MSO 201: Probability and Statistics

II Semester

Assignment No. 7

Hints/ Solutions

Problem 1 (a) M.L.E = M.M.E $= \frac{1}{\bar{X}}$. By WLLN $\bar{X} \xrightarrow{P} E(X_1) = \frac{1}{\theta} \Rightarrow$ M.L.E = M.M.E = $\frac{1}{\bar{X}} \xrightarrow{P} \theta$, so consistent. ($g(x) = \frac{1}{x}$ is continuous for $x > 0$)

(b) M.L.E. = M.M.E $= e^{\bar{X}}$. Using W.L.L.N $\bar{X} \xrightarrow{P} \theta \Rightarrow$ M.M.E = M.L.E $= e^{\bar{X}} \xrightarrow{P} e^{\theta}$, so consistent ($g(x) = e^x$ is continuous).

(c) Let $S =$ # of $X_1, \ldots, X_n$ that are one. Then

$$L_{\underset{\sim}{x}}(\theta_1, \theta_2) = \begin{cases} \dfrac{\theta_1^S (1-\theta_1)^{n-S}}{(\theta_2-1)^{n-S}}, & 0<\theta_1<1,\ \theta_2 \geq x_{(n)},\ \theta_2 \in \{2, 3, \ldots\} \\ 0, & \text{o.w.} \end{cases}$$

$$\text{M.L.E} = \underset{\sim}{\delta}_{ML} = (\delta_{1ML}, \delta_{2ML}) = \begin{cases} \left(\frac{S}{n}, X_{(n)}\right), & \text{if } 0 \leq S < n \\ (1,1) \text{ or } (1,2) \text{ or } (1,3) \text{ or} \ldots, & \text{if } S = n \end{cases}$$

M.L.E. is not unique. In particular $\left(\frac{S}{n}, X_{(n)}\right)$ is a M.L.E.

$$E_{\underset{\sim}{\theta}}(X_1) = 1 + \frac{\theta_2(1-\theta_1)}{2}, \quad E_{\underset{\sim}{\theta}}(X_1^2) = \theta_1 + \frac{(1-\theta_1)}{6}(2\theta_2^2 + 5\theta_2 + 6)$$

$$\Rightarrow \hat{\theta}_1 = 1 - \frac{2(A_1-1)}{\hat{\theta}_2}, \quad \hat{\theta}_2 = 0 \text{ or } \hat{\theta}_2 = \frac{3(A_2-1)}{2(A_1-1)} - \frac{5}{2}.$$

Thus M.M.E: $\underset{\sim}{\delta}_{MM} = (\delta_{1MM}, \delta_{2MM}) = \left(1 - \frac{2(A_1-1)}{\delta_{2MM}}, \frac{3(A_2-1)}{2(A_1-1)} - \frac{5}{2}\right)$,

where $A_1 = \bar{X}$ and $A_2 = \frac{(n-1)S^2}{n} + A_1^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2$.

$S \sim Bin(n, \theta_1) \Rightarrow \frac{S}{n} \xrightarrow{P} \theta_1 \Rightarrow \frac{S}{n}$ is consistent for θ_1.

Since $E_{\underset{\sim}{\theta}}(X_1)$ and $E_{\underset{\sim}{\theta}}(X_1^2)$ are continuous functions of (θ_1, θ_2), δ_{1MM} and δ_{2MM} are consistent.

For fix $\varepsilon > 0$,

$$0 \leq P[|X_{(n)} - \theta_2| > \varepsilon] = P[X_{(n)} < \theta_2 - \varepsilon] \leq P[X_{(n)} \leq \theta_2 - \varepsilon]$$

$$= \begin{cases} 0, & 2 \leq \theta_2 < 1+\varepsilon,\ \varepsilon > 1 \\ \left[\theta_1 + \frac{1-\theta_1}{\theta_2 - 1}\left(\frac{[\theta_2-\varepsilon]([\theta_2-\varepsilon]+1)}{2} - 1\right)\right]^n, & \text{if } \theta_2 \geq 1+\varepsilon \end{cases}$$

$$\longrightarrow 0, \text{ as } n \to \infty.$$

Thus δ_{2ML} is consistent for θ_2.

Note that δ_{2MM} may take non-integer values whereas θ_2 takes integer values.

(d) $K(\theta) = (\theta+1)(\theta+2)$, $\delta_{ML}(\underline{X}) = \dfrac{-\sqrt{\bar{L}^2+4} - (3\bar{L}+2)}{2\bar{L}}$, where

$\bar{L} = \frac{1}{n}\sum_{i=1}^{n} \ln x_i$.

$E_\theta(X_1) = \dfrac{\theta+1}{\theta+3}$, $A_1 = \dfrac{\hat{\theta}+1}{\hat{\theta}+3}$ $\Rightarrow$ $\hat{\theta} = \dfrac{3A_1-1}{1-A_1}$

$\Rightarrow$ $\delta_{MM}(\underline{X}) = \dfrac{3\bar{X}-1}{1-\bar{X}}$ (M.M.E.)

Since $E_\theta(X_1)$ is a continuous function of θ, $\delta_{MM}(\underline{X})$ is consistent.

$E(\ln X_1) = -\dfrac{(2\theta+3)}{(\theta+1)(\theta+2)}$ $\Rightarrow$ $\bar{L} \xrightarrow{P} -\dfrac{(2\theta+3)}{(\theta+1)(\theta+2)}$

$\Rightarrow$ $\delta_{ML} \xrightarrow{P} \theta$ $\Rightarrow$ δ_{ML} is consistent

(e) $L_{\underline{x}}(\alpha,\mu) = \left(\dfrac{1}{\Gamma\alpha\, \mu^\alpha}\right)^n e^{-\frac{\sum_{i=1}^n x_i}{\mu}} \left(\prod_{i=1}^n x_i\right)^{\alpha-1}$, $\alpha>0$, $\mu>0$

$\ln L_{\underline{x}}(\alpha,\mu) = -n\ln(\Gamma\alpha) - n\alpha\ln(\mu) - \frac{1}{\mu}\sum_{i=1}^n x_i + (\alpha-1)\sum_{i=1}^n \ln x_i$, $\alpha>0$ $\mu>0$

$\frac{\partial}{\partial\alpha}\ln L_{\underline{x}}(\alpha,\mu)=0$, $\frac{\partial}{\partial\mu}\ln L_{\underline{x}}(\alpha,\mu)=0$

$$\Rightarrow \begin{cases} n\dfrac{\Gamma'(\alpha)}{\Gamma(\alpha)} + n\ln(\mu) - \sum_{i=1}^n \ln(x_i) = 0 \\ n\mu\alpha - \sum_{i=1}^n x_i = 0 \end{cases} \Rightarrow \quad \hat{\mu} = \frac{\bar{X}}{\hat{\alpha}}, \quad n\frac{\Gamma'(\hat{\alpha})}{\Gamma(\hat{\alpha})} + n\ln\left(\frac{\bar{X}}{\hat{\alpha}}\right) - \sum_{i=1}^n \ln(x_i) = 0,$$

$\Gamma'(\alpha) = \int_0^\infty (\ln x)\, e^{-x} x^{\alpha-1}\,dx$ $\longrightarrow$ to be solved numerically. (M.L.E.)

Proving consistency of M.L.E. is a difficult problem.

