MSO201A
Hints to Solution of Quiz-III

March 20, 2021

1. Use Jensen's inequality to show $\mathbb{E}(X)\mathbb{E}\left(\frac{1}{X}\right) \geq 1$. Note that $P[\frac{1}{X} \leq 1] = 1$, so $\mathbb{E}\left(\frac{1}{X}\right) \leq 1$.

$$\text{Suppose } 0 < a \leq x \leq b, \text{ then } 0 \leq (b-x)(x-a) = (a+b-x)x - ab$$

$$\implies \frac{1}{x} \leq \frac{a+b-x}{ab} \implies \mathbb{E}(X)\mathbb{E}\left(\frac{1}{X}\right) \leq \frac{\mathbb{E}(X)\big(a+b-\mathbb{E}(X)\big)}{ab}$$

$$\because a \leq \mathbb{E}(X) \leq b, \quad \therefore \mathbb{E}(X)\big(a+b-\mathbb{E}(X)\big) \leq \frac{(a+b)^2}{4}$$

$$\implies \mathbb{E}(X)\mathbb{E}\left(\frac{1}{X}\right) \leq \frac{(a+b)^2}{4ab}$$

Equality can be achieved for $P[X = a] = P[X = b] = 1/2$.

2. $\dfrac{\partial^2 F(x,y)}{\partial x \partial y}$ represents the joint density function and it has to be positive for a cdf $F(x,y)$, but $\dfrac{\partial^2 F_1(x,y)}{\partial x \partial y} < 0$ for $x > 0, y > 0$, so $F_1(x,y)$ is not a cdf.

   $F_2(x,y)$ satisfies all properties of a proper cdf (it is known as Morgenstern's bivariate distribution).

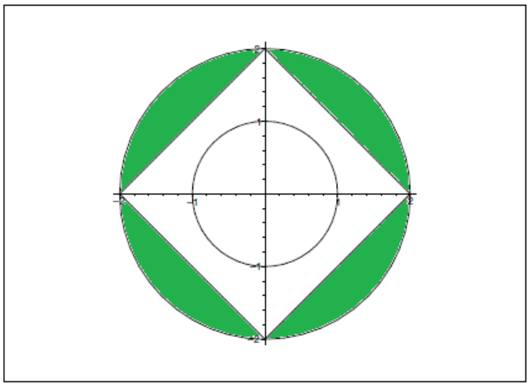
3. $$\mathbb{E}(X) = \int_0^{\infty} \Big(1 - F_X(x)\Big)dx - \int_{-\infty}^{0} F_X(x)dx$$

   where F_X is the cdf of random variable X. Hence, $\mathbb{E}(X) \geq \mathbb{E}(Y)$.

   Let $X \sim Unif(a,1)$, and $Y \sim Unif(0,a)$ independent of X. So, $F_X(x) \leq F_Y(x)\ \forall\ x \in \mathbb{R}$. But, $V(X) < V(Y)$ for $a \in (0.5, 1)$ and $V(X) > V(Y)$ for $a \in (0, 0.5)$.

   Observe that $P[h(X) \leq z] = P[X \leq h^{-1}(z)] \leq P[Y \leq h^{-1}(z)] = P[h(Y) \leq z]$

4.
Green colored area represents the region such that $|X| + |Y| \geq 2$ and the joint density $h(x, y)$ is positive.



It follows by considering the figure that $P\{|X| + |Y| \geq 2\}$ is equal to the integral of $h(x, y)$ over the four circular segments, thus equal to $\frac{1}{3\pi}$ times the area of these four circular segments, hence

$$P\{|X| + |Y| \geq 2\} = \frac{1}{3\pi}\left\{\pi \cdot 2^2 - (2\sqrt{2})^2\right\} = \frac{4}{3\pi}(\pi - 2) = \frac{4}{3} - \frac{8}{3\pi} \approx 0.485.$$

