EE 200: Problem Set 4

1. Determine the exponential Fourier series representations of the periodic analog signal whose one period is given by

$$\tilde{x}(t) = \begin{cases} 1 \ ; & 0 \leq t \leq 1 \\ 0 \ ; & 1 < t \leq 2 \end{cases}$$

Solution 1: The fundamental period is $T_0 = 2$, and the fundamental angular frequency $\Omega_0 = 2\pi/T_0 = \pi$. The Fourier coefficients are

$$\begin{aligned} X_k &= \frac{1}{T_0}\int_0^{T_0} x(t)e^{-jk\Omega_0 t}dt \\ &= \frac{1}{2}\int_0^{2} x(t)e^{-jk\pi t}dt \\ &= \frac{1}{2}\int_0^{1} e^{-jk\pi t}dt = \frac{-1}{j2k\pi}e^{-jk\pi t}\Big|_0^1 \\ &= \frac{-1}{j2k\pi}(e^{-jk\pi} - 1) = \frac{j}{2k\pi}[(-1)^k - 1] \end{aligned} \tag{1}$$

2. Determine the exponential form of the Fourier series of the periodic square wave signal $\tilde{x}(t)$ of slide 10, ch 3-1 (Notes). **Solution 2:** The Fourier coefficients are

$$\begin{aligned} X_0 &= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \tilde{x}(t)dt = \frac{1}{T_0}\int_{-T_0/4}^{T_0/4} dt \\ &= \frac{1}{T_0}\left(t\Big|_{-T_0/4}^{T_0/4} \right) = \frac{1}{T_0}\frac{T_0}{2} = \frac{1}{2}. \end{aligned} \tag{2}$$

$$
\begin{aligned}
X_k &= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \tilde{x}(t)e^{-jk\Omega_0 t}dt = \frac{1}{T_0}\int_{-T_0/4}^{T_0/4} e^{-j2\pi kt/T_0}dt \\
&= \frac{-1}{T_0}\times\frac{T_0}{j2}\left(e^{-j2\pi kt/T_0}\Big|_{-T_0/4}^{T_0/4}\right) \\
&= \frac{-1}{j2\pi k}\left(e^{-j\pi kt/2} - e^{j\pi kt/2}\right) = \frac{1}{k\pi}\sin(k\pi t/2) \qquad (3)
\end{aligned}
$$

3. Prove the Parseval's identity given by

$$
\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} |\tilde{x}(\tau)|^2 d\tau = \sum_{k=-\infty}^{\infty} |X_k|^2
$$

where T_0 is the fundamental period.
Solution 3: Proof of the Parseval's identity:
Let $\tilde{x}(t)$ and $\tilde{h}(t)$ be two periodic signals of period $T_0 = 2\pi/\Omega_0$. Their exponential Fourier series representations are given by:

$$
\tilde{x}(t) = \sum_{-\infty}^{\infty} X_k e^{jk\Omega_0 t}
$$

and

$$
\tilde{h}(t) = \sum_{-\infty}^{\infty} H_k e^{jk\Omega_0 t}
$$

The exponential Fourier series representation of $\tilde{y}(t) = \tilde{x}(t)\tilde{h}(t)$ is given by:

$$
\tilde{y}(t) = \sum_{-\infty}^{\infty} Y_k e^{jk\Omega_0 t}
$$

where

$$
\begin{aligned}
Y_k &= \frac{1}{T_0}\int_0^{T_0} \tilde{x}(t)\tilde{h}(t)e^{-jk\Omega_0 t}dt \\
&= \frac{1}{T_0}\int_0^{T_0} \left[\sum_{l=-\infty}^{\infty} X_l e^{jl\Omega_0 t}\right] \tilde{h}(t)e^{-jk\Omega_0 t}dt \\
&= \sum_{l=-\infty}^{\infty} X_l \left[\frac{1}{T_0}\int_0^{T_0} \tilde{h}(t)e^{-j(k-l)\Omega_0 t}dt\right] \\
&= \sum_{l=-\infty}^{\infty} X_l H_{k-l} \qquad (4)
\end{aligned}
$$

