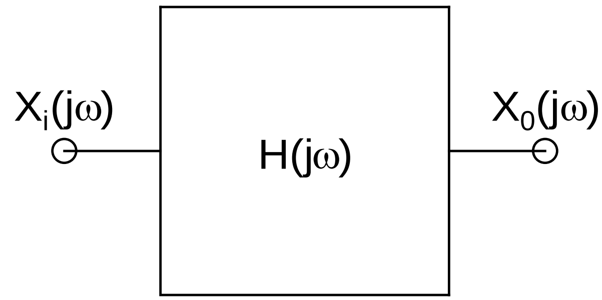
Sinusoidal Frequency Response

- Behavior of circuits under sinusoidal excitation having ***varying frequency***
- Since inductive and capacitive reactances change with frequency, hence, the circuit response would also change with frequency
- Need to have the concept of ***Transfer Function*** and ***Bode Plot***
- Also, to understand various types of ***filter*** topologies: ***Low-Pass***, ***High-Pass***, ***Band-Pass***

Transfer Function:

* Expressed as:

$$H(j\omega) = \frac{X_0(j\omega)}{X_i(j\omega)}$$



X: either voltage (V) or current (I)

* ***Four*** different topologies possible:

- ***Voltage Gain*** = V_0/V_i (dimensionless)
- ***Current Gain*** = I_0/I_i (dimensionless)
- ***Transresistance Gain*** = V_0/I_i (Ω)
- ***Transconductance Gain*** = I_0/V_i (℧)

* In ***polar*** form:

$$H(j\omega) = |H| \angle \theta$$

$|H|$: ***Magnitude*** and θ: ***Phase*** of the transfer function

* ***Note***: $H(j\omega)$ is ***not*** a phasor

It simply is a ***complex number***

* ***Example***: Consider a ***voltage amplifier*** $(X = V)$

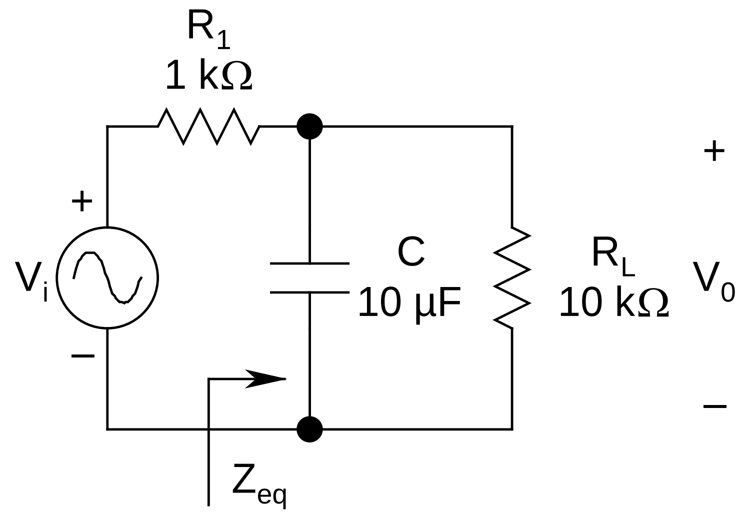
Let V_i and V_0 be expressed in phasor form as:

$$\bar{V}_i(j\omega) = |V_i| \angle \phi_i \text{ and } \bar{V}_0(j\omega) = |V_0| \angle \phi_0$$

$$\Rightarrow \bar{V}_0(j\omega) = H(j\omega)\bar{V}_i(j\omega)$$

$$\Rightarrow |V_0| = |H||V_i| \qquad \text{and} \qquad \phi_0 = \theta + \phi_i$$

Sinusoidal Frequency Response:



$$Z_{eq} = \left(\frac{1}{R_L} + j\omega C\right)^{-1} = \frac{R_L}{1 + j\omega R_L C}$$

$$H(j\omega) = \frac{V_0}{V_i} = \frac{Z_{eq}}{Z_{eq} + R_1}$$

$$= \frac{R_L}{(R_L + R_1) + j\omega R_L R_1 C} = \frac{K}{1 + j\omega/\omega_c}$$

$$\left(K = R_L \big/ (R_L + R_1) = 10/11,\ \omega_c = 1\big/\left[(R_1 \parallel R_L)C\right] = 110\ \text{rad/sec}\right)$$

$$|H| = \frac{K}{\sqrt{1 + (\omega/\omega_c)^2}},\ \angle\theta = -\tan^{-1}(\omega/\omega_c)$$

- ***Filters***:
  - Pass signals of desired frequency and block all others
  - Immensely useful module in all kinds of circuit design
  - Four Types:
    - ***Low-Pass Filter*** (***LPF***)
    - ***High-Pass Filter*** (***HPF***)
    - ***Band-Pass Filter*** (***BPF***)
    - ***Band-Reject*** (or ***Notch***) ***Filter*** (***BRF***)

Low-Pass Filter (LPF):

* As $\omega \to 0$, $X_C = 1/(j\omega C) \to \infty$

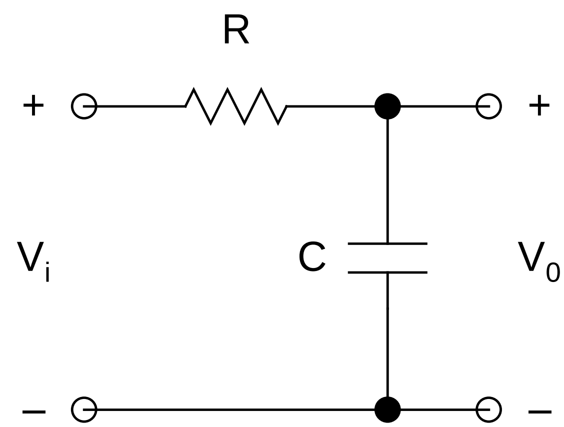
  $\Rightarrow$ C behaves like an ***open-circuit***

  $\Rightarrow V_0$ ***follows*** V_i

* As $\omega \to \infty$, $X_C \to 0$

  $\Rightarrow$ C behaves like a ***short-circuit***

  $\Rightarrow V_0 = 0$



* Thus, low frequency signals are ***passed*** (known as ***pass-band***) while high frequency signals are ***blocked*** (known as ***stop-band***) $\Rightarrow$ ***LPF***

* Transfer function:

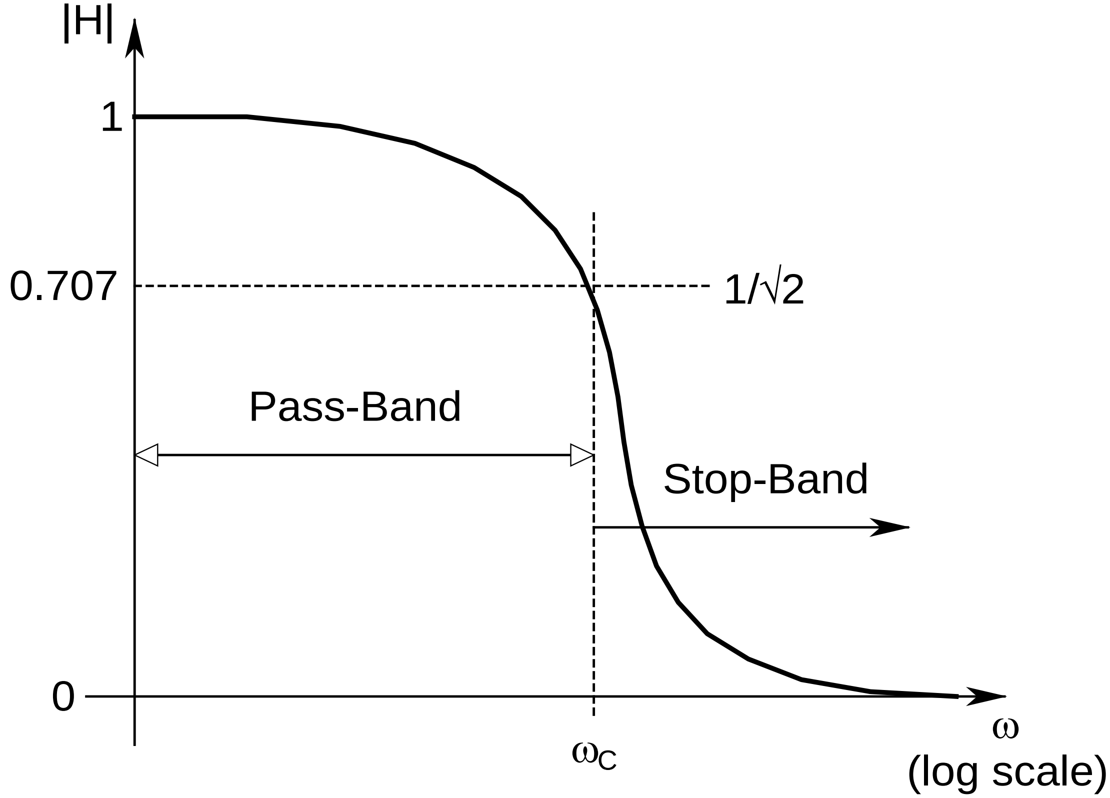
$$H(j\omega) = \frac{V_0}{V_i} = \frac{1/(j\omega C)}{R + 1/(j\omega C)} = \frac{1}{1 + j\omega/\omega_C}$$

$\omega_C = 1/(RC)$ is known as the ***upper cutoff frequency***

* ***Note***: As $\omega \to 0$, $|H| \to 1$; as $\omega \to \infty$, $|H| \to 0$.

