Lecture Note: 2
Block Diagrams and Signal Flow Graphs

1 Block Diagrams

A block diagram can be used to describe the composition and interconection of a system, Or it can be used, together with transfer functions, to describe the cause-and-effect relationships throughout the system.

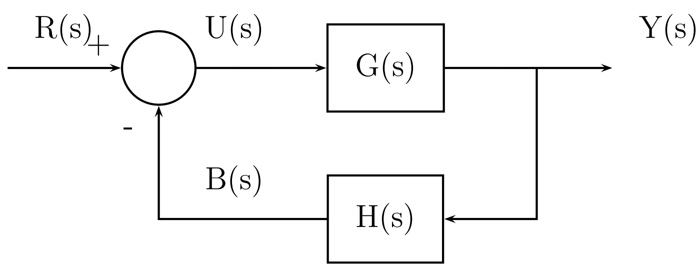


Figure 1: Basic Block diagram of a feedback control system

$R(s)$ = reference input (command)
$Y(s)$ = output (controlled variable)
$B(s)$ = feedback signal
$U(s)$ = actuating signal=error signal $E(s)$, when $H(s) = 1$
$H(s)$ = feedback transfer function
$-G(s)H(s) = L(s)$ = loop transfer function
$G(s)$ = forward-path transfer function
$M(s) = Y(s)/R(s)$ = closed-loop transfer function or system transfer function

From Fig. 1, we write

$$Y(s) = G(s)U(s) \tag{1}$$

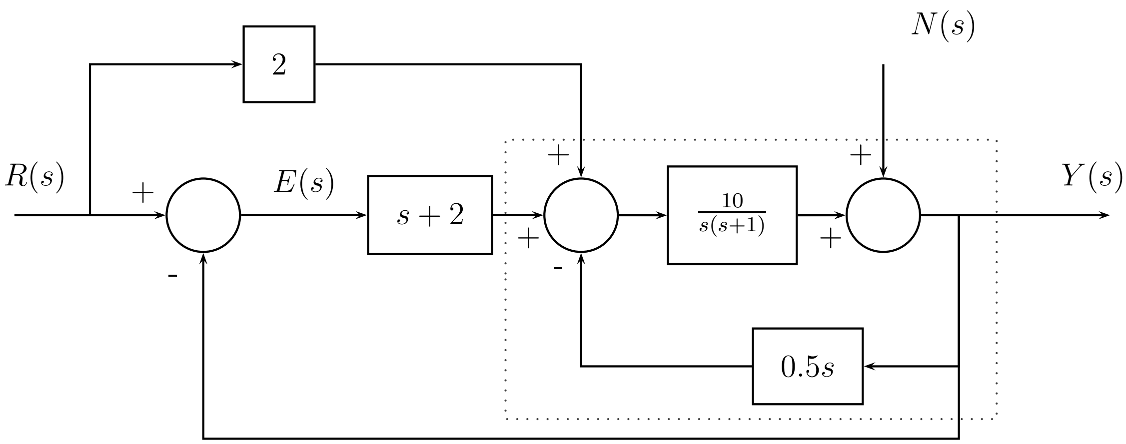


Figure 2: Block diagram of a feedback control system

and

$$B(s) = H(s)Y(s) \tag{2}$$

The actuating signal is written

$$U(s) = R(s) - B(s) \tag{3}$$

Substituting Eq. (3) into Eq. (1) yields

$$Y(s) = G(s)R(s) - G(s)B(s) \tag{4}$$

Substituting Eq. (2) into Eq. (4) and then solving for $Y(s)/R(s)$ gives the closed loop transfer function:

