MSO202A COMPLEX VARIABLES
Solutions–1

Problems for Discussion:

1. For any $z, w \in \mathbb{C}$, show that (a) $\overline{z+w} = \overline{z} + \overline{w}$,(b) $\overline{zw} = \overline{z}\overline{w}$, (c) $\overline{\overline{z}} = z$, (d) $|\overline{z}| = |z|$ and (e) $|zw| = |z||w|$.

   Solution: Easy.

2. Show that $(a)|z+w|^2 = |z|^2 + |w|^2 + 2\text{Re}(zw)$

   Solution: $|z+w|^2 = (z+w)\overline{(z+w)} = |z|^2 + |w|^2 + (z\overline{w} + \overline{z}w) = |z|^2 + |w|^2 + 2\text{Re}(zw)$

   $(b)|z+w|^2 + |z-w|^2 = 2(|z|^2 + |w|^2)$.

   Solution: Follows from (b). $(c)|z+w| = |z|+|w|$ if and only if either $zw = 0$ or $z = cw$ for some positive real number c. Solution:(c) If $|z+w| = |z| + |w|$ and $zw \neq 0$, then we see that $\text{Re}(zw) = |zw|$. Hence, $z\overline{w}$ must be a positive real,say c. Thus $z = c\frac{w}{|w|^2}$. Conversely, if $zw = 0$, then either $z = 0$ or $w = 0$. If $z = cw$, then $|z+w| = (1+c)|w| = |z| + |w|$.

3. Let z be a non zero complex number and n a positive integer. If $z = r(\cos\theta + i\sin\theta)$, show that $z^{-n} = r^{-n}(\cos n\theta - \sin n\theta)$.

   Solution: $z = r(\cos\theta + i\sin\theta)$. For $n > 0, z^n = r^n(\cos n\theta + i\sin n\theta)$. $z^{-n} = \dfrac{1}{z^n} = \dfrac{1}{r^n(\cos n\theta + i\sin\theta)} = r^{-n}(\cos n\theta - \sin n\theta)$.

4. Let α be any of the n th roots of unity except 1. Show that $1+\alpha+\alpha^2+\ldots+\alpha^{n-1} = 0$.

   Solution: For any $z \neq 1$, we know that $1 + z + z^2 + \ldots + z^k = \frac{z^{k+1}-1}{z-1}$. Let α be any root different from 1. The result follows from the above observation.

5. Find the roots of each of the following in the form $x+iy$. Indicate the principal root (a) $\sqrt{2i}$, (b) $(-1)^{1/3}$ and (c) $(-16)^{1/4}$.

   Solution: (a) $2i = 2e^{i(\frac{\pi}{2}+2k\pi)} \Rightarrow \sqrt{2i} = \sqrt{2}e^{i(\frac{\pi}{4}+k\pi)} = 1+i$, when $k = 0$ and is $-1-i$ when $k = 1$. $k = 0$ corresponds to the principal root. (b) $-1 = e^{i(\pi+2k\pi)} \Rightarrow (-1)^{\frac{1}{3}} = e^{i(\frac{\pi}{3}+2k\frac{\pi}{3})}$. When $k = 0$ this is $\frac{1+i\sqrt{3}}{2}$, which corresponds to the principal root and when $k = 1$ this is -1, when $k = 2$ this is $\frac{1-i\sqrt{3}}{2}$.

   (c) $(-16) = 16e^{i(\pi+2k\pi)} \Rightarrow (-16)^{\frac{1}{4}} = 2e^{i(\pi/4+k\pi/2)}$. For $k = 0$ this is $\sqrt{2}(1+i)$, when $k = 1$ this is $\sqrt{2}(-1+i)$, when $k = 2$ this is $\sqrt{2}(-1-i)$, when $k = 3$ this is $\sqrt{2}(1-i)$. $k = 0$ corresponds to the principal root.

6. Determine the values of the following:

   (a) $(1+i)^{20} - (1-i)^{20}$.

   Solution : $1+i = \sqrt{2}e^{i\pi/4}$, so $(1+i)^{20} = \sqrt{2}^{20}e^{i5\pi} = \sqrt{2}^{20}$, thus $(1+i)^{20} - (1-i)^{20} = 0$.

   (b) $\cos\pi 4 + i\cos\dfrac{3}{4}\pi + \ldots + i^n\cos\dfrac{2n+1}{4}\pi + \ldots + i^{40}\cos\dfrac{81}{4}\pi$.

Solution : Let $a_n = i^n \cos \frac{2n+1}{4}\pi$ Then $a_{n+2} = -i^n \cos\left(\frac{2n+1}{4}\pi + \pi\right) = a_n$. Thus, $a_0 = a_2 = \ldots = a_{40}$ and $a_1 = a_3 = \ldots = a_{39}$. So, $a_0 + \ldots + a_{40} = 21a_0 + 20a_1 = \frac{\sqrt{2}}{2}(21 - 20i)$.

7. Find the roots of $z^4 + 4 = 0$. Use these roots to factor $z^4 + 4$ as a product of two quadratics with real coefficients.

