EE250: Lecture Note 1
Linear Dynamical Systems

1 Introduction

In this course, we will learn about control systems. In this age of automation, control systems are all pervading from aircraft to washing machine, and from robotics to power systems. Lets take an example of an automatic sprinkler system.

Automatic Sprinkler System

Figure 1 shows an automatic sprinkler system for watering a garden. There are many electrodes placed in the garden which provide information regarding the dryness of the soil to a properly decision making system which either turns on the sprinkler or turns of the sprinkler.

Various operation of different sub-systems of the automatic sprinkler system as shown in Figure 1 are as follows.

1. The soil moisture is measured through sensors placed in the various parts of the garden. These sensor data are used as feedback to the Controller.

2. A computer can work as the Controller which is a decision making process. You can also use a micro-controller. This unit reads all sensor data and learns the average moisture contents of the soil. Based on a pre-decided threshold value, the controller actuates an ON signal or OFF signal to the driver of the sprinkler which is usually a pump unit. You all can write a simple code for this sub-system.

3. The actuator is the device that automatically powers the pump or turns off the pump. You need to design this subsystem using an electronic logic unit.

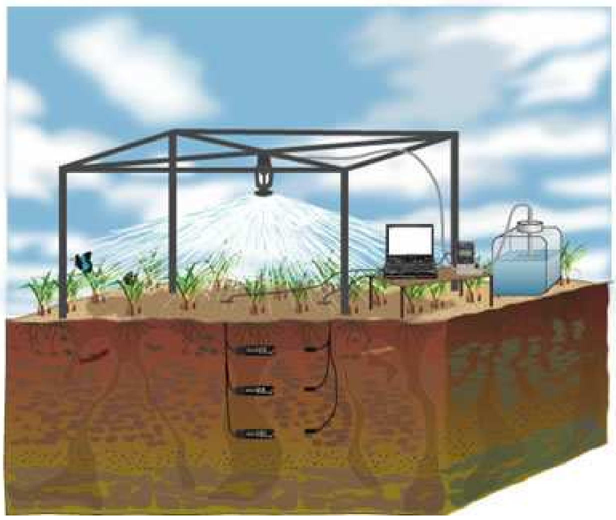

Figure 1: Automatic sprinkler system

4. The sprinkler along with the soil can be thought of as the Plant or system to be controlled. Changes in the Plant input (sprinkler turn ON signal) will effectively change the Plant output (moisture level of soil). So the soil will be watered as and when necessary.

I hope that you can design an automatic sprinkler system for your home garden. Here is an assignment for you:

Assignment 1: Design an automatic alarm system for ATM Machine.

2 A Generic Control System

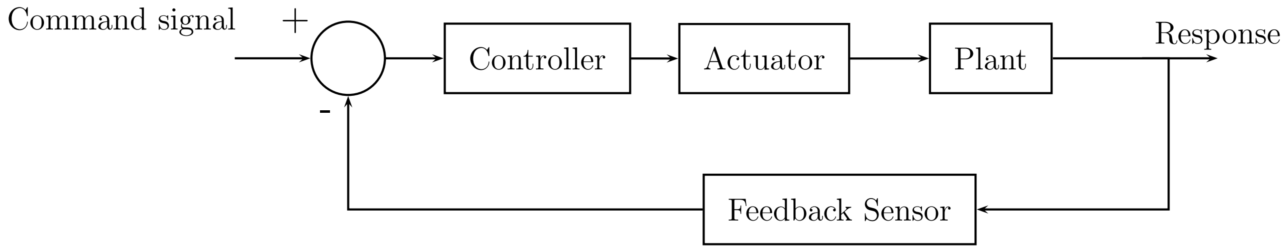

Figure 2: A general control system

In Figure 2, a schematic block-diagram is given that depicts a generic control System which consists of following components:

- Plant
- Controller
- Actuator
- Feedback Sensor

Within the scope of this course, we would mainly deal with mathematical representation of Systems (Plants) and focus on Controller algorithms, not on the hardware implementation. We would assume the Actuator and Sensors to be available and having ideal behaviours.

A control engineer designs a controller to control some of the physical variables of a plant or a system. The plant can be a DC motor, or a Power plant generation unit, or it can be a robot manipulator. In general, a plant that is made up from various physical components follows some physical laws. The plant dynamics is expressed mathematically which describes the plant behaviour in terms of input and response/output. For example consider a mass M resting on a friction-less surface. A horizontal force $u(t)$ is acting on the mass M. The position of the mass x is given by the following equation:

$$M\frac{d^2x}{dt^2} = u(t) \tag{1}$$

Thus the dynamics of a force-mass system as described above is governed by second order differential equation. In general plant dynamics are expressed in terms of differential equations. Further plant dynamics can be either linear or nonlinear.

Example: Separately excited DC Motor

Figure 3 shows the circuit diagram explaining the operation of this motor. The motor armature receives a DC volt as input. The Armature rotates in the magnetic field set up by the stator field. The current carrying conductors in the armature are subjected to an induced emf which is also called as back emf $e_b = K_b\omega$. The motor shaft is connected to a load parametrized by J, the moment of inertia and B, the viscous coefficient. It turns out that this motorgenerates a torque that is proportional to armature current i_a. Looking at the circuit, you should be able to write corresponding dynamic equations.

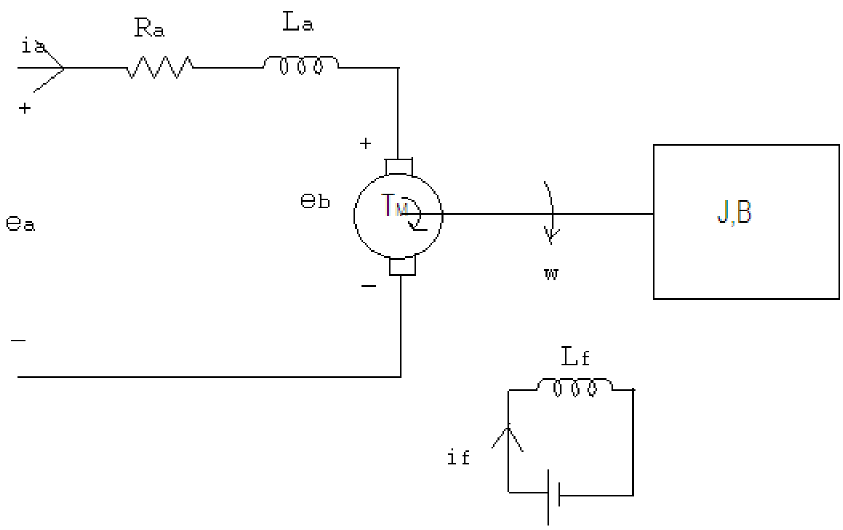

Figure 3: Separately excited DC motor

$$T_m = K_T i_a \tag{2}$$

$$e_b = K_b\omega \tag{3}$$

$$L_a\frac{di_a}{dt} + R_a i_a + e_b = e_a \tag{4}$$

$$J\frac{d\omega}{dt} + B\omega + T_\omega = T_m = K_T i_a \tag{5}$$

Please note that this system has two state variables: i_a and ω
Let $x_1 = \omega$, $x_2 = i_a$