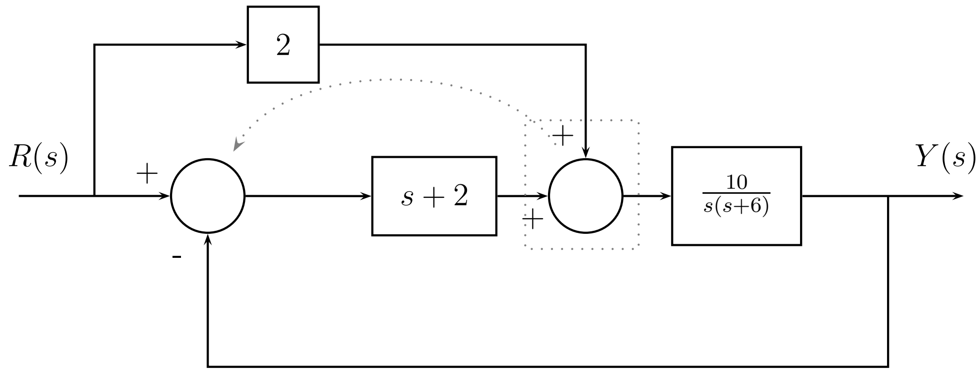
$$M(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} \tag{5}$$

In general, a control system may contain more than one feedback loop, and the evaluation of the tranfer fucntion from the block diagram by the algebraic method just described may be tedious. Although, in principle, the block diagram of a system with one input and one output can always be reduced to the basic single-loop form of Fig. 1. We demonstrate this in the following example.

Example 1. The block diagram of a feedback control system is shown in Fig. 2. Find the following transfer functions:

$$\text{(a)}\ \left.\frac{Y(s)}{R(s)}\right|_{N=0} \quad \text{(b)}\ \left.\frac{Y(s)}{E(s)}\right|_{N=0} \quad \text{(c)}\ \left.\frac{Y(s)}{N(s)}\right|_{R=0}$$

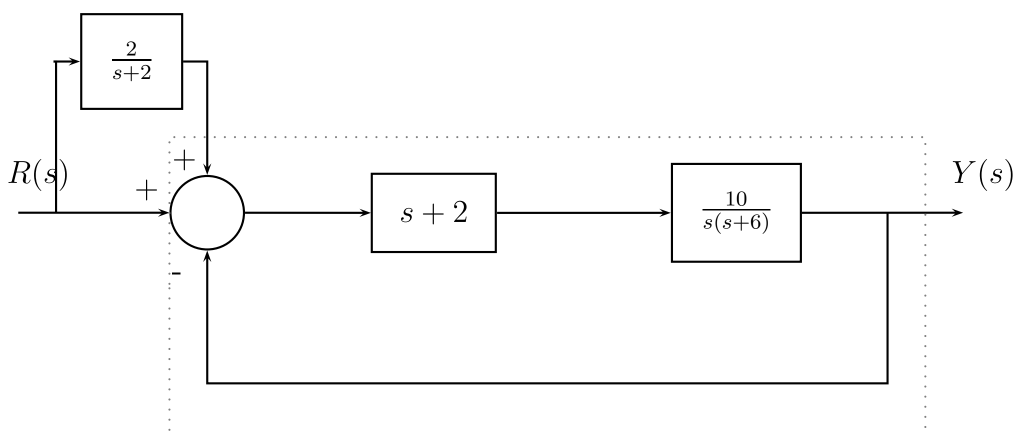
Solution: **(a) step 1**: By making $N(s) = 0$ and reducing the inner block (shown in dotted box), we get



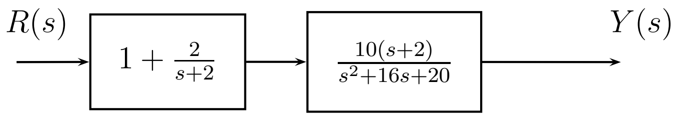
The inner block transfer function is reduced as follows:

$$\frac{\frac{10}{s(s+1)}}{1+\frac{s}{2}\frac{10}{s(s+1)}} = \frac{10}{s(s+6)}$$

step 2: For simplifying the block diagram, we would shift the summation point inside the square dotted box in a direction shown by dotted arrow. This gives us



step 3: Reduce the part of block diagram within the dotted box. We get



Step 4: Hence, the transfer function between $R(s)$ and $Y(s)$ is given by

$$\left.\frac{Y(s)}{R(s)}\right|_{N=0} = \frac{10(s+4)}{s^2+16s+20}$$

Putting $R(s) = \frac{1}{s}$,

$$Y(s)|_{N=0} = \frac{10(s+4)}{s(s^2+16s+20)} = \frac{2}{s} - \frac{1.45}{s+1.36} - \frac{0.54}{s+14.63}$$

