MSO 201a: Probability & Statistics

Assignment-III

Solutions

Problem No. 1 (a) We have $S_X = \{0, 1, 2, \ldots\}$ and

$$F_X(x) = P(X \le x) = \begin{cases} 0, & x < 0 \\ \sum_{j=0}^{i} \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^j, & i \le x < i+1,\ i = 0, 1, \ldots \end{cases}$$

$$= \begin{cases} 0, & \\ 1 - \left(\frac{2}{3}\right)^{i+1}, & i \le x < i+1,\ i = 0, 1, \ldots \end{cases}$$

$Y = \frac{X}{X+1} \uparrow \Rightarrow S_Y = \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\}$

$$F_Y(y) = P(Y \le y) = P\left(\frac{X}{1+X} \le y\right) = \begin{cases} F_X\left(\frac{y}{1-y}\right), & y < 1 \\ 1, & y \ge 1 \end{cases}$$

$$= \begin{cases} 0, & y < 0 \\ 1 - \left(\frac{2}{3}\right)^{i+1}, & \frac{i}{i+1} \le y < \frac{i+1}{i+2},\ i = 0, 1, 2, \ldots \\ 1, & y \ge 1 \end{cases}$$

For $y \notin S_Y = \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\}$, $P(Y = y) = 0$, and for $y \in S_Y$

$$P(Y = y) = F_Y(y) - F_Y(y^-) = \left(1 - \left(\frac{2}{3}\right)^{\frac{y}{1-y}+1}\right) - \left(1 - \left(\frac{2}{3}\right)^{\frac{y}{1-y}}\right)$$

$$= \frac{1}{3}\left(\frac{2}{3}\right)^{\frac{y}{1-y}}$$

Thus

$$f_Y(y) = \begin{cases} \frac{1}{3}\left(\frac{2}{3}\right)^{\frac{y}{1-y}}, & \text{if } y \in S_Y = \{0, \frac{1}{2}, \frac{2}{3}, \ldots\} \\ 0, & \text{otherwise.} \end{cases}$$

(b) As in (a) $S_Y = \{0, \frac{1}{2}, \frac{2}{3}, \dots\}$. For $y \in S_Y$

$$P(Y=y) = P\left(\frac{X}{1+X} = y\right) = P\left(X = \frac{y}{1-y}\right) = \frac{1}{3}\left(\frac{2}{3}\right)^{\frac{y}{1-y}}$$

$$\Rightarrow f_Y(y) = \begin{cases} \frac{1}{3}\left(\frac{2}{3}\right)^{\frac{y}{1-y}}, & y \in \{0, \frac{1}{2}, \frac{2}{3}, \dots\} \\ 0, & \text{otherwise} \end{cases}$$

$$F_Y(y) = P(Y \le y) = \begin{cases} 0, & y < 0 \\ \sum_{x \in \{0, \frac{1}{2}, \dots, \frac{i}{i+1}\}} \frac{1}{3}\left(\frac{2}{3}\right)^{\frac{x}{1-x}}, & \frac{i}{i+1} \le y < \frac{i+1}{i+2}, \; i = 0, 1, 2, \dots \\ 1, & y \ge 1 \end{cases}$$

$$= \begin{cases} 0, & y < 0 \\ \sum_{t=0}^{i} \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^{t}, & \frac{i}{i+1} \le y < \frac{i+1}{i+2}, \; i = 0, 1, 2, \dots \\ 1, & y \ge 1 \end{cases}$$

$$= \begin{cases} 0, & y < 0 \\ 1 - \left(\frac{2}{3}\right)^{i+1}, & \frac{i}{i+1} \le y < \frac{i+1}{i+2}, \; i = 0, 1, 2, \dots \\ 1, & y \ge 1 \end{cases}$$

(c). We will use the fact that for $|t| < 1$ and $r \in \{1, 2, \dots\}$

$$(1-t)^{-r} = \sum_{k=0}^{\infty} \binom{r+k-1}{k} t^k.$$

$$E(X) = \sum_{x \in S_X} x f_X(x) = \sum_{k=0}^{\infty} k \frac{1}{3}\left(\frac{2}{3}\right)^k = \frac{1}{3} \sum_{k=1}^{\infty} k \left(\frac{2}{3}\right)^k = \frac{1}{3} \sum_{k=0}^{\infty} (k+1)\left(\frac{2}{3}\right)^{k+1}$$

$$= \frac{1}{3} \times \frac{2}{3} \sum_{k=0}^{\infty} \binom{1+k}{k}\left(\frac{2}{3}\right)^k = \frac{2}{9}\left(1 - \frac{2}{3}\right)^{-2} = 2$$

$$E(X(X-1)) = \sum_{k=0}^{\infty} k(k-1) \frac{1}{3}\left(\frac{2}{3}\right)^k = \frac{1}{3} \sum_{k=2}^{\infty} k(k-1)\left(\frac{2}{3}\right)^k = \frac{1}{3} \sum_{k=0}^{\infty} (k+2)(k+1)\left(\frac{2}{3}\right)^{k+2}$$

$$= \frac{1}{3} \times \left(\frac{2}{3}\right)^2 \times 2 \sum_{k=0}^{\infty} \binom{2+k}{k}\left(\frac{2}{3}\right)^k = \frac{8}{27}\left(1 - \frac{2}{3}\right)^{-3} = 8$$

$$E(X^2) = E(X(X-1)) + E(X) = 10$$

$$Var(X) = E(X^2) - (E(X))^2 = 10 - 4 = 6.$$

Problem No. 2

(a) We have $Y = X^2$ and $P(X \in (-2, -1) \cup (0, 3)) = 1$. Thus $P(Y \in (0, 9)) = 1$, $F_Y(y) = 0$, for $y < 0$ and $F_Y(y) = 1$, for $y \ge 9$.

