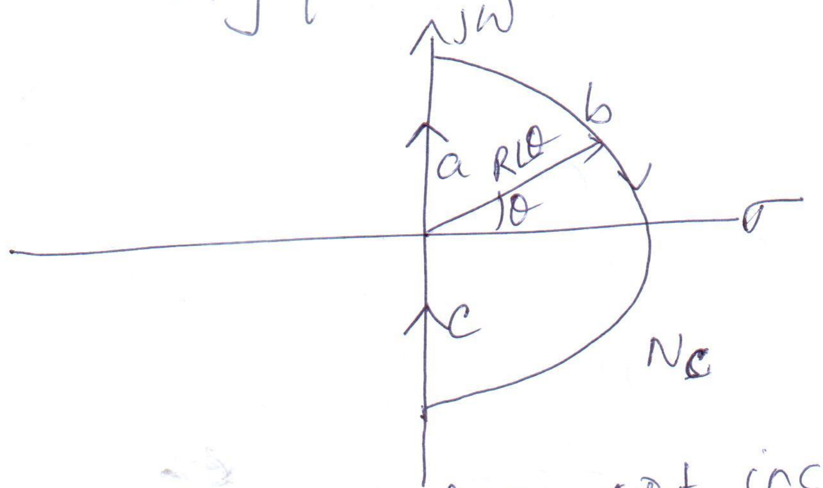
Tutorial Sheet 7

EE 250 24 March 2021

Q1 Given $L(s) = \frac{K}{(s+1)^3}$. Draw the Nyquist plot for $K=4$ and $K=10$. Comment on the stability of the closed loop system.

Soln

Nyquist Contour



The contour does not include any pole of $L(s)$.

Let's evaluate $L(s)$ for part a of N_C.

$s = j\omega$, $K = 4$

$$L(j\omega) = \frac{4}{(j\omega+1)^3} = \frac{4}{(1-3\omega^2)+j(3\omega-\omega^3)}$$

$$= \frac{4(1-3\omega^2) - j\,4(3\omega-\omega^3)}{(1-3\omega^2)^2+(3\omega-\omega^3)^2}$$

$$\angle L(j\omega) = -3\tan^{-1}\omega$$

ω	$\lvert L(j\omega)\rvert$	$\angle L(j\omega)$
0	4	0
1	$2\sqrt{2}$	-135
∞	0	-270

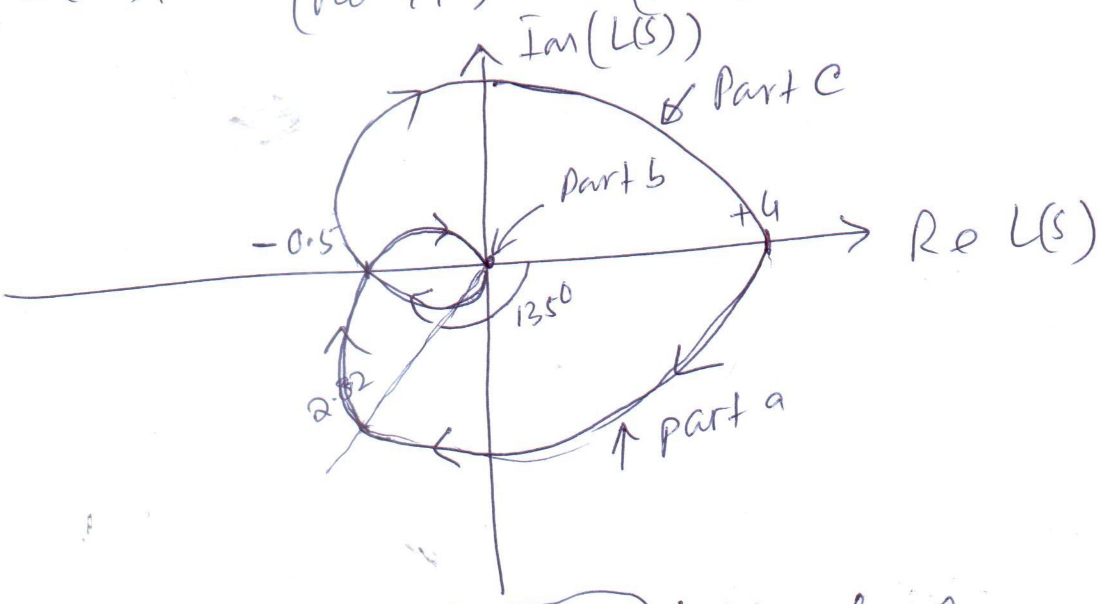
Since the Nyquist plot would cross real axis, to find out the crossing point, $\operatorname{Im}(L(j\omega)) = 0$

