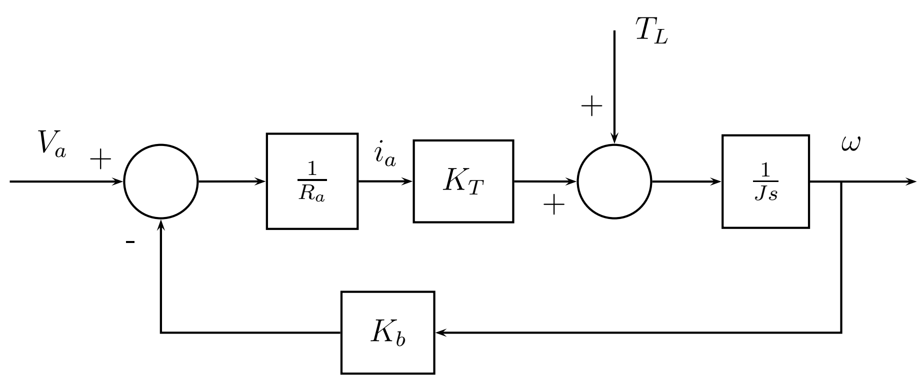
Lecture Note: 3
S-plane analysis - Part I

1 Need for feedback control

1. **Disturbance Rejection:**

Example 1 (DC motor speed control)**.**

$$\frac{\omega(s)}{T_L(s)} = \frac{\frac{1}{Js}}{1+\frac{K_T K_b}{R_a Js}} = \frac{1}{R_a Js + K_T K_b} \tag{1}$$



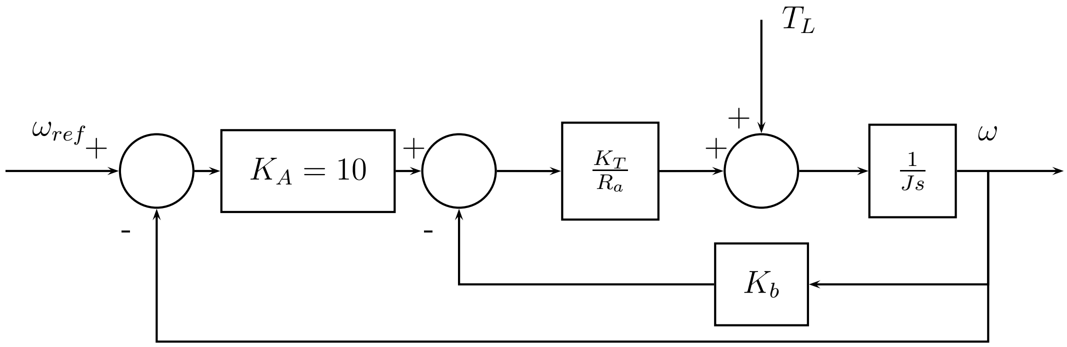
Given, $K_T = 1.5$, $R_a = 1\Omega$ and $J = 1$, we have

$$\frac{\omega(s)}{T_L(s)} = \frac{1}{s+2.25}$$

For a unit step disturbance torque $T_L(s) = \frac{1}{s}$, output is evaluated as follows:

$$\begin{aligned} \omega(s) &= \frac{1}{s(s+2.25)} \\ \Rightarrow w(t) &= 0.444(1-e^{-2.25t}) \end{aligned}$$

The steady state value of the output is $\omega_{ss} = 0.444$. When feedback is introduced, the above block diagram becomes



Now, the closed loop transfer function between $\omega(s)$ and $T_L(s)$ turns out to be

$$\frac{\omega(s)}{T_L(s)} = \frac{1}{s + 17.25} \tag{2}$$

The response for a unit step disturbance torque is computed as follows:

$$\begin{aligned}\omega(s) &= \frac{1}{s(s + 17.25)} \\ \Rightarrow w(t) &= 0.0571(1 - e^{-17.25t})\end{aligned}$$

with a steady state value $\omega_{ss} = 0.0571$. If the block K_A is replaced by an integrator $G_c(s) = \frac{10}{s}$, we have

$$\begin{aligned}\frac{\omega(s)}{T_L(s)} &= \frac{\frac{1}{Js}}{1 + \frac{K_T K_b}{R_a J s} + \frac{K_T K_A}{R_a J s}} \\ &= \frac{s}{s^2 + 2.25s + 15}\end{aligned}$$

The response for a unit step disturbance becomes

$$\begin{aligned}\omega(s) &= \frac{1}{s^2 + 2.25s + 15} \\ \Rightarrow \omega(t) &= 0.2698e^{-1.125t} sin(3.7059t)\end{aligned}$$

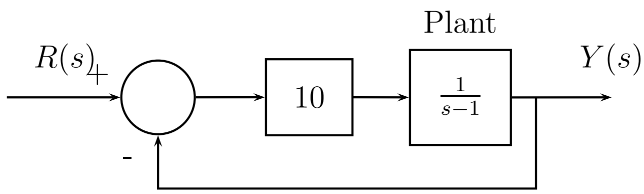
with a steady state error $\omega_{ss} = 0$. This example demonstrates that feedback can be used to eliminate the effect of external disturbances.

■

2. **Reduces the effect of parameter uncertainties** Please solve problems 3 and 4 to understand this aspect.

3. **Provides stability to an open-loop unstable system**

   Example 2. Consider the following block diagram where the plant is unstable.



   The closed loop transfer function is obtained as

$$\frac{Y(s)}{R(s)} = \frac{10}{s+9}$$

   which is a stable system.

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2 Poles and zeros

A generic transfer function of a plant is given as

$$G(s) = \frac{n(s)}{d(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \ldots + b_{m-1}s + b_m}{s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n} \tag{3}$$

where $m \leq n$.

Definition 1 (Poles)**.** Poles are given by the roots of denominator polynomial $d(s)$, i.e., $d(s) = 0$.

Definition 2 (Zeros)**.** Zeros are given by roots of polynomial $n(s)$, i.e., $n(s) = 0$. Zeros are the points in s-plane where the transfer function becomes zero.

Example 3. Consider the following transfer function

$$G(s) = \frac{s+3}{(s-4)(s^2+12s+52)}$$

It has a single zero, $s = -3$ and three poles, $s = 4$, $s = -6 \pm 4j$.

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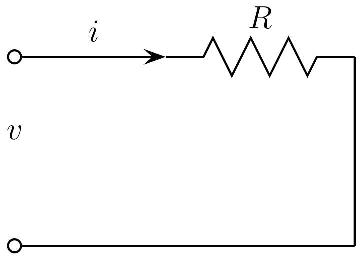
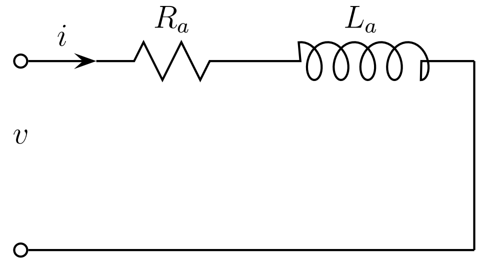
Origin of Poles:

Poles represent internal dynamics of a system.