$E_{\underline{\theta}}(X_1) = \alpha\mu$, $E_{\underline{\theta}}(X_1^2) = \alpha(\alpha+1)\mu^2$ $\Rightarrow$ $\hat{\alpha}\hat{\mu} = A_1$, $\hat{\alpha}(\hat{\alpha}+1)\hat{\mu}^2 = A_2$

$\Rightarrow$ $\hat{\alpha} = \dfrac{A_1^2}{A_2 - A_1^2}$, $\hat{\mu} = \dfrac{A_2 - A_1^2}{A_1}$, i.e. $\underset{\sim}{\delta}_{MM} = (\delta_{1M}, \delta_{2M})$, where

$\delta_{1M} = \dfrac{\bar{X}^2}{\frac{n-1}{n}S^2}$, $\delta_{2M} = \dfrac{\left(\frac{n-1}{n}\right)S^2}{\bar{X}}$.

Since $E_{\underline{\theta}}(X_1)$ and $E_{\underline{\theta}}(X_1^2)$ are continuous functions of (α,μ), δ_{1M} and δ_{2M} are consistent.

(f) $\underset{\sim}{\delta}_{ML} = (\delta_{1ML}, \delta_{2ML})$, where $\delta_{1ML} = \frac{1}{n}\sum_{i=1}^n \ln x_i$, $\delta_{2ML} = \frac{1}{n}\sum_{i=1}^n (\ln x_i - \delta_{1ML})^2$.

$E(\ln X_1) = \mu$, $Var(\ln X) = \sigma^2$ $\Rightarrow$ $\delta_{1ML} \xrightarrow{P} \mu$, $\delta_{2ML} \xrightarrow{P} \sigma^2$,

so δ_{1ML} and δ_{2ML} are consistent estimators of μ and σ^2.

(g) $g(\theta) = P_\theta(X \le 1) = 1 - e^{-\frac{1}{\theta}}$

M.L.E of θ is $\bar{X}$ $\Rightarrow$ M.L.E of $g(\theta)$ is $\delta_{ML}(\underline{X}) = 1 - e^{-\frac{1}{\bar{X}}}$

By W.L.L.N $\bar{X} \xrightarrow{P} \theta$ and since $g(\theta)$ is a continuous function of θ

$\Rightarrow$ $\delta_{ML}(\underline{X}) \xrightarrow{P} 1 - e^{-\frac{1}{\theta}} = g(\theta)$ $\Rightarrow$ $\delta_{ML}(\underline{X})$ is consistent for estimating $g(\theta)$

$E_\theta(X_1) = \theta$. So M.M.E of $g(\theta)$ is $\delta_{MM}(\underline{X}) = 1 - e^{-\frac{1}{\bar{X}}} = \delta_{ML}(\underline{X})$

(h) $g(\theta) = P_\theta(X_1 + X_2 + X_3 = 0) = e^{-3\theta}$

M.L.E of θ is $\bar{X}$ $\Rightarrow$ M.L.E. of $g(\theta)$ is $\delta_{ML}(\underline{X}) = e^{-3\bar{X}}$.

By W.L.L.N $\bar{X} \xrightarrow{P} E(X_1) = \theta$ and since $g(\theta)$ is a continuous function of θ $\Rightarrow$ $\delta_{ML}(\underline{X}) \xrightarrow{P} e^{-3\theta} = g(\theta) \Rightarrow \delta_{ML}(\underline{X})$ is consistent for $g(\theta)$.

$E_\theta(X_1) = \theta$. So M.M.E. of $g(\theta)$ is $\delta_{MM}(\underline{X}) = e^{-3\bar{X}} = \delta_{ML}(\underline{X})$.

(i) $l_{\underline{x}}(\theta) = \begin{cases} \frac{1}{\theta^n}, & \text{if } \theta \ge 2T \\ 0, & \text{o.w} \end{cases}$, where $T = \max(|X_1|, \ldots, |X_n|)$.

So M.L.E. of θ is $2T$ $\Rightarrow$ M.L.E. of $g(\theta) = (1+\theta)^{-1}$ is $\delta_{ML}(\underline{X}) = (1+2T)^{-1}$.

Fix $\varepsilon > 0$ then

$P(|2T - \theta| > \varepsilon) = P(\frac{\theta}{2} - T > \frac{\varepsilon}{2}) = P(T < \frac{\theta}{2} - \frac{\varepsilon}{2}) = \begin{cases} 0, & \text{if } \varepsilon > 2\theta \\ (1 - \frac{\varepsilon}{2\theta})^n, & \text{if } 0 < \varepsilon \le 2\theta \end{cases}$

$\to 0$, as $n \to \infty$. Thus $2T \xrightarrow{P} \theta$ $\Rightarrow$ $\delta_{ML}(\underline{X}) \xrightarrow{P} (1+\theta)^{-1}$ (since $g(\theta) = (1+\theta)^{-1}$ is a continuous function for $\theta > 0$). So M.L.E is consistent

$E_\theta(X_1) = 0$. So method of moment for estimation fails.

But $E_\theta(X_1^2) = \frac{\theta^2}{12}$. So modified M.M.E of θ can be obtained from

$A_2 = \frac{\hat{\theta}^2}{12} \Rightarrow \hat{\theta} = \sqrt{12 A_2}$. So modified M.M.E. $\delta_{MM}(\underline{X}) = (1 + \sqrt{12 A_2})^{-1}$; here $A_2 = \frac{1}{n}\sum_{i=1}^n X_i^2$. By W.L.L.N $A_2 \xrightarrow{P} \frac{\theta^2}{12}$

$\Rightarrow$ $\delta_{MM}(\underline{X}) \xrightarrow{P} g(\theta) = (1+\theta)^{-1}$. So modified M.M.E $\delta_{MM}(\underline{X})$ is consistent.

(j) M.L.E. of (μ, σ^2) is $(\hat{\mu}, \hat{\sigma}^2) = (\bar{X}, \frac{1}{n}\sum_{i=1}^n (X_i - \bar{X})^2) = (A_1, A_2 - A_1^2)$.

So M.L.E of $g(\theta)$ is $\delta_{ML}(\underline{X}) = \left(\frac{\bar{X}^2}{\frac{1}{n}\sum(X_i - \bar{X})^2}\right)$.

$\bar{X} \xrightarrow{P} \mu$, $\frac{1}{n}\sum_{i=1}^n (X_i - \bar{X})^2 \xrightarrow{P} \sigma^2$ $\Rightarrow$ $\delta_{ML}(\underline{X}) \xrightarrow{P} \frac{\mu^2}{\sigma^2} = g(\theta)$

$\Rightarrow$ M.L.E. $\delta_{MM}(\underline{X})$ is consistent.

$E(X_1) = \mu$, $E(X_1^2) = \sigma^2 + \mu^2$ $\Rightarrow$

M.M.E of (μ, σ^2) is $(\hat{\mu}, \hat{\sigma}^2) = (A_1, A_2 - A_1^2)$

$\Rightarrow$ M.M.E of $g(\theta)$ is $\delta_{MM}(\underline{X}) = \frac{\bar{X}^2}{\frac{1}{n}\sum_{i=1}^n (X_i - \bar{X})^2} = \delta_{ML}(\underline{X})$.

(b) $$L^*_{\underset{\sim}{x}}(\mu,\sigma) = \ln L_{\underset{\sim}{x}}(\mu,\sigma) = \begin{cases} -n\ln\sigma - \dfrac{\sum_{i=1}^{n}(x_i-\mu)}{\sigma}, & \mu \le x_{(1)}, \sigma>0 \\ 0, & \text{o.w.} \end{cases}$$

Clearly

$$L^*_{\underset{\sim}{x}}(\mu,\sigma) \le L^*_{x}(x_{(1)},\sigma), \quad \forall\ \mu\le x_{(1)},\ \sigma>0.$$

$$L^*_{x}(x_{(1)},\sigma) = -n\ln\sigma - \frac{1}{\sigma}\sum_{i=1}^{n}(x_i-x_{(1)}), \quad \sigma>0$$

$$\frac{\partial}{\partial\sigma} L^*_{\underset{\sim}{x}}(x_{(1)},\sigma) = -\frac{n}{\sigma} + \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-x_{(1)})$$

$\Rightarrow$ $L^*_{\underset{\sim}{x}}(x_{(1)},\sigma)\ \uparrow (\downarrow)$ if $\sigma < (>) \frac{1}{n}\sum_{i=1}^{n}(x_i-x_{(1)}) = \hat{\sigma}$, say.