$$\frac{dx_1}{dt} = -\frac{B}{J}x_1 + \frac{K_T}{J}x_2 \tag{6}$$

$$\frac{dx_2}{dt} = -\frac{R_a}{L_a}x_2 - \frac{K_b}{L_a}x_1 + \frac{1}{L_a}e_a \tag{7}$$

The system is thus represented in the form of vector differential equation which can be generalised as

$$\dot{x} = Ax + Bu \tag{8}$$

where

$$x = [x_1 \quad x_2]^T \tag{9}$$

$$A = \begin{pmatrix} -\frac{B}{J} & \frac{K_T}{J} \\ -\frac{R_a}{L_a} & -\frac{K_b}{L_a} \end{pmatrix} \tag{10}$$

$$B = \begin{pmatrix} 0 \\ \frac{1}{L_a} \end{pmatrix} \tag{11}$$

You can note that the response of the motor is speed ω

$$\begin{aligned} y &= x_1 & (12) \\ &= [1 \quad 0] \quad [x_1 \quad x_2]^T & (13) \\ &= Cx & (14) \end{aligned}$$

where

$$C = [1 \quad 0] \tag{15}$$

The step response of the system is shown in the Figure 4.
The corresponding Matlab code is given below:

```
clc;
% Separately Excited DC Motor
% Dated: 2nd Jan 2014
% Governing Equations:
% d(x1)/dt = - (B/J)*x1 + (K_T/J)*x2   .......  (1)
% d(x2)/dt = - (K_b/L_a)*x1 - (R_a/L_a)*x2 - (e_a/L_a)  .......  (2)
% Y = x1  ........  (3)
% e_a is the input and Y is the output
% State Space form of these equations:
% d(X)/dt = A*X + B*U
% Y = C*X + D*U
% where A = [(-B/J) (K_T/J) ; (-K_b/L_a) (-R_a/L_a)]
% B = [0 ; 1/L_a]    C = [1 0]    D = 0

% Suppose we fix the values of B,J,K_T,K_b,L_a,R_a
B = 0.1;
J = 1.2;
K_T = 0.8;
K_b = 1.6;
L_a = 0.045;
R_a = 20;

% State Space Parameters
A = [(-B/J) (K_T/J) ; (-K_b/L_a) (-R_a/L_a)];
B = [0 ; 1/L_a];
C = [1 0];
D = 0;

%[Num,Den] = ss2tf(A,B,C,D,1); % state-space to transfer function
%tf_motor = tf(Num,Den);
tf_motor = ss(A,B,C,D);
[y,t] = step(tf_motor); % step response
plot(t,y)
```

Example: An Electrical Circuit

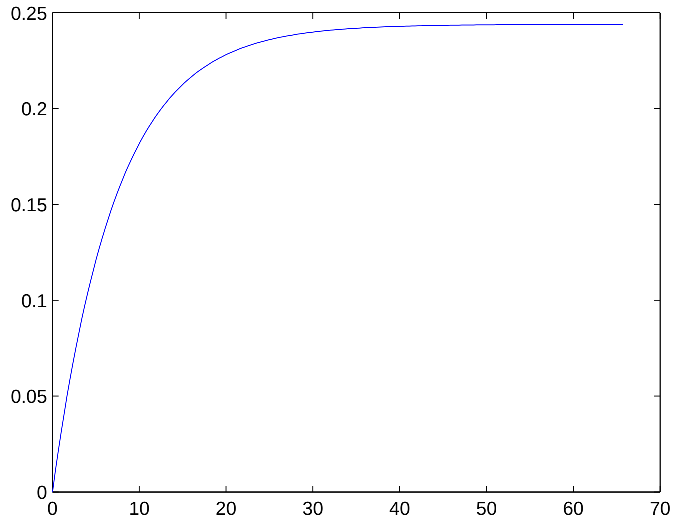

Figure 4: Step response of the DC motor system

Figure 5 shows a RLC circuit. Input for this system is the voltage given as v_S. We are interested in the capacitor voltage v_C - lets term it as output variable. You probably have figured out that the current through inductor i_L is another state variable. The dynamic

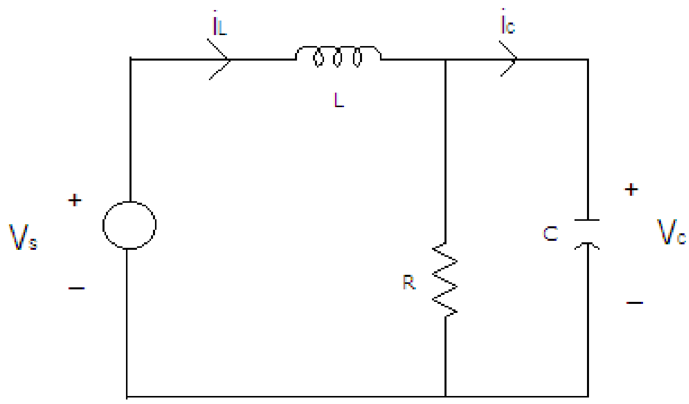

Figure 5: Electrical circuit

equations for this circuit can be written as:

$$C\frac{dv_C}{dt} = i_C = i_L - \frac{v_C}{R} \tag{16}$$

$$L\frac{di_C}{dt} + v_C = v_S \tag{17}$$

By selecting states as $x_1 = v_C$ and $x_2 = i_L$ with initial conditions as $i_L(0) = i_L^0, \quad v_C(0) = v_C^0$, the state equations can be written as

$$\frac{dx_1}{dt} = -\frac{1}{RC}x_1 + \frac{1}{C}x_2 \tag{18}$$

$$\frac{dx_2}{dt} = -\frac{1}{L}x_1 + \frac{1}{L}v_s \tag{19}$$

The step response of the system is shown in the Figure 6. The corresponding Matlab code is given below:

```
% Electrical Circuit
clc;
C = 0.03;
R = 10;
L = 0.0045;

A = [(-1/(C*R)) (1/C); (-1/L) 0];
B = [0; (1/L)];
C = [1 0];
D = 0;


sys = ss(A,B,C,D);

% Initial Conditions
i_L_0 = 0.02;
v_C_0 = 2.1;

x0 = [i_L_0 v_C_0];
T = 0:0.01:10;
U = ones(size(T));
[y,t,x] = lsim(sys,U,T,x0);
plot(t,y)
```

Example: Single Link Manipulator

Figure 7 shows a single link manipulator that rotates in the vertical plane under the influence of gravity.