$\frac{I}{V} = \frac{1}{R}$. Thus, there is no dynamics.

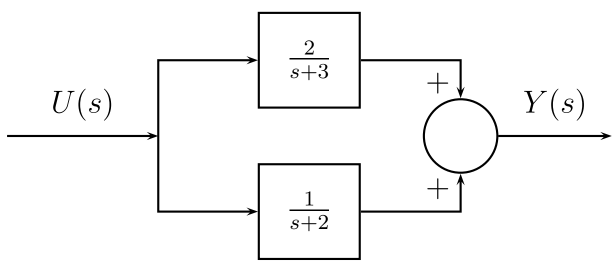


$\frac{I}{V}(s) = \frac{1}{R_a + L_a s}$. It has a single pole at $s = -\frac{R_a}{L_a}$.

Consider a first order system $G(s) = \frac{1}{1+\tau s}$. It has a pole at $s = -\frac{1}{\tau}$, then its impulse response is given as $g(t) = e^{-t/\tau}$.

Origin of Zero:

Let us consider the following block diagram where each individual subsystem has a single pole. However the overall transfer function has a zero.



The transfer function between $Y(s)$ an $U(s)$ is given by

$$\frac{Y(s)}{U(s)} = \frac{s+1}{(s+2)(s+3)}$$

Zero can affect input Signal
Let us consider a transfer function

$$G(s) = \frac{Y(s)}{U(s)} = \frac{s+2}{s(s+3)}$$

Let $U(s) = \frac{1}{s+2}$, i.e., input signal has a pole at the 'zero' of $G(s)$. Then $Y(s) = \frac{1}{s(s+3)}$, i.e., input signal does not appear at the output.Here is another example to make this point very clear.

Example 4 (Electrical Circuit)**.** The transfer function is

$$G(s) = \frac{V_o}{V_i} = \frac{s+1}{(s+2)(s+3)}$$

Let $V_i = \frac{10}{s}$, then

$$\begin{aligned} V_o(s) &= \frac{s+1}{s(s+2)(s+3)} = \frac{1}{6s} + \frac{1}{2(s+2)} - \frac{2}{3(s+3)} \\ \Rightarrow v_o(t) &= \frac{1}{6}(1 + 3e^{-2t} - 4e^{-3t}), \quad t > 0 \end{aligned}$$

Thus, the output voltage contains three components

1. due to input term $\frac{1}{s}$.
2. due to pole at $s = -2$ and
3. due to pole at $s = -3$.

Take $v_i(t) = 10e^{-t}$, i.e., $V_i(s) = \frac{10}{s+1}$. Now, the output is given by

$$\begin{aligned} V_o(s) &= \frac{10}{(s+2)(s+3)} = \frac{10}{s+2} - \frac{10}{s+3} \\ \Rightarrow v_o(t) &= 10e^{-2t} - 10e^{-3t} \end{aligned}$$

Thus, the input signal $v_i(t) = 10e^{-t}$ appear to be lost inside the circuit. Hence, a zero can block the transmission of a signal.

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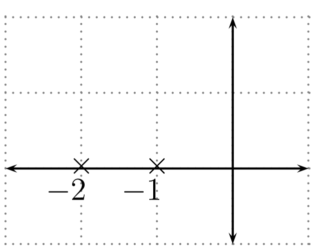
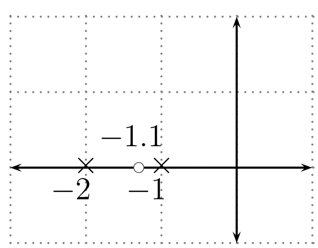
Zero can reduce the effect of a Pole





$$\begin{aligned} H_1(s) &= \frac{2}{(s+1)(s+2)} \\ &= \frac{2}{s+1} - \frac{2}{s+2} \\ or,\ h_1(t) &= 2e^{-t} - 2e^{-2t} \end{aligned}$$

$$\begin{aligned} H_2(s) &= \frac{2(s+1.1)}{1.1(s+1)(s+2)} \\ &= \frac{0.18}{s+1} + \frac{1.64}{s+2} \\ or,\ h_2(t) &= 0.18e^{-t} + 1.64e^{-2t} \end{aligned}$$

Conclusion:

1. Zero at $s = -1.1$ reduces the effect of the pole near to it ($s = -1$) drastically
2. We can use a zero to cancel a pole completely if the pole is in the left half of the s-plane.

3 Physical systems vs Simulators

All physical devices are causal by nature. An aircraft, an automobile, and a DC servo motor are examples of physical devices. By definition, an input u at time $t = \tau$, can have influence on the response of the system y only at $t \geq \tau$. All these systems will have transfer functions $G(s) = \frac{N(s)}{D(s)}$, where the denominator order is greater than or equal to numerator order.

Any system whether causal or non-causal can be simulated using analogue circuits or digital circuits - respectively called as analogue computer or digital computer. In the class, we have shown how to simulate following transfer functions:

$$G(s) = \frac{s+2}{s^3+2s^2+3s+2} \tag{4}$$

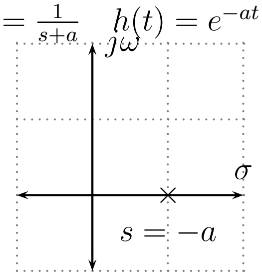
$$G(s) = \frac{s^2+2s+3}{s+1} \tag{5}$$

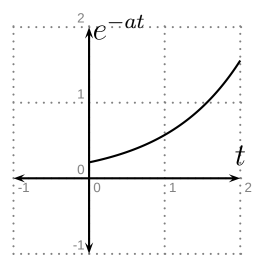
Exercise: Realize these transfer functions using analogue computers.

4 First and Second order Systems

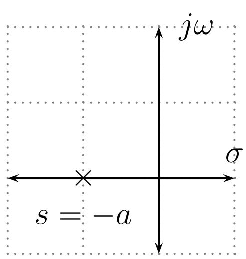
First order response

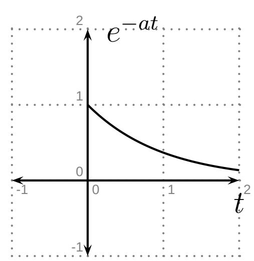
$H(s) = \frac{1}{s+a}$ $h(t) = e^{-at}$



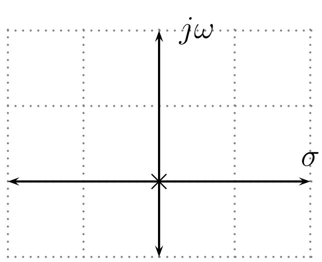


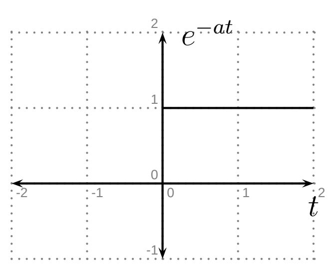
$a < 0$ $-a$ is positive





$a > 0$ $-a$ is negative





$a = 0$ $a = 0$

$\tau = \frac{1}{a}$ is the **time constant**, the time when the response is 63.2% of the initial value.

