Network Analysis Techniques

- ***Goal***:
  - To determine individual branch currents and node voltages
- ***Procedure***:
  - Define all relevant variables in a clear and systematic fashion
  - Identify all known and unknown variables
  - Construct a set of equations relating these variables
  - Solve these equations

- **Techniques**:
  - ***Node Voltage Method***
  - ***Mesh Current Method***
  - ***Superposition Principle***
  - ***Thevenin Equivalent***
  - ***Norton Equivalent***

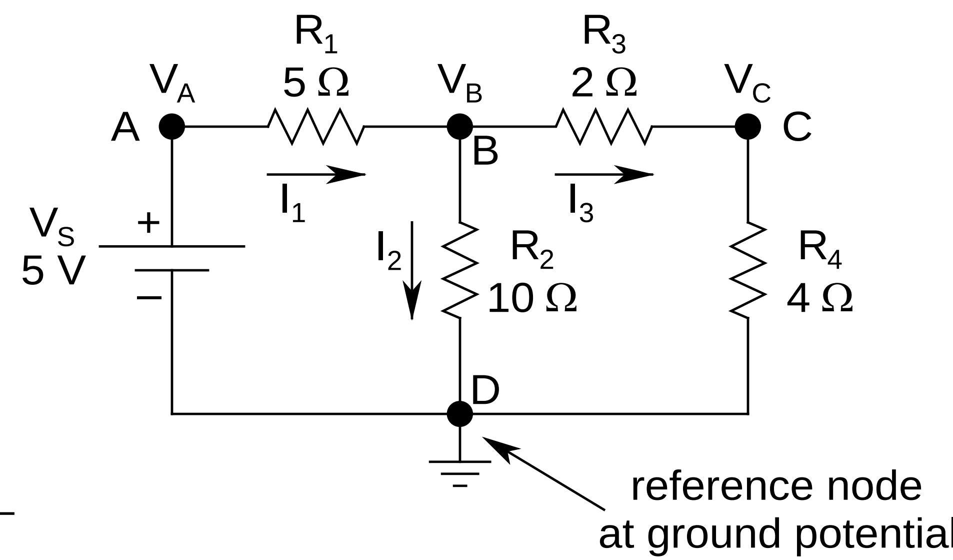
- ***Node Voltage Method***:
  - By far, the simplest and the most widely used
  - Consider a circuit having N non-trivial nodes
  - Pick a reference node, and define all other node voltages with respect to this reference node
  - Apply Ohm's Law between any two adjacent nodes, and write the current equations
  - Thus, we arrive at a set of (N – 1) equations
  - Note that the final set of equations does not contain any current variable
  - Solve them to find the node voltages and currents

- ***Example***:

Node D: Reference Node

Node C: Trivial Node

KCL at node B: $I_1 = I_2 + I_3$



$$\Rightarrow \frac{V_A - V_B}{R_1} = \frac{V_B}{R_2} + \frac{V_B}{R_3 + R_4}$$

$$\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3 + R_4}\right)V_B = \frac{V_S}{R_1}$$

$$\Rightarrow \left(\frac{1}{5} + \frac{1}{10} + \frac{1}{6}\right)V_B = \frac{5}{5} = 1$$

$$V_A = V_S = 5\ \text{V}, V_B = 2.143\ \text{V}$$

$$I_1 = (5 - 2.143)/5 = 0.571\ \text{A}$$

$$I_2 = 2.143/10 = 0.214\ \text{A}$$

$$I_3 = 2.143/6 = 0.357\,\text{A}$$

$$V_C = I_3 R_4 = V_B - I_3 R_3 = 1.428\,\text{V}$$

- ***Mesh Current Method***:
  - Branch currents are taken to be independent variables
    - Known as the complement of node voltage method
  - Find the minimum number of independent meshes in the network
  - Using KVL, write the mesh equations in terms of the voltage drop across each element
  - Repeat for all the meshes
  - Number of equations would equal the number of independent meshes
    - Individual mesh currents can be evaluated

- Once the mesh currents are known, all the node voltages can be evaluated
- The unknown mesh currents are always considered to be positive in the ***clockwise*** direction
- This technique becomes quite involved for networks having three or more independent meshes, since the number of equations that has to solved is always equal to the number of independent meshes
- Not much used for large networks, where the node voltage method is preferred