Thus

$$L^*_{\underset{\sim}{x}}(\mu,\sigma) \le L^*_{\underset{\sim}{x}}(x_{(1)},\sigma) \le L^*_{x}\left(x_{(1)}, \frac{1}{n}\sum_{i=1}^{n}(x_i-x_{(1)})\right), \quad \forall\ \mu\le x_{(1)},\ \sigma>0$$

$\Rightarrow$ M.L.E. of (μ,σ) is $\underset{\sim}{\delta}_{ML}(\underset{\sim}{X}) = (\delta_{1ML}(\underset{\sim}{X}), \delta_{2ML}(\underset{\sim}{X})) = (X_{(1)}, \frac{1}{n}\sum_{i=1}^{n}(X_i-X_{(1)}))$.

Fix $\varepsilon>0$. Then

$$P_{\underset{\sim}{\theta}}(|X_{(1)}-\mu|>\varepsilon) = P(X_{(1)}>\mu+\varepsilon) = \prod_{i=1}^{n} P(X_i>\mu+\varepsilon) = e^{-\frac{n\varepsilon}{\sigma}} \to 0 \text{ as } n\to\infty.$$

Thus $\delta_{1ML}(\underset{\sim}{X}) = X_{(1)} \xrightarrow{P} \mu$. So δ_{1ML} is consistent for estimating μ.

Also $E_{\underset{\sim}{\theta}}(X_1) = \mu+\sigma$. So by W.L.L.N $A_1 = \frac{1}{n}\sum_{i=1}^{n}X_i \xrightarrow{P} \mu+\sigma$

$$\Rightarrow \delta_{2ML}(\underset{\sim}{X}) = \frac{1}{n}\sum_{i=1}^{n}X_i - X_{(1)} \xrightarrow{P} \mu+\sigma-\mu = \sigma.$$

$\Rightarrow$ δ_{2ML} is consistent for σ.

$E_{\underset{\sim}{\theta}}(X_1) = \mu+\sigma$, $E_{\underset{\sim}{\theta}}(X_1^2) = (\mu+\sigma)^2+\sigma^2$.

So M.M.E. $\delta_{MM}(\underset{\sim}{X}) = (\delta_{1MM}(\underset{\sim}{X}), \delta_{2MM}(\underset{\sim}{X}))$ is given by,

$$A_1 = \delta_{1MM}+\delta_{2MM}, \quad A_2 = (\delta_{1MM}+\delta_{2MM})^2 + \delta_{2MM}^2$$

$$\Rightarrow \delta_{2MM} = \sqrt{\frac{n-1}{n}}\,S \quad \text{and} \quad \delta_{1MM} = \bar{X} - \sqrt{\frac{n-1}{n}}\,S.$$

$S^2 \xrightarrow{P} V(X_1) = \sigma^2$, $\bar{X} \xrightarrow{P} \mu+\sigma \Rightarrow \delta_{1MM} \xrightarrow{P} \mu+\sigma-\sqrt{\sigma^2} = \mu$

$\delta_{2MM} \xrightarrow{P} \sigma$ $\Rightarrow$ δ_{1MM} and δ_{2MM} are consistent for μ and σ, respectively.

(d) $$L_{\underset{\sim}{x}}(\theta_1,\theta_2) = \begin{cases} \dfrac{1}{(\theta_2-\theta_1)^n}, & \text{if } \theta_1\le x_{(1)} \text{ and } \theta_2\ge x_{(n)} \\ 0, & \text{o.w.} \end{cases}$$

Clearly M.L.E. of $g(\underset{\sim}{\theta}) = (\theta_1,\theta_2)$ is $\delta_{ML}(\underset{\sim}{X}) = (\delta_{1ML}(\underset{\sim}{X}), \delta_{2ML}(\underset{\sim}{X})) = (X_{(1)}, X_{(n)})$

Fix $\varepsilon > 0$. Then

$$P_{\underline{\theta}}(|X_{(1)}-\theta_1|>\varepsilon) = P_{\underline{\theta}}(X_{(1)} > \theta_1+\varepsilon) = \prod_{i=1}^{n} P_{\underline{\theta}}(X_i > \theta_1+\varepsilon) = \begin{cases} 0, & \text{if } \varepsilon \ge \theta_2-\theta_1 \\ \left(1-\frac{\varepsilon}{\theta_2-\theta_1}\right)^n, & \text{if } 0<\varepsilon<\theta_2-\theta_1 \end{cases} \longrightarrow 0 \text{ as } n\to\infty.$$

So $S_{1ML}(\underline{X}) = X_{(1)}$ is consistent for θ_1.

$$P_{\underline{\theta}}(|X_{(n)}-\theta_2|>\varepsilon) = P_{\underline{\theta}}(\theta_2 - X_{(n)} > \varepsilon) = P_{\underline{\theta}}(X_{(n)} < \theta_2-\varepsilon) = \prod_{i=1}^{n} P_{\underline{\theta}}(X_i < \theta_2-\varepsilon)$$

$$= \begin{cases} 0, & \text{if } \varepsilon \ge \theta_2-\theta_1 \\ \left(1-\frac{\varepsilon}{\theta_2-\theta_1}\right)^n, & \text{if } 0<\varepsilon<\theta_2-\theta_1 \end{cases}$$

Thus $\longrightarrow 0$ as $n\to\infty$.

$S_{2ML}(\underline{X}) = X_{(n)}$ is consistent for θ_2.

$$E_{\underline{\theta}}(X_1) = \frac{\theta_1+\theta_2}{2}, \quad E_{\underline{\theta}}(X_1^2) = \frac{\theta_2^2+\theta_1^2+\theta_1\theta_2}{3} = \frac{(\theta_1+\theta_2)^2-\theta_1\theta_2}{3}.$$

Thus the M.M.E. $\underset{\sim}{S}_{MM} = (S_{1MM}, S_{2MM})$ is given by

$$\frac{S_{1MM}+S_{2MM}}{2} = A_1, \quad \frac{(S_{1MM}+S_{2MM})^2 - S_{1MM}S_{2MM}}{3} = A_2$$

$$\Rightarrow S_{1MM} = A_1 - \sqrt{3(A_2-A_1^2)} = \bar{X} - \sqrt{\frac{3(n-1)}{n}}\, S,$$

$$S_{2MM} = A_1 + \sqrt{3(A_2-A_1^2)} = \bar{X} + \sqrt{\frac{3(n-1)}{n}}\, S.$$

Since $E_{\underline{\theta}}(X_1)$ and $E_{\underline{\theta}}(X_1^2)$ are continuous functions of (θ_1, θ_2), it follows that S_{1MM} and S_{2MM} are consistent for estimating θ_1 and θ_2, respectively.

Problem 2

$g(\theta) = P_\theta(X_1 \ge 4) = (1-\theta)^3$

M.L.E. of θ is $\frac{1}{\bar{X}}$, so M.L.E. of $g(\theta)$ is $\hat{g}_{ML} = \left(1-\frac{1}{\bar{X}}\right)^3$.

$$\bar{X} = \frac{2+7+6+5+9}{5} = \frac{29}{5}.$$

So m.l.e. of $g(\theta) = \left(1-\frac{5}{29}\right)^3 = \left(\frac{24}{29}\right)^3$.

Problem 3

Let X = # of items that have failed in less than 100 hours.

