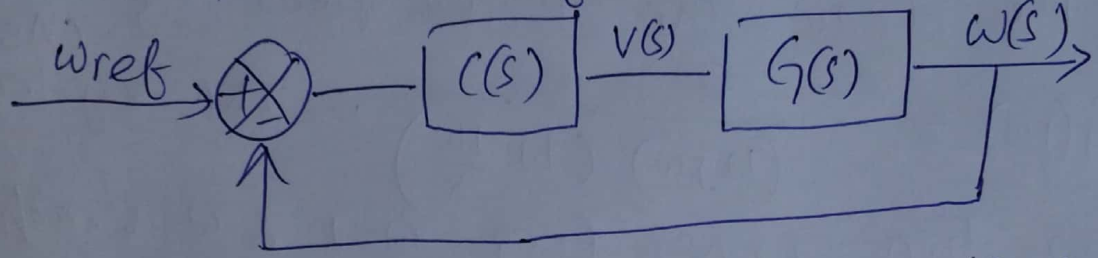
Tutorial sheet 8

EE 250   8 April 2021

Q1 A certain DC Motor has the transfer function $G(s) = \frac{\omega(s)}{V(s)} = \frac{10}{(s+1)(s+5)}$.

A speed controller is to be designed so that $\omega(t)$ follows ω_{ref} as given:



Design a lag compensator that would ensure 70^0 of phase margin. What is your observation from the unity step response of the compensated system? Redesign the compensator for PM = 50^0. Comment on the speed of response.

Solution Without the compensator

$$\frac{\omega(s)}{\omega_{ref}(s)} = \frac{G(s)}{1+G(s)} = \frac{10}{s^2+6s+15}$$

$$\omega_n = \sqrt{15} = 3.87 \text{ rad/sec.}$$

$$\zeta = 6/2\omega_n = 0.774.$$

$$t_s = \frac{4}{\zeta\omega_n} = 1.33 \text{ sec.}$$

The system has fast response.

But DC gain $= \frac{10}{15}$

Steady state error $= \left(1 - \frac{10}{15}\right) 100\,\%$

$= 33.3\,\%$

Thus we need a lag compensator to improve the steady state response.

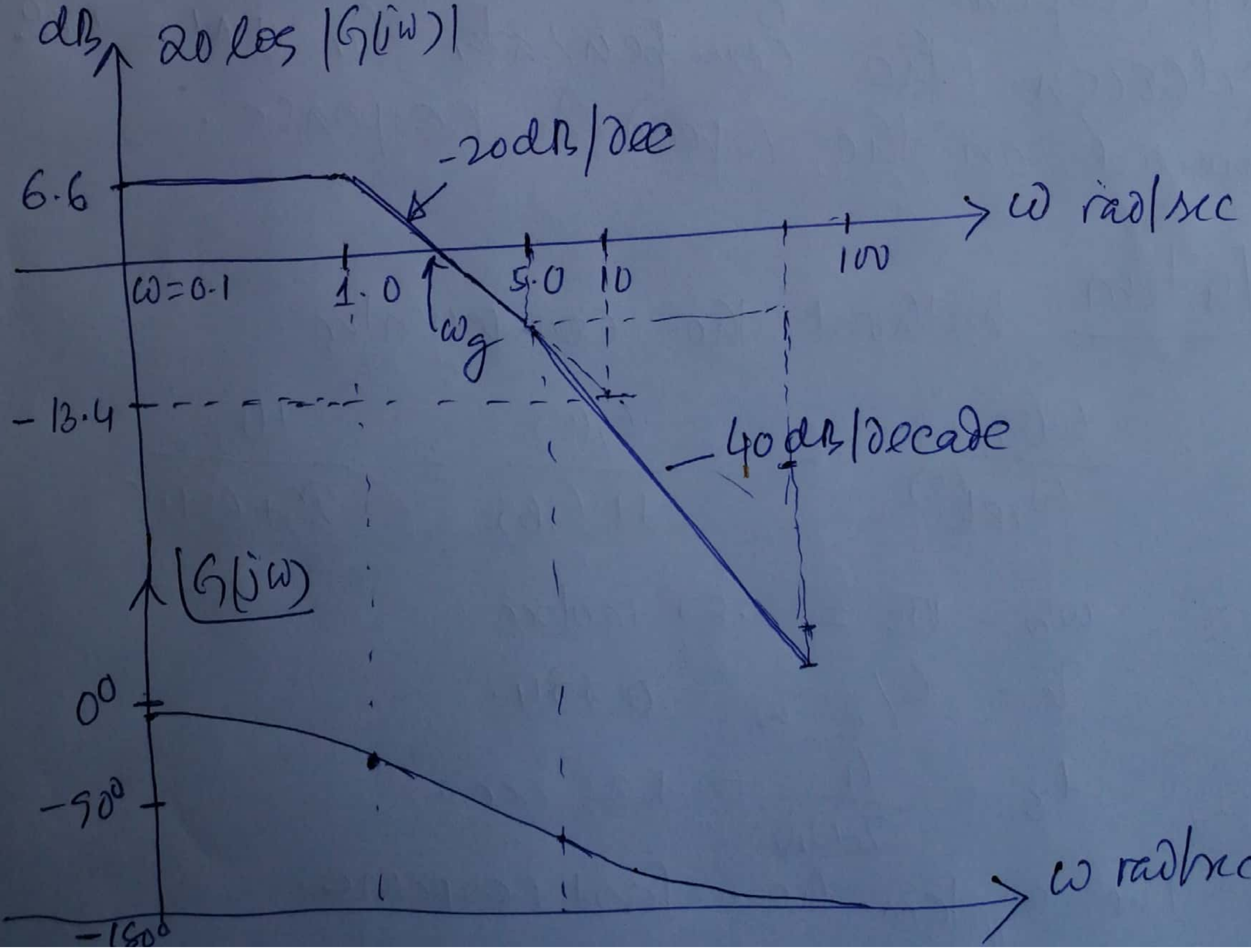
Let's draw the Bode plot of $G(s)$ (Asymptotic)

