MSO 201a: Probability and Statistics

Assignment - I

Solutions

Problem No. 1

(a) Required probability $= P((A\cap B^c) + (A^c \cap B))$

$= P(A\cap B^c) + P(A^c \cap B)$

$= (P(A) - P(A\cap B)) + (P(B) - P(A\cap B))$

$= P(A) + P(B) - 2P(A\cap B)$

$= 0.2 + 0.4 - 2\times 0.1 = 0.4$

(b) Required probability $= P(A\cup B)$

$= P(A) + P(B) - P(A\cap B) = 0.5$

(c) Required probability $= P(A^c \cap B^c)$

$= P((A\cup B)^c)$

$= 1 - P(A\cup B) = 1 - 0.5$ (using (b))

$= 0.5$

Problem No. 2

Total number of ways in which $P_1, \dots, P_n$ can stand in a row $= \lfloor\underline{n}$

Total number of possible positions for P_1 and P_2 such that there are exactly r positions between P_1 and P_2

$= \lfloor\underline{2} \times (n-r-1)$

Corresponds to permutation of positions for P_1 and P_2

(Corresponds to positions $\{1, r+2\}$, $\{2, r+3\}, \dots, \{n-r-1, n\}$)

Thus total number of ways in which $P_1, \dots, P_n$ can stand in a row such that there are exactly r persons between P_1 and P_2

$= (\lfloor\underline{2} \times (n-r-1)) \times \lfloor\underline{n-2}$

Corresponds to permutations of $(n-2)$ persons other than P_1 and P_2

$$\text{Required probability} = \frac{(\lfloor\underline{2} \times (n-r-1)) \times \lfloor\underline{n-2}}{\lfloor\underline{n}} = \frac{2(n-r-1)}{n(n-1)}.$$

$\boxed{\text{Problem No. 3}}$ For a, b and c $(a<b<c)$ to be in GP we must have $b = ar$ and $c = ar^2$ for some $r>1$ and $a, b, c \in \{1, \dots, 50\}$. Thus we have

$1 \le a < ar < ar^2 \le 50$; a, ar and ar^2 are integers

$$\Rightarrow \quad 1 \le a \le \frac{50}{r^2}, \; r>1 \quad \Rightarrow \boxed{1 \le a \le \frac{50}{r^2}, \; 1 < r \le \sqrt{50}}$$

$\Downarrow$

$(\; r^2 \le 50 \;)$ Also r is rational (as $r = \frac{b}{a}$, $a, b \in \{1 \dots 50\}$)

The following cases arise.

<u>Case I.</u> r is an integer

$$1 < r \le \sqrt{50} \Rightarrow r \in \{2, 3, \dots, 7\}$$

For each $r \in \{2, 3, \dots, 7\}$ $\quad 1 \le a \le \left[\frac{50}{r^2}\right]$

$\hookrightarrow$ maximum integer contained in $\frac{50}{r^2}$

$\Rightarrow$ total # of favorable cases with r as an integer

$$= \sum_{r=2}^{7} \left[\frac{50}{r^2}\right] = 12 + 5 + 3 + 2 + 1 + 1 = 24$$

<u>Case II</u> $r = \frac{m}{n}$, where m and n are coprimes, $m > n > 1$.

We have

$1 \le a < \frac{m}{n} a < \frac{m^2}{n^2} a \le 50$; a, $\frac{m}{n} a$ and $\frac{m^2}{n^2} a$ are integers

$\Rightarrow$ a is an integer and a is a multiple of n^2 (m and n are coprimes)

Thus for each fixed $r = \frac{m}{n}$ $(m > n > 1$; m and n coprimes) we have

$$1 \le a \le \frac{50 n^2}{m^2} \quad \text{and } a \text{ is a multiple of } n^2$$