Response of a Standard 2nd order System

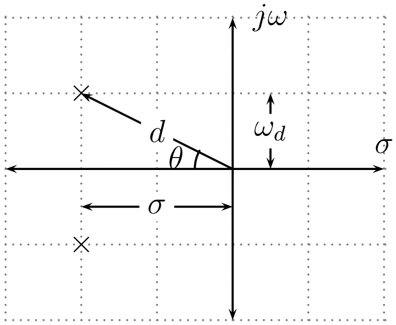
The transfer function of a second order system is often given as follows:

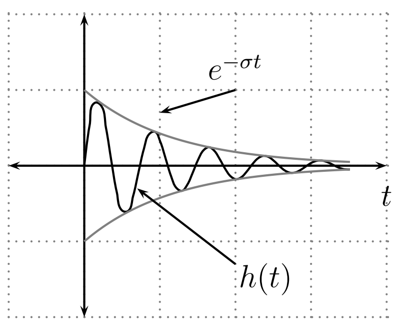
$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

It has two poles, $s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$. Here ω_n is termed as natural frequency while ζ is known as damping ratio or damping coefficient. Taking inverse Laplace transform, we get the following *impulse* response:

$$h(t) = \mathscr{L}^{-1}H(s) = \frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\sigma t}sin(\omega_d t)u(t)$$

where $\sigma = \zeta\omega_n$ and $\omega_d = \omega_n\sqrt{1-\zeta^2}$. ω_d is also called as frequency of transient oscillation.





$$\begin{aligned}
d &= \sqrt{\zeta^2\omega_n^2 + \omega_n^2(1-\zeta^2)} \\
&= \omega_n^2 \\
cos\ \theta &= \frac{\zeta\omega_n}{\omega_n} \ = \ \zeta \\
\theta &= cos^{-1}\zeta
\end{aligned}$$

Step response: The output of the system with above transfer function is given by

$$\begin{aligned}
Y(s) &= H(s)\frac{1}{s} \\
e9 + or,\ y(t) &= 1 - e^{-\zeta\omega_n t}(cos\ \omega_d t + \frac{\zeta\omega_n}{\omega_d}sin\ \omega_d t) \\
&= \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}sin(\omega_d t + \theta) \\
where\quad \theta &= cos^{-1}\zeta
\end{aligned}$$

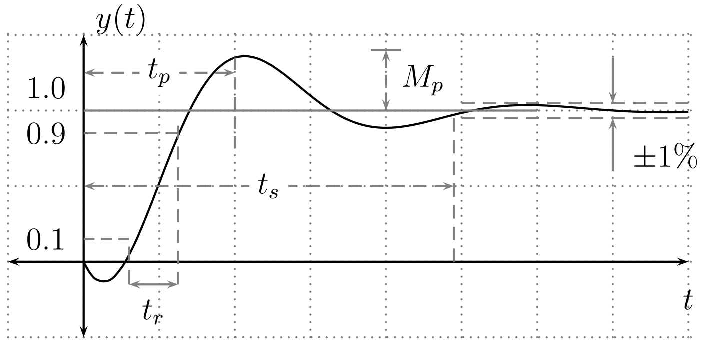


Figure 1: Time response of a second order system

Rise time (t_r): It is the time required by the output to go from 10% to 90% of the final value. The time it takes the response to reach the value 1 can be considered to be upper bound of t_r. This value can be obtained by making the sin term of the response vanish (zero).

$$\begin{aligned}
1 - e^{-\zeta\omega_n t_r}(cos\ \omega_d t_r + \frac{\zeta\omega_n}{\omega_d} sin\ \omega_d t_r) &= 1 \\
sin(\omega_n\sqrt{1-\zeta^2}t_r + \theta) &= sin\pi = 0 \\
\omega_n\sqrt{1-\zeta^2}t_r + \theta &= \pi
\end{aligned}$$

$$t_r = \frac{\pi - cos^{-1}\zeta}{\omega_n\sqrt{1-\zeta^2}} \quad (maximum \quad value)$$

This is an upperbound. The actual value is obtained through empirical observation. For a system with $\zeta = 0.5$, t_r can be approximated as $t_r = \frac{1.8}{\omega_n}$.

Settling time (t_s): It is the time taken by the output to enter ±1% band around the final value.

$$\begin{aligned}
t_s &= \frac{4.6}{\zeta\omega_n} \quad \text{for} \quad \pm 1\%\ band \\
&= \frac{4}{\zeta\omega_n} \quad \text{for} \quad \pm 2\%\ band
\end{aligned}$$

Peak Time (t_p): The time taken by the output to reach the maximum overshoot point.

$$t_p = \frac{\pi}{\omega_d}$$

The output of the system is given by

$$y(t) = 1 - e^{-\sigma t}\sqrt{1 + \frac{\sigma^2}{\omega_d^2}} cos(\omega_d t - \beta)$$

At peak overshoot, $\dot{y}(t) = 0$

$$\dot{y} = \frac{\zeta\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n\ t} sin(\omega_n\sqrt{1-\zeta^2}\ t + \theta) - \frac{e^{-\zeta\omega_n t}}{1-\zeta^2} cos(\omega_n\sqrt{1-\zeta^2}t + \theta)$$

$$= \omega_n \frac{e^{-\zeta\omega_n t}}{1-\zeta^2}(sin(\omega_d t+\theta)cos(\theta) - cos(\omega_d t+\theta)sin(\theta)) = 0$$

$$\Rightarrow \ \omega_n \frac{e^{-\zeta\omega_n t}}{1-\zeta^2}(sin(\omega_d t)) = 0$$
$$which\ gives$$
$$\omega_d t_p = \pi$$
$$\Rightarrow \ t_p = \frac{\pi}{\omega_d}$$

Peak Overshoot (M_p): It is the ratio of maximum amount of the overshoot of the output over its final value to its final value. The maximum value is given by

$$\begin{aligned} y(t_p) &= 1 - e^{-\sigma t}(cos\ \omega_d t + \frac{\sigma}{\omega_d} sin\ \omega_d t)\Big|_{t=t_p} \\ &= 1 - e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}\left(cos\pi + \frac{\sigma}{\omega_d} sin\ \pi\right) \\ &= 1 + e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \end{aligned}$$

Thus, the maximum overshoot is given as

$$M_p = \frac{y(t_p)-1}{1} = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}$$

Please note that the peak overshoot is ONLY a function of damping coefficient ζ.

Example 5. Consider the following position control system where $K = 10$, $R_a = 1\Omega$, $K_T = 1.5N - m/amp$, $J = 5Kg - m^2$ and $B = 1.75N - ms/rad$.

(a) Determine the frequency of the transient oscillation, the peak overshoot, the peak time, settling time and the steady state error when the command signal θ_R is a unity step.

(b) Determine the steady state error when the command signal $\theta_R = t$, a ramp signal.

(c) Determine the steady state error when a step disturbance torque T_L is applied.