$$G(j\omega) = \frac{2}{(1+j\omega)\left(1+\frac{j\omega}{5}\right)}$$

. The system has two corner freq $\omega = 1$ & $\omega = 5$ rad/sec

$\angle G(j\omega) = -\tan^{-1}\omega - \tan^{-1}\omega/5$

	$\omega = 0.1$	$\omega = 1$	$\omega = 5$	$\omega = \infty$
$\angle G(j\omega)$	-6.8^0	-56.3^0	-123.7^0	-180^0



For the Bode plot

At ω_g, $\dfrac{2}{\sqrt{\omega^2+1}\,\sqrt{1+\frac{\omega^2}{25}}} = 1$

The uncompensated system has PM 104° at $\omega_g = 1.62$ rad/sec

$$(\omega^2+1)\left(1+\frac{\omega^2}{25}\right) = 4$$

$$\omega^4 + 26\omega^2 - 75 = 0$$

$$x^2 + 26x - 75 = 0 \Rightarrow x = 2.62 \Leftarrow \omega^2$$

$$\omega = \sqrt{2.62} = 1.62 \text{ rad/sec.}$$

For the lag compensator design, the general structure is:

$$C(s) = \alpha \frac{Ts+1}{\alpha Ts+1}, \quad \alpha > 1$$

As the compensator adds high gain (α) at steady state (low freq/zero freq), the PM is retained from the $G(j\omega)$.

Find ω for which $\angle G(j\omega) = -110^0$ such that PM $= 70^0$.

$$-\tan^{-1}\omega - \tan^{-1}\frac{\omega}{5} = -110$$

$$\frac{\omega + \frac{\omega}{5}}{1 - \frac{\omega^2}{5}} = \tan 110 = -2.75$$

$$\frac{6\omega}{5-\omega^2} = -2.75$$

$$2.75\omega^2 - 6\omega - 13.75 = 0$$

$$\omega = 3.58 \text{ rad/sec.}$$

At this ω, $|G(j\omega)| = \dfrac{10}{(3.58^2+1)^{1/2}(3.58^2+25)^{1/2}}$

$$= -7.2 \text{ dB.}$$

To ensure PM = 70°, $\omega_g = \omega \Rightarrow$

a gain 7.2 dB must be added

$$20 \log k = 7.2$$

$$k = 10^{\frac{7.2}{20}} = 2.29.$$

$$C(s) = \alpha \frac{Ts+1}{\alpha Ts+1}$$

Let $\alpha = 10$. The higher corner freq $\frac{1}{T}$ (that of zero) be placed at least a decade to the left of ω_g.

Let $\frac{1}{T} = 0.3$ rad/sec.

$$T = 3.33 \text{ sec}.$$

$$\alpha T = 33.3$$

$$C(s) = 10 \frac{3.33s+1}{33.3s+1}$$

Since we have added a gain of 2.29 to make $\omega = 3.58$ rad/sec as ω_g,

$$C(s) = 10 \times 2.29 \frac{3.33s+1}{33.3s+1}$$

$$= 22.9 \frac{3.33s+1}{33.3s+1}$$

Using matlab command margin(C(s).G(s)), we can verify that the compensated system has PM = 65.5° at 3.59 rad/sec. There is a slight reduction in required PM as C(s) has contributed negative phase at ω_g.