Taking inverse Laplace transform,

$$y(t) = 2u(t) - 1.45e^{-1.36}u(t) - 0.54e^{-14.63}u(t)$$

Proceeding similarly, the transfer function between $N(s)$ and $R(s)$ is given by

$$\left.\frac{Y(s)}{N(s)}\right|_{R=0} = \frac{s(s+1)}{s^2+16s+20} = 1 + \frac{0.03}{s+1.36} - \frac{15.03}{s+14.63}$$

Taking inverse Laplace transform,

$$y(t) = u(t) + 0.03e^{-1.36}u(t) - 15.03e^{-14.63}u(t)$$

$\square$

2 Physical Models

2.1 DC motor Model (Separately Excited)

Two main physical principles by which an armature controlled DC motor works as we set the field current to a constant value:

1. The back emf e_b is directly proportional to the speed of the armature/rotor: $e_b \propto \omega$
2. Electrical torque developed in the motor is directly proportional to the armature current: $T_M \propto i_a$.

System Dynamics

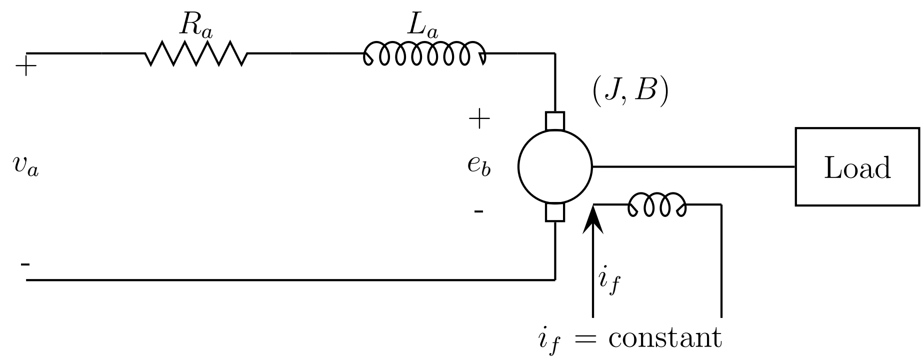


Figure 3: Separately Excited DC motor

The governing equations for the DC motor can be written by looking at figure 3:

$$
\begin{aligned}
v_a - e_b &= R_a i_a + L_a \frac{di_a}{dt} && (6) \\
e_b &= k_b \dot{\theta} && (7) \\
\tau &= k_T i_a && (8) \\
\tau &= J\ddot{\theta} + B\dot{\theta} && (9)
\end{aligned}
$$

where motor inertia is neglected. Here the first equation is derived by applying KVL to the armature circuit. Next two equations stand for the principle by which the DC motor works. The last equation describes angular motion dynamics.

State Variable Model

Choose the state variables as $x_1 = i_a$, $x_2 = \theta$ and $x_3 = \dot{\theta}$. The state equations can be written as

$$
\begin{aligned}
\dot{x}_1 &= -\frac{R_a}{L_a} x_1 - \frac{K_b}{L_a} x_3 + \frac{1}{L_a} v_a && (10) \\
\dot{x}_2 &= x_3 && (11) \\
\dot{x}_3 &= \frac{K_T}{J} x_1 - \frac{B}{J} x_3 && (12)
\end{aligned}
$$

and the output equation is given by

$$
y = \theta = x_2 = [0 \quad 1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad (13)
$$

The above state model can be written in a short format as follows:

$$
\begin{aligned}
\dot{\boldsymbol{x}} &= A\boldsymbol{x} + Bu \\
y &= C\boldsymbol{x} && (14)
\end{aligned}
$$

where

$$A = \begin{bmatrix} -\frac{R_a}{L_a} & 0 & -\frac{K_b}{L_a} \\ 0 & 0 & 1 \\ \frac{K_T}{J} & 0 & -\frac{B}{J} \end{bmatrix} \quad B = \begin{bmatrix} \frac{1}{L_a} \\ 0 \\ 0 \end{bmatrix} \quad C = [0 \quad 1 \quad 0]$$

Please note that $K_b = K_T$ (numerically) for separately excited DC motor.

