MSO201a: Probability and Statistics
Summer Term: 2019
End Semester Examination
Instructor: Neeraj Misra

Time Allowed: 180 Minutes | Maximum Marks: 100

NOTE: (i) Start answer of every question on a new page. Moreover, attempt all the parts of a question at one place.
(ii) Answer each question legibly, clearly and concisely. Illegible answers will not be graded.
(ii) For each question, provide details of all the steps involved in arriving at the final answer(s)/conclusion(s) and box your final answer(s)/conclusion(s).

1. Let $\underline{X} = (X_1, X_2, X_3)$ be a random vector having the joint p.d.f.

$$f_{X_1,X_2,X_3}(x_1, x_2, x_3) = \begin{cases} cx_1x_2x_3, & \text{if } x_i > 0, i = 1,2,3, x_1 + x_2 + x_3 \leq 2 \\ 0, & \text{otherwise} \end{cases},$$

where c is a positive real constant.

(i) Find the value of the constant c;

(ii) Find the joint p.d.f. of (X_1, X_2);

(iii) Find the marginal p.d.f. of X_3;

(iv) Find the value of the conditional correlation between X_1 and X_2 given that $X_3 = 1$;

(v) Find the value of $\Pr(X_1 + X_3 \leq 1, X_2 + X_3 \leq 1)$. $\boxed{3+3+3+4+4 = 17 \text{ MARKS}}$

2. Let X and Y be jointly distributed random variables with $E(X) = 1, E(X^2) = 2, E(X^3) = 4, E(X^4) = 10, E(Y|X = x) = 3x + 4$ and $\text{Var}(Y|X = x) = x^2 + 2x$. Find the mean and the variance of $Z = XY$. $\boxed{4 + 5 = 9 \text{ MARKS}}$

3. Let $\underline{X} = (X_1, X_2)$ be a random vector with joint m.g.f.

$$M_{X_1,X_2}(t_1, t_2) = \frac{e^{2t_2}(1 + e^{t_1 - t_2})^2}{4}, \quad \underline{t} = (t_1, t_2) \in \mathbb{R}^2.$$

(i) Show that $X_1 \sim \text{Bin}(2, \frac{1}{2})$;

(ii) Show that $\frac{X_1 - X_2 + 2}{2} \stackrel{d}{=} X_1$;

(iii) Find the value of $\text{Cov}(X_1, X_2)$. $\boxed{3 + 3 + 4 = 10 \text{ MARKS}}$

4. Let X_1, X_2 and X_3 be i.i.d. Exp(1) r.v.s and let $X_{(1)} \leq X_{(2)} \leq X_{(3)}$ be the corresponding order statistics. Let $Y_1 = 3X_{(1)}, Y_2 = 2(X_{(2)} - X_{(1)})$ and $Y_3 = X_{(3)} - X_{(2)}$.

(i) Show that Y_1, Y_2 and Y_3 are i.i.d. Exp(1) r.v.s;

(ii) Using (a), find the value of $\mathrm{Var}(X_{(2)})$;

(iii) Using (a), find the value of $\mathrm{Cov}(X_{(1)}, X_{(2)})$. $\boxed{6+3+3=12 \text{ MARKS}}$

5. (i) A person plays a game 2000 times. If the probability of the person winning any game is 0.0005 and the games are played independently, using suitable approximation, find the probability that the person will win at least 2 games;

(ii) A pair of fair dice is independently rolled four times. Find the probability that on two occasions we get a a sum of 7;

(iii) A person keeps rolling a fair die until six is observed thrice. If the rolls are independent, find the expected number of trails required;

(iv) An urn contains 3000 balls out of which 2000 balls are red and remaining balls are black. Five balls are drawn at random and without replacement from the urn. Using suitable approximation, find the probability of getting 3 red balls. $\boxed{3\times 4=12 \text{ MARKS}}$

6. (i) Let $X \sim \mathrm{N}(2,4)$. Find the values of $\Pr(|X| \geq 2), \Pr(1 < X \leq 3)$ and $\Pr(X \leq 3 | X \geq 1)$ (Given that $\Phi(0.5) = 0.6915, \Phi(1) = 0.8413, \Phi(2) = 0.9772$);