Input is motor torque τ that rotates the link and the response is angular position θ. It is assumed that the link is mass-less and there is a point mass at the top. The dynamics of motion of this rigid link is given as

$$ml^2\frac{d^2\theta}{dt^2} + mgl\sin\theta = \tau \tag{20}$$

Taking system states as $x_1 = \theta, \quad x_2 = \dot{\theta}$, the state models are derived as

$$\dot{x_1} = x_2 \tag{21}$$

$$\dot{x_2} = -\frac{g}{l}\sin x_1 + \frac{1}{ml^2}\tau \tag{22}$$

The step response of the system is shown in the Figure 2. The corresponding Matlab code is given below:

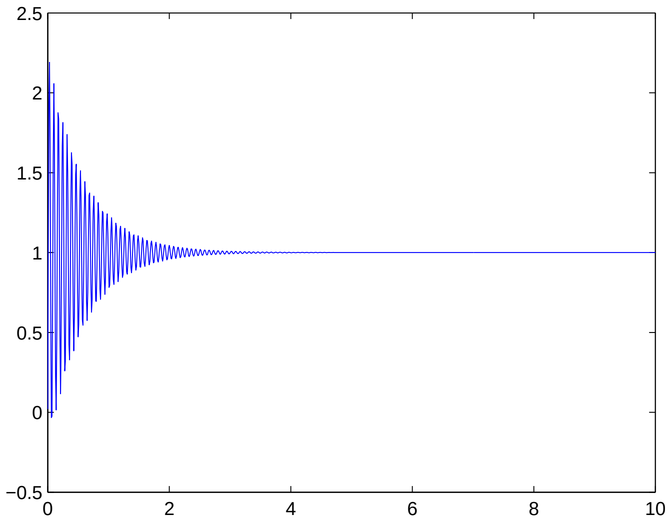

Figure 6: Step response of the circuit

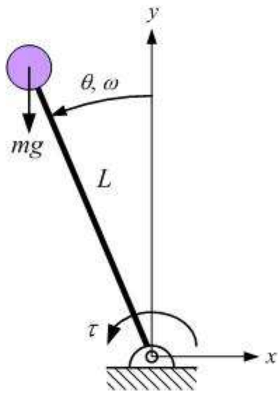

Figure 7: Single link manipulator

```
% Single Link Manipulator
[T,Y] = ode15s(@vdp1000,[0 10],[0 0]);
```

```
function dy = vdp1000(t,y)
g = 1;
m = 1;
l = 1;
dy = zeros(2,1);    % a column vector
dy(1) = y(2);
dy(2) = -g/l*sin(y(1)) + sin(3*t)/(m*l*l);
```

From these three examples, you can notice that first two systems are linear while the third one is nonlinear. How will you ascertain a system is linear?

3 Linear System

A linear system follows two primary properties:

1. It obeys the principle of superposition.
2. The response of a linear time-invariant (LTI) system can be expressed as the convolution of the input signal with unit impulse response.

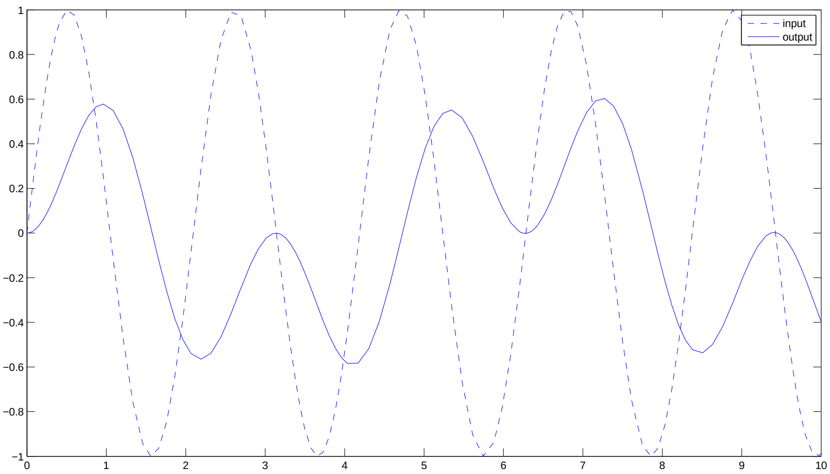

Figure 8: Step response of the single link manipulator system

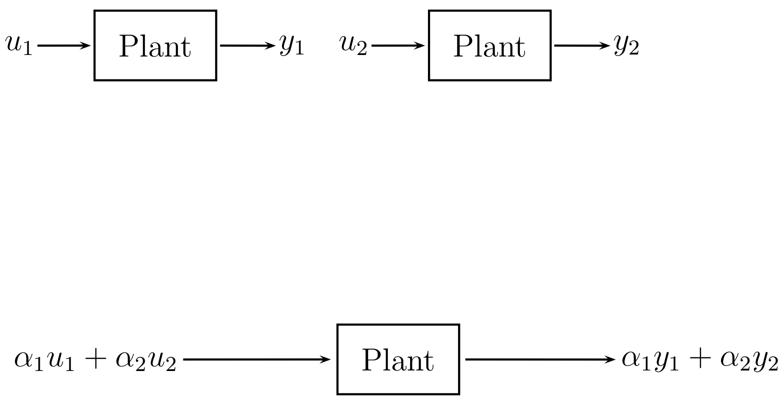

Figure 9: Superposition Principle

3.1 Principle of Superposition

This is explained in Figure 9. Suppose that the plant produces a response y_1 for an input u_1 and response y_2 for input u_2. It is said to follow the superposition principle if the plant produces a response $\alpha_1 y_1 + \alpha_2 y_2$ for an input $\alpha_1 u_1 + \alpha_2 u_2$.

