1. Countable and uncountable

Definition 1. A set Ω is said to be *finite* if there exists an $n \in \mathbb{N}$ and a bijection from Ω onto $[n]$ [1]. An infinite set Ω is said to be countable if there is a bijection from $\mathbb{N}$ onto Ω.

Generally, the word countable also includes finite sets. If Ω is an infinite countable set, then using any bijection $f : \mathbb{N} \to \Omega$, we can list the elements of Ω as a sequence

$$f(1), f(2), f(3), \ldots$$

so that each element of Ω occurs exactly once in the sequence. Conversely, if you can write the elements of Ω as a sequence, it defines an injective function from natural numbers onto Ω (send 1 to the first element of the sequence, 2 to the second element, etc.).

Example 2. The set of integers $\mathbb{Z}$ is countable. Define $f : \mathbb{N} \to \mathbb{Z}$ by

$$f(n) = \begin{cases} \frac{1}{2}n & \text{if } n \text{ is even,} \\ -\frac{1}{2}(n-1) & \text{if } n \text{ is odd.} \end{cases}$$

It is clear that f maps $\mathbb{N}$ into $\mathbb{Z}$. Thus, we have found a bijection from $\mathbb{N}$ onto $\mathbb{Z}$ which shows that $\mathbb{Z}$ is countable. This function is a formal way of saying the we can list the elements of $\mathbb{Z}$ as

$$0, +1, -1, +2, -2, +3, -3, \ldots.$$

It is obvious, but good to realize there are wrong ways to try writing such a list. For example, if you list all the negative integers first, as $-1, -2, -3, \ldots$, then you will never arrive at 0 or 1, and hence the list is incomplete!
Exercise: Check that f is one-one and onto.

Example 3. The set $\mathbb{N} \times \mathbb{N}$ is countable. Rather than give a formula, we list the elements of $\mathbb{N} \times \mathbb{N}$ as follows;

$$(1,1),\ (1,2),(2,1),\ (1,3),(2,2),(3,1),\ (1,4),(2,3),(3,2),(4,1),\ \ldots$$

The pattern should be clear. Use this list to define a bijection from $\mathbb{N}$ onto $\mathbb{N} \times \mathbb{N}$, and hence show that $\mathbb{N} \times \mathbb{N}$ is countable.

Example 4. The set $\mathbb{Z} \times \mathbb{Z}$ is countable. This follows from the first two examples. Indeed, we have a bijection $f : \mathbb{N} \to \mathbb{Z}$ and a bijection $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. Define a bijection $F : \mathbb{N} \times \mathbb{N} \to \mathbb{Z} \times \mathbb{Z}$ by composing them, i.e., $F(n, m) = f(g(n))$. Then, F is one-one and onto. This shows that $\mathbb{Z} \times \mathbb{Z}$ is countable.

Example 5. The set of rational numbers $\mathbb{Q}$ is countable. Recall that rational numbers other than 0 can be written uniquely in the form p/q where p is a non-zero integer and q is a strictly positive

[1] We use the notation $[n]$ to denote the set $\{1, 2, \ldots, n\}$

integer, and there are no common factors of p and q (this is called the *lowest form* of the rational number r). Consider the map $f : \mathbb{Q} \to \mathbb{Z} \times \mathbb{Z}$ defined by

$$f(r) = \begin{cases} (0,1) & \text{if } r = 0, \\ (p,q) & \text{if } r = \frac{p}{q} \text{ in the lowest form.} \end{cases}$$

Clearly, f is injective and hence, it appears that "$\mathbb{Z} \times \mathbb{Z}$ is a bigger set than $\mathbb{Q}$". Next, define the function $g : \mathbb{Z} \to \mathbb{Q}$ by setting $g(n) = n$. This is also injective and hence we may say that "$\mathbb{Q}$ is a bigger set than $\mathbb{N}$". But, we have already seen that $\mathbb{N}$ and $\mathbb{Z} \times \mathbb{Z}$ are in bijection with each other, in that sense, they are of equal size. Since $\mathbb{Q}$ is sandwiched between the two it should be true that $\mathbb{Q}$ has the same size as $\mathbb{N}$, and thus countable. This reasoning is not incorrect, but an argument is needed to make it an honest proof.
Exercise: Try to directly find a bijection between $\mathbb{Q}$ and $\mathbb{N}$.

Example 6. The set of real numbers $\mathbb{R}$ is not countable. The extraordinary proof of this fact is due to Cantor, and the core idea, called the *diagonalization trick* is one that can be used in many other contexts.

