MSO201a: Probability and Statistics

Summer Term: 2019
Quiz I

Time Allowed: 45 Minutes **Maximum Marks: 20**

1. Let A, B and C be three events in a probability space $(\Omega, \mathcal{F}, P)$ such that $P(A) = P(A \cap B) = P(A \cap C) = P(A \cap B \cap C) = \frac{1}{4}$, $P(B) = P(B \cap C) = \frac{1}{2}$ and $P(C) = \frac{3}{4}$. Find the probability that exactly one of the events A or B or C occur. [6 Marks]

2. Let X be a random variable with d.f.

$$F(x) = \begin{cases} 0, & \text{if } x < -1 \\ c, & \text{if } -1 \leq x < 0 \\ \frac{1}{3}, & \text{if } 0 \leq x < 2 \\ \frac{7c^2 - 9c + 6}{4}, & \text{if } 2 \leq x \leq 3 \\ 1, & \text{if } x > 3 \end{cases},$$

where c is a real constant. Find the values of constant c, $\Pr(-\frac{1}{2} \leq X < 2 | X \geq 0)$ and $\text{Var}(X)$. [2+2+3=7 Marks]

3. Let X be a r.v. with p.d.f.

$$f(x) = \begin{cases} \frac{1}{4}, & \text{if } -2 < x < 0 \\ x, & \text{if } 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}.$$

Find the p.d.f. of $Y = X + 2|X|$ and hence find the variance of Y. [4+3=7 Marks]

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Problem No. 1

Required probability $= P(A \cap \bar{B} \cap \bar{C}) + P(\bar{A} \cap B \cap \bar{C}) + P(\bar{A} \cap \bar{B} \cap C)$

$= P(A \cap (\overline{B \cup C})) + P(B \cap (\overline{A \cup C})) + P(C \cap (\overline{A \cup B}))$

$= P(A) - P(A \cap (B \cup C)) + P(B) - P(B \cap (A \cup C)) + P(C) - P(C \cap (A \cup B))$

$= P(A) + P(B) + P(C) - P((A \cap B) \cup (A \cap C)) - P((A \cap B) \cup (B \cap C))$
$\quad - P((A \cap C) \cup (B \cap C))$

$= P(A) + P(B) + P(C) - 2P(A \cap B) - 2P(A \cap C) - 2P(B \cap C)$
$\quad + 3P(A \cap B \cap C)$

$= \frac{1}{4} + \frac{1}{2} + \frac{3}{4} - \frac{1}{2} - \frac{1}{2} - 1 + \frac{3}{4} = \frac{1}{4}$

Answer: $\frac{1}{4}$ ..... 5 MARKS

Problem No. 2

F is right continuous at $x = 3$

$\Rightarrow 7c^2 - 9c + 2 = 0 \Rightarrow c = \frac{2}{7}, 1$

$F \uparrow \Rightarrow F(0-) \le F(0) \Rightarrow c \le \frac{1}{3} \Rightarrow \boxed{c = 2/7}$ .... 2 MARKS

$$F(x) = \begin{cases} 0, & x < -1 \\ 2/7, & -1 \le x < 0 \\ 4/3, & 0 \le x < 2 \\ 1 & x \ge 2 \end{cases}$$

$$Pr\left(-\tfrac{1}{2} \le X < 2 \mid X \ge 0\right) = \frac{Pr\left(-\frac{1}{2} \le X < 2,\ X \ge 0\right)}{Pr(X \ge 0)} = \frac{Pr(0 \le X < 2)}{Pr(X \ge 0)} = \frac{F(2-) - F(0-)}{1 - F(0-)}$$

$$= \frac{\frac{1}{3} - \frac{2}{7}}{1 - \frac{2}{7}} = \frac{1}{15}$$

$\boxed{Pr\left(-\frac{1}{2} \le X < 2 \mid X \ge 0\right) = \frac{1}{15}}$ .... 2 MARKS

Clearly X is a discrete r.v. with support $S = \{-1, 0, 2\}$ and p.m.f.

$$f(x) = \begin{cases} F(x) - F(x-), & x \in S \\ 0, & \text{o.w} \end{cases} = \begin{cases} 2/7, & x = -1 \\ \frac{1}{21}, & x = 0 \\ 2/3, & x = 2 \\ 0, & \text{o.w.} \end{cases}$$

$$E(X) = -\frac{2}{7} + 0 + \frac{4}{3} = \frac{22}{21}$$

$$E(X^2) = \frac{2}{7} + \frac{8}{3} = \frac{62}{21}$$

$$\text{Var}(X) = E(X^2) - (E(X))^2 = \frac{62}{21} - \left(\frac{22}{21}\right)^2 = \frac{818}{441}$$

$\boxed{\text{Var}(X) = \frac{818}{441}}$ ... 3 MARKS

Problem No. 3 $\quad S_X = [-2, 1]$, $\quad h(x) = x + 2|x|$, $x \in \mathbb{R}$

$$h(x) = \begin{cases} -x, & -2 < x < 0 \\ 3x, & 0 < x < 1 \end{cases} \qquad h(S_X) = (0, 3)$$

$S_1 = (-2, 0)$

$h(S_1) = (0, 2)$, $h(\cdot) \downarrow$

$h^{-1}(y) = -y$

$S_2 = (0, 1)$

$h(S_2) = (0, 3)$, $h(\cdot) \uparrow$

$h^{-1}(y) = \frac{y}{3}$

Thus the p.d.f. of Y is

$$g(y) = f(h^{-1}(y)) \left|\frac{d}{dy} h^{-1}(y)\right| I_{(0,2)}(y) + f(h^{-1}(y)) \left|\frac{d}{dy} h^{-1}(y)\right| I_{(0,3)}(y)$$

$$= f(-y)\, I_{(0,2)}(y) + \frac{1}{3} f\left(\frac{y}{3}\right) I_{(0,3)}(y)$$

$$\boxed{g(y) = \begin{cases} \frac{1}{4} + \frac{y}{9}, & 0 < y < 2 \\ y/9, & 2 < y < 3 \\ 0, & \text{o.w.} \end{cases}} \quad \text{... 4 MARKS}$$

$$E(Y) = \int_0^2 y\left(\frac{1}{4} + \frac{y}{9}\right) dy + \int_2^3 \frac{y^2}{9}\, dy = \frac{1}{4}\int_0^2 y\, dy + \frac{1}{9}\int_0^3 y^2\, dy = \frac{3}{2}$$

$$E(Y^2) = \frac{1}{4}\int_0^2 y^2 dy + \frac{1}{9}\int_0^3 y^3 dy = \frac{35}{12}$$

$$\text{Var}(Y) = E(Y^2) - (E(Y))^2 = \frac{35}{12} - \frac{9}{4} = \frac{2}{3}$$

$\boxed{\text{Var}(Y) = \frac{2}{3}}$ ... 3 MARKS