- ***Example***: Simple resistive circuit

Meshes marked 1 & 2

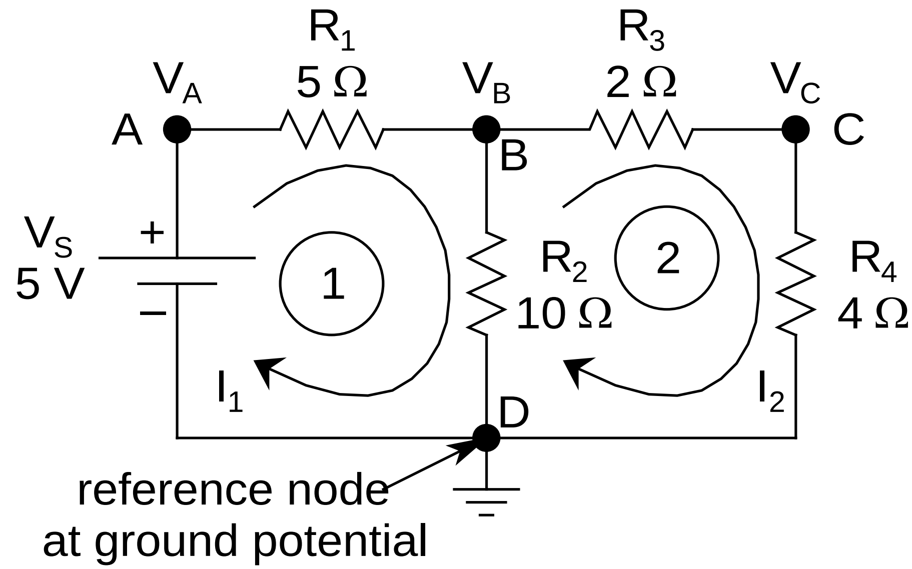
Note: Current flowing through $R_2 = I_1 - I_2$

$V_S = I_1R_1 + (I_1 - I_2)R_2$

$\Rightarrow\ 15I_1 - 10I_2 = 5\ \cdots\ (1)$

$(I_1 - I_2)R_2 = I_2R_3 + I_2R_4$

$\Rightarrow\ 10I_1 - 16I_2 = 0\ \cdots\ (2)$



Solving (1) and (2) simultaneously, we get $I_1 = 0.571$ A and $I_2 = 0.357$ A

Since $I_1 > I_2$, therefore the actual current (= 0.214 A) through R_2 is flowing from node B to ground

- ***Example***: Mesh containing an independent current source

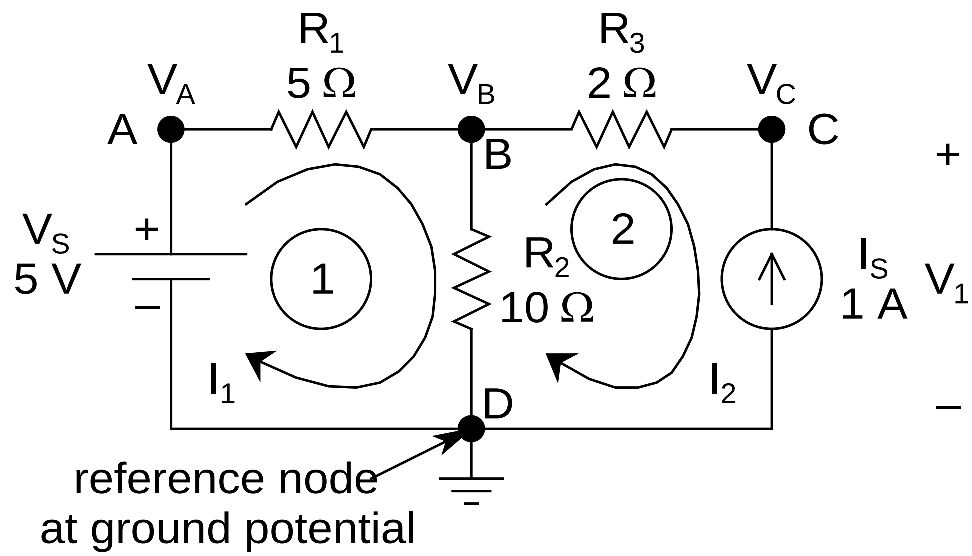
$$I_2 = -I_S$$

$$V_S = I_1R_1 + \left(I_1 - I_2\right)R_2$$

$$= \left(R_1 + R_2\right)I_1 + I_SR_2$$

$$I_1 = \frac{V_S - I_SR_2}{R_1 + R_2} = \frac{5 - 1\times10}{5+10}$$

$$= -0.333 \text{ A}$$



Negative sign implies that actual direction of I_1 is opposite to that shown in the figure. For computation of V_1 :

$$\left(I_1 - I_2\right)R_2 = I_2R_3 + V_1$$

$$V_1 = I_1R_2 + \left(R_2 + R_3\right)I_S = -0.333\times10 + \left(10+2\right)\times1 = 8.67 \text{ V}$$