Example

Consider a system which obeys following dynamics:

$$\dot{y} + ky = u$$

Show that this system follows superposition principle.

Solution:

Let y_1 and y_2 are solutions for inputs u_1 and u_2 respectively. This implies that $\dot{y_1} + ky_1 = u_1$ and $\dot{y_2} + ky_2 = u_2$ are valid identity. Let $y = \alpha_1 y_1 + \alpha_2 y_2$. Then its derivative can be reduced as follows:

$$\begin{aligned}
\dot{y} &= \alpha_1 \dot{y}_1 + \alpha_2 \dot{y}_2 \\
&= \alpha_1 (u_1 - ky_1) + \alpha_2 (u_2 - ky_2) \\
&= -k(\alpha_1 y_1 + \alpha_2 y_2) + (\alpha_1 u_1 + \alpha_2 u_2)
\end{aligned}$$

This shows that if u_1 triggers response y_1 and u_2 triggers response y_2, then $\alpha_1 u_1 + \alpha_2 u_2$ triggers response $\alpha_1 y_1 + \alpha_2 y_2$. Thus superposition principle holds.

Q. *What is the benefit of superposition?*
A. *Response to a general signal is the sum of responses to elementary signals where the general signal is expressed as sum of elementary signals.* In general, impulse and exponential signals are considered as elementary signals.

3.2 Response of a Linear System

Definition of an impulse: A very intense force for a very short duration. *Paul Dirac* provided mathematical definition of an impulse $\delta(t)$ as

$$\int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)d\tau = f(t)$$

where

$$\int_{-\infty}^{\infty} \delta(t-\tau)d\tau = 1$$

Superposition Integral:
Let $h(t,\tau)$ be the impulse response of a linear system at t to an impulse applied at τ

$$y(t) = \int_{-\infty}^{\infty} u(\tau)h(t,\tau)d\tau$$

if the system is LTI, then

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} u(\tau)h(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} u(t-\tau)h(\tau)d\tau \end{aligned}$$

This implies that the response of a LTI system to an exponential is also an exponential.

Example

Find the response of the following dynamic system

$$\dot{y} + ky = u, \quad y(0) = 0$$

Solution:

First find the impulse response.

$$\dot{y} + ky = \delta(t)$$

$$\int_{0^-}^{0^+} \dot{y}dt + k\int_{0^-}^{0^+} ydt = \int_{0^-}^{0^+} \delta(t)dt$$

This gives,

$$y(o^+) - y(0^-) = 1$$

Thus the system becomes

$$\dot{y} + ky = 0 \text{with } y(0^+) = 1$$

Let the solution to this differential equation be $y = Ae^{st}$. Then substituting it in the above equation, we get

$$\begin{aligned} Ase^{st} + kAe^{st} &= 0 \\ (s+k)Ae^{st} &= 0 \end{aligned}$$

This implies $s + k = 0$ or $s = -k$. At $t = 0$, $y(0^+) = 1$. This implies that $y(0^+) = 1 = Ae^{-k.0} => A = 1$. Thus impulse resonse becomes $h(t) = e^{-kt}$.

The final response due to any input $u(t)$ is given by the convolution integral:

$$y(t) = \int_0^\infty u(\tau)e^{-k(t-\tau)}d\tau \tag{23}$$

.

In control system, the dynamic analysis of a linear plant is done using two approaches,

1. Transfer function approach
2. State space approach

4 Transfer Function

Before we introduce the concept of transfer function, the concept of Laplace Transform is revisited here.

4.1 Laplace Transform

The Laplace Transform is one of the mathematical tools used to solve linear ordinary differential equations.

Given the real function $f(t)$ that satisfies the condition

$$\int_0^\infty |f(t)e^{-\sigma t}|dt < \infty$$

for some finite, real σ, the Laplace Transform of $f(t)$ is defined as

$$F(s) = \int_0^\infty f(t)e^{-st}dt$$

where s is referred to as a laplace operator which is a complex variable, i.e, $s = \sigma + j\omega$.

Please refer any reference book to go through important theorems on Laplace Transforms. However, we will present two important theorems here.

Theorem 1 (Final-Value Theorem) *If the Laplace Transform of $f(t)$ is $F(s)$ and if $sF(s)$ is analytic on the imaginary axis and in the right half of the s-plane then*

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

Theorem 2 (Real Convolution) *Let $F_1(s)$ and $F_2(s)$ be the Laplace Transforms of $f_1(t)$ and $f_2(t)$ respectively and $f_1(t) = 0$, $f_2(t) = 0$ for $t < 0$, then*

$$\begin{aligned} F_1(s)F_2(s) &= \mathcal{L}[f_1(t) * f_2(t)] \\ &= \mathcal{L}\left[\int_0^t f_1(\tau)f_2(t-\tau)d\tau\right] \\ &= \mathcal{L}\left[\int_0^t f_2(\tau)f_1(t-\tau)d\tau\right] \end{aligned}$$

Example: (Application of the Laplace Transform to the solution of linear ordinary differential equations)

Consider the differential equation

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = 5u(t)$$

where $u(t)$ is a unit step function. The initial conditions are $y(0) = -1$ and $\frac{dy}{dt} = 0$ at $t = 0$. Find the solution using Laplace Transform. Apply final value theorem to determine the steady-state solution.

Solution: Applying Laplace Transform to both sides

$$s^2Y(s) - sy(0) - \frac{dy}{dt}(0) + 3sY(s) - 3y(0) + 2Y(s) = \frac{5}{s}$$

Substituting initial values,

$$Y(s) = \frac{-s^2 - 3s + 5}{s(s+1)(s+2)} = \frac{5}{2s} - \frac{7}{s+1} + \frac{7}{2(s+2)}$$

Taking the inverse Laplace Transform, we get the complete solution as

$$y(t) = \frac{5}{2} - 7e^{-t} + \frac{7}{2}e^{-2t}$$

Now applying final value theorem,

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} s\frac{-s^2 - 3s + 5}{s(s+1)(s+2)} = \frac{5}{2}$$

This result can be obtained from the solution of y(t) by putting $t = \infty$.