$$\frac{4(3\omega-\omega^3)}{(1-3\omega^2)^2+(3\omega-\omega^3)^2} = 0$$

$$3\omega-\omega^3 = 0$$

$$\omega(3-\omega^2)=0 \Rightarrow \omega = \pm\sqrt{3}$$

$$\lvert L(j\omega)\rvert = \frac{4}{(\sqrt{\omega^2+1})^3} = \frac{4}{(\sqrt{3+1})^3} = \frac{4}{8} = 0.5$$

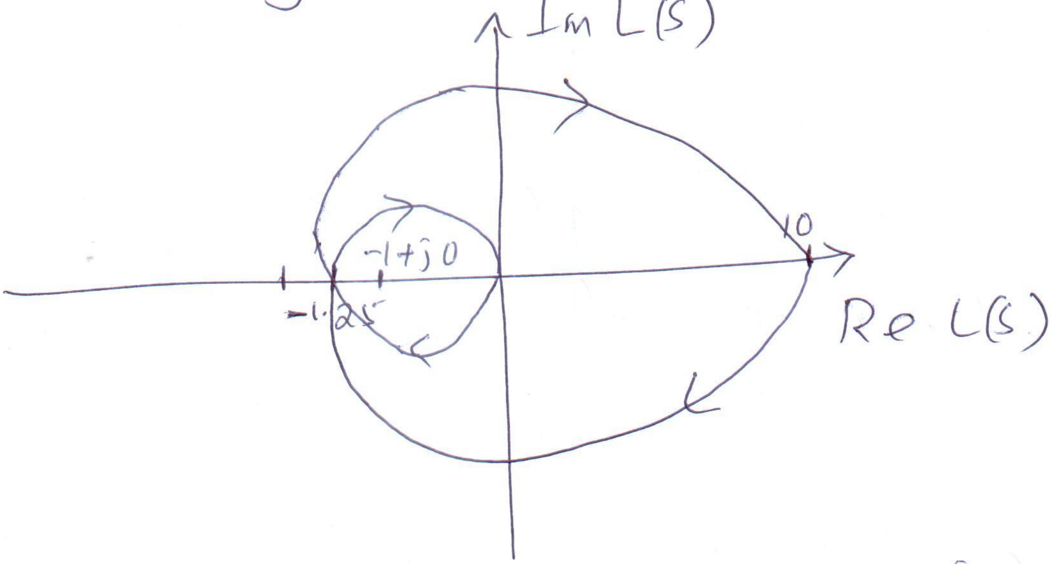


Corresponding to L(s) plane

Nyquist plot of part c is the mirror image around the real axis. The outer semicircle would map to origin. Since the plot does not encircle $-1+j0$, the system in closed loop is stable.

If we take $K = 10$, $\angle L(j\omega)$ will not change. The crossing of the real axis will take place at the same $\omega = \pm\sqrt{3}$. The gain $|L(j\omega)|$ at $\omega = +\sqrt{3}$ is $\frac{10}{(\omega^2+1)^{3/2}} = \frac{10}{8} = 1.25$.

The Nyquist plot would look as



The plot encircles $-1+j0$ two times clock wise

$N = Z - P$, $N = 2$, $P = 0$

$2 = Z - 0 \Rightarrow Z = 2$.

There are two closed loop poles in the RHS-plane. System is thus unstable.

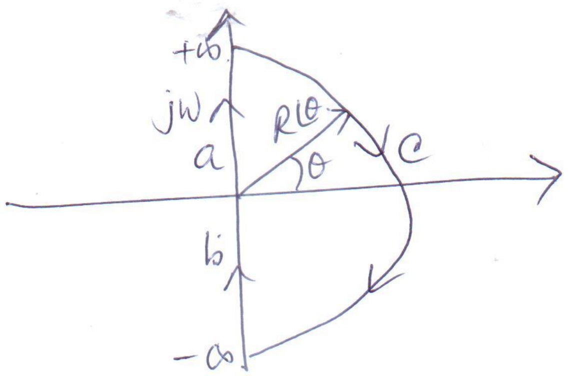
Q2 The open loop transfer function of a system in a unity feedback configuration is given as

$$G(s) = K \frac{s+2}{(s-2)^2}$$

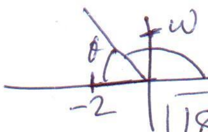
a. Draw the Nyquist plot for $k = 1$. Comment on the stability. Compute the gain margin.

b. How does this plot change when $K = 10$. Comment on the stability. Compute the phase margin.

