EE 200: Solution to Problem Set 3

1. Show that the analog system with an input/output relation given by

$$y(t) = \int_{t-t_0}^{t+t_0} x(\tau)d(\tau)$$

where $y(t)$ and $x(t)$ are, respectively, the output and input signals, is a linear, non-causal, and time-invariant system.

Solution 1: For an input $x_1(t)$, the output is

$$y_1(t) = \int_{t-t_0}^{t+t_0} x_1(\tau)d\tau$$

for an input $x_2(t)$, the output is

$$y_2(t) = \int_{t-t_0}^{t+t_0} x_2(\tau)d\tau$$

For an input $\alpha x_1(t) + \beta x_2(t)$, the output is

$$\begin{aligned} y(t) &= \int_{t-t_0}^{t+t_0} [\alpha x_1(\tau) + \beta x_2(\tau)]d\tau \\ &= \alpha y_1(t) + \beta y_2(t) \quad \text{(Superposition principle)} \end{aligned}$$

Hence it is a linear system.

It follows from the equation $y(t) = \int_{t-t_0}^{t+t_0} x(\tau)d\tau$ that the system is noncausal, as $y(t) \neq 0$ for $t_0 \leq t < 0$.

The system is time-invariant as for an input $x(t-T)$, the output is

$$
\begin{aligned}
y_d(t) &= \int_{t-t_0}^{t+t_0} x(\tau - T)d\tau \\
&= \int_{t-t_0-T}^{t+t_0-T} x(\xi)d\xi = y(t-T)
\end{aligned}
$$

2. Evaluate the following convolution integrals:
   (a) $y_1(t) = [\mu(t) - \mu(t-1)] \circledast [\mu(t) - \mu(t-1)]$
   (b) $y_2(t) = \mu(t) \circledast e^{-\alpha t}\mu(t), \quad \alpha > 0$

   Solution 2(a):

$$
\begin{aligned}
y_1(t) &= \int_{-\infty}^{\infty} [\mu(\tau) - \mu(\tau - 1)][\mu(t-\tau) - \mu(t-1-\tau)]d\tau \\
&= \int_{-\infty}^{\infty} \mu(\tau)\mu(t-\tau)d\tau - \int_{-\infty}^{\infty} \mu(\tau)\mu(t-1-\tau)d\tau \\
&- \int_{-\infty}^{\infty} \mu(\tau-1)\mu(t-\tau)d\tau + \int_{-\infty}^{\infty} \mu(\tau-1)\mu(t-1-\tau)d\tau
\end{aligned}
$$

Now,

$$
\mu(\tau)\mu(t-\tau) = \begin{cases} 1 \ ; & 0 < \tau < t, t > 0 \\ 0 \ ; & \text{otherwise} \end{cases}
$$

$$
\mu(\tau)\mu(t-1-\tau) = \begin{cases} 1 \ ; & 0 < \tau < t-1, t > 1 \\ 0 \ ; & \text{otherwise} \end{cases}
$$

$$
\mu(\tau-1)\mu(t-\tau) = \begin{cases} 1 \ ; & 1 < \tau < t, t > 1 \\ 0 \ ; & \text{otherwise} \end{cases}
$$

$$
\mu(\tau-1)\mu(t-1-\tau) = \begin{cases} 1 \ ; & 0 < \tau < t-1, t > 2 \\ 0 \ ; & \text{otherwise} \end{cases}
$$

Therefore,

$$\begin{aligned} y_1(t) &= \left(\int_0^t d\tau\right)\mu(t) - \left(\int_0^{t-1} d\tau\right)\mu(t-1) \\ &- \left(\int_1^t d\tau\right)\mu(t-1) + \left(\int_1^{t-1} d\tau\right)\mu(t-2) \\ &= t\mu(t) - (t-1)\mu(t-1) - (t-1)\mu(t-1) + (t-2)\mu(t-2) \end{aligned}$$

2(b)

$$\begin{aligned} y_2(t) &= \int_{-\infty}^{\infty} e^{-\alpha\tau}\mu(\tau)\mu(t-\tau)d\tau \\ &= \int_0^t e^{-\alpha\tau} d\tau = -\frac{1}{\alpha}e^{-\alpha\tau}\bigg|_0^t \\ &= \frac{1}{\alpha}(1 - e^{-\alpha t})\mu(t) \end{aligned}$$

3. The periodic convolution integral of two periodic signals $\tilde{g}(t)$ and $\tilde{h}(t)$ with fundamental period T_0 is given by

$$y(t) = \tilde{g}(t) \circledast \tilde{h}(t) = \int_0^{T_0} \tilde{g}(\tau)\tilde{h}(t-\tau)d\tau$$

Show that $y(t)$ is also a periodic signal with a fundamental period T_0.

Solution 3: Now $\tilde{h}(t)$ being a periodic signal with a fundamental period T_0, we note that $\tilde{h}(t-\tau) = \tilde{h}(t + T_0 - \tau)$.

Hence,

$$\begin{aligned} y(t+T_0) &= \int_0^{T_0} \tilde{g}(\tau)\tilde{h}(t+T_0-\tau)d\tau \\ &= \int_0^{T_0} \tilde{g}(\tau)\tilde{h}(t-\tau)d\tau \\ &= y(t) \end{aligned}$$

4. The cross-correlation function $r_{xy}(\tau)$ of two real analog signals $x(t)$ and $y(t)$ is defined by

$$r_{xy}(\tau) = \int_{-\infty}^{\infty} x(\xi)y(\xi - \tau)d\xi$$

and is a measure of the similarity between two analog signals as function of time lag τ.

Evaluate the cross-correlation function for $x(t) = e^{-\alpha t}\mu(t)$, $y(t) = e^{-\beta t}\mu(t)\ \alpha > 0,\ \beta > 0$.