(ii) Let Z_1 and Z_2 be i.i.d. $\mathrm{N}(0,1)$ r.v.s. Show that $T = \frac{Z_1+Z_2}{|Z_1-Z_2|}$ has the Cauchy distribution (Student-t distribution with one degree of freedom);

(iii) Let Z_1, Z_2, Z_3 and Z_4 be i.i.d. $N(0,1)$ r.v.s. $F = 2\frac{(Z_1+Z_2)^2}{(Z_1-Z_2)^2+(Z_3+Z_4)^2}$ has Snedcor's F distribution with $(1,2)$ degrees of freedom;

(iv) Let $X_1, \ldots, X_5$ be i.i.d. $\mathrm{N}(1,1)$ r.v.s, $\overline{X} = \frac{1}{5}\sum_{i=1}^{5} X_i$ and $S^2 = \frac{1}{4}\sum_{i=1}^{5}(X_i - \overline{X})^2$. Find the value of $E(\frac{\overline{X}}{S})$. $\boxed{3\times 4=12 \text{ MARKS}}$

7. (a) Let $\underline{X} = (X_1, X_2, X_3) \sim \mathrm{Mult}(6, \frac{1}{4}, \frac{1}{2}, \frac{1}{8})$.

(i) Find the value of $\Pr(X_1 + X_2 = 3)$;

(ii) Find the value of $\mathrm{Var}(X_1 + X_3)$;

(iii) Find the value of $\mathrm{Cov}(X_2, X_3)$;

(iv) Find the value of $\Pr(X_1 = 3 | X_2 + X_3 = 1)$. $\boxed{3\times 4=12 \text{ MARKS}}$

(b) A bag contains twelve marbles, out of which 3 are red, 4 are blue and 5 are green. Five marbles are drawn with replacement from the bag. Find the probability that, among the marbles drawn, exactly two are red and exactly two are blue. $\boxed{4 \text{ MARKS}}$

8. Let $\underline{X} = (X_1, X_2) \sim N_2(0,0,1,1,\frac{1}{2})$.

(i) For any $t \in \mathbb{R}$ and $x_1 \in \mathbb{R}$, find the conditional expectation $E(e^{tX_1X_2} | X_1 = x_1)$;

(ii) Using (i), find $E(e^{tX_1X_2})$;

(iii) Using (ii), find the mean and variance of $Y = X_1X_2$;

(iv) Let $Y_1 = X_1$ and $Y_2 = X_1 + X_2$. Find the expression for joint p.d.f. of $\underline{Y} = (Y_1, Y_2)$. $\boxed{3\times 4=12 \text{ MARKS}}$

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Model Solutions

Problem No. 1

(i) $$c\int_0^2\int_0^{2-x_1}\int_0^{2-x_1-x_2} x_1x_2x_3\,dx_3\,dx_2\,dx_1 = 1$$

$$\Rightarrow \frac{c}{2}\int_0^2\int_0^{2-x_1} x_1x_2(2-x_1-x_2)^2\,dx_2\,dx_1 = 1$$

$$\Rightarrow \frac{c}{2}\int_0^2 x_1\left\{\int_0^{2-x_1} x_2^2(2-x_1-x_2)\,dx_2\right\}dx_1 = 1$$

$$\Rightarrow \frac{c}{2}\int_0^2 x_1\left[\frac{(2-x_1)^3}{3}-\frac{(2-x_1)^4}{4}\right]dx_1 = 1$$

$$\Rightarrow \frac{c}{24}\int_0^2 x_1(2-x_1)^4\,dx_1 = 1$$

$$\Rightarrow \frac{c}{24}\int_0^2 x_1^4(2-x_1)\,dx_1 = 1 \quad \Rightarrow \quad \boxed{c=\frac{45}{4}}$$

.... 3 MARKS

(ii) For $x_1>0$, $x_2>0$ and $x_1+x_2\le 2$

$$f_{X_1,X_2}(x_1,x_2) = c\,x_1x_2\int_0^{2-x_1-x_2} x_3\,dx_3 = \frac{45}{8}x_1x_2(2-x_1-x_2)^2.$$

Thus

$$\boxed{f_{X_1,X_2}(x_1,x_2) = \begin{cases}\frac{45}{8}x_1x_2(2-x_1-x_2)^2, & x_1>0,\ x_2>0,\ x_1+x_2\le 2\\ 0, & \text{otherwise.}\end{cases}}$$

