MSO201a: Probability and Statistics
Summer Term: 2019
Mid Semester Examination
Instructor: Neeraj Misra

Time Allowed: 120 Minutes | **Maximum Marks: 60**

NOTE: **(i) Start answer of every question on a new page. Moreover, attempt all the parts of a question at one place.**
(ii) Answer each question legibly, clearly and concisely. Illegible answers will not be graded.
(ii) For each question, provide details of all the steps involved in arriving at the final answer(s)/conclusion(s) and box your final answer(s)/conclusion(s).

1. At a party m (≥ 3) men take off their hats and the m hats are then mixed up. After the party these m hats are distributed to m men randomly. Find the probability that:

   (a) no man gets his own hat; $\boxed{\text{5 MARKS}}$

   (b) exactly k ($k \in \{2, 3, \dots, m-1\}$) men get their own hat. $\boxed{\text{5 MARKS}}$

2. Let X be a random variable having the distribution function

$$F(x) = \begin{cases} 0, & \text{if } x < 0 \\ \frac{x}{4}, & \text{if } 0 \leq x < 1 \\ \frac{x^2}{4}, & \text{if } 1 \leq x < 2 \\ 1, & \text{if } x \geq 2 \end{cases},$$

   Find the p.m.f./p.d.f. of $Y = \sqrt{X}$. Hence find the mean and variance of Y^2. $\boxed{\text{5+2+3=10 MARKS}}$

3. Let X be a random variable with m.g.f.

$$M(t) = \frac{e^{-t}}{8} + \frac{1}{4} + \frac{5e^{2t}}{8}, \; t \in \mathbb{R},$$

   and let $Y = |X|$. Find the m.g.f. of Y and hence (using the m.g.f. of Y) find the mean and variance of Y. $\boxed{\text{5+2+3=10 MARKS}}$

4. (a) Let X be a r.v. with $E(X) = -2$ and $E(X^2) = 5$. Find a lower bound for the probability $Pr(-5 < X < 1)$. $\boxed{\text{5 MARKS}}$

   (b) Let X be a nonnegative r.v. (i.e., $Pr(X \geq 0) = 1$). Show that, for any $\alpha \geq 1$, $E(X^\alpha) \geq (E(X))^\alpha$, provided the expectations exist. Hence, for any nonnegative r.v. Y, show that

$$((E(Y^n))^{\frac{1}{n}} \leq ((E(Y^{n+1}))^{\frac{1}{n+1}}, \; n = 1, 2, \dots,$$

   provided the expectations exist. $\boxed{\text{5 MARKS}}$

5. (a) Does the function

$$F(x,y) = \begin{cases} 0, & \text{if } x < 0 \text{ or } x+y < 1 \text{ or } y < 0 \\ 1, & \text{otherwise} \end{cases}$$

defines a distribution functions? $\boxed{\text{3 MARKS}}$

(b) Let $\underline{X} = (X_1, X_2)$ be a bivariate random vector having the d.f.

$$F(x,y) = \begin{cases} 0, & \text{if } x < 0 \text{ or } y < 0 \\ \frac{1+xy}{2}, & \text{if } 0 \le x < 1, 0 \le y < 1 \\ \frac{1+x}{2}, & \text{if } 0 \le x < 1, y \ge 1 \\ \frac{1+y}{2}, & \text{if } x \ge 1, 0 \le y < 1 \\ 1 & \text{if } x \ge 1, y \ge 1 \end{cases}.$$

(i) Find the marginal distribution function of X_1;

(ii) Find $Pr(\frac{1}{2} \le X_1 \le 1, \frac{1}{4} < X_2 < \frac{1}{2})$;

(iii) Find $Pr(X_1 = 0, X_2 = 1)$.

$\boxed{\text{2+2+3=7 MARKS}}$

6. Let (X, Y) be a r.v. with p.m.f.

$$f(x,y) = \begin{cases} cy, & \text{if } x \in \{1,2,3\}, y \in \{1,2,3,4\}, \ \& \ x \le y \\ 0, & \text{otherwise} \end{cases}.$$

(a) Find the value of constant c;

(b) Find the marginal p.m.f.s of X and Y;

(c) Are X and Y independent?

(d) Find $Pr(X + Y > 4)$.

