + K = 0$$

s^4	1	6	K.
s^3	4	4	0
s^2	$-\frac{4-24}{4}$	K	0
s^1	$-\frac{4K-20}{5}$	0	
s^0	K		

For cross over

$$4K - 20 = 0$$

$$K = \frac{20}{4} = 5$$

$$5s^2 + 5 = 0$$

$$s^2 + 1 = 0$$

$$s_{1,2} = \pm j1$$

part (c)

$$G(s) = \frac{K(s+1)}{s(s+2)(s^2+2s+2)}$$

$n = 4$, $m = 1$, 3 root locus branches

Asymptote angles $60^0, 180^0, 300^0$

$$\text{Centroid} = \frac{-2-1-1+1}{4-1} = -1$$

Break away point

$$K = -\frac{s(s+2)(s^2+2s+20)}{s+1}$$

$$\frac{dK}{ds} = -\frac{4s^4+12s^3+12s^2+4s}{(s+1)^2} = 0$$

$$s = 0,\ s = -1$$

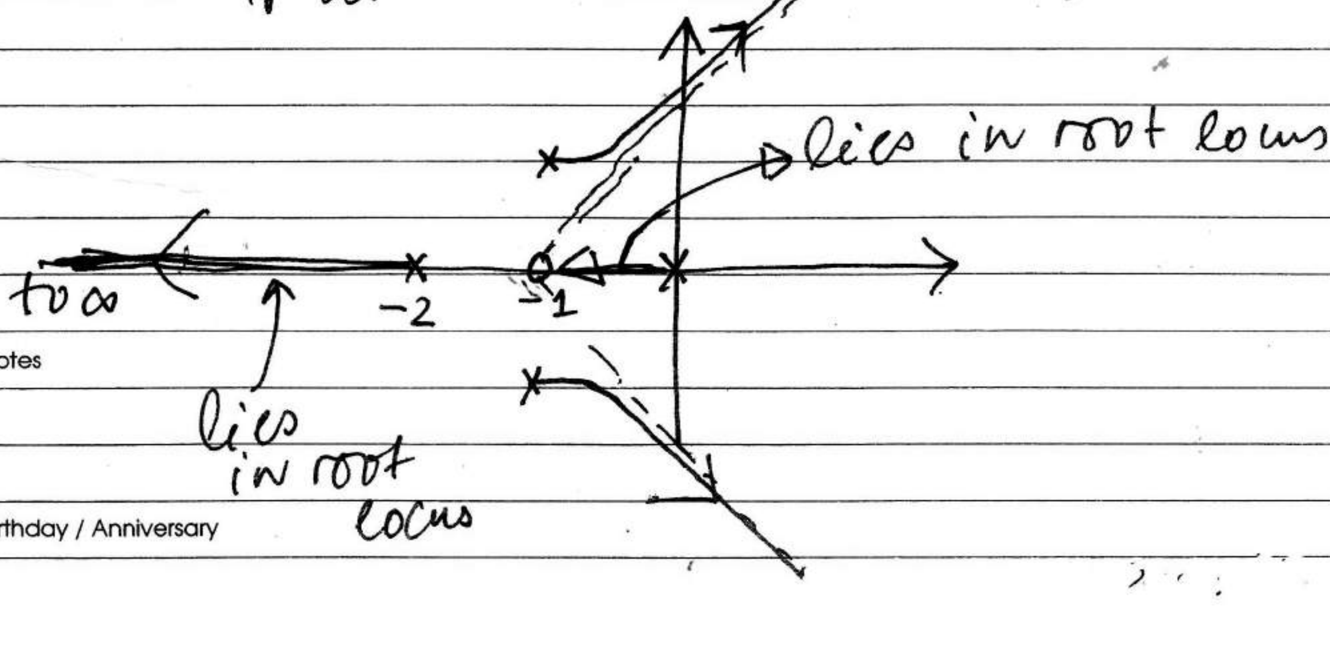
These are not break away point

Angle of departure at pole $s = -1+j1$

$$\phi_p = 180^0 + (90-45-90-135)$$

$$= 0^0$$

ϕ_p at $s = -1-j1 = 0^0$



Notes

Birthday / Anniversary