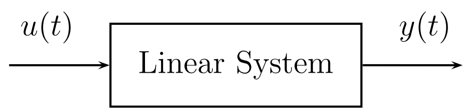
Lecture Note: 5
Frequency Response

1 Introduction

Frequency response is the study of behaviour of a linear system when excited by sinusoidal inputs of different frequencies.



Impulse response of the system is $h(t) = \mathscr{L}^{-1}G(s)$ where $G(s)$ is the transfer function of the linear system. The output of the system is given by

$$y(t) = \int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau$$

Let $u(t) = e^{s_o t}$ where $s_o = \sigma_o + j\omega_o$. This is a generic exponential function. Then

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(\tau)e^{s_o(t-\tau)}d\tau \\ &= \left[\int_{-\infty}^{\infty} h(\tau)e^{-s_o\tau}d\tau\right]e^{s_o t} \\ &= G(s_o)e^{s_o t} \end{aligned}$$

Therefore, the output $y(t)$ is equal to the input exponential multiplied by a complex number $G(s_o)$. That is,

$$y(t) = |G(s_o)|e^{(s_o t + j\angle G(s_o))}$$

Let $u(t) = Asin\omega_o t = \frac{A}{2j}(e^{j\omega_o t} - e^{-j\omega_o t})$. Then

$$\begin{aligned} y(t) &= \frac{A}{2j}(G(j\omega_o)e^{j\omega_o t} - G(-j\omega_o)e^{-j\omega_o t}) \\ &= |G(j\omega_o)|\frac{A}{2j}\left[e^{j(\omega_o t + \angle G(j\omega_o))} - e^{-j(\omega_o t + \angle G(j\omega_o))}\right] \end{aligned}$$

Using fact that $|G(j\omega_o)| = |G(-j\omega_o)|$ and $\angle G(j\omega_o) = -\angle G(-j\omega_o)$, one can show that

$$y(t) = A|G(j\omega_o)|sin(\omega_o t + \angle G(j\omega_o))$$

That is, the output $y(t)$ is also a sinusoid at the same frequency as the input with gain $|G(j\omega_o)|$ and phase $\angle G(j\omega_o)$.

In general, $Y(s) = G(s)U(s)$. Suppose $u(t) = Asin\omega_o t$, then $U(s) = \frac{A\omega_o}{s^2+\omega_o^2}$. Let $G(s)$ has n-poles at $-p_1, -p_2, \ldots, -p_n$, then

$$\begin{aligned} Y(s) &= \frac{\alpha_1}{s+p_1} + \frac{\alpha_2}{s+p_2} + \ldots + \frac{\alpha_n}{s+p_n} + \frac{\beta_o}{s+j\omega_o} + \frac{\beta_o^*}{s-j\omega_o} \\ &= \alpha_1 e^{-p_1 t} + \alpha_2 e^{-p_2 t} + \ldots + \alpha_n e^{-p_n t} + A|G(j\omega_o)|sin(\omega_o t + \angle G(j\omega_o)) \end{aligned}$$

If the system is BIBO stable, then all exponential responses will die down leaving sinusoidal response alone to prevail in the steady state.

2 Bode Plot (Bode, Bell Laboratories, 1932-42)

If $\frac{Y(s)}{U(s)} = G(s)$ and $u(t) = Asin\omega_o t$, then

$$y(t) = A|G(j\omega_o)|sin(\omega_o t + \phi)$$

where $\phi = tan^{-1}(\frac{Im[G(j\omega_o)]}{Re[G(j\omega_o)]}) = \angle G(j\omega_o)$. Bode plot is the plot of magnitude $|G(j\omega_o)|$ and phase $\angle G(j\omega_o)$ versus frequency.

Example 1.

$$\begin{aligned} G(j\omega_o) &= K_o \frac{j\omega\tau_1 + 1}{(j\omega)^2(j\omega\tau_a + 1)} \\ \angle G(j\omega_o) &= \angle K_o + \angle(j\omega\tau_1 + 1) - \angle(j\omega)^2 - \angle(j\omega\tau_a + 1) \\ log|G(j\omega)| &= log|K_o| + log|(j\omega\tau_1 + 1)| - log|(j\omega)^2| - log|(j\omega\tau_a + 1)| \\ |G(j\omega)|_{dB} &= 20log|K_o| + 20log|(j\omega\tau_1 + 1)| - 20log|(j\omega)^2| - 20log|(j\omega\tau_a + 1)| \end{aligned}$$

■

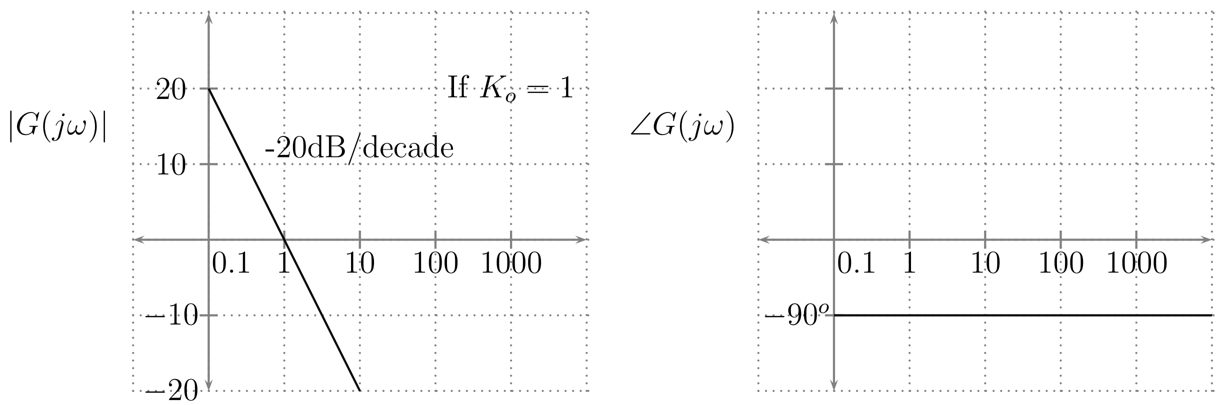
Any transfer function is composed of 3 classes of terms

1. $K_o(j\omega)^n$
2. $(j\omega\tau + 1)^{\pm 1}$
3. $\left[(\frac{j\omega}{\omega_n})^2 + 2\zeta\frac{j\omega}{\omega_n} + 1\right]^{\pm 1}$

$\boldsymbol{K_o(j\omega)^n}$**:** Taking logarithm, we get

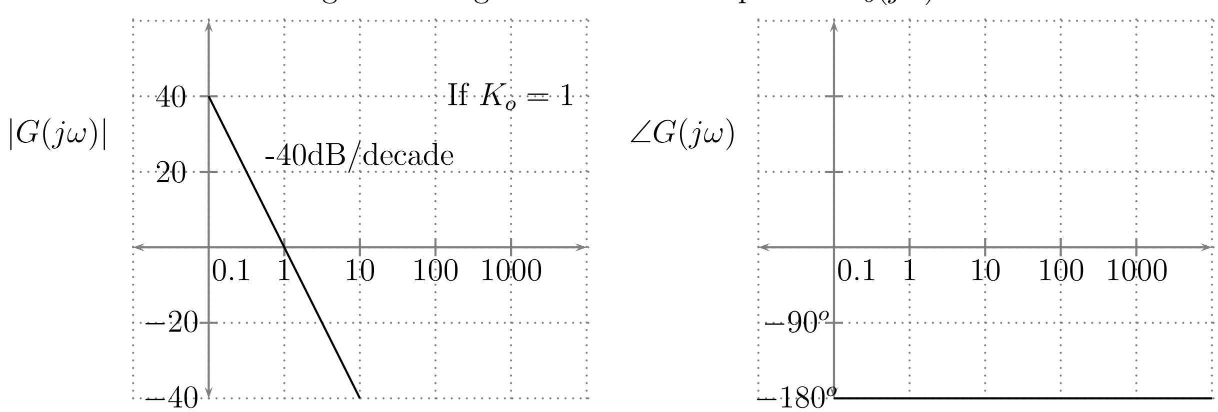
$$20logK_o(j\omega)^n = 20logK_o + 20nlog|j\omega|$$

The corresponding magnitude and phase plots are shown in Fig.1 for the case $n = -1$. If $n = 1$, then magnitude plot will have slope 20dB per decade while angle plot will be a constant line passing through $+90^0$.
If $n = -2$, then magnitude plot will have slope -40dB per decade while angle plot will be a constant line passing through -180^0 which is shown in Fig.2.



(a) Magnitude Plot (b) Phase Plot

Figure 1: Magnitude and Phase plot of $K_o(j\omega)^{-1}$



(a) Magnitude Plot (b) Phase Plot

Figure 2: Magnitude and Phase plot of $K_o(j\omega)^{-2}$

$(\boldsymbol{j\omega\tau + 1})^{-1}$**:** We can see that

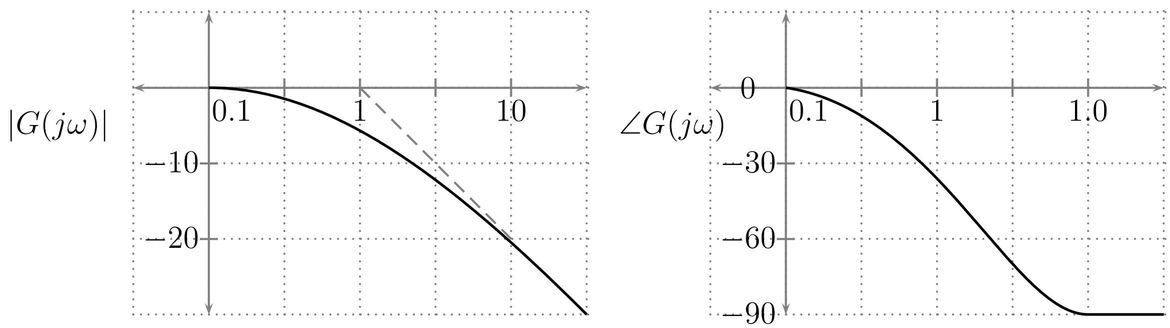
(a) $\omega\tau << 1$, $|j\omega\tau + 1| \approx 1$

(b) $\omega\tau >> 1$, $|j\omega\tau + 1| \approx \omega\tau$

$\omega = \frac{1}{\tau}$ is called break point. When $\omega\tau = 1$ i.e at break point the gain is $-3db$. The corresponding phase and magnitude plots are shown in Fig.3 . for $\tau = 1$, the break point is at $\omega = 1$. The phase angle is given as follows:

$$\begin{aligned}\angle G(j\omega) &= 0 \;\text{ if } \omega\tau << 1 \\ &= -90^o \;\text{ if } \omega\tau >> 1 \\ &= -45^o \;\text{ if } \omega\tau = 1\end{aligned}$$

The exact angle plot is done by using the formula $\angle G(j\omega) = -tan^{-1}\omega\tau$. One just has to compute this angle for various values of ω.



(a) Magnitude plot (b) Phase plot

Figure 3: Magnitude and phase plot of $(j\omega\tau + 1)^{-1}$

$\boldsymbol{j\omega\tau + 1}$**:** We can see that

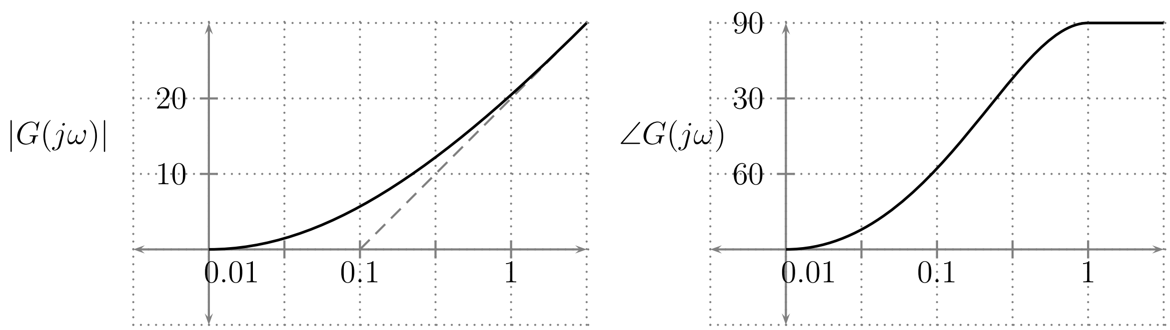
(a) $\omega\tau << 1$, $|j\omega\tau + 1| \approx 1$

(b) $\omega\tau >> 1$, $|j\omega\tau + 1| \approx \omega\tau$

$\omega = \frac{1}{\tau}$ is called break point. The corresponding phase and magnitude plots are shown in Fig. 4. for $\tau = 10$, the break point is at $\omega = 0.1$. The phase angle is given as follows:

$$\begin{aligned}\angle G(j\omega) &= 0 \;\text{ if } \omega\tau << 1 \\ &= 90^o \;\text{ if } \omega\tau >> 1 \\ &= 45^o \;\text{ if } \omega\tau = 1\end{aligned}$$

The exact angle plot is done by using the formula $\angle G(j\omega) = tan^{-1}\omega\tau$. One just has to compute this angle for various values of ω.



(a) Magnitude plot (b) Phase plot

Figure 4: Magnitude and phase plot of $j\omega\tau + 1$

$\boldsymbol{\left((\frac{j\omega}{\omega_n})^2 + 2\zeta\frac{j\omega}{\omega_n} + 1\right)^{\pm 1}}$**:**

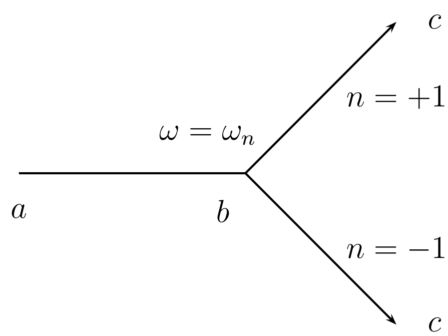
$$(\frac{j\omega}{\omega_n})^2 + 2\zeta\frac{j\omega}{\omega_n} + 1 = (1 - u^2) + j(2\zeta u) \tag{1}$$

where $u = \frac{\omega}{w_n}$.
As you can see if u is much much less than 1 then the gain of the expression given below in dB is zero (0) in both the cases.

$$G(s) = ((\frac{j\omega}{\omega_n})^2 + 2\zeta\frac{j\omega}{\omega_n} + 1)^{\pm 1} \tag{2}$$

Now if u is much much greater than 1 then the gain in dB is either $-40log(u)$ (when exponent is -1) or $40log(u)$ (when exponent is $+1$).
Hence the break point occurs at $u = 1$ i.e $\omega = \omega_n$. The magnitude at break point is given by

$$|G(j\omega)| = 2\zeta\ \frac{j\omega}{\omega_n} = |-j2\zeta|^{\pm 1}$$



The asymptote before break point (ab) is a straight line along 0 dB. The asymptote after break point (bc) has slope 40 dB/decade (+ve for $n = +1$, -ve for $n = -1$). At break point,

$$\begin{aligned} |G(j\omega)|_{dB} &= \left(\frac{1}{2\zeta}\right)_{dB} \quad \text{when} \;\; n = -1 \\ &= (2\zeta)_{dB} \quad \text{when} \;\; n = +1 \end{aligned}$$

3 Practice Problems

Q1. Plot the Bode magnitude and phase for the following transfer functions:

(i)

$$G(s) = \frac{2000(s + 0.5)}{s(s + 10)(s + 50)}$$

(ii)

$$G(s) = \frac{10}{s(s^2 + 0.4s + 4)}$$

Solution:

(i)

$$G(s) = \frac{2000(s+0.5)}{s(s+10)(s+50)}$$
$$G(j\omega) = \frac{2(\frac{j\omega}{0.5}+1)}{j\omega(\frac{j\omega}{10}+1)(\frac{j\omega}{50}+1)}$$

Three break points: $\omega = 0.5,\ 10,\ 50$. At $\omega = 0.1$, $|G(j\omega)|_{dB} = 26.02$. (**Note:** While calculating the dB gains for drawing the asymptotic plots we must only consider those terms which are contributing in that region)

We have the following 4 observations from the structure of $G(j\omega)$:
1. Before $\omega = 0.5$, only the $(j\omega)$ term contributes a slope of $-20dB/decade$.
2. Between $0.5 \leq \omega \leq 10$ the term $(j\omega)$ contributes a slope of $-20dB/decade$ and the term $(1 + j\omega/0.5)$ contributes a slope of $+20dB/decade$. Hence the plot will be a straight line horizontal to the ω axis.
3. Between $10 \leq \omega \leq 50$ the term $(1 + j\omega/10)$ contributes a slope of $-20dB/decade$. The terms $(j\omega)$ and $(1 + j\omega/0.5)$ being in the denominator and numerator respectively has *zero* contribution to the slope of the graph. Hence the plot will be a straight line with a slope of $-20dB/decade$.
4. Following fom the same logic as above, for $\omega \geq 50$ the term $(1 + j\omega/10)$ and $(1 + j\omega/50)$ contributes a slope of $-20dB/decade$ each. Hence the plot will be a straight line with a slope of $-40dB/decade$.