.... 3 MARKS

(iii) For $0<x_3<2$

$$f_{X_3}(x_3) = c\,x_3\int_0^{2-x_3}\int_0^{2-x_1-x_3} x_1x_2\,dx_2\,dx_1 = \frac{15}{32}x_3(2-x_3)^4$$

Thus

$$\boxed{f_{X_3}(x_3) = \begin{cases}\frac{15}{32}x_3(2-x_3)^4, & 0<x_3<2\\ 0, & \text{otherwise.}\end{cases}}$$

... 3 MARKS

(IV) $$f_{X_1,X_2|X_3}(x_1,x_2|1) = \frac{f_{X_1,X_2,X_3}(x_1,x_2,1)}{f_{X_3}(1)}$$

$$= \begin{cases} \dfrac{\frac{45}{4}x_1x_2}{\frac{15}{32}} = 24x_1x_2, & x_1>0,\ x_2>0,\ x_1+x_2\le 1 \\ 0, & \text{otherwise} \end{cases}$$

$$\boxed{f_{X_1,X_2|X_3}(x_1,x_2|1) = \begin{cases} 24x_1x_2, & x_1>0,\ x_2>0,\ x_1+x_2\le 1 \\ 0, & \text{otherwise.} \end{cases}}$$

(V) For $r, s = 0, 1, 2, \ldots$

$$E(X_1^r X_2^s \mid X_3=1) = 24\int_0^1 x_1^{r+1}\left(\int_0^{1-x_1} x_2^{s+1}\,dx_2\right)dx_1$$

$$= \frac{24}{s+2} B(r+2, s+3)$$

$$= \frac{24}{s+2} \times \frac{\lfloor r+1 \ \lfloor s+2}{\lfloor r+s+4}$$

$$E(X_1|X_3=1) = E(X_2|X_3=1) = \frac{2}{5}$$

$$E(X_1^2|X_3=1)\ E(X_2^2|X_3=1) = \frac{1}{5}$$

$$Var(X_1|X_3=1) = Var(X_2|X_3=1) = \frac{1}{5} - \frac{4}{25} = \frac{1}{25}$$

$$E(X_1X_2|X_3=1) = \frac{2}{15}$$

$$Cov(X_1,X_2|X_3=1) = \frac{2}{15} - \frac{4}{25} = -\frac{2}{75}$$

$$Corr(X_1,X_2|X_3=1) = \frac{-2/75}{1/25} \Rightarrow \boxed{Cov(X_1,X_2|X_3=1) = -\frac{2}{3}}$$

... $\boxed{\text{4 MARKS}}$

(VI) $$Pr(X_1+X_3\le 1,\ X_2+X_3\le 1) = Pr(X_3 \le \min\{1-X_1, 1-X_2\})$$

$$= c\int_0^1\int_0^1\int_0^{\min(1-x_1,1-x_2)} x_1x_2x_3\,dx_3\,dx_2\,dx_1$$

$$= \frac{c}{2}\int_0^1\int_0^1 x_1x_2\{\min\{1-x_1, 1-x_2\}\}^2\,dx_1\,dx_2$$

$$= c\iint_{0<x_1<x_2<1} x_1x_2(1-x_2)^2\,dx_1\,dx_2$$

$$= c\int_0^1\int_0^{x_2} x_1x_2(1-x_2)^2\,dx_1\,dx_2$$

$$= \frac{c}{2}\int_0^1 x_2^3(1-x_2)^2\,dx_2$$

$\boxed{2/9}$

$$= \frac{c}{2} B(4,3) = \frac{45}{8} \times \frac{6 \times 2}{720} = \frac{3}{32}$$

$$\boxed{\Pr(X_1 + X_3 \le 1,\ X_2 + X_3 \le 1) = \frac{3}{32}}$$ ... 4 MARKS

Problem No. 2

$E(Z) = E(XY)$

$= E(E(XY|X))$ .... 2 MARKS

$= E(X\,E(Y|X))$

$= E(X(3X+4))$

$$\boxed{E(Z) = 10}$$ ... 2 MARKS

$\mathrm{Var}(Z) = \mathrm{Var}(XY)$ $E(\mathrm{Var}(XY|X))$

$= \mathrm{Var}(E(XY|X)) + E(X^2\,\mathrm{Var}(Y|X))$

$= \mathrm{Var}(X\,E(Y|X)) + E(X^2(X^2+2X))$ ... 3 MARKS

$= \mathrm{Var}(X(3X+4)) + E(X^4) + 2E(X^3)$

$= \mathrm{Var}(3X^2+4X) + 10 + 8$

$= 9\,\mathrm{Var}(X^2) + 16\,\mathrm{Var}(X) + 24\,\mathrm{Cov}(X^2, X) + 10 + 8$