For $0 \le y < 9$,

$$F_Y(y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

(X is of continuous type)

We have

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt = \begin{cases} 0, & x < -2 \\ \frac{x+2}{2}, & -2 \le x < -1 \\ \frac{1}{2}, & -1 \le x < 0 \\ \frac{1}{2} + \frac{x}{6}, & 0 \le x < 3 \\ 1, & x \ge 3 \end{cases}$$

$$\Rightarrow F_Y(y) = F_X(\sqrt{y}) - F_X(-\sqrt{y}) = \begin{cases} 0, & y < 0 \\ \frac{\sqrt{y}}{6}, & 0 \le y < 1 \\ \frac{2}{3}\sqrt{y} - \frac{1}{2}, & 1 \le y < 4 \\ \frac{\sqrt{y}}{6} + \frac{1}{2}, & 4 \le y < 9 \\ 1, & y \ge 9 \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{1}{12\sqrt{y}}, & 0 < y < 1 \text{ or } 4 < y < 9 \\ \frac{1}{3\sqrt{y}}, & 1 < y < 4 \\ 0, & \text{otherwise} \end{cases}$$

(b) $S_X = [-2, -1] \cup [0, 3]$ and $\{x \in \mathbb{R} : f_X(x) > 0\} = (-2, -1) \cup (0, 3)$; $h(x) = x^2$ is strictly $\downarrow$ on $(-2, -1)$ and strictly $\uparrow$ on $(0, 3)$; $h_1^{-1}(y) = -\sqrt{y}$, $h_2^{-1}(y) = \sqrt{y}$.

Clearly Y is of continuous type with pdf

$$f_Y(y) = f_X(h_1^{-1}(y)) \left| \frac{d}{dy} h_1^{-1}(y) \right| I_{h(-2,-1)}(y) + f_X(h_2^{-1}(y)) \left| \frac{d}{dy} h_2^{-1}(y) \right| I_{h(0,3)}(y)$$

$$= f_X(-\sqrt{y}) \left| -\frac{1}{2\sqrt{y}} \right| I_{(1,4)}(y) + f_X(\sqrt{y}) \left| \frac{1}{2\sqrt{y}} \right| I_{(0,9)}(y)$$

$$= \begin{cases} \frac{1}{12\sqrt{y}}, & y \in (0,1) \cup (4,9) \\ \frac{1}{3\sqrt{y}}, & y \in (1,4) \\ 0, & \text{otherwise.} \end{cases}$$

$$F_Y(y) = \int_{-\infty}^{y} f_X(x)\,dx = \begin{cases} 0, & y < 0 \\ \int_0^y \frac{1}{12\sqrt{x}}\,dx, & 0 \le y < 1 \\ \int_0^1 \frac{1}{12\sqrt{x}}\,dx + \int_1^y \frac{1}{3\sqrt{x}}\,dx, & 1 \le y < 4 \\ \int_0^1 \frac{1}{12\sqrt{x}}\,dx + \int_1^4 \frac{1}{3\sqrt{x}}\,dx + \int_4^y \frac{1}{12\sqrt{x}}\,dx, & 4 \le y < 9 \\ 1, & y \ge 9 \end{cases}$$

$$= \begin{cases} 0, & y < 0 \\ \frac{\sqrt{y}}{6}, & 0 \leq y < 1 \\ \frac{2}{3}\sqrt{y} - \frac{1}{2}, & 1 \leq y < 4 \\ \frac{\sqrt{y}}{6} + \frac{1}{2}, & 4 \leq y < 9 \\ 1, & y \geq 9 \end{cases}.$$

(c) $E(X) = \int_{-\infty}^{\infty} x f_X(x)\, dx = \int_{-2}^{-1} \frac{x}{2} dx + \int_{0}^{3} \frac{x}{6} dx = 0$

$E(X^2) = \int_{-2}^{-1} \frac{x^2}{2} dx + \int_{0}^{3} \frac{x^2}{6} dx = \frac{8}{3}$

$Var(X) = E(X^2) - (E(X))^2 = \frac{8}{3}.$

$\boxed{\text{Problem No. 3}}$ (a) We will use a result from MTH-101 that the series $\sum_{n=1}^{\infty} \frac{1}{n^r}$ converges for $r > 1$ and diverges for $r \leq 1$ ($\sum_{n=1}^{\infty} \frac{1}{n^r} = \infty$, for $r \leq 1$).

$0 < \sum_{n=1}^{\infty} \frac{1}{n^r} < \infty$, for $r > 1$,

Now let X be a random variable with p.m.f.

$$f_X(x) = \begin{cases} \frac{c}{x^3}, & x \in \{1, 2, 3, \dots\} \\ 0, & \text{otherwise} \end{cases},$$

where $c = \left(\sum_{n=1}^{\infty} \frac{1}{n^3}\right)^{-1}$. Then X is of discrete type with

$S_X = \{1, 2, \dots\}$

$E(X) = \sum_{x \in S_X} x f_X(x) dx = c \sum_{x=1}^{\infty} \frac{1}{x^2} < \infty$

and $E(X^2) = \sum_{x \in S_X} x f_X(x) dx = c \sum_{x=1}^{\infty} \frac{1}{x} = \infty$

(b) Note that

$\int_1^{\infty} \frac{1}{x^r} dx = \frac{1}{r-1}$, for $r > 1$, and $\int_1^{\infty} \frac{1}{x^r} dx = \infty$, for $r \leq 1$.