From the above informations we derived we can now plot the Bode Magnitude plot of the transfer function.
For $\omega \leq 0.5$ there will be a straight line with a slope of $-20dB/decade$ having a gain of 26.02 at $\omega = 0.1$ (as calculated above). From $\omega = 0.5$ to $\omega = 10$ the plot will be a horizontal line. Again from $\omega = 10$ to $\omega = 50$ the plot will be a straight line with a slope of $-20dB/decade$. Finally from $\omega = 50$ onwards the plot will be a straight line with a slope of $-40dB/decade$. For plotting the $-40dB/decade$ line from $\omega = 50$ we can just join the point we have at $\omega = 50$ (suppose gain here is x) with the point of gain $(x - 40)$ at $\omega = 500$.

Now if we want the exact Bode Magnitude plot we will have to make the necessary $3dB$ error compensation at each break points. For a break point we found from an expression occuring in the denominator of the transfer function the error to be compensated is $-3dB$ and $+3dB$ if it's corresponding expression is in the numerator.

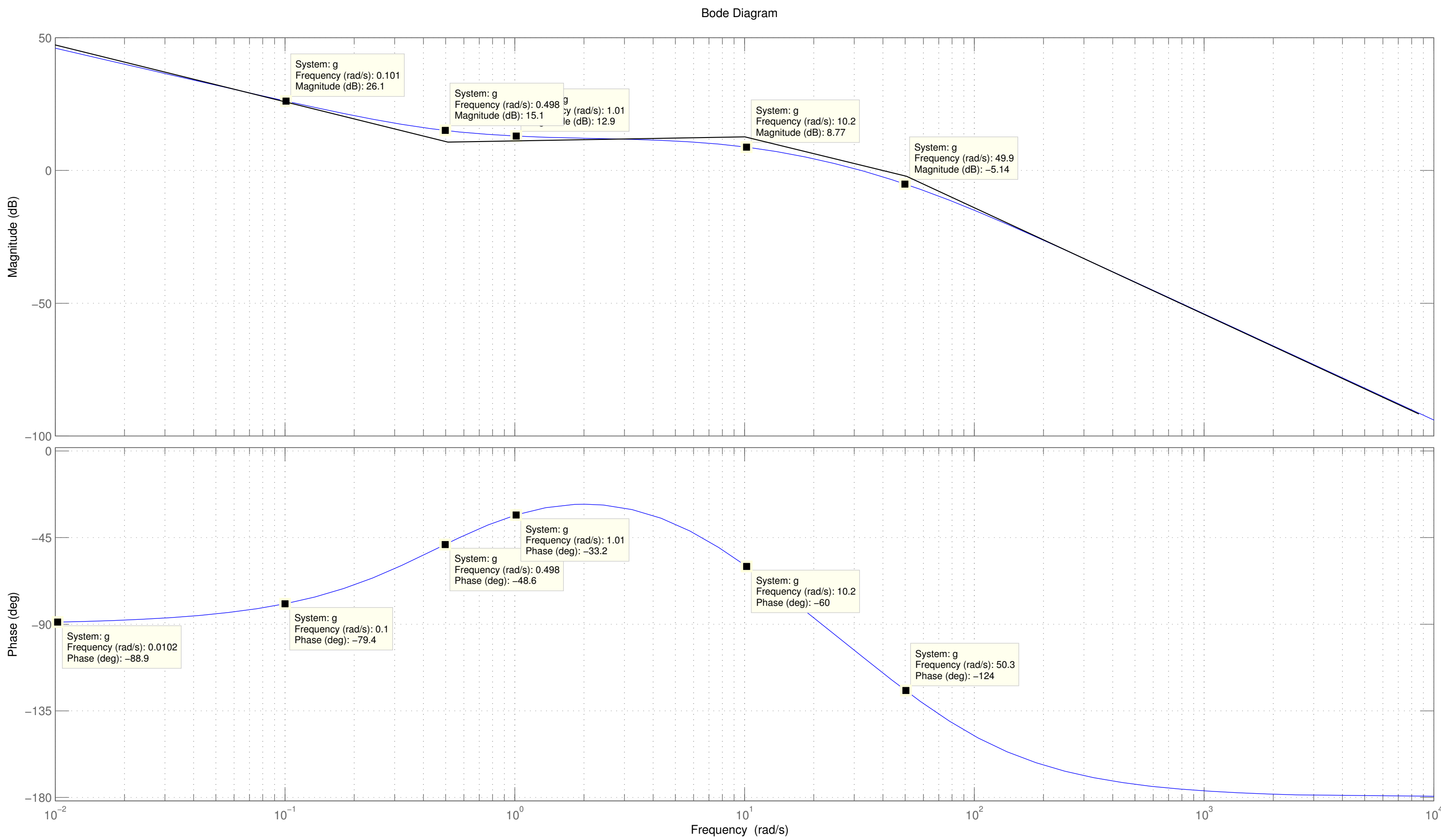


Figure 5: Bode plot for Tutorial Problem 1

The bode plot of the above system is shown in Fig.27.
Drawing the phase plot is fairly straight forward. Draw a table and corresponding to some some points in ω (including the break points) just calculate the phase from the equation below. The table 6 shows phase angles $\angle G(j\omega)$ at various frequencies. The angle of the transfer function at a particular frequency can be calculated as follows:

$$\angle G(j\omega) = -90 + tan^{-1}\frac{\omega}{0.5} - tan^{-1}\frac{\omega}{10} - tan^{-1}\frac{\omega}{50}$$

ω	0.01	0.1	0.5	1.0	5.0	10.0	100.0	1000
$\angle G(j\omega)$	-88.9230	-79.3776	-48.4353	-33.4214	-37.9862	-59.1723	-148.0108	-176.5933

Table 1: Phase angle $\angle G(j\omega)$

In the magnitude plot, there are four asymptotes:

Asymptote ab: -20 dB/decade due to the term $\frac{2}{j\omega}$

Asymptote bc: 0 dB/decade due to $\frac{2}{j\omega}(\frac{j\omega}{0.5}+1)$

Asymptote cd: -20 dB/decade due to $\frac{2}{j\omega}(\frac{j\omega}{0.5}+1)\frac{1}{(\frac{j\omega}{10}+1)}$

Asymptote de: -40 dB/decase due to complete $G(j\omega)$

Solution:

(ii)

$$KG(j\omega) = \frac{10}{4}\frac{1}{j\omega(\frac{(j\omega)^2}{\omega_n^2}+2\zeta\frac{(j\omega)}{\omega_n}+1)}$$

where $\omega_n = 2$, $\zeta = 0.1$. Break point is at $\omega = \omega_n = 2$. Before the breakpoint, the asymptote passes through the point $20log\frac{2.5}{\omega}$ at $\omega = 1$. This value is 8 dB. This asymptote has $-20\ dB/decade$ slope. The second asymptote has a slope $-60\ dB/decade$. The corresponding Bode plot is shown in Fig.6.

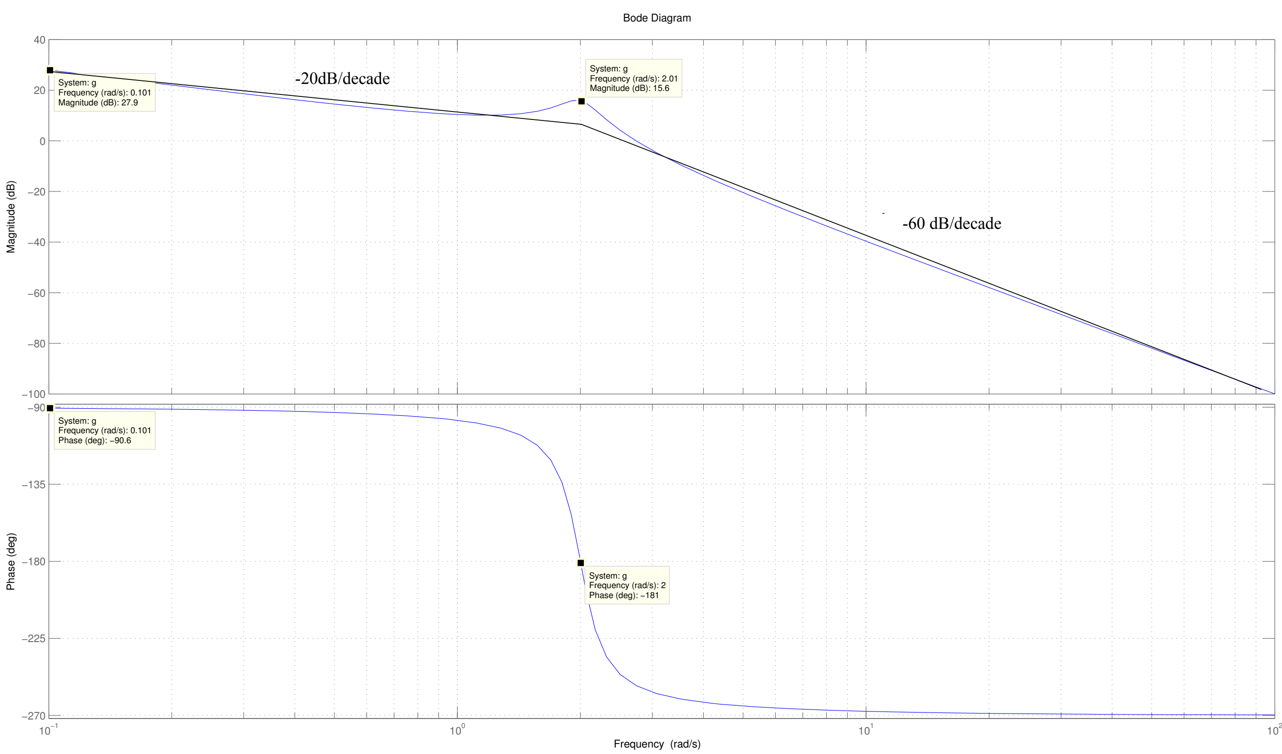


Figure 6: Bode plot for Tutorial problem 2

The correction at the break point $\omega = \omega_n$ is made by noting that

$$|G(j\omega)| = \frac{1}{2\zeta} = \frac{1}{0.2} = 5$$

or $|G(j\omega)|_{dB} = 20\ log\ 5 = 14$. Therefore, the gain at the break point after correction is $|G(j\omega)|_{dB} = 14 + 20\ log\ \frac{2.5}{2} = 16$. The phase angle values at various frequencies is shown in the table 2. The phase angles are calculated as follows:

$$\begin{aligned} \angle G(j\omega) &= -90 - tan^{-1}\frac{0.4\omega}{4-\omega^2} && \text{if } \angle G(j\omega) \geq -180^o \text{ or } \omega < 2 \\ &= -90 + tan^{-1}\frac{0.4\omega}{\omega^2-4} - 180^o && \text{if } \angle G(j\omega) < -180^o \text{ or } \omega \geq 2 \end{aligned}$$

ω	0.1	0.2	0.4	0.6	0.8	1.0	5.0	10.0
$\angle G(j\omega)$	-90.5744	-91.1573	-92.3859	-93.7723	-95.4403	-97.5946	-264.5597	-267.6141

Table 2: Phase angle $\angle G(j\omega)$

4 Non-Minimum Phase system

$$\begin{aligned} G_1(s) &= 10\frac{s+1}{s+10} \\ G_2(s) &= 10\frac{s-1}{s+10} \\ |G_1(j\omega)| &= |G_2(j\omega)| \end{aligned}$$

The magnitude plots for two systems are same.

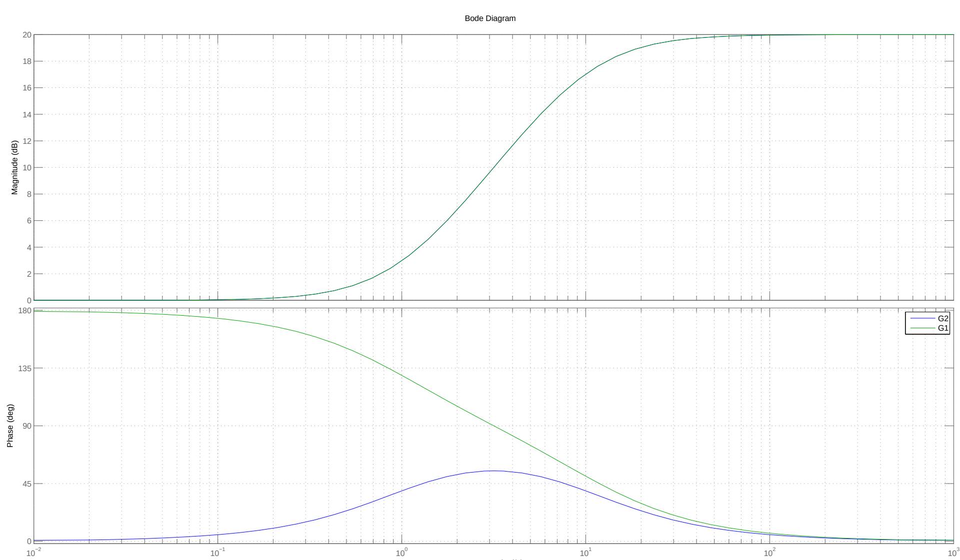


Figure 7: Bode Plot

However,

$$
\begin{aligned}
\angle G_1(j\omega) &\neq \angle G_2(j\omega) \\
\angle G_1(j\omega) &= tan^{-1}\omega - tan^{-1}\frac{\omega}{10} \\
\angle G_2(j\omega) &= 180 - tan^{-1}\omega - tan^{-1}\frac{\omega}{10}
\end{aligned}
$$

The corresponding phase plots are shown in Fig.7. One can observe that G_2 induces maximum phase angle while G_1 has minimum phase angle. Therefore, G_2 is called a *non-minimum phase* system.

5 Steady-state errors

At low frequencies, $KG(j\omega) \approxeq K_0(j\omega)^n$. We have earlier discussed that steady state errors are eliminated if the system gain at the low frequency is very high.

- For **type 0** system, $KG(j\omega) \approx K_0$ at low frequency. Therefore,

$$e_{ss} = \frac{1}{1+K_0}$$

- For **type 1** system,

$$KG(j\omega) \approx \frac{K_0}{\omega} \approx \infty \text{ as } \omega \to 0$$

. Thus, the steady state error due to step input is always zero. However, the steady state error to a ramp input is

$$e_{ss} = \frac{1}{K_v}$$

, where, $K_v = K_0$; K_v is the velocity error constant.

6 Stability:Phase and Gain Margin

During root locus, it is observed that for some systems, system becomes unstable as gain K is increased. Such systems are common and are stable for $K < K_{max}$ where K_{max} is the gain at which the root locus crosses $j\omega$ axis from left half s-plane into right half s-plane. For example:

$$G(s) = \frac{100}{s(s+2)(s+5)} \tag{3}$$

See the Root-Locus plot.