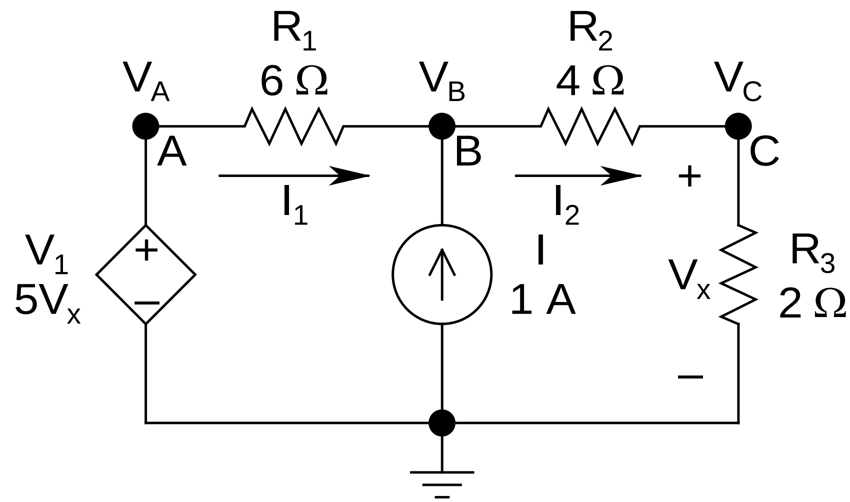
- ***Example***: Mesh containing a controlled source

$V_A = V_1 = 5V_x$ and $V_C = V_x$

Writing KCL at Node B:

$I_1 + I = I_2$

$$\Rightarrow \quad \frac{V_A - V_B}{R_1} + I = \frac{V_B}{R_2 + R_3}$$



$$V_A = R_1\left[\left(\frac{1}{R_1} + \frac{1}{R_2 + R_3}\right)V_B - I\right] = 6\left[\left(\frac{1}{6} + \frac{1}{4+2}\right)V_B - 1\right] = 2V_B - 6$$

$$V_C = V_x = R_3 V_B / \left(R_2 + R_3\right) = V_B / 3$$

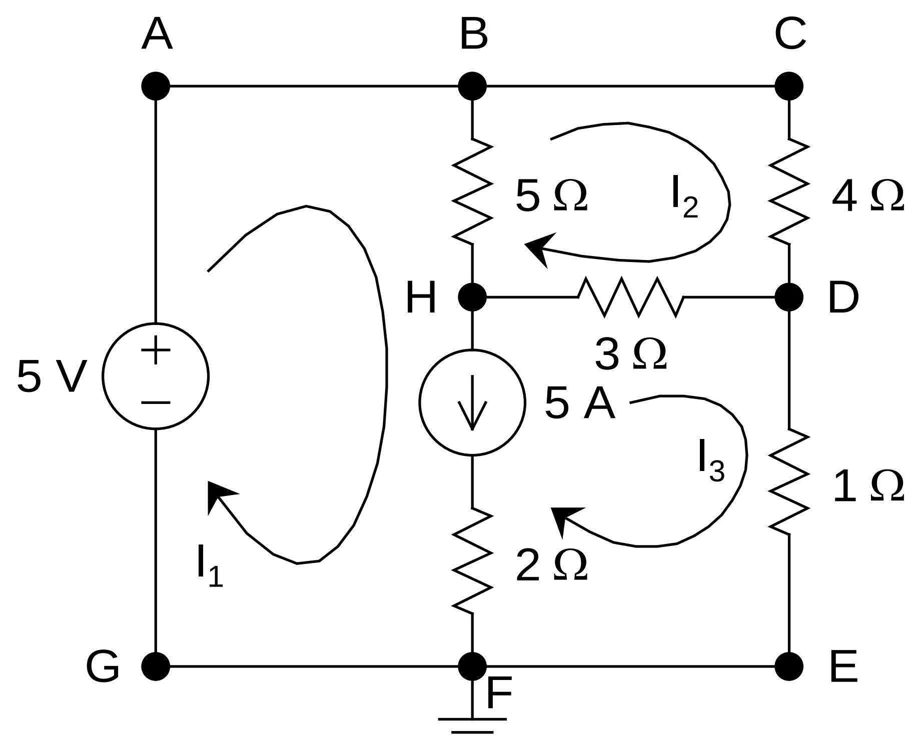
$$V_A = V_1 = 5V_x = 5V_B / 3 = 1.667 V_B$$

