1.a. $\displaystyle\int_{-4}^{5} c(x+4)\,dx = 1$

$t = x+4$

$\Rightarrow c \displaystyle\int_{0}^{9} t\,dt = 1$

$\Rightarrow c \cdot \dfrac{t^2}{2}\Big|_0^9 = c \cdot \dfrac{81}{2} = 1$

$\Rightarrow c = \dfrac{2}{81}$. $\boxed{\text{1 mark}}$

$y = x|x| = \begin{cases} -x^2, & x<0 \\ x^2, & x<0 \end{cases}$ $\uparrow$ on $\mathbb{R}$

$S_Y = (-16, 25) = (-16, 0) \cup (0, 25)$

- For $-16 < y < 0$, we have

$P(X|X| \leq y) = P(X<0, X^2 \geq -y) + P(X>0, X^2 \leq y)$

$= P(X \leq -\sqrt{-y}) + 0$

$= \dfrac{2}{81} \displaystyle\int_{-4}^{-\sqrt{-y}} (x+4)\,dx$

$= \dfrac{(4-\sqrt{-y})^2}{81}$. $\boxed{\text{1 mark}}$

- For $0 < y < 25$, we have

$$P(X|X| \leq y) = P(X<0, X^2 \geq -y) + P(X>0, X^2 \leq y)$$

$$= P(X<0) + P(0 < X \leq \sqrt{y})$$

$$= P(X \leq \sqrt{y})$$

$$= \frac{2}{81} \int_{-4}^{\sqrt{y}} (x+4)\, dx$$

$$= \frac{(4+\sqrt{y})^2}{81}.$$ $\boxed{1 \text{ mark}}$

So,

$$F_Y(y) = \begin{cases} 0 & , y < -16 \\ \dfrac{(4-\sqrt{-y})^2}{81} & , -16 \leq y < 0, \\ \dfrac{(4+\sqrt{y})^2}{81} & , 0 \leq y < 25, \\ 1 & , y \geq 25. \end{cases}$$

$\boxed{1 \text{ mark}}$

F_Y is differentiable (except at a finite number of points).

So, the pdf is

$$f_Y(y) = F'_Y(y)$$

$$= \begin{cases} \dfrac{(4-\sqrt{-y})}{81 \cdot \sqrt{-y}}, & -16 < y < 0 \\ \dfrac{(4+\sqrt{y})}{81 \cdot \sqrt{y}}, & 0 < y < 25 \\ 0, & \text{otherwise.} \end{cases}$$

$\boxed{\text{1 mark}}$

1\. b. (i) Let $x > 0$.

$$F(x) = \int_0^x 2\beta t e^{-\beta t^2} dt$$

$$z = \beta t^2$$

$$= \int_0^{\beta x^2} e^{-z} dz$$

$$= 1 - e^{-\beta x^2}$$

$$F(x) = \begin{cases} 0 & , x < 0 \\ 1 - e^{-\beta x^2} & , x \geq 0. \end{cases}$$

1 mark

Let the median be m_0.

$$F(m_0) = \frac{1}{2}$$

$$\Rightarrow 1 - e^{-\beta m_0^2} = \frac{1}{2}$$

$$\Rightarrow e^{-\beta m_0^2} = \frac{1}{2}$$

$$\Rightarrow \beta m_0^2 = \log_e 2$$

$$\Rightarrow m_0 = \sqrt{\frac{\log_e 2}{\beta}}.$$

1 mark

Now,

$$\log_e f(t) = \log_e 2\beta + \log_e t - \beta t^2$$

$$\Rightarrow f'(t) = f(t)\left[\frac{1}{t} - 2\beta t\right] = 0$$

$$\Rightarrow t_0 = \frac{1}{\sqrt{2\beta}}$$ $\boxed{\text{1 mark}}$

$$f''(t) = f(t)\left[-\frac{1}{t^2} - 2\beta\right] +$$

$$f(t)\left[\frac{1}{t} - 2\beta t\right]^2$$

$$= f(t)\left[-4\beta + 4\beta^2 t^2\right]$$

$$f''(t_0) = f(t_0)\cdot\left[-4\beta + 2\beta\right]$$

$$= -2\beta f(t_0) < 0 \,..$$

$[\because \beta > 0 \;\&\; f > 0]$ $\boxed{\text{1 mark}}$

1. b. (ii) From $M_X(t)$, it is clear that

$$p(x) = \begin{cases} \frac{1}{8} & x = -1, \\ \frac{1}{4} & x = 0, \\ \frac{5}{8} & x = 2. \end{cases}$$

1 mark

$$E(X) = -\frac{1}{8} + \frac{10}{8} = \frac{9}{8}$$

$$E(X^2) = \frac{1}{8} + \frac{20}{8} = \frac{21}{8}$$

$$Var(X) = \frac{21}{8} - \frac{81}{64} = \frac{87}{64}$$

1 mark