$= 9\,[E(X^4) - (E(X^2))^2] + 16\,[E(X^2) - (E(X))^2] + 18$

$\quad + 24\,[E(X^3) - E(X^2)\,E(X)]$

$= 136$

$$\boxed{\mathrm{Var}(Z) = 136}$$ ... 2 MARKS

Problem No 3

(i) $M_{X_1}(t_1) = M_{X_1, X_2}(t_1, 0)$

$= \frac{(1+e^{t_1})^2}{4} = \left(\frac{1}{2} + \frac{1}{2}e^{t_1}\right)^2, \quad \forall\, t_1 \in \mathbb{R}$

$=$ m.g.f. of $\mathrm{Bin}(2, \frac{1}{2})$ at point t_1 $\forall\, t_1 \in \mathbb{R}$

$\Rightarrow \boxed{X_1 \sim \mathrm{Bin}(2, \tfrac{1}{2})}$ ... 3 MARKS

(ii) Let $Y = \frac{X_1 - X_2 + 2}{2}$. Then

$M_Y(t) = e^t\, M_{X_1, X_2}\left(\frac{t}{2}, -\frac{t}{2}\right)$

$$= e^{t} \times e^{-t} \frac{(1+e^{t})^2}{4}$$

$$= \left(\frac{1}{2} + \frac{1}{2} e^{t}\right)^2$$

$= $ m.g.f. of $Bin(2, \frac{1}{2})$ at the point t_1, $\forall\, t_1 \in \mathbb{R}$

$\Rightarrow \quad Y \sim Bin(2, \frac{1}{2})$

$\Rightarrow \quad \boxed{Y \overset{d}{=} X_1}$ ... $\boxed{3 \text{ MARKS}}$

(iii) $\ln M_{X_1, X_2}(\underline{t}) = 2t_2 + 2\ln(1+e^{t_1-t_2}) - \ln 4$

$$\frac{\partial}{\partial t_1} \ln M_{X_1, X_2}(\underline{t}) = 2 \frac{e^{t_1-t_2}}{1+e^{t_1-t_2}}$$

$$\frac{\partial^2}{\partial t_2 \partial t_1} \ln M_{X_1, X_2}(\underline{t}) = 2 \frac{-e^{t_1-t_2}(1+e^{t_1-t_2}) + e^{2(t_1-t_2)}}{(1+e^{t_1-t_2})^2}, \quad (t_1, t_2) \in \mathbb{R}^2$$

$$Cov(X_1, X_2) = \left[\frac{\partial^2}{\partial t_2 \partial t_1} \ln M_{X_1, X_2}(\underline{t})\right]_{\underline{t}=\underline{0}} = -\frac{1}{2}$$

$\boxed{Cov(X_1, X_2) = -\frac{1}{2}}$ ... $\boxed{4 \text{ MARKS}}$

$\boxed{\text{Problem No. 4.}}$ (i) The joint p.d.f. of $(X_{(1)}, X_{(2)}, X_{(3)})$ is

$$f_{X_{(1)}, X_{(2)}, X_{(3)}}(x_1, x_2, x_3) = \underline{|3}\, f(x_1) f(x_2) f(x_3), \quad -\infty < x_1 < x_2 < x_3 < \infty,$$

where $f(x) = \begin{cases} e^{-x}, & \text{if } x > 0 \\ 0, & \text{otherwise.} \end{cases}$

The transformation

$(X_{(1)}, X_{(2)}, X_{(3)}) \longrightarrow (Y_1, Y_2, Y_3) = (3X_{(1)}, 2(X_{(2)} - X_{(1)}), X_{(3)} - X_{(1)})$ is 1-1

with inverse transformation

$$X_{(1)} = \frac{Y_1}{3}, \quad X_{(2)} = \frac{Y_1}{3} + \frac{Y_2}{2}, \quad X_{(3)} = \frac{Y_1}{3} + \frac{Y_2}{2} + Y_1$$

the Jacobian

$$J = \begin{vmatrix} 1/3 & 0 & 0 \\ 1/3 & 1/2 & 0 \\ 1/3 & 1/2 & 1 \end{vmatrix} = \frac{1}{6}. \quad \ldots$$