4.2 Transfer function of a single-input single-output (SISO) system

Let's take a simple plant,

$$\dot{y} + 2y = u$$

where initial conditions are zero. The impulse response is found to be

$$h(t) = e^{-2t}$$

The actual response is given by the convolution

$$y(t) = \int_{-\infty}^{\infty} h(t)u(t-\tau)d\tau = \int_{0}^{\infty} h(t)u(t-\tau)d\tau$$

Using theorem 2, we get

$$\begin{aligned} Y(s) &= H(s)U(s) \\ \frac{Y(s)}{U(s)} &= H(s) \end{aligned}$$

where $H(s)$ is the transfer function. In this case,

$$H(s) = \mathcal{L}(e^{-2t}) = \frac{1}{s+2}$$

Formally, the transfer function of a linear time-invariant system is defined as the Laplace transform of the impulse response, with all the initial conditions set to zero.

4.2.1 Incorrect use of final value theorem

Given,

$$Y(s) = \frac{3}{s(s-2)}$$

Then, Note that

$$y(\infty) \neq \lim_{s\to 0} sY(s) = \frac{-3}{2}$$

Rather,

$$y(\infty) = \lim_{t\to\infty} y(t) = \lim_{t\to\infty} [-\frac{3}{2} + \frac{3}{2}e^{2t}] = \infty$$

DC gain

It is the ratio of the output of a system to its input at steady state i.e., all the transients have decayed.

$$\text{DC gain} = \lim_{s\to 0} G(s)$$

Example 1. Consider a transfer function

$$G(s) = \frac{3}{s^2 + 2s - 3}$$

It has two poles namely, $s = 1$ and $s = -3$. Since one of them is unstable, we can not find the dc gain. The final value theorem is not applicable in this case.

Example 2. Consider another transfer function

$$G(s) = \frac{s+1}{s^2 + s + 3}$$

It has two stable poles ($s = -\frac{1}{2} \pm i\frac{\sqrt{11}}{2}$). By applying final value theorem, we get

$$\text{dc gain} = \frac{1}{3} = 0.333$$

Laplace transform has been introduced for representing linear dynamical systems because of following two important reasons:

1. The diferential equations are converted to algebraic equations. More of this aspect will be covered in control system representation through block diagram and signal flow graph.

2. The convolution integral in time domain gets represented as simple multiplication in s-plane.

4.2.2 Exercise

1. Find the transfer function $\frac{Y(s)}{U(s)}$ for the following systems:

   (a) $\frac{d^3y(t)}{dt^3} + 4\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 2y(t) = 6\frac{du(t)}{dt} + u(t)$

   (b) $3\frac{d^2y(t)}{dt^2} - \frac{dy(t)}{dt} + 3y(t) = 2u(t) - u(t-1)$

   (c) $\frac{d^4y(t)}{dt^4} + 10\frac{d^2y(t)}{dt^2} + \frac{dy(t)}{dt} + 5y(t) = 5u(t)$

   (d) $\frac{d^3y(t)}{dt^3} + 10\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + y(t) + 2\int_0^t y(\tau)d\tau = \frac{du(t)}{dt} + 2u(t)$

2. Find the response of the following systems to a unity step input:

(a) $G(s) = \frac{10(s+1)}{s(s+4)(s+6)}$

(b) $G(s) = \frac{1}{s(s^2+1)(s+0.5)}$

(c) $G(s) = \frac{(s+1)}{s(s+3)(s^2+4s+8)}$

Verify your solutions using MATLAB commands.

3. Solve the following ordinary differential equation using Laplace transforms

(a) $\ddot{y}(t) - 2\dot{y}(t) + 4y(t) = 0; \quad y(0) = 1,\ \dot{y}(0) = 2$

(b) $\ddot{y}(t) + 3y(t) = sin(t); \quad y(0) = 1,\ \dot{y}(0) = 2$

(c) $\ddot{y}(t) + y(t) = t; \quad y(0) = 1,\ \dot{y}(0) = -1$

Verify your results using MATLAB.

4. Find the time function corresponding to each of the following Laplace transforms using partial fraction expansions:

(a) $F(s) = \frac{2}{s(s+2)}$

(b) $F(s) = \frac{3s+2}{s^2+4s+20}$

(c) $F(s) = \frac{1}{s^2+4}$

(d) $F(s) = \frac{s+1}{s^2}$

5 State Variable Model

Example

Consider a nth order differential equation

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_n y = u$$

Define following variables,

$$\begin{aligned}
y &= x_1 \\
\frac{dy}{dt} &= x_2 \\
\vdots &= \vdots \\
\frac{d^{n-1}y}{dt^{n-1}} &= x_n \\
\frac{d^n y}{dt^n} &= -a_1 x_{n-1} - a_2 x_{n-2} - \ldots - a_n x_1 + u
\end{aligned}$$

The nth order differential equation may be written in the form of n first order differential equations as

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\vdots &= \vdots \\
\dot{x}_n &= -a_1 x_{n-1} - a_2 x_{n-2} - \ldots - a_n x_1 + u
\end{aligned}$$

or in matrix form as,

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + Bu$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

The output can be one of states or a combination of many states. Since, $y = x_1$,

$$y = [1\ 0\ 0\ 0\ \dots\ 0]\boldsymbol{x}$$

Transfer function approach of system modeling provides final relation between output variable and input variable. However, a system may have many other internal variables of concerns to a control engineer. State variable representation takes into account of all such internal variables.

- State: The state of a dynamic system is the smallest set of variables, $\boldsymbol{x} \in R^n$, such that given $\boldsymbol{x}(t_0)$ and $u(t),\ t > t_0,\ \boldsymbol{x}(t),\ t > t_0$ can be uniquely determined.
- Usually a system governed by a n^{th} order differential equation or n^{th} order transfer function is expressed in terms of n state variables: $x_1, x_2, \dots, x_n$.
- A generic structure of a state-space model of a n^{th} order dynamical system is given by:

$$\begin{aligned} \dot{x}_1 &= a_{11}x_1(t) + a_{12}x_2(t) + \dots + a_{1n}x_n(t) + b_1u(t) \\ \dot{x}_2 &= a_{21}x_1(t) + a_{22}x_2(t) + \dots + a_{2n}x_n(t) + b_2u(t) \\ \vdots &= \vdots \\ \dot{x}_n &= a_{n1}x_1(t) + a_{n2}x_2(t) + \dots + a_{nn}x_n(t) + b_nu(t) \\ \\ y &= c_1x_1(t) + c_2x_2(t) + \dots + c_nx_n(t) + du(t) \end{aligned}$$