This can be rectified by placing the Corner freq $\frac{1}{T}$ left of 0.3 rad/sec.

$$T(s) = \frac{C(s)G(s)}{1+C(s)G(s)}$$

$$= \frac{762.6s+229}{33.3s^3+200.8s^2+935.1s+234}$$

from Matlab, step(T(s)) shows that although the steady state error has been eliminated, the settling time has increased to 8 sec (too high as compared to 1.3 sec of the uncompensated system).

Speed of response is generally increased by increasing the bandwidth or ω_g. This would imply that we reduce the gain margin. Let the revised gain margin is 50^0.

$$\frac{6\omega}{5-\omega^2} = \tan 130^0 = -1.19$$

$$1.19\omega^2 - 6\omega - 5.95 = 0 \Rightarrow \omega = 5.89 \text{ rad/sec.}$$

$$|G(j\omega)|_{\omega=5.89} = 0.2166 = -13.28 \text{ dB}.$$

The gain needs to be added 13.28 dB to make this ω as ω_g.

$$K = 10^{13.28/20} = 4.61$$

If we retain the same C(s),

$$C(s) = 4.61 \times 10 \frac{3.33s+1}{33.3s+1} = 46.1 \frac{3.33s+1}{33.3s+1}$$

From MATLAB, margin(C(s)G(s)) show

PM = 47.3^0 at $\omega_g = 5.89$ rad/sec.

$$T(s) = \frac{C(s)G(s)}{1+C(s)G(s)}$$

$$= \frac{1535s + 461}{33.3s^3 + 200.8s^2 + 1708s + 466}$$

Step(T(s)) shows that the settling time is less than 3 sec..

Q2 Design a lead compensator for a system in unity feedback configuration whose open loop transfer fn is