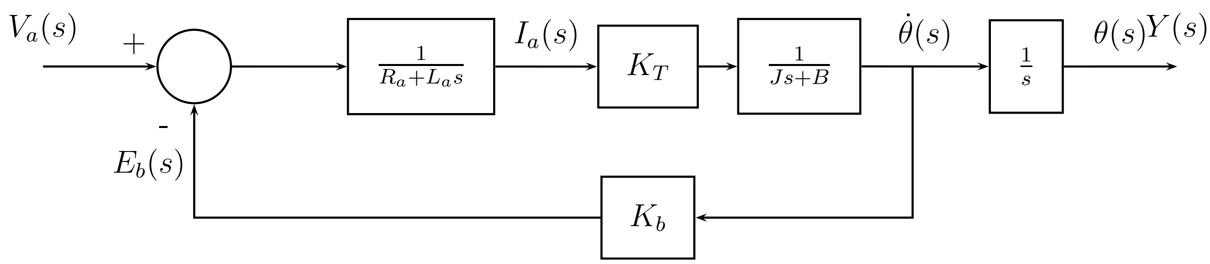
Exercise Show that $K_b = K_T$ (numerically). Please note that Power developed in the armature $P = e_b i_a = T_m \omega$.

Transfer function model of DC motor using Block Diagram approach

Taking Laplace transform on both sides of Eqns. (6), (7) and (8), we get,

$$\begin{aligned} V_a(s) - E_b(s) &= (R_a + L_a s) I_a(s) && (15) \\ E_b(s) &= K_b s \theta(s) && (16) \\ K_T I_a(s) &= (Js^2 + Bs)\theta(s) && (17) \end{aligned}$$

In block diagrammatic representation of these equations looks as:



Simplying the above block diagram, the closed loop transfer function can be written as

$$\frac{\theta(s)}{V_a(s)} = \frac{K_T}{s(R_a + sL_a)(Js + B) + sK_b K_T} \tag{18}$$

3 Signal-Flow Graphs (SFGs)

SFGs may be defined as a graphical means of portraying the input-output relationships among the variables of a set of linear algebraic equations. This is an alternative to block-diagrammatic representation.

Input Node (source) An input node is one that has only outgoing branches.

Output Node (sink) An output node is a node that has only incoming branches.

Path A path is any collection of a continuous succession of branches traversed in the same direction.

Forward Path A forward path is a path that starts at an input node and ends at an output node, and along which no node is traversed more than once.

Loop A loop is a path that originates and terminates on the same node, and along which no other node is encountered more than once.

Path Gain The product of the branch gains encountered in traversing a path is called the path gain.

Forward-Path Gain The forward-path gain is the path gain of a forward path.

Loop Gain The loop gain is the path gain of a loop.

Nontouching Loops Two parts of an SFG are nontouching if they don't share a common node.

4 Mason's Gain Formula

Given an SFG with N forward paths and K loops, the gain between the input node y_{in} and output node y_{out} is

$$M = \frac{y_{out}}{y_{in}} = \sum_{k=1}^{N} \frac{M_k \triangle_K}{\triangle} \tag{19}$$

where
y_{in} = input-node variable
y_{out} = Output-node variable
M = gain between y_{in} and y_{out}.
N = total number of forward paths between y_{in} and y_{out}.
M_k = gain of the kth forward path between y_{in} and y_{out}

$$\triangle = 1 - \sum_i L_{i1} + \sum_j L_{j2} - \sum_k L_{k3} + \ldots$$

L_{mr} = gain product of mth$(m = i, j, k, \ldots)$ possible combinations of r nontouching loops $(1 \leq r < K)$.
or
$\triangle$ = 1-(sum of the gains of **all individual loops**)+(sum of products of gains of all possible combinations of **two** nontouching loops)-(sum of products of gains of all possible combinations of **three** nontouching loops)+ ...
$\triangle_k$ = the $\triangle$ for that part of the SFG that is nontouching with the kth forward path.

