MSO201A
Hints for solution of Quiz II

February 15, 2021

Q.1 Let $X \sim Binomial(3, p)$. For $Y = g(X)$,

$$\mathbb{E}(Y) = \sum_{x=0}^{3} g(x) \binom{3}{x} p^x (1-p)^{3-x},$$
$$\mathbb{E}(Y^2) = \sum_{x=0}^{3} [g(x)]^2 \binom{3}{x} p^x (1-p)^{3-x}.$$

Q.2 Let X be a non-negative integer valued random variable having probability mass function:

$$f(x) = \begin{cases} 0.25u(x) + 0.75\dfrac{\exp(-6)6^x}{x!} & x = 0, 1, 2, 3, \ldots \\ 0 & \text{otherwise,} \end{cases}$$

where $u(x) = \begin{cases} 1 & x = a \\ 0 & \text{otherwise} \end{cases}$, for a fixed number $a \in \{0, 1, 2, \ldots\}$.
Then,

$$\mathbb{E}(X) = \sum_{x \in \{0,1,2,\ldots\}} x f(x) = 0.25 \times a + 0.75 \times 6,$$
$$\mathbb{E}(X^2) = \sum_{x \in \{0,1,2,\ldots\}} x^2 f(x) = 0.25 \times a^2 + 0.75 \times 42.$$

Q.3 Let X be a random variable taking values in the interval $[a,b]$, where $-\infty < a < b < \infty$. Observe that

$$\begin{aligned} Var(X) &= \mathbb{E}[\{X - \mathbb{E}(X)\}^2] \\ &\leq \mathbb{E}[\{X - t\}^2], \text{ for any } t \in \mathbb{R} \\ \Rightarrow \; Var(X) &\leq \mathbb{E}\left[\left\{X - \frac{a+b}{2}\right\}^2\right]. \end{aligned}$$

Since $X - a \geq 0$ and $X - b \leq 0$, we have

$$((X-a)+(X-b))^2 \leq ((X-a)-(X-b))^2 = (b-a)^2.$$

Q.4 A point Q is chosen at random along a rod of length l. The rod is bent at Q to form a right-angled triangle. Let θ be the smallest angle. Without loss of generality, suppose rod to be of unit length, then suppose X be the distance of Q to one of the ends, say O, then:

$$\tan(\theta) = \begin{cases} \dfrac{X}{1-X} & ,0 \leq X \leq 1/2 \\ \\ \dfrac{1-X}{X} & ,1/2 < X \leq 1 \end{cases}$$

Since X follows uniform distribution $U[0,1]$, hence:

$$\mathbb{E}\big(\tan(\theta)\big) = \int_0^{1/2} \frac{x}{1-x}dx + \int_{1/2}^{1} \frac{1-x}{x}dx$$

$$\approx 0.39$$

Similarly, second moment also exists.

Cumulative distribution of $Y = \cot(\theta)$, for $y \geq 1$, is as follows:

$$F_Y(y) = P[Y \leq y] = P\left[\tan(\theta) \geq \frac{1}{y}\right]$$

$$= P\left[X \leq \frac{y}{1+y} \cap X \geq \frac{1}{1+y}\right]$$

$$= \frac{y-1}{y+1}$$

Check that the first and second moments of Y do not exist.

Q.5 Suppose that we are given three events A, B and C such that:

(i) A and B are independent,

(ii) B and C are independent.

A and C even may not be independent, for example take $A = C$.

Consider $A = \{1, 2, 3, 4\}, B = \{1, 2, 8, 7\}, C = \{4, 3, 7, 8\}, S = \{1, 2, 3, 4, 5, 6, 7, 8\}$, where S is the sample space, now consider all outomes as equally likely.

$$P[A] = P[B] = P[C] = 1/2,$$

$$P[B \cap (A \cup C)] = 1/2, P[B \cap A \cap C] = 0.$$

Q.6 In the box of a product, there is a coupon with a number from the set $\{1, 2, \ldots, n\}$. A person gets a free box if s/he succeeds in getting all numbers of this set. Let N be the number of boxes that one needs to buy before getting a free box.

Let

$$N = X_1 + X_2 + \cdots + X_n,$$

where X_i is the number of purchases needed to get i^{th} coupon after collecting $i-1$ coupons.

Every X_i is distributed geometrically, with probability of success $\dfrac{n-i}{n}$.

Now, due to linearity of expectations:

$$\begin{aligned}\mathbb{E}[N] =& \mathbb{E}[X_1] + \mathbb{E}[X_2] + \cdots + \mathbb{E}[X_n] \\ =& 1 + \frac{n}{n-1} + \frac{n}{n-2} + \cdots + n, \\ =& n\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right).\end{aligned}$$

Q.7 Suppose $X \sim N(\mu, \sigma^2)$. Define $Y = \exp(X)$ and

$$\phi(y) = \int_0^y \frac{1}{u}\exp\left(-\frac{(\ln u - \mu)^2}{2\sigma^2}\right)du.$$

Find $100 \times P\big[\phi(Y) \le a\big]$.
Observe that

$$\frac{\phi(y)}{\sigma\sqrt{2\pi}} = P[Y \le y] = F_Y(y).$$

Note that F_Y^{-1} exists, then

$$\begin{aligned}P[\phi(Y) \le a] &= P\left[\frac{\phi(Y)}{\sigma\sqrt{2\pi}} \le \frac{a}{\sigma\sqrt{2\pi}}\right] \\ &= P\left[F_Y(Y) \le \frac{a}{\sigma\sqrt{2\pi}}\right] \\ &= P\left[Y \le F_Y^{-1}\left(\frac{a}{\sigma\sqrt{2\pi}}\right)\right] \\ &= F_Y\left(F_Y^{-1}\left(\frac{a}{\sigma\sqrt{2\pi}}\right)\right) = \frac{a}{\sigma\sqrt{2\pi}}.\end{aligned}$$

Q.8 A point Q is picked at random from a triangle with height h and with base of length b. Let X be the perpendicular distance from Q to the base, then calculate $100 \times P[X \le kh]$.

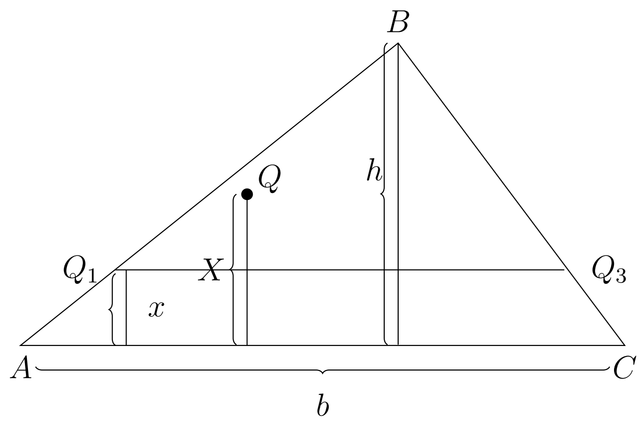

Figure 1:

For $x \leq h$, event $X > x$ is equivalent to the event that Q lies inside the triangle BQ_1Q_3. So,

$$P[Q \in BQ_1Q_3] = \left(\frac{h-x}{h}\right)^2.$$