Date
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Department of Electrical Engineering, IIT Kanpur
EE250: Control Systems Analysis
Tutorial 1

Question 1

We found out the dynamics of a servo-motor in the class as

$$\frac{dx_1}{dt} = -\frac{B}{J}x_1 + \frac{K_T}{J}x_2$$

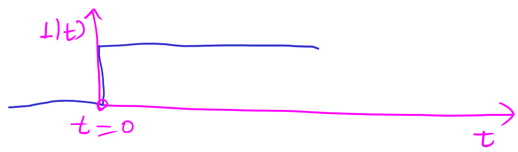
$$\frac{dx_2}{dt} = -\frac{K_b}{L_a}x_1 - \frac{R_a}{L_a}x_2 + \frac{1}{L_a}u$$

where

x_1 = speed (ω) of the motor
x_2 = Armature current I_a
u = Armature voltage e_a

u(t)

Parameters are:



$B = 0.25$ N-m/(rad/sec)
$R_a = 5\ \Omega$
$L = 0.1$ H
$J = 2$ N-M/(rad/sec)
$K_b = 1$ volt/(rad/sec)

= Y(s)
$Y(s) = X_1(s)$

(i) Given output $\boxed{y = x_1}$, find $\frac{Y(s)}{U(s)}$

$u(t) = 100 \cdot 1(t)$ → unit step signal.

(ii) Given $u(t) = 100$ volt (sudden), find $y(t)$. Compute y_{ss}

$u(t) = 100 \times 1(t)$

Input signal

(iii) Find state transition matrix e^{At}
(iv) Find $y(t)$ using e^{At}

Torque constant

Solution

(i) For a servo-motor, K_T=K_b
Hence $K_T = 1$ N-m/Amp
The equations with parameter values given:

$\mathcal{L}\left[\frac{dx}{dt}\right] = sX(s) - x(0)$

$x(0) = x(t)\big|_{t=0}$

$$\frac{dx_1}{dt} = -0.125x_1 + 0.5x_2$$

$$\frac{dx_2}{dt} = -10x_1 - 50x_2 + 10u$$

Setting all initial conditions to zero, Laplace transform of these equations are:

$$\boxed{\begin{aligned} sX_1(s) &= -0.125X_1(s) + 0.5X_2(s) \\ sX_2(s) &= -10X_1(s) - 50X_2(s) + 10U(s) \end{aligned}} \qquad \begin{aligned} (1) \\ (2) \end{aligned}$$

From (1),

$$X_2(s) = (2s + 0.25)X_1(s) \tag{3}$$

By setting $X_1(s)$ as $Y(s)$ and using (3), equation (2) becomes

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + Bu$$

$$(2s^2 + 0.25s + 10 + 100s + 12.5)Y(s) = 10U(s)$$

$$\dot{x}_1 = \frac{dx_1}{dt} = -0.125x_1 + 0.5x_2 + 0 \cdot u$$

$$\dot{x}_2 = \frac{dx_2}{dt} = -10x_1 - 50x_2 + 10u$$

$$y = x_1 + 0.$$

$$\begin{aligned} \frac{Y(s)}{U(s)} &= \frac{10/2}{s^2 + 50.125s + 11.25} \\ &= \frac{5}{(s + 49.8995)(s + 0.2255)} \end{aligned}$$

$$\boxed{\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}} \qquad D = 0, \quad C = [1, 0]$$

Second approach:

$$\boxed{\frac{Y(s)}{U(s)} = C(sI - A)^{-1}B}$$

$$A = \begin{pmatrix} -0.125 & 0.5 \\ -10 & -50 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 10 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

Putting the values,

$$\frac{Y(s)}{U(s)} = \begin{pmatrix} 1 & 0 \end{pmatrix} \left[\begin{pmatrix} s + 0.125 & -0.5 \\ 10 & s + 50 \end{pmatrix}^{-1}\right] \begin{pmatrix} 0 \\ 10 \end{pmatrix}$$

$$\left[\begin{pmatrix} s + 0.125 & -0.5 \\ 10 & s + 50 \end{pmatrix}^{-1}\right] = \frac{1}{s^2 + 50.125s + 11.25} \begin{pmatrix} s + 50 & 0.5 \\ -10 & 0.125(8s + 1) \end{pmatrix}$$