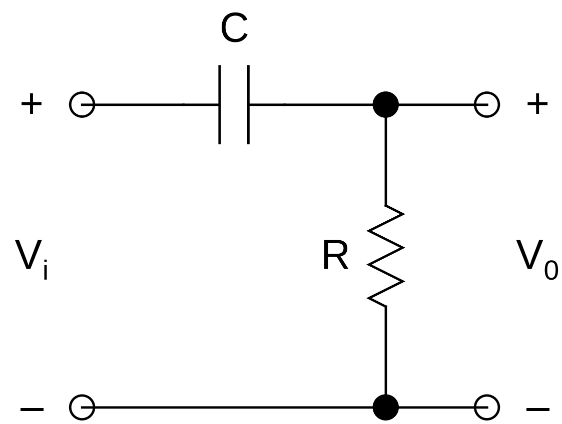
and at $\omega = \omega_C$, $|H| = |H|_{max}/\sqrt{2} = 0.707|H|_{max}$

$\Rightarrow$ This is the definition of ***cutoff frequency***



For LPFs, pass-band extends from 0 to ω_c
and stop-band extends from ω_c to ∞

High-Pass Filter (***HPF***):



* Just ***interchanging*** R and C ***converts*** an ***LPF*** to an ***HPF***

* As $\omega \to 0$, $X_C = 1/(j\omega C) \to \infty$

$\Rightarrow$ The capacitor behaves like an ***open-circuit***, and V_i is ***blocked***

$\Rightarrow V_0 = 0$

* As $\omega \to \infty$, $X_C \to 0$ (***short-circuit***)

$\Rightarrow V_0$ ***follows*** V_i

* Thus, low frequency signals are ***blocked*** (***stop-band***) while high frequency signals are ***passed*** (***pass-band***)

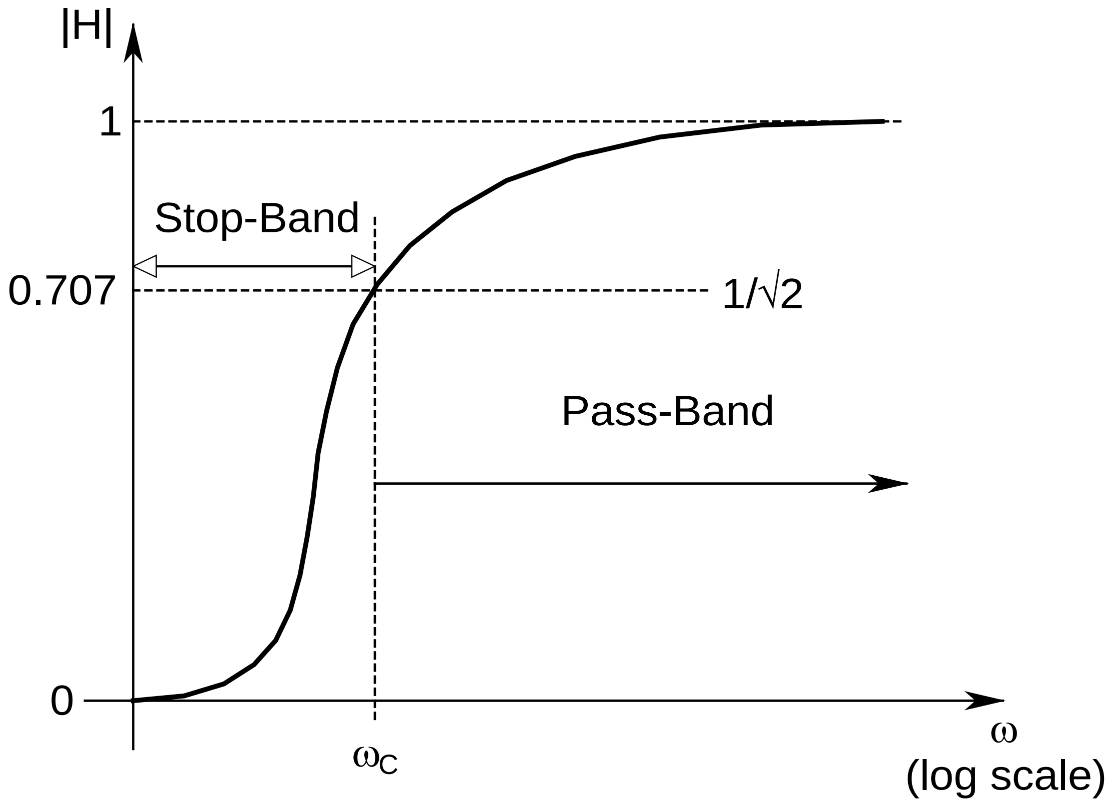
$\Rightarrow$ ***HPF***

* Transfer function:

$$H(j\omega) = \frac{V_0}{V_i} = \frac{R}{R + 1/(j\omega C)} = \frac{j\omega/\omega_C}{1 + j\omega/\omega_C}$$

$\omega_C = 1/(RC)$ is known as the ***lower cutoff frequency***

* ***Note***: As $\omega \to 0$, $|H| \to 0$; as $\omega \to \infty$, $|H| \to 1$; and at $\omega = \omega_c$, $|H| = 1/\sqrt{2}$



For HPFs, pass-band extends from ω_c to ∞
and stop-band extends from 0 to ω_c

***Band-Pass Filter* (*BPF*):**

* Extremely useful circuit to pass signals within a certain range of frequencies bounded between two ***cutoff frequencies*** ω_1 and ω_2, with $\omega_2 > \omega_1$

ω_1: ***Lower Cutoff Frequency*** (0 to ω_1: ***stop-band***)

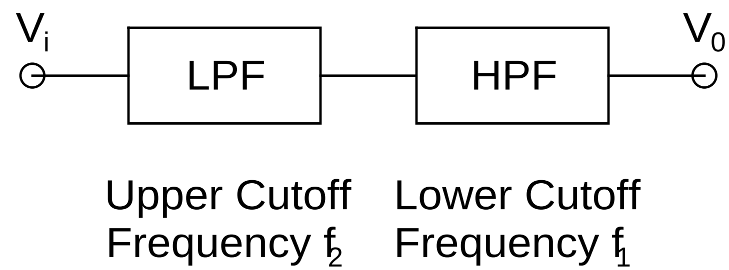
ω_2: ***Upper Cutoff Frequency*** (ω_2 to ∞: ***stop-band***)

ω_1 to ω_2: ***pass-band***

* Can be constructed simply by putting an ***LPF*** and an ***HPF*** in ***series***!

Simplest Construction of a BPF:

* Connect an ***LPF*** (with ***upper cutoff frequency*** f_2) in ***series*** with an ***HPF*** (with ***lower cutoff frequency*** f_1)

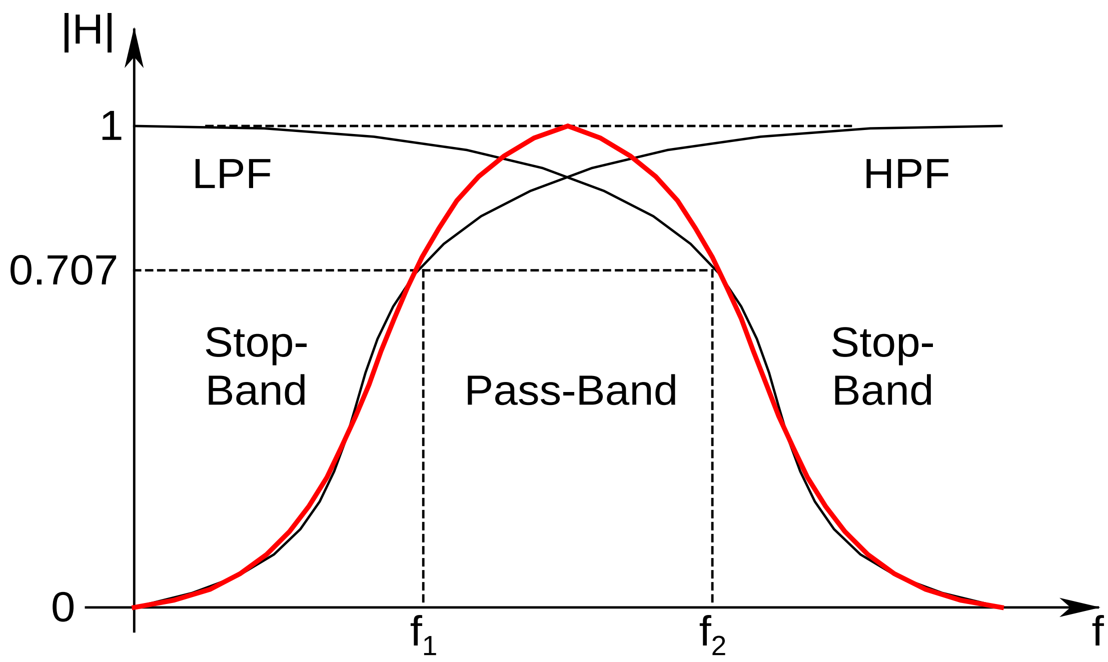


* Make sure that $f_2 > f_1$

* The ***LPF*** would ***pass*** all signals ***till*** f_2

* The ***HPF*** would ***block*** all signals ***below*** f_1

* Thus, all signals ***below*** f_1 and ***above*** f_2 will be ***blocked*** (***stop-bands***), while ***passing*** signals ***within a range between*** f_1 ***and*** f_2 (***pass-band***) $\Rightarrow$ ***BPF***



