MSO201a: Probability and Statistics

Summer Term: 2019

Quiz II

Time Allowed: 45 Minutes | Maximum Marks: 20

1. Let X_1 and X_2 be independent random variables with X_i having the p.d.f.

$$f_i(x) = \begin{cases} e^{-x}x^{i-1}, & \text{if } x > 0 \\ 0, & \text{otherwise} \end{cases}, \quad i = 1, 2.$$

Define $Y_1 = X_1 + X_2$ and $Y_2 = \frac{X_1}{X_1+X_2}$. Show that Y_1 and Y_2 are independently distributed and find their marginal p.d.f.s. **8 Marks**

2. Let X and Y be random variables with joint p.m.f.

$$f(x, y) = \begin{cases} \frac{1}{8}, & \text{if } (x, y) \in \{(0,0), (0,1), (1,0)\} \\ \frac{5}{8}, & \text{if } (x, y) = (1, 1) \\ 0, & \text{otherwise} \end{cases}.$$

Define $Z_1 = X + 2Y$ and $Z_2 = 2X - 3Y$. Find the correlation between Z_1 and Z_2. **6 Marks**

3. A mathematician carries at all times two match boxes, one in his left pocket and one in his right pocket. To begin with each match box contains 10 matches. Each time the mathematician needs a match he is equally likely to take it from either pocket. Consider the moment when the mathematician for the first time discovers that one of the match boxes is empty. Show that the probability that at that moment the other box contains exactly 5 matches is $\frac{3003}{32768}$. **6 Marks**

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Model Solutions

Problem No. 1 The joint p.d.f. of $\underline{X} = (X_1, X_2)$ is:

$$f_{X_1,X_2}(x_1,x_2) = f_1(x_1)\, f_2(x_2) = \begin{cases} e^{-(x_1+x_2)}\, x_2, & \text{if } x_1>0,\ x_2>0 \\ 0, & \text{otherwise}. \end{cases}$$

$S_{X_1,X_2} = (0,\infty) \times (0,\infty)$.

The transformation $(X_1, X_2) \to (Y_1, Y_2)$ is 1-1 on $S^{\circ}_{X_1,X_2} = (0,\infty)\times(0,\infty)$ with inverse transformation $(x_1, x_2) = (y_1 y_2,\ y_1(1-y_2))$ and Jacobian

$$J = \begin{vmatrix} y_2 & y_1 \\ 1-y_2 & -y_1 \end{vmatrix} = -y_1$$

$x_1>0,\ x_2>0 \iff y_1 y_2 > 0,\ y_1(1-y_2)>0 \iff y_1>0,\ 0<y_2<1$

Thus, under transformation $(X_1, X_2) \to (Y_1, Y_2)$, $S^{\circ}_{X_1,X_2} \to (0,\infty)\times(0,1) = S^{\circ}_{Y_1,Y_2}$.

Consequently the joint p.d.f. of (Y_1, Y_2) is

$$f_{Y_1,Y_2}(y_1,y_2) = f_{X_1,X_2}(y_1y_2,\ y_1(1-y_2))\, |J|\, I_{(0,\infty)\times(0,1)}(y_1,y_2)$$

$$= \begin{cases} e^{-y_1}\, y_1(1-y_2) \times |y_1|, & \text{if } y_1>0,\ 0<y_2<1 \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} y_1^2 e^{-y_1}(1-y_2), & \text{if } y_1>0,\ 0<y_2<1 \\ 0, & \text{otherwise} \end{cases}$$

Clearly Y_1 and Y_2 are independently distributed with pdfs ... $\boxed{\text{4 MARKS}}$

$$f_{Y_1}(y_1) = \begin{cases} \dfrac{y_1^2 e^{-y_1}}{2}, & \text{if } y_1>0 \\ 0, & \text{otherwise} \end{cases} \quad \dots \boxed{\text{2 MARKS}}$$

and

$$f_{Y_2}(y_2) = \begin{cases} 2(1-y_2), & \text{if } 0<y_2<1 \\ 0, & \text{otherwise} \end{cases}, \quad \dots \boxed{\text{2 MARKS}}$$

respectively.

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$\boxed{\text{Problem No. 2}}$

$$E(X) = E(Y) = E(X^2) = E(Y^2) = \frac{1}{8} + \frac{5}{8} = \frac{3}{4}$$

$$Var(X) = Var(Y) = \frac{3}{4} - \left(\frac{3}{4}\right)^2 = \frac{3}{16}$$

$$E(XY) = \frac{5}{8}$$

$$Cov(X, Y) = \frac{5}{8} - \frac{9}{16} = \frac{1}{16}$$

$$\rho(X, Y) = Corr(X, Y) = \frac{Cov(X, Y)}{\sqrt{Var(X)\ Var(Y)}} = \frac{1}{3}$$

$$Cov(Z_1, Z_2) = Cov(X + 2Y,\ 2X - 3Y)$$

$$= 2\,Var(X) - 3\,Cov(X, Y) + 4\,Cov(X, Y) - 6\,Var(Y)$$

$$= Cov(X, Y) - 4\,Var(X) = \frac{1}{16} - \frac{3}{4} = \frac{-11}{16} \quad \ldots \quad \boxed{\text{2 MARKS}}$$

$$Var(Z_1) = Var(X + 2Y) = Var(X) + 4\,Var(Y) + 4\,Cov(X, Y)$$

$$= \frac{15}{16} + \frac{4}{16} = \frac{19}{16} \quad \ldots \quad \boxed{\text{1 MARK}}$$

$$Var(Z_2) = Var(2X - 3Y) = 4\,Var(X) + 9\,Var(Y) - 12\,Cov(X, Y)$$

$$= \frac{39}{16} - \frac{12}{16} = \frac{27}{16} \quad \ldots \quad \boxed{\text{1 MARK}}$$

$$\Rightarrow \rho(Z_1, Z_2) = \frac{Cov(Z_1, Z_2)}{\sqrt{Var(Z_1)\ Var(Z_2)}} = \frac{-11}{\sqrt{19 \times 27}} = \frac{-11\sqrt{513}}{513} = -0.4857$$

$\boxed{\text{2 MARK}}$

Problem No. 3 Let us call the two pockets as left and right pockets. In each trial, define

Success: Choosing left pocket

Failure: Choosing right pocket

We have a sequence of Bernoulli trials with probability of success in each trial being $p = \frac{1}{2}$.

Required probability

= Pr (when left pocket is found empty, the right pocket has 5 matches)

+ Pr (when right pocket is found empty the left pocket has 5 matches)

= Pr (5th failure precedes the 11th success)

+ Pr (5th success precedes the 11th failure) ... $\boxed{3 \text{ MARKS}}$

$$= \binom{15}{5} \times \left(\frac{1}{2}\right)^5 \times \left(\frac{1}{2}\right)^{10} \times \frac{1}{2} + \binom{15}{5} \times \left(\frac{1}{2}\right)^5 \times \left(\frac{1}{2}\right)^{10} \times \frac{1}{2}$$

$$= \binom{15}{5} \times \frac{1}{2^{15}} = \frac{3003}{32768}$$ ... $\boxed{3 \text{ MARKS}}$

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