$X \sim Bin(10, \mu)$, where $\mu = \frac{1}{\theta}\int_0^{100} e^{-x/\theta}\,dx = 1 - e^{-\frac{100}{\theta}}$

$\Rightarrow \theta = \frac{-100}{\ln(1-\mu)}$. Given $x=3$, $\hat{\mu} = \frac{3}{10} = 0.3$ is the m.l.e. of μ

$\Rightarrow$ m.l.e. of θ is $\hat{\theta} = \frac{-100}{\ln(0.7)}$.

$\boxed{\text{Problem 4}}$ (a) $L_{\underset{\sim}{x}}(\theta) = (2\pi)^{-\frac{n}{2}} e^{-\frac{1}{2}\sum_{i=1}^{n}(x_i-\theta)^2}$, $\theta \geq 0$

$L^*_{\underset{\sim}{x}}(\theta) = \ln L_{\underset{\sim}{x}}(\theta) = -\frac{n}{2}\ln(2\pi) - \frac{1}{2}\sum_{i=1}^{n}(x_i-\theta)^2$, $\theta \geq 0$

$\frac{d}{d\theta} L^*_{\underset{\sim}{x}}(\theta) = \sum_{i=1}^{n}(x_i-\theta) \geq 0 \iff \theta < \bar{x}$

Thus $L^*_{\underset{\sim}{x}}(\theta) \uparrow (\downarrow)$ if $\theta < \bar{x}$ $(\theta > \bar{x})$

Case I $\bar{x} < 0$

$L^*_x(\theta)$, $\theta \in [0, \infty)$, is maximized at $\theta = 0$.

Case II $\bar{x} \geq 0$

$L^*_x(\theta)$, $\theta \in [0, \infty)$, is maximized at $\theta = \bar{x}$.

Thus, the M.L.E. of θ $(\theta \in \Theta = [0, \infty))$ is $\delta_{ML}(\underset{\sim}{x}) = \max(\bar{x}, 0)$.

(b) $L_{\underset{\sim}{x}}(\theta) = \prod_{i=1}^{n}\binom{1}{x_i}\theta^{x_i}(1-\theta)^{1-x_i} = c_{\underset{\sim}{x}}\, \theta^{\sum x_i}(1-\theta)^{n-\sum x_i}$, $\frac{1}{4} \leq \theta \leq \frac{3}{4}$,

where $c_{\underset{\sim}{x}} = \prod_{i=1}^{n}\binom{1}{x_i}$.

$L^*_{\underset{\sim}{x}}(\theta) = \ln L_{\underset{\sim}{x}}(\theta) = \ln c_{\underset{\sim}{x}} + \left(\sum_{i=1}^{n} x_i\right)\ln\theta + (n-\sum x_i)\ln(1-\theta)$.

$\frac{d}{d\theta} L^*_{\underset{\sim}{x}}(\theta) = \frac{\sum_{i=1}^{n} x_i}{\theta} - \frac{(n-\sum_{i=1}^{n} x_i)}{1-\theta} \geq 0 \iff \theta < \bar{x}$.

Thus $L^*_{\underset{\sim}{x}}(\theta) \uparrow (\downarrow)$ if $\theta < \bar{x}$ $(\theta > \bar{x})$.

Case I $0 \leq \bar{x} \leq \frac{1}{4}$

$L^*_{\underset{\sim}{x}}(\theta)$ is maximized at $\theta = \frac{1}{4}$.

Case II. $\frac{1}{4} \leq \bar{x} \leq \frac{3}{4}$

$L^*_{\underset{\sim}{x}}(\theta)$ is maximized at $\theta = \bar{x}$

Case III $\bar{x} \geq 3/4$

$L^*_{\underset{\sim}{x}}(\theta)$ is maximized at $\theta = 3/4$.

Thus, the M.L.E. of θ $(\theta \in \Theta = [\frac{1}{4}, \frac{3}{4}])$ is

$$\delta_{ML}(\underset{\sim}{x}) = \begin{cases} \frac{1}{4}, & \text{if } 0 \leq \bar{x} < \frac{1}{4} \\ \bar{x} & \text{if } \frac{1}{4} \leq \bar{x} \leq 3/4 \\ 3/4 & \text{if } \bar{x} \geq 3/4 \end{cases} = \psi(\bar{x}), \text{ say.}$$

By W.L.L.N $\bar{X} \xrightarrow{P} \theta \Rightarrow \psi(\bar{X}) \xrightarrow{P} \begin{cases} \frac{1}{4}, & \text{if } 0 \le \theta < 1/4 \\ \theta, & \text{if } \frac{1}{4} \le \theta \le 3/4 \\ 3/4, & \text{if } \theta > 3/4 \end{cases}$ (since

$\psi(\theta)$ is a continuous function of θ). Thus $\delta_{ML}(\underline{X}) \xrightarrow{P} \theta$, $\forall \theta \in \Theta = [\frac{1}{4}, \frac{3}{4}] \Rightarrow \delta_{ML}$ is consistent for estimating θ ($\theta \in \Theta = [\frac{1}{4}, \frac{3}{4}]$).

$\boxed{\text{Problem 5}}$ Let $\underline{\theta} = (\mu, \sigma^2)$. Then

$$E_{\underline{\theta}}(\bar{X}) = E_{\underline{\theta}}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n}\sum_{i=1}^{n} E_{\underline{\theta}}(X_i) = \frac{1}{n}\sum_{i=1}^{n}\mu = \mu$$

$$E_{\underline{\theta}}\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2\right) = \frac{1}{n}\sum_{i=1}^{n} E_{\underline{\theta}}(X_i^2) = E_{\underline{\theta}}(X_1^2) = V_{\underline{\theta}}(X_1) + (E_{\underline{\theta}}(X_1))^2 = \sigma^2 + \mu^2$$

$$\Rightarrow E_{\underline{\theta}}(S^2) = E_{\underline{\theta}}\left(\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2\right) = \frac{n}{n-1} E_{\underline{\theta}}\left[\frac{1}{n}\sum_{i=1}^{n} X_i^2 - \bar{X}^2\right].$$

$$V_{\underline{\theta}}(\bar{X}) = V_{\underline{\theta}}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\sum_{i=1}^{n} V_{\underline{\theta}}(X_i) = \frac{1}{n^2}\sum_{i=1}^{n}\sigma^2 = \frac{\sigma^2}{n}.$$

$$\Rightarrow E_{\underline{\theta}}(\bar{X}^2) = V_{\underline{\theta}}(\bar{X}) + (E_{\underline{\theta}}(\bar{X}))^2 = \frac{\sigma^2}{n} + \mu^2$$

Thus

$$E_{\underline{\theta}}(S^2) = \frac{n}{n-1}\left[\sigma^2 + \mu^2 - \frac{\sigma^2}{n} - \mu^2\right] = \sigma^2.$$

$\boxed{\text{Problem 6}}$ (a) M.L.E of θ is $\bar{X}$. So M.L.E of $g(\theta)$ is $\delta_M(\underline{X}) = \bar{X}^r = \frac{T^r}{n^r}$, where $T = \sum_{i=1}^{n} X_i \sim$ Gamma(n, θ)

$$E_\theta(T^r) = n(n+1)\cdots(n+r-1)\,\theta^r$$

$$\Rightarrow E_\theta\left(\frac{T^r}{n(n+1)\cdots(n+r-1)}\right) = \theta^r$$

$$\Rightarrow \delta_U(\underline{X}) = \frac{T^r}{n(n+1)\cdots(n+r-1)} = \frac{n^r}{n(n+1)\cdots(n+r-1)}\bar{X}^r.$$