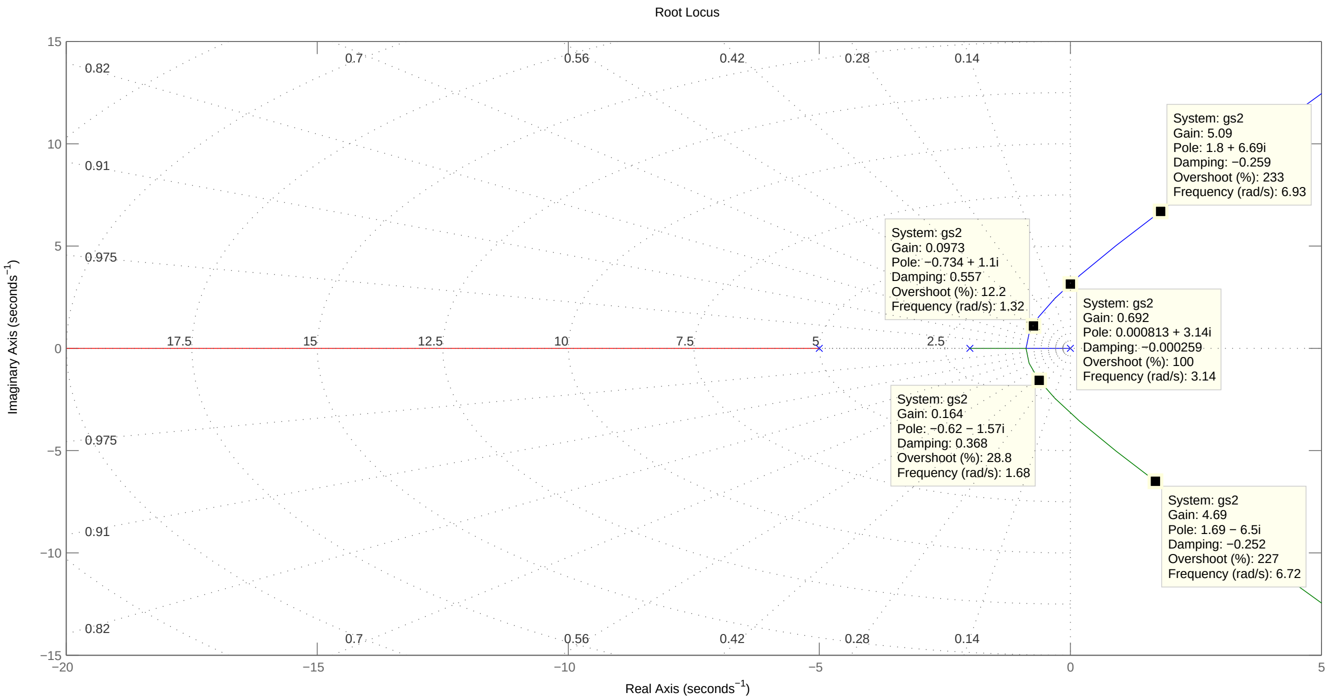


Figure 8: Root Locus Plot

We have also discussed that there exists system for which higher values of gain K make the system stable. In this case, the root locus crosses $j\omega$ axis from right half s-plane to right half s-plane. For the moment, we will only consider the **former case** which is more general.

Along the root locus following condition holds good:

$$1 + KG(s) = 0 \quad and \quad \angle KG(s) = -180^0 \tag{4}$$

This equation is also true when root locus crosses $j\omega$ axis. At this point:

$$|KG(j\omega)| = 1 \quad and \quad \angle KG(j\omega) = -180^0 \tag{5}$$

It should be noted that this condition pertains to neutral stability. Thus stability can be defined using this condition as the starting point:

The system is stable if following condition is true: $|KG(j\omega)| < 1 \quad at \quad \angle G(j\omega) = -180^0$.. This stability condition is also defined in terms of phase margin (PM) and Gain Margin (GM).

Phase Margin: It is the amount by which the phase $\angle G(j\omega)$ exceeds -180^0 when $|KG(j\omega)| = 1$.

Gain Margin: It is the factor by which the gain can be raised before instability results (i.e. negative phase margin).

See the P.M and G.M for the same system whose root-locus plot has been shown below.

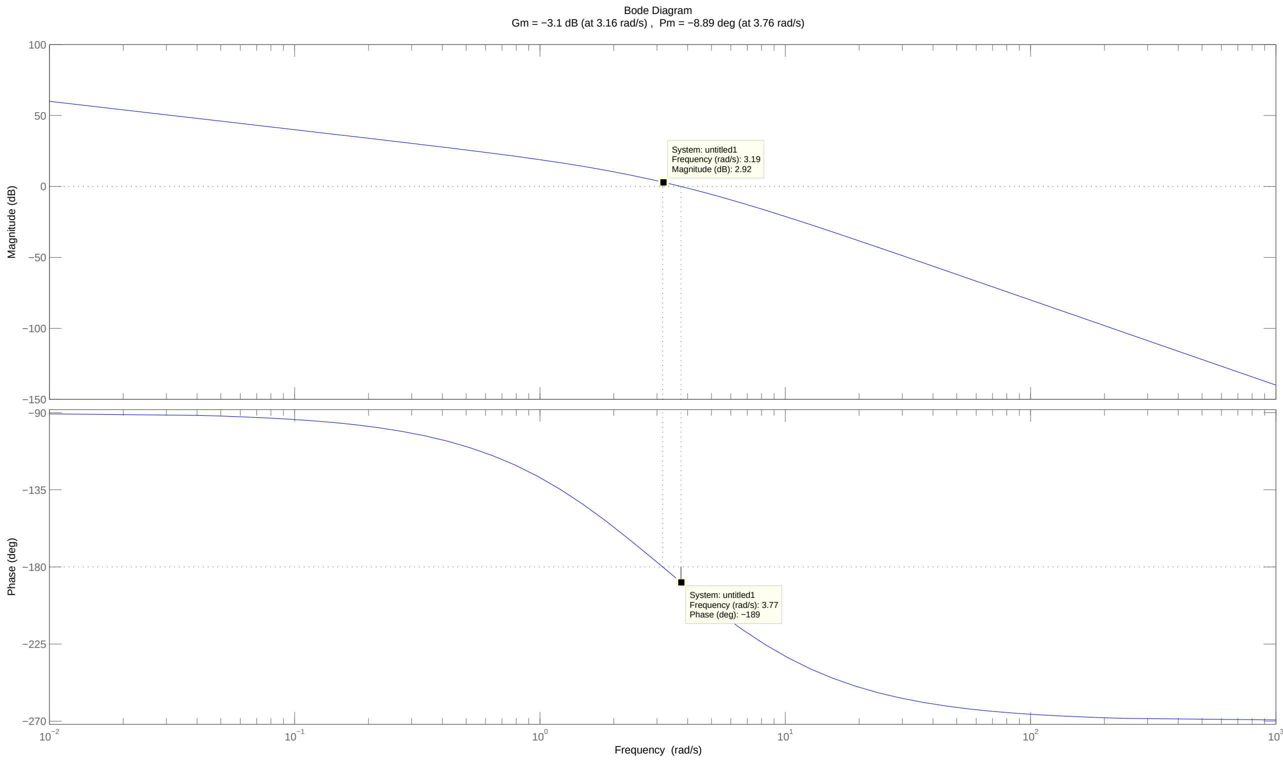


Figure 9: Root Locus Plot

7 Design of Compensators using Bode Plot technique

The bode plots which will be used for the design purpose have to be drawn in semi-log graph. It looks like this:

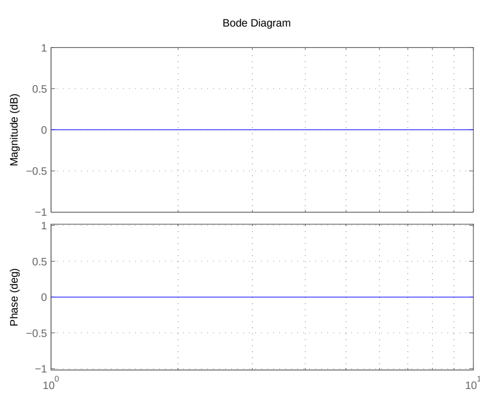


Figure 10: Semi-Log Graph

In Bode Plot based design we first adjust the open loop gain in such a way that the steady

state response accuracy is met. The magnitude and phase graph of the open loop system with an added gain required in the first step is then plotted. If the required P.M and G.M are not satisfied then we design a compensator to reshape the bode plot.
There are two types of compensator:
1. Lag or PI compensator
2. Lead or PD compensator

7.1 Lag or PI compensator

The PI controller has the following form:

$$G_c(s) = \alpha \frac{Ts + 1}{\alpha Ts + 1}$$

where $\alpha > 1$. One can easily see that this compensator has gain 1 at high frequency and the gain becomes α in the low frequency zone. The break frequencies are at $\frac{1}{\alpha T}$ and $\frac{1}{T}$. The Bode Plot of a lag compensator is shown in Fig.11 below.

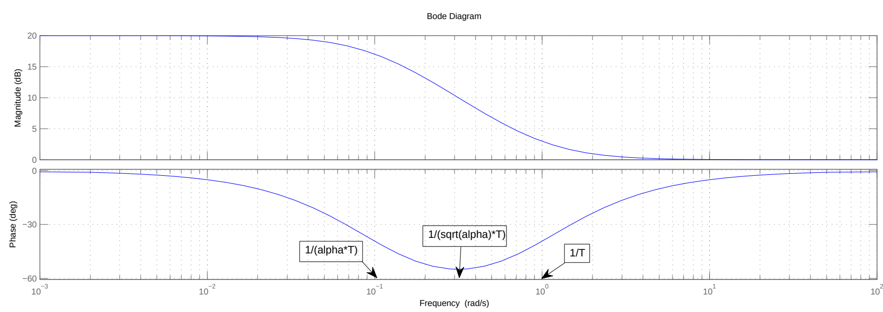


Figure 11: Bode Plot of $G_c(s) = 10\frac{s+1}{10s+1}$

Now, phase angle of the lag compensator

$$\phi = tan^{-1}(\omega T) - tan^{-1}(\omega \alpha T) \tag{6}$$

At minimum slope is zero. Hence,

$$\frac{d\phi}{d\omega} = 0 \tag{7}$$

$$\omega_m = \frac{1}{\sqrt{\alpha}T} \tag{8}$$

Hence,

$$\phi_m = tan^{-1}(\omega_m T) - tan^{-1}(\omega_m \alpha T) \tag{9}$$

Now putting the value of $\omega_m = \frac{1}{\sqrt{\alpha}T}$ we get

$$sin(\phi_m) = \frac{1-\alpha}{1+\alpha} \tag{10}$$

Rearranging we get

$$\alpha = \frac{1 - sin(\phi_m)}{1 + sin(\phi_m)} \tag{11}$$

7.2 Lead or PD compensator

Lead compensator provides positive phase to a system. Not only that it provides high gain at high frequencies and zero gain (dB gain) at low frequencies. A generic PD compensator is:

$$C(s) = \frac{1 + Ts}{1 + \alpha Ts} \tag{12}$$

Where $\alpha < 1$.
The bode plot is shown below.

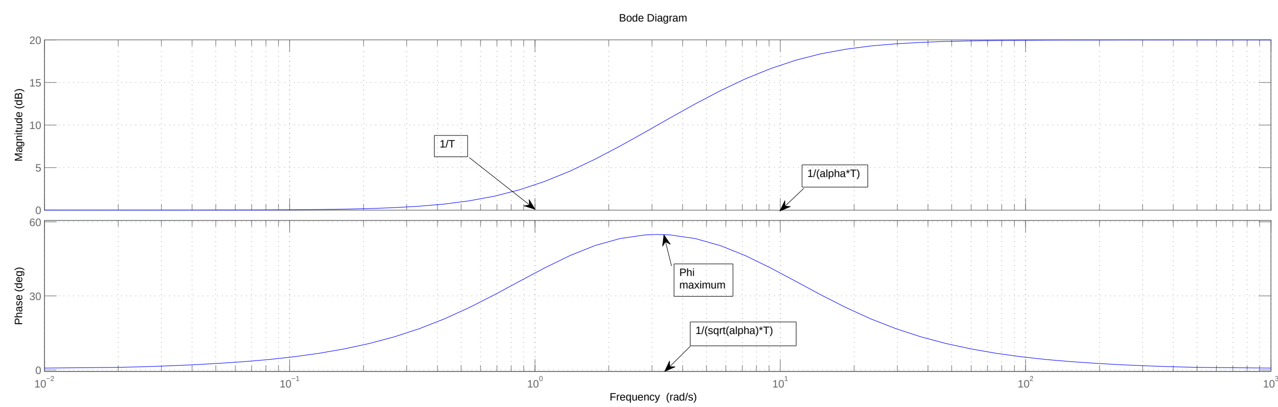


Figure 12: Bode PLot

As you can see above $\frac{1}{T}$ is the first break point. $\frac{1}{\alpha T}$ is the second break point. And $\frac{1}{\sqrt{\alpha}T}$ is the angle where maximum phase lead (ϕ_m) occurs.
The relationship between α and ϕ_m is **exactly** same as that of the lag compensator. Now, phase angle of the lead compensator

$$\phi = tan^{-1}(\omega T) - tan^{-1}(\omega\alpha T) \tag{13}$$

At minimum slope is zero. Hence,

$$\frac{d\phi}{d\omega} = 0 \tag{14}$$

$$\omega_m = \frac{1}{\sqrt{\alpha}T} \tag{15}$$

Hence,

$$\phi_m = tan^{-1}(\omega_m T) - tan^{-1}(\omega_m \alpha T) \tag{16}$$

Now putting the value of $\omega_m = \frac{1}{\sqrt{\alpha T}}$ we get

$$sin(\phi_m) = \frac{1-\alpha}{1+\alpha} \tag{17}$$

Rearranging we get

$$\alpha = \frac{1-sin(\phi_m)}{1+sin(\phi_m)} \tag{18}$$

7.3 Design of Compensators for speed control of a DC motor

Consider a separately excited DC motor with following parameters:

$$\begin{aligned} &J = 0.01; \\ &b = 0.1; \\ &Kt = Kb = 0.01; \\ &R = 1; \\ &L = 0.5; \end{aligned}$$

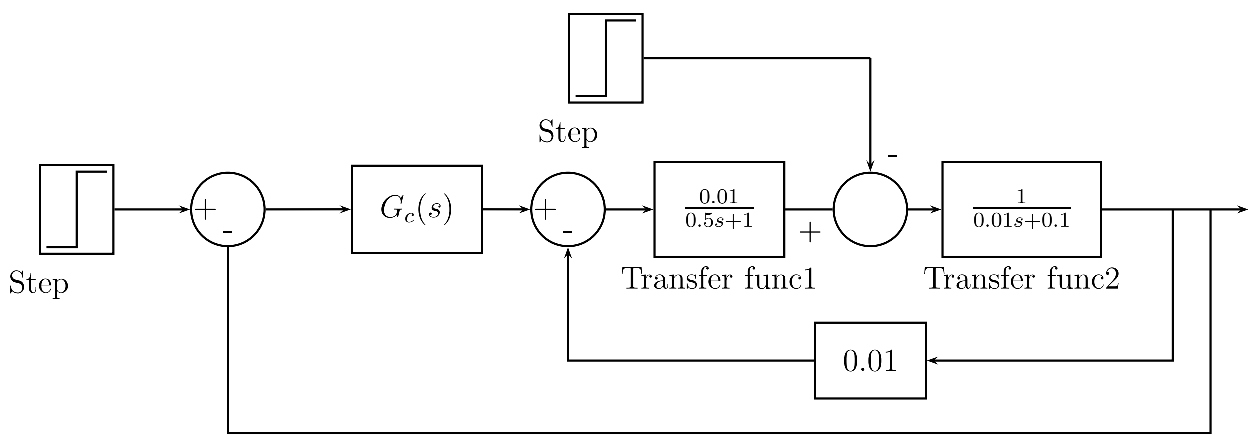


Figure 13: DC motor

1. Design a PI (Lag) so that with a 1 rad/sec step reference, the design criteria are:
   - Settling time less than equal to 2 seconds
   - Overshoot less than 5%
   - Steady-state error less than 5%

2. Design a PID (Lag-lead compensator) so that settling time is approximately 1 sec while other specifications remain same as that for PI and PD compensator.