Solution: The first part of solution would be to find out the transfer functions between output $\theta(s)$ and inputs $\theta_R(s)$ and T_L,

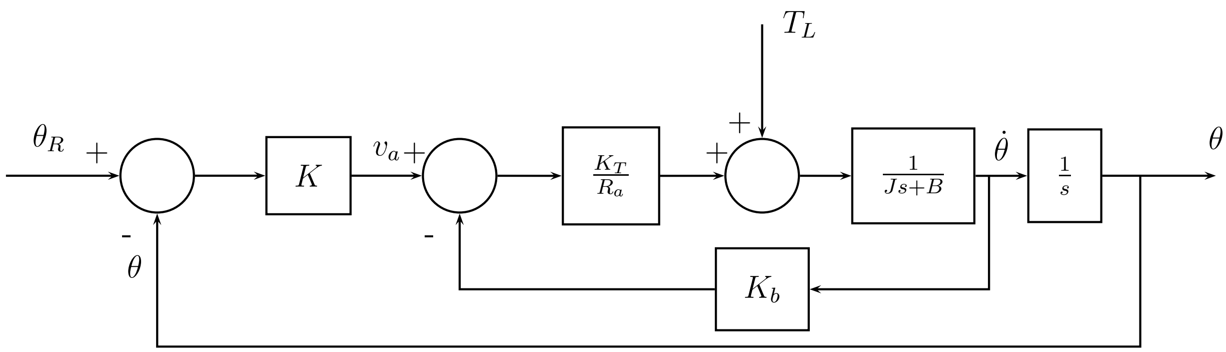


Figure 2: Position Control System

$$
\begin{aligned}
&\textit{The forward gain between } \theta(s) \textit{ and } \theta_R(s) \textit{ is} \\
M_1 &= \frac{15}{s(5s+1.75)} \\
&\textit{The forward gain between } \theta(s) \textit{ and } T_L(s) \textit{ is} \\
M_2 &= \frac{1}{s(5s+1.75)} \\
&\textit{The loop gains are as follows}: \\
L_{11} &= \frac{-2.25}{(5s+1.75)} \\
L_{21} &= \frac{-15}{(5s+1.75)} \\
Thus, \frac{\theta(s)}{\theta_R(s)} &= \frac{15}{(5s^2+4s+15)} = \frac{3}{s^2+0.8s+3} \\
and\ \frac{\theta(s)}{T_L(s)} &= \frac{1}{(5s^2+4s+15)}
\end{aligned}
$$

Now, the transfer function between $\theta(s)$ and $\theta_R(s)$ is given by

$$
\begin{aligned}
\frac{\theta(s)}{\theta_R(s)} &= \frac{3}{s^2+0.8s+3} \\
&= \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}
\end{aligned}
$$

where the parameter $\omega_n = \sqrt{3} = 1.732$ and $2\zeta\omega_n = 0.8$ which gives $\zeta = 0.2309$.

(a) The frequency of transient oscillation

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 1.6852$$

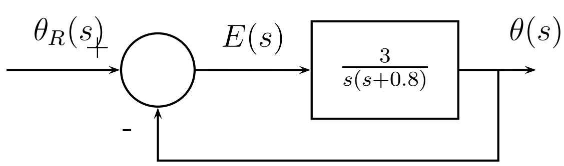
The peak time $t_p = \frac{\pi}{\omega_d} = 1.8645$, the peak overshoot

$$M_p = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} = 0.4744$$

The steady state value of the output is given to be $\theta_{ss} = \lim_{s\to 0} s\frac{3}{s^2+0.8s+3} = 1$ and thus the steady state error is $e_{ss} = 1 - \theta_{ss} = 0$.

$$The\ settling\ time\ t_s = \frac{4}{\zeta\omega_n} = \frac{4}{0.2309 * 1.732} = 10sec$$

(b) $\theta_R(s) = \frac{1}{s^2}$. Please see that you can't apply the same procedure.



$$\begin{aligned} E &= \theta_R - \frac{3}{s(s+0.8)}E \\ or,\ E(s) &= \frac{1}{1+\frac{3}{s(s+0.8)}}\theta_R(s) \end{aligned}$$

where $\theta_R(s) = \frac{1}{s^2}$. Therefore, the steady state error

$$\begin{aligned} E_{ss} &= \lim_{s\to 0} sE(s) \\ &= \lim_{s\to 0} \frac{1}{s+\frac{3}{s+0.8}} \\ &= 0.266\ radian \end{aligned}$$

(c) The transfer function between $\theta(s)$ and $T_L(s)$ is given by

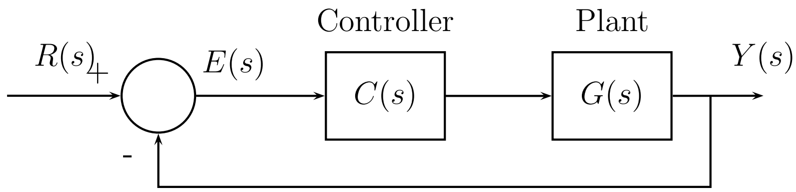
$$\frac{\theta(s)}{T_L(s)} = \frac{1}{5s^2+4s+15}$$

For a unit step disturbance, the steady state value of output is given as

$$\theta_{ss} = \lim_{s\to 0} s\frac{1}{5s^2+4s+15}\frac{1}{s} = \frac{1}{15} = 0.067$$

□

4.1 Steady state error computation



The transfer function between $Y(s)$ and $R(s)$ is given by

$$\frac{Y(s)}{R(s)} = \frac{C(s)G(s)}{1 + C(s)G(s)}$$

The error

$$\begin{aligned} E(s) &= R(s) - Y(s) \\ &= R(s) - \frac{C(s)G(s)}{1 + C(s)G(s)} R(s) \\ &= \frac{R(s)}{1 + C(s)G(s)} \\ \therefore \; E_{ss} &= \lim_{s \to 0} sE(s) \end{aligned}$$

Define the following constants

- K_p (position error constant) $= \lim_{s\to 0} C(s)G(s)$
- K_v (velocity error constant) $= \lim_{s\to 0} sC(s)G(s)$
- K_a (acceleration error constant) $= \lim_{s\to 0} s^2C(s)G(s)$

Please verify this table and satisfy yourself:

	Input		
Type of $DG(s)$	Step	Ramp	Parabola
type 0	$\frac{1}{1+K_p}$	∞	∞
type 1	0	$\frac{1}{K_v}$	∞
type 2	0	0	$\frac{1}{K_a}$

Table 1: Steady State Error Values(E_{ss})

Eg (i): when G(s) is a Type 0 system

$$
\begin{aligned}
If\ \ C(s) &= 1\ \ and\ \ G(s) = \frac{1}{(s+2)} \\
E(s) &= \frac{R(s)}{1+C(s)G(s)} \\
&= \frac{\frac{1}{s}}{1+\frac{1}{s+2}} \\
Hence,\ e_{ss} &= \lim_{s\to 0} sE(s) = \frac{2}{3} = \frac{1}{1+K_p} = \frac{1}{1+\frac{1}{2}} \\
where\ \ K_p &= \lim_{s\to 0} C(s)G(s) = \lim_{s\to 0}\frac{1}{(s+2)} = \frac{1}{2}
\end{aligned}
$$