Consider any function $f : \mathbb{N} \to [0,1]$. We show that it is not onto, and hence not a bijection. Indeed, use the decimal expansion to write a number $x \in [0,1]$ as $0.x_1x_2x_3\ldots$ where $x_i \in \{0,1,\ldots,9\}$. Write the decimal expansion for each of the numbers $f(1), f(2), f(3), \ldots$. as follows:

$$f(1) = 0.X_{1,1}X_{1,2}X_{1,3}\ldots$$

$$f(2) = 0.X_{2,1}X_{2,2}X_{2,3}\ldots$$

$$f(3) = 0.X_{3,1}X_{3,2}X_{3,3}\ldots$$

$$\ldots\ldots\;\ldots\;\ldots$$

Let $Y_1, Y_2, Y_3, \ldots$ be any numbers in $\{0,1,\ldots,9\}$ with the only condition that $Y_i \neq X_{i,i}$. Clearly, it is possible to choose Y_i like this. Now, consider the number $y = 0.Y_1Y_2Y_3\ldots$ which is a number in $[0,1]$. However, it does not occur in the above list. Indeed, y disagrees with $f(1)$ in the first decimal place, disagrees with $f(2)$ in the second decimal place, etc. Thus, $y \neq f(i)$ for any $i \in \mathbb{N}$ which means that f is not onto $[0,1]$.

Thus, no function $f : \mathbb{N} \to [0,1]$ is onto, and hence there is no bijection from $\mathbb{N}$ onto $[0,1]$ and hence $[0,1]$ is not countable. Obviously, if there is no onto function onto $[0,1]$, there cannot be an onto function onto $\mathbb{R}$. Thus, $\mathbb{R}$ is also uncountable.

Example 7. Let $A_1, A_2, \ldots$ be subsets of a set Ω. Suppose each A_i is countable (finite is allowed). Then, $\cup_i A_i$ is also countable. We leave it as an exercise.
[*Hint:* If each A_i is countably infinite and pairwise disjoint, then $\cup A_i$ can be thought of as $\mathbb{N} \times \mathbb{N}$].

2. ON INFINITE SUMS

There are some subtleties in the definition of probabilities which we address now. The definition of $\mathbf{P}(A)$ for an event A, and $\mathbf{E}[X]$ for a random variable X involves infinite sums (when Ω is countably infinite). In fact, in the very definition of probability space, we had the condition that $\sum_\omega p_\omega = 1$, but what is the meaning of this sum when Ω is infinite? In this section, we make precise the notion of infinite sums.

Let Ω be a countable set, and let $f : \Omega \to \mathbb{R}$ be a function. We want to give a meaning to the infinite sum $\sum_{\omega\in\Omega} f(\omega)$.

The idea: By definition of countability, there is a bijection $\varphi : \mathbb{N} \to \Omega$ which allows us to list the elements of Ω as $\omega_1 = \varphi(1), \omega_2 = \varphi(2), \ldots$. Consider the partial sums $x_n = f(\omega_1) + f(\omega_2) + \cdots + f(\omega_n)$. Since f is non-negative, these numbers are non-decreasing, i.e., $x_1 \le x_2 \le x_3 \le \cdots$. Hence, they converge to a finite number or to $+\infty$ (which is just another phrase for saying that the partial sums grow without bound). We would like to simply define the sum $\sum_{\omega\in\Omega} f(\omega)$ as the limit $L = \lim_{n\to\infty}(f(\omega_1) + \cdots + f(\omega_n))$, which may be finite or $+\infty$.

The problem is that this may depend on the bijection Ω chosen. For example, if $\psi : \mathbb{N} \to \Omega$ is a different bijection, we would write the elements of Ω in a different sequence $\omega'_1 = \psi(1), \omega'_2 = \psi(2), \ldots$, the partial sums $y_n = f(\omega'_1) + \cdots + f(\omega'_n)$ and then define $\sum_{\omega\in\Omega} f(\omega)$ as the limit $L' = \lim_{n\to\infty}(f(\omega'_1) + \cdots + f(\omega'_n))$.

Question: Is it necessarily true that $L = L'$?