$$\Rightarrow V_B = 18 \text{ V}, \; V_A = 30 \text{ V}, \; V_C = 6 \text{ V}$$

and $I_1 = \left(V_A - V_B\right) / R_1 = 2 \text{ A}, \; I_2 = V_B / \left(R_2 + R_3\right) = 3 \text{ A}$

- ***Example***: Loop containing an independent current source: Concept of ***Supermesh***

Nodes A, B, and C are same node, similarly, nodes E, F, and G are same KVL around the mesh ABHFGA cannot be written, since the potential dropped across the 5 A current source is not known



Similarly, KVL around HDEFH cannot be written

Hence, need to invoke the concept of supermesh that constitutes parts of different meshes

Supermesh: a mesh containing parts of other meshes

Example: ABHDEFGA

KVL around this loop:

$$5 = \left(I_1 - I_2\right) \times 5 + \left(I_3 - I_2\right) \times 3 + I_3 \times 1 = 5I_1 - 8I_2 + 4I_3$$

with $I_1 - I_3 = 5 \text{ A} \Rightarrow 9I_1 - 8I_2 = 25 \cdots (1)$

KVL around BCDHB:

$$I_2 \times 4 + \left(I_2 - I_3\right) \times 3 + \left(I_2 - I_1\right) \times 5 = 0$$

$$\Rightarrow 8I_1 - 12I_2 = 15 \cdots (2)$$

Thus, $I_1 = 4.09 \text{ A}$, $I_2 = 1.48 \text{ A}$, and $I_3 = -0.91 \text{ A}$

- ***Example***: The Ultimate: Having Everything!

$I_1 = 5$ A, and mesh equations for I_1 and I_3 cannot be written

Note: $I_3 - I_1 = V_x / 5$

Also, $V_x = 3(I_3 - I_2)$

$$\Rightarrow I_3 - 5 = \frac{3}{5}(I_3 - I_2)$$

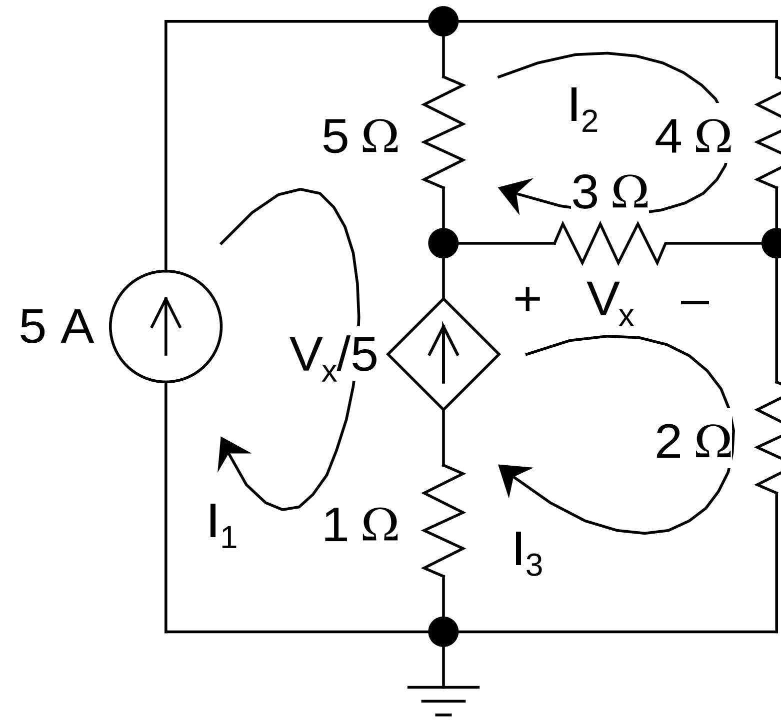
$$\Rightarrow 3I_2 + 2I_3 = 25 \cdots (1)$$