Also

$$0 < x_1 < x_2 < x_3 < \infty \iff 0 < \frac{y_1}{3} < \frac{y_1}{3} + \frac{y_2}{2} < \frac{y_1}{3} + \frac{y_2}{2} + y_1$$

$\iff y_1 > 0, \; y_2 > 0, \; y_3 > 0$ ... $\boxed{3 \text{ MARKS}}$

Thus the joint p.d.f. of $\underline{Y} = (Y_1, Y_2, Y_3)$ is

$$f_{Y_1, Y_2, Y_3}(y_1, y_2, y_3) = 6 f\left(\frac{y_1}{3}\right) f\left(\frac{y_1}{3}+\frac{y_2}{2}\right) f\left(\frac{y_1}{3}+\frac{y_2}{2}+y_3\right) I_{[0,a)^3}(y_1, y_2, y_3)$$

$$= \begin{cases} e^{-y_1} e^{-y_2} e^{-y_3}, & \text{if } y_1>0,\ y_2>0,\ y_3>0 \\ 0, & \text{otherwise} \end{cases}$$

Clearly Y_1, Y_2 and Y_3 are iid Exp(1) r.v.s. ... $\boxed{\text{3 MARKS}}$

(ii) $Var(X_{(2)}) = Var\left(\frac{Y_1}{3}+\frac{Y_2}{2}\right)$

$$= \frac{Var(Y_1)}{9} + \frac{Var(Y_2)}{4} = \frac{1}{9}+\frac{1}{4} = \frac{13}{36}$$

$\boxed{Var(X_{(2)}) = \frac{13}{36}}$ ... $\boxed{\text{3 MARKS}}$

(iii) $Cov(X_{(1)}, X_{(2)}) = Cov\left(\frac{Y_1}{3}, \frac{Y_1}{3}+\frac{Y_2}{2}\right)$

$$= \frac{Var(Y_1)}{9} = \frac{1}{9}$$

$\boxed{Cov(X_{(1)}, X_{(2)}) = \frac{1}{9}}$ ... $\boxed{\text{3 MARKS}}$

$\boxed{\text{Problem No.5}}$ (i) Let X = # of wins in 2000 games. Then $X \sim Bin(2000, .0005)$

$n = 2000$ is large and $p = 0.0005$ is small. $\lambda = np = 1$. We have

Using Poisson approximation to binomial distribution

$$X \overset{\text{approx}}{\approx} Po(1).$$

Required probability $= Pr(X \geq 2) = 1 - Pr(X=0) - Pr(X=1)$

$\boxed{\text{Required prob.} = 1 - \frac{2}{e} = .2642}$ ... $\boxed{\text{3 MARKS}}$

(ii) Pr(getting a sum of 7 in a single trial) $= \frac{6}{36} = \frac{1}{6}$

X = # of times we get a sum of 7 in 4 trials $\sim Bin(4, \frac{1}{6})$

$$Pr(X=2) = \binom{4}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^2 = \frac{25}{216} \simeq 0.1157$$

$\boxed{\text{Required prob.} = \frac{25}{216} \simeq 0.1157}$ ... $\boxed{\text{3 MARKS}}$

$\boxed{5/9}$

(iii) Designate getting a six as success. Let

X = # of trials to get 3rd success

Y = # of failures preceding 3rd success

$= X-3 \sim NB(3, \frac{1}{6})$

$E(Y) = \frac{rq}{p} = 3 \times \frac{5/6}{1/6} = 15$

$\boxed{E(X) = 18}$ ... $\boxed{\text{3 MARKS}}$

(iv) Designate getting a red ball as success.

$X_{a\,n\,N}$ = # of red balls in a sample of $n \sim Hyp(a\, n\, N)$;

here $a = 2000$ (large), $N = 3000$ (large), $n = 5$, $\frac{a}{N} = \frac{2}{3}$.