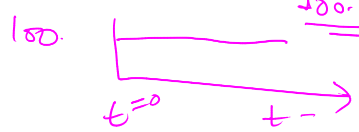
$$\begin{aligned} \frac{Y(s)}{U(s)} &= \frac{1}{s^2 + 50.125s + 11.25} \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} s + 50 & 0.5 \\ -10 & 0.125(8s + 1) \end{pmatrix} \begin{pmatrix} 0 \\ 10 \end{pmatrix} \\ &= \frac{1}{s^2 + 50.125s + 11.25} \begin{pmatrix} s + 50 & 0.5 \end{pmatrix} \begin{pmatrix} 0 \\ 10 \end{pmatrix} \\ &= \frac{5}{s^2 + 50.125s + 11.25} \end{aligned}$$

(ii) In Laplace domain,

$$\left[U(s) = \frac{100}{s} \right]$$

$u(t) = 100V.$

$100 \cdot 1(t)$



Hence,

$$\begin{aligned} Y(s) &= \frac{100}{s} \frac{5}{(s + 49.8995)(s + 0.2255)} \\ &= -\frac{44.6369}{s + 0.2255} + \frac{0.201718}{s + 49.8995} + \frac{44.4352}{s} \end{aligned}$$

[partial fraction method]

Taking inverse Laplace transform,

$$\mathcal{L}[f(t)] = \int_{0^-}^{\infty} f(t)\, e^{-st}\, dt \qquad y(t) = \Big(-44.6369\overset{\to 0}{e^{-0.2255t}} + 0.201718\overset{\to 0}{e^{-49.8995t}} + 44.4352\underset{=1}{u(t)}\Big)$$

$1(t) \to t = 0,\ t > 0$

Now,

$$\begin{aligned} y_{ss} &= \lim_{s\to 0} sY(s) \\ &= \lim_{s\to 0} s\frac{500}{s(s+49.8995)(s+0.2255)} \\ &= 44.4352 \end{aligned}$$

(iii)

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{aligned} e^{At} &= \mathcal{L}^{-1}[(sI-A)^{-1}] \\ &= \mathcal{L}^{-1}\begin{pmatrix} \frac{s+50}{s^2+50.125s+11.25} & \frac{0.5}{s^2+50.125s+11.25} \\ \frac{-10}{s^2+50.125s+11.25} & \frac{0.125(8s+1)}{s^2+50.125s+11.25} \end{pmatrix} \\ &= \mathcal{L}^{-1}\begin{pmatrix} -\frac{0.0223}{s+49.89}+\frac{1.00221}{s+0.22} & \frac{0.01006}{s+0.22}-\frac{0.01006}{s+49.89} \\ \frac{0.2013}{s+49.89}-\frac{0.2013}{s+0.22} & \frac{1.0019}{s+49.89}-\frac{0.0019}{s+0.22} \end{pmatrix} \end{aligned}$$

$$e^{At} = \begin{pmatrix} -0.0223e^{-49.89t}+1.0022e^{-0.22t} & 0.01006e^{-0.22t}-0.01006e^{-49.89t} \\ 0.2013e^{-49.89t}-0.2013e^{0.22t} & 1.0019e^{-49.89t}-0.0019e^{-0.22t} \end{pmatrix} \cdot 1(t)$$

(iv)

$$\boxed{\begin{gathered} \mathbf{x}(t) = e^{At}\overset{\to 0}{\mathbf{x}(0)} + \int_0^t \boxed{e^{A(t-\tau)}}Bu(\tau)d\tau \\ \mathbf{x}(0) = 0 \end{gathered}}$$

