1.a. Two gamblers A and B agree to play as follows. They throw two fair dice. If the sum $S$ of the outcomes is $< 10$, then B receives Rs. $S$ from A, otherwise B pays A Rs. $x$. Determine $x$ so that the game is fair. [3]

1.b. Consider the following mgf:

$$M_X(t) = \frac{(e^{2t} - e^t)}{2t} + (1-\alpha)\frac{3}{\pi^2}\sum_{n=1}^{\infty}\frac{e^{t/n}}{n^2}$$

for $t \neq 0$, and $M_X(0) = 1$.
Find (i) $P(X = 1/2)$, and (ii) $P(X \in (1.1, 1.25))$. [1.5+1.5]

1.a.

The probability distribution of S:

| $S=s$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $P[S=s]$ | $\frac{1}{36}$ | $\frac{2}{36}$ | $\frac{3}{36}$ | $\frac{4}{36}$ | $\frac{5}{36}$ | $\frac{6}{36}$ | $\frac{5}{36}$ | $\frac{4}{36}$ | $\frac{3}{36}$ | $\frac{2}{36}$ | $\frac{1}{36}$ |

if $S < 10 \equiv S \le 9$

For fair game we must have

$$2 \cdot \frac{1}{36} + 3 \cdot \frac{2}{36} + 4 \cdot \frac{3}{36} + \cdots + 9 \cdot \frac{4}{36}$$ (1 mark)

$$= x \cdot \frac{3}{36} + x \cdot \frac{2}{36} + x \cdot \frac{1}{36}$$ (1 mark)

$$\Rightarrow 2 + 6 + 12 + 20 + 30 + 42 + 40 + 36 = 6x$$

$$\Rightarrow 188 = 6x$$

$$\Rightarrow \boxed{x = \frac{188}{6} = 31.33}$$ (1 mark)

1.b. Here $M_X(t) = \frac{1}{2}\left(\frac{e^{2t} - e^{t}}{t}\right) + \left(1 - \frac{1}{2}\right) \cdot \frac{6(1-\alpha)}{\pi^2} \sum_{n=1}^{\infty} \frac{e^{t/n}}{n^2}$

$\therefore$ $F = p F_c + (1-p) F_D$ with $p = \frac{1}{2}$

where $F_c$ is $U(1,2)$

and $F_D$ has pmf : $X = \frac{1}{n}$ w.p. $\frac{6(1-\alpha)}{\pi^2} \cdot \frac{1}{n^2}$

$\therefore X \sim F$

(i). $P(X = \frac{1}{2}) = (1 - \frac{1}{2}) \frac{6(1-\alpha)}{\pi^2} \cdot \frac{1}{4} = \frac{3(1-\alpha)}{4\pi^2}$ (1.5 mark)

(ii). $P[X \in (1.1, 1.25)] = \frac{1}{2} \times (0.15) = 0.075$ (1.5 mark)