Thus $X_{a\,n\,N} \overset{approx}{\sim} Bin(5, 2/3)$

and $Pr(X_{a\,n\,N} = 3) = \binom{5}{3}\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^2 = \frac{80}{243} = 0.3292$

$\boxed{\text{Required prob.} = \frac{80}{243} = 0.3292}$ ... $\boxed{\text{3 MARKS}}$

$\boxed{\text{Problem No. 6}}$ (i) $Pr(|X| \geq 2) = Pr(X \leq -2) + Pr(X \geq 2)$

$= \Phi\left(\frac{-2-2}{2}\right) + 1 - \Phi\left(\frac{2-2}{2}\right)$

$= 2 - \Phi(2) - \Phi(0) = \frac{3}{2} - 0.9772$

$\boxed{Pr(|X| \geq 2) = 0.5228}$ ... $\boxed{\text{1 MARK}}$

$Pr(1 < X \leq 3) = Pr(X \leq 3) - Pr(X \leq 1) = \Phi\left(\frac{3-2}{2}\right) - \Phi\left(\frac{1-2}{2}\right) = \Phi\left(\frac{1}{2}\right) - \Phi\left(-\frac{1}{2}\right)$

$= 2\Phi\left(\frac{1}{2}\right) - 1$

$= 2 \times .6915 - 1 = .383$

$\boxed{Pr(1 < X \leq 3) = 0.383}$ ... $\boxed{\text{1 MARK}}$

$Pr(X \leq 3 \mid X \geq 1) = \frac{Pr(1 \leq X \leq 3)}{Pr(X \geq 1)} = \frac{.383}{1 - \Phi\left(\frac{1-2}{2}\right)} = \frac{.383}{\Phi(1/2)} = \frac{.383}{.6915} = .5539$

$\boxed{Pr(X \leq 3 \mid X \geq 1) = .5539}$ ... $\boxed{\text{1 MARK}}$

(ii) $\frac{Z_1+Z_2}{\sqrt{2}} \sim N(0, 1)$ and $\frac{Z_1-Z_2}{\sqrt{2}} \sim N(0, 1)$ are independent

$\Rightarrow \frac{Z_1+Z_2}{2} \sim N(0, 1)$ and $\frac{(Z_1-Z_2)^2}{2} \sim \chi^2_1$ are independent

$$\Rightarrow \frac{(Z_1+Z_2)/\sqrt{2}}{\sqrt{\frac{(Z_1-Z_2)^2/2}{1}}} = \frac{N(0,1)}{\sqrt{\chi^2_1/1}} \Big\} \text{ independent } \sim t_1$$

$$\Rightarrow \boxed{T = \frac{Z_1+Z_2}{|Z_1-Z_2|} \sim t_1 \equiv \text{Cauchy distribution}}$$

... $\boxed{\text{3 MARKS}}$

(iii) $\frac{Z_1+Z_2}{\sqrt{2}}$, $\frac{Z_1-Z_2}{\sqrt{2}}$ and $\frac{Z_3+Z_4}{\sqrt{2}}$ are iid $N(0,1)$

$\Rightarrow \frac{(Z_1+Z_2)^2}{2}$, $\frac{(Z_1-Z_2)^2}{2}$, and $\frac{(Z_3+Z_4)^2}{2}$ are iid χ^2_1

$\Rightarrow \frac{(Z_1+Z_2)^2}{2} \sim \chi^2_1$ and $\frac{(Z_1-Z_2)^2}{2} + \frac{(Z_3+Z_4)^2}{2} \sim \chi^2_2$ are independent

$$\Rightarrow \frac{\frac{(Z_1+Z_2)^2/2}{1}}{\frac{(Z_1-Z_2)^2/2 + (Z_3+Z_4)^2/2}{2}} = \frac{\chi^2_1/1}{\chi^2_2/2} \Big\} \text{ independent } \sim F_{1,2}$$

$$\Rightarrow \boxed{F = 2\,\frac{(Z_1+Z_2)^2}{(Z_1-Z_2)^2+(Z_3+Z_4)^2} \sim F_{1,2}}$$