(b) M.L.E. of (μ, σ^2) is $(\bar{X}, \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2)$. So M.L.E. of $g(\underline{\theta})$ is

$$\delta_M(\underline{X}) = \bar{X} + \frac{T}{\sqrt{n}}, \quad \text{where } T = \sqrt{\sum_{i=1}^{n}(X_i - \bar{X})^2}.$$

$$\frac{T^2}{\sigma^2} \sim \chi^2_{n-1} \Rightarrow E_{\underline{\theta}}\left(\frac{T}{\sigma}\right) = \frac{2^{n/2}\,\Gamma\frac{n}{2}}{2^{\frac{n-1}{2}}\,\Gamma\frac{n-1}{2}}$$

$$\Rightarrow E_{\underline{\theta}}\left(\frac{\Gamma\frac{n-1}{2}}{\sqrt{2}\,\Gamma\frac{n}{2}} T\right) = \sigma$$

$$\Rightarrow \delta_U(\underline{X}) = \bar{X} + \frac{\Gamma\frac{n-1}{2}}{\sqrt{2}\,\Gamma\frac{n}{2}} T.$$

(c) M.L.E. of θ is $\bar{X}$. So M.L.E. of $g(\theta)$ is $\delta_M(\underline{X}) = e^{\bar{X}} = e^{\frac{T}{n}}$,

where $T = \sum_{i=1}^{n} X_i \sim \text{Poisson}(n\theta)$.

We need to find $h(T)$ (a function of T or equivalently of $\bar{X}$) s.t.,

$$E_\theta(h(T)) = e^\theta, \quad \forall \theta > 0$$

$$\Rightarrow \sum_{j=0}^{\infty} h(j) \frac{e^{-n\theta}(n\theta)^j}{j!} = e^\theta \quad \forall \theta > 0$$

$$\Rightarrow \sum_{j=0}^{\infty} \frac{h(j)\, n^j}{j!} \theta^j = e^{(n+1)\theta}, \quad \forall \theta > 0$$

$$= \sum_{j=0}^{\infty} \frac{(n+1)^j \theta^j}{j!}, \quad \forall \theta > 0$$

Since the two power series (on L.H.S and R.H.S.) match $\forall \theta > 0$, the coefficients of θ^j in two power series are same, i.e.

$$\frac{h(j)\, n^j}{j!} = \frac{(n+1)^j}{j!}, \quad j = 0, 1, 2, \ldots$$

$$\Rightarrow h(j) = \left(1 + \frac{1}{n}\right)^j, \quad j = 0, 1, 2, \ldots$$

$$\Rightarrow h(T) = \left(1 + \frac{1}{n}\right)^T = \left(1 + \frac{1}{n}\right)^{n\bar{X}}$$

$$\Rightarrow \delta_U(\underline{X}) = \left(1 + \frac{1}{n}\right)^{n\bar{X}}.$$

$\boxed{\text{Problem 7}}$ (a) M.L.E. of $g(\theta) = \theta$ is $\delta_M(\underline{X}) = X_{(1)}$.

$$f_{X_{(1)}}(x) = \begin{cases} e^{-n(x-\theta)}, & \text{if } x > 0 \\ 0, & \text{o.w.} \end{cases}$$

$E_\theta(X_{(1)}) = \theta + \frac{1}{n}, \quad \forall \theta \in (-\infty, \infty) \quad \Rightarrow E_\theta\left(X_{(1)} - \frac{1}{n}\right) = \theta \quad \forall \theta \in (-\infty, \infty)$

$$\Rightarrow \delta_U(\underline{X}) = X_{(1)} - \frac{1}{n}.$$

$$M_{\delta_M}(\underset{\sim}{\theta}) - M_{\delta_U}(\underset{\sim}{\theta}) = E_{\underset{\sim}{\theta}}[(X_{(1)} - \theta)^2] - E_{\underset{\sim}{\theta}}[(X_{(1)} - \tfrac{1}{n} - \theta)^2]$$

$$= +\frac{2}{n} E_\theta(X_{(1)} - \theta) - \frac{1}{n^2} = \frac{1}{n^2} > 0, \quad \forall \theta \in (-\infty, \infty)$$

Thus, in terms of M.S.E., δ_U is better than δ_M.

(b) M.L.E. of $\underline{\theta} = (\mu, \sigma)$ is $\left(X_{(1)}, \frac{1}{n}\sum_{i=1}^{n}(X_i - X_{(1)})\right)$. So the M.L.E. of $g(\underline{\theta})$ is $\delta_M(\underline{X}) = X_{(1)}$.

Let $T = \sum_{i=1}^{n}(X_i - X_{(1)})$. Then $X_{(1)}$ and T are independent (see the Problem 4 of Mid Sem Exam-II), with

$$f_{X_{(1)}}(x) = \begin{cases} \frac{n}{\sigma} e^{-\frac{n}{\sigma}(x-\mu)} & , \text{ if } x \geq \mu \\ 0, & \text{o.w.} \end{cases}$$

$$f_T(t) = \frac{e^{-\frac{t}{\sigma}} t^{n-2}}{\Gamma_{n-1} \sigma^{n-1}}, \quad t > 0.$$

$$E_{\underline{\theta}}(\delta_n) = \mu + \frac{\sigma}{n}, \quad E_\theta(T) = (n-1)\sigma \Rightarrow E_\theta\left(\frac{T}{n-1}\right) = \sigma, \ \forall \underset{\sim}{\theta}.$$

$$\Rightarrow \delta_U(\underline{x}) = \delta_n - \frac{T}{n(n-1)} = x_{(1)} - \frac{1}{n(n-1)} \sum_{i=1}^{n} (x_i - x_{(1)})$$

$$M_{\delta_n}(\underline{\theta}) - M_{\delta_U}(\underline{\theta}) = E_{\underline{\theta}}[(x_{(1)}-\mu)^2] - E_\theta[(x_{(1)} - \frac{T}{n(n-1)} - \mu)^2]$$

$$= \frac{2}{n(n-1)} E_\theta[(x_{(1)}-\mu)T] - \frac{1}{n^2(n-1)^2} E_{\underline{\theta}}[T^2]$$

$$= \frac{2}{n(n-1)} E_\theta[(x_{(1)}-\mu)] E_\theta[T] - \frac{1}{n^2(n-1)^2} E_\theta(T^2)$$

$$E_\theta[(x_{(1)}-\mu)] = \frac{\sigma}{n} \quad E_\theta(T) = (n-1)\sigma, \quad E_\theta(T^2) = n(n-1)\sigma^2$$

$$M_{\delta_n}(\underline{\theta}) - M_{\delta_U}(\underline{\theta}) = \frac{2}{n(n-1)} \times \frac{\sigma}{n} \times (n-1)\sigma - \frac{1}{n^2(n-1)^2} \times n(n-1)\sigma^2$$

$$= \frac{2}{n^2}\sigma^2 - \frac{\sigma^2}{n(n-1)} > 0,$$

$\Rightarrow$ Under the M.S.E criterion, δ_U is better than δ_n.