Solution 4: The cross-correlation function is

$$\begin{aligned} r_{xy}(\tau) &= \int_{-\infty}^{\infty} x(\xi)y(\xi - \tau)d\xi \\ &\int_{-\infty}^{\infty} x(\xi)y(-(\tau - \xi))d\xi \\ &= x(\tau) \circledast y(-\tau) \end{aligned}$$

For $x(t) = e^{-\alpha t}\mu(t)$, $y(t) = e^{-\beta t}\mu(t)$, $\alpha > 0,\ \beta > 0:$

$$\begin{aligned} r_{xy}(\tau) &= \int_{-\infty}^{\infty} e^{-\alpha\xi}\mu(\xi)e^{-\beta(\xi-\tau)}\mu(\xi - \tau)d\xi \\ &= e^{\beta\tau}\int_{-\infty}^{\infty} e^{-(\alpha+\beta)\xi}\mu(\xi)\mu(\xi - \tau)d\xi \\ &= \begin{cases} e^{\beta\tau}\left(\frac{-1}{\alpha+\beta}\right)e^{-(\alpha+\beta)\xi}\Big|_0^{\infty};\ \tau < 0 \\ \\ e^{\beta\tau}\left(\frac{-1}{\alpha+\beta}\right)e^{-(\alpha+\beta)\xi}\Big|_{\tau}^{\infty};\ \tau \geq 0 \end{cases} \\ &= \begin{cases} \frac{e^{\beta t}}{\alpha+\beta},\ \tau < 0 \\ \\ \frac{e^{-\alpha t}}{\alpha+\beta},\ \tau \geq 0 \end{cases} \end{aligned}$$

5. The auto-correlation function $r_{xx}(\tau)$ of a real analog signal $x(t)$ is defined by

$$r_{xy}(\tau) = \int_{-\infty}^{\infty} x(\xi)x(\xi - \tau)d\xi$$

which is a cross-correlation of $x(t)$ with itself.

Evaluate the auto correlation function for $x(t) = \mu(t-\alpha) - \mu(t),\ \alpha > 0$.

Solution 5:

$$r_{xx}(\tau) = \begin{cases} 0,\ \tau \le -\alpha \\ \alpha+\tau,\ -\alpha < \tau \le 0 \\ \alpha-\tau,\ 0 < \tau \le \alpha \\ 0,\ \tau > \alpha \end{cases}$$

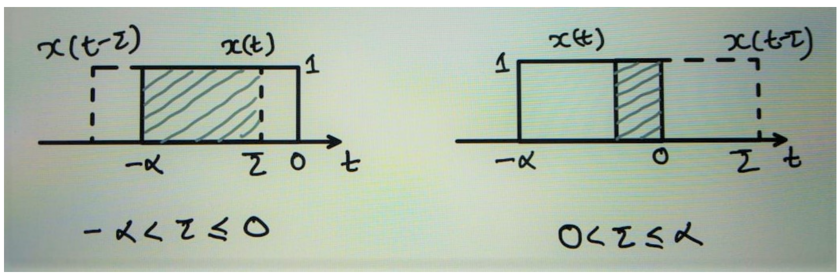

Figure 1: Diagram for Solution 5

6. Show that the inverse of a causal LTI analog system with an impulse response $g(t) = A\delta(t) + Be^{-\alpha t}\mu(t)$ is a causal LTI analog system with an impulse response given by

$$h(t) = \frac{1}{A}\delta(t) - \frac{B}{A^2}e^{-(\alpha+\frac{B}{A})t}\mu(t)$$

Solution 6:

$$\begin{aligned} g(t) \circledast h(t) &= \left[A + Be^{-\alpha t}\mu(t)\right] \circledast \left[\frac{1}{A}\delta(t) - \frac{B}{A^2}e^{-(\alpha+\frac{B}{A})t}\mu(t)\right] \\ &= A\delta(t) \circledast \frac{1}{A}\delta(t) - A\delta(t) \circledast \left[\frac{B}{A^2}e^{-(\alpha+\frac{B}{A})t}\mu(t)\right] \\ &\quad + Be^{-\alpha t}\mu(t) \circledast \frac{1}{A}\delta(t) - Be^{-\alpha t}\mu(t) \circledast \left[\frac{B}{A^2}e^{-(\alpha+\frac{B}{A})t}\mu(t)\right] \\ &= \delta(t) - \frac{B}{A}e^{-(\alpha+\frac{B}{A})t}\mu(t) + \frac{B}{A}e^{-\alpha t}\mu(t) \\ &\quad - \frac{B^2}{A^2}\left[e^{-\alpha t}\mu(t) \circledast e^{-(\alpha+\frac{B}{A})t}\mu(t)\right] \end{aligned}$$

Now,

$$
\begin{aligned}
e^{-\alpha t}\mu(t) \circledast e^{-(\alpha+\frac{B}{A})t}\mu(t) &= \frac{e^{-\alpha t} - e^{-(\alpha+\frac{B}{A})t}}{\alpha + \frac{B}{A} - \alpha}\mu(t) \\
&= \frac{A}{B}e^{-\alpha t}\mu(t) - \frac{A}{B}e^{-(\alpha+\frac{B}{A})t}\mu(t)
\end{aligned}
$$

hence,

$$
\begin{aligned}
g(t) \circledast h(t) &= \delta(t) - \frac{B}{A}e^{-(\alpha+\frac{B}{A})t}\mu(t) + \frac{B}{A}e^{-\alpha t}\mu(t) \\
&\quad - \frac{B}{A}e^{-\alpha t}\mu(t) + \frac{B}{A}e^{-(\alpha+\frac{B}{A})t}\mu(t) \\
&= \delta(t)
\end{aligned}
$$