Band-Pass Filter Response

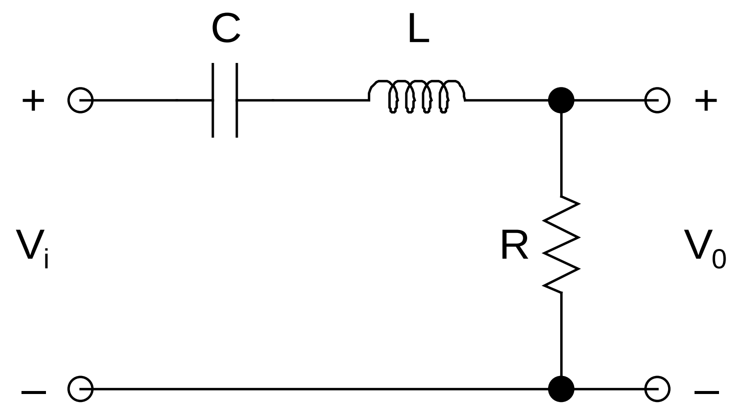
Implementation of a BPF Using RLC 2nd-Order Circuit:

* For ***very low frequency***, C behaves like ***open-circuit***, and V_i gets blocked

$\Rightarrow V_0 = 0$

* For ***very high frequency***, L behaves like ***open-circuit***, and again V_i gets blocked

$\Rightarrow V_0 = 0$



RLC Circuit Implementation for a Band-Pass Filter

* In between, V_0 ***increases*** with ω initially, reaches a ***peak***, and then starts to ***drop*** again

Series Resonance:

* ***Transfer function***: $H(j\omega) = \dfrac{V_0}{V_i} = \dfrac{R}{R + j(X_L - X_C)}$

with $X_L = \omega L$ and $X_C = 1/(\omega C)$

$$\Rightarrow \quad |H(j\omega)| = \frac{R}{\sqrt{R^2 + (X_L - X_C)^2}}$$

and $\angle\theta = -\tan^{-1}\left(\dfrac{X_L - X_C}{R}\right)$

* If $X_L > X_C$, the circuit is ***inductive*** and $\angle\theta$ is ***negative***

* If $X_C > X_L$, the circuit is ***capacitive*** and $\angle\theta$ is ***positive***

* ***Note***: At a certain frequency ω_0, X_L would become ***equal*** to X_C

$\Rightarrow \left|H(j\omega)\right| = 1$ and $\angle\theta = 0°$

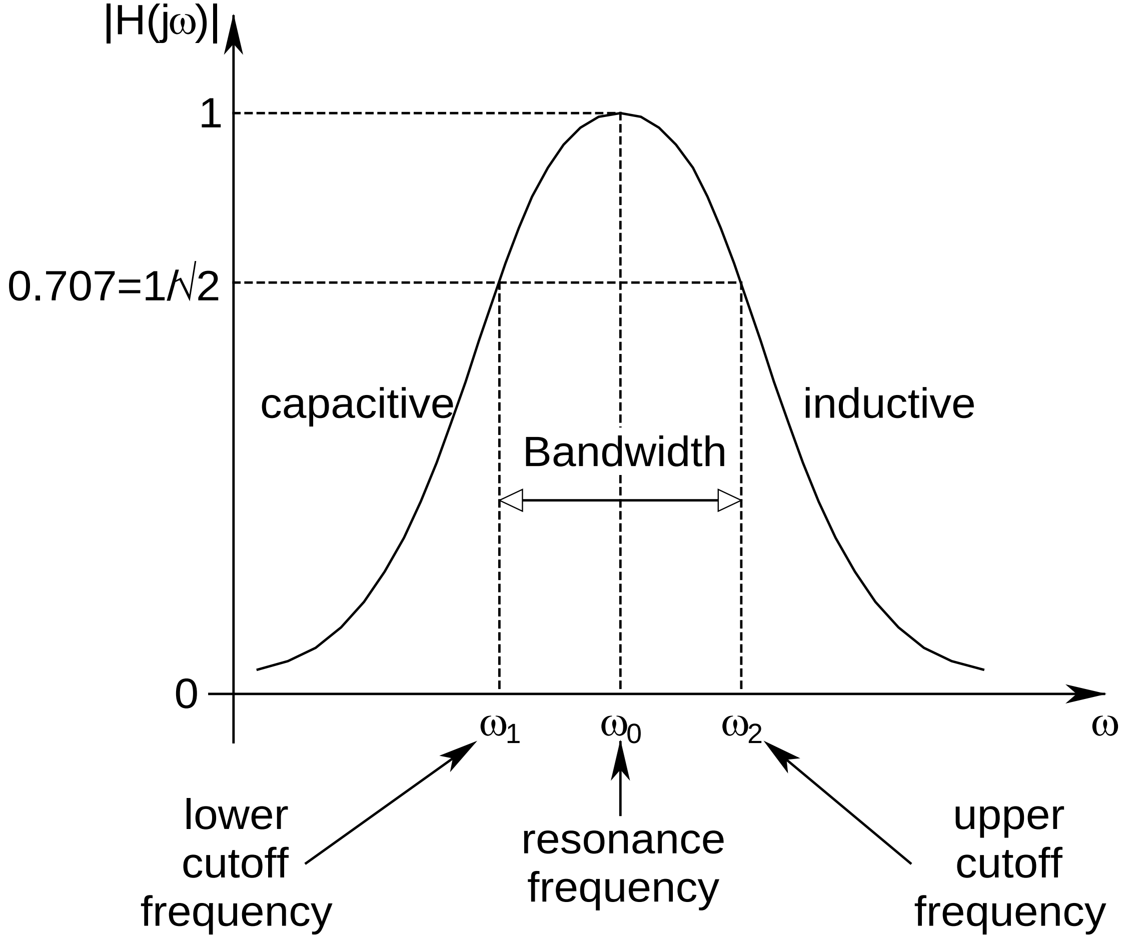
* Thus, the circuit becomes ***purely resistive***

$\Rightarrow$ Known as ***series resonance***

* Under this condition, V_0 attains its ***maximum*** value and occurs at ω_0 = ***resonance frequency***

$= 1/\sqrt{LC}$

* For $\omega < \omega_0$, the circuit response is ***capacitive***
* For $\omega > \omega_0$, the circuit response is ***inductive***


|H(jω)|
1
0.707=1/√2
0
capacitive
inductive
Bandwidth
ω1
ω0
ω2
ω
lower
cutoff
frequency
resonance
frequency
upper
cutoff
frequency

* ***Second-order circuit*** with ***two roots*** (ω_1 and ω_2)

* To find them, put $\left|H(j\omega)\right| = 1/\sqrt{2}$

$$\Rightarrow \omega_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

$$\text{and } \omega_2 = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

* ***Note***: ω_1 and ω_2 are ***perfectly symmetric*** around ω_0

$\Rightarrow$ The response is a ***mirror image*** around ω_0

* Actually, ω_0 is a ***geometric mean*** of ω_1 and ω_2

$$\Rightarrow \omega_0 = \sqrt{\omega_1 \omega_2}$$

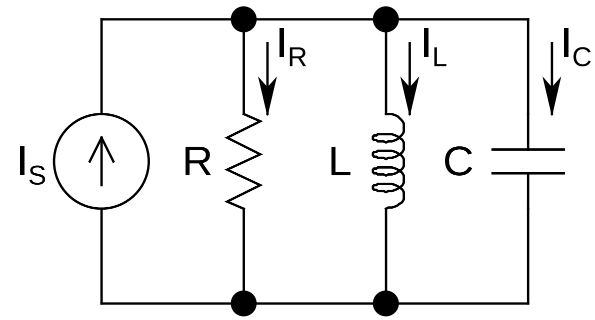
* ***Bandwidth*** $BW = \omega_2 - \omega_1 = R/L$
* The ***sharpness*** of the peak is defined by the ***Quality Factor*** (QF):

$$QF = \frac{\omega_0}{BW} = \frac{\omega_0 L}{R} = \frac{1}{R}\sqrt{\frac{L}{C}}$$