(c) M.L.E of $g(\underline{\theta})$ is $\delta_n(\underline{x}) = \frac{1}{n} \sum_{i=1}^{n} (x_i - x_{(1)}) = \frac{T}{n}$, Now

$$E_{\underline{\theta}}(\delta_n(\underline{x})) = E_\theta\left(\frac{T}{n}\right) = \frac{n-1}{n}\sigma, \quad \forall \underset{\sim}{\theta}$$

$$\Rightarrow E_\theta\left(\frac{T}{n-1}\right) = \sigma, \quad \forall \underset{\sim}{\theta} \in \Theta$$

$$\Rightarrow \delta_U(\underline{x}) = \frac{\sum_{i=1}^{n} (x_i - x_{(1)})}{n-1}.$$

$$M_{\delta_n}(\underline{\theta}) - M_{\delta_U}(\theta) = E_{\underline{\theta}}\left[\left(\frac{T}{n} - \sigma\right)^2\right] - E_\theta\left[\left(\frac{T}{n-1} - \sigma\right)^2\right]$$

$$= \left(\frac{1}{n^2} - \frac{1}{(n-1)^2}\right) E_{\underline{\theta}}(T^2) - 2\sigma\left[E_\theta(T)\right]\left[\frac{1}{n} - \frac{1}{n-1}\right]$$

$$= \frac{-2n+1}{n^2(n-1)^2} \times n(n-1)\sigma^2 + 2\sigma^2 \times \frac{n-1}{n(n-1)}$$

$$= -\frac{2n-1}{n(n-1)}\sigma^2 + \frac{2\sigma^2}{n} < 0 \Rightarrow$$ δ_n is better than δ_U under the M.S.E. criterion

(d) M.L.E. of $g(\theta)$ is $\delta_M(\underline{X}) = \bar{X} = \frac{T}{n}$, where $T = \sum_{i=1}^{n} X_i \sim \text{Gamma}(n, \theta)$

$E_\theta(T) = n\theta, \quad \forall \theta > 0 \Rightarrow E_\theta\left(\frac{T}{n}\right) = \theta, \quad \forall \theta \in \Theta$

$\Rightarrow \delta_U(\underline{X}) = \delta_M(\underline{X}) = \bar{X}$.

(e) M.L.E. of θ is $X_{(n)}$. So M.L.E. of $g(\theta)$ is $\delta_M(\underline{X}) = X_{(n)}^r$.

$$f_{X_{(n)}}(x) = \begin{cases} \frac{n x^{n-1}}{\theta^n}, & 0 < x < \theta \\ 0, & \text{o.w.} \end{cases}$$

$E_\theta(X_{(n)}^r) = \frac{n}{n+r}\theta^r, \quad \forall \theta \in \Theta \Rightarrow E_\theta\left(\frac{n+r}{n} X_{(n)}^r\right) = \theta^r, \ \theta > 0.$

$\Rightarrow \delta_U(\underline{X}) = \frac{n+r}{n} X_{(n)}^r$.

$$M_{\delta_M}(\theta) - M_{\delta_U}(\theta) = E_\theta\left[(X_{(n)}^r - \theta^r)^2\right] - E_\theta\left[\left(\frac{n+r}{n}X_{(n)}^r - \theta^r\right)^2\right]$$

$$= \left[1 - \frac{(n+r)^2}{n^2}\right] E_\theta(X_{(n)}^{2r}) - 2\theta^r\left[1 - \frac{n+r}{n}\right] E_\theta(X_{(n)}^r)$$

$$= \left[1 - \frac{(n+r)^2}{n^2}\right] \cdot \frac{n}{n+2r}\theta^{2r} - 2\theta^r\left(1 - \frac{n+r}{n}\right)\frac{n}{n+r}\theta^r$$

$$= \frac{2r\,\theta^{2r}}{n+r} - \frac{2nr + r^2}{n(n+2r)}\theta^{2r}$$

$$= \frac{r^2(n-r)}{n(n+r)(n+2r)} \quad \begin{matrix} > 0, \text{ if } n > r, & = 0 \text{ if } n = r, \\ & < 0 \text{ if } n < r. \end{matrix}$$

Thus for $n > r$, δ_U is better than δ_M (under m.s.e.), for $n < r$, δ_M is better than δ_U and for $n = r$ δ_M and δ_U have the same m.s.e.

(f) M.L.E. of θ is $\bar{X}$. So M.L.E. of $g(\theta)$ is $\delta_M(\underline{X}) = \bar{X}^2$.

$E_\theta(\bar{X}^2) = \frac{1}{n} + \theta^2 \Rightarrow E_\theta\left(\bar{X}^2 - \frac{1}{n}\right) = \theta^2, \quad \forall \theta$

$\Rightarrow \delta_U(\underline{X}) = \bar{X}^2 - \frac{1}{n}$.

$$M_{\delta_M}(\theta) - M_{\delta_U}(\theta) = E_\theta\left[(\bar{X}^2 - \theta^2)^2\right] - E_\theta\left[\left(\bar{X}^2 - \frac{1}{n} - \theta^2\right)^2\right]$$

$$= \frac{2}{n} E_\theta\left[\bar{X}^2 - \theta^2\right] - \frac{1}{n^2}$$

$$= \frac{1}{n^2} > 0$$

$\Rightarrow$ δ_U is better than δ_M under the m.s.e. criterion.

Problem 8 Suppose that $\delta(x_1, x_2)$ is unbiased, i.e. $E_{\underline{\theta}}(\delta(x_1, x_2)) = g(\underline{\theta})$ $\forall \underline{\theta} \in \Theta$, and $P_{\underline{\theta}_0}(\delta(x_1, x_2) = \delta(x_2, x_1)) < 1$, for some $\underline{\theta}_0 \in \Theta$.

Define

$$\delta_U(x_1, x_2) = \frac{1}{2}[\delta(x_1, x_2) + \delta(x_2, x_1)]. \text{ Then } \delta_U(x_1, x_2) = \delta_U(x_2, x_1), \text{ w.p. } 1.$$

x_1, x_2 is a random sample $\Rightarrow (x_1, x_2) \stackrel{d}{=} (x_2, x_1) \Rightarrow E_{\underline{\theta}}[\delta(x_2, x_1)] =$

$$E_{\underset{\sim}{\theta}}(\delta(x_1, x_2)) = g(\underline{\theta}), \ \forall \underline{\theta} \in \Theta.$$

Thus,

$$E_{\underline{\theta}}(\delta_U(x_1, x_2)) = g(\underline{\theta}), \ \forall \underline{\theta} \in \Theta.$$

Also

$$V_\theta(\delta_U(x_1, x_2)) = \frac{1}{4}[V_\theta(\delta(x_1, x_2)) + V_\theta(\delta(x_2, x_1)) + 2\,Cov_\theta(\delta(x_1, x_2), \delta(x_2, x_1))]$$

$$\leq \frac{1}{4}[V_\theta(\delta(x_1, x_2)) + V_\theta(\delta(x_2, x_1)) + 2\sqrt{V_\theta(\delta(x_1, x_2))V_\theta(\delta(x_2, x_1))}]$$

$$= V_{\underline{\theta}}(\delta(x_1, x_2)), \ \forall \theta \in \Theta \quad (\text{since } (x_1, x_2) \stackrel{d}{=} (x_2, x_1) \Rightarrow V_\theta(\delta(x_1, x_2)) = V_\theta(\delta(x_2, x_1)))$$

and we have equality at $\theta \in \Theta$ iff

$$P_\theta\left(\frac{\delta(x_1, x_2) - g(\underline{\theta})}{\sqrt{V_\theta(\delta(x_1, x_2))}} = \frac{\delta(x_2, x_1) - g(\theta)}{\sqrt{V_\theta(\delta(x_2, x_1))}}\right) = 1$$

$$\Leftrightarrow P_\theta(\delta(x_1, x_2) = \delta(x_2, x_1)) = 1$$

Since $P_{\underline{\theta}_0}(\delta(x_1, x_2) = \delta(x_2, x_1)) < 1$, it follows that

$$V_{\underline{\theta}}(\delta_U(x_1, x_2)) \leq V_{\underset{\sim}{\theta}}(\delta(x_1, x_2)), \ \forall \underline{\theta} \in \Theta$$

and $V_{\underset{\sim}{\theta}_0}(\delta_U(x_1, x_2)) < V_{\underset{\sim}{\theta}_0}(\delta(x_1, x_2))$

$\Rightarrow$ δ_U is permutation symmetric and better than δ.