Eg (ii): when G(s) is a Type 1 system

$$
\begin{aligned}
If\ \ C(s) &= 1\ \ and\ \ G(s) = \frac{1}{s(s+2)} \quad R(s) = \frac{1}{s^2} \\
E(s) &= \frac{\frac{1}{s^2}}{1+\frac{1}{s(s+2)}} = \frac{s+2}{s^2+2s+1} = 2 \\
K_v &= \lim_{s\to 0} sC(s)G(s) = \lim_{s\to 0}\frac{s}{s(s+2)} = \frac{1}{2} \\
Hence,\ e_{ss} &= \tfrac{1}{K_v}
\end{aligned}
$$

5 Stability of Linear Time Invariant Systems

$$
\begin{aligned}
H(s) &= \frac{Y(s)}{R(s)} \\
&= \frac{b_o s^m + b_1 s^{m-1} + \ldots + b_m}{s^n + a_1 s^{n-1} + \ldots + a_n}, \quad m \le n && (6) \\
&= \frac{k\prod_{i=1}^{m}(s-z_i)}{\prod_{j=1}^{n}(s-p_j)} = \sum_i \frac{K_i}{s-p_i} && (7) \\
or,\ h(t) &= \sum_i K_i e^{p_i t} && (8)
\end{aligned}
$$

The system is **stable** if and only if every term in (8) go to zero as $t \to \infty$, i.e.,

$$e^{p_i t} \to 0 \quad \forall p_i \quad \text{as} \quad t \to \infty$$

This will happen if all the poles of the system are strictly in the left half of the s-plane, i.e., $Re\{p_i\} < 0$. This is called *internal stability*. The system is **unstable** if there exists a pole for which $Re\{p_i\} > 0$.

Neutral/Marginal stability:

The system is said to be marginally stable if it has *non-repeated* roots on $j\omega$ axis. This implies that

(i) A pole at origin will result in a non-decaying transient, i.e., $\mathscr{L}^{-1}[\frac{1}{s}] = 1$.

(ii) A pair of conjugate pole $\pm j\omega$ will result in oscillatory response with constant magnitude.

However, repeated poles on $j\omega$ axis makes the system unstable. Examples of such unstable systems are

$$\begin{array}{ll} G(s) = \frac{1}{s^2}, & h(t) = t \\ G(s) = \frac{4^2}{(s^2+4)^2}, & h(t) = -2t\ cos\ 2t + sin\ 2t \end{array}$$

5.1 Routh's Stability Criterion

The *characteristic polynomial* of a transfer function is represented as

$$d(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_{n-1} s + a_n \tag{9}$$

For stability, it is desired that all the roots of $d(s)$ have negative real parts. Without explicitly computing roots of polynomial (9), can we comment about the nature of roots? Routh's stability criterion helps us to do that. This criterion consists of following two conjectures:

1. The **necessary** (but not sufficient) condition for stability is that all the coefficients of the characteristic polynomial $d(s)$ be positive.

2. Sufficient condition for stability:A system is stable if and only if all the elements in the first column of the Routh array are positive.

How do we obtain a Routh array? Consider the characteristic polynomial given in (9). Routh array follows as:

$$\begin{array}{ccccc} s^n & 1 & a_2 & a_4 & \ldots \\ s^{n-1} & a_1 & a_3 & a_5 & \ldots \\ s^{n-2} & b_1 & b_2 & b_3 & \ldots \\ s^{n-3} & c_1 & c_2 & c_3 & \ldots \\ \vdots & & & & \\ s^0 & & & & \end{array}$$

For a system of order 'n', the Routh array has $n+1$ rows, where

$$b_1 = -\frac{\begin{vmatrix} 1 & a_2 \\ a_1 & a_2 \end{vmatrix}}{a_1} = \frac{a_1 a_2 - a_3}{a_1}$$
$$b_1 = -\frac{\begin{vmatrix} 1 & a_4 \\ a_1 & a_5 \end{vmatrix}}{a_1} = \frac{a_1 a_4 - a_5}{a_1}$$
$$\vdots$$
$$c_1 = -\frac{\begin{vmatrix} a_1 & a_3 \\ b_1 & b_2 \end{vmatrix}}{b_1} = \frac{a_3 b_1 - a_1 b_2}{b_1}$$
$$c_2 = -\frac{\begin{vmatrix} a_1 & a_5 \\ b_1 & b_3 \end{vmatrix}}{b_1} = \frac{a_5 b_1 - a_1 b_3}{b_1}$$
$$\vdots$$

Example 6 (Stability)**.** The characteristic polynomial is given as

$$d(s) = s^6 + 4s^5 + 3s^4 + 2s^3 + s^2 + 4s + 4$$

Constructing the Routh table, we have

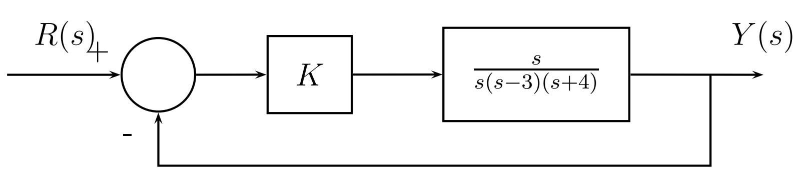
s^6	1	3	1	4
s^5	4	2	4	0
s^4	2.5	0	4	0
s^3	2	-12/5	0	0
s^2	3	4	0	0
s^1	-76/15	0	0	0
s^0	4	0	0	0

Since, the elements in the first column corresponding to row s^1 is negative, the system is unstable. The numbe of roots in the right half of the s-plane are given by the number of sign changes in the first column. In this example, the number of roots in the right half s-plane is 2.

■

5.2 Application of Routh array to find parameter range

Example 7. Consider the following system



Find the range of K for which the closed loop system is stable. The transfer function is given by

$$\frac{Y(s)}{R(s)} = \frac{K(s+2)}{s(s-3)(s+4)+K(s+2)} = \frac{n(s)}{d(s)}$$

The characteristic polynomial is given by

$$d(s) = s^3 + s^2 + (K-12)s + 2K$$

The Routh array is constructed as follows:

$$\begin{array}{ccc}
s^3 & 1 & \text{K-12} \\
s^2 & 1 & \text{2K} \\
s^1 & \text{-(K+12)} & 0 \\
s^0 & \text{2K} &
\end{array}$$

From s^1 row,

$$\begin{aligned}
-K-12 &> 0 \\
K+12 &< 0 \\
K &< -12 \qquad (10)
\end{aligned}$$

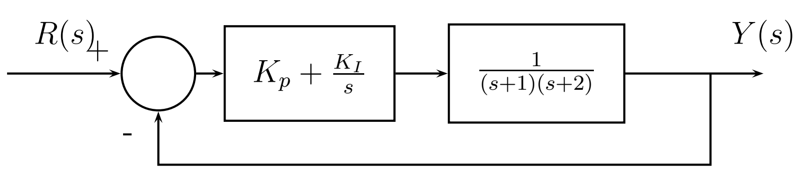
and from s^0 row, we have

$$K > 0 \qquad (11)$$

Since, equations (10) and (11) have no common solution, the system is unstable for all values of K.