$\boxed{\text{2+4+2+2=10 MARKS}}$

MSO201a: Probability and Statistics

Summer Term: 2019

Mid Semester Examination

Model Solutions

Problem No. 1

(a) Define events

A_i: i-th man gets his hat $i=1,2,\ldots,n$

Then

Required probability $= P(\bar{A}_1 \cap \bar{A}_2 \cap \ldots \cap \bar{A}_n)$

$= 1 - P(A_1 \cup A_2 \cup \ldots \cup A_n)$

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = \sum_{i=1}^{n} P(A_i) - \sum\sum_{1 \le i < j \le n} P(A_i \cap A_j) + \cdots + (-1)^{k-1} \sum_{1 \le i_1 < \cdots < i_k \le n} \cdots \sum P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) + \cdots + (-1)^{n-1} P(A_1 \cap A_2 \cap \cdots \cap A_n)$$

For $1 \le i_1 < i_2 < \cdots < i_k \le n$ $(k=1,2,\ldots,n)$

$$P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) = \frac{\lfloor n-k}{\lfloor n}, \quad k=1,2,\ldots,n$$

Therefore

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = \binom{n}{1} \frac{\lfloor n-1}{\lfloor n} - \binom{n}{2} \frac{\lfloor n-2}{\lfloor n} + \cdots + (-1)^{k-1} \binom{n}{k} \frac{\lfloor n-k}{\lfloor n} + \cdots + (-1)^{n-1} \frac{1}{\lfloor n} \quad \ldots \boxed{3 \text{ MARKS}}$$

$$= 1 - \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \cdots + (-1)^{k-1} \frac{1}{\lfloor k} + \cdots + (-1)^{n-1} \frac{1}{\lfloor n}$$

$$\text{Required probability} = \frac{1}{\lfloor 2} - \frac{1}{\lfloor 3} + \cdots + (-1)^{k} \frac{1}{\lfloor k} + \cdots + (-1)^{n} \frac{1}{\lfloor n} \quad \ldots \boxed{2 \text{ MARKS}}$$

(b) From (a) it is clear that in a group of m people if their hats are distributed to them after mixing, the number of ways in which no one gets his hat is

$$\lfloor m \left(\frac{1}{\lfloor 2} - \frac{1}{\lfloor 3} + \cdots + \frac{(-1)^m}{\lfloor m} \right). \quad \ldots \boxed{3 \text{ MARKS}}$$

Thus

$$\Pr(k \text{ men get their hat}) = \frac{\binom{n}{k} \lfloor n-k \left(\frac{1}{\lfloor 2} - \frac{1}{\lfloor 3} + \cdots + \frac{(-1)^{n-k}}{\lfloor n-k} \right)}{\lfloor n}$$

$$= \frac{1}{\lfloor k} \left(\frac{1}{\lfloor 2} - \frac{1}{\lfloor 3} + \cdots + \frac{(-1)^{n-k}}{\lfloor n-k} \right)$$

$\ldots \boxed{2 \text{ MARKS}}$

— o —

$\boxed{\text{Problem No. 2}}$ Clearly F is differentiable everywhere except at $x = 0, 1, 2$; with

$$F'(x) = \begin{cases} \frac{1}{4}, & 0 < x < 1 \\ \frac{3}{2}, & 1 < x < 2 \\ 0, & x < 0 \text{ or } x > 2 \end{cases}$$

... $\boxed{2 \text{ MARKS}}$

and $\int_{-\infty}^{\infty} F'(x)\,dx = \frac{1}{4} + \frac{3}{4} = 1$

Thus X is a continuous r.v. with a p.d.f.

$$f(x) = \begin{cases} \frac{1}{4}, & 0 \le x < 1 \\ \frac{3}{2}, & 1 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$$

$S = [0, 2)$; $g(x) = \sqrt{x}$ is strictly increasing in $(0, 2)$ with $g^{-1}(y) = y^2$.

$g(S) = [0, \sqrt{2})$. The p.d.f. of $Y = \sqrt{X}$ is

$$g(y) = f(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| I_{[0, \sqrt{2})}(y)$$

$$= f(y^2)\,|2y|\,I_{[0, \sqrt{2})}(y)$$

$$= \begin{cases} \frac{y}{2}, & 0 \le y < 1 \\ y^3 & 1 \le y < \sqrt{2} \\ 0, & \text{otherwise} \end{cases}$$

... $\boxed{3 \text{ MARKS}}$

$$E(Y^2) = \int_0^1 \frac{y^3}{2}\,dy + \int_1^{\sqrt{2}} y^5\,dy = \frac{31}{24}$$