- Thus the generalized state variable representation in compact form looks as:

$$\begin{aligned} \dot{\boldsymbol{x}} &= A\boldsymbol{x} + Bu \\ y &= C\boldsymbol{x} + du \end{aligned} \tag{24}$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & & & \dots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix} \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \quad C = \begin{bmatrix} c_1 & c_2 & c_3 & \dots & c_n \end{bmatrix}$$

6 System Response

Given a system in state-space form, how can we compute the state response and output response of such systems? From the scope of this course, you will find that Laplace Transform will play a BIG role. Lets take an example of a simple scalar differential equation:

$$\dot{x}(t) = ax, \quad x(t_0) = x_0$$

You already know the generic solution:

$$x(t) = ke^{at} \quad where \quad k = x_0e^{-at_0}$$

What happens to Vector differential equation?

$$\dot{\mathbf{x}} = A\mathbf{x}, \quad \mathbf{x}(t_0) = \mathbf{x_0}$$

Intuitively, the generic solution can be taken as: $\mathbf{x}(t) = e^{At}\mathbf{k}$. But what is this entity e^{At}?

$$e^{At} = I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + ...$$

$$\begin{aligned}
\frac{d}{dt}(e^{At}) &= A + A^2t + \frac{1}{2!}A^3t^2 + ... \\
&= A(I + At + \frac{1}{2!}A^2t^2 + ...) \\
&= Ae^{At}
\end{aligned}$$

So, taking the solution as $\mathbf{x}(t) = e^{At}\mathbf{k}$,

$$\dot{\mathbf{x}}(t) = \mathbf{k}\frac{d}{dt}(e^{At}) = A\mathbf{k}e^{At} = A\mathbf{x}$$

This solution thus satisfies the vector differential equation.

$$\mathbf{k} = e^{-At_0}\mathbf{x_0}$$

$$\mathbf{x}(t) = e^{A(t-t_0)}\mathbf{x_0}$$

If $t_0 = 0$, $\mathbf{x}(t) = e^{At}\mathbf{x_0}$

How does one compute e^{At} ?

$$\dot{x} = Ax, \quad x(0) = x_0$$

Taking Laplace Transform,

$$sX(s) - x_0 = AX(s)$$

$$(sI - A)X(s) = x_0$$

$$X(s) = (sI - A)^{-1}x_0$$

$$x(t) = \mathcal{L}^{-1}[(sI - A)^{-1}x_0] = e^{At}x_0$$

So,

$$e^{At} = \mathcal{L}^{-1}[(sI - A)^{-1}]$$

This expression e^{At} is called as state transition matrix. We will learn more about this state transition matrix later in this course.

Example 1

Given

$$A = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$$

Compute e^{At}

Solution

$$
\begin{aligned}
sI - A &= \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \\
&= \begin{pmatrix} s & -1 \\ 2 & s+3 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
[sI - A]^{-1} &= \frac{1}{s(s+3)+2} \begin{pmatrix} s+3 & 1 \\ -2 & s \end{pmatrix} \\
&= \frac{1}{s^2+3s+2} \begin{pmatrix} s+3 & 1 \\ -2 & s \end{pmatrix} \\
&= \frac{1}{(s+2)(s+1)} \begin{pmatrix} s+3 & 1 \\ -2 & s \end{pmatrix} \\
&= \begin{pmatrix} \frac{2}{s+1} - \frac{1}{s+2} & \frac{1}{s+1} - \frac{1}{s+2} \\ & \\ -\frac{2}{s+1} + \frac{2}{s+2} & \frac{2}{s+2} - \frac{1}{s+1} \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
e^{At} &= \mathcal{L}^{-1}[sI - A]^{-1} \\
&= \mathcal{L}^{-1} \begin{pmatrix} \frac{2}{s+1} - \frac{1}{s+2} & \frac{1}{s+1} - \frac{1}{s+2} \\ & \\ -\frac{2}{s+1} + \frac{2}{s+2} & \frac{2}{s+2} - \frac{1}{s+1} \end{pmatrix} \\
&= \begin{pmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{pmatrix}
\end{aligned}
$$

We will now investigate the solution of the state model with external input:

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + Bu$$

Taking Laplace transform,

$$sX(s) - x_0 = AX(s) + BU(s)$$

$$(sI - A)X(s) = x_0 + BU(s)$$

$$X(s) = (sI - A)^{-1}x_0 + (sI - A)^{-1}BU(s)$$

$$\boldsymbol{x}(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}BU(\tau)d\tau$$

$$y(t) = Cx(t) = Ce^{At}x_0 + \int_0^t Ce^{A(t-\tau)}BU(\tau)d\tau$$

6.1 Solution of State Equation - Another Approach

Consider the state equation

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + Bu(t); \quad x(t_0) = x_0 \tag{6}$$

We will find the solution of this state equation using following identity

$$\frac{d}{dt}e^{At} = Ae^{At} = e^{At}A \qquad \text{(\textbf{Exercise:} Verify this)}$$

Multiplying both sides of Eqn. (6.1) with e^{-At}, we get

$$
\begin{aligned}
e^{-At}\dot{\boldsymbol{x}}(t) &= e^{-At}A\boldsymbol{x}(t) + e^{-At}Bu(t) \\
e^{-At}\dot{\boldsymbol{x}}(t) - e^{-At}A\boldsymbol{x}(t) &= e^{-At}Bu(t)
\end{aligned}
$$

Since,

$$\frac{d}{dt}\left[e^{-At}\boldsymbol{x}(t)\right] = e^{-At}\dot{\boldsymbol{x}}(t) - e^{-At}A\boldsymbol{x}(t)$$

we have,

$$\frac{d}{dt}\left[e^{-At}\boldsymbol{x}(t)\right] = e^{-At}Bu(t)$$

Integrating both sides,

$$\begin{aligned} \int_0^t d(e^{-A\tau}\boldsymbol{x}(\tau)) &= \int_0^t e^{-A\tau}Bu(\tau)d\tau \\ e^{-At}\boldsymbol{x}(t) - \boldsymbol{x}(0) &= \int_0^t e^{-A\tau}Bu(\tau)d\tau \end{aligned}$$

Thus, the solution of state equation is given as

$$\boldsymbol{x}(t) = e^{At}\boldsymbol{x}(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