■

Example 8. Consider the following system



The characteristic equation is given by

$$d(s) = s^3 + 3s^2 + (2+K_p)s + K_I = 0$$

The routh array is constructed as

$$\begin{array}{ccc}
s^3 & 1 & 2+K_p \\
s^2 & 3 & K_I \\
s^1 & \frac{6+3K_p-K_I}{3} & 0 \\
s^0 & K_I &
\end{array}$$

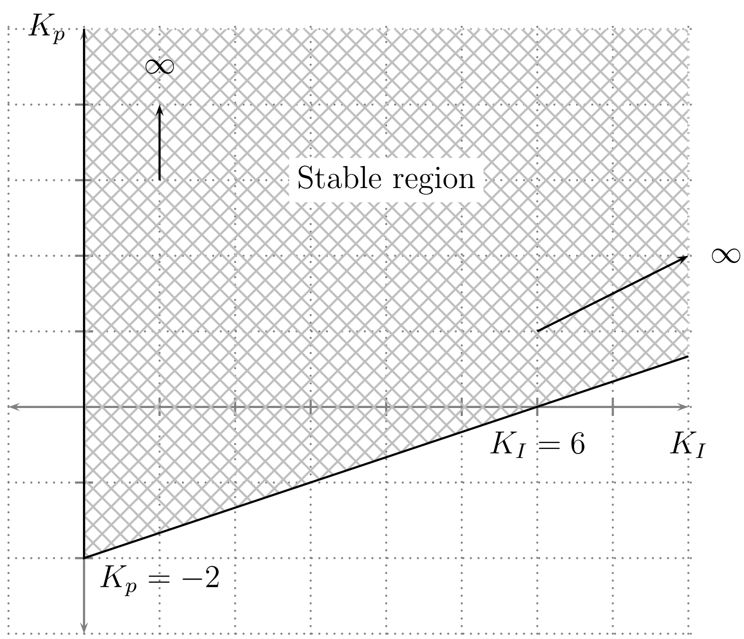
For stability,

(i) from s^1 row,

$$3K_p > K_I - 6$$
$$or, \quad K_p > \frac{K_I}{3} - 2$$

(ii) from s^0 row, $K_I > 0$.

The shaded region in the following figure indicates stable region.



■

Routh test can find if roots are on $j\omega$ axis. This happens when an entire row becomes zero.

Example 9 (C)**.** onsider the following characteristic polynomial

$$d(s) = s^5 + 5s^4 + 11s^3 + 23s^2 + 28s + 12$$

The corresponding Routh array

s^5	1	11	28
s^4	5	23	12
s^3	6.4	25.6	0
s^2	3	12	0
s^1	0	0	0
new s^1	6	0	
s^0	12		

The auxiliary equation

$$\begin{aligned} d_1(s) &= 3s^2 + 12 \\ \frac{d}{ds}d_1(s) &= 6s \end{aligned}$$

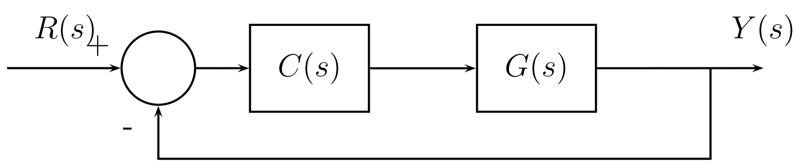
Since, all the elements in column 1 are positive, no roots are in the right half of s-plane. There exist a pair of conjugate poles on the $j\omega$ axis which can be obtained from the following equation

$$\begin{aligned} 3s^2 + 12 &= 0 \\ s^2 + 4 &= 0 \\ s_{1,2} &= \pm j2 \end{aligned}$$

■

6 Control Design in s-plane

Consider the system as shown below:



In s-plane analysis, the control problem may be stated as follows: *Given $G(s)$, design $C(s)$ such that the closed loop transfer function $\frac{C(s)G(s)}{1+C(s)G(s)}$ can be approximated by an ideal second order transfer function $\frac{{\omega_n}^2}{s^2+2\zeta\omega_n s+{\omega_n}^2}$. The ideal second order transfer function is derived from the desired specifications such as peak overshoot and settling time..* In a sense, desired specifications provides us a pair of dominant poles and other desired poles may be placed so that over all transient follows the behavior attributed to the pair of dominant poles. It should be clear that a designer starts from the end result, i.e. the desired polynomial $F(s)$ is known where $F(s)$ is the denominator of the closed loop transfer function $T(s) = \frac{C(s)G(s)}{1+C(s)G(s)}$.

Let $G(s) = \frac{N(s)}{D(s)}$ and $C(s) = \frac{B(s)}{A(s)}$. Thus the characteristic polynomial is derived from the identity:$L(s) = 1 + G(s)C(s) = 1 + \frac{N(s)}{D(s)}\frac{B(s)}{A(s)} = 0$

$$F(s) = D(s)A(s) + N(s)B(s) \tag{12}$$

Formally the control problem can be stated as follows: *Find $B(s)$ and $A(s)$, given $D(s)$, $N(s)$ and $F(s)$ such that equation* (12) *is satisfied.* . First we will show that equation (12) has many possible solutions.

Generic Solution

Step 1: Find $\bar{A}$ and $\bar{B}$ such that

$$D(s)\bar{A}(s) + N(s)\bar{B}(s) = 1 \tag{13}$$

For instance, if $D(s) = s^2 - 1$ and $N(s) = s - 2$, then it can be shown that $\bar{A}(s) = \frac{1}{3}$ and $\bar{B}(s) = -\frac{s+2}{3}$ satisfy the equation (13). Multiplying both sides of (13) with $F(s)$, we get

$$F(s)D(s)\bar{A}(s) + F(s)N(s)\bar{B}(s) = F(s) \tag{14}$$

Step 2: There exist $\hat{A}(s)$ and $\hat{B}(s)$ such that

$$D(s)\hat{A}(s) + N(s)\hat{B}(s) = 0$$

One possible solution is $\hat{A}(s) = -N(s)$ and $\hat{B}(s) = D(s)$. Then for any polynomial $Q(s)$

$$\hat{A}(s)Q(s)D(s) + \hat{B}(s)Q(s)N(s) = 0 \tag{15}$$

Adding (14) and (15), we get

$$\begin{array}{ll} & (F(s)\bar{A}(s) + \hat{A}(s)Q(s))D(s) + (\hat{B}(s)Q(s) + F(s)\bar{B}(s))N(s) = F(s) \\ \Rightarrow & A(s) = F(s)\bar{A}(s) - N(s)Q(s) \\ & B(s) = F(s)\bar{B}(s) + D(s)Q(s) \end{array}$$

This implies for each selection of $Q(s)$, a new controller is derived.

Example 10. Given $G(s) = \frac{s-2}{s^2-1}$, design a controller so that desired poles are located at -2 and $-1 \pm j1$.