Example 2. Mason's gain formula can be used to simplify the block-diagrams directly. We illustrate this with this example. Let's consider the system in Fig. 2. Here we have two forward paths between $R(s)$ and $Y(s)$ with gains

$$M_1 = \frac{10(s+2)}{s(s+1)} \quad \text{and } M_2 = \frac{20}{s(s+1)} \qquad \text{respectively.}$$

There are two individual loops with following loop gains:

$$L_{11} = -\frac{5}{s+1} \qquad L_{21} = -\frac{10(s+2)}{s(s+1)}$$

There are no non-touching loops and thus we have

$$\begin{aligned} \triangle &= 1 - (L_{11} + L_{21}) \\ &= \frac{s^2 + 16s + 20}{s(s+1)} \end{aligned}$$

Also, $\triangle_1 = 1$ and $\triangle_2 = 1$. Thus, the transfer function between $R(s)$ and $Y(s)$ is given by the Mason's gain formula (19) as follows:

$$\begin{aligned} M(s) &= \left.\frac{Y(s)}{R(s)}\right|_{N=0} \\ &= \frac{M_1\triangle_1 + M_2\triangle_2}{\triangle} \\ &= \frac{10(s+4)}{s^2 + 16s + 20} \end{aligned}$$

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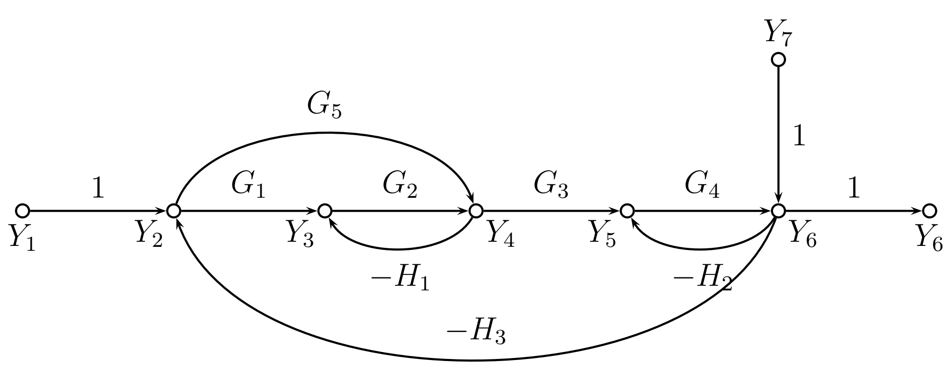

Figure 4: An SFG example

Example 3. Find the following transfer functions for the SFG shown in Fig. 4

$$\left.\frac{Y_6}{Y_1}\right|_{Y_7=0} \qquad \left.\frac{Y_6}{Y_7}\right|_{Y_1=0}$$

Solution: (a) There are two forward paths between Y_1 and Y_6 and the forward-path gains are

$$\begin{array}{ll} M_1 = G_1G_2G_3G_4 & \text{Forward Path: } Y_1 - Y_2 - Y_3 - Y_4 - Y_5 - Y_6 \\ M_2 = G_5G_3G_4 & \text{Forward Path: } Y_1 - Y_2 - Y_4 - Y_5 - Y_6 \end{array}$$

There are four loops. The loop gains are

$$\begin{array}{ll} L_{11} = -G_2H_1 & L_{21} = -G_4H_2 \\ L_{31} = -G_5G_4G_3H_3 & L_{41} = -G_1G_2G_3G_4H_3 \end{array}$$