Soln N_c: Nyquist contour



$$G(j\omega) = K \frac{j\omega+2}{(j\omega-2)^2}$$



$\tan^{-1}(\omega/2)$
$\Downarrow$
$\boxed{180^\circ - \tan^{-1}\omega/2}$

For part a & b, $s = j\omega$, $K = 1$

$$|G(j\omega)| = \frac{j\omega+2}{(j\omega-2)^2} = \frac{(j\omega+2)((4-\omega^2)+j4\omega)}{(4-\omega^2)^2+(4\omega)^2}$$

$$= \frac{1}{\sqrt{\omega^2+4}} = \frac{1/2}{\sqrt{\frac{\omega^2}{4}+1}}$$

$$\underline{/G(j\omega)} = \tan^{-1}\omega/2 - \tan^{-1}\left(-\frac{\omega}{2}\right) - \tan^{-1}(-\omega/2)$$

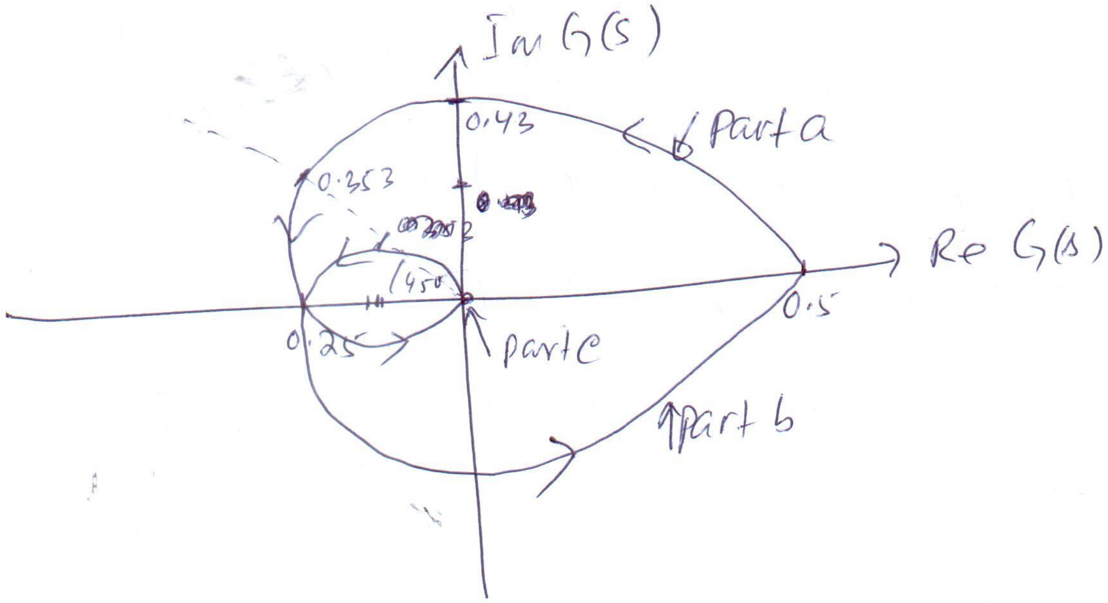
$$= \tan^{-1}\omega/2 - \left(180^\circ - \tan^{-1}\frac{\omega}{2}\right) - \left(180^\circ - \tan^{-1}\omega/2\right)$$

$$\angle G(j\omega) = 3\tan^{-1}\frac{\omega}{2} - 360^0$$

$$= 3\tan^{-1}\frac{\omega}{2}$$

For part a of Nc

ω	$\lvert G(j\omega)\rvert$	$\angle G(j\omega)$
0	0.5	0
1	0.45	79.5^0
1.15	0.43	90^0
2	0.353	135^0
3.46	0.25	180^0
∞	0	270^0