We are given following: $N(s) = s - 2$, $D(s) = s^2 - 1$, and $F(s) = s^3 + 4s^2 + 6s + 4$. From Step 1, one can obtain $\bar{A}(s) = \frac{1}{3}$ and $\bar{B}(s) = -\frac{s+2}{3}$. Then

$$\begin{aligned} A(s) &= \frac{1}{3}(s^3 + 4s^2 + 6s + 4) + Q(s)(-s + 2) \\ B(s) &= -\frac{1}{3}(s + 2)(s^3 + 4s^2 + 6s + 4) + Q(s)(s^2 - 1) \end{aligned}$$

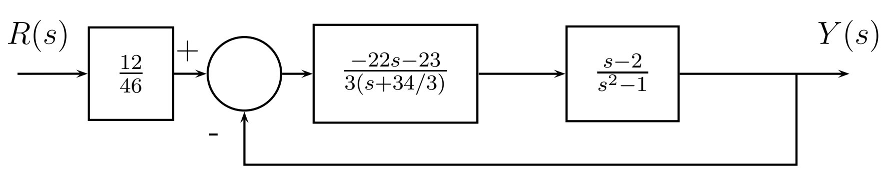
Intelligent guess for $Q(s)$ while intending least degrees for the compensator:

$$\begin{aligned} Q(s) &= \frac{s^2 + 6s + 15}{3} \\ A(s) &= s + \frac{34}{3} \\ B(s) &= -\frac{22s + 23}{3} \end{aligned}$$

The closed loop transfer function reads as:

$$\begin{aligned} \frac{Y(s)}{R(s)} &= \frac{N(s)B(s)}{s^3 + 4s^2 + 6s + 4} \\ &= \frac{\frac{1}{3}(s - 2)(-22s - 23)}{s^3 + 4s^2 + 6s + 4} \end{aligned}$$

However, the DC gain of this transfer function is not 1, i.e.$\frac{Y(0)}{R(0)} = \frac{46}{12}$. Thus the complete compensation system looks as:



Therefore,

$$\frac{Y(s)}{R(s)} = \frac{\frac{2}{23}(s-2)(-22s-23)}{s^3+4s^2+6s+4}$$

which has a DC gain=1.

■

6.1 Solution by Linear Algebra

$$F(s) = D(s)A(s) + N(s)B(s) \tag{16}$$

where,

$$\begin{aligned}
D(s) &= D_0 + D_1 s + D_2 s^2 + \ldots + D_n s^n \\
N(s) &= N_0 + N_1 s + N_2 s^2 + \ldots + N_n s^n \\
A(s) &= A_0 + A_1 s + A_2 s^2 + \ldots + A_m s^m \\
B(s) &= B_0 + B_1 s + B_2 s^2 + \ldots + B_m s^m \\
F(s) &= F_0 + F_1 s + F_2 s^2 + \ldots + F_{m+n} s^{m+n}
\end{aligned}$$

Equating the coefficients on both sides of (16), we get

$$\begin{aligned}
A_0 D_0 + B_0 N_0 &= F_0 \\
A_0 D_1 + B_0 N_1 + A_1 D_0 + B_1 N_0 &= F_1 \\
\vdots &= \vdots \\
A_m D_n + B_m N_n &= F_{n+m}
\end{aligned}$$

Rewriting the above equations in matrix form, we get

$$[A_0 \quad B_0 \quad A_1 \quad B_1 \quad \ldots \quad A_m \quad B_m] S_m = [F_0 \quad F_1 \quad F_2 \quad \ldots \quad F_{n+m}]$$

where,

$$S_m = \begin{bmatrix}
D_0 & D_1 & \ldots & D_n & 0 & \ldots & 0 \\
N_0 & N_1 & \ldots & N_n & 0 & \ldots & 0 \\
 & \vdots & & \vdots & & & \vdots \\
0 & D_0 & \ldots & D_{n-1} & D_n & \ldots & 0 \\
0 & N_0 & \ldots & N_{n-1} & N_n & \ldots & 0 \\
\vdots & & & & \vdots & & \vdots \\
0 & 0 & \ldots & 0 & 0 & D_0 & D_n \\
0 & 0 & \ldots & 0 & 0 & N_0 & N_n
\end{bmatrix}$$

Example 11. Consider following plant

$$G(s) = \frac{s-2}{s^2-1}$$

It is desired that the closed loop system have three poles -2, $-1 \pm j1$, then the characteristic polynomial becomes

$$F(s) = (s+2)(s^2+2s+2) = s^3 + 4s^2 + 6s + 4$$

Now,

$$\begin{aligned} N(s) = s - 2 &= -2 + 1.s + 0.s^2 \\ D(s) = s^2 - 1 &= -1 + 0.s + 1.s^2 \end{aligned}$$

Thus, we have

$$\begin{bmatrix} A_0 & B_0 & A_1 & B_1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & -2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 6 & 4 & 1 \end{bmatrix}$$

Its solution,

$$A_1 = 1, \quad A_0 = \frac{34}{3}, \quad B_1 = -\frac{22}{3}, \quad B_0 = -\frac{22}{3}$$

Thus the controller is given by

$$D(s) = \frac{B(s)}{A(s)} = -\frac{22s + 23}{3s + 34}$$

■

7 Exercise

1. Consider the system shown in figure 3 with PI control.

   (a) Determine the transfer function from R to Y.

   (b) Determine the transfer function from W to Y.

   (c) Use Routh's criteria to find the range of (k_P, k_I) for which the system is stable.

   (d) What is the system type and error constant with respect to reference tracking?

   (e) What is the system type and error constant with respect to disturbance rejection?

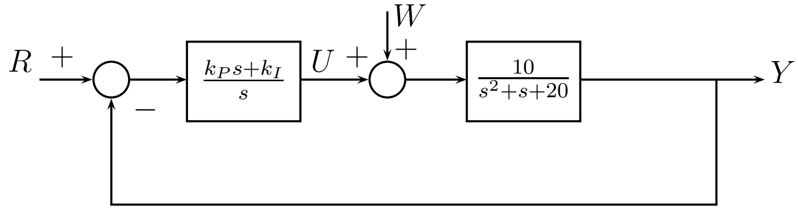


Figure 3:

2. Consider the second-order system

$$G(s) = \frac{1}{s^2 + 2\zeta s + 1}$$

We would like to add a transfer function of the form $D(s) = K(s + a)/(s + b)$ in series with $G(s)$ in a unity-feedback structure.

(a) Ignoring stability for the moment, what are the constraints on K, a, and b so that the system type 1?

(b) What are the constraints placed on K, a, and b so that the system is stable and type 1?

(c) What are the constraints on a and b so that the system is type 1 and remains stable for every positive value of K?

3. Given a plant with transfer function

$$G(s) = \frac{(s - 1)}{(s^2 - 4)}$$

Find a compensator in the unity-feedback configuration so that the overall system has desired poles at -2 and $-1 \pm j1$. Also find a feedforward gain so that the resulting system will track any step reference input.