Let X be a random variable with p.d.f.

$$f_X(x) = \begin{cases} \frac{2}{x^3}, & x > 1 \\ 0, & \text{otherwise} \end{cases}$$

Then

$E(X) = \int_{-\infty}^{\infty} x f_X(x) dx = 2 \int_1^{\infty} \frac{1}{x^2} dx = 2$

$E(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x) \cdot dx = 2 \int_1^{\infty} \frac{1}{x} dx = \infty.$

Problem No. 4 $S_X = \{-2, -1, 0, 1, 2, 3\}$, $Y = X^2$, $S_Y = \{0, 1, 4, 9\}$

$P(Y=0) = P(X=0) = \frac{1}{7}$; $P(Y=1) = P(X \in \{-1, 1\}) = \frac{2}{21} + \frac{4}{21} = \frac{2}{7}$

$P(Y=4) = P(X \in \{-2, 2\}) = \frac{1}{21} + \frac{5}{21} = \frac{2}{7}$; $P(Y=9) = P(X \in \{-3, 3\}) = 0 + \frac{2}{7} = \frac{2}{7}$.

The p.m.f. of Y is

$$f_Y(y) = P(Y=y) = \begin{cases} \frac{1}{7}, & y=0 \\ \frac{2}{7}, & y=1, 4, 9 \\ 0, & \text{otherwise} \end{cases}$$

The distribution function of $Y = X^2$ is

$$F_Y(y) = \begin{cases} 0, & y<0 \\ 1/7, & 0 \le y < 1 \\ 3/7, & 1 \le y < 4 \\ 5/7, & 4 \le y < 9 \\ 1, & y \ge 9 \end{cases}$$

Problem No. 5 $\{x \in \mathbb{R} : f_X(x) > 0\} = (0, 1)$

(a) $h(x) = \sqrt{x}$ is strictly $\uparrow$ on $(0,1)$; $h(0,1) = (0,1)$, $h^{-1}(y) = y^2$, $y \in (0,1)$.
Thus Y_1 is of continuous type with p.d.f.

$$f_{Y_1}(y) = f_X(h^{-1}(y)) \left| \frac{d}{dy} h^{-1}(y) \right| I_{h(0,1)}(y)$$

$$= \begin{cases} 2y, & 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$

(b) $h(x) = x^2$ is strictly $\uparrow$ on $(0,1)$; $h(0,1) = (0,1)$, $h^{-1}(y) = \sqrt{y}$, $y \in (0,1)$.
Thus Y_2 is of continuous type with p.d.f.

$$f_{Y_2}(y) = f_X(h^{-1}(y)) \left| \frac{d}{dy} h^{-1}(y) \right| I_{h(0,1)}(y) = \begin{cases} \frac{1}{2\sqrt{y}}, & 0<y<1 \\ 0, & \text{otherwise} \end{cases}$$

(c) $h(x) = 2x+3$ is strictly $\uparrow$ on $(0,1)$; $h(0,1) = (3,5)$, $h^{-1}(y) = \frac{y-3}{2}$. Thus
Y_3 is of continuous type with p.d.f.

$$f_{Y_3}(y) = f_X(h^{-1}(y)) \left| \frac{d}{dy} h^{-1}(y) \right| I_{h(0,1)}(y) = \begin{cases} \frac{1}{2}, & 3<y<5 \\ 0, & \text{otherwise} \end{cases}$$

(d) $h(x) = -\ln x$ is strictly $\downarrow$ on $(0,1)$; $h(0,1) = (0, \infty)$, $h^{-1}(y) = e^{-y}$, $y>0$.
Thus $Y_4 = -\ln X$ is of continuous type with p.d.f.

$$f_{Y_4}(y) = f_X(h^{-1}(y)) \left| \frac{d}{dy} h^{-1}(y) \right| I_{h(0,1)}(y) = \begin{cases} e^{-y} & y>0 \\ 0 & \text{otherwise} \end{cases}$$

Problem No. 6 $\{x \in \mathbb{R} : f_X(x) > 0\} = (0, 1)$, $h(x) = x^2(3-2x)$; $h'(x) = 6x(1-x) > 0 \ \forall x \in (0,1)$

$\Rightarrow h$ is strictly $\uparrow$ on $(0,1)$; $h((0,1)) = (0,1)$.

(a) $F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt = \begin{cases} 0 & x < 0 \\ x^2(3-2x), & 0 \le x < 1 \\ 0 & \text{Otherwise} \end{cases} = \begin{cases} 0 & x < 0 \\ h(x), & 0 \le x < 1 \\ 1, & x \ge 1 \end{cases}$

For $y \in (0,1)$ so that $h^{-1}(y) \in (0,1)$

$F_Y(y) = P(h(X) \le y) = P(h(X) \le y,\ X \in (0,1)) = P(X \le h^{-1}(y),\ X \in (0,1)) = F_X(h^{-1}(y))$

$= h(h^{-1}(y)) = y.$

Obviously, for $y < 0$, $F_Y(y) = 0$ and, for $y \ge 1$, $F_Y(y) = 1$. Thus

$$F_Y(y) = \begin{cases} 0, & y < 0 \\ y, & 0 \le y < 1 \\ 1 & y \ge 1 \end{cases}.$$

F_Y is continuous everywhere and differentiable everywhere except at 0 and 1. Thus Y is of continuous type with pdf

$$f_Y(y) = \begin{cases} 1 & 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

(b) Clearly Y is of continuous type with p.d.f.