There is only one pair of nontouching loops; that is, the two loops are $Y_3 - Y_4 - Y_3$ and $Y_5 - Y_6 - Y_5$. Thus the product of the gains of the two nontouching loops is

$$L_{12} = G_2G_4H_1H_2$$

All loops are in touch with forward paths M_1 and M_2. Thus, $\triangle_1 = \triangle_2 = 1$. Therefore,

$$\begin{aligned} \triangle &= 1 - (L_{11} + L_{21} + L_{31} + L_{41}) + L_{12} \\ &= 1 + G_2H_1 + G_4H_2 + G_3G_4G_5H_3 + G_1G_2G_3G_4H_3 + G_2G_4H_1H_2 \end{aligned}$$

From (19), we have

$$\begin{aligned} \left.\frac{Y_6}{Y_1}\right|_{Y_7=0} &= \frac{M_1\triangle_1 + M_2\triangle_2}{\triangle} \\ &= \frac{(G_1G_2 + G_5)G_3G_4}{1 + G_2H_1 + G_4H_2 + G_3G_4G_5H_3 + G_1G_2G_3G_4H_3 + G_2G_4H_1H_2} \end{aligned}$$

(b) There is only one forward path between Y_7 and Y_6 with following gains:

$$M_1 = 1 \qquad \triangle_1 = 1 + G_2H_1$$

Its because there is only one loop ($Y_3 - Y_4 - Y_3$) that is nontouching with the forward path $Y_7 - Y_6$ and its loop gain is $L_{11} = -G_2H_1$. Thus, we have

$$\begin{aligned} \left.\frac{Y_7}{Y_6}\right|_{Y_1=0} &= \frac{M_1\triangle_1}{\triangle} \\ &= \frac{1 + G_2H_1}{1 + G_2H_1 + G_4H_2 + G_3G_4G_5H_3 + G_1G_2G_3G_4H_3 + G_2G_4H_1H_2} \end{aligned}$$

□

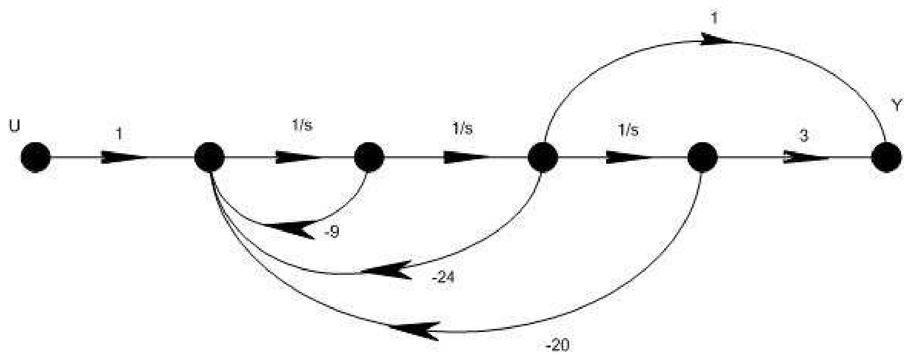

Figure 5:

Example 4. Find the transfer function $\frac{Y(s)}{U(s)}$ for the SFG shown in Figure 5.

Solution: There are two forward paths between the input node and the output node, and the forward path gains are

$$M_1 = 1 * \frac{1}{s} * \frac{1}{s} * 1 = \frac{1}{s^2}$$
$$M_2 = 1 * \frac{1}{s} * \frac{1}{s}\frac{1}{s} * 3 = \frac{3}{s^3}$$

There are three loops. The loop gains are

$$-\frac{9}{s}, \quad -\frac{24}{s^2}, \quad -\frac{20}{s^3}$$

All of the loops are touching. Hence,

$$\triangle = 1 + \frac{9}{s} + \frac{24}{s^2} + \frac{20}{s^3}$$

So, the transfer function between input node and output node is given by the Mason's gain formula as

$$\begin{aligned} M(s) &= \frac{\frac{3}{s^3} + \frac{1}{s^2}}{1 + \frac{9}{s} + \frac{24}{s^2} + \frac{20}{s^3}} \\ &= \frac{\frac{s+3}{s^3}}{\frac{s^3+9s^2+24s+20}{s^3}} \end{aligned}$$

□

4.1 Tutorial Problems

1. A Signal Flow Graph is shown in Figure 6. Find the transfer function between the input node and the output node, using Mason's gain formula.