4. Suppose the plant transfer function in problem 3 changes to

$$G(s) = \frac{(s - 0.9)}{(s^2 - 4.1)},$$

after the design is completed. Can the overall system still track asymptotically any step reference input? If not, design a compensator that will track asymptotically and robustly any step reference input. Do you need additional desired poles? If yes, place them at -3

5. Consider the unity-feedback system shown in Figure 4. Is the transfer function from r to y BIBO stable? Is the system totally stable? If not, find an input-output pair whose closed loop transfer function is not BIBO stable.

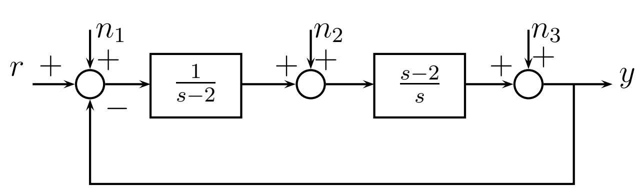


Figure 4:

6. The DC-motor speed control shown in Fig. 5 is described by the differential equation

$$\dot{y} + 60y = 600v_a - 1500\omega$$

where y is the motor speed, v_a is the armature voltage, and ω is the load torque. Assume the armature voltage is computed using the PI control law

$$v_a = \left(k_p e + K_I \int_0^t e dt\right)$$

(a) Compute the transfer function from W to Y as a function of k_p and k_I.

(b) Compute values for k_p and k_I so that the characteristic equation of the closed loop system will have roots at $-60 \pm 60j$.

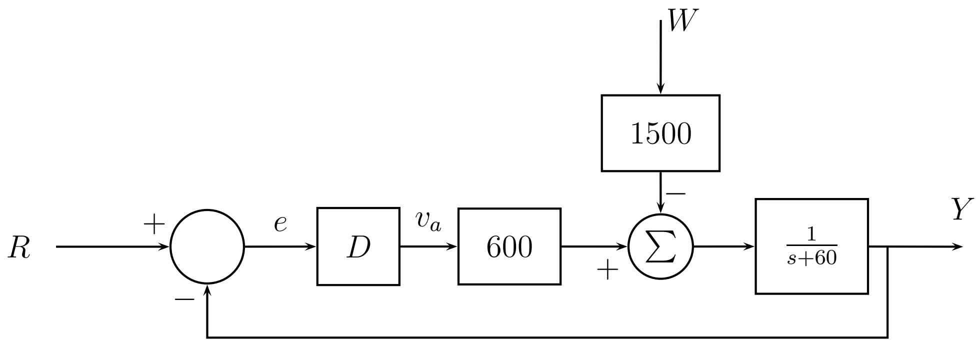


Figure 5:

7. A certain control system has the following specifications: rise time $t_r \leq 0.01$ sec, overshoot $M_p \leq 16\%$, and steady-state error to unit ramp $e_{ss} \leq 0.005$.

(a) Sketch the allowable region in the s-plane for the dominant second-order poles of an acceptable system.

(b) If $Y/R = G/(1+G)$, what conditions must $G(s)$ satisfy near $s = 0$ for the closed-loop system to meet specifications; that is, what is the required asymptotic low-frequency behaviour of $G(s)$?

8. Consider the system shown in Fig. 6, where

$$D(s) = K\frac{(s+\alpha)^2}{s^2+\omega^2}$$

(a) Prove that if the system is unstable, it is capable of tracking a sinusoidal reference input $r = sin\ \omega_0 t$ with zero steady-state error. (Look at the transfer function from R to E and consider the gain at ω_0.

(b) Use Routh's criteria to find the range of K such that the closed-loop system remains stable if $\omega_0 = 1$ and $\alpha = 0.25$.

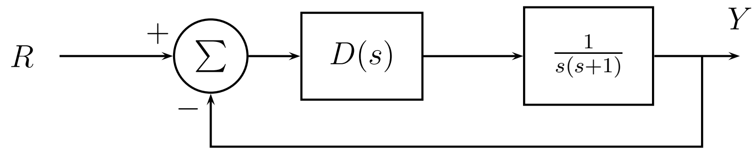


Figure 6:

7.1 Matlab Simulations: Submission dead line 24 Feb 2005

1. Consider the second-order plant

$$G(s) = \frac{1}{(s+1)(5s+1)}$$

   (a) Determine the system type and error constant with respect to tracking polynomial reference inputs of the system for P, PD, and PID controllers (as configured in Fig. 7). Let $k_p = 19$, $k_I = 0.5$, $k_D = \frac{4}{19}$.

   (b) Determine the system type and error constant of the system with respect to disturbance inputs for each of the three regulators in part (a) with respect to rejecting polynomial disturbances $\omega(t)$ at the input to the plant.

   (c) Is this system better at tracking references or rejecting disturbances? Explain your response briefly.

   (d) Verify your results for parts (a) and (b) using MATLAB by plotting unit step and ramp responses for both tracking and disturbance rejection.

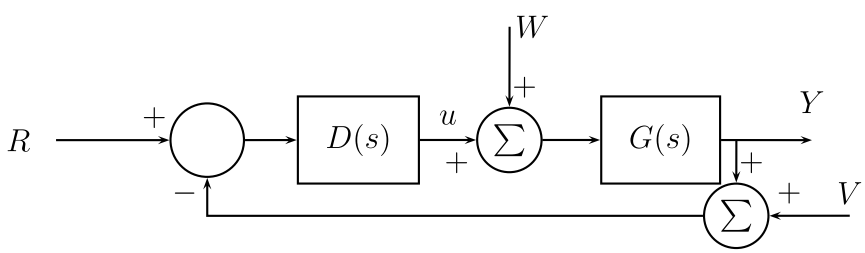


Figure 7:

2. Consider a system with the plant transfer function $G(s) = 1/s(s+1)$. You wish to add a dynamic controller so that $\omega_n = 2\ rad/s$ and $\zeta \geq 0.5$. Several dynamic controllers have been proposed:

   (a) $D(s) = (s+2)/2$

   (b) $D(s) = 2\frac{s+2}{s+4}$

   (c) $D(s) = 5\frac{s+2}{s+10}$

   (d) $D(s) = 5\frac{(s+2)(s+0.1)}{(s+10)(s+0.01)}$

(i) Using MATLAB, compare the resulting transient and steady-state responses to reference step inputs for each controller choice. Which controller is best for the smallest rise time and smallest overshoot?

(ii) Which system would have the smallest steady-state error to a ramp reference input?

(iii) Compare each system for peak control effort, that is, measure the peak magnitude of the plant input $u(t)$ for a unit reference step input.

(iv) Based on your results from parts (i) to (iii), recommend a dynamic controller for the system from the four candidate designs.

3. A position control system has the closed loop transfer function (meter/meter) given by

$$\frac{Y(s)}{R(s)} = \frac{b_0 s + b_1}{s^2 + a_1 s + a_2}$$

(a) Choose the parameters (a_1, a_2, b_0, b_1) so that the following specifications are satisfied simultaneously:

i. The rise time $t_r < 0.1$ *sec.*

ii. The overshoot $M_p < 20$

iii. The settling time $t_s < 0.5$ *sec.*

iv. The steady-state error to a step reference is zero.

v. The steady-state error to a ramp reference input of 0.1 m/sec is not more than 1 mm.

(b) Verify your answer via MATLAB simulation.

***