* A ***high*** QF implies a ***very narrow*** and ***sharp*** response
* ***Note***: For given values of L and C, QF is an ***inverse*** function of R

  $\Rightarrow$ ***Small*** R gives ***sharp*** response, and vice versa
* ***Application***: Selectively picking up a ***narrow band*** of signal $\Rightarrow$ ***tuning circuits***

Parallel RLC Circuit (***Parallel Resonance***):



- ***Tank Circuit***: L and C in parallel
- ***Admittance method*** works best for this circuit
- ***Note***: All currents are ***sinusoids***
- Net admittance $Y = G + j\left(B_C - |B_L|\right)$

$$G = 1/R,\ |B_L| = 1/(\omega L),\ B_C = \omega C$$

$$\Rightarrow Y = G + j\left(\omega C - \frac{1}{\omega L}\right)$$

- ***Note***: For $\omega C = 1/(\omega L)$, imaginary part of Y ***vanishes***

* Known as ***parallel resonance***, with the ***parallel resonance frequency*** $\omega_0 = 1/\sqrt{LC}$ (***same as series resonance***)

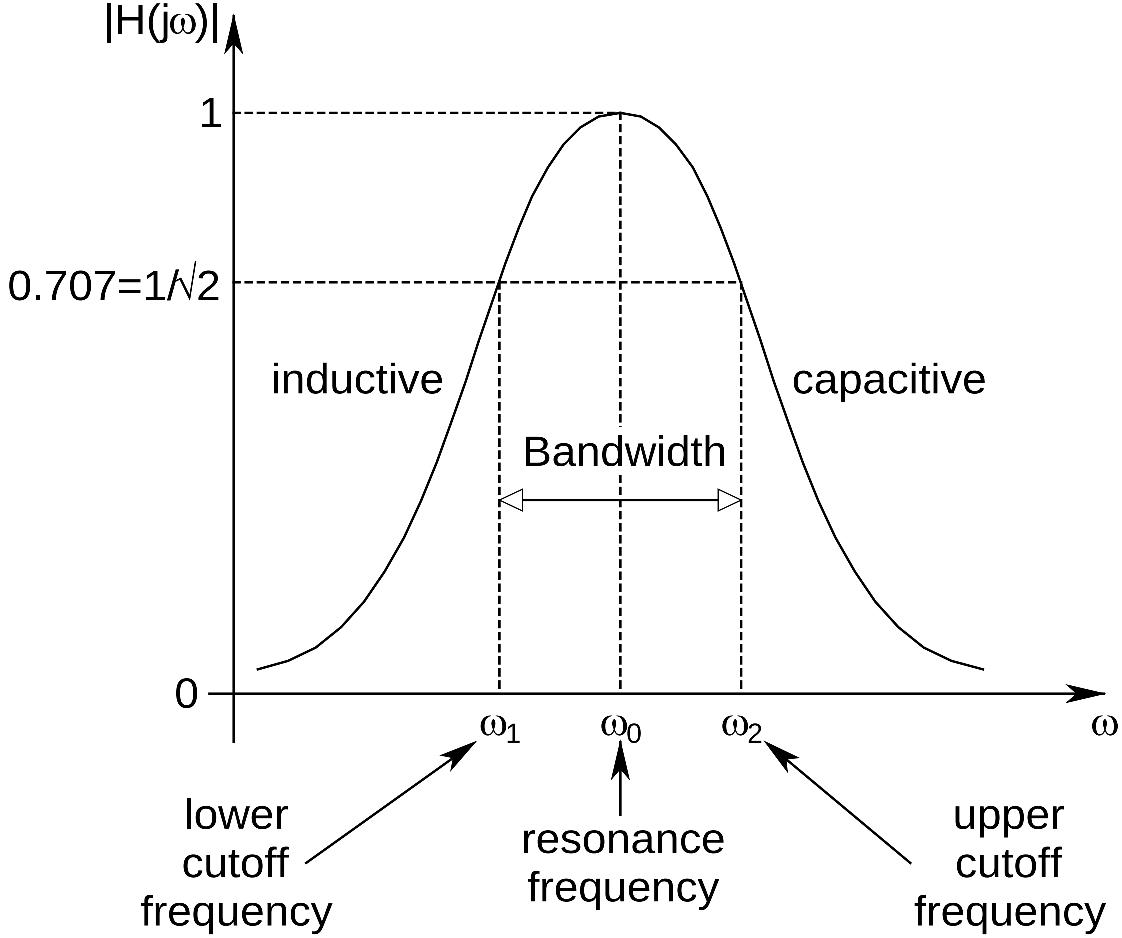
* ***Transfer function***:

$$H(j\omega) = \frac{I_R}{I_S} = \frac{1/R}{1/R + 1/(j\omega L) + j\omega C} = \frac{j\omega L}{R(1-\omega^2 LC) + j\omega L}$$

$$\Rightarrow \left|H(j\omega)\right| = \frac{\omega L}{\sqrt{R^2(1-\omega^2 LC)^2 + (\omega L)^2}}$$

* ***Note***: At resonance, $\left|H(j\omega)\right| = 1$

$\Rightarrow$ Current through R is ***maximum*** under this condition



|H(jω)|
1
0.707=1/√2
inductive
capacitive
Bandwidth
0
ω1
ω0
ω2
ω
lower cutoff frequency
resonance frequency
upper cutoff frequency

* Again a ***second-order circuit*** with two roots (ω_1 and ω_2)

* To find them, put $\left|H(j\omega)\right| = 1/\sqrt{2}$

$$\Rightarrow \omega_1 = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}}$$

$$\text{and } \omega_2 = \frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}}$$

* Again note that ω_1 and ω_2 are perfectly ***symmetric*** around ω_0

* ***BW*** $= \omega_2 - \omega_1 = 1/(RC)$

* ***QF*** $= \omega_0/BW = \omega_0 RC = R\sqrt{\frac{C}{L}}$

Observations:

* In ***series*** RLC circuit at ***resonance***:

- Net ***impedance*** is ***minimum***

  ⇒ ***Current*** drawn from the voltage source is ***maximum***

  ⇒ ***Voltage*** drop across R is ***maximum***

- Voltage drops across L and C are ***equal in magnitude*** but ***opposite in phase***

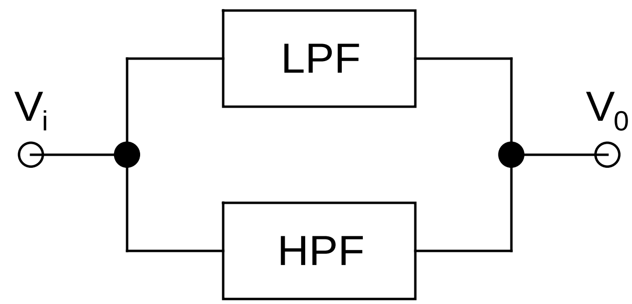
  ⇒ They ***cancel out***, making the entire source voltage ***drop across R***

- As R ***decreases***, the response becomes ***sharper***

* In ***parallel*** RLC circuit at ***resonance***:
  - Net ***admittance*** is ***minimum***
    - $\Rightarrow$ Net ***impedance*** is ***maximum***
    - $\Rightarrow$ ***Current*** drawn from the voltage source is ***minimum***
    - $\Rightarrow$ ***Current*** through R is ***maximum***
  - Currents flowing through L and C are ***equal in magnitude*** but ***opposite in phase***
    - $\Rightarrow$ known as ***circulating current***
    - $\Rightarrow$ Entire current supplied by source ***flows through R***
  - As R ***increases***, the response becomes ***sharper***

Band-Reject (or ***Notch***) ***Filter***:

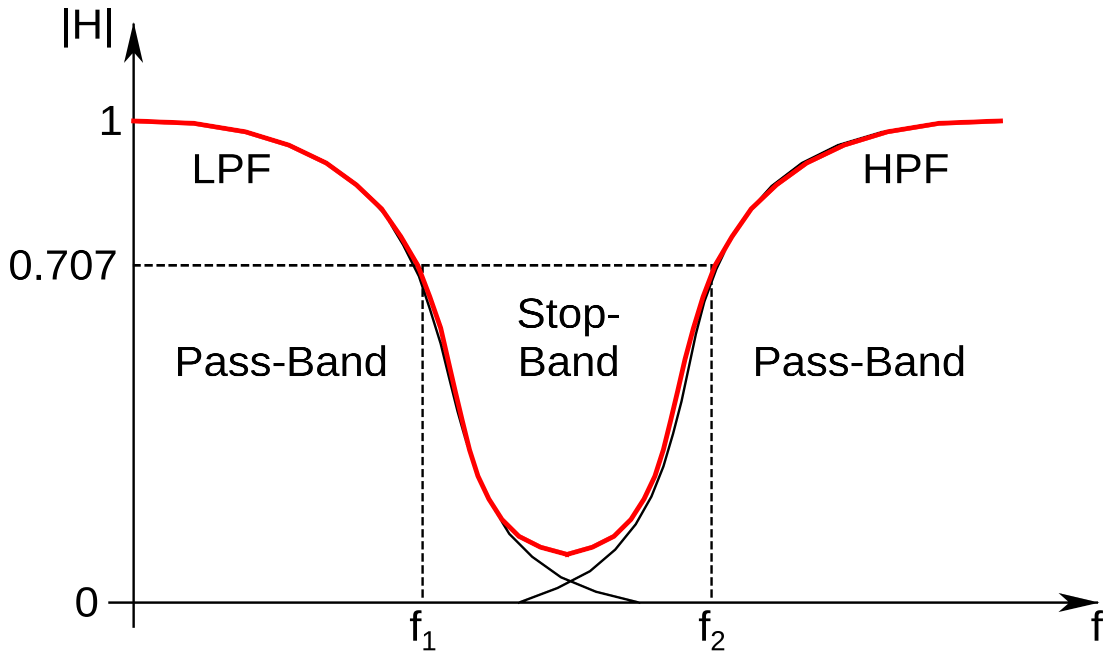
* ***Simplest implementation***: put an ***LPF*** having upper cutoff frequency f_1 and an ***HPF*** having lower cutoff frequency f_2 (with $f_2 > f_1$) in ***parallel***



* It will ***pass*** frquencies below f_1 and above f_2 (***pass-bands***), while ***blocking*** frequencies within the band $(f_2 - f_1)$ (***stop-band***)

$\Rightarrow$ ***Band-reject filter***

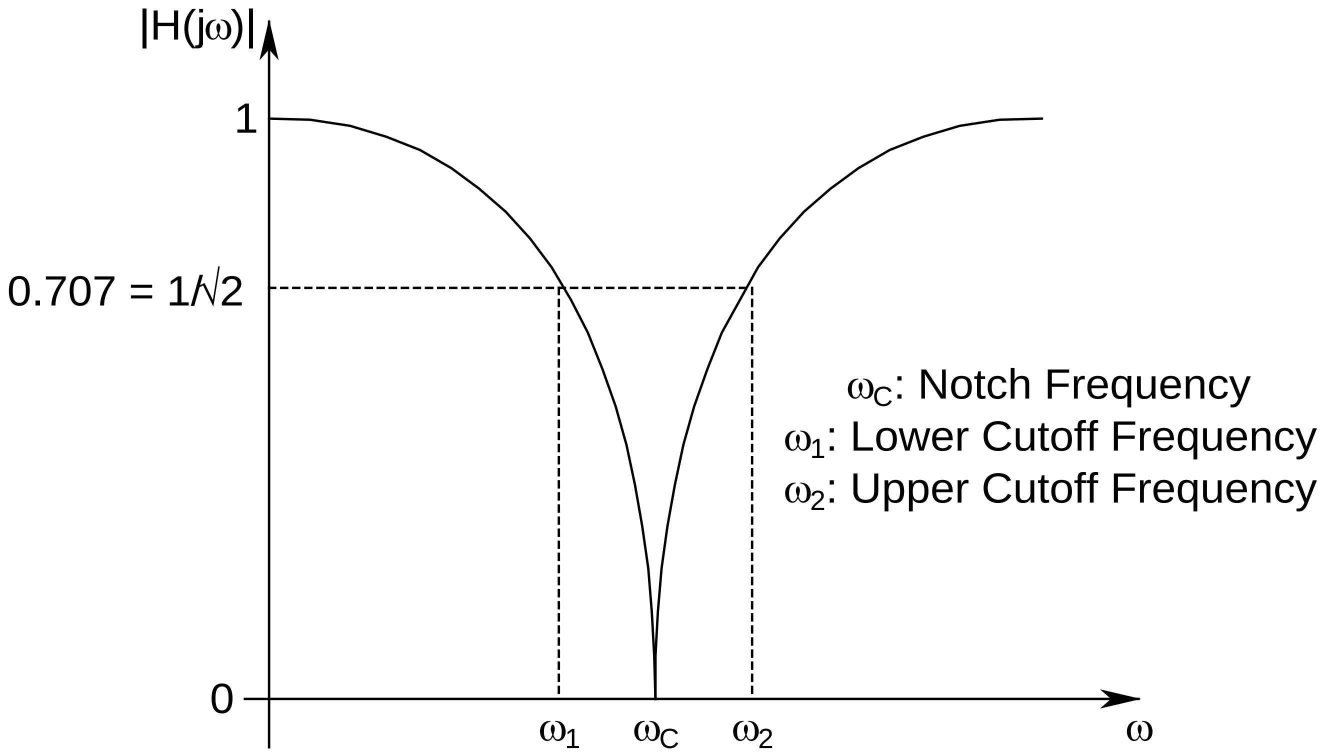
* If $(f_2 - f_1)$ is made extremely ***narrow***, then we arrive at the ***notch filter***



Band-Reject Filter Response

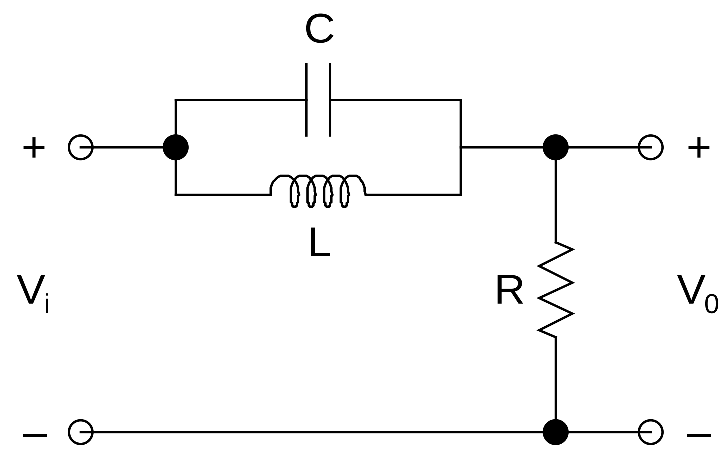
Note: Upper cutoff frequency of LPF = Lower cutoff frequency of BRF

Lower cutoff frequency of HPF = Upper cutoff frequency of BRF



Notch Filter Response

Simplest RLC Implementation of a Notch Filter:



* $Z =$ Impedance of parallel combination of L and C

$$= \frac{j\omega L}{1-\omega^2 LC}$$

* ***Note***: For $\omega^2 LC = 1,\ Z \rightarrow \infty$

$\Rightarrow V_i$ gets ***blocked*** and $V_0 = 0$

* This frequency is known as the ***notch frequency***

ω_C, and is given by $\omega_C = \dfrac{1}{\sqrt{LC}}$

* ***Transfer function***:

$$H(j\omega) = \frac{V_0}{V_i} = \frac{R}{R+Z} = \frac{R}{R + \dfrac{j\omega L}{1-\omega^2 LC}}$$

* ***Ex.***: Find ω_1 and ω_2 by substituting $\left|H(j\omega)\right| = \frac{1}{\sqrt{2}}$

and show that $\omega_C = \sqrt{\omega_1 \omega_2}$

* ***BW*** $= \omega_2 - \omega_1$

* ***QF*** $= \dfrac{\omega_C}{BW}$

Bode Plot:

* The most convenient way to plot a ***transfer function***
* Conceived by ***Hendrik W. Bode*** of the ***Bell Telephone Laboratories***
* Provides a very easy and convenient method of extracting information regarding variation of ***magnitude*** and ***phase*** of any transfer function as a function of ***frequency***
* Can be made even simpler by a technique known as the ***Asymptotic Bode Plot***

* $|H|$ is plotted in ***decibels*** $(\boldsymbol{dB})$, while $\angle\theta$ is plotted in ***degrees*** $(°)$, both in ***linear scale***, while the ***frequency*** is plotted in ***log scale***, in a ***semilog graph***
* $|H|(\text{dB}) = 20\log_{10}|H(j\omega)|$
* ***Inverse operation***:

  $$|H(j\omega)| = 10^{[|H|(\text{dB})]/20}$$

* ***Note***: In Bode plot, the ***frequency*** ω is always plotted in ***log scale***

$\lvert H(j\omega) \rvert$	$\lvert H \rvert (dB)$	$\lvert H(j\omega) \rvert$	$\lvert H \rvert (dB)$
0.001	–60	0.01	–40
0.1	–20	$1/\sqrt{2}$	–3
1	0	$\sqrt{2}$	3
5	14	10	20
100	40	1000	60

Bode Magnitude Plot:

* Consider the transfer function:

$$H(j\omega) = 1 + j\frac{\omega}{a}$$

$$\Rightarrow \left|H(j\omega)\right| = \sqrt{1 + \frac{\omega^2}{a^2}} \quad \text{and}$$

$$\left|H\right|(dB) = 20\log_{10}\left(\sqrt{1 + \frac{\omega^2}{a^2}}\right)$$

Observations:

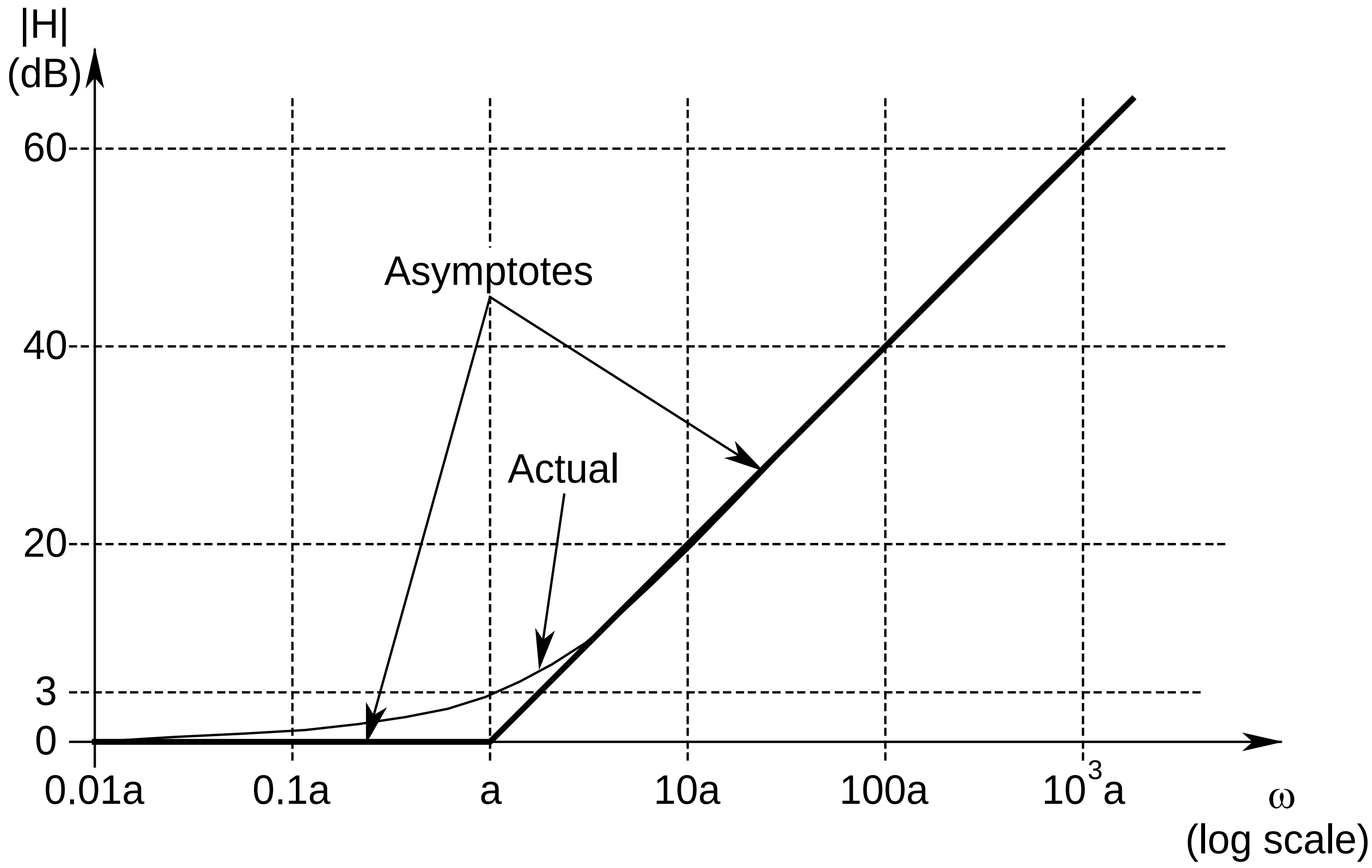
* For $\omega \ll a$, $|H(j\omega)| \simeq 1$, and $\left|H\right|(dB) = 0$

* For $\omega \gg a$, $|H(j\omega)| \simeq \omega / a$, and $|H|(dB) =$ $20\log_{10}(\omega / a)$, which is ***linear*** with respect to ω ($\because$ ω is plotted in ***log scale***), and increases at the rate of $+20$ dB / decade
* At $\omega = a$, $|H(j\omega)| = \sqrt{2}$, and $|H|(dB) = 3$
* The angular frequency a has several names: ***corner frequency***, ***cutoff frequency***, ***break-point frequency***, ***+3-dB frequency***, etc.
* It is also known as the ***zero*** of the transfer function

Concept of Asymptotes:

* ***Simplification*** of the actual Bode plot
* For $\omega \le a$, it is taken to be equal to ***zero***
* For $\omega > a$, it is assumed to be a ***straight line***, starting from a, with a slope of ***+ 20 dB/decade***
* ***Note***: Maximum error occurs at $\omega = a$, which is equal to 3 dB

$\Rightarrow$ a is known as the ***3-dB frequency***


|H|
(dB)
60
40
20
3
0
Asymptotes
Actual
0.01a
0.1a
a
10a
100a
10^3 a
ω
(log scale)

Bode Phase Plot:

* ***Phase*** ϕ of the transfer function:

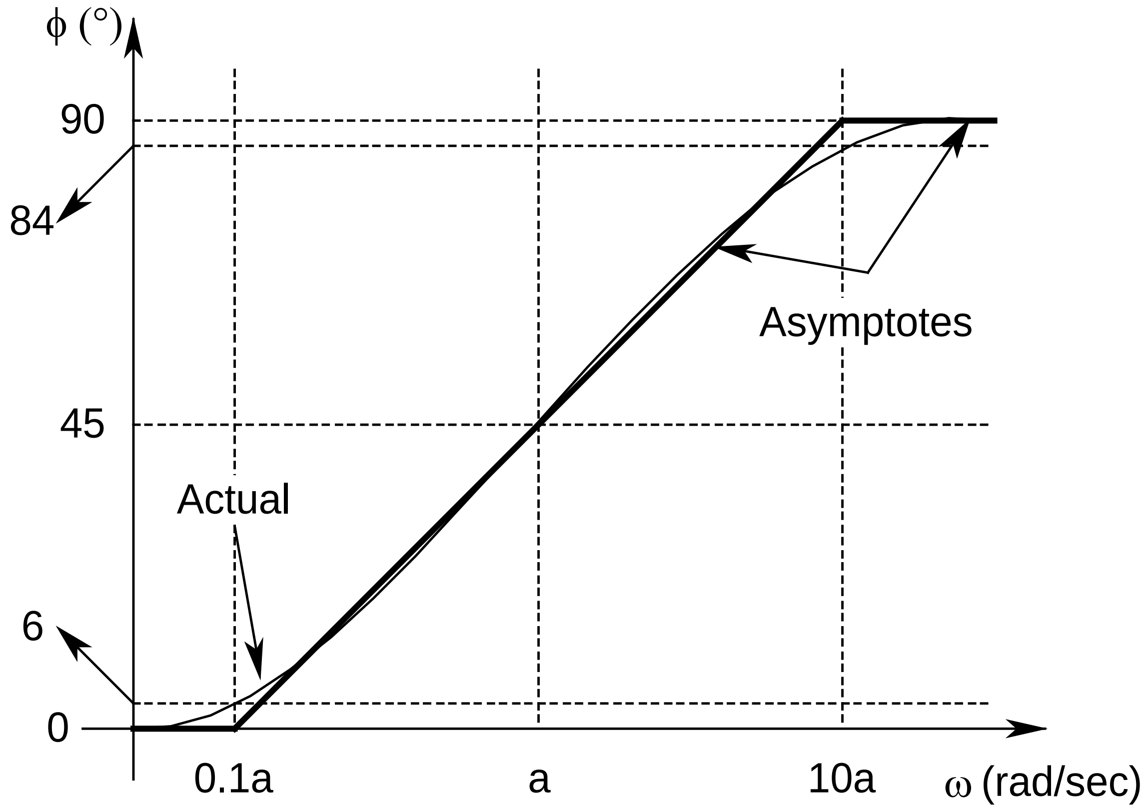
$$\phi = \tan^{-1}\left(\frac{\omega}{a}\right)$$

Observations:

* At $\omega = a$, $\phi = 45°$; $\omega = 0.1a$, $\phi = 6°$; $\omega = 10a$, $\phi = 84°$; and as $\omega \to \infty$, $\phi = 90°$

* Can be adequately described by ***three asymptotes***:

1. $0°$ for $\omega \le 0.1a$
2. $90°$ for $\omega \ge 10a$
3. Between $0.1a$ and $10a$, changing ***linearly*** with a slope of $+$***45°/decade***



Note: Actual phase at 0.1a and 10a is equal to 6° and 84° respectively, hence, these two frequencies are known as ***lower and upper 6° frequencies***

* Consider another transfer function:

$$H\left(j\omega\right)=\frac{1}{1+j\omega/b}$$

$$\Rightarrow \left|H\left(j\omega\right)\right|=1\Big/\sqrt{1+\omega^2/b^2} \text{ and}$$