$f_Y(y) = f_X(h^{-1}(y)) \left| \frac{d}{dy} h^{-1}(y) \right| I_{h((0,1))}(y) = \begin{cases} 1, & 0 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$

(c) $E(Y) = \int_{-\infty}^{\infty} y f_Y(y)\,dy = \int_0^1 y\,dy = \frac{1}{2}$

$E(Y^2) = \int_{-\infty}^{\infty} y^2 f_Y(y)\,dy = \int_0^1 y^2\,dy = \frac{1}{3}$

$Var(Y) = E(Y^2) - (E(Y))^2 = \frac{1}{12}.$

Problem No. 7 (a) $S_X = \{1, 2, \ldots, N\}$. For $x \in S_X$, $P(X \le x) = \left(\frac{x}{N}\right)^n$. Thus,

for $x \in S_X$,

$f_X(x) = P(X = x) = P(X \le x) - P(X \le x-1) = \left(\frac{x}{N}\right)^n - \left(\frac{x-1}{N}\right)^n$

$E(X) = \sum_{x=1}^{N} x \left[\left(\frac{x}{N}\right)^n - \left(\frac{x-1}{N}\right)^n \right] = N - \frac{1}{N^n} \sum_{x=1}^{N-1} x^n.$

(b) Let X = # of throws required to get a 6. Then $S_X = \{1, 2, 3, \ldots\}$

$f_X(x) = \left(\frac{5}{6}\right)^{x-1} \frac{1}{6}$, $x \in S_X$. Thus

$E(X) = \sum_{x=1}^{\infty} x \left(\frac{5}{6}\right)^{x-1} \frac{1}{6} = 6.$

Problem No. 8 Let Z = score on a shot. Then $S_Z = \{0, 2, 3, 4\}$

$$P(Z=0) = P(X > \sqrt{3}) = \int_{\sqrt{3}}^{\infty} \frac{2}{\pi(1+x^2)}\,dx = \frac{2}{\pi}\left[\tan^{-1}x\right]_{\sqrt{3}}^{\infty} = \frac{1}{3}$$

$$P(Z=2) = P(1 < X < \sqrt{3}) = \int_{1}^{\sqrt{3}} \frac{2}{\pi(1+x^2)}\,dx = \frac{1}{6}$$

$$P(Z=3) = P\left(\frac{1}{\sqrt{3}} < X < 1\right) = \int_{\frac{1}{\sqrt{3}}}^{1} \frac{2}{\pi(1+x^2)}\,dx = \frac{1}{6}$$

$$P(Z=4) = P\left(X < \frac{1}{\sqrt{3}}\right) = \int_{0}^{\frac{1}{\sqrt{3}}} \frac{2}{\pi(1+x^2)}\,dx = \frac{1}{3}$$

$$E(Z) = \sum_{z \in S_Z} z\,P(Z=z) = 0 \times \frac{1}{3} + 2 \times \frac{1}{6} + 3 \times \frac{1}{6} + 4 \times \frac{1}{3} = \frac{13}{6}.$$

Problem No. 9 (a) We have $F_X(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \le x < 1 \\ 1, & x \ge 1 \end{cases}.$

$$F_Y(y) = P\left(\min\{X, \tfrac{1}{2}\} \le y\right) = 1 - P\left(\min\{X, \tfrac{1}{2}\} > y\right) = 1 - P\left(X > y, \tfrac{1}{2} > y\right)$$

$$= 1 - \begin{cases} P(X > y) & y < \frac{1}{2} \\ 0 & y \ge \frac{1}{2} \end{cases} = \begin{cases} F_X(y), & y < \frac{1}{2} \\ 1, & y \ge \frac{1}{2} \end{cases}.$$

D = set of discontinuity pts of $F_Y = \{\frac{1}{2}\}$;

sum of jump sizes $= F_Y(\frac{1}{2}) - F_Y(\frac{1}{2}-) = 1 - F_X(\frac{1}{2}) = \frac{1}{2} \in (0, 1)$

$\Rightarrow$ X is neither of continuous type $(D \neq \phi)$ nor

X is of discrete type (sum of jump sizes $\neq 1$).

(b) $$P(Y=-1) = P(X<0) = \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-x^2/2}\,dx = \frac{1}{2\sqrt{\pi}} \int_{0}^{\infty} e^{-t}\, t^{1/2-1}\,dt = \frac{1}{2}$$

$P(Y=\frac{1}{2}) = P(X=0) = 0$ (since X is of type)

$$P(Y=1) = P(X>0) = \int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx = \frac{1}{2}.$$

Let $S_Y = \{-1, 1\}$. Then $P(Y=y) > 0 \ \forall y \in S_Y$ and $P(Y \in S_Y) = 1$. Thus Y is of discrete type with support $S_Y = \{-1, 1\}$ and p.m.f.

$$f_Y(y) = \begin{cases} \frac{1}{2}, & y = -1, 1 \\ 0, & \text{otherwise}. \end{cases}$$

Problem No. 10 (a) We have, for $0<\alpha\le\beta<\infty$,

$$E(|X|^{\alpha})=\int_{|x|\le 1}|x|^{\alpha}f_X(x)\,dx+\int_{|x|>1}|x|^{\alpha}f_X(x)\,dx$$

$$\le \int_{|x|\le 1}f_X(x)\,dx+\int_{|x|>1}|x|^{\beta}f_X(x)\,dx$$

$$\le 1+\int_{-\infty}^{\infty}|x|^{\beta}f_X(x)\,dx$$

$$=1+E(|X|^{\beta})<\infty,$$

provided $E(|X|^{\beta})<\infty$.

