Solution set

Assignment 5

1. In lecture it is done that a radial function 'u' satisfying
$$\Delta u = 0 \text{ on } \mathbb{R}^2 \setminus \{(0,0)\}$$
has to be $u(x,y) = A \log (x^2+y^2)^{1/2} + B$

$\therefore$ Clearly (a) and (c) are only true options

at $|\log r| \to \infty$ as $r \to 0$ and ∞.

— x —

2. $Y = AX$ where $A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$\Rightarrow$ $y_1 = x_1 \cos\theta + x_2 \sin\theta$

$y_2 = -x_1 \sin\theta + x_2 \cos\theta$

$v(y_1, y_2) = u(x_1, x_2)$ By chain rule

$$u_{x_1} = v_{y_1}\frac{\partial y_1}{\partial x_1} + v_{y_2}\frac{\partial y_2}{\partial x_1}$$

$$= \cos\theta\, v_{y_1} - \sin\theta\, v_{y_2}$$

$$u_{x_1x_1} = \cos\theta\left[\cos\theta\, v_{y_1y_1} - \sin\theta\, v_{y_1y_2}\right]$$
$$-\sin\theta\left[\cos\theta\, v_{y_1y_2} - \sin\theta\, v_{y_2y_2}\right]$$

$$= \cos^2\theta\, v_{y_1y_1} - 2\sin\theta\cos\theta\, v_{y_1y_2}$$
$$+\sin^2\theta\, v_{y_2y_2}$$

$$\longrightarrow (*)$$

Similarly.

$$u_{x_2} = v_{y_1}\frac{\partial y_1}{\partial x_2} + v_{y_2}\frac{\partial y_2}{\partial x_2}$$

$$= \sin\theta\, v_{y_1} + \cos\theta\, v_{y_2}$$

$$\Rightarrow u_{x_2x_2} = \sin\theta\left(\sin\theta\, v_{y_1y_1} + \cos\theta\, v_{y_1y_2}\right)$$
$$+\cos\theta\left(\sin\theta\, v_{y_1y_2} + \cos\theta\, v_{y_2y_2}\right)$$

$$= \sin^2\theta\, v_{y_1y_1} + 2\sin\theta\cos\theta\, v_{y_1y_2}$$
$$+\cos^2\theta\, v_{y_2y_2}$$

$$\longrightarrow (*)(*)$$

$\therefore$ from $(*)$ and $(*)(*)$, we obtain

$$\Delta u = \Delta v = 0$$

4. If possible let v_1 and v_2 be two distinct solution of the problem $\begin{Bmatrix} \Delta u = f & \text{on } \Omega \\ u = g & \text{on } \partial\Omega \end{Bmatrix}$

Consider the function $w = v_1 - v_2$

$\Rightarrow \Delta w = 0$ on Ω

$w = 0$ on $\partial\Omega$

Now by applying max principals we know that, both maximum and the minimum value of 'w' takes place on $\partial\Omega$, which is '0' here.

$\Rightarrow w \equiv 0$ on Ω

$\Rightarrow v_1 \equiv v_2$ on Ω (contradiction)

5. $(x,y) \leftrightarrow (r,\theta)$

$\left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right\}$ are polar co-ordinates

$\Rightarrow x^2+y^2 = r^2$

$\theta = \tan^{-1}(y/x)$

$v(r,\theta) = u(x,y)$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$$= v_r \cos\theta - v_\theta \left(\frac{\sin\theta}{r}\right)$$

(Calculate)

$$u_{xx} = \left(v_{rr}\cos\theta - v_{\theta r}\frac{\sin\theta}{r} + v_\theta \frac{\sin\theta}{r^2}\right)\cos\theta$$

$$+ \left(u_{r\theta}\cos\theta - v_r \sin\theta - v_{\theta\theta}\sin\theta/r - u_\theta \frac{\cos\theta}{r}\right)\left(-\frac{\sin\theta}{r}\right)$$

Similarly calculating,

u_{yy} we get finally

$$\Delta u = \underbrace{v_{rr} + v_r/r + \frac{v_{\theta\theta}}{r^2}}_{\text{Expression of Laplacian in Polar Coordinates}} = 0.$$