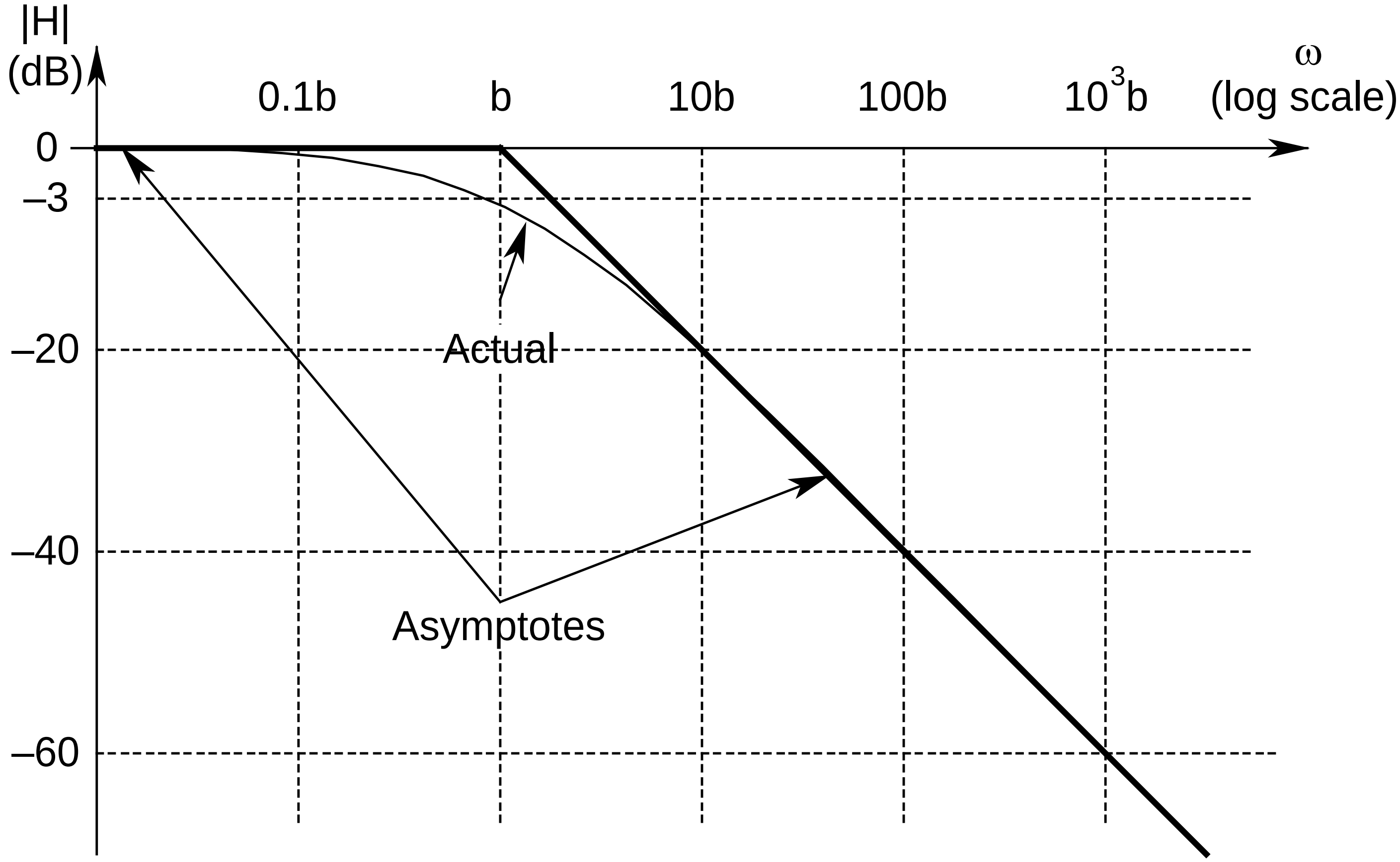
$$\left|H\right|\left(dB\right)=-20\log_{10}\left(\sqrt{1+\omega^2/b^2}\right)$$

<u>Observations</u>:

* For $\omega \ll b$, $|H(j\omega)| \simeq 1$, and $\left|H\right|\left(dB\right)=0$

* For $\omega \gg b$, $|H(j\omega)| \simeq b/\omega$, and $\left|H\right|\left(dB\right)=$ $20\log_{10}\left(b/\omega\right)$, which is ***linear*** with respect to ω, and changes at the rate of $-$ ***20 dB/decade***

* Also, at $\omega = b$, $|H(j\omega)| = 1/\sqrt{2}$, and $|H|(dB) = -3$
* Similar to a, the angular frequency b also has several names: ***corner frequency***, ***cutoff frequency***, ***break-point frequency***, ***–3-dB frequency***, etc.
* It is also known as the ***pole*** of the transfer function
* ***Asymptotes***:
  - For $\omega \le b$, it is taken to be equal to ***zero***
  - For $\omega > b$, it is assumed to be a ***straight line***, starting at b, with a slope of $-$ ***20 dB/decade***



Note: Maximum error occurs at ω = b, which is equal to -3 dB

Hence, b is known as the $-$***3-dB frequency***

* ***Phase*** ϕ of the transfer function:

$$\phi = -\tan^{-1}\left(\frac{\omega}{b}\right)$$

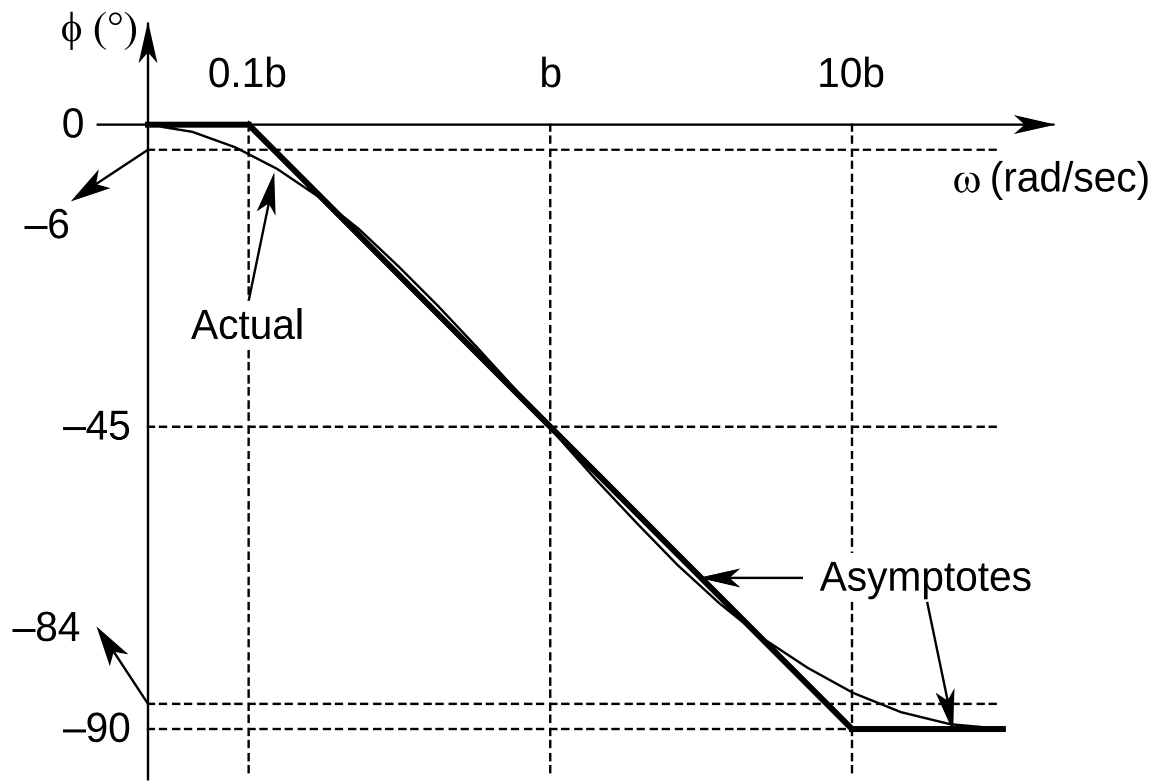
Observations:

* At $\omega = b$, $\phi = -45°$; $\omega = 0.1b$, $\phi = -6°$;

$\omega = 10b$, $\phi = -84°$; and as $\omega \to \infty$, $\phi = -90°$

* Can be adequately described by ***three asymptotes***:

1. $0°$ for $\omega \le 0.1b$
2. $-90°$ for $\omega \ge 10b$
3. Between $0.1b$ and $10b$, changing ***linearly*** with a slope of $-$***45°/decade***



Note: Actual phase at 0.1b and 10b is equal to $-6°$ and –84° respectively, hence, these two frequencies are known as ***lower and upper 6° frequencies***

* Consider another simple transfer function: $H(j\omega) = j\omega$

$\Rightarrow |H| = \omega$, and $|H|(dB) = 20\log(\omega)$, and a constant phase of ***90°***