Solution 1: The transfer function is given as:

$$G(s) = \frac{2}{(s+2)(s+10)}$$

Step 1: Draw the Bode Plot
Bode Plot of $G(s)$ is :

```
s = tf('s');
gs = 2/((s+2)*(s+10));
bode(gs)
grid on
```

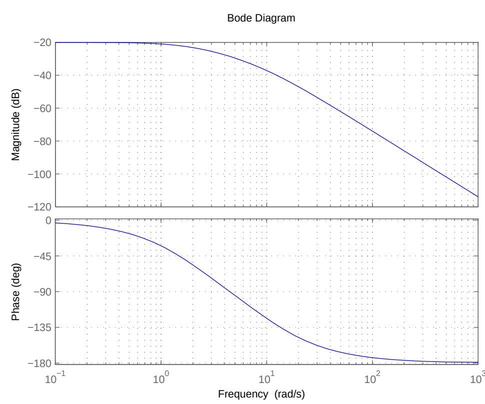


Figure 14: Bode Plot

Step 2: Calculate the Phase Margin
From given values of $t_s = 2\ sec$ and $M_p = 0.05$, We have

$$0.05 = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \quad t_s = \frac{4.6}{\omega_n}$$

This gives $\zeta = 0.6901$ and $\omega_n = 3.3328$.
Using the approximate relation that ζ=P.M/100, we get a value of **P.M. = 70 deg** approximately.

Step 3: Adding Gain to the system
As can be seen from the above graph, the system is always stable. However a phase margin of 70 degree can be attained at $\omega = 7.4\ rad/s$ if the gain plot is shifted by 33.7 dB.

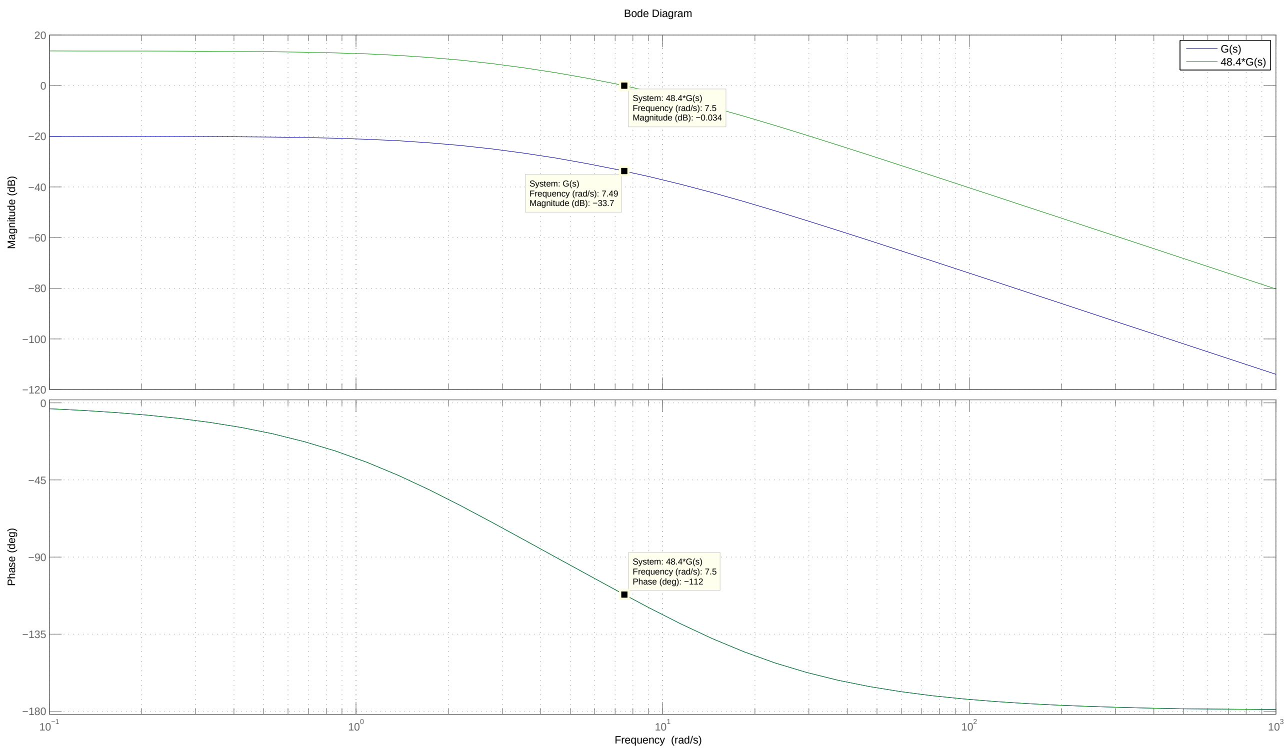


Figure 15: Bode Plot

Thus the open loop gain necessary to achieve this objective is calculated as follows:
20 $log|G(j\omega)|$ = -33.7dB, so, we get $log|G(j\omega)|$ = -1.6850.
Therefore, the magnitude of $G(j\omega)$ at $\omega = 7.26$ is $10^{-1.6850}$, which is approximately 0.0207. The open loop gain= $1/0.0207 = 48.4172$. This is the additional gain that must be added to the uncompensated system $G(s)$ so that the system may have a phase margin of 70 degrees.

Obviously the phase margin of 70 degree is achieved as we increased the gain as shown in the bode plot above. The step response of this system in closed loop is shown in Fig.16 below.

```
step(feedback(48.4*gs,1));
grid on
```

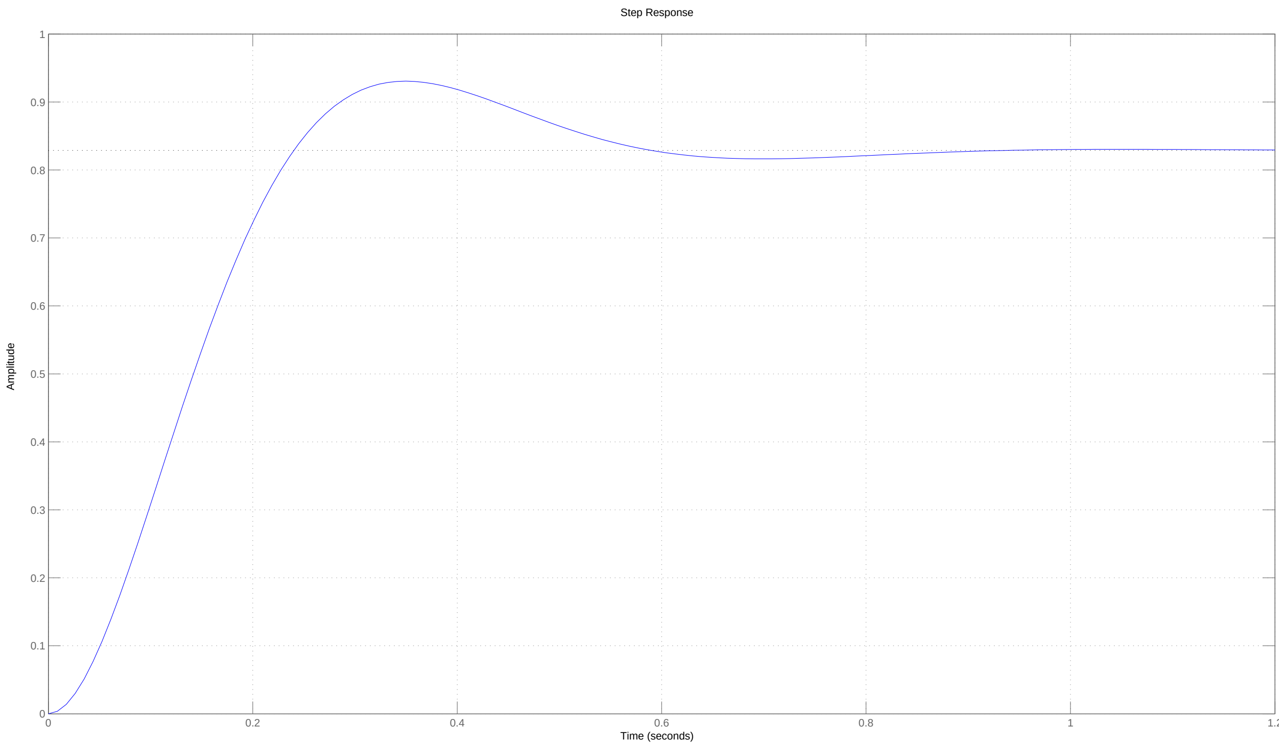


Figure 16: Step Response

Step 4: Design the PI Compensator
It should be very clear from the response that system steady state error is very large. The objective of PI compensator is to introduce high gain at low frequency so that the steady state error is reduced.

PI control Design:

Since the gain cross over frequency for required phase margin is 7.4 rad/sec, this PI compensator should be placed such that the corresponding negative phase contribution only affects low frequency zone.
By selecting $G_c(s) = 10\frac{s+1}{10s+1}$, the low frequency gain has been increased by 10 times. Please note the choice of **T**. It is so adjusted that the lag compensator does not change the gain cross over frequency. The corresponding bode plot is shown in Fig.17 below.

```
gcs = 10*(s+1)/(10*s+1);
margin(48.4*gs*gcs)
grid on
```

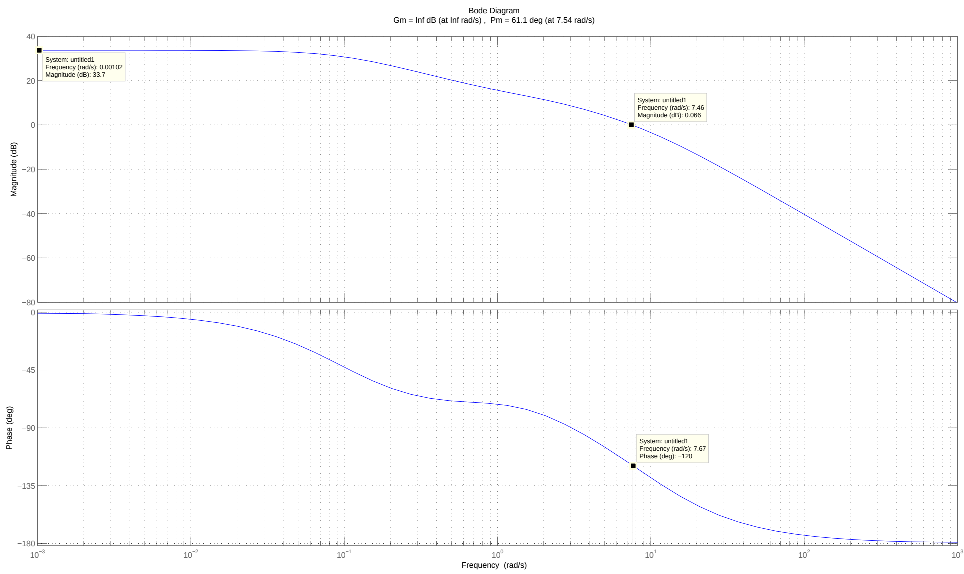


Figure 17: Bode Plot

The new gain crossover frequency is at 7.54 rad/sec which is slightly shifted from the original value of 7.4 rad/sec. And the new phase margin is 61.1 deg which is slightly lower than the required 70 degree. One can repeat this experiment by providing a little more phase margin while selecting the open loop gain K which is taken as 48.12 at present. The step response of the compensated system is shown in Fig.18.

```
step(feedback(48.4*gs*gcs,1))
grid on
```

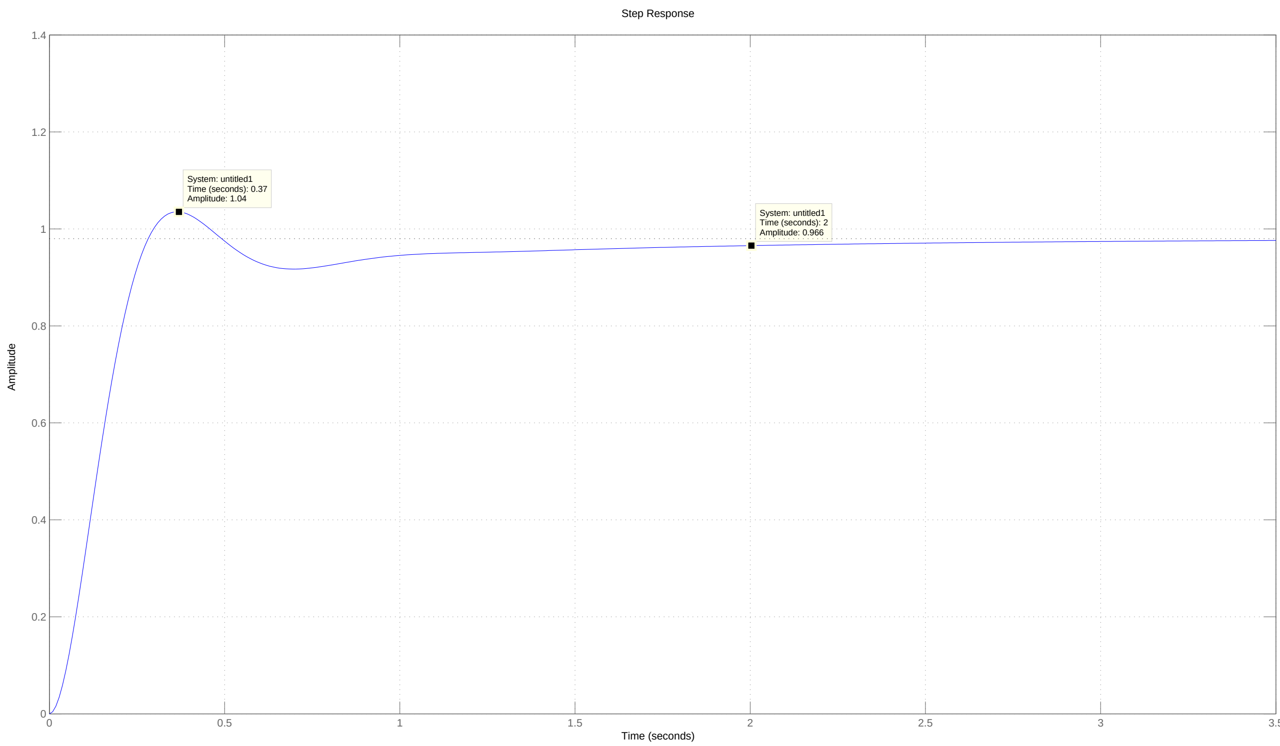


Figure 18: Step Response

The transfer function of the open loop compensated system is thus

$$G_c(s)KG(s) = 10\frac{s+1}{10s+1}48.4172G(s) = 484.1\frac{s+1}{10s+1}G(s)$$

This compensator has a different structure than the one we obtained using root locus method. Let's look at the closed loop poles of the above transfer function:

```
% First find the closed loop transfer function
feedback(48.4*gs*gcs,1)
```

The denominator is a polynomial equation. Now to find the roots of the polynomial equation, we type:

```
roots([10 121 1180 988])
```

-5.5916 + 8.7485i; -5.5916 - 8.7485i; -0.9168

The first one has no relevance with root locus based design compensator while the second one is exactly the same design obtained using root locus. It should be noted that root locus based design keeps required dominant poles as the main design criteria while the frequency domain approach ensures the required phase margin. [Notice that with the newer design we get a complex-conjugate system of poles which are the dominant poles of the system. In the previous design the complex conjugate poles were not dominant; in fact the dominant pole was a real pole.]

***Solution* 2:**
PID control Design:

We have to design a PID or Lag-Lead compensator.

Since the desired PM is 70 degree, the first step is to divide this required PM into two parts: both Lag and Lead compensator will provide 40 degree PM each.(The margin of error is taken to be 10 degrees. Hence $70 + 10 = 2 * 40$).