Around the I_2 mesh:

$$5(I_2 - I_1) + 4I_2 + 3(I_2 - I_3) = 0$$

$$\Rightarrow 12I_2 - 3I_3 = 25 \cdots (2)$$

Thus, $I_2 = 3.788$ A, and $I_3 = 6.818$ A



- ***Superposition Principle***:
  - Very powerful technique of analyzing linear electrical networks, having multiple sources
- ***Linearity and Linear Networks***:
  - ***Linear function***: $y = kx$, $k =$ constant
  - For the function to be linear, $y_{12} = y_1 + y_2$ must equal kx_{12}, where $x_{12} = x_1 + x_2$
  - Consider $y = kx^2$ – it's a ***nonlinear function***, since for this case $(y_1 + y_2) \neq y_{12}$, because $\left(x_1^2 + x_2^2\right) \neq \left(x_1 + x_2\right)^2$

- Other examples of non-linear functions: $y = kx_1x_2$, $y = ke^x$, etc.
- Resistive networks follow linearity (***Ohm's Law***)
- Hence, superposition principle can be applied for analysis of resistive circuits

- ***Technique***:

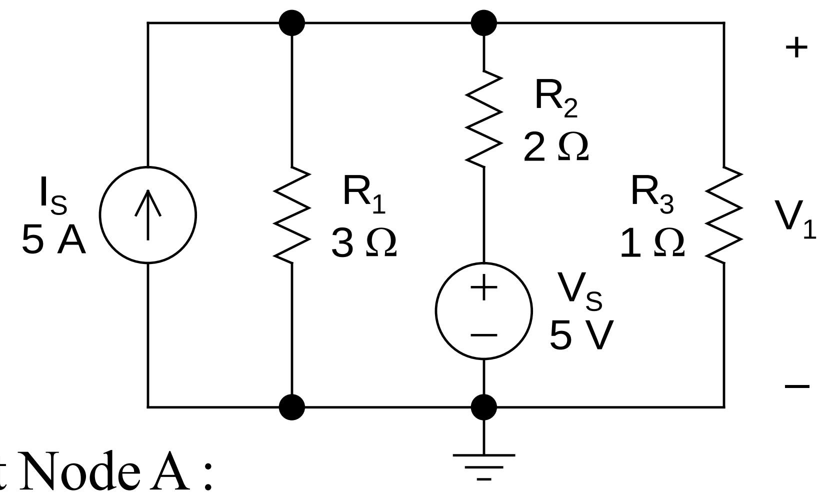
1. Take one source at a time and null all other independent sources
    - Nulling of sources: for voltage sources, short them; for current sources, open them
    - Be careful about dependent sources, they should not be nulled

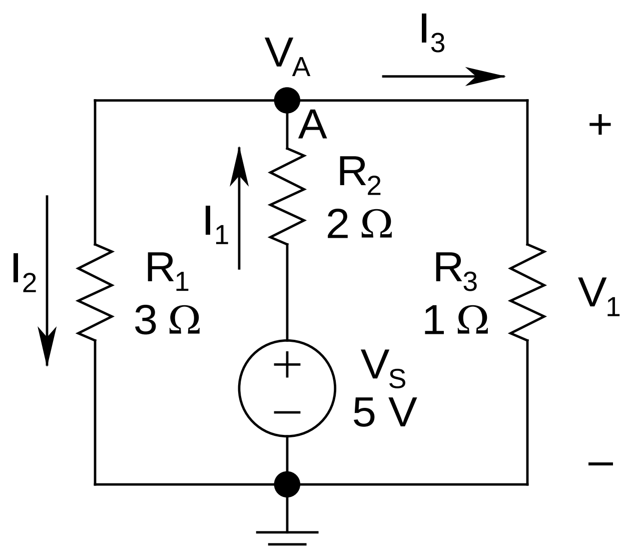
2. By adopting KCL, KVL, node voltage, or mesh current method, evaluate the currents through all branches, as well as the node voltages
3. Repeat steps 1 and 2 till all the sources are exhausted
4. Then, the current through any branch or the voltage at any node is evaluated as a linear superposition of all the currents flowing through that branch or the voltages appearