System Response

System is represented as TF

Example

$$\frac{Y(s)}{U(s)} = \frac{1}{\tau s+1}$$

$$u(t) = t\,u(t) \Rightarrow U(s) = \frac{1}{s^2}$$

$$Y(s) = \frac{1}{s^2(\tau s+1)} = \frac{1}{s^2} - \frac{\tau}{s} + \frac{\tau^2}{\tau s+1}$$

$$y(t) = t\,u(t) - \tau\,u(t) + \tau e^{-\frac{1}{\tau}t}$$

$$y_{ss} = \lim_{t\to\infty} y(t) = (t-\tau)$$

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State space model

$$\dot{x} = Ax + Bu, \quad x(t_0) = x_0$$

$$x \in R^n, \; A \in R^{n \times n}, \; B \in R^n$$

$$\boxed{\dot{x} = Ax}$$ Find $x(t)$ given $x(t_0) = x_0$

$$\dot{x} = ax, \quad x(t_0) = x_0$$

Let $x(t)$ is a solution which is

$$\boxed{x(t) = k e^{at}}$$

LHS $= k a e^{at}$

RHS $= a k e^{at}$

$$x(t_0) = x_0 = k e^{a t_0}$$

$$k = x_0 e^{-a t_0}$$

$$x(t) = x_0 e^{a(t-t_0)}$$

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$$\dot{x} = ax, \quad x(t_0) = x_0$$

$$x(t) = x_0 e^{a(t-t_0)} \rightarrow \text{solution}$$

$$\dot{x} = Ax, \quad x \in R^n, \quad \underline{x}(t_0) = \underline{x}_0$$

$$e^{At} = I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots$$

$$\frac{d}{dt}e^{At} = A + A^2t + \frac{1}{2!}A^3t^2 + \cdots$$

$$= A\left[I + At + \frac{1}{2!}A^2t^2 + \cdots\right]$$

$$= A e^{At} \quad \Leftrightarrow \quad \frac{d}{dt}e^{at} = a e^{at}$$

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$$\boxed{\frac{d}{dt} e^{At} = A e^{At}}$$

$$\dot{x} = Ax, \quad x(t_0) = x_0$$

$$\underset{n\times 1}{x(t)} = \underset{n\times n}{e^{At}} \cdot \underset{n\times 1}{K}$$

$$A e^{At} K = A e^{At} K$$

$$x(t_0) = e^{At_0} K = x_0$$

$$K = e^{-At_0} x_0$$

$$x(t) = e^{At} e^{-At_0} x_0$$

$$= e^{A(t-t_0)} x_0$$

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$$\dot{x} = at$$

$$\Downarrow$$

$$x(t) = x_0 e^{a(t-t_0)}$$

$$\dot{x} = Ax$$

$$\Downarrow$$

$$\underset{n\times 1}{x(t)} = \underset{n\times n}{e^{A(t-t_0)}} \underset{n\times 1}{x_0}$$

$$\dot{x} = Ax + Bu \,, \quad x(t_0) = x_0$$

$$\frac{d}{dt} e^{At} = Ae^{At} = e^{At} \cdot A \quad \& \text{ verify}$$

$$e^{-At}\dot{x} = e^{-At}Ax + e^{-At}Bu$$

$$\frac{d}{dt}\left[e^{-At}x(t)\right] = \underbrace{e^{-At}\dot{x} - A^{-At}Ax}_{} = e^{-At}Bu$$

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$$\frac{d}{dt}\left(e^{-At}x(t)\right) = e^{-At}Bu(t)$$

$$\int_0^t d\left(e^{-A\tau}x(\tau)\right) = \int_0^t e^{-A\tau}Bu(\tau)\,d\tau$$

$$e^{-A\tau}x(\tau)\Big]_0^t = \int_0^t e^{-A\tau}Bu(\tau)\,d\tau$$

$$e^{At} \times e^{-At}x(t) - e^{At}x_0 = \int_0^t e^{At}e^{-A\tau}Bu(\tau)\,d\tau$$

$$x(t) - e^{At}x_0 = \int_0^t e^{A(t-\tau)}Bu(t)\,d\tau$$

$$\underbrace{x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)} B u(\tau)\, d\tau}_{\text{Generic Response}}$$

$$\dot{x} = Ax + Bu, \quad x(t_0) = x_0$$

First challenge $\boxed{\text{Compute } e^{At}}$

$$\mathcal{L}\,\dot{x} = \mathcal{L}\,Ax \qquad x(0) = x_0$$

$$sX(s) - x(0) = AX(s)$$

$$(sI - A)X(s) = x_0$$

$$X(s) = (sI - A)^{-1} x_0$$

$$x(t) = e^{A(t-t_0)} x_0 \;, \quad t_0 = 0$$

$$= e^{At} x_0$$

$$e^{At} = \mathcal{L}^{-1}(sI - A)^{-1}$$

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Ex $\quad A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

Compute e^{At}

$$(sI - A)^{-1} = \left\{ \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \right\}^{-1}$$

$$= \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}^{-1} = \frac{1}{s(s+3)+2} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

$$= \frac{1}{(s+1)(s+2)} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} = \Big[$$

$$(SI - A)^{-1} = \begin{bmatrix} \frac{2}{s+1} - \frac{1}{s+2} & \frac{1}{s+1} - \frac{1}{s+2} \\ -\frac{2}{s+1} + \frac{2}{s+2} & \frac{2}{s+2} - \frac{1}{s+1} \end{bmatrix}$$

$$e^{At} = \mathcal{L}^{-1}(SI - A)^{-1} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & 2e^{-2t} - e^{-t} \end{bmatrix}$$

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$$\dot{x} = Ax + Bu, \quad x(t_0) = x_0$$

$$x(t) = ?$$

$$= \underline{e^{At}x_0 + \int_0^t e^{A(t-\tau)} Bu(\tau)\,d\tau}$$

$$sX(s) - x_0 = AX(s) + BU(s)$$

$$(sI-A)X(s) = x_0 + BU(s)$$

$$X(s) = (sI-A)^{-1}x_0 + (sI-A)^{-1}BU(s)$$

$$\underline{x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)} Bu(\tau)\,d\tau}$$

Another example

$$\dot{x} = \begin{bmatrix}-1 & 1\\ -1 & 10\end{bmatrix} x + \begin{bmatrix}0\\ 10\end{bmatrix} u$$

Find out $x(t)$ under zero initial condition

$$x(t) = \int_0^t e^{A(t-\tau)} B\, u(\tau)\, d\tau$$

$$e^{At} = \mathcal{L}^{-1}(sI-A)^{-1} = \mathcal{L}^{-1}\frac{1}{s^2+11s+11}\begin{bmatrix}s+10 & 1\\ -1 & s+1\end{bmatrix}$$

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$$s^2 + 11s + 11 = (s + 1.1125)(s + 9.8875)$$

$$e^{At} = \begin{bmatrix} \underbrace{1.0128\, e^{-1.1125t} - 0.0128\, e^{-9.8875t}}_{1st} & 0.114\, e^{-1.1125t} - 0.114\, e^{-9.8875t} \\ -0.114\, e^{-1.1125t} + 0.114\, e^{-9.8875t} & -0.0128\, e^{-1.1125t} + 1.0128\, e^{-9.8875t} \end{bmatrix}$$

$$x(t) = \int e^{A(t-\tau)} B\, \cancel{u}(\tau) \, d\tau \qquad \cancel{u} \to 1$$

Find out step response

$$x(t) = \begin{bmatrix} 0.5054 - 1.0247e^{-1.1125t} + 0.1153e^{-5.88t} \\ -0.0132 + 0.1153e^{-1.1125t} - 0.1019e^{-5.88t} \end{bmatrix}$$

Summary

1. Response given TF

$$TF = \frac{Y(s)}{U(s)}$$

$$Y(s) = TF \times U(s)$$

$$y(t) = \mathcal{L}^{-1}\; TF \times U(s)$$

Response of a system in state space representation

1. $\dot{x} = Ax + Bu, \quad x(t_0) = x_0$

$$x(t) = e^{At} x_0 + \int_0^t e^{A(t-\tau)} B \underline{u(\tau)}\, d\tau$$

$$\underline{\underline{e^{At}}} = \mathcal{L}^{-1} (sI - A)^{-1}$$