It follows from the above equation that

$$
\frac{1}{T_0}\int_0^{T_0} \tilde{x}(t)\tilde{h}(t)dt = \sum_{-\infty}^{\infty} X_l H_{-l} \qquad (5)
$$

Now,

$$
\begin{aligned}
\tilde{x}^*(t) &= \left(\sum_{k=-\infty}^{\infty} X_k e^{jk\Omega_0 t}\right)^* = \sum_{k=-\infty}^{\infty} X_k^* e^{-jk\Omega_0 t} \\
&= \sum_{k=-\infty}^{\infty} X_{-k}^* e^{jk\Omega_0 t} \qquad (6)
\end{aligned}
$$

Substituting $\tilde{h}(t) = \tilde{x}^*(t)$ in the above equation, we get:

$$
\frac{1}{T_0}\int_0^{T_0} |\tilde{x}(t)|^2 dt = \sum_{\infty}^{\infty} |X_l|^2 \qquad (7)
$$

4. Consider two periodic analog signals $\tilde{g}(t)$ and $\tilde{h}(t)$ with a fundamental period $T_0 = 2\pi/\Omega_0$ represented in the complex exponential Fourier series representation given by,

$$
\tilde{g}(t) = \sum_{k=-\infty}^{\infty} G_k e^{jk\Omega_0 t}
$$

and

$$
\tilde{h}(t) = \sum_{k=-\infty}^{\infty} H_k e^{jk\Omega_0 t}
$$

(a) Show that the analog signal $\tilde{g}(t)\tilde{h}(t)$ is also a periodic signal with the same period and has the Fourier series representation of the form

$$\tilde{g}(t)\tilde{h}(t) = \tilde{x}(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\Omega_0 t}$$

where the Fourier coefficients X_k are given by

$$X_k = \sum_{l=-\infty}^{\infty} G_l H_{k-l}$$

(b) Show that

$$\frac{1}{T_0}\int_0^{T_0} \tilde{g}(t)\tilde{h}(t)dt = \sum_{k=-\infty}^{\infty} G_k H_{-k}$$

Solution 4: Note that $\tilde{x}(t+T_0) = \tilde{g}(t+T_0)\tilde{h}(t+T_0) = \tilde{g}(t)\tilde{h}(t)$ Thus, $\tilde{x}(t)$ is a periodic signal with a fundamental period T_0. For the rest of the problem see solution 3.

5. Verify the following CTFT pairs,
(a) $1 \leftrightarrow 2\pi\delta(\Omega)$
(b) $e^{j\Omega_0 t} \leftrightarrow 2\pi\delta(\Omega - \Omega_0)$
(c) $\sin(\Omega_0 t) \leftrightarrow j\pi[\delta(\Omega + \Omega_0) - \delta(\Omega - \Omega_0)]$
Solution 5(a): We determine the inverse CTFT of $2\pi\delta(\Omega)$:

$$\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\Omega)e^{j\Omega t}dt \\ &= \int_{-\infty}^{\infty} \delta(\Omega)e^{j\Omega t}dt \\ &= 1 \end{aligned}$$

5(b): Recall that the CTFT of of $x(t) = 1$ is $X(j\Omega) = 2\pi\delta(\Omega)$ Applying the frequency-shifting property of the CTFT, we note that the inverse CTFT of $2\pi\delta(\Omega - \Omega_0)$ is $e^{j\Omega_0 t}$

5(c):

$$\begin{aligned}\sin(\Omega_0 t) &= \frac{e^{j\Omega_0 t} - e^{-j\Omega_0 t}}{2j} \\ &= \frac{1}{2j}e^{j\Omega_0 t} - \frac{1}{2j}e^{-j\Omega_0 t}\end{aligned}$$

Thus using the result of part (b), the CTFT of $\sin(\Omega_0 t)$ is,

$$\frac{1}{2j}2\pi\delta(\Omega - \Omega_0) - \frac{1}{2j}2\pi\delta(\Omega + \Omega_0) = j\pi[\delta(\Omega + \Omega_0) - \delta(\Omega - \Omega_0)]$$

6. (a) Prove the integration property of the CTFT:
If $x(t) \leftrightarrow X(j\Omega)$, then

$$\int_{-\infty}^{t} x(\tau)d\tau \leftrightarrow \frac{1}{j\Omega}X(j\Omega) + \pi X(0)\delta(\Omega)$$