Case I - Non-negative f: We claim that for any two bijections φ and ψ as above, the limits are the same (this means that the limits are $+\infty$ in both cases, or the same finite number in both cases). Indeed, fix any n and recall that $x_n = f(\omega_1) + \cdots + f(\omega_n)$. Now, ψ is surjective, hence there is some m (possibly very large) such that $\{\omega_1, \ldots, \omega_n\} \subseteq \{\omega'_1, \ldots, \omega'_m\}$. Now, we use the non-negativity of f to observe that

$$f(\omega_1) + \cdots + f(\omega_n) \le f(\omega'_1) + \cdots + f(\omega'_m).$$

This is the same as $x_n \le y_m$. Since y_k are non-decreasing, it follows that $x_n \le y_m \le y_{m+1} \le y_{m+2} \le \cdots$, which implies that $x_n \le L'$. Now, let $n \to \infty$ and conclude that $L \le L'$. Repeat the argument with the roles of φ and ψ reversed to conclude that $L' \le L$. Hence $L = L'$, as we desired to show.

In conclusion, for non-negative functions f, we can assign an unambiguous meaning to $\sum_\omega f(\omega)$ by setting it equal to $\lim_{n\to\infty}(f(\varphi(1) + \cdots + f(\varphi(n)))$, where $\varphi : \mathbb{N} \to \Omega$ is any bijection (the point being that the limit does not depend on the bijection chosen), and the limit here may be allowed to be $+\infty$ (in which case we say that the sum does not converge).

Case II - General $f : \Omega \to \mathbb{R}$: The above argument fails if f is allowed to take both positive and negative values (why?). In fact, the answers L and L' from different bijections may be completely different. An example is given later to illustrate this point. For now, here is how we deal with this problem.

For a real number x we introduce the notations, $x_+ = x \vee 0$ and $x_- = (-x) \vee 0$ [2]. Then $x = x_+ - x_-$, while $|x| = x_+ + x_-$. Define the non-negative functions $f_+, f_- : \Omega \to \mathbb{R}_+$ by $f_+(\omega) = (f(\omega))_+$ and $f_-(\omega) = (f(\omega))_-$. Observe that $f_+(\omega) - f_-(\omega) = f(\omega)$, while $f_+(\omega) + f_-(\omega) = |f(\omega)|$ for all $\omega \in \Omega$.

Example 8. Let $\Omega = \{a, b, c, d\}$ and let $f(a) = 1$, $f(b) = -1$, $f(c) = -3$, $f(4) = -0.3$. Then, $f_+(a) = 1$ and $f_+(b) = f_+(c) = f_+(d) = 0$, while $f_-(1) = 0$ and $f_-(b) = 1$, $f_-(c) = 3$, $f_-(d) = 0.3$.

Since f_+ and f_- are non-negative functions, we know how to define their sums. Let $S_+ = \sum_\omega f_+(\omega)$ and $S_- = \sum_\omega f_-(\omega)$. Recall that one or both of S_+, S_- could be equal to $+\infty$, in which case we say that $\sum_\omega f(\omega)$ *does not converge absolutely* and do not assign it any value. If both S_+ and S_- are finite, then we define $\sum_\omega f(\omega) = S_+ - S_-$. In this case, we say that $\sum f$ *converges absolutely*.

This completes our definition of absolutely convergent sums. A few exercises to show that when working with absolutely convergent sums, the usual rules of addition remain valid. For example, we can add the numbers in any order.

Exercise 9. Show that $\sum_{\omega \in \Omega} f(\omega)$ converges absolutely if and only if $\sum_{\omega \in \Omega} |f(\omega)|$ is finite (since $|f(\omega)|$ is a non-negative function, this latter sum is always defined, and may equal $+\infty$).

For non-negative f, we can find the sum by using any particular bijection and then taking limits of partial sums. What about general f?

Exercise 10. Let $f : \Omega \to \mathbb{R}$. Suppose $\sum_{\omega \in \Omega} f(\omega)$ be summable and let the sum be S. Then, for any bijection $\varphi : \mathbb{N} \to \Omega$, we have $\lim_{n\to\infty}(f(\varphi(1)) + \cdots + f(\varphi(n))) = S$.

Conversely, if $\lim_{n\to\infty}(f(\varphi(1)) + \cdots + f(\varphi(n)))$ exists and is the same finite number for any bijection $\varphi : \mathbb{N} \to \mathbb{R}$, then f must be absolutely summable and $\sum_{\omega \in \Omega} f(\omega)$ is equal to this common limit.

The usual properties of summation (without which life would not be worth living) are still valid.

Exercise 11. Let $f, g : \Omega \to \mathbb{R}_+$ and $a, b \in \mathbb{R}$. If $\sum f$ and $\sum g$ converge absolutely, then $\sum(af + bg)$ converges absolutely and $\sum(af + bg) = a\sum f + b\sum g$. Further, if $f(\omega) \le g(\omega)$ for all $\omega \in \Omega$, then $\sum f \le \sum g$.