> **Design Step 1:** So **first step** of the design is very similar where one designs a lag compensator that provides PM = 40 degree. From Fig.19, the open loop gain K is found to be 156.67 at 16 rad/sec.(Since around 43.9 dB gain has to be injected into the system to have gain cross over frequency at 16 rad/sec)

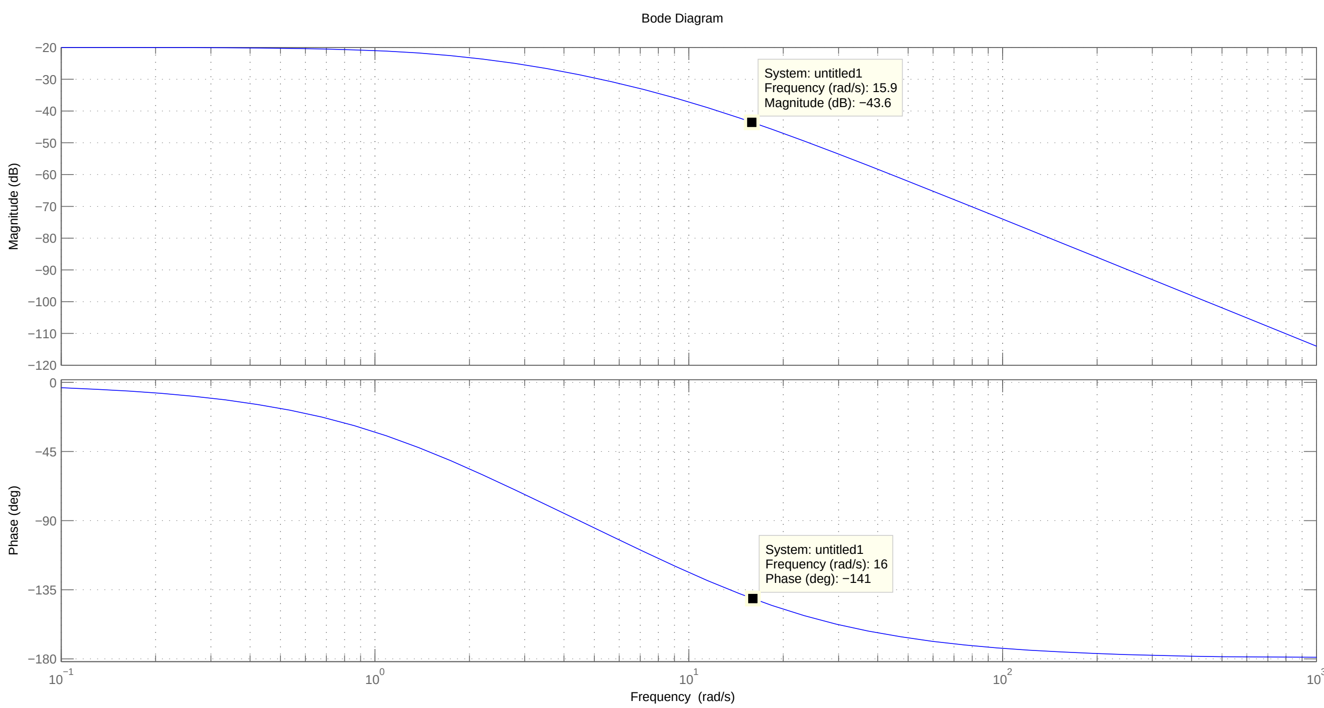


Figure 19: Bode PLot of G(s)

After adding the gain the bode plot looks like:

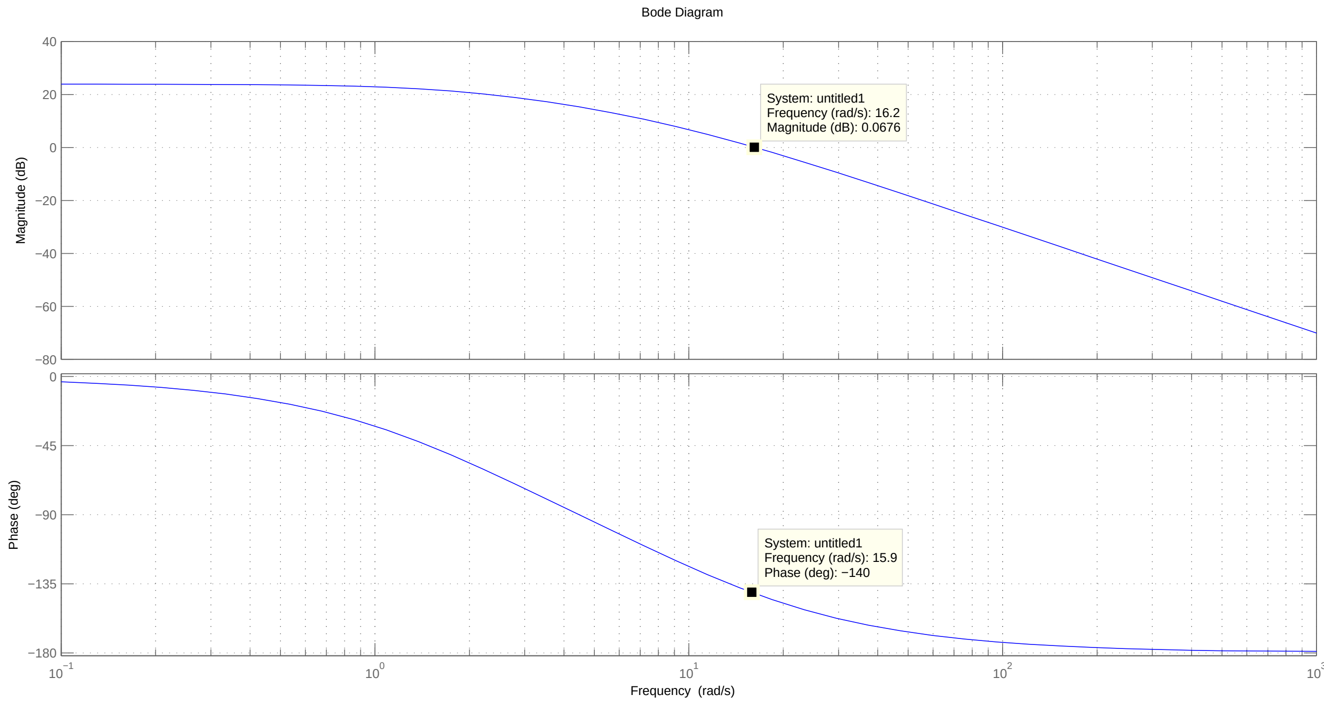


Figure 20: Bode PLot of K*G(s)

Design Step 2: We select the **same lag compensator**

$$G_c(s) = 10\frac{s+1}{10s+1}$$

.

Design Step 3: Now we want to add an additional 40 deg of positive phase using the lead compensator. We use the following relation:

$$\alpha = \frac{1 - sin(\phi_{max})}{1 + sin(\phi_{max})}$$

where $\phi_{max} = 40$ deg. This gives us a value of $\alpha = 0.2174$. The phase lead compensator is then of the form:

$$D(s) = \frac{1+Ts}{1+\alpha Ts}$$

where $\alpha < 1$. The two corner frequencies of the lead compensator should be so selected that the maximum phase of 40 degree is contributed approximately at 16 rad sec.
We can suppose that the lead compensator we are going to choose provides mamimum phase as at gain cross-over frequency i.e at 16 rad/sec. Hence,

$$\omega_m = \frac{1}{\sqrt{\alpha}T} = 16 \tag{19}$$

$$T = 0.1340 \tag{20}$$

Thus one of the possible structure of the compensator is as follows:

$$D(s) = \frac{0.134s + 1}{0.0291s + 1}$$

The bode plot of the c system using lead-lag $(K * G(s) * G_c(s) * D(s))$is shown in Fig.21 below.

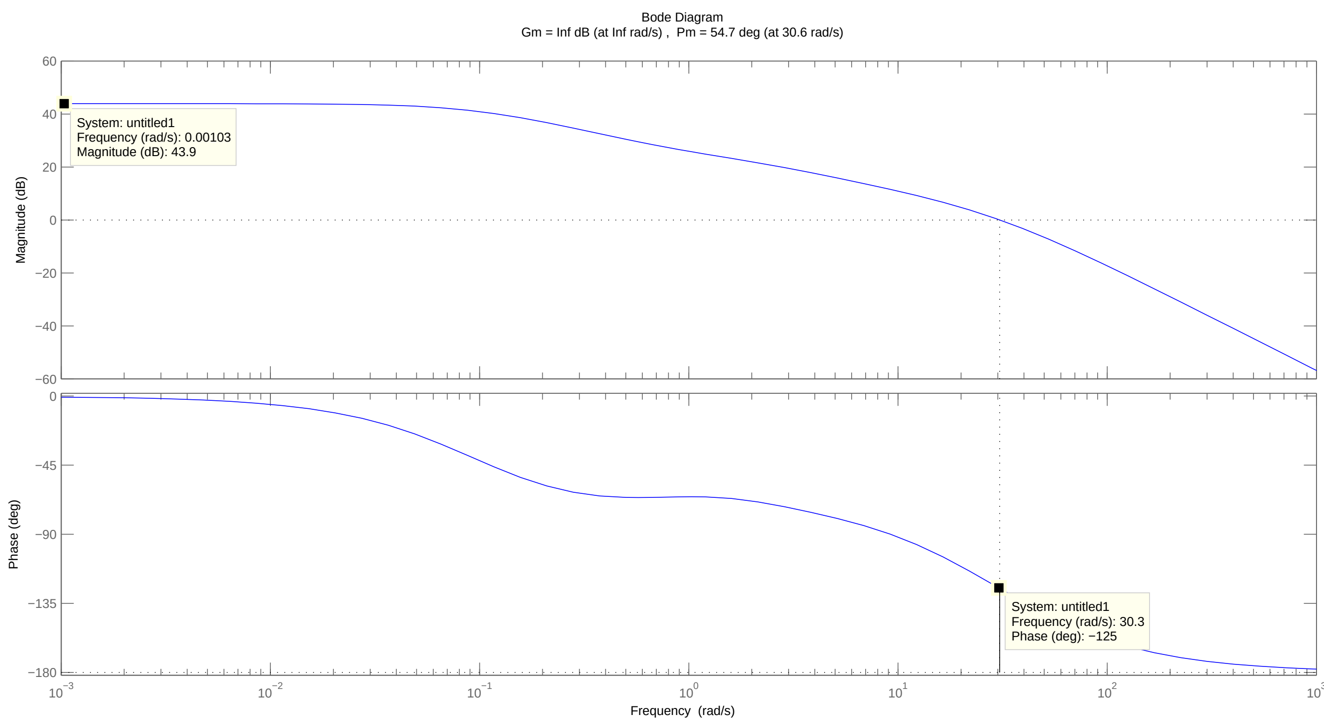


Figure 21: Bode PLot of $K * G(s) * G_c(s) * D(s)$

The phase margin is now approx 54.7 and the new gain crossover frequency is 30 rad/sec. The step response of the closed loop compensated system is shown in Fig.22 below.

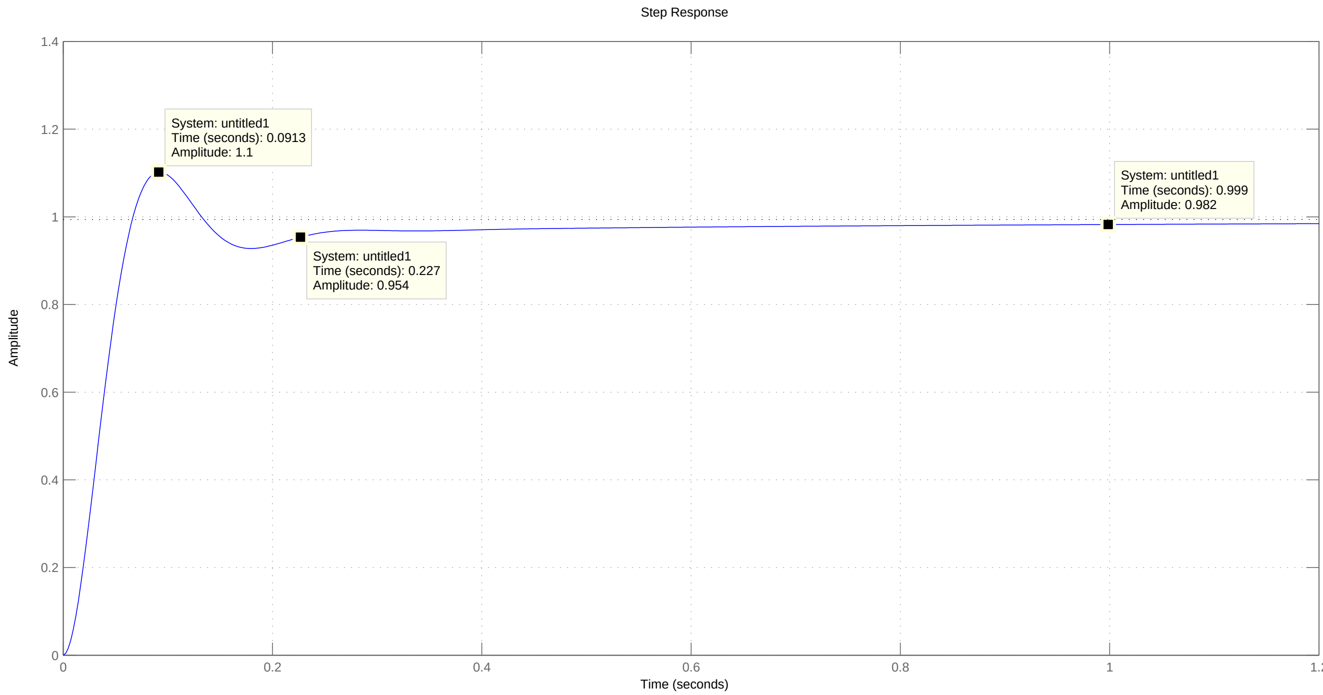


Figure 22: Step Response ofthe Closed Loop Lag-Lead Compensated System

As seen above our design objectives have been met. Compare this step response with that of PI alone. Notice that the rise time and setting time has improved however the overshoot has gone up a bit. But all the requirements have been satisfied.

Now if instead of the lead compensator

$$G1(s) = \frac{0.134s + 1}{0.0291s + 1}$$

which we used above, we had a compensator

$$G2(s) = \frac{0.1s + 1}{0.0217s + 1}$$

the difference in response would be:

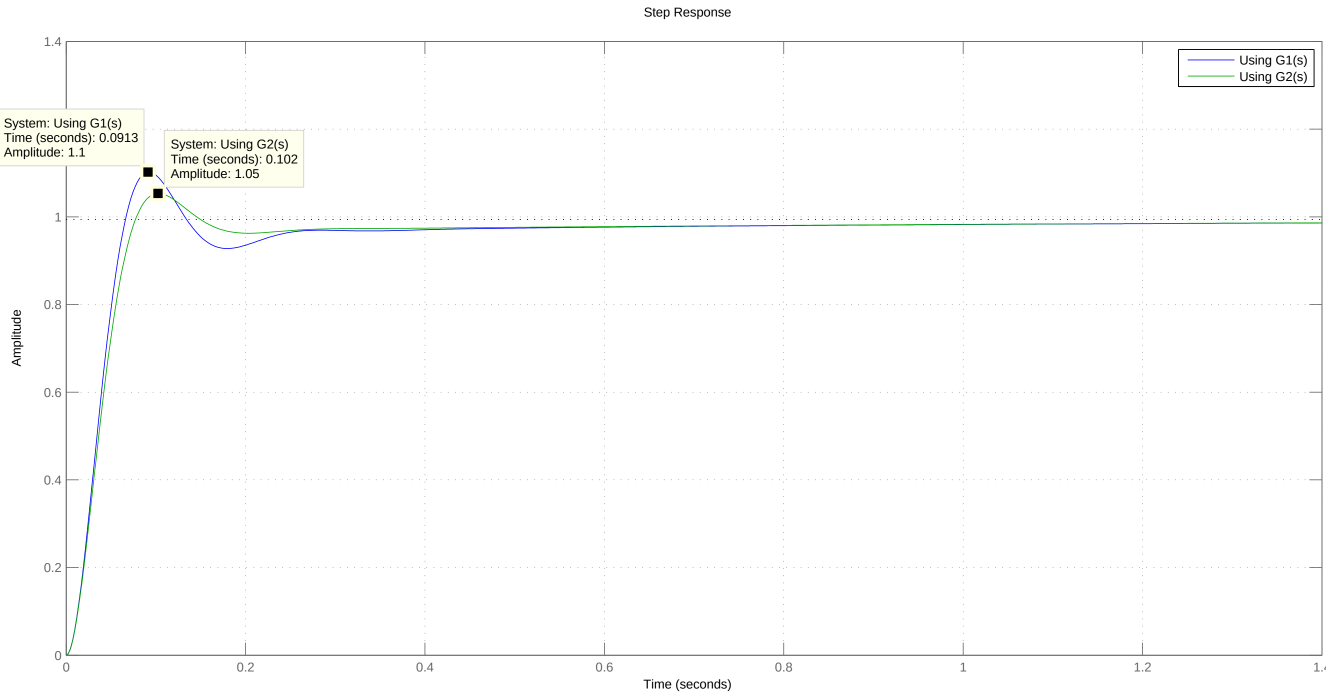


Figure 23: Step Response ofthe Closed Loop Lag-Lead Compensated System for different lead compensator

As we can see, the second lead compensator's (G2(s)) behavior is far more better than that of the first lead compensator (G1(s)) in terms of peak overshoot. But why is there such a difference ?
This is because the value of **T** is 0.1 in case of G2(s) and it is equal to 0.134 in case of G1(s). Hence the position where maximum phase occurs in case of a lead compensator has shifted more towards the right i.e towards high frequency, in case of G2(s), than G1(s). So we can see that adjusting the value of **T** we can vary the peak overshoot of the step response of the system.