Alternately, we can derive this solution by using Laplace transform as follows. We know that $\boldsymbol{X}(s) = (sI-A)^{-1}\boldsymbol{x}_0 + (sI-A)^{-1}BU(s)$. Taking inverse Laplace transform on both sides, we get

$$\boldsymbol{x}(t) = \mathcal{L}^{-1}[(sI-A)^{-1}\boldsymbol{x}_0] + \mathcal{L}^{-1}[(sI-A)^{-1}BU(s)]$$

Since, it can be verified that $x(t) = e^{At}x_0$ is a solution of $\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t)$, $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$, by comparison, we can write

$$\mathcal{L}^{-1}(sI-A)^{-1} = e^{At}$$

Since $\mathcal{L}^{-1}[(sI-A)^{-1}BU(s)]$ is a convolution integral, the complete solution becomes

$$\boldsymbol{x}(t) = e^{At}\boldsymbol{x}_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \tag{1}$$

Example 2

Find the solution of the system described by

$$\begin{aligned} \dot{\boldsymbol{x}} &= A\boldsymbol{x} + Bu \\ y(t) &= C\boldsymbol{x} \end{aligned}$$

where

$$A = \begin{bmatrix} -1 & 1 \\ -1 & 10 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 10 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 10 \end{bmatrix}$$

and thus the response of the system to a unit step input under zero initial condtions.

Solution:

$$\begin{aligned} (sI-A)^{-1} &= \begin{bmatrix} s+1 & -1 \\ 1 & s+10 \end{bmatrix}^{-1} \\ &= \frac{1}{s^2+11s+11}\begin{bmatrix} s+10 & 1 \\ -1 & s+1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{s+10}{(s+a_1)(s+a_2)} & \frac{1}{(s+a_1)(s+a_2)} \\ \frac{-1}{(s+a_1)(s+a_2)} & \frac{s+1}{(s+a_1)(s+a_2)} \end{bmatrix} \end{aligned}$$

where $a_1 = 1.1125$ and $a_2 = 9.8875$.

$$\begin{aligned} e^{At} &= \mathcal{L}^{-1}[(sI-A)^{-1}] \\ &= \begin{bmatrix} 1.0128e^{-a_1t} - 0.0128e^{-a_2t} & 0.114e^{-a_1t} - 0.114e^{-a_2t} \\ -0.114e^{-a_1t} + 0.114e^{-a_2t} & -0.0128e^{-a_1t} + 1.0128e^{-a_2t} \end{bmatrix} \end{aligned}$$

The input is $u(t) = 1;\ t \geq 0$ and $\boldsymbol{x}(0) = \mathbf{0}$. Therefore,

$$\begin{aligned}
\boldsymbol{x}(t) &= \int_0^t e^{A(t-\tau)} b d\tau \\
&= \int_0^t \begin{bmatrix} 1.14\left(e^{-a_1(t-\tau)} - e^{-a_2(t-\tau)}\right) \\ 1.14\left(-0.1123 e^{-a_1(t-\tau)} + 8.8842 e^{-a_2(t-\tau)}\right) \end{bmatrix} \\
&= \begin{bmatrix} 0.9094 - 1.0247 e^{-a_1 t} + 0.1153 e^{-a_2 t} \\ -0.0132 + 0.1151 e^{-a_1 t} - 0.1019 e^{-a_2 t} \end{bmatrix}
\end{aligned}$$

The output

$$y(t) = 0.9094 - 1.0247 e^{-1.1125t} + 0.1153 e^{-9.8875t}$$

6.2 Conversion of state variable models to transfer function

Consider a state variable model

$$\begin{aligned}
\dot{\boldsymbol{x}}(t) &= A\boldsymbol{x}(t) + Bu(t) \\
y(t) &= C\boldsymbol{x}(t) + du(t)
\end{aligned} \tag{-9}$$

Taking the Laplace Transform on the both sides of Eqn. (-8), we get

$$\begin{aligned}
sX(s) - \boldsymbol{x}_0 &= AX(s) + BU(s) \\
Y(s) &= CX(s) + dU(s)
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \quad & (sI - A)X(s) = \boldsymbol{x}_0 + BU(s) \\
\text{or,} \quad & X(s) = (sI - A)^{-1}\boldsymbol{x}_0 + (sI - A)^{-1}BU(s)
\end{aligned}$$

If we let $x_0 = 0$, then

$$Y(s) = \Big(C(sI - A)^{-1}B + d\Big)U(s)$$

Therefore, the transfer function becomes

$$\frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + d \tag{-12}$$

6.2.1 Exercise

1. Construct state models for the following differential equations.
   (a) $\dddot{y} + 3\ddot{y} + 2\dot{y} = \dot{u} + u$
   (b) $\dddot{y} + 6\ddot{y} + 11\dot{y} + 6y = u$
   (c) $\dddot{y} + 6\ddot{y} + 11\dot{y} + 6y = \dddot{u} + 8\ddot{u} + 17\dot{u} + 8u$

2. Consider the system

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\boldsymbol{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u; \quad \boldsymbol{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix}\boldsymbol{x}$$

   Find the output response of the system to unit-step function.

3. Consider the following system

$$\dot{\boldsymbol{x}} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\boldsymbol{x} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}u; \quad \boldsymbol{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

   Find the solution of the state equation.

4. An oscillation can be generated by

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \boldsymbol{x}$$

Show that its solution is

$$\dot{\boldsymbol{x}} = \begin{bmatrix} cos\ t & sin\ t \\ -sin\ t & cos\ t \end{bmatrix} \boldsymbol{x}$$

5. Use two different methods to find the unit-step response of

$$\begin{aligned} \dot{\boldsymbol{x}} &= \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u \\ y &= [2 \quad 3]\, \boldsymbol{x} \end{aligned}$$

7 Numerical Integration

The state model of an autonomous non-linear system is given as follows

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}(t), u(t)), \quad \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \tag{-12}$$

It is desired to find solution of this differential equation in the interval $[t_0, t_1]$ using various numerical integration techniques.