G(s) plane

The Nyquist plot would cross the real axis at -180^0 if

$$Im(G(j\omega)) = 0 \quad \text{or}$$

$$3\tan^{-1}\omega/2 = 180^0$$

$$G(j\omega) = \frac{2(4-\omega^2) - 4\omega^2 + j(8\omega + \omega(4-\omega^2))}{(4-\omega^2)^2 + (4\omega)^2}$$

$$Im(G(j\omega)) = \frac{8\omega + \omega(4-\omega^2)}{(4-\omega^2)^2 + (4\omega)^2} = 0$$

$$8\omega + \omega(4-\omega^2) = 0$$

$$\omega(8+4-\omega^2) = 0$$

$$\omega = \pm\sqrt{12} = \pm 2\sqrt{3}$$

$$\omega = 3.46 \text{ rad/sec}$$

$$\tan^{-1}\omega/2 = 60$$

$$\omega/2 = \tan 60 = 1.732$$

$$\omega = 3.46 \text{ rad/sec}$$

At $\omega = 3.46$ rad/sec, $|G(j\omega)| = \dfrac{0.5}{\sqrt{\frac{3.46^2}{4}+1}} = 0.25$

$\boxed{\text{part c} \quad \text{Converges to origin}}$

In the Nyquist plot, encirclement of $-1+j0 = 0$

$N = 0$, $P = 2$,

$N = Z - P \Rightarrow Z = 2$

No of closed loop poles in RHS plane = 2

The system is unstable. For unstable system, gain margin has no meaning or it does not exist.

Q2b $K = 10$, $\angle G(j\omega)$ is unchanged.

The crossing of the real axis at 180^0 is also unchanged at $\omega = 3.46$ rad/sec.

$$|G(j\omega)|\Big|_{\omega=3.46} = \frac{10}{\sqrt{\omega^2+4}}$$

$$= \frac{10}{4} = 2.5$$

The Nyquist plot would encircle $-1+j0$ 2 times anti clockwise.

$$N = -2$$
$$= Z - P \qquad \text{Given } P = 2$$
$$= Z - 2$$

$\Rightarrow Z = 0 \Rightarrow$ There are no closed loop poles in RHS plane. The system is stable.

At phase margin, $|G(j\omega)| = 1$

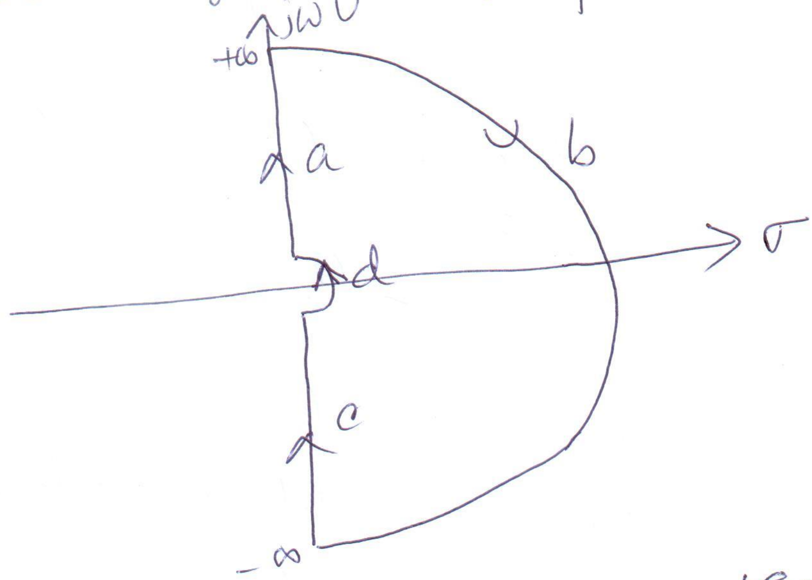
$$\frac{10}{(\omega^2+4)^{1/2}} = 1$$

$$\omega^2 + 4 = 100$$
$$\omega^2 = 96 \Rightarrow \omega_g = \sqrt{96} = 9.798 \text{ rad/sec.}$$

$$\angle G(j\omega) = 3\tan^{-1}\left(\frac{\omega_g}{2}\right) = 235.38^0$$
$$PM = -180 + 235.38^0 = 55.38^0$$

Q3 Draw the Nyquist plot of the system $L(s) = K\dfrac{s+2}{s(s-1)}$ ($\Leftarrow$ OLTF in unity feedback).
Find the range of K for stability.

Soln



N_c has four parts as there is a pole at origin.