i.e., $1 \le a \le \left[\frac{50 n^2}{m^2}\right]$ and a is a multiple of n^2

$r = \frac{m}{n}$	range of a $1 \le a \le \left[\frac{50n^2}{m^2}\right]$	Possible a's that are multiple of n^2	# of cases
$\frac{3}{2}$	$[1, 22]$	$\{4, 8, 12, 16, 20\}$	5
$\frac{5}{2}$	$[1, 8]$	$\{4, 8\}$	2
$\frac{7}{2}$	$[1, 4]$	$\{4\}$	1
$\frac{4}{3}$	$[1, 28]$	$\{9, 18, 27\}$	3
$\frac{5}{3}$	$[1, 18]$	$\{9, 18\}$	2
$\frac{7}{3}$	$[1, 9]$	$\{9\}$	1
$\frac{5}{4}$	$[1, 32]$	$\{16, 32\}$	2
$\frac{7}{4}$	$[1, 16]$	$\{16\}$	1
$\frac{6}{5}$	$[1, 34]$	$\{25\}$	1
$\frac{7}{5}$	$[1, 25]$	$\{25\}$	1
$\frac{7}{6}$	$[1, 36]$	$\{36\}$	1
			Total = 20

Thus total # of cases with r fractional = 20

$\Rightarrow$ total # of favorable cases (Case I + Case II) = 24 + 20 = 44

Required probability = $\dfrac{44}{\binom{50}{3}}$.

$\boxed{\text{Problem No. 4}}$ Define events

E_i : i-th letter is in right envelope, $i=1,\dots,n$.

Then

Required probability $= P\left(\bigcup_{i=1}^{n} E_i\right)$

$= p_{1,n} - p_{2,n} + p_{3,n} + \dots + (-1)^{n-1} p_{n,n}$,

(Inclusion-Exclusion principle)

where

$$p_{k,n} = \underbrace{\sum \cdots \sum_{1 \le i_1 < i_2 < \dots < i_k \le n}}_{\text{this has } \binom{n}{k} \text{ terms}} P(E_{i_1} \cap E_{i_2} \cap \dots \cap E_{i_k}), \quad k=1,\dots,n$$

We have, for $1 \le i_1 < i_2 < \dots < i_k \le n$

$E_{i_1} \cap E_{i_2} \cap \dots \cap E_{i_k} =$ letters $i_1, \dots, i_k$ go to right envelopes

$\Rightarrow$ # of favorable cases to $E_{i_1} \cap E_{i_2} \cap \dots \cap E_{i_k} = \lfloor n-k$

$\Rightarrow P(E_{i_1} \cap \dots \cap E_{i_k}) = \dfrac{\lfloor n-k}{\lfloor n}$

$\Rightarrow p_{k,n} = \binom{n}{k} \dfrac{\lfloor n-k}{\lfloor n} = \dfrac{1}{\lfloor k}, \quad k=1,\dots,n$

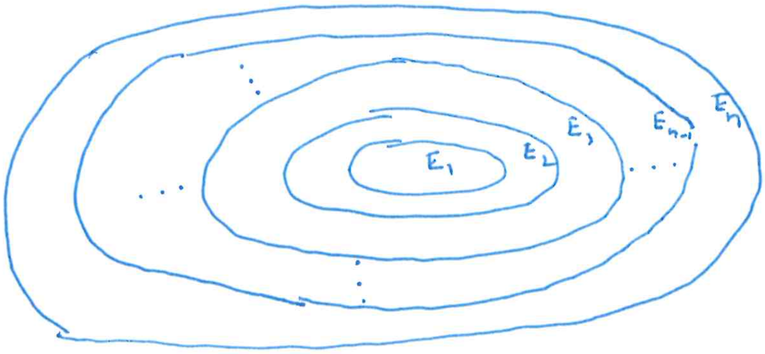
Required probability $= \dfrac{1}{\lfloor 1} - \dfrac{1}{\lfloor 2} + \dfrac{1}{\lfloor 3} + \dots + \dfrac{(-1)^{n-1}}{\lfloor n}$

For large n $(n=50)$

Required prob. $\approx = 1 - \dfrac{1}{\lfloor 2} + \dfrac{1}{\lfloor 3} + \dots = 1 - e^{-1} = 0.632$

Problem No. 5

(a)



Define events

$$B_1 = E_1, \quad B_n = E_n - E_{n-1}, \quad n = 2, 3 \ldots$$

Then B_i's are disjoint, $\bigcup_{k=1}^{n} B_k = E_n$, $n = 1, 2 \ldots$

$\bigcup_{n\geq 1}^{\infty} B_n = \bigcup_{n\geq 1}^{\infty} E_n$, and

$$P\left(\bigcup_{n\geq 1}^{\infty} E_n\right) = P\left(\bigcup_{n\geq 1}^{\infty} B_n\right)$$

$$= \sum_{n=1}^{\infty} P(B_n) \quad (B_n\text{'s are disjoint})$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} P(B_k)$$

$$= \lim_{n\to\infty} P\left(\bigcup_{k=1}^{n} B_k\right) = \lim_{n\to\infty} P(E_n)$$