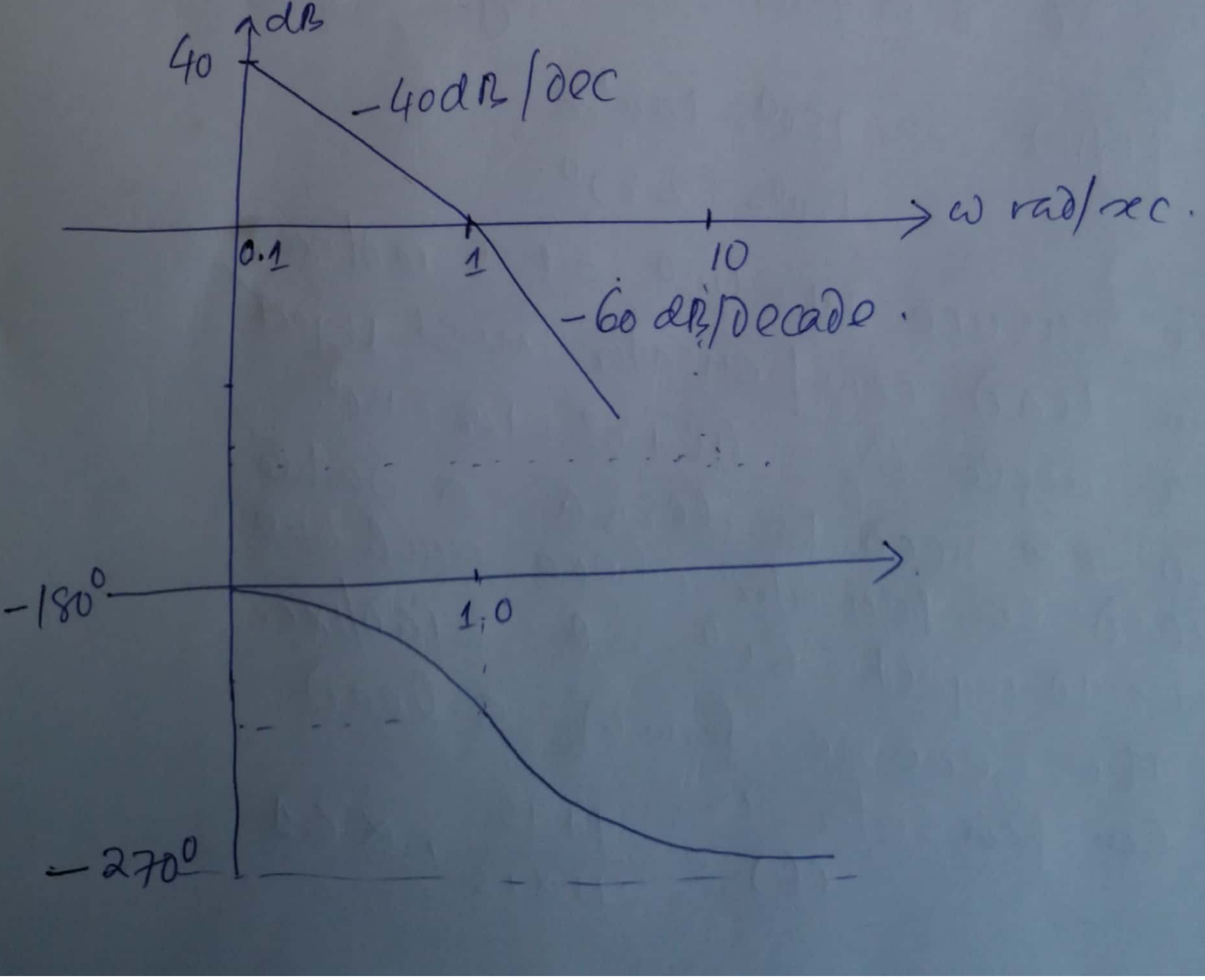
$$G(s) = \frac{1}{s^2(s+1)}.$$

The compensator should ensure a phase margin of 45°.

Soln

$$G(j\omega) = \frac{1}{(j\omega)^2(1+j\omega)}$$

$$\angle G(j\omega) = -180^0 - \tan^{-1}\omega.$$



$$|G(j\omega)| = \frac{1}{\omega^2(1+\omega^2)^{1/2}} = 1$$

$$\omega^4(1+\omega^2) = 1$$

$$\omega = 0.8688 \text{ rad/sec.}$$

$$\angle G(j\omega) = -180^0 - \tan^{-1} 0.8688$$

$$= -180^0 - 41^0$$

So the system has $PM = -41^0$ at $\omega_g = 0.8688$ rad/sec. (Please validate through MATLAB as well)

If we keep $\omega_g = 0.87$ rad/sec, then the system response will be very sluggish. So Let's fix ω_g at 8 rad/sec. At $\omega_g = 8$ rad/sec,

$$\angle G(j\omega) = -180^0 - \tan^{-1} 8$$

$$= -180^0 - 82.87^0.$$

To ensure $PM = 45^0$ at 8 rad/sec, the lead compensator must inject a angle of $45 + 83 = 128 \simeq 130^0$.

So ~~we~~ need to design a double lead compensator where each one would inject 65^0 at 8 rad/sec.

The generic form of a lead compensator is

$$C(s) = \frac{1+Ts}{1+\alpha Ts}, \ \alpha < 1$$

where the max angle contribution takes place at

$$\frac{1}{\sqrt{\alpha}\,T} = 8$$

$$\alpha = \frac{1-\sin\phi_m}{1+\sin\phi_m}, \quad \phi_m = 65^0$$

$$\cong 0.05$$

$$\frac{1}{\sqrt{0.05}\,T} = 8 \Rightarrow T = 0.56$$

Thus $C(s) = \dfrac{0.56s+1}{0.028s+1} \triangleq \dfrac{1+Ts}{1+\alpha Ts}$.

We have to design K such that

$$\left| K\, C(s)\, C(s)\, G(s) \right|_{\omega = 8\,\text{rad/sec}} = 1$$

$$\left| C^2(j\omega)\, G(j\omega) \right| = \left[\frac{((0.56\omega)^2+1)}{((0.028\omega)^2+1)} \cdot \frac{1}{\omega^2(1+\omega^2)^{1/2}} \right]_{\omega=8\,\text{rad/sec}}$$

$$= \frac{21.06}{1.0502} \times \frac{1}{8^2 \times 8.06}$$

$$= 0.0389$$

$$= -28.2\ \text{dB}.$$

We have to add 28.2 dB to achieve $\omega_c = 8$ rad/sec.

$$20 \log k = 28.2$$

$$k = 10^{28.2/20}$$

$$= 25.7$$

Thus the compensator is

$$C_D(s) = 25.7 \left(\frac{0.56s+1}{0.028s+1}\right)^2$$

In MATLAB, one can verify that the phase margin of the compensated system is PM = 46.7° at 8 rad/sec.

The closed loop transfer function is

$$T(s) = \frac{8.06s^2 + 28.78s + 25.7}{0.000784s^5 + 0.05678s^4 + 1.056s^3 + 9.06s^2 + 28.78s + 25}$$

The step response using MATLAB shows that $t_s = 1$ sec with peak overshoot of 34%. If the desired peak overshoot is 20%, then we have to reduce ω_g and redesign.