... $\boxed{\text{3 MARKS}}$

(iv) $\bar{X} \sim N(1, \frac{1}{5})$ and $4S^2 \sim \chi^2_4$ are independent

$\Rightarrow E(\frac{\bar{X}}{S}) = E(\bar{X} \cdot \frac{1}{S}) = E(\bar{X})\,E(\frac{1}{S}) = E(\frac{1}{S})$

$$E(\frac{1}{2S}) = E\left(\frac{1}{\sqrt{\chi^2_4}}\right) = \int_0^\infty \frac{1}{\sqrt{x}} \cdot \frac{1}{2^{4/2}\,\Gamma 4/2}\, e^{-x/2}\, x^{4/2-1}\, dx$$

$$= \frac{1}{4}\int_0^\infty e^{-x/2} x^{3/2-1}\, dx = \frac{2^{3/2}\,\Gamma 3/2}{4}$$

$E(\frac{1}{S}) = \sqrt{\frac{\pi}{2}} \Rightarrow \boxed{E(\frac{\bar{X}}{S}) = \sqrt{\frac{\pi}{2}}}$ ... $\boxed{\text{3 MARKS}}$

Problem No. 7

(a) (i) $X_1 + X_2 \sim Bin(6, 3/4)$

$$Pr(X_1+X_2 = 3) = \binom{6}{3}\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^3 = \frac{405}{512}$$

$$\boxed{Pr(X_1+X_2=3) = \frac{135}{1024}}$$

... $\boxed{\text{3 MARKS}}$

(ii) $X_1 + X_3 \sim Bin(6, 3/8)$

$\Rightarrow Var(X_1 + X_3) = 6 \times 3/8 \times 5/8 = \frac{45}{32}$: $\boxed{Var(X_1+X_3) = \frac{45}{32}}$ ... $\boxed{3 \text{ MARKS}}$

(iii) $X_2 + X_3 \sim Bin(6, 5/8)$, $X_2 \sim Bin(6, \frac{1}{2})$, $X_3 \sim Bin(6, 1/8)$

$Var(X_2 + X_3) = Var(X_2) + Var(X_3) + 2\,Cov(X_2, X_3)$

$6 \times \frac{5}{8} \times \frac{3}{8} = 6 \times \frac{1}{2} \times \frac{1}{2} + 6 \times \frac{1}{8} \times \frac{7}{8} + 2\,Cov(X_2, X_3)$

$\Rightarrow \boxed{Cov(X_2, X_3) = -3/8}$ ... $\boxed{3 \text{ MARKS}}$

(Alt. $Cov(X_2, X_3) = -n p_2 p_3 = -6 \times \frac{1}{2} \times \frac{1}{8} = -3/8$)

(iv) $Pr(X_1 = 3 \mid X_2 + X_3 = 1) = \frac{Pr(X_1 = 3,\ X_2 + X_3 = 1)}{Pr(X_2 + X_3 = 1)}$

$(X_1, X_2 + X_3) \sim Mult(6, \frac{1}{4}, \frac{5}{8})$ and $X_2 + X_3 \sim Bin(6, \frac{5}{8})$

$\Rightarrow Pr(X_1 = 3 \mid X_2 + X_3 = 1) = \frac{\frac{\underline{|6}}{\underline{|3}\ \underline{|1}\ \underline{|2}} \left(\frac{1}{4}\right)^3 \left(\frac{5}{8}\right)^1 \left(\frac{1}{8}\right)^2}{\binom{6}{1}\left(\frac{5}{8}\right)\left(\frac{3}{8}\right)^5} = \frac{80}{243}$

$\boxed{Pr(X_1 = 3 \mid X_2 + X_3 = 1) = \frac{80}{243} = 0.3292}$ ... $\boxed{3 \text{ MARKS}}$

(b) Pr (drawing red ball in each draw) $= \frac{3}{12} = \frac{1}{4}$

Pr (drawing blue ball in each draw) $= \frac{4}{12} = \frac{1}{3}$

X_1 = # of red balls in the balls drawn

X_2 = # of blue balls in the balls drawn

Then $\underline{X} = (X_1, X_2) \sim Mult(5, \frac{1}{4}, \frac{1}{3})$ ... $\boxed{2 \text{ MARKS}}$

Required prob. $= Pr(X_1 = 2, X_2 = 2) = \frac{\underline{|5}}{\underline{|2}\ \underline{|2}\ \underline{|1}} \left(\frac{1}{4}\right)^2 \left(\frac{1}{3}\right)^2 \left(1 - \frac{1}{4} - \frac{1}{3}\right)^1 = \frac{25}{288}$

$\boxed{\text{Required prob.} = \frac{25}{288} = 0.0868}$ ... $\boxed{2 \text{ MARKS}}$