(b) $E_n \downarrow \Rightarrow E_n^c \uparrow$. Thus, by (a)

$$P\left(\bigcup_{n\geq 1}^{\infty} E_n^c\right) = \lim_{n\to\infty} P(E_n^c)$$

$$1 - P\left(\left(\bigcup_{n\geq 1}^{\infty} E_n^c\right)^c\right) = \lim_{n\to\infty} [1 - P(E_n)]$$

$$P\left(\bigcap_{n\geq 1}^{\infty} E_n\right) = \lim_{n\to\infty} P(E_n) \quad \left(\text{De-Morgan's law, } \left(\bigcup_{\alpha\in\Lambda} F_\alpha\right)^c = \bigcap_{\alpha\in\Lambda} F_\alpha^c\right)$$

Problem No. 6 (a) Define

$$B_n = \bigcup_{k=1}^{n} E_k, \quad n = 1, 2, \dots$$

Then $B_n \uparrow$ and $\lim B_n = \bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{n} E_k = \bigcup_{k=1}^{\infty} E_k$.

By Continuity of probability measures (Problem 5)

$$P(\lim B_n) = \lim P(B_n)$$

$$P\left(\bigcup_{k=1}^{\infty} E_k\right) = \lim_{n\to\infty} P\left(\bigcup_{k=1}^{n} E_k\right) \quad \dots\dots \quad (1)$$

But using Boole's inequality

$$P\left(\bigcup_{k=1}^{n} E_k\right) \le \sum_{k=1}^{n} P(E_k) \quad \forall\, n = 1, 2, \dots$$

$$\Rightarrow \lim_{n\to\infty} P\left(\bigcup_{k=1}^{n} E_k\right) \le \lim_{n\to\infty} \sum_{k=1}^{n} P(E_k) = \sum_{k=1}^{\infty} P(E_k)$$

$$\Rightarrow P\left(\bigcup_{k=1}^{\infty} E_k\right) \le \sum_{k=1}^{\infty} P(E_k) \quad (\text{using } (1)).$$

(b)(i) Clearly if $P\left(\bigcup_{\alpha\in\Delta} E_\alpha\right) = 0$ then

$$0 \le P(E_\beta) \le P\left(\bigcup_{\alpha\in\Delta} E_\alpha\right) = 0 \quad \forall\, \beta\in\Delta$$

$$\left(E_\beta \subseteq \bigcup_{\alpha\in\Delta} E_\alpha \quad \forall\, \beta\in\Delta\right)$$

$$\Rightarrow P(E_\beta) = 0, \quad \forall\, \beta\in\Delta.$$

Conversely if $P(E_\beta) = 0$, $\forall\, \beta\in\Delta$ then using (a)

$$0 \le P\left(\bigcup_{\alpha\in\Delta} E_\alpha\right) \le \sum_{\alpha\in\Delta} P(E_\alpha) = 0$$

$$\Rightarrow P\left(\bigcup_{\alpha\in\Delta} E_\alpha\right) = 0.$$

(ii) $P(E_\beta) = 1 \quad \forall\, \beta\in\Delta \iff P(E_\beta^c) = 0, \quad \forall\, \beta\in\Delta$

$$\iff P\left(\bigcup_{\alpha\in\Delta} E_\alpha^c\right) = 0, \quad (\text{using } (i))$$

$$\iff P\left(\left(\bigcup_{\alpha\in\Delta} E_\alpha^c\right)^c\right) = 1$$

$$\iff P\left(\bigcap_{\alpha\in\Delta} E_\alpha\right) = 1.$$

Problem No. 7 Define events

E_i: ith machine produced Code 1 $i = 1, 2, 3, 4$

Required probability $= P(E_1^c \mid E_4) = 1 - P(E_1 \mid E_4)$

We have $P(E_1) = \frac{3}{4}$. By Baye's Theorem

$$P(E_1 \mid E_4) = \frac{P(E_4 \mid E_1)\, P(E_1)}{P(E_4 \mid E_1) P(E_1) + P(E_4 \mid E_1^c)\, P(E_1^c)}$$