The result can be extended to a sample size of $n (\geq 3)$, by considering

$$\delta_U(x_1, x_2, \ldots, x_n) = \frac{1}{n!}\sum_{(i_1 \ldots i_n)} \cdots \sum \delta(x_{i_1}, \ldots, x_{i_n}),$$

where sum is over all $n!$ permutations $(i_1, \ldots, i_n)$ of $(1, \ldots, n)$.

Problem 9 δ is an unbiased estimator of θ iff

$$E_\theta(\delta(x)) = \theta, \quad \forall \theta \in (0, 1)$$

$$\Leftrightarrow \delta(-1)\theta + (1-\theta)^2 \sum_{x=0}^{\infty} \delta(x)\theta^x = \theta, \quad \forall \theta \in (0, 1)$$

$\iff \sum_{x\geq 0}^{\infty} \delta(x)\,\theta^{x} = (1-\delta(-1))\,\theta\,(1-\theta)^{-2}, \quad \forall\ 0<\theta<1$

$\iff \sum_{x\geq 0}^{\infty} \delta(x)\,\theta^{x} = (1-\delta(-1))\,\theta\,(1+2\theta+3\theta^2+4\theta^3+\cdots), \quad \forall\ 0<\theta<1$

$= (1-\delta(-1))\,(\theta+2\theta^2+3\theta^3+4\theta^4+\cdots), \quad \forall\ 0<\theta<1$

Thus we have two power series in θ matching on interval $(0,1)$

$\iff \delta(0)=0, \quad \delta(x)=(1-\delta(-1))\,x, \quad x=1,2,\ldots$

Thus the class of unbiased estimators of θ is

$$\mathcal{D}_U = \{\delta : \delta(-1)=a,\ \delta(k)=(1-a)k,\ k=0,1,2,\ldots,\ a\in\mathbb{R}\}.$$

$\boxed{\text{Problem 10}}$ M.L.E. of (μ,σ) is $(X_{(1)}, T)$. We ~~know~~ have

$$\left.\begin{array}{l} \dfrac{n(X_{(1)}-\mu)}{\sigma} \sim \text{Exp}(1) \\[2ex] \dfrac{T}{\sigma} \sim \text{Gamma}(n-1,1) \end{array}\right\rangle \text{ independent. (Show this)}$$

Thus, $\forall\ \underline{\theta}\in\Theta$,

$$E_{\underline{\theta}}\left[\frac{n(X_{(1)}-\mu)}{\sigma}\right]=1, \quad E_{\underline{\theta}}\left(\frac{T}{\sigma}\right)=n-1$$

$$\Rightarrow E_{\underline{\theta}}[X_{(1)}] = \mu+\frac{\sigma}{n}, \quad E_{\underline{\theta}}[T]=\sigma(n-1)$$

$$\Rightarrow E_{\underline{\theta}}\left[X_{(1)}-\frac{T}{n(n-1)}\right]=\mu, \quad \forall\ \underline{\theta}\in\Theta$$

$$\Rightarrow \delta_U(\underline{X}) = X_{(1)} - \frac{1}{n(n-1)}\sum_{i=1}^{n}(X_i-X_{(1)}). \quad \text{(Also see Problem 7(b))}.$$

For $\underline{\theta}\in\Theta$ and $c\in(-\infty,\infty)$

$$MS_c(\underline{\theta}) = E_{\underline{\theta}}[(X_{(1)}-cT-\mu)^2]$$

$$= c^2 E_{\underline{\theta}}[T^2] - 2c\,E_{\underline{\theta}}[(X_{(1)}-\mu)T] + E_{\underline{\theta}}[(X_{(1)}-\mu)^2]$$

$$= c^2 E_{\underline{\theta}}[T^2] - 2c\,E_{\underline{\theta}}[X_{(1)}-\mu]\,E_{\underline{\theta}}[T] + E_{\underline{\theta}}[(X_{(1)}-\mu)^2]$$

$=$ ($X_{(1)}$ and T are independent).

Fix $\underline{\theta}\in\Theta$. Then

$$\frac{\partial}{\partial c} MS_c(\underline{\theta}) = 2c\,E_{\underline{\theta}}(T^2) - 2\,E_{\underline{\theta}}[(X_{(1)}-\mu)]\,E_{\underline{\theta}}[T]$$

$$\frac{\partial^2}{\partial c^2} MS_c(\underline{\theta}) = 2\,E_{\underline{\theta}}(T^2) > 0.$$

Thus, for fixed $\underline{\theta} \in \Theta$, $M_{\delta_c}(\underline{\theta})$ is minimized at

$$c = \frac{E_{\underline{\theta}}[X_{(1)}-\mu]\, E_{\underline{\theta}}(T)}{E_{\underline{\theta}}(T^2)} = \frac{\frac{\sigma}{n} \times \sigma(n-1)}{n(n-1)\sigma^2}$$

$$= \frac{1}{n^2} \longrightarrow \text{does not dependent on } \underline{\theta}.$$

Thus for every $\underline{\theta} \in \Theta$, $M_{\delta_c}(\underline{\theta})$ is minimized at $c = \frac{1}{n^2} = c_0$ (say).

$$\Rightarrow \delta_{c_0}(\underline{X}) = X_{(1)} - \frac{1}{n^2} \sum_{i=1}^{n} (X_i - X_{(1)})$$

is the best estimator, with respect to the M.S.E. criterion in the class $\mathcal{D}$.

$\boxed{\text{Problem 11}}$ M.L.E. of θ is $\delta_{ML}(\underline{X}) = X_{(n)}$
M.M.E. of θ is $\delta_{MM}(\underline{X}) = 2\overline{X}$ (since $E(X_1) = \frac{\theta}{2}$)

$$E_{\underline{\theta}}(X_{(n)}) = \frac{n}{n+1}\theta, \quad E_{\underline{\theta}}(X_{(n)}^2) = \frac{n}{n+2}\theta^2, \quad \theta \in \Theta$$

$$E_{\underline{\theta}}(X_1) = \frac{\theta}{2}, \quad E_\theta(X_1^2) = \frac{\theta^2}{3}, \quad V_\theta(X_1) = \frac{\theta^2}{12}, \quad \theta \in \Theta.$$

$$M_{\delta_{ML}}(\underline{X}) = E_\theta[(X_{(n)}-\theta)^2] = E_\theta[X_{(n)}^2] - 2\theta E_\theta(X_{(n)}) + \theta^2$$

$$= \theta^2\left[\frac{n}{n+2} - \frac{2n}{n+1} + 1\right] = \frac{2}{(n+1)(n+2)}\theta^2.$$

$$M_{\delta_{MM}}(\underline{X}) = E_\theta[(2\overline{X}-\theta)]^2 = V_\theta(2\overline{X}) = 4V_\theta(\overline{X}) = 4 \times \frac{\theta^2}{12n} = \frac{\theta^2}{3n}$$

$$> \frac{2}{(n+1)(n+2)}\theta^2, \quad \forall\, \theta \in \Theta,\ n \geq 2.$$

Thus, for $n \geq 2$, δ_{ML} is preferable over δ_{MM}.

Fix $\theta \in \Theta$. Then, for $c \in (0, \infty)$,

$$M_{\delta_c}(\theta) = E_\theta[cX_{(n)}-\theta]^2 = c^2 E_\theta(X_{(n)}^2) - 2c\theta E_\theta(X_{(n)}) + \theta^2$$

$$\frac{\partial}{\partial c} M_{\delta_c}(\theta) = 2c\, E_\theta(X_{(n)}^2) - 2\theta\, E_\theta(X_{(n)})$$

$$\frac{\partial^2}{\partial c^2} M_{\delta_c}(\theta) = 2E_\theta(X_{(n)}^2) > 0.$$

Thus, for fixed $\theta \in \Theta$, $M_{\delta_c}(\theta)$ is minimized at

$$c = \frac{\theta E_\theta(X_{(n)})}{E_\theta(X_{(n)}^2)} = \frac{n+2}{n+1} \longrightarrow \text{does not depend on } \theta.$$

Thus, for every $\theta \in \Theta$, $M_{\delta_c}(\theta)$ is minimized at $c = \frac{n+2}{n+1} = c_0$ say.