$\boxed{\text{Problem No. 8}}$ (i) $X_2 | X_1 = x_1 \sim N\left(\frac{x_1}{2}, \frac{3}{4}\right)$

$$E\left(e^{tX_1X_2} \mid X_1 = x_1\right) = M_{X_2|X_1=x_1}(t x_1)$$

$$= e^{\frac{x_1^2}{2}\left(t + \frac{3}{4}t^2\right)} \qquad t \in \mathbb{R}\ \ x_1 \in \mathbb{R}$$

$$\boxed{E\left(e^{tX_1X_2} \mid X_1 = x_1\right) = e^{\frac{x_1^2}{2}\left(t + \frac{3}{4}t^2\right)}, \quad t \in \mathbb{R},\ x_1 \in \mathbb{R}}$$

... $\boxed{\text{3 MARKS}}$

(ii) $E\left(e^{tX_1X_2}\right) = E\left(E\left(e^{tX_1X_2} | X_1\right)\right)$

$$= E\left(e^{\frac{X_1^2}{2}\left(t + \frac{3}{4}t^2\right)}\right) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}\left(1 - \left(t + \frac{3}{4}t^2\right)\right)} dz$$

$$\boxed{E\left(e^{tX_1X_2}\right) = \left[1 - \left(t + \frac{3}{4}t^2\right)\right]^{-\frac{1}{2}}, \quad -2 < t < \frac{2}{3}}$$

... $\boxed{\text{3 MARKS}}$

(iii) $M_Y(t) = E\left(e^{tX_1X_2}\right) = \left[1 - \left(t + \frac{3}{4}t^2\right)\right]^{-\frac{1}{2}}$

$$\ln M_Y(t) = -\frac{1}{2} \ln\left[1 - \left(t + \frac{3}{4}t^2\right)\right]$$

$$\frac{d}{dt} \ln M_Y(t) = \frac{1}{2} \frac{1 + \frac{3}{2}t}{1 - \left(t + \frac{3}{4}t^2\right)}$$

$$\frac{d^2}{dt^2} \ln M_Y(t) = \frac{1}{2} \frac{\frac{3}{2}\left[1 - \left(t + \frac{3}{4}t^2\right)\right] + \left(1 + \frac{3}{2}t\right)^2}{\left[1 - \left(t + \frac{3}{4}t^2\right)\right]^2}$$

$$E(Y) = \left[\frac{d}{dt} \ln M_Y(t)\right]_{t=0} = \frac{1}{2} \qquad \boxed{E(Y) = \frac{1}{2}} \ \ldots \ \boxed{\text{1 MARK}}$$

$$Var(Y) = \left[\frac{d^2}{dt^2} \ln M_Y(t)\right]_{t=0} = \frac{5}{4} \qquad \boxed{Var(Y) = \frac{5}{4}} \ \ldots \ \boxed{\text{2 MARK}}$$

(IV) Consider

$c_1Y_1 + c_2Y_2 = (c_1 + c_2)X_1 + c_2X_2 \sim N_1(\cdot, \cdot)$ (since $(X_1, X_2) \sim N_2(\cdot, \cdot)$)

$\Rightarrow \underline{Y} = (Y_1, Y_2) \sim N_2(\cdot, \cdot, \cdot, \cdot, \cdot)$

$E(Y_1) = E(X_1) = 0$; $E(Y_2) = E(X_1 + X_2) = 0$; $Var(Y_1) = Var(X_1) = 1$,

$Var(Y_2) = Var(X_1 + X_2) = 2+1 = 3$, $Cov(Y_1, Y_2) = \frac{3}{2}$, $\rho = Corr(Y_1, Y_2) = \frac{\sqrt{3}}{2}$

Thus $\underline{Y} = (Y_1, Y_2) \sim N_2\left(0, 0, 1, 3, \frac{1}{\sqrt{3}}\right)$ and the p.d.f. of $\underline{Y} = (Y_1, Y_2)$ is

$$\boxed{f_{Y_1, Y_2}(y_1, y_2) = \frac{1}{\pi\sqrt{3}} e^{-2\left(y_1^2 + \frac{y_2^2}{3} - y_1y_2\right)}, \quad -\infty < y_i < \infty \quad i = 1, 2.}$$

... $\boxed{\text{3 MARKS}}$

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