   Solution : $z = \sqrt{2}e^{i(\frac{\pi}{4}+\frac{k\pi}{2})}, k = 0,1,2,3$. So the roots are $z_0 = 1+i, z_1 = -1+i, z_2 = -1-i, z_3 = 1-i$. Thus $z^4 + 4 = (z - z_0)(z - z_1)(z - z_2)(z - z_3) = (z^2 - 2z + 2)(z^2 + 2z + 2)$.

8. Determine whether the following sets describe domains (open and connected sets) in $\mathbb{C}$: (*a*) Re $z > 1$ (*b*) $0 \leq Arg z \leq \frac{\pi}{4}$ (*c*) Im $(z) = 1$, (*d*) $|z - 2 + i| < 1$ (*e*) $|2z + 3| > 4$.

   Solution:

   (a) Re $z > 1$. This implies $x > 1$, the half plane, which is open and connected.
   (b) (b) $0 \leq Arg z \leq \frac{\pi}{4}$. This is connected but not open and hence not a domain.
   (c) Im $(z) = 1$. This is the line $y = 1$ which is not open and hence not a domain.
   (d) $|z - 2 + i| < 1$. Interior of the circle with center $(2, -1)$ and has radius 1. Hence, it is a domain.
   (e) $|2z + 3| > 4$. The exterior of the circle of radius 2 and center $(-3/2, 0)$. This is a domain.

Problem for Tutorial:

1. Give a geometric description of the following sets:

   (a) $\{z \in \mathbb{C} : |z + i| \geq |z - i|\}$
   Solution : This is the upper half plane.
   (b) $\{z \in \mathbb{C} : |z - i| + |z + i| = 2\}$.
   Solution: Note that the distance between i and $-i$ is 2. Thus the points on the line joining i and $-i$ have the sum of distances from i and $-i$ equal to 2 and these are all as otherwise triangle inequality is violated for the triangle with vertices on $\pm i, z$ for z outside this line.

2. Discuss the convergence of the following sequences: (a) (z^n), (b) $(\frac{z^n}{n!})$, (c) $(i^n \sin \frac{n\pi}{4})$ and (d) $(\frac{1}{n} + i^n)$.

   Solution : (a) If (z^n) converges, then so does $|(z^n)|$ and hence $|z| \leq 1$. If $|z| < 1, z^n \to 0$ and if $z = 1$, then $z^n \to 1$. If $|z| = 1, z \neq 1$, then $\lim_{n\to\infty} z^n = l \Rightarrow |l| = 1$. Now $z^{n+1} - z^n \to 0 \Rightarrow l(1 - z) = 0 \Rightarrow l = 0$, a contradiction. Alt: $z^n = r^n e^{in\theta}$ which has a limit if $r < 1$, for other r, consider the behaviour of $\cos n\theta$ and $\sin n\theta$. (b) converges to 0, using tests for real sequences applied to $|\frac{z^n}{n!}|$. (c) and (d) do not converge.

3. Discuss the behaviour of $e^{1/z}$ as z approaches 0.

   Solution : The limit does not exist as the limit along the positive x axis is ∞ and 0 along the negative x axis.

4. Find all the values in $[0, 2\pi)$ where $\lim_{r\to\infty} e^{re^{i\theta}}$ exists.

   Solution : Since $e^{re^{i\theta}} = e^{r\cos\theta}e^{ir\sin\theta}$, if this limit exists, then so must $lim_{r\to\infty}e^{re^{i\theta}} = lim_{r\to\infty}e^{r\cos\theta}$. Hence, $\cos\theta \le 0$. If $\cos\theta = 0$, then $\theta = \pi/2, 3\pi/2$ in which case $\lim_{r\to\infty} e^{re^{i\theta}} = \lim_{r\to\infty} e^{\pm ir}$ which does not exist. For $\pi/2 < \theta < 3\pi/2, \lim_{r\to\infty} |e^{re^{i\theta}}| = 0$. Thus, $\pi/2 < \theta < 3\pi/2$.

5. Verify if the following functions can be given a value at $z = 0$, so that they become continuous: $f(z) = \frac{|z|^2}{z}$, $f(z) = \frac{z+1}{|z|-1}$, $f(z) = \frac{\bar{z}}{z}$, $\frac{\text{Im }(z^2)}{|z|}$, $\frac{\text{Im }z}{1-|z|}$.

   Solution: $(a) \lim\limits_{z\to 0} f(z) = 0, \quad (b) \ -1$ and for part (c) the limit does not exist In (d)),

$$f(z) = \frac{2xy}{\sqrt{x^2+y^2}} + i0 \to 0 + i0 = \frac{r^2 \sin 2\theta}{r} + i0 \quad r \to 0,$$

   hence assigning $f(0) = 0$ makes f continuous at $z = 0$.

   In case of (e), we have

$$f(z) = \frac{y}{1 - \sqrt{x^2+y^2}} + i0 = \frac{r\sin\theta}{1-r} + i0 \to 0 + i0, \quad r \to 0,$$

   hence assigning $f(0) = 0$ makes f continuous at $z = 0$.