[2] $a \vee b = \max\{a, b\}$

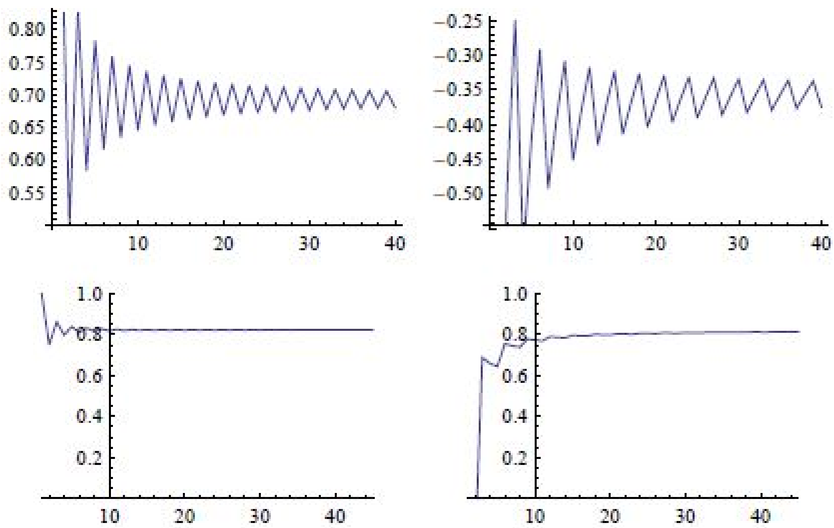

FIGURE 1. On the top left we plot the successive partial sums of the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$. On the top right we plot the successive partial sums of $1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{7} - \cdots$. Although we are adding the same numbers in a different order, they converge to different limits! In contrast, as the bottom pictures how, the same kind of rearrangement for the series $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$ does not change the limit.

Example 12. This example will illustrate why we refuse to assign a value to $\sum_\omega f(\omega)$ in some cases. Let $\Omega = \mathbb{Z}$ and define $f(0) = 0$ and $f(n) = 1/n$ for $n \neq 0$. At first one may like to say that $\sum_{n\in\mathbb{Z}} f(n) = 0$, since we can cancel $f(n)$ and $f(-n)$ for each n. However, following our definitions

$$f_+(n) = \begin{cases} \frac{1}{n} & \text{if } n \geq 1 \\ 0 & \text{if } n \leq 0, \end{cases} \qquad \text{and} \qquad f_-(n) = \begin{cases} \frac{1}{n} & \text{if } n \leq -1 \\ 0 & \text{if } n \geq 0. \end{cases}$$

Hence, S_+ and S_- are both $+\infty$ which means our definition does not assign any value to the sum $\sum_{\omega\in\Omega} f(\omega)$.

Indeed, by ordering the numbers appropriately, we can get any value we like! For example, here is how to get 10. We know that $1 + \frac{1}{2} + \cdots + \frac{1}{n}$ grows without bound. Just keep adding these positive number till the sum exceeds 10 for the first time. Then, start adding the negative numbers $-1 - \frac{1}{2} - \cdots - \frac{1}{m}$ till the sum comes below 10. Then, add the positive numbers $\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n'}$ till the sum exceeds 10 again, and then negative numbers till the sum falls below 10 again, etc. Using the fact that the individual terms in the series are going to zero, it is easy to see that the partial sums then converge to 10. There is nothing special about 10, we can get any number we want!

A remark on why we assumed Ω to be countable.

Remark 13. What if Ω is uncountable? Take any $f : \Omega \to \mathbb{R}_+$. Define the sets $A_n = \{\omega : f(\omega) \geq 1/n\}$ [3] with $n \in \mathbb{N}$. For some n, if A_n has infinitely many elements, then clearly the only reasonable value that we can assign to $\sum f(\omega)$ is $+\infty$ (since the sum over elements of A_n itself is larger than any finite number). Therefore, for $\sum f(\omega)$ to be a finite number it is essential that A_n is a finite set for each set.

Now, a countable union of finite sets is countable (or finite). Therefore $A = \bigcup_n A_n$ is a countable set. But note that A is also the set $\{\omega : f(\omega) > 0\}$ (since, if $f(\omega) > 0$ it must belong to some A_n). Consequently, even if the underlying set Ω is uncountable, our function will have to be equal to zero except on a countable subset of Ω. In other words, we are reduced to the case of countable sums!

[3] Note that $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$, i.e., A_n is an increasing sequence of sets