$P(E_4 \mid E_1) =$ P(machines M_2, M_3 and M_4 either make no code change or make 2 code changes)

$$= \left(\frac{1}{4}\right)^3 + \binom{3}{2}\left(\frac{3}{4}\right)^2 \times \frac{1}{4} = \frac{7}{16}.$$

$P(E_4 \mid E_1^c) =$ P(machines M_2, M_3 and M_4 either make 1 Code Change or make 3 Code changes)

$$= \binom{3}{1}\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^2 + \left(\frac{3}{4}\right)^3 = \frac{9}{16}.$$

Thus

$$\text{Required probability} = 1 - \frac{\frac{7}{16} \times \frac{3}{4}}{\frac{7}{16} \times \frac{3}{4} + \frac{9}{16} \times \frac{1}{4}} = \frac{3}{10}.$$

Problem No. 8 Define the events

B: student clears Biology examination

C: student clears Chemistry examination

P: student clears Physics examination

M: student clears Mathematics examination

Then $P(B) = \frac{1}{2}$, $P(C) = \frac{1}{3}$, $P(P) = \frac{1}{4}$, $P(M) = \frac{1}{4}$ and

B, C, P and M are independent events.

(a) Required probability $= P(B \cap C \cap P \cap M)$

$= P(B)\, P(C)\, P(P)\, P(M)$ (independence)

$= \frac{1}{2} \times \frac{1}{3} \times \frac{1}{4} \times \frac{1}{5} = \frac{1}{120}$

(b) Required probability $= P(B^c \cap C^c \cap P^c \cap M^c)$

$= P(B^c)\, P(C^c)\, P(P^c)\, P(M^c)$ (independence)

$= (1-\frac{1}{2})(1-\frac{1}{3})(1-\frac{1}{4})(1-\frac{1}{5}) = \frac{1}{5}$

(c) Required probability $= P(B \cap C^c \cap P^c \cap M^c) + P(B^c \cap C \cap P^c \cap M^c)$

$+ P(B^c \cap C^c \cap P \cap M^c) + P(B^c \cap C^c \cap P^c \cap M)$

$= \frac{1}{2} \times (1-\frac{1}{3}) \times (1-\frac{1}{4}) \times (1-\frac{1}{5}) + (1-\frac{1}{2}) \times \frac{1}{3} \times (1-\frac{1}{4}) \times (1-\frac{1}{5})$

$+ (1-\frac{1}{2}) \times (1-\frac{1}{3}) \times \frac{1}{4} \times (1-\frac{1}{5}) + (1-\frac{1}{2}) \times (1-\frac{1}{3}) \times (1-\frac{1}{4}) \times \frac{1}{5}$

(using independence)

$= \frac{5}{12}$

(d) Required probability $= P(B \cap C \cap P^c \cap M^c) + P(B \cap C^c \cap P \cap M^c)$

$+ P(B \cap C^c \cap P^c \cap M) + P(B^c \cap C \cap P \cap M^c)$

$+ P(B^c \cap C \cap P^c \cap M) + P(B^c \cap C^c \cap P \cap M)$

$= \frac{1}{2} \times \frac{1}{3} \times (1-\frac{1}{4}) \times (1-\frac{1}{5}) + \frac{1}{2} \times (1-\frac{1}{3}) \times \frac{1}{4} \times (1-\frac{1}{5})$

$+ \frac{1}{2} \times (1-\frac{1}{3}) \times (1-\frac{1}{4}) \times \frac{1}{5} + (1-\frac{1}{2}) \times \frac{1}{3} \times \frac{1}{4} \times (1-\frac{1}{5})$

$+ (1-\frac{1}{2}) \times \frac{1}{3} \times (1-\frac{1}{4}) \times \frac{1}{5} + (1-\frac{1}{2}) \times (1-\frac{1}{3}) \times \frac{1}{4} \times \frac{1}{5}$

$= \frac{7}{24}$.

