Problem 5

$$u(x,t) = v\left(\frac{x}{\sqrt{t}}\right)\frac{1}{\sqrt{t}}$$

$$\Rightarrow \quad u_t(x,t) = \frac{1}{\sqrt{t}}\, v'\left(\frac{x}{\sqrt{t}}\right)\left(\frac{x}{-2\,t^{3/2}}\right) + v\left(\frac{x}{\sqrt{t}}\right)\left(-\frac{1}{2}\right)\frac{1}{t^{3/2}}$$

$$= -\frac{x}{2t^2}\, v'\left(\frac{x}{\sqrt{t}}\right) - \frac{1}{2t^{3/2}}\, v\left(\frac{x}{\sqrt{t}}\right)$$

$$u_x = \frac{1}{\sqrt{t}}\, v'\left(\frac{x}{\sqrt{t}}\right)\frac{1}{\sqrt{t}} = \frac{1}{t}\, v'\left(\frac{x}{\sqrt{t}}\right)$$

$$u_{xx} = \frac{1}{t^{3/2}}\, v''\left(\frac{x}{\sqrt{t}}\right)$$

$$\therefore \quad u_t = u_{xx}$$

$$\Rightarrow \quad -\frac{x}{2t^2}\, v'\left(\frac{x}{\sqrt{t}}\right) - \frac{1}{2t^{3/2}}\, v\left(\frac{x}{\sqrt{t}}\right) = \frac{1}{t^{3/2}}\, v''\left(\frac{x}{\sqrt{t}}\right)$$

$$\because t > 0$$

$$\Rightarrow \quad -\frac{x}{2\sqrt{t}}\, v'\left(\frac{x}{\sqrt{t}}\right) - \frac{1}{2}\, v\left(\frac{x}{\sqrt{t}}\right) = v''\left(\frac{x}{\sqrt{t}}\right)$$

Let $\frac{x}{\sqrt{t}} = r$

$$\Rightarrow \frac{r}{2} v'(r) + \frac{1}{2} v(r) + v''(r) = 0$$

$$\Rightarrow v'' + \frac{1}{2} (r v(r))' = 0$$

Integrating,

$$\Rightarrow v'(r) + \frac{1}{2} r v(r) = a$$

for some constant 'a'

Given $v'(r) \to 0$ as $r \to \infty$
$v(r) \to 0$

$$\Rightarrow a = 0$$

Integrating further

$$\boxed{v(r) = b e^{-r^2/4}.}$$

$$\Rightarrow u(x,t) = \frac{b}{\sqrt{t}} e^{-\left(x^2/4t\right)}, \quad t > 0$$

—x—

Problem 2:

Consider the function

$$v(x,t) = e^{-t} \sin x .$$

Let us calculate the Eqn that 'v' satisfies.

$$\left.\begin{aligned} v_t &= -e^{-t} \sin x \\ v_x &= e^{-t} \cos x \\ v_{xx} &= -e^{-t} \sin x \end{aligned}\right\} \Rightarrow \begin{array}{l} v_t = v_{xx} \\ \text{is satisfied} \\ \text{on } \underbrace{(0,\pi)\times(0,T)}_{A} \end{array}$$

and $v(x,0) = \sin x .$

$$\Rightarrow \left\{\begin{aligned} v_t &= v_{xx} \\ v(x,0) &= \sin x \\ v(0,t) &= v(\pi,t) = 0 \end{aligned}\right. \quad \Bigg| \quad \left\{\begin{aligned} u_t &= u_{xx} \\ u(x,0) &= \sin^2 x . \\ u(0,t) &= u(\pi,t) = 0 \end{aligned}\right.$$

$\because \sin^2 x \leq \sin x \quad \forall x \in (0,\pi)$

$\Rightarrow$ From "comparison principal"

$$u(x,t) \leq v(x,t) = e^{-t} \sin x . \longrightarrow (1)$$

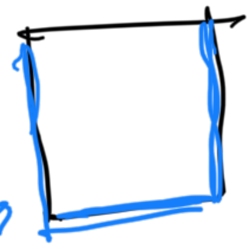
we also show that

We are left to show that

$$u(x,t) \geq 0 \quad \text{on } (0,\pi)\times(0,\pi)$$

But this is a direct application of maximum principal.

$$\min_{A} u = \min_{\Gamma} u$$



where Γ = part in blue $(0,\pi)\times(0,\pi)$

From the the Boundary and initial conditions in the problem we know $u \geq 0$ on Γ, $\Rightarrow \min_{\Gamma} u \geq 0$

$$\Rightarrow \min_{A} u \geq 0$$

$$\Rightarrow u(x,t) \geq 0 \quad (x,t) \in A.$$

— X —

Problem 3 :-

This is an application of sturm-comparison principal.

Since $q_1(t) = \sin t + 5 \geq 1 \quad \forall t \in \mathbb{R}$, $q_2(t)$

Problem 4 :=

$$\begin{cases} u_t - 9u_{xx} = 0 \quad \text{on } (0,l)\times(0,\infty) \\ u(0,t) = 10, \quad u(l,t) = 100 \\ u(x,0) = \sin\left(\frac{\pi}{l}x\right) \end{cases} \rightarrow (*)$$

Define the fn.

$$v_1(x,t) = \frac{90}{l}x + 10$$

and Define

$$w(x,t) = u(x,t) - v_1$$

where 'u' solves $(*)$

Then
$$\begin{cases} w_t = 9w_{xx} \\ w(0,t) = 0, \quad w(l,t) = 0 \\ w(x,0) = \sin\left(\frac{\pi}{l}x\right) - \frac{90x}{l} - 10 \end{cases}$$

This is usual homogeneous problem.

2 The other problem is also as done in lecture video.