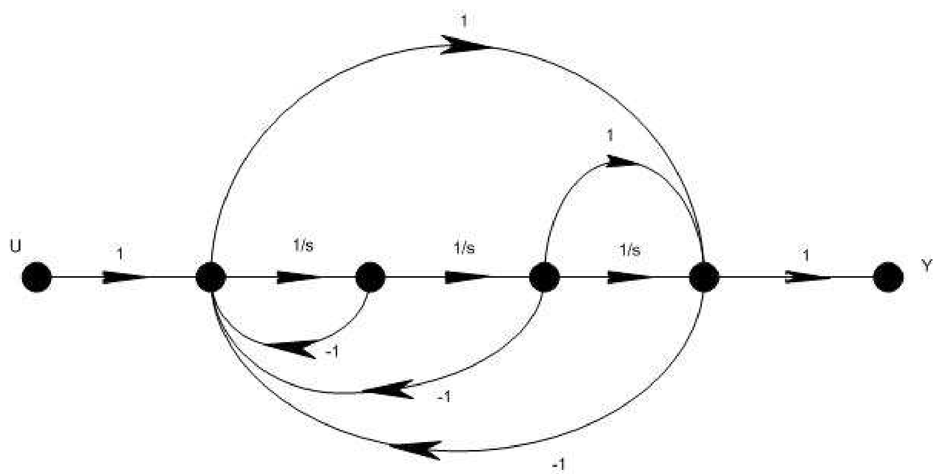

Figure 6: Problem 1

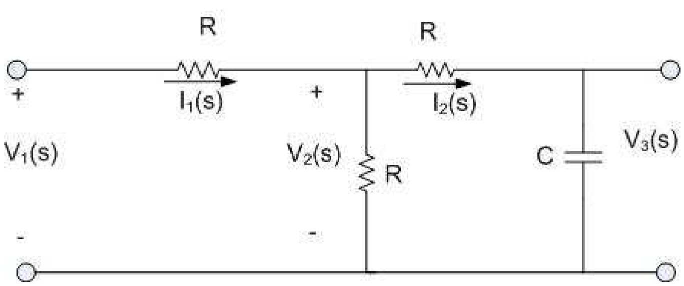

Figure 7: Problem 2

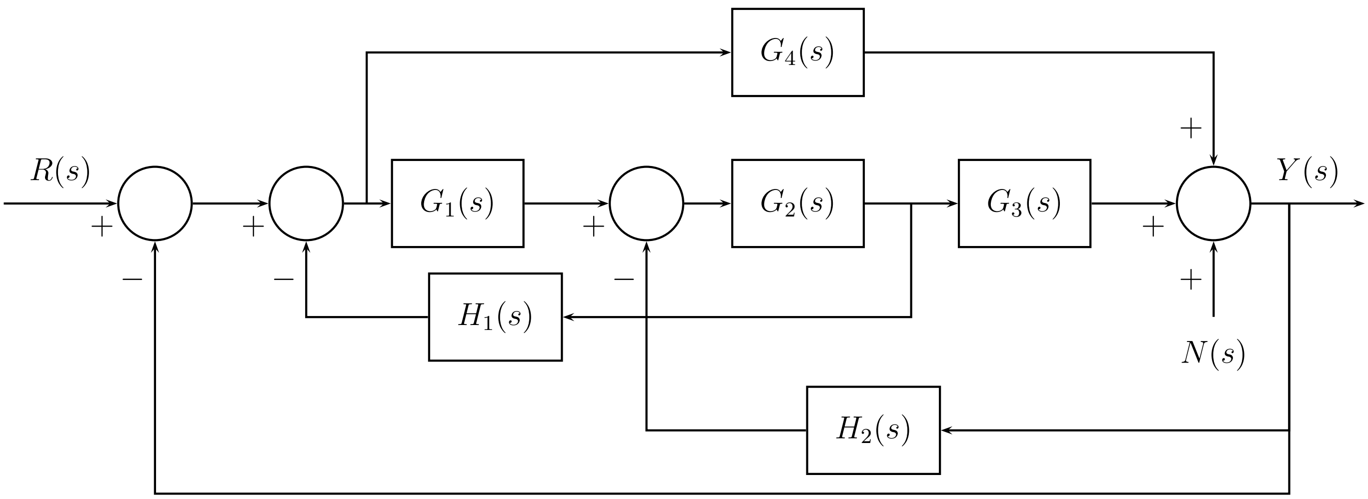

Figure 8: Problem 3

2. A low-pass filter is shown in Figure 7. $R = 1k\Omega$, $C = 1\mu F$
   Construct a signal flow graph connecting input $V_1(s)$ and output $V_2(s)$ and showing internal signals $F_1(s)$, $F_2(s)$, and $V_2(s)$.
   Find $T(s) = \frac{V_3(s)}{V_1(s)}$

3. The block diagram of a feedback control system is shown in Fig. 8.

   (a) Apply the SFG formula directly to the block diagram to find the transfer functions

   $$\left.\frac{Y(s)}{R(s)}\right|_{N=0} \qquad \left.\frac{Y(s)}{N(s)}\right|_{R=0}$$