$$\begin{aligned} y(t) &= C\mathbf{x}(t) = x_1(t) \\ &= \int_0^t \begin{pmatrix} 1 \\ 0 \end{pmatrix}^T \begin{pmatrix} -0.0223e^{-49.89(t-\tau)}+1.0022e^{-0.22(t-\tau)} & 0.01006e^{-0.22(t-\tau)}-0.01006e^{-49.89(t-\tau)} \\ 0.2013e^{-49.89(t-\tau)}-0.2013e^{0.22(t-\tau)} & 1.0019e^{-49.89(t-\tau)}-0.0019e^{-0.22(t-\tau)} \end{pmatrix} \begin{pmatrix} 0 \\ 10 \end{pmatrix} 100d\tau \\ &= \int_0^t \begin{pmatrix} -0.0223e^{-49.89(t-\tau)}+1.0022e^{-0.22(t-\tau)} & 0.01006e^{-0.22(t-\tau)}-0.01006e^{-49.89(t-\tau)} \end{pmatrix} \begin{pmatrix} 0 \\ 1000 \end{pmatrix} d\tau \\ &= \int_0^t \left(10.06e^{-0.22(t-\tau)}-10.06e^{-49.89(t-\tau)}\right)d\tau \\ &= \frac{10.06}{0.22}e^{-0.22t}[e^{0.22\tau}]_0^t - \frac{10.06}{49.89}e^{-49.89t}[e^{49.89\tau}]_0^t \\ &= 45.52u(t) - 45.72e^{-0.22t} + 0.20e^{-49.89t} \end{aligned}$$

$X(0)$

$x(t)\big|_{t=0}$

$$\mathcal{L}\left[\frac{d^n x}{dt^n}\right] = s^n X(s) - s^{n-1}x(0) - s^{n-2}x'(0) - \cdots - x^{n-1}(0)$$

Question 2

$$\frac{d^3y(t)}{dt^3} + 4\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 2y(t) = 6\frac{du(t)}{dt} + u(t)$$

Find $\frac{Y(s)}{U(s)}$

Solution

Taking Laplace transform,

$$s^3Y(s) + 4s^2Y(s) + 5sY(s) + 2Y(s) = 6sU(s) + U(s)$$

[assuming initial conditions zero]

$$(s^3 + 4s^2 + 5s + 2)Y(s) = (6s + 1)U(s)$$
$$\frac{Y(s)}{U(s)} = \frac{6s + 1}{s^3 + 4s^2 + 5s + 2}$$

Question 3

$$\frac{d^3y(t)}{dt^3} + 10\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + y(t) + 2\int_0^t y(\tau)d\tau = \frac{du(t)}{dt} + 2u(t)$$

Find $\frac{Y(s)}{U(s)}$

Solution

Taking Laplace transform,

$$s^3Y(s) + 10s^2Y(s) + 2sY(s) + Y(s) + 2\frac{Y(s)}{s} = sU(s) + 2U(s)$$

[assuming initial conditions zero]

$$(s^3 + 10s^2 + 2s + 1 + \frac{2}{s})Y(s) = (s + 2)U(s)$$
$$\frac{Y(s)}{U(s)} = \frac{s^2 + 2s}{s^4 + 10s^3 + 2s^2 + s + 2}$$

Question 4

Solve the following ordinary differential equation using Laplace transform

$$\ddot{y}(t) - 2\dot{y}(t) + 4y(t) = 0; y(0) = 1, \dot{y}(0) = 2$$

Solution

Taking Laplace transform,

$$s^2Y(s) - sy(0) - \dot{y}(0) - 2(sY(s) - y(0)) + 4Y(s) = 0$$

$$\Rightarrow (s^2 - 2s + 4)Y(s) - (s-2)y(0) - \dot{y}(0) = 0.$$

$$\Rightarrow Y(s) = \frac{s-2}{s^2-2s+4}\overset{1}{y(0)} + \frac{1}{s^2-2s+4}\overset{2}{\dot{y}(0)}$$

$$= \frac{(s-1)}{(s-1)^2 + (\underbrace{\sqrt{3}}_{a})^2} + \frac{1}{\sqrt{3}}\,\frac{\overset{a}{\sqrt{3}}}{\underbrace{(s-1)^2}_{(s-1)^2} + \underbrace{(\sqrt{3})^2}_{a^2}}$$

$$\boxed{y(t) = \left\{e^t \cos\sqrt{3}t + \frac{e^t}{\sqrt{3}}\sin\sqrt{3}t\right\}1(t)}$$

$\mathcal{L}[\sin at\, 1(t)] = \frac{a}{s^2+a^2}$

$\mathcal{L}[\cos at\, 1(t)] = \frac{s}{s^2+a^2}$

$\mathcal{L}[e^{-at}f(t)] = \underbrace{F(s+a)}_{s-1}$

$\mathcal{L}[e^{at}f(t)] = F(s+a)$

$\mathcal{L}[\sin at \cdot 1(t)] = \frac{a}{s^2+a^2}$

$\mathcal{L}[\cos at \cdot 1(t)] = \frac{s}{s^2+a^2}$