(b) Show that

$$\mu(t) \leftrightarrow \frac{1}{j\Omega} + \pi\delta(\Omega)$$

Solution 6(a): Consider the convolution integral,

$$\begin{aligned}x(t) \circledast \mu(t) &= \int_{-\infty}^{\infty} x(\tau)\mu(t-\tau)d\tau \\ &= \int_{-\infty}^{t} x(\tau)d\tau\end{aligned}$$

From the convolution property of the CTFT, we note $x(t) \circledast \mu(t) \leftrightarrow X(j\Omega)M(j\Omega)$
Using $M(j\Omega) = \frac{1}{j\Omega} + \pi\delta(\Omega)$, we have

$$X(j\Omega)M(j\Omega) = \frac{1}{j\Omega}X(j\Omega) + \pi X(0)\delta(\Omega)$$

Hence,

$$\int_{-\infty}^{t} x(\tau)d\tau \leftrightarrow \frac{1}{j\Omega}X(j\Omega) + \pi X(0)\delta(\Omega)$$

6(b): Since $\mu(t)$ is discontinuous at $t = 0$, we consider,
$\mu(0) = \frac{1}{2}$ and $\mu(0+) = 1$.
Now, $\mu(0) = \frac{1}{2} \leftrightarrow \frac{1}{2}2\pi\delta(\Omega) = \pi\delta(\Omega)$ and
$\mu(0+) \leftrightarrow \int_{0+}^{\infty} e^{-j\Omega t}dt = \frac{1}{j\Omega}$.
Hence, $\mu(t) \leftrightarrow \frac{1}{j\Omega} + \pi\delta(\Omega)$

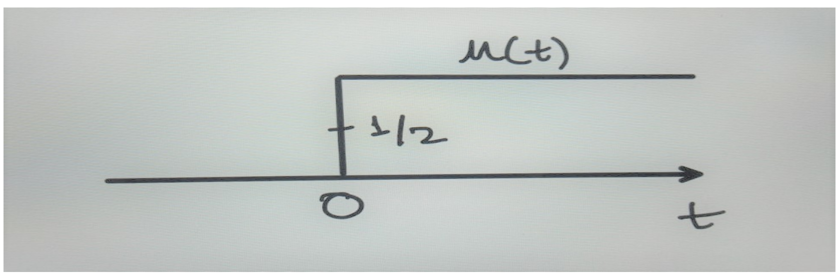

Figure 1: Diagram for Solution 6(b)

7. Determine the CTFT of the following analog signals:
(a) $x_1(t) = \mu(-t)$
(b) $x_2(t) = e^{\alpha t}\mu(-t), \alpha < 0$
(c) $x_3(t) = e^{-\alpha|t|}, \alpha > 0$
Solution 7(a): We note that,

$$\mu(t) \leftrightarrow \frac{1}{j\Omega} + \pi\delta(\Omega)$$

Using the time-reversal property of the CTFT: $x(-t) \leftrightarrow X(-j\Omega)$.

We have $\mu(-t) \leftrightarrow -\frac{1}{j\Omega} + \pi\delta(\Omega)$

7(b):

$$e^{-\alpha t}\mu(t) \leftrightarrow \frac{1}{\alpha + j\Omega}$$

Using the time-reversal property,

$$e^{\alpha t}\mu(-t) \leftrightarrow \frac{1}{\alpha - j\Omega}, \alpha < 0$$

7(c):

$$e^{-\alpha|t|}, \alpha > 0$$
$$= e^{-\alpha t}\mu(t) + e^{\alpha t}\mu(-t)$$

Now, $e^{-\alpha t}\mu(t) \leftrightarrow \frac{1}{\alpha+j\Omega}$ and $e^{\alpha t}\mu(-t) \leftrightarrow \frac{1}{\alpha-j\Omega}$
Hence, $e^{-\alpha|t|} \leftrightarrow \frac{1}{\alpha+j\Omega} + \frac{1}{\alpha-j\Omega} = \frac{2\alpha}{\alpha^2+\Omega^2}$