(e) Required probability $= 1 - P(\text{no subject is cleared})$

$= 1 - \frac{1}{5} = \frac{4}{5}$ (using (b))

Problem No. 9 (a) Let $B_n = \bigcup_{k=n}^{\infty} E_k$, $n = 1, 2, \ldots$ Then $B_n \downarrow$,

$$\bigcap_{n=1}^{\infty} B_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k$$

$$P\left(\bigcap_{n=1}^{\infty} B_n\right) = \lim P(B_n) \quad \left(\text{Problem 5 (b)}, \text{ Continuity of probability measure}\right)$$

$$\Rightarrow P\left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k\right) = \lim P\left(\bigcup_{k=n}^{\infty} E_k\right)$$

$$\leq \lim_{n\to\infty} \sum_{k=n}^{\infty} P(E_k) \quad \left(\text{Boole's Inequality of Problem 6}\right)$$

$$= 0 \quad \left(\text{since } \sum_{k=1}^{\infty} P(E_k) < \infty\right)$$

$$\Rightarrow P\left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k\right) = 0$$

$$\Rightarrow P\left(\left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k\right)^c\right) = 1$$

$$\Rightarrow P\left(\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} E_k^c\right) = 1$$

Note that

$\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} E_k^c = \{\omega \in \Omega :$ there exists an $n \geq 1$ such that $\omega \notin E_k \ \forall k \geq n\}$

$= \{\omega \in \Omega :$ ω belongs to only finitely many E_k's$\}$.

(b)

$$P\left(\bigcap_{i=1}^{n} E_i^c\right) = \prod_{i=1}^{n} P(E_i^c) \quad (\text{Independence})$$

$$= \prod_{i=1}^{n} (1 - P(E_i))$$

$$\leq \prod_{i=1}^{n} e^{-P(E_i)} \quad \left(e^{-x} \geq 1 - x \ \ \forall x \in \mathbb{R}\right)$$

$$= e^{-\sum_{i=1}^{n} P(E_i)}$$

(c) Let $B_n = \bigcap_{i=1}^{n} E_i^c$, $n = 1, 2, \ldots$ Then $B_n \downarrow$

$$\bigcap_{n=1}^{\infty} B_n = \bigcap_{n=1}^{\infty} \bigcap_{i=1}^{n} E_i^c = \bigcap_{i=1}^{\infty} E_i^c$$

$$P\left(\bigcap_{n=1}^{\infty} B_n\right) = \lim_{n\to\infty} P(B_n) \quad \text{(Problem 5 (b))}$$

$$\Rightarrow P\left(\bigcap_{i=1}^{\infty} E_i^c\right) = \lim_{n\to\infty} P\left(\bigcap_{i=1}^{n} E_i^c\right)$$

$$\leq \lim_{n\to\infty} e^{-\sum_{i=1}^{n} P(E_i)} \quad \text{(using (b))}$$

$$= e^{-\sum_{i=1}^{\infty} P(E_i)}$$

(d) We will show that $P\left(\left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k\right)^c\right) = 0$, i.e.

$$P\left(\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} E_k^c\right) = 0.$$

Let $D_n = \bigcap_{k=n}^{\infty} E_k^c$, $n = 1, 2, \ldots$ Then $D_n \uparrow$, $\bigcup_{n=1}^{\infty} D_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} E_k^c$ and

$$P\left(\bigcup_{n=1}^{\infty} D_n\right) = \lim_{n\to\infty} P(D_n) \quad \text{(Problem 5(a))}$$

$$\Rightarrow P\left(\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} E_k^c\right) = \lim_{n\to\infty} P\left(\bigcap_{k=n}^{\infty} E_k^c\right)$$

$$\leq \lim_{n\to\infty} e^{-\sum_{k=n}^{\infty} P(E_k)}$$

$$= 0 \quad \left(\text{as } \sum_{k=n}^{\infty} P(E_k) = \infty \ \forall n \geq 1\right)$$

$$\Rightarrow P\left(\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} E_k^c\right) = 0$$

$$\Rightarrow P\left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k\right) = 1 - P\left(\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} E_k^c\right) = 1$$

Note that

$$\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k = \{\omega \in \Omega : \forall n \geq 1, \exists k \geq n \text{ such that } \omega \in E_k\}$$

$$= \{\omega \in \Omega : \omega \text{ belongs to infinitely many } E_n\text{'s}\}$$

10/1

Problem No. 10 (a) $P(A) = P(B) = P(C) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

$$P(A\cap B) = P(A\cap C) = P(B\cap C) = P(\{4\}) = \frac{1}{4}$$

Clearly $P(A\cap B) = P(A)\,P(B)$; $P(A\cap C) = P(A)\,P(C)$ and $P(B\cap C) = P(B)\,P(C)$

$\Rightarrow$ A, B and C are pairwise independent.