□

8 Summary

1. Frequency response design is based on adding extra gain or extra phase to the uncompensated system. Concepts of poles and zeros are not used in this design.

2. A lag compensator has the following form: $G_c(s) = \alpha\frac{1+Ts}{1+\alpha Ts}$ where $\alpha > 1$. One can observe that this compensator has gain $|G_c(0)| = \alpha$ at low frequency and the gain becomes one at high frequency, i.e.$|G_c(s)|_{\omega=\infty} = 1$. This makes it very clear that if design specification demands better steady state error performance, then a lag compensator should be used. The

compensator is so named because angle contribution due to a lag compensator is always negative. Thus the bandwidth of this compensator is fixed in lowfrequency zone in such a way that its angle contribution does not affect the over all phase angle of the compensated system at gain cross over frequency.

3. A lead compensator has the following form: $G_c(s) = \frac{1+Ts}{1+\alpha Ts}$ where $\alpha < 1$. One can observe that this compensator has gain $|G_c(0)| = 20log\frac{1}{\alpha}$ at high frequency and the gain becomes one at low frequency. The compensator is so named because angle contribution due to a lead compensator is always positive. This makes it very clear that if the uncompensated system has negative phase margin at the desired gain cross over frequency, then a lead compensator should be designed. The design should be such that maximum phase angle contribution takes place at the gain cross over frequency of the compensated system. Since this compensator adds extra gain at high frequency, band width of the compensated system increases.

4. Higher the band width, faster is the response. Lag can reduce the band width while lead always increases the band width.

5. Extra gain can be added everywhere by increasing gain K_0.

6. Extra gain is added at low frequency by designing a suitable lag.

7. Extra gain is added at high frequency by designing a suitable lead.

8. Extra phase can be added at any frequency only by suitable design of lead.

9 Problems

1. Sketch the asymptotes of the Bode plot magnitude and exact angle plot for the following cases:

   (a) $KG(s) = \frac{4000}{s(s+4000)}$
   (b) $KG(s) = \frac{10}{s(s+1)(s+3)}$
   (c) $KG(s) = \frac{1}{s^2+3s+10}$
   (d) $KG(s) = \frac{s+1}{(s+2)(s^2+3s+10)}$
   (e) $KG(s) = \frac{s+1}{(s-1)^2}$
   (f) $KG(s) = \frac{1}{s^2(s+8)}$
   (g) $KG(s) = \frac{1}{s^2-1}$
   (h) $KG(s) = \frac{s+2}{s(s+10)(s^2-1)}$

2. Find phase and gain margins for the following systems:

   (a) $KG(s) = \frac{1}{s^2}$
   (b) $KG(s) = \frac{s+2}{s+10}\frac{1}{s^2}$

(c) $KG(s) = \frac{s+2}{s(s+10)} \frac{1}{s^2}$

(d) $KG(s) = \frac{s+2}{s(s+10)} \frac{1}{s^2-1}$

3. Design lead compensator for the following systems so that $PM > 45^0$.

   (a) $KG(s) = \frac{1}{s^2}$

   (b) $KG(s) = \frac{1}{s^2-1}$

4. Design lag compensator for the following systems so that $PM > 45^0$ and steady state error is less than 5%.

   (a) $KG(s) = \frac{1}{(s+1)(s+2)}$

   (b) $KG(s) = \frac{s-1}{(s+2)^2}$

Tutorial Problems

Sketch the asymptotic Bode plot magnitude along with error corrections and exact angle plot for the following cases. Also find the Phase Margin and Gain Margin in each case.

1. $G(s) = \frac{10}{s(s+1)(s+3)}$
2. $G(s) = \frac{(s+1)}{(s+2)(s^2+3s+10)}$
3. $G(s) = \frac{(s+2)}{s(s+10)(s^2-1)}$
4. $G(s) = \frac{(s+2)}{s^3(s+10)}$

Design a Lead Compensator for the following system so that P.M $> 45^0$

1. $G(s) = \frac{1}{s^2(s+1)}$

Solutions:

Asymptotic Bode plot magnitude along with error corrections and exact angle plot:

$$G(s) = \frac{10}{s(s+1)(s+3)} \tag{21}$$

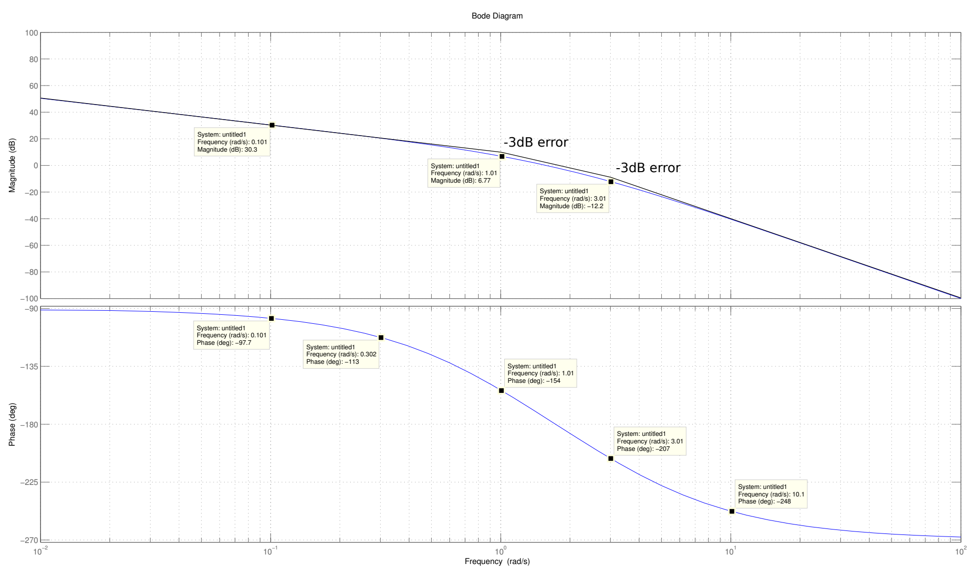


Figure 24: Bode plot for $G(s) = \frac{10}{s(s+1)(s+3)}$

Explanation:

$$G(s) = \frac{10}{s(s+1)(s+3)}$$
$$G(j\omega) = \frac{10/3}{(j\omega)(1+j\omega)(1+j\omega/3)}$$

There are two(2) break points: $\omega = 1$ and 3.

At $\omega = 0.01$, $|G(j\omega)|_{dB} = 50.3$ and at $\omega = 0.1$, $|G(j\omega)|_{dB} = 30.4095$. (**Note:** While calculating the dB gains for drawing the asymptotic plots we must only consider those terms which are contributing in that region)

We have the following 3 observations from the structure of $G(j\omega)$:
1. Before $\omega = 1$, only the $(j\omega)$ term contributes a slope of $-20dB/decade$.

2. Between $1 \leq \omega \leq 3$ the term $(j\omega)$ contributes a slope of $-20dB/decade$ and the term $(1 + j\omega)$ contributes a slope of $-20dB/decade$. Hence the plot will be a straight line with a slope of $-40dB/decade$.
3. After $\omega = 3$ the term $(1 + j\omega/3)$ contributes a slope of $-20dB/decade$. The terms $(j\omega)$ and $(1 + j\omega)$ both being in the denominator have a total contribution of $-40dB/decade$ to the slope of the graph. Hence the plot will be a straight line with a slope of $-60dB/decade$.

From the above informations we derived we can now plot the Asymptotic Bode Magnitude plot of the transfer function.
For $\omega \leq 1$ there will be a straight line with a slope of $-20dB/decade$ having a gain of 30.40 at $\omega = 0.1$ (as calculated above). From $\omega = 1$ to $\omega = 3$ the plot will be a $-40dB/decade$ sloped line. Again from $\omega = 3$ onwards the plot will be a straight line with a slope of $-60dB/decade$.

Now if we want the exact Bode Magnitude plot we will have to make the necessary $3dB$ error compensation at each break points. For a break point we found from an expression occurring in the denominator of the transfer function the error to be compensated is $-3dB$ and $+3dB$ if it's corresponding expression is in the numerator.
Hence at $\omega = 1\&3$, $-3dB$ error compensations have to be made in each of them.

Drawing the phase plot is fairly straight forward. Draw a table and corresponding to some some points in ω (including the break points) just calculate the phase from the equation below. The table 6 shows phase angles $\angle G(j\omega)$ at various frequencies. The angle of the transfer function at a particular frequency can be calculated as follows:

$$\angle G(j\omega) = -90 - tan^{-1}\frac{\omega}{1} - tan^{-1}\frac{\omega}{3}$$

ω	0.01	0.1	0.3	1.0	3.0	10.0
$\angle G(j\omega)$	-90.76	-97.7	-113	-154	-207	-248

Table 3: Phase angle $\angle G(j\omega)$

Phase Margin : 4.6^0 at 1.58 rad/sec
Gain Margin : $1.58dB$ at 1.73 rad/sec

Asymptotic Bode plot magnitude along with error corrections and exact angle plot:

$$G(s) = \frac{(s+1)}{(s+2)(s^2+3s+10)} \tag{22}$$

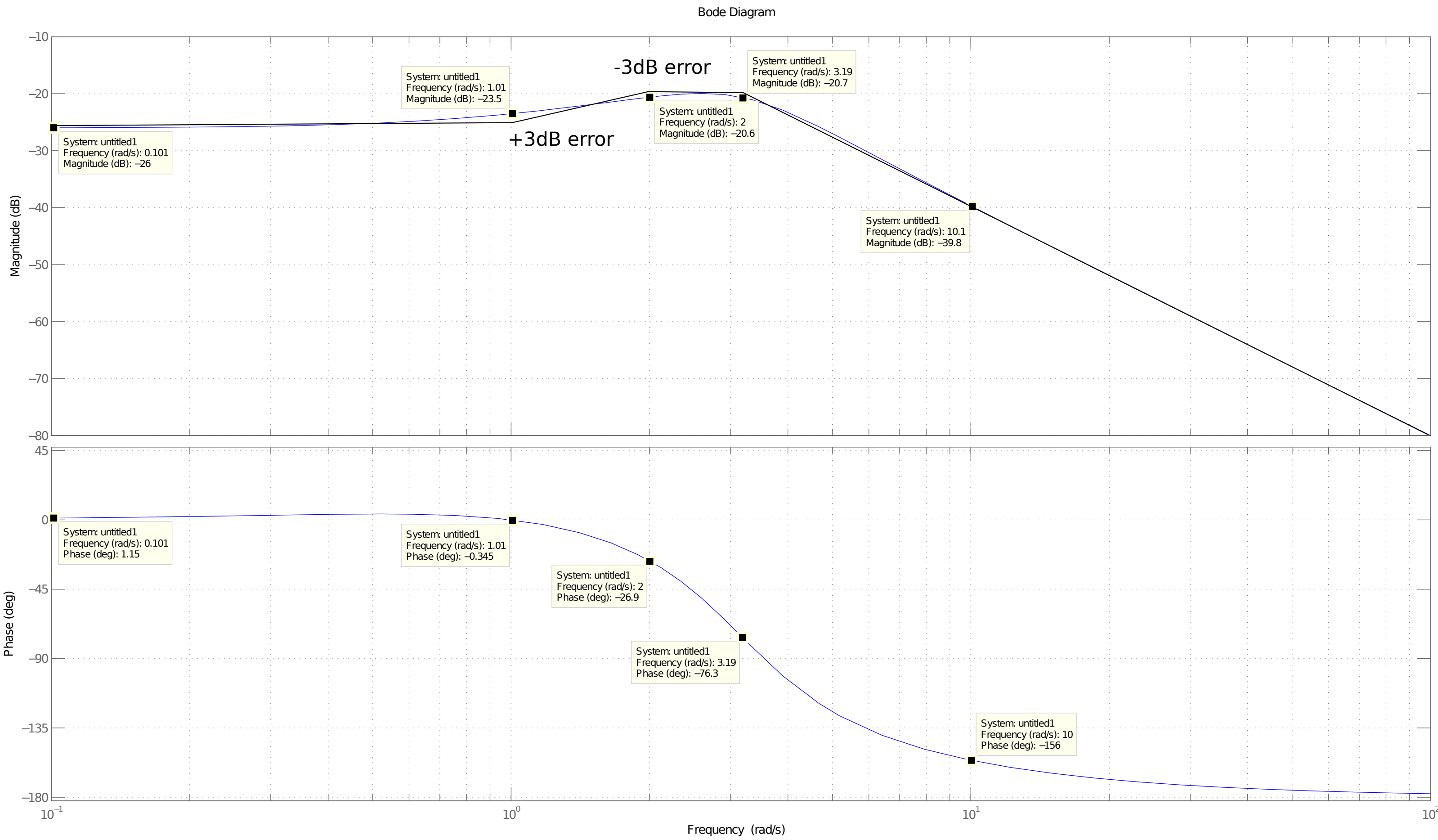


Figure 25: Bode plot for $G(s) = \frac{(s+1)}{(s+2)(s^2+3s+10)}$

Explanation:

$$G(s) = \frac{(s+1)}{(s+2)(s^2+3s+10)}$$

$$G(s) = \frac{0.05(1+j\omega)}{(1+j\omega/2)(1+(\frac{3}{10}j\omega)-(\frac{\omega}{\sqrt{10}})^2)}$$

Three break points: $\omega = 1$, 2 and 3.1623.

The denominator has a quadratic expression:

$$1 + 2\zeta(\frac{j\omega}{\omega_n}) - (\frac{\omega}{\omega_n})^2 \tag{23}$$

$$1 + (\frac{3}{10}j\omega) - (\frac{\omega}{\sqrt{10}})^2 \tag{24}$$

From here we get the value of ω_n and ζ

$$\omega_n = \sqrt{10} = 3.1623 \tag{25}$$

$$\zeta = \frac{3}{2\sqrt{10}} = 0.4743 \tag{26}$$

At $\omega = 0.1$, $|G(j\omega)|_{dB} = 26.0206$. (**Note:** While calculating the dB gains for drawing the asymptotic plots we must only consider those terms which are contributing in that region)

We have the following 6 observations from the structure of $G(j\omega)$:
1. We must first see the quadratic term. It has a contribution of 0dB/decade line upto 3.1623 rad/sec and then a contribution of -40dB/decade after $\omega = 3.1623$.
2. Before $\omega = 1$, there is only a constant gain line of 26.0206 in dB scale, since there is no contribution from any term.
3. Between $1 \leq \omega \leq 2$ the term $(1 + j\omega)$ contributes a slope of $+20dB/decade$ and the the term $(1 + j\omega/2)$ contributes a slope of $-20dB/decade$. Hence the net contribution is $0dB/decade$. Therefore the graph would be a straight line horizontal to the ω axis.
5.Between $2 \leq \omega \leq 3.1623$ the term $(1 + j\omega)$ contributes a slope of $+20dB/decade$.
6. After $\omega = 3$ the term $(1 + (\frac{3}{10}j\omega) - (\frac{\omega}{\sqrt{10}})^2)$ contributes a slope of $-40dB/decade$. The terms $(1+j\omega)$ and $(1+j\omega/2)$ both has an effective contribution of $0dB/decade$ to the slope of the graph. Hence the plot will be a straight line with a slope of $0dB/decade$.