Thus among the estimators in class $\mathcal{D}$, $\delta_{c_0}(\underline{x}) = \frac{n+2}{n+1} X_{(n)}$ has the smallest m.s.e. for each $\theta \in \Theta$

$\boxed{\text{Problem 12}}$ Fix $\theta \in \Theta$. Then, for $\alpha \in [0,1]$,

$$M_{\delta_\alpha}(\theta) = E_\theta\left[\left(\alpha\left(X_{(n)} - \tfrac{1}{2}\right) + (1-\alpha)\left(X_{(1)} + \tfrac{1}{2}\right) - \theta\right)^2\right]$$

$$= E_\theta\left[\left(\alpha\left(X_{(n)} - X_{(1)} - 1\right) + X_{(1)} + \tfrac{1}{2} - \theta\right)^2\right]$$

$$= \alpha^2 E_\theta\left[\left(X_{(n)} - X_{(1)} - 1\right)^2\right] + 2\alpha E_\theta\left[\left(X_{(n)} - X_{(1)} - 1\right)\left(X_{(1)} + \tfrac{1}{2} - \theta\right)\right]$$

$$+ E_\theta\left[\left(X_{(1)} + \tfrac{1}{2} - \theta\right)^2\right].$$

$$\frac{\partial}{\partial\alpha} M_{\delta_\alpha}(\theta) = 2\alpha E_\theta\left[\left(X_{(n)} - X_{(1)} - 1\right)^2\right] + 2 E_\theta\left[\left(X_{(n)} - X_{(1)} - 1\right)\left(X_{(1)} + \tfrac{1}{2} - \theta\right)\right]$$

$$\frac{\partial^2}{\partial\alpha^2} M_{\delta_\alpha}(\theta) = 2 E_\theta\left[\left(X_{(n)} - X_{(1)} - 1\right)^2\right] > 0.$$

Thus, for fixed $\theta \in \Theta$, $M_{\delta_\alpha}(\theta)$ is minimized at

$$\alpha = \frac{E_\theta\left[\left(1 - X_{(n)} + X_{(1)}\right)\left(X_{(1)} + \tfrac{1}{2} - \theta\right)\right]}{E_\theta\left[\left(X_{(n)} - X_{(1)} - 1\right)^2\right]}$$

$$f_{X_{(1)}, X_{(n)}}(x, y) = n(n-1)(y-x)^{n-2}, \quad \theta - \tfrac{1}{2} < x < y < \theta + \tfrac{1}{2}.$$

$$E_\theta\left[\left(1 - X_{(n)} + X_{(1)}\right)\left(X_{(1)} + \tfrac{1}{2} - \theta\right)\right]$$

$$= \iint_{\theta - \frac{1}{2} < x < y < \theta + \frac{1}{2}} (1 - y + x)\left(x + \tfrac{1}{2} - \theta\right) n(n-1)(y-x)^{n-2}\, dy\, dx$$

$$= n(n-1) \iint_{0 < s < t < 1} s\, t\, (1-t)^{n-2}\, ds\, dt \qquad \left(\begin{array}{c} x + \frac{1}{2} - \theta = s \\ 1 - y + x = t \end{array}\right)$$

$$= \frac{n(n-1)}{2} B(4, n-1)$$

$$E_\theta\left[\left(X_{(n)} - X_{(1)} - 1\right)^2\right] = n(n-1) \iint_{0 < s < t < 1} t^2 (1-t)^{n-2}\, dt = n(n-1) B(4, n-1)$$

Thus, for fixed $\theta \in \Theta$, $M_{\delta_\alpha}(\theta)$ is minimised at

$$\alpha = \tfrac{1}{2} = \alpha_0 \text{ say}$$

$\Rightarrow$ Among the estimators in the class $\mathcal{D}$, $\delta_{\alpha_0}(X) = \frac{X_{(n)} + X_{(1)}}{2}$ has the smallest m.s.e at each parametric point.

Problem 13 $T = \sum_{i=1}^{n} X_i \sim \text{Gamma}(n, \theta)$

$$E(T^k) = \frac{\Gamma(n+k)\, \theta^{k+n}}{\Gamma(n)\, \theta^n} = \frac{\Gamma(n+k)}{\Gamma(n)} \theta^k, \quad k > 0.$$

Fix $\theta \in \Theta$. Then

$$M_{\delta_c}(\theta) = E_\theta[(\delta_c(\underline{X}) - \theta^r)^2] = E_\theta[(c\bar{X}^r - \theta^r)^2]$$

$$= c^2 E_\theta(\bar{X}^{2r}) - 2c\theta^r E_\theta(\bar{X}^r) + \theta^{2r}.$$

$$\frac{\partial}{\partial c} M_{\delta_c}(\theta) = 2c\, E_\theta(\bar{X}^{2r}) - 2\theta^r E_\theta(\bar{X}^r)$$

$$\frac{\partial^2}{\partial c^2} M_{\delta_c}(\theta) = 2 E_\theta(\bar{X}^{2r}) > 0.$$

Thus, for fixed $\theta \in \Theta$, $M_{\delta_c}(\theta)$ is minimized at

$$c = \frac{\theta^r E_\theta(\bar{X}^r)}{E_\theta(\bar{X}^{2r})} = n^r \theta^r \frac{E_\theta(T^r)}{E_\theta(T^{2r})}$$

$$= n^r \theta^r \frac{\frac{\Gamma(n+r)}{\Gamma(n)} \theta^r}{\frac{\Gamma(n+2r)}{\Gamma(n)} \theta^{2r}}$$

$$= n^r \frac{\Gamma(n+r)}{\Gamma(n+2r)} = \frac{n^r}{(n+2r-1)(n+2r-2)\cdots(n+r)} = c_0, \text{ say}$$

$\longrightarrow$ does not depend on θ

Thus, for every $\theta \in \Theta$, $M_{\delta_c}(\theta)$ is minimized at $c = c_0 = n^r \dfrac{\Gamma(n+r)}{\Gamma(n+2r)}$.

$\Rightarrow$ Among the estimators in the class $\mathcal{D}$, the estimator $\delta_{c_0}(\underline{X})$

$$= \frac{n^r \Gamma(n+r)}{\Gamma(n+2r)} \left(\sum_{1}^{n} X_i\right)^r = \frac{n^{2r}\, \Gamma(n+r)}{\Gamma(n+2r)} \bar{X}^r$$ has the smallest m.s.e

at each parametric point.

$$\frac{n^{2r}\, \Gamma(n+r)}{\Gamma(n+2r)} = \frac{n^{2r}}{(n+2r-1)(n+2r-2)\cdots(n+r)} = \frac{1}{[1+\frac{2r-1}{n}][1+\frac{2r-2}{n}]\cdots[1+\frac{r}{n}]}$$

$\longrightarrow 1$ as $n \to \infty$.

Also, by W.L.L.N $\bar{X}_n \xrightarrow{P} E(X_1) = \theta \Rightarrow \bar{X}_n^r \xrightarrow{P} \theta^r$. Thus

$\delta_{c_0}(\underline{X}) \xrightarrow{P} \theta^r$. $\Rightarrow \delta_{c_0}(\underline{X})$ is consistent for estimating θ.