5. The random variables $X + Y$ and $|X - Y|$ are dependent as

$$P(X + Y = 0, |X - Y| = 0) \neq P(X + Y = 0)P(|X - Y| = 0).$$

The random variables $X + Y$ and $|X - Y|$ are uncorrelated since

$$\mathbb{E}[X|X - Y|] = 1 \times P(X = 1, Y = 0) = \frac{1}{4} = \mathbb{E}[Y|X - Y|],$$

and

$$\begin{aligned}\operatorname{cov}\big(X + Y, |X - Y|\big) &= \mathbb{E}\big\{(X + Y)|X - Y|\big\} - \mathbb{E}(X + Y)\mathbb{E}(|X - Y|)\\ &= \tfrac{1}{4} + \tfrac{1}{4} - 1 \cdot \tfrac{1}{2} = 0.\end{aligned}$$

6. Order statistics are equally likely by symmetry.

(a)

$$P(X_1 < X_2 > X_3) = P(X_2 \text{ is largest the observation among } \{X_1, X_2, X_3\}) = \frac{1}{3}.$$

Each observation among $\{X_1, X_2, X_3\}$ is equally likely to be the largest.

(c) Observe that each of the arrangements $\{X_1 < X_2 < X_3\}$, $\{X_1 < X_3 < X_2\}$, $\{X_2 < X_1 < X_3\}$, $\{X_2 < X_3 < X_1\}$, $\{X_3 < X_1 < X_2\}$ and $\{X_3 < X_2 < X_1\}$ are equally likely. Hence,

$$P(X_1 < X_2 < X_3) = \frac{1}{6}.$$

7. $F_U(u) = 1 - (1 - u)^2$, $0 < u < 1$. This follows, $\mathbb{E}(U) = 1/3$, and hence $\mathbb{E}(V) = 1 - \mathbb{E}(U) = 1 - 1/3$ as $U + V = X + Y$. Observe that

$UV = XY$, so that $\mathbb{E}(UV) = \mathbb{E}(X)\mathbb{E}(Y) = \frac{1}{4}$. Hence

$$\mathrm{cov}(U, V) = \mathbb{E}(UV) - \mathbb{E}(U)\mathbb{E}(V) = \frac{1}{4} - \frac{1}{3}(1 - \frac{1}{3}) = \frac{1}{36}$$

8. Here $Z \in \{2, 3, \ldots, 12\}$ and $W \in \{0, 1, 2, 3, 4, 5\}$ and observe that
   - $P(Z = 2, W = 0) = 0 \neq P(Z = 2)P(W = 0)$.
   - $P(X = 1, W = 0) = P(X = 1, Y = 5) = \frac{1}{36} = P(X = 1)P(W = 0)$, similarly it can be verified that $P(X = a, W = b) = P(X = a)P(W = b)$ for all possible values of a and b.
   - $P\big((X, Y) = (1, 1), Z = 2\big) \neq P\big((X, Y) = (1, 1)\big)P(Z = 2)$.
   - $P\big((X, Y) = (1, 1), W = 2\big) \neq P\big((X, Y) = (1, 1)\big)P(W = 2)$.
9. For n throws, $(X, Y, n - X - Y)$ follows trinomial distribution with parameters $(n, p, q, 1 - p - q)$ with $p = q = 1/6$.

$$\begin{aligned}\mathsf{Pr}(X = x \mid Y = y) &= \frac{\mathsf{Pr}(X = x \cap Y = y)}{\mathsf{Pr}(Y = y)} \\ &= \frac{\frac{n!}{x!y!(n-x-y)!}p^x q^y (1 - p - q)^{n-x-y}}{\frac{n!}{y!(n-y)!}q^y (1 - q)^{n-y}} \\ &= \frac{(n - y)!}{x!(n - x - y)!} \frac{p^x (1 - p - q)^{n-x-y}}{(1 - q)^{n-y}} \\ &= \binom{n - y}{x} \left(\frac{p}{1 - q}\right)^x \left(\frac{1 - p - q}{1 - q}\right)^{n-y-x},\end{aligned}$$

   which is the binomial distribution with parameters $(n - y, \frac{p}{1-q})$.

   Here $n = 90$, $y = 10$ and $p = q = 1/6$, and hence

$$\mathbb{E}(X|Y = 10) = (90 - 10) \times \frac{1}{5} = 16.$$

10. $a = \mathbb{E}(X) = \frac{\theta}{1+\theta}$ and $b = \mathbb{E}\big(\log(X)\big) = -\frac{1}{\theta}$.