From the above informations we derived we can now plot the Asymptotic Bode Magnitude plot of the transfer function.
For $\omega \leq 1$ there will be a horizontal straight line having a gain of 26 (as calculated above). From $\omega = 1$ to $\omega = 2$ the plot will be a $+20dB/decade$ sloped line. Again from $\omega = 2$ to $\omega = 3.1623$ the plot will be a horizontal straight line. Now from $\omega = 3.1623$ onwards the plot will be a line with a slope of $-40dB/decade$

Now if we want the exact Bode Magnitude plot we will have to make the necessary $3dB$ error compensation at each break points for **linear terms** like $(1 + Ts)$. For quadratic term we have to calculate the necessary dB error at the break point.For a break point we found from an linear expression occurring in the denominator of the transfer function the error to be compensated is $-3dB$ and $+3dB$ if it's corresponding expression is in the numerator.

Hence at $\omega = 1$ there is a $+3dB$ error and at $\omega = 2$ there is a $-3dB$ error.
At $\omega = 3.1623$ we can see that the gain is:

$$20log|G(j\omega)|_{\omega=3.1623} = 20log|\frac{0.05*(1+(j\omega)}{(1+j\omega/2)(1+(\frac{3j\omega}{10}-(\frac{\omega}{\sqrt{10}})^2))}| = -20.5898 \tag{27}$$

Drawing the phase plot is fairly straight forward. Draw a table and corresponding to some some points in ω (including the break points) just calculate the phase from the equation below. The table 6 shows phase angles $\angle G(j\omega)$ at various frequencies. The angle of the transfer function at a particular frequency can be calculated as follows:

$$\angle G(j\omega) = +tan^{-1}\frac{\omega}{1} - tan^{-1}\frac{\omega}{2} - tan^{-1}\frac{1-(\frac{\omega}{\sqrt{10}})^2}{\frac{3\omega}{10}}$$

ω	0.1	0.2	0.5	1.0	2.0	3.0	9.0	60
$\angle G(j\omega)$	1.16	2.15	3.78	-0.34	-27.5	-68	-153	-176

Table 4: Phase angle $\angle G(j\omega)$

Phase Margin : Inf
Gain Margin : Inf

Asymptotic Bode plot magnitude along with error corrections and exact angle plot:

$$G(s) = \frac{(s+2)}{s(s+10)(s^2-1)} \tag{28}$$

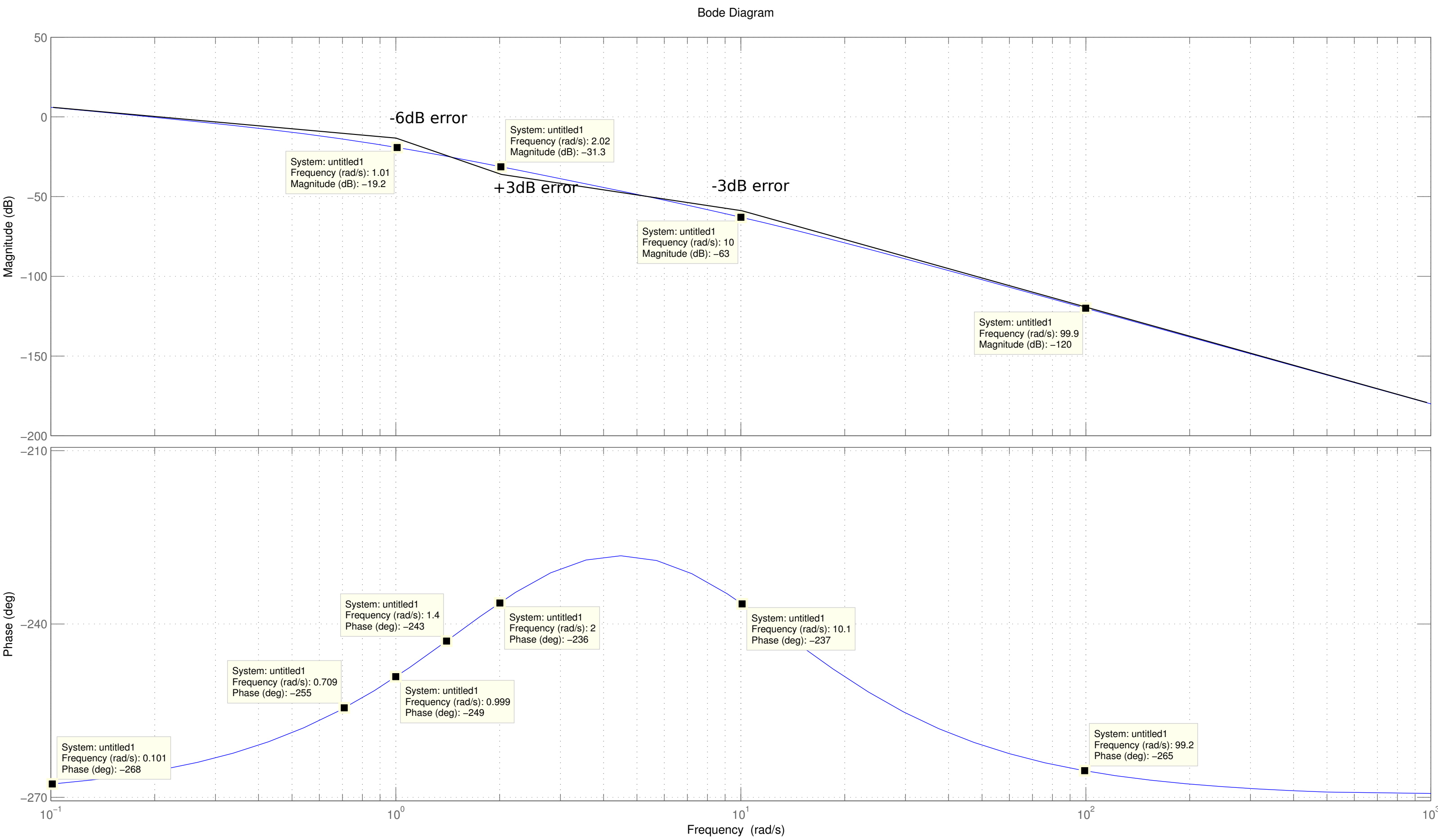


Figure 26: Bode plot for $G(s) = \frac{(s+2)}{s(s+10)(s^2-1)}$

Explanation:

$$G(s) = \frac{(s+2)}{s(s+10)(s^2-1)}$$
$$G(s) = \frac{(s+2)}{s(s+10)(s+1)(s-1)}$$
$$G(j\omega) = \frac{0.2(1+j\omega/2)}{(j\omega)(1+j\omega/10)(j\omega+1)(-j\omega+1)}$$

There are three(3) break points: $\omega = 1$, 2 and 10.

At $\omega = 0.1$, $|G(j\omega)|_{dB} = 5.75$ and at $\omega = 0.4$, $|G(j\omega)|_{dB} = -7.25$. (**Note:** While calculating the dB gains for drawing the asymptotic plots we must only consider those terms which are contributing in that region)

We have the following 5 observations from the structure of $G(j\omega)$:
1. There are two terms:$(j\omega+1)$ and $(j\omega-1)$. These two terms have their magnitude plot exactly same. **2.** Before $\omega = 1$, only the $(j\omega)$ term contributes a slope of $-20dB/decade$.
3. Between $1 \leq \omega \leq 2$ the term $(j\omega)$ contributes a slope of $-20dB/decade$ and the term $(1+j\omega)*(j\omega-1)$ contributes a slope of $-40dB/decade$. Hence the plot will be a straight line with a slope of $-60dB/decade$.
4. Between $2 \leq \omega \leq 10$ the term $(j\omega)$ contributes a slope of $-20dB/decade$ and the term $(1+j\omega)*(j\omega-1)$ contributes a slope of $-40dB/decade$. The term $(1+j\omega/2)$ contributes a slope of $+20dB/decade$. Hence the plot will be a straight line with a slope of $-40dB/decade$.
5. After $\omega = 10$ the term $(1+j\omega/10)$ contributes a slope of $-20dB/decade$. The other terms combined have a total contribution of $-40dB/decade$ to the slope of the graph. Hence the plot will be a straight line with a slope of $-60dB/decade$.

From the above informations we derived we can now plot the Asymptotic Bode Magnitude plot of the transfer function.
For $\omega \leq 1$ there will be a straight line with a slope of $-20dB/decade$ having a gain of 5.75 at $\omega = 0.1$ (as calculated above). From $\omega = 1$ to $\omega = 2$ the plot will be a $-60dB/decade$ sloped line. Again from $\omega = 2$ to $\omega = 10$ the plot will be a straight line with a slope of $-40dB/decade$. From $\omega = 10$ onwards the plot will be a line with a slope of $-60dB/decade$

Now if we want the exact Bode Magnitude plot we will have to make the necessary $3dB$ error compensation at each break points. For a break point we found from an expression occurring in the denominator of the transfer function the error to be compensated is $-3dB$ and $+3dB$ if it's corresponding expression is in the numerator.
Hence at $\omega = 10$ there is a $-3dB$ error. At $\omega = 2$ there is a $+3dB$ error and at $\omega = 1$ there is a $(+3dB + 3dB) = 6dB$ error.

Drawing the phase plot is fairly straight forward. Draw a table and corresponding to some some points in ω (including the break points) just calculate the phase from the equation below. The table 6 shows phase angles $\angle G(j\omega)$ at various frequencies. The angle of the transfer function at a particular frequency can be calculated as follows:

$$\angle G(j\omega) = +tan^{-1}\frac{\omega}{2} - 90 - tan^{-1}\frac{\omega}{10} - tan^{-1}\frac{\omega}{1} - tan^{-1}\frac{\omega}{-1}$$

ω	0.1	0.4	1	2	4	10	20	40	100	400
$\angle G(j\omega)$	-268	-261	-249	-236	-229	-236	-249	-259	-265	-269

Table 5: Phase angle $\angle G(j\omega)$

Phase Margin : -85.6^0 at 0.194 rad/sec
Gain Margin : Inf

Asymptotic Bode plot magnitude along with error corrections and exact angle plot:

$$G(s) = \frac{(s+2)}{s^3(s+10)} \tag{29}$$

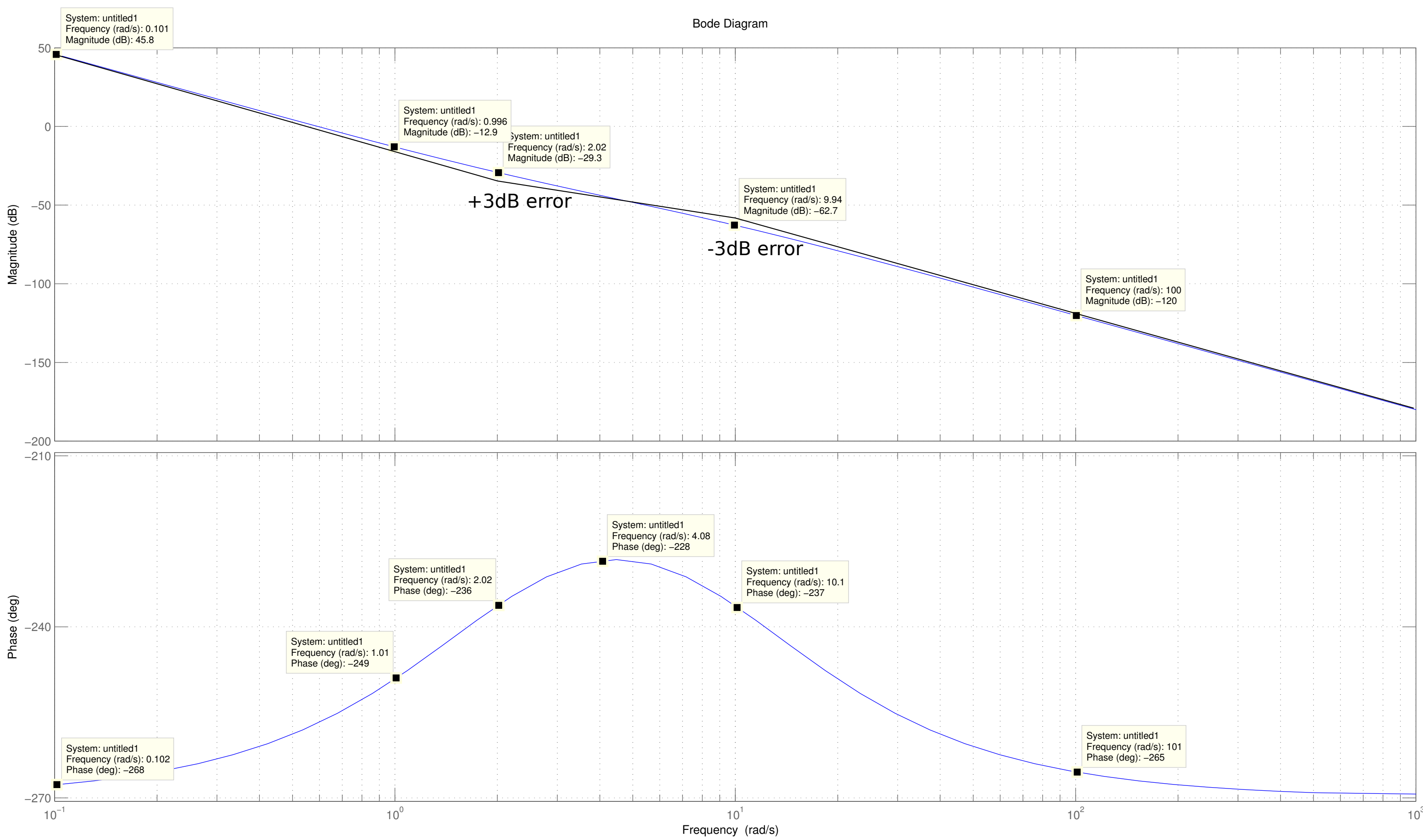


Figure 27: Bode plot for $G(s) = \frac{(s+2)}{s^3(s+10)}$

Explanation:

$$G(s) = \frac{(s+2)}{s^3(s+10)}$$
$$G(j\omega) = \frac{0.2(1+j\omega/2)}{(j\omega)^3(1+j\omega/10)}$$

There are two(2) break points: $\omega = 2$ and 10.

At $\omega = 0.1$, $|G(j\omega)|_{dB} = 45.4$ and at $\omega = 1$, $|G(j\omega)|_{dB} = -13.4$. (**Note:** While calculating the dB gains for drawing the asymptotic plots we must only consider those terms which are contributing in that region)

We have the following 5 observations from the structure of $G(j\omega)$:
1. Before $\omega = 2$, only the $(j\omega)^3$ term contributes a slope of $-60dB/decade$.
2. Between $2 \leq \omega \leq 10$ the term $(j\omega)^3$ contributes a slope of $-60dB/decade$ and the term $(1 + j\omega/2)$ contributes a slope of $+20dB/decade$. Hence the plot will be a straight line with a slope of $-40dB/decade$.
3. After $\omega = 10$ the term $(1 + j\omega/10)$ contributes a slope of $-20dB/decade$. The other terms combined have a total contribution of $-40dB/decade$ to the slope of the graph. Hence the plot will be a straight line with a slope of $-60dB/decade$.

From the above informations we derived we can now plot the Asymptotic Bode Magnitude plot of the transfer function.
Upto $\omega \leq 2$ there will be a straight line with a slope of $-60dB/decade$ having a gain of 45.4 at $\omega = 0.1$ and a gain of -13.4 at $\omega = 1$ (as calculated above). From $\omega = 2$ to $\omega = 10$ the plot will be a $-40dB/decade$ sloped line. Again from $\omega = 10$ onwards the plot will be a straight line with a slope of $-60dB/decade$.