(b) $E(X)$ is finite $\Rightarrow \lim_{x\to-\infty}\int_{-\infty}^{x}t f_X(t)\,dt=0$ and $\lim_{x\to\infty}\int_{x}^{\infty}t f_X(t)\,dt=0$

$$0=\lim_{x\to-\infty}\int_{-\infty}^{x}t f_X(t)\,dt\le\lim_{x\to-\infty}\left[\int_{-\infty}^{x}x f_X(t)\,dt\right]=\lim_{x\to-\infty}[xF_X(x)]\le 0$$

$$\Rightarrow\quad \lim_{x\to-\infty}xF_X(x)=0$$

Also

$$0\le\lim_{x\to\infty}[x(1-F_X(x))]\le\lim_{x\to\infty}\int_{x}^{\infty}t f_X(t)\,dt=0$$

$$\Rightarrow\quad \lim_{x\to\infty}[x(1-F_X(x))]=0$$

(c) Let X be a rv with p.d.f. $f_X(x)=\begin{cases}\frac{c}{x^2\ln x}, & x>2\\ 0 & \text{otherwise}\end{cases}$. Then, for $\alpha=1$,

$x^{\alpha}P(|X|>x)=cx\int_{x}^{\infty}\frac{1}{t^2\ln t}\,dt\to 0$, as $x\to\infty$. Thus $E(|X|^{\beta})<\infty$, $\forall\ 0<\beta<\alpha=1$. However

$$E(|X|^{\alpha})=E(|X|)=c\int_{2}^{\infty}\frac{1}{x\ln x}\,dx=\infty.$$

Problem No. 11 (a) Let X be the random variable corresponding to m.g.f. $M(t)=(1-t)^{-3}$, $t<1$. We have $M^{(1)}(t)=3(1-t)^{-4}$, $M^{(2)}(t)=12(1-t)^{-5}$, ..., $M^{(r)}(t)=\frac{\lfloor r+2}{2}(1-t)^{-(r+3)}$, ...

$$\Rightarrow\quad M(t)=\sum_{r=0}^{\infty}\frac{t^r}{\lfloor r}M^{(r)}(0)=\sum_{r=0}^{\infty}\frac{t^r}{\lfloor r}\,\frac{\lfloor r+2}{2},\quad t<1$$

$\Rightarrow\quad \mu_r'=E(X^r)=$ Coeff. of $\frac{t^r}{\lfloor r}$ in the expansion of $M(t)$ $=\frac{\lfloor r+2}{2}$.

(b) Clearly, $M(t) = E(e^{tX}) = \frac{e^{-t}}{8} + \frac{e^{t}}{4} + \frac{e^{2t}}{8} + \frac{e^{3t}}{2}$, $t \in \mathbb{R}$, is the m.g.f. of random variable X having the p.m.f.

$$f_X(x) = \begin{cases} \frac{1}{8}, & x = -1, 2 \\ \frac{1}{4}, & x = 1 \\ \frac{1}{2}, & x = 3 \\ 0, & \text{otherwise} \end{cases}$$

The d.f. of X is

$$F_X(x) = \begin{cases} 0, & x < -1 \\ 1/8, & -1 \le x < 1 \\ 3/8, & 1 \le x < 2 \\ 1/2, & 2 \le x < 3 \\ 1, & x \ge 3 \end{cases}.$$

$$P(X^2 = 1) = P(X = -1) + P(X = 1) = \frac{1}{8} + \frac{1}{4} = 3/8.$$

(c) Clearly $M(t) = \frac{e^{t} - e^{-2t}}{3t}$, $t \neq 0$, is the m.g.f. of a rv X having the p.d.f.

$$f_X(x) = \begin{cases} \frac{1}{3}, & -2 < x < 1 \\ 0, & \text{otherwise} \end{cases}.$$

$S_X^* = \{x \in \mathbb{R} : f_X(x) > 0\} = (-2, 1)$, $Y = X^2$; $h(x) = x^2$, $x \in S_X^*$.