(b)

$$P(A\cap B\cap C) = P(\{4\}) = \frac{1}{4} \neq P(A)P(B)P(C) = \frac{1}{8}$$

$\Rightarrow$ A, B and C are not independent.

(c) pairwise independence $\not\Rightarrow$ independence.

Problem No. 11 (a) $P(A\cap B\mid C) = \dfrac{P(A\cap B\cap C)}{P(C)} = P(A\mid B\cap C)\,\dfrac{P(B\cap C)}{P(C)}$

$$= P(A\mid B\cap C)\,P(B\mid C)$$

(b) Consider the example of Problem 10.

$$P(A\cap B\mid C) = \frac{P(A\cap B\cap C)}{P(C)} = \frac{1/4}{1/2} = \frac{1}{2}$$

$$P(A\mid C) = \frac{P(A\cap C)}{P(C)} = \frac{1/4}{1/2} = \frac{1}{2}$$

$$P(B\mid C) = \frac{P(B\cap C)}{P(C)} = \frac{1/4}{1/2} = \frac{1}{2}$$

Clearly

$$P(A\cap B\mid C) \neq P(A\mid C)\,P(B\mid C).$$

(c) Events A and B may be independent in pair but they may not be independent given any other event C

Problem No. 12

Let $P(A|B\cap D) = \alpha_1$, $P(A|B^c\cap D) = \beta_1$, $P(A|B\cap D^c) = \alpha_2$, $P(A|B^c\cap D^c) = \beta_2$, $P(D|B) = p_1$ and $P(D|B^c) = p_2$. Then

$$P(A|B) = P(A\cap D|B) + P(A\cap D^c|B)$$
$$= P(A|D\cap B)\, P(D|B) + P(A|D^c\cap B)\, P(D^c|B)$$
$$= p_1\alpha_1 + (1-p_1)\alpha_2.$$

$$P(A|B^c) = p_2\beta_1 + (1-p_2)\beta_2.$$

We have to choose real numbers α_1, α_2, β_1, β_2, p_1 and p_2 such that $0<\alpha_1<\beta_1<1$, $0<\alpha_2<\beta_2<1$, $0<p_1<1$, $0<p_2<1$ and

$$p_1\alpha_1 + (1-p_1)\alpha_2 > p_2\beta_1 + (1-p_2)\beta_2.$$

i.e. $$\alpha_2 + (\alpha_1-\alpha_2)p_1 > \beta_2 + (\beta_1-\beta_2)p_2.$$

Let us take $\alpha_1 = 0.2$, $\alpha_2 = 0.6$, $\beta_1 = 0.4$ and $\beta_2 = 0.8$. Then

$$0.6 - 0.4p_1 > 0.8 - 0.4p_2 \Rightarrow p_2 - p_1 > \frac{1}{2}$$

Thus one may take, for example, $\alpha_1 = 0.2$, $\alpha_2 = 0.6$, $\beta_1 = 0.4$, $\beta_2 = 0.8$, $p_1 = 0.2$ and $p_2 = 0.8$.

Problem No. 13 Let the doors be numbered 1 to N and WLOG assume that car is behind door No. 1. Define events,

D_i : Contestant chooses door No. i, $i = 1, \ldots, N$

W : 'A' wins the car.

Case I : Contestant decides to switch

$P(W \mid D_1) = 0, \quad P(W \mid D_i) = 1 \quad i = 2, 3, \ldots, N.$

$$P(W) = \sum_{i=1}^{N} P(W \mid D_i) P(D_i) \qquad \text{(Theorem of Total Probability)}$$

$$= \frac{1}{N}[0 + N - 1] = \frac{N-1}{N}$$

For $N = 3$, $P(W) = \frac{2}{3}$

Case II : Contestant decides not to switch

$P(W \mid D_1) = 1, \quad P(W \mid D_i) = 0, \quad i = 2, 3, \ldots N$

$$P(W) = \sum_{i=1}^{N} P(W \mid D_i) P(D_i)$$

$$= \frac{1}{N}.$$

For $N = 3$, $P(W) = \frac{1}{3}$

Thus the contestant should ~~~~ switch the door as the probability of win doubles by doing so in case of $N = 3$ and increases $N-1$ times in case of N doors.