   Express $Y(s)$ in terms of $R(s)$ and $N(s)$ when both inputs are applied simultaneously.

   (b) Find the desired relation among the transfer functions $G_1(s)$, $G_2(s)$, $G_3(s)$, $G_4(s)$, $H_1(s)$ and $H_2(s)$ so that the output $Y(s)$ is not affected by the disturbance signal $N(s)$ at all.

4. The block diagram of the position-control system of the electronic word processor is shown in Fig. 9.

   (a) Find the loop transfer function $\frac{\Theta_0(s)}{\Theta_e(s)}$ (the outer feedback path is open)

   (b) Find the closed-loop transfer function $\frac{\Theta_o(s)}{\Theta_r(s)}$.

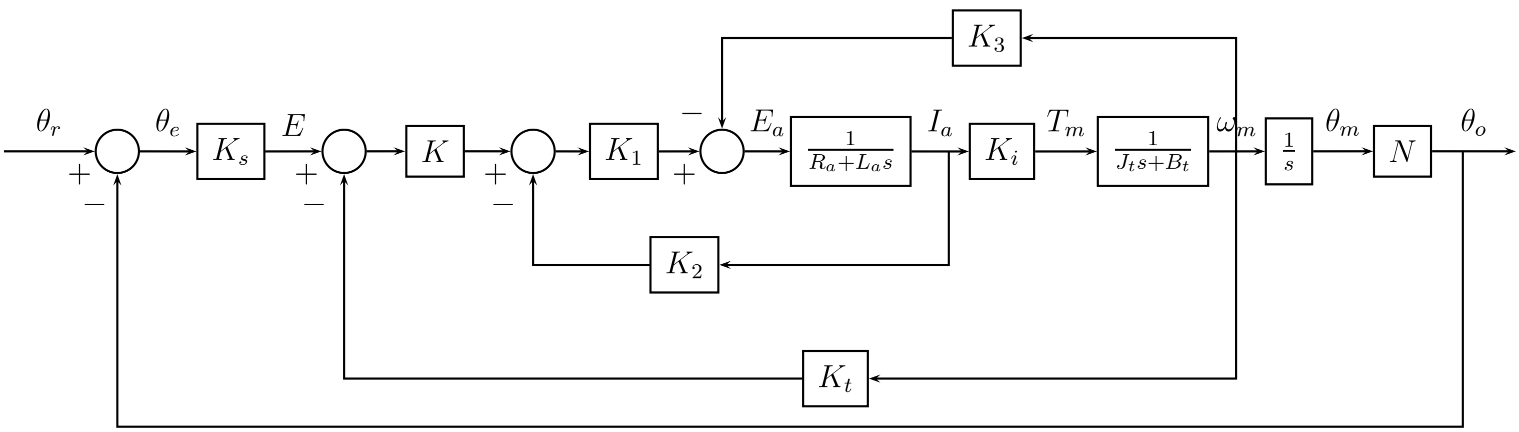

Figure 9: Problem 4