$S_{1,X}^* = (-2, 0)$ $\qquad$ $S_{2,X}^* = (0, 1)$

$h_1^{-1}(y) = -\sqrt{y}$ $\qquad$ $h_2^{-1}(y) = \sqrt{y}$

$h(S_{1,X}) = (0, 4)$ $\qquad$ $h(S_{2,X}) = (0, 1)$

Clearly Y is of continuous type with p.d.f.

$$f_Y(y) = f_X(h_1^{-1}(y)) \left| \frac{d}{dy} h_1^{-1}(y) \right| I_{h(S_{1,X}^*)}(y) + f_X(h_2^{-1}(y)) \left| \frac{d}{dy} h_2^{-1}(y) \right| I_{h(S_{2,X}^*)}(y)$$

$$= f_X(-\sqrt{y}) \frac{1}{2\sqrt{y}} I_{(0,4)}(y) + f_X(\sqrt{y}) \frac{1}{2\sqrt{y}} I_{(0,1)}(y)$$

$$= \begin{cases} \frac{1}{3\sqrt{y}}, & 0 < y < 1 \\ \frac{1}{6\sqrt{y}}, & 1 < y < 4 \\ 0, & \text{otherwise} \end{cases}$$

Problem No. 13 (a) $M_{X_p}(t) = E(e^{tX_p}) = \sum_{k=0}^{n} e^{tk} \binom{n}{k} p^k q^{n-k}$

$$= \sum_{k=0}^{n} \binom{n}{k} (pe^t)^k q^{n-k}$$

$$= (q + pe^t)^n, \quad t \in \mathbb{R}$$

For $t \in \mathbb{R}$,

$$\psi_{X_p}(t) = \ln M_{X_p}(t) = n \ln (q + pe^t)^n$$

$$\psi^{(1)}_{X_p}(t) = \frac{npe^t}{q+pe^t}, \quad \psi^{(2)}_{X_p}(t) = np \frac{(q+pe^t)e^t - pe^{2t}}{(q+pe^t)^2}$$

$$\Rightarrow E(X) = \psi^{(1)}_{X_p}(0) = np$$

$$Var(X) = \psi^{(2)}_{X_p}(0) = npq.$$

(b) For $t \in \mathbb{R}$,

$$M_{Y_p}(t) = E(e^{tY_p}) = e^{nt} E(e^{-tX_p}) = e^{nt} M_{X_p}(-t)$$

$$= e^{nt}(1-p+pe^{-t})^n = (p + (1-p)e^t)^n = M_{X_{1-p}}(t)$$

$$\Rightarrow Y_p \stackrel{d}{=} X_{1-p}$$

$$\Rightarrow f_{Y_p}(y) = f_{X_{1-p}}(y)$$

$$= \begin{cases} \binom{n}{y} q^y (1-q)^{n-y}, & y \in \{0, 1, \cdots, n\} \\ 0, & \text{otherwise}. \end{cases}$$

Problem No. 13 Let $h: \mathbb{R} \to \mathbb{R}$ be defined by

$$h(c) = E((X-c)^2) = c^2 - 2cE(X) + E(X^2), \quad c \in \mathbb{R}$$

Then

$$h'(c) = 2c - 2E(X) \quad \text{and} \quad h''(c) = 2 > 0.$$

It follows that has has a minimum at $c = E(X) = \mu$

$$\Rightarrow h(c) \geq h(\mu), \quad \forall c \in \mathbb{R}$$

$$\Rightarrow E((X-c)^2) \geq E((X-\mu)^2), \quad \forall c \in \mathbb{R}.$$

(b) Consider $\Delta = E(|X-c|) - E(|X-\mu|)$.

Case I $\quad -\infty < c < m$

$$\Delta = \int_{-\infty}^{c} (c-x) f_X(x)\,dx + \int_{c}^{\infty} (x-c) f_X(x)\,dx - \int_{-\infty}^{m} (m-x) f_X(x)\,dx - \int_{m}^{\infty} (x-m) f_X(x)\,dx$$

$= 2c F_X(c) - c + 2\int_{c}^{m} x f_X(x)\,dx \quad$ (using $F_X(m) = \frac{1}{2}$)

$\geq 2c F_X(c) - c + 2c\,[F_X(m) - F_X(c)] = 0 \quad$ (again using $F_X(m) = \frac{1}{2}$)

Case II $\quad -\infty < m < c < \infty$

$$\Delta = 2cF_X(c) - c - 2\int_{m}^{c} x f_X(x)\,dx \geq 2cF_X(c) - c - 2c\,[F_X(c) - F_X(m)] = 0.$$

Problem No. 14 $\quad$ (a) $E(\psi(X)) = \int_0^\infty \psi(x) f_X(x)\,dx = \int_0^\infty \int_0^x h(t) f_X(x)\,dt\,dx$

$= \int_0^\infty \int_t^\infty h(t) f_X(x)\,dx\,dt \quad$ (change of order of integration is allowed as integrand is non-negative)

$= \int_0^\infty h(t) \left(\int_t^\infty f_X(x)\,dx\right) dt = \int_0^\infty h(t)\, P(X>t)\,dt.$

(b) Taking $h(t) = \alpha t^{\alpha-1}$, $t \in (0, \infty)$ in (a) we have

$$E(X^\alpha) = \alpha \int_0^\infty t^{\alpha-1} P(X>t)\,dt.$$

(c) $F(t) \geq G(t)$, $\forall t>0 \;\cdots \Rightarrow P(Y>t) \geq P(X>t)$, $\forall t>0$

$\Rightarrow E(Y^k) = k\int_0^\infty t^{k-1} P(Y>t)\,dt \geq k\int_0^\infty t^{k-1} P(X>t)\,dt = E(X^k)$

(Note that $F(0) = G(0) = 0 \Rightarrow S_X, S_Y \subseteq [0, \infty)$ or $P(X \geq 0) = P(Y \geq 0) = 1$).