Remark: In light of new additional information it is advisable to update prior probabilities

$\longrightarrow$ Bayesian Approach.

Problem No. 14 (a) Let

$p_i = P(\text{A wins all the money})$, $i = 1, 2, \dots N$

Here the probability of win depends on the initial capital 'i' available with 'A'. Then by theorem of total probability (on the result of first flip)

$p_i = P(\text{A wins all the money} \mid \text{first flip is head}) \times p + P(\text{A wins all the money} \mid \text{first flip is tail}) \times (1-p)$

$$\Rightarrow \quad p_i = p\, p_{i+1} + q\, p_{i-1}, \quad i = 2, 3, \dots, N-1 \qquad \dots \quad (I)$$

$p_1 = p\, p_2 \Rightarrow p_2 - p_1 = \frac{q}{p} p_1$

$p_N = 1$

$p_{i+1} - p_i = \frac{q}{p}(p_i - p_{i-1})$, $i = 2, 3 \dots N-1$ (using (I))

$p_2 - p_1 = \frac{q}{p} p_1$

$p_3 - p_2 = \frac{q}{p}(p_2 - p_1) = \left(\frac{q}{p}\right)^2 p_1$

$p_4 - p_3 = \frac{q}{p}(p_3 - p_2) = \left(\frac{q}{p}\right)^3 p_1$

$\vdots$

$p_i - p_{i-1} = \left(\frac{q}{p}\right)^{i-1} p_1$

Summing the last $(i-1)$ rows we get

$p_i - p_1 = \left[\frac{q}{p} + \left(\frac{q}{p}\right)^2 + \dots + \left(\frac{q}{p}\right)^{i-1}\right]$

$$p_i = p_1\left[1 + \frac{q}{p} + \left(\frac{q}{p}\right)^2 + \dots + \left(\frac{q}{p}\right)^{i-1}\right] = \begin{cases} \dfrac{1-\left(\frac{q}{p}\right)^i}{1 - q/p}\, p_1, & \text{if } \frac{q}{p} \neq 1 \\ i\, p_1, & \text{if } \frac{q}{p} = 1 \end{cases}$$

We have $p_N = 1$. Thus

$$p_i = \begin{cases} \dfrac{1 - (q/p)^i}{1 - (q/p)^N}, & \text{if } p \neq \frac{1}{2} \\ \dfrac{i}{N}, & \text{if } p = \frac{1}{2} \end{cases}, \quad i = 1 \dots, N$$

$\left(\frac{q}{p} \neq (=) 1 \iff p \neq (=) \frac{1}{2}\right)$

(b) Let q_i be the probability that B will win all the money.

By symmetry

$$q_i = \begin{cases} \dfrac{1-(p/q)^{N-i}}{1-p/q}, & \text{if } q \neq \frac{1}{2} \ (\text{or } p \neq \frac{1}{2}) \\ \dfrac{N-i}{N}, & \text{if } q = \frac{1}{2} \ (\text{or } p = \frac{1}{2}) \end{cases}$$

Clearly $p_i + q_i = 1 \quad \forall \ i = 1, \ldots, N$.

(c) For $i = 10$, $N = 20$ & $p = 0.49$, $p_i = 0.4$, $q_i = 0.6$

For $i = 50$, $N = 100$ & $p = 0.49$, $p_i = 0.12$, $q_i = 0.88$

For $i = 100$, $N = 200$ & $p = 0.49$, $p_i = 0.02$, $q_i = 0.98$

In Casino even if the game may look fair ($p \approx 0.49$), the gambler is bound to be ruined.

For $i = 5$, $N = 15$ and $p = 0.5$, $p_i = \frac{1}{3}$, $q_i = \frac{2}{3}$

For $i = 5$, $N = 15$ and $p = 0.6$, $p_i = 0.87$, $q_i = 0.13$

Small variations in p effect p_i significantly