... $\boxed{2 \text{ MARKS}}$

$$E(Y^4) = \int_0^1 \frac{y^5}{2}\,dy + \int_1^{\sqrt{2}} y^7\,dy = \frac{47}{24}$$

$$\mathrm{Var}(Y^2) = E(Y^4) - (E(Y^2))^2 = \frac{47}{24} - \left(\frac{31}{24}\right)^2 = \frac{167}{576}$$

... $\boxed{3 \text{ MARKS}}$

—o—

Problem No. 3

Clearly X has the p.m.f.

$$f(x) = \begin{cases} \frac{1}{8}, & x=-1 \\ \frac{1}{4}, & x=0 \\ \frac{5}{8}, & x=2 \\ 0, & \text{o.w.} \end{cases}$$

The m.g.f. of $Y=|X|$ is

$$M_Y(t) = E(e^{tY}) = E(e^{t|X|})$$
$$= \frac{e^t}{8} + \frac{1}{4} + \frac{5}{8}e^{2t}, \quad t \in \mathbb{R} \quad \ldots \boxed{5 \text{ MARKS}}$$

$$M_Y^{(1)}(t) = \frac{e^t}{8} + \frac{5}{4}e^{2t}, \quad M_Y^{(2)}(t) = \frac{e^t}{8} + \frac{5}{2}e^{2t}$$
$$\ldots \boxed{2 \text{ MARKS}}$$

$$E(Y) = M_Y^{(1)}(0) = \frac{1}{8} + \frac{5}{4} = \frac{11}{8}$$

$$E(Y^2) = M_Y^{(2)}(0) = \frac{1}{8} + \frac{5}{2} = \frac{21}{8}$$

$$Var(Y) = E(Y^2) - (E(Y))^2 = \frac{21}{8} - \frac{121}{64} = \frac{47}{64} \quad \ldots \boxed{3 \text{ MARKS}}$$

Alt:
$$M_Y(t) = \frac{1}{4} + \frac{1}{8}\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right) + \frac{5}{8}\left(1 + 2t + \frac{4t^2}{2!} + \frac{8t^3}{3!} + \cdots\right)$$
$$= 1 + \frac{11}{8}t + \frac{21}{8}\frac{t^2}{2!} + \frac{41}{8}\frac{t^3}{3!} + \cdots, \quad t \in \mathbb{R}$$
$$\boxed{2 \text{ MARKS}}$$

$\Rightarrow$ $E(Y) =$ coefficient of t in $M_Y(t) = \frac{11}{8}$

$E(Y^2) =$ coefficient of $\frac{t^2}{2!}$ in $M_Y(t) = \frac{21}{8}$

$$Var(Y) = E(Y^2) - (E(Y))^2 = \frac{21}{8} - \frac{121}{64} = \frac{147}{64}. \quad \boxed{3 \text{ MARKS}}$$

— o —

Problem No. 4

(a) $\mu = E(X) = -2$, $\sigma^2 = Var(X) = 5 - 4 = 1$. Thus

$$P(|X-\mu| < k\sigma) \geq 1 - \frac{1}{k^2}$$ ... $\boxed{\text{2 MARKS}}$

$$P(\mu - k\sigma < X < \mu + k\sigma) \geq 1 - \frac{1}{k^2}$$

$$P(-2-k < X < -2+k) \geq 1 - \frac{1}{k^2}$$

For $k=3$, we get

$$P(-5 < X < 1) \geq 1 - \frac{1}{9} = \frac{8}{9}$$ ... $\boxed{\text{3 MARKS}}$

(b) For $\alpha \geq 1$, $g(x) = x^\alpha$ is a convex function on the support $S \subseteq [0, \infty)$ of r.v. X. Thus, by the Jensen inequality,

$$E(X^\alpha) \geq (E(X))^\alpha.$$ ... $\boxed{\text{2 MARKS}}$

Taking $X = Y^n$ in the last example (so that $Pr(X \geq 0) = Pr(Y \geq 0) = 1$), we get

$$E(Y^{n\alpha}) \geq (E(Y^n))^\alpha$$

Taking $\alpha = \frac{n+1}{n}$, we get

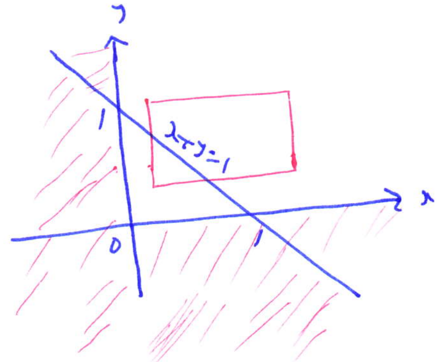
$$E(Y^{n+1}) \geq (E(Y^n))^{\frac{n+1}{n}}$$

$$\Rightarrow (E(Y^n))^{1/n} \leq (E(Y^{n+1}))^{\frac{1}{n+1}}, \quad n = 1, 2, \ldots$$ ... $\boxed{\text{3 MARKS}}$