Problem No. 15 $\quad$ (a) $P(X \geq 2\mu) = P(|X| \geq 2\mu) \leq \frac{E(|X|)}{2\mu} = \frac{E(X)}{2\mu} = \frac{1}{2}$

(b) $\mu = E(X) = 3$, $\sigma^2 = Var(X) = E(X^2) - (E(X))^2 = 4$. Thus

$P(-2 < X < 8) = P\left(-\frac{5}{2} < \frac{X-\mu}{\sigma} < \frac{5}{2}\right) = P\left(\left|\frac{X-\mu}{\sigma}\right| < \frac{5}{2}\right) =$

$= 1 - P\left(\left|\frac{X-\mu}{\sigma}\right| \geq 5/2\right) = 1 - P\left(|X-\mu| \geq \frac{5}{2}\sigma\right) \geq 1 - \frac{4}{25}$

$= \frac{21}{25}.$

Problem No. 16 (a) Let $X =$ # of telephone calls received on a typical day; Then $P(X \geq 0) = 1$, $\mu = E(X) = 25000$. Therefore

$$P(X \geq 30000) \leq \frac{E(X)}{30000} = \frac{5}{6} \approx 0.83..$$

(b) Let $X =$ # of telephone calls received on a typical day. Then $\mu = E(X) = 20{,}000$ and $\sigma^2 = Var(X) = 2500$. Therefore

$$P(19900 \leq X \leq 20100) = P(-100 \leq X - \mu \leq 100) = P(|X-\mu| \leq 100)$$

$$\geq 1 - \frac{\sigma^2}{100^2} = 0.75.$$

$$P(X \geq 20{,}200) = P(X-\mu \geq 200) \leq P(|X-\mu| \leq 200) \leq \frac{\sigma^2}{(200)^2} = \frac{1}{16}$$

(using Chebyshev's Ineq.)

Using Markov's inequality we have

$$P(X \geq 20{,}200) \leq \frac{E(X)}{20{,}200} = \frac{100}{101}.$$

Thus the knowledge of variance substantially improves the bound.

Problem No. 17 (For continuous case)

(a) + (b) $M(t) = \int_{-\infty}^{\infty} e^{tx} f_X(x)\,dx \geq \int_a^{\infty} e^{tx} f_X(x)\,dx$. Also $M(t) \geq \int_{-\infty}^{a} e^{tx} f_X(x)\,dx$

Thus, for $0 < t < h$, $M(t) \geq \int_a^{\infty} e^{tx} f_X(x)\,dx \geq e^{at} \int_a^{\infty} f_X(x)\,dx$

$$= e^{at} P(X \geq a)$$

and, for $-h < t < 0$, $M(t) \geq \int_{-\infty}^{a} e^{tx} f_X(x)\,dx \geq e^{at} \int_{-\infty}^{a} f_X(x)\,dx$

$$= e^{at} P(X \leq a).$$

For discrete case replace $\int$ by Σ.

(c) Clearly $M(t)$ is the m.g.f. of r.v. X having the p.d.f

$$f(x) = \frac{3}{4} e^{-3x} + \frac{6}{4} e^{-2x}, \quad x > 0.$$

Thus

$$P(X > 1) = \int_1^{\infty} f(x)\,dx = \frac{e^{-3}}{4} + \frac{3e^{-2}}{4}.$$

Problem No. 18 (a) Clearly $f_{X_{\mu,\sigma}}(x) \geq 0 \ \forall x \in \mathbb{R}$. Also

$$\int_{-\infty}^{\infty} f_{X_{\mu,\sigma}}(x)\,dx = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}}\,dz = I, \text{ (say)}.$$

Clearly $I \geq 0$ and

$$I^2 = \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\,dy \right)\left(\frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\,dz \right)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{y^2+z^2}{2}}\,dy\,dz$$

$$= \frac{1}{2\pi} \int_{0}^{\infty}\int_{0}^{2\pi} r e^{-\frac{r^2}{2}}\,d\theta\,dr$$

(on making the transformation $y = r\cos\theta$, $z = r\sin\theta$, $r>0$, $0 \leq \theta \leq 2\pi$, so that the Jacobian of the transformation is r)

$$= \int_{0}^{\infty} r e^{-\frac{r^2}{2}}\,dr = \int_{0}^{\infty} e^{-z}\,dz = 1$$

$\Rightarrow I = 1$ (as $I \geq 0$).

(b) Clearly, $f_{X,\mu,\sigma}(\mu - x) = f_{X_{\mu,\sigma}}(\mu + x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \ \forall x \in \mathbb{R}$

$\Rightarrow$ distribution of $X_{\mu,\sigma}$ is symmetric about μ and

$E(X_{\mu,\sigma}) = \mu$ (it can be shown that $E(X_{\mu,\sigma})$ is finite)

(c) $$M_{X_{\mu,\sigma}}(t) = E(e^{tX_{\mu,\sigma}}) = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t(\mu+\sigma z)} e^{-\frac{z^2}{2}}\,dz$$

$$= \frac{e^{\mu t + \frac{\sigma^2 t^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(z-\sigma t)^2}{2}}\,dz = e^{\mu t + \frac{\sigma^2 t^2}{2}}, \ t \in \mathbb{R}$$

(since by (a) $\int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx = \sigma\sqrt{2\pi}, \ \forall \mu \in \mathbb{R}, \sigma > 0$)

$$\psi_{X_{\mu,\sigma}}(t) = \ln(M_{X_{\mu,\sigma}}(t)) = \mu t + \frac{\sigma^2 t^2}{2}$$

$E(X) = \psi^{(1)}_{X_{\mu,\sigma}}(0) = \mu$; $\quad Var(X) = \psi^{(2)}_{X_{\mu,\sigma}}(0) = \sigma^2$.