Euler's Method

- Divide the interval $[t_0, t_1]$ into N equal sub-intervals of width $h = \frac{t_f - t_0}{N}$. h is called *step length.*

- Set $t_k = t_0 + kh$, $k = 1, 2, \ldots, N$. The derivative $\frac{d\boldsymbol{x}(t_k)}{dt}$ at time $t = t_k$ is approximated as

$$\frac{d\boldsymbol{x}(t_k)}{dt} \approx \frac{\boldsymbol{x}(t_{k+1}) - \boldsymbol{x}(t_k)}{h} \tag{-12}$$

  From Eqn. (7) and Eqn. (7), we get

$$\frac{\boldsymbol{x}(t_{k+1}) - \boldsymbol{x}(t_k)}{h} = \boldsymbol{f}(\boldsymbol{x}(t_k), u(t_k))$$

- This gives the update equation

$$\boldsymbol{x}(t_{k+1}) = \boldsymbol{x}(t_k) + h\boldsymbol{f}(\boldsymbol{x}(t_k), u(t_k)) \tag{-12}$$

Runge-Kutta 4th Order Algorithm

The update equation is given as follows:

$$\boldsymbol{x}(t_{k+1}) = \boldsymbol{x}(t_k) + \frac{h}{6}(\boldsymbol{m}_0 + 2\boldsymbol{m}_1 + 2\boldsymbol{m}_2 + \boldsymbol{m}_3) \tag{-12}$$

where

$$\begin{aligned} \boldsymbol{m}_0 &= \boldsymbol{f}(t_k, \boldsymbol{x}(t_k), u(t_k)) \\ \boldsymbol{m}_1 &= \boldsymbol{f}(t_k + \frac{h}{2}, \boldsymbol{x}(t_k) + \frac{\boldsymbol{m}_0 h}{2}, u(t_k)) \\ \boldsymbol{m}_2 &= \boldsymbol{f}(t_k + \frac{h}{2}, \boldsymbol{x}(t_k) + \frac{\boldsymbol{m}_1 h}{2}, u(t_k)) \\ \boldsymbol{m}_3 &= \boldsymbol{f}(t_k + h, \boldsymbol{x}(t_k) + \boldsymbol{m}_2 h, u(t_k)) \end{aligned}$$

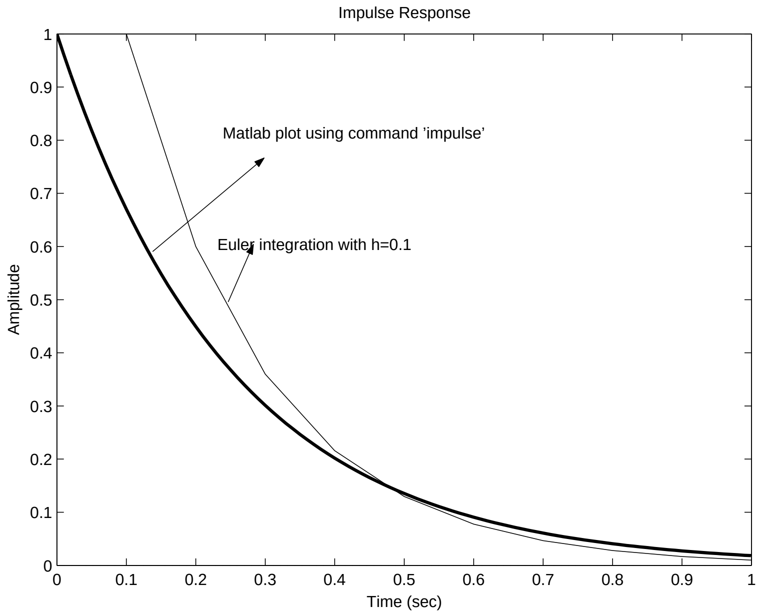

Figure 10: Numerical Solution of state equation using Euler's method

Example

Find impulse response of the following system in the interval [0,1] using Euler's method

$$\dot{x} + 4x = \delta(t), \quad x(t_0) = 0$$

Solution:

Take $h = 0.1$. Using the update equation (7), we have

$$x(t_{k+1}) = x(t_k) + h(-4x(t_k) + \delta(t_k))$$

Thus, for $t = 0.1$,

$$x(0.1) = 0 + 0.1(-4x(0) + \delta(0)) = 1.0$$
$$x(0.2) = 1 + 0.1(-4x(0.1)) = 1 - 0.4 = 0.6$$

Similarly, we can calculate other values as shown in the table 7. The result is shown graphically in the Figure 10 where it is compared with the simulation obtained using a MATLAB function that uses a very small step size.

t_k	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$x(t_k)$	0	1.0	0.6	0.36	0.216	0.1296	0.0776	0.0467	0.0280	0.0168	0.01

Table 1: Values of $x(t)$ obtained from Euler's integration

8 MATLAB Assignment

Consider the following nonlinear system:

$$\dot{x} = -x + x^2$$

Find the solution of the above state equation using Euler, Runge-Kutta methods and Matlab command (ode45) for following intial conditions $x_0 = 0.5$, -0.5 and 1.5. For Euler and Runge-Kutta method, write your own code and verify your result using ode45 routine in MATLAB.

Department of Electrical Engineering, IIT Kanpur
EE250: Control Systems Analysis
Tutorial 1
08 Jan 2014

Question 1

We found out the dynamics of a servo-motor in the class as

$$\frac{dx_1}{dt} = -\frac{B}{J}x_1 + \frac{K_T}{J}x_2$$
$$\frac{dx_2}{dt} = -\frac{K_b}{L_a}x_1 - \frac{R_a}{L_a}x_2 + \frac{1}{L_a}u$$

where

x_1 = speed (ω) of the motor
x_2 = Armature current I_a
u = Armature voltage e_a

Parameters are:

$B = 0.25$ N-m/(rad/sec)
$R_a = 5\ \Omega$
$L = 0.1$ H
$J = 2$ N-M/(rad/sec)
$K_b = 1$ volt/(rad/sec)

(i) Given output $y = x_1$, find $\frac{Y(s)}{U(s)}$
(ii) Given $u(t) = 100$ volt (sudden), find $y(t)$. Compute y_{ss}
(iii) Find state transition matrix e^{At}
(iv) Find $y(t)$ using e^{At}

Question 2

$$\frac{d^3y(t)}{dt^3} + 4\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 2y(t) = 6\frac{du(t)}{dt} + u(t)$$

Find $\frac{Y(s)}{U(s)}$

Question 3

$$\frac{d^3y(t)}{dt^3} + 10\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + y(t) + 2\int_0^t y(\tau)d\tau = \frac{du(t)}{dt} + 2u(t)$$

Find $\frac{Y(s)}{U(s)}$

Question 4

Solve the following ordinary differential equation using Laplace transform

$$\ddot{y}(t) - 2\dot{y}(t) + 4y(t) = 0; y(0) = 1, \dot{y}(0) = 2$$