—o—

Problem No. 5

(a) For rectangle $(\frac{1}{4}, 1] \times (\frac{1}{4}, 1]$



$$\Delta = F(1, 1) - F(\tfrac{1}{4}, 1) - F(1, \tfrac{1}{4}) + F(\tfrac{1}{4}, \tfrac{1}{4})$$

$$= -1 < 0$$

$\Rightarrow$ F is not a d.f. ... 3 MARKS

(b)

(i) $F_{X_1}(x) = \lim_{y \to \infty} F(x, y) = \begin{cases} 0, & x < 0 \\ \frac{1+x}{2}, & 0 \le x < 1 \\ 1, & x \ge 1 \end{cases}$ ... 2 MARKS

(ii) $Pr(\frac{1}{2} \le X_1 \le 1, \frac{1}{4} < X_2 < \frac{1}{2}) = F(1, \frac{1}{2}-) - F(\frac{1}{2}-, \frac{1}{2}-) - F(1, \frac{1}{4}) + F(\frac{1}{2}-, \frac{1}{4})$

$$= \frac{3}{4} - \frac{5}{8} - \frac{5}{8} + \frac{9}{16} = \frac{1}{16}$$ ... 2 MARKS

(iii) $Pr(X_1 = 0, X_2 = 1) = Pr(X_1 = 0, X_2 \le 1) - Pr(X_1 = 0, X_2 < 1)$

$$= P(X_1 \le 0, X_2 \le 1) - P(X_1 < 0, X_2 \le 1) - P(X_1 \le 0, X_2 < 1) + P(X_1 < 0, X_2 < 1)$$

$$= F(0, 1) - F(0-, 1) - F(0, 1-) + F(0-, 1-)$$

$$= \frac{1}{2} - 0 - \frac{1}{2} + 0 = 0$$ ... 3 MARKS

—o—

Problem No. 6

(a) $S = \{(x,y) : (x,y) \in \{1,2,3\} \times \{1,2,3,4\},\ x \le y\}$

$= \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4)\}$

$$\sum_{(x,y)\in S} f(x,y) = 1 \Rightarrow c \sum_{(x,y)\in S} y = 1$$

$$\Rightarrow c[1+2+3+4+2+3+4+3+4] = 1 \Rightarrow c = \frac{1}{26}$$ ... 2 MARKS

(b) For $x \in S_x = \{1,2,3\}$

$$f_X(x) = \sum_{y=x}^{4} \frac{y}{26} = \frac{(5-x)(x+4)}{52}$$

Thus

$$f_X(x) = \begin{cases} \frac{(5-x)(x+4)}{52}, & x = 1,2,3 \\ 0, & \text{o.w.} \end{cases} = \begin{cases} \frac{5}{13}, & x=1 \\ \frac{9}{26}, & x=2 \\ \frac{7}{26}, & x=3 \\ 0, & \text{o.w.} \end{cases}$$ ... 2 MARKS

For $y \in S_y = \{1,2,3,4\}$

$$f_Y(y) = \sum_{x=1}^{\min\{3,y\}} \frac{y}{26} = \frac{y \min\{3,y\}}{26}$$

Thus

$$f_Y(y) = \begin{cases} \frac{y^2}{26}, & y = 1,2,3 \\ \frac{3y}{26}, & y=4 \\ 0, & \text{o.w.} \end{cases} = \begin{cases} \frac{1}{26}, & y=1 \\ \frac{2}{13}, & y=2 \\ \frac{9}{26}, & y=3 \\ \frac{6}{13}, & y=4 \\ 0, & \text{o.w.} \end{cases}$$ ... 2 MARKS

(c) $S \neq S_x \times S_y \Rightarrow X$ and Y are not independent. ... 2 MARKS

(d) $\Pr(X+Y>4) = \Pr(X=1, Y>3) + P(X=2, Y>2) + \Pr(X=3, Y\ge 3)$

$= \Pr(X=1, Y=4) + \Pr(X=2, Y=3) + \Pr(X=2, Y=4) + \Pr(X=3, Y=3) + \Pr(X=3, Y=4)$

$= \frac{1}{26}[4+3+4+3+4] = \frac{9}{13}$ ... 2 MARKS

—o—