(d) $$M_{Y_{\mu,\sigma}}(t) = E\left(e^{t(aX_{\mu,\sigma}+b)}\right) = e^{tb} M_{X_{\mu,\sigma}}(at)$$

$$= e^{(a\mu+b)t + \frac{a^2\sigma^2t^2}{2}} = M_{X_{a\mu+b,|a|\sigma}}(t), \ \forall t \in \mathbb{R}$$

$\Rightarrow Y_{\mu,\sigma} \stackrel{d}{=} X_{a\mu+b,|a|\sigma}$

$$\Rightarrow f_{Y_{\mu,\sigma}}(x) = f_{X_{a\mu+b,\, |a|\sigma}}(x), \quad \forall x \in \mathbb{R}$$

$$= \frac{1}{|a|\sigma\sqrt{2\pi}} e^{-\frac{(y-(a\mu+b))^2}{2a^2\sigma^2}}, \quad y \in \mathbb{R}.$$

Problem No. 19 : Clearly

$$f\left(\tfrac{1}{2}+x\right) = f\left(\tfrac{1}{2}-x\right) = \begin{cases} \dfrac{1}{\pi\sqrt{\left(\frac{1}{2}+x\right)\left(\frac{1}{2}-x\right)}}, & -\frac{1}{2} < x < \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$$

$\Rightarrow X - \frac{1}{2} \stackrel{d}{=} \frac{1}{2} - X$, i.e., distribution of X is symmetric about $\frac{1}{2} = \mu$.

$\Rightarrow E(X) = \frac{1}{2}$; $P(X > \frac{1}{2}) = P(X > \mu) = \frac{1}{2}$.

Problem No. 20 : Clearly, $f(x) = f(-x)$ $\forall x \in \mathbb{R}$.

$\Rightarrow X \stackrel{d}{=} -X$ (i.e. the distribution of X is symmetric about 0)

$\Rightarrow E(X^3) = E((-X)^3)$

$\Rightarrow E(X^3) = 0$ (It can be shown that $E(X^3)$ is finite)

Also $P(X > 0) = \frac{1}{2}$.

Problem No. 21 (a) By Jensen's inequality ($g(x) = e^{-x}$, $x \in \mathbb{R}$, is a convex function), we have

$$E(g(X)) \geq g(E(X))$$

$$\Rightarrow E(e^{-X}) \geq e^{-E(X)}$$

$$\geq 1 - E(X) + \frac{(E(X))^2}{2} - \frac{(E(X))^3}{\lfloor 3}$$

$\left(e^{-x} \geq 1 - x + \frac{x^2}{2} - \frac{x^3}{\lfloor 3}, \ \forall x \in \mathbb{R}\right)$

$$= \frac{1}{3}.$$

(b) Let X be a random variable with p.m.f.

$$f_X(x) = \begin{cases} \dfrac{b_i}{\sum_{j=1}^{n} b_j}, & \text{if } x = a_i, \; i=1,\dots,n \\ 0, & \text{otherwise} \end{cases}.$$

Then f_X is a proper p.m.f. with range $S_X = \{a_1, a_2, \dots\}$ $\left(f_X(x) \geq 0 \;\; \forall x \in \mathbb{R} \text{ and } \sum_{x \in S_X} f_X(x) = \sum_{i=1}^{n} \frac{b_i}{\sum_{j=1}^{n} b_j} = 1 \right)$

Also $P(X>0)=1$. By Jensen's inequality ($g(x) = x^r$, $x>0$ is a convex function provided $r>1$)

$$E(g(X)) \geq g(E(X))$$

$$\Rightarrow E(X^r) \geq (E(X))^r$$

$$\Rightarrow \sum_{i=1}^{n} a_i^r \frac{b_i}{\sum_{j=1}^{n} b_j} \geq \left(\sum_{i=1}^{n} a_i \frac{b_i}{\sum_{j=1}^{n} b_j} \right)^r$$

$$\Rightarrow \left(\sum_{i=1}^{n} a_i^r b_i \right) \left(\sum_{j=1}^{n} b_j \right)^{r-1} \geq \left(\sum_{i=1}^{n} a_i b_i \right)^r$$

On taking $r=2$, $a_i \equiv a_i^m$, $b_i \equiv a_i$, $i=1,\dots,n$, we get

$$\left(\sum_{i=1}^{n} a_i^{2m+1} \right) \left(\sum_{i=1}^{n} a_i \right) \geq \left(\sum_{i=1}^{n} a_i^{m+1} \right)^2.$$

$\boxed{\text{Problem No. 22}}$ (a) We have $P(X>0)=1$. Using Jensen's inequality ($g(x) = x^{2m+1}$, $x>0$ is a convex function)

$$E(X^{2m+1}) \geq (E(X))^{2m+1}.$$

(b) Let $g(x) = (x-1)e^x$, $x>0$. Then $g'(x) = xe^x \uparrow$ and therefore g is convex on $(0,\infty)$. Consequently

$$E(g(X)) \geq g(E(X))$$

$$\Rightarrow E((X-1)e^X) \geq (E(X)-1)e^{E(X)}$$

$$\Rightarrow E(Xe^X) + e^{E(X)} \geq E(X)e^{E(X)} + E(e^X).$$

$\boxed{15/\text{III}}$