Now if we want the exact Bode Magnitude plot we will have to make the necessary $3dB$ error compensation at each break points. For a break point we found from an expression occurring in the denominator of the transfer function the error to be compensated is $-3dB$ and $+3dB$ if it's corresponding expression is in the numerator.
Hence at $\omega = 10$ there is a $-3dB$ error and at $\omega = 2$ there is a $+3dB$ error .

Drawing the phase plot is fairly straight forward. Draw a table and corresponding to some some points in ω (including the break points) just calculate the phase from the equation below. The table 6 shows phase angles $\angle G(j\omega)$ at various frequencies. The angle of the transfer function at a particular frequency can be calculated as follows:

$$\angle G(j\omega) = +tan^{-1}\frac{\omega}{2} - 270 - tan^{-1}\frac{\omega}{10}$$

ω	0.1	1	2	4	10	30	100
$\angle G(j\omega)$	-268	-249	-236	-228	-236	-255	-265

Table 6: Phase angle $\angle G(j\omega)$

Phase Margin : -76.9^0 at 0.593 rad/sec
Gain Margin : Inf

Lead compensator for the following system so that P.M > 45^0:

$$G(s) = \frac{1}{s^2(s+1)} \tag{30}$$

The bode plot is shown below.

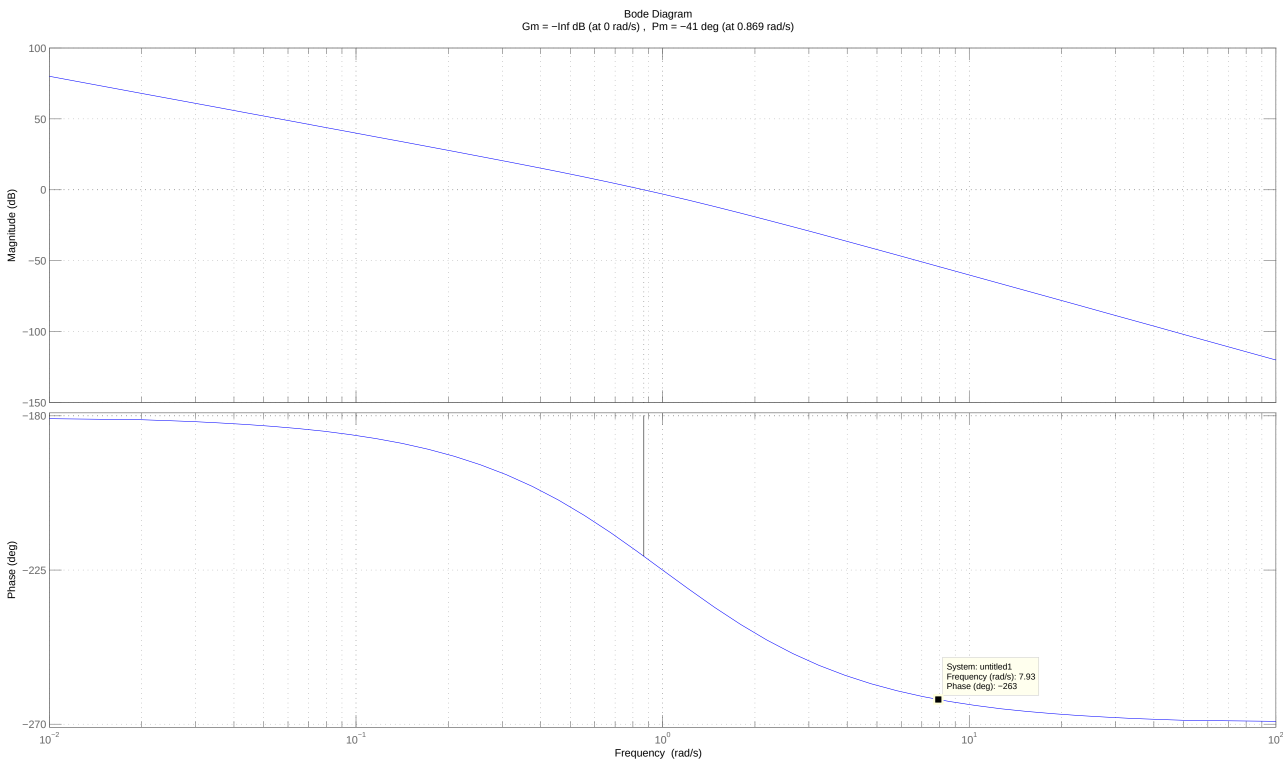


Figure 28: Bode PLot of $G(s) = \frac{1}{s^2(s+1)}$

Lead compensator provides positive phase to a system. Not only that it provides high gain at high frequencies and zero gain (dB gain) at low frequencies. A generic PD compensator is:

$$C(s) = \frac{1+Ts}{1+\alpha Ts} \tag{31}$$

Where $\alpha < 1$.

The above Bode Plot shows that the uncompensated system is having a phase margin of -41 degree at 0.8 rad/sec. If we design a compensator at this frequency (low frequency) the response of the system will be very sluggish. So we need to design a compensator in such a way so that it has the desired phase margin at around 8 rad/sec (high frequency).
From the Bode Plot shown above we see that at around 8 rad/sec the **Phase** of the system is

$-263 + 180 = -83^0$.
Hence to have a **P.M** of $+45^0$ we need to inject $+83 + 45 = 130$ degrees phase into the system.

For this purpose we will use a **double** lead compensator. The two identical phase compensator will provide $+65^0$ phase each.

Now, the value of α for the lead compensator:

$$\phi_m = 65^0 \tag{32}$$

$$\alpha = \frac{1 - sin(65^0)}{1 + sin(65^0)} = 0.05 \tag{33}$$

Hence the single lead compensator providing a P.M of 65^0 at around 8 rad/sec is:

$$C(s) = \frac{0.5s + 1}{0.025s + 1} \tag{34}$$

The Bode Plot of the open loop system with double compensator has been given below:

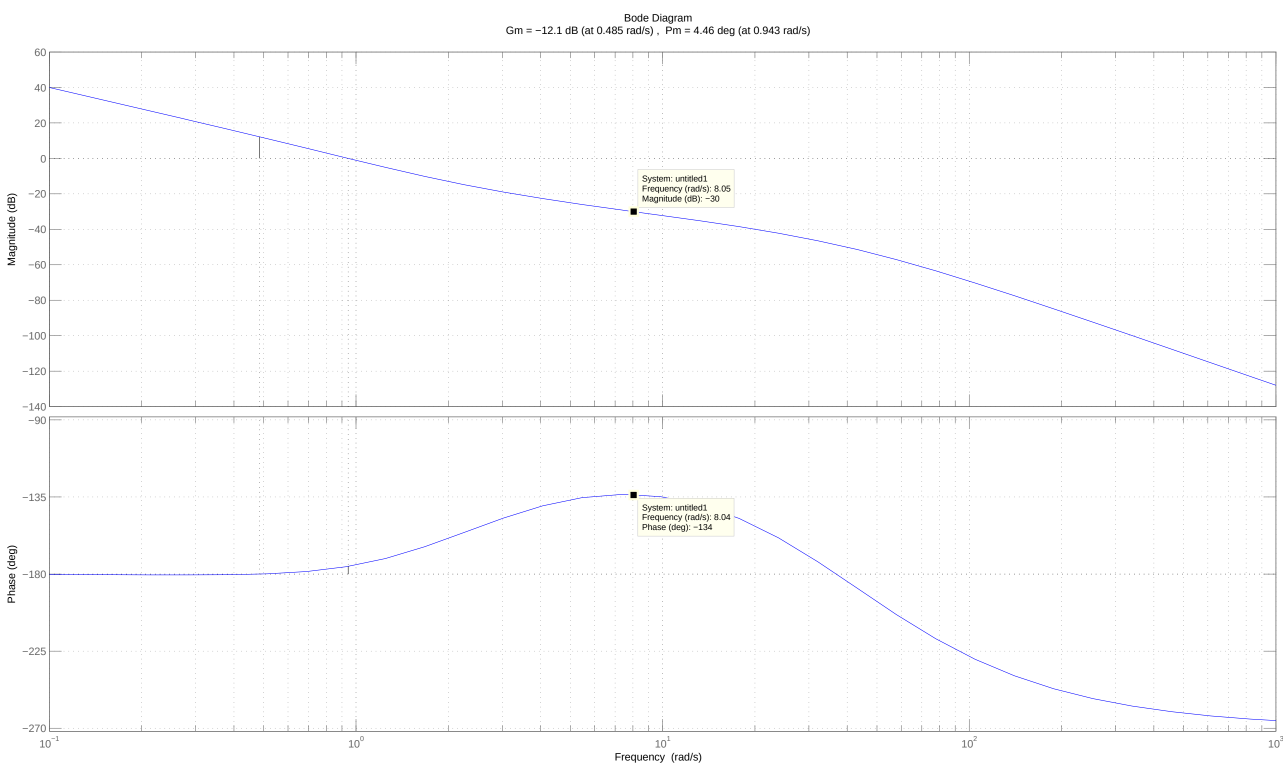


Figure 29: Bode PLot of $C(s) * C(s) * G(s)$

We can see from the above plot that to have a P.M of 45^0 at 8 rad/sec we need to add 30 dB gain to the system.

$$20logk = 30 \tag{35}$$

$$k = 10^{1.5} = 31.6 \tag{36}$$

Hence the resultant compensator becomes:

$$G_l = k * C(s) * C(s) \tag{37}$$

The Bode Plot of the compensated system is given below:

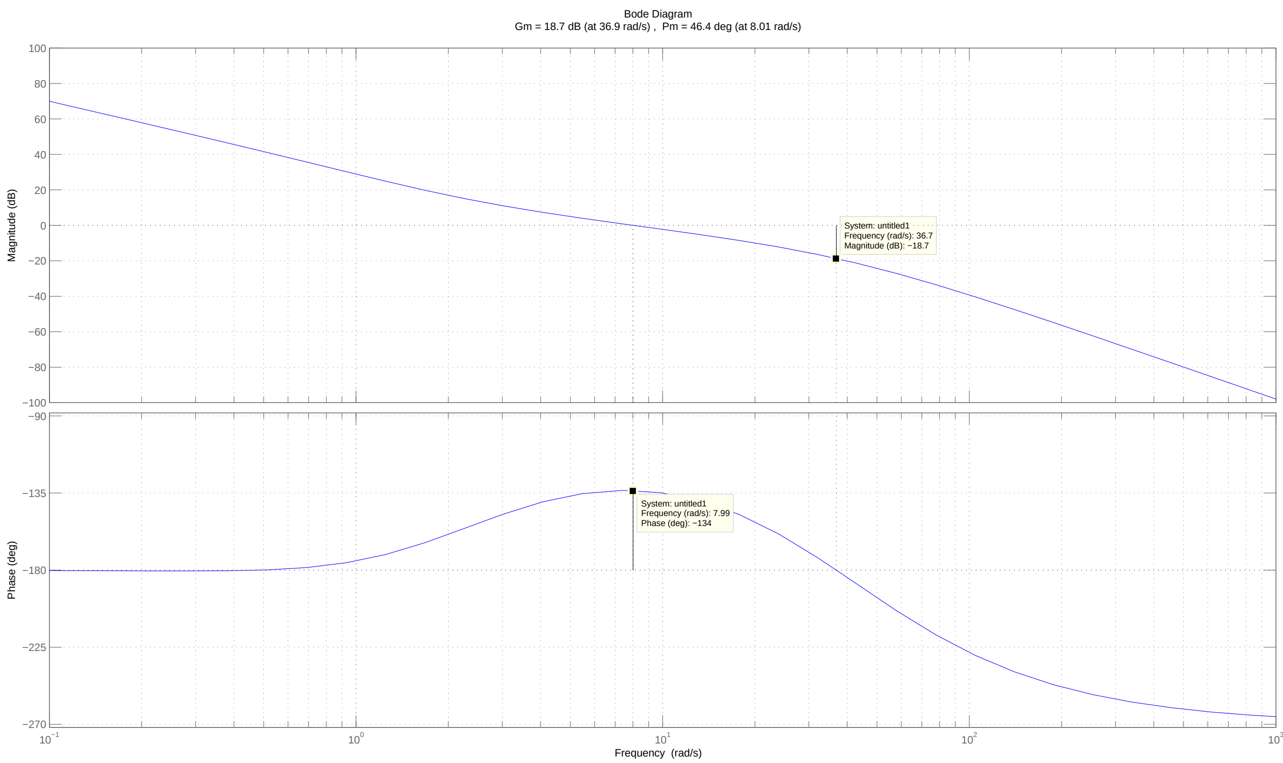


Figure 30: Bode PLot of $k * C(s) * C(s) * G(s)$

The closed loop system becomes:

$$H(s) = \frac{7.9s^2 + 31.6s + 31.6}{0.000625s^5 + 0.05063s^4 + 1.05s^3 + 8.9s^2 + 31.6s + 31.6} \tag{38}$$

The step response of the closed loop compensated system is shown below:

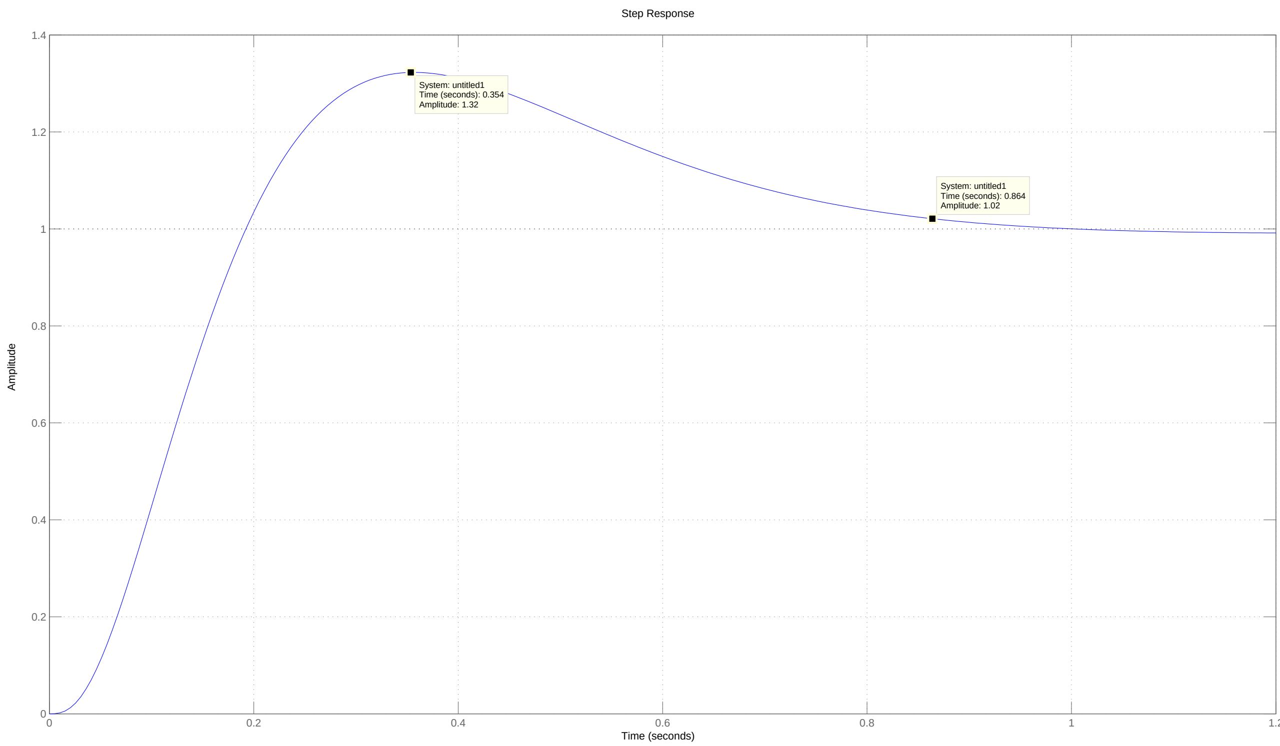


Figure 31: Bode PLot of $H(s)$

We can see the settling time is below 1 sec. To adjust the peak overshoot we need to vary **T** of the lead compensator.