$\Rightarrow |H|(dB)$ versus ω is a ***straight line*** in the asymptotic Bode plot, having a positive slope of ***20 dB/decade***, and crossing the 0 dB line at $\omega = 1$ rad/sec

* Similarly, the asymptotic Bode plot of the function $H(j\omega) = (j\omega)^{-1}$ is another ***straight line*** having a slope of – ***20 dB/decade***, crossing the 0 dB line at $\omega = 1$ rad/sec, and having a constant phase of – ***90°***

Ex: Asymptotic magnitude and phase Bode plots for the function: $H(j\omega) = \dfrac{j\omega(1 + j\omega/100)}{(1 + j\omega/10)(1 + j\omega/1000)}$

Magnitude Plot:

$\boldsymbol{j\omega}$: straight line with a slope of +20 dB/decade, crossing 0 dB at $\omega = 1$ rad/sec (***plot a***)

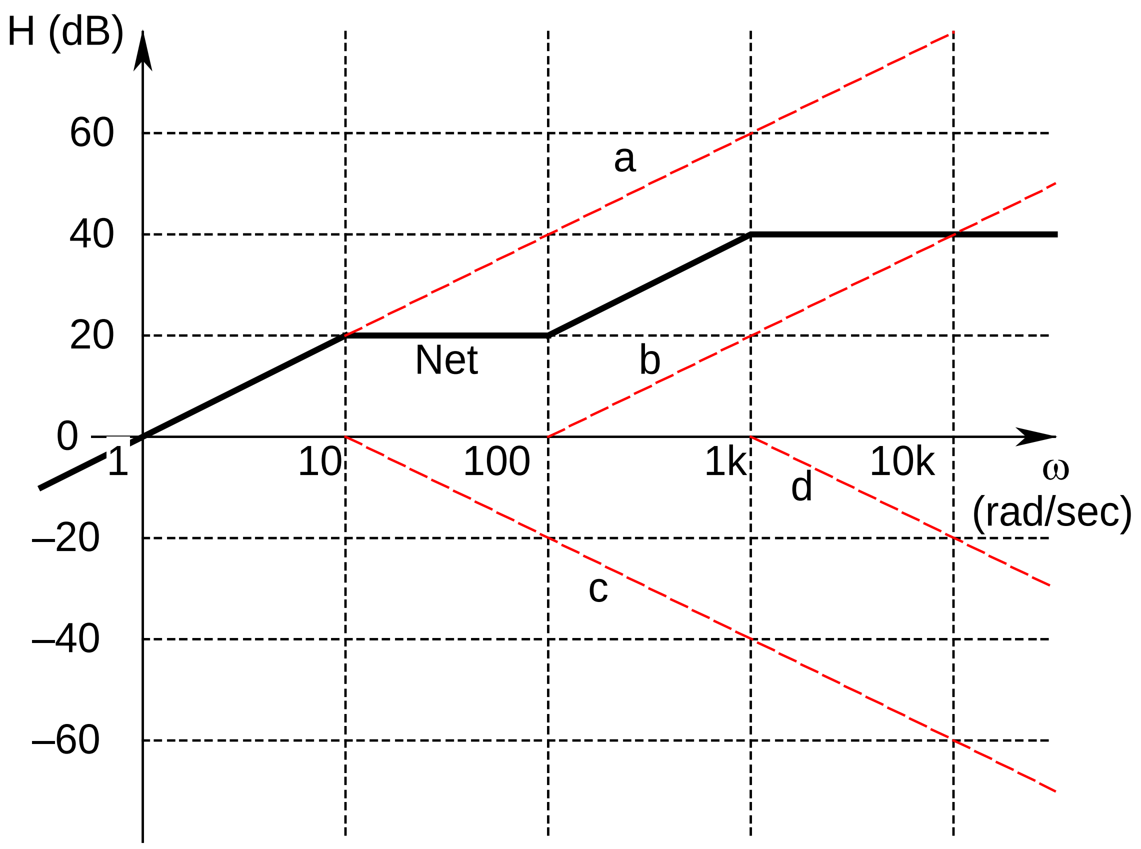
$\boldsymbol{(1 + j\omega/100)}$: 0 for $\omega \le 100$ rad/sec, then increasing @+20 dB/decade (***plot b***)

$\boldsymbol{(1 + j\omega/10)}$: 0 for $\omega \le 10$ rad/sec, then decreasing @ – 20 dB/decade (***plot c***)

$\left(\boldsymbol{1 + j\omega/1000}\right)$: 0 for $\omega \leq 1000$ rad/sec, then

decreasing @ – 20 dB/decade (***plot d***)

The net plot is an ***algebraic sum*** of all the individual plots



Phase Plot:

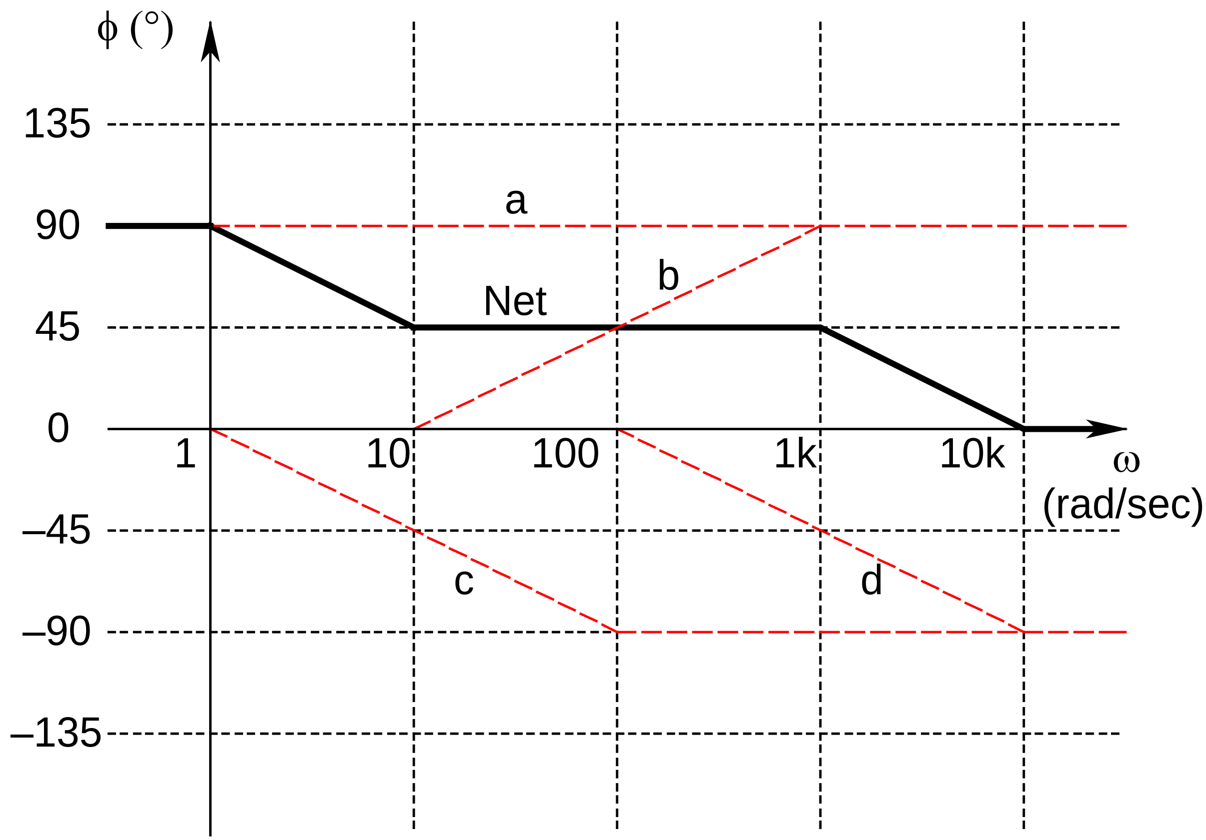
$j\omega$: constant phase of 90° (***a***)

$\left(1 + j\omega/100\right)$: 0° for $\omega \leq 10$ rad/sec, 45° at $\omega = 100$ rad/sec, and 90° for $\omega \geq 1000$ rad/sec (***b***)

$\left(1 + j\omega/10\right)$: 0° for $\omega \leq 1$ rad/sec, $-45°$ at $\omega = 10$ rad/sec, and $-90°$ for $\omega \geq 100$ rad/sec (***c***)

$\left(1 + j\omega/1000\right)$: 0° for $\omega \leq 100$ rad/sec, $-45°$ at $\omega = 1000$ rad/sec, and $-90°$ for $\omega \geq 10^4$ rad/sec (***d***)

The net plot is an ***algebraic sum*** of all the individual plots


ϕ (°)
135
90
45
0
–45
–90
–135
a
b
Net
c
d
1
10
100
1k
10k
ω
(rad/sec)

Note: If the transfer function contains any ***constant*** K, then it will add a ***constant offset*** of $20\log_{10}K$ to the ***magnitude plot***, while the ***phase plot*** will remain ***unaffected***