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Q4) $\begin{cases} u_t = u_{xx} & ; (x,t) \in (0,1)\times(0,\infty) \\ u(x,0) = x(1-x) & ; x \in (0,1) \\ u(0,t) = u(1,t) = 0 \end{cases}$

$\boxed{\text{Pg } 1}$

So, let us use Separation of Variables,
So,
$$u(x,t) = F(x).\,G(t)$$

$$\therefore \quad F(x)\,G'(t) = F''(x)\,G(t)$$
$$\Rightarrow \quad \frac{F''(x)}{F(x)} = \frac{G'(t)}{G(t)} = \lambda \quad \text{(say)} \quad \text{——— (1)}$$

Also, $u(x,0) = x(1-x)$
$$\Rightarrow \quad F(x).\,G(0) = x(1-x) \quad \text{——— (2)}$$

Also, $u(0,t) = u(1,t) = 0 \quad (t>0)$
$$\Rightarrow \quad F(0)\,G(t) = F(1)\,G(t) = 0$$
$$\Rightarrow \quad F(0) = F(1) = 0 \quad \text{——— (3)}$$

from (1) & (3),
$$\begin{cases} F''(x) = \lambda F(x) \\ F(0) = F(1) = 0 \end{cases}$$

We know the solution for the SLEVP from the lectures.
$$\therefore \quad \boxed{\lambda_n = -n^2\pi^2 \; , \; F_n(x) = \sin(n\pi x)}$$

~~Now, using the value of $\lambda_n$ in (1),~~

$\cancel{G'(t) = \lambda_n G(t) \Rightarrow G_n = A_n \cos(n\pi t) + B_n \sin(n\pi t)}$

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Now, using the values of $\lambda_n$ in ①,

$$G'(t) = -n^2\pi^2 G(t) \Rightarrow G_n(t) = A_n e^{-n^2\pi^2 t}$$

Using superposition principle,

$$u_n(x,t) = F_n(x) \cdot G_n(t) = A_n e^{-n^2\pi^2 t} \sin(n\pi x)$$

$$\therefore u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} A_n e^{-n^2\pi^2 t} \sin(n\pi x)$$

Now,

let $f(x) = x(1-x)$

$\therefore u(x,0) = f(x) \quad \forall x \in (0,1)$

$$\therefore f(x) = \sum_{n=1}^{\infty} A_n \sin(n\pi x)$$

from Fourier Series' knowledge,

$$A_n = \frac{2}{1}\int_0^1 f(x) \sin\left(\frac{n\pi}{1} x\right) dx$$

$$= 2\int_0^1 x \sin(n\pi x)\, dx - 2\int_0^1 x^2 \sin(n\pi x)\, dx$$

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$$= 2\left\{ -\frac{x\cos(n\pi x)}{n\pi} + \cancel{\frac{\sin(n\pi x)}{n^2\pi^2}} \right\}\Bigg|_0^1$$

$$-2\left\{ -\frac{x^2\cos n\pi x}{n\pi} + \cancel{2x\frac{\sin(n\pi x)}{n^2\pi^2}} + 2\frac{\cos(n\pi x)}{n^3\pi^3} \right\}\Bigg|_0^1$$

$$= 2\left\{ -\cancel{\frac{\cos n\pi}{n\pi}} + \cancel{\frac{\cos n\pi}{n\pi}} - \left(\frac{2\cos n\pi - 2}{n^3\pi^3}\right) \right\}$$

$$= \frac{4}{n^3\pi^3}(1 - \cos n\pi)$$

$$= \begin{cases} 0 & ; \text{ n is even} \\ \frac{8}{n^3\pi^3} & ; \text{ n is odd} \end{cases}$$

$$\therefore \boxed{u(x,t) = \sum_{n=0}^{\infty} \frac{8}{(2n+1)^3\pi^3} e^{-(2n+1)^2\pi^2 t} \sin\{(2n+1)\pi x\}}$$

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[Pg-4]

Q5) a) from D-Alembert's Principle,

- $u(x,t) = \frac{1}{2}[f(x+t) + f(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} g(s)ds$

Now, $|x+y| \le |x| + |y|$

$\Rightarrow |u(x,t)| \le \frac{1}{2}|f(x+t)| + \frac{1}{2}|f(x-t)| + \frac{1}{2}\left|\int_{x-t}^{x+t} g(s)ds\right|$

$\le \frac{1}{2}\times 1 + \frac{1}{2}\times 1 + \frac{1}{2}\int_{x-t}^{x+t} |g(s)|ds$

$\le \frac{1}{2}+\frac{1}{2} + \frac{1}{2}\int_{x-t}^{x+t} 2\,ds$

$= 1 + \frac{1}{2}\times 2 \times (x+t-x+t)$

$= 1 + 2t$

$\therefore |u(x,t)| \le 1+2t < 5+4t$

$\Rightarrow |u(x,t)| \le 5+4t \quad \forall (x,t) \in A$

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[Pg 5]

b) Now, we know that $t > 0$

$\therefore$ if $x + t \leq 0$

$\Rightarrow x + t - 2t \leq -2t$

$\Rightarrow x - t \leq -2t \leq 0$

$\therefore x + t \leq 0 \Rightarrow x - t \leq 0$

So, $u(x,t) = \frac{1}{2}[f(x+t) + f(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} g(s)\,ds$

Now, we know that $f(s) = g(s) = 0 \;\forall\; s \leq 0$

$\therefore$ for $x + t \leq 0$,

$$u(x,t) = \frac{1}{2}[0 + 0] + \frac{1}{2}\int_{x-t}^{x+t} 0\,ds = 0$$

So, $u(x,t) = 0$

$\forall\; (x,t) \in \{(x,t) \mid x + t < 0\} \cap A.$

$\square$