For part a & c $s = jw$

$$L(jw) = K\frac{jw+2}{jw(jw-1)}$$

$$= K\left[\frac{-3w^2}{w^4+w^2} + j\,\frac{2w-w^3}{w^4+w^2}\right]$$

$$= K\left[\frac{-3}{w^2+1} + j\,\frac{2w-w^2}{w^4+w^2}\right]$$

$$\angle L(jw) = \tan^{-1} w/2 - 90^0 - (180^0 - \tan^{-1} w)$$

$$= -270^0 + \tan^{-1} w/2 + \tan^{-1} w$$

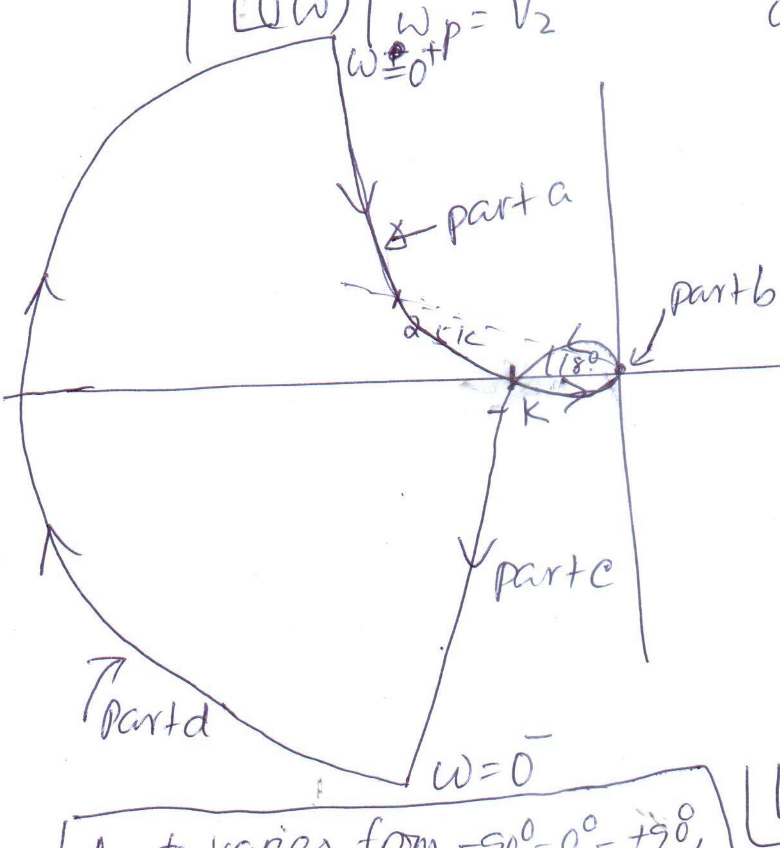
ω	$\lvert L(j\omega)\rvert$	$\angle L(j\omega)$
0	$-3+j\infty = \infty$	-270^0
1	$K\sqrt{2.5}$	-198.5^0
∞	0	-90^0

The polar plot for part a would cross the real axis at -180^0 for which

$$\frac{2\omega-\omega^3}{\omega^4+\omega^2} = 0 \Rightarrow \omega(2-\omega^2)=0$$

$$\Rightarrow \omega_p = \sqrt{2} = 1.414 \text{ rad/sec}.$$

$$\lvert L(j\omega)\rvert_{\omega_p=\sqrt{2}} = -3\frac{K}{\omega^2+1} = -K$$



Nyquist plot corresponding to part a is thus plotted as shown. It's mirror image is part c. part b would map to origin.

For part d

$$s = \epsilon\angle\phi = \epsilon e^{j\phi}, \epsilon\to 0$$

$$L(s) = K\frac{s+2}{s(s+1)}\Big|_{s=\epsilon e^{j\phi}}$$

$$= -\frac{2K}{\epsilon e^{j\phi}} = -\infty e^{j\phi}$$

$$\angle L(s)\big|_{\text{at } \omega=0^+} = -270^0$$

$$\angle L(s)\big|_{\text{at } \omega=0^-} = -90^0$$

$\boxed{\text{As } \phi \text{ varies from } -90^0 - 0^0 - +90^0,\ \angle L(s) \text{ varies from } -90^0 - 180^0 - 270^0}$

This semicircle of infinite radius will cross the real axis at angle -180^0.

$N = Z - P$ where $P=1$. For stability $Z=0$ so $N=-1$. This will be satisfied if $-K<-1$ i.e $K>1$. So system is stable for $K>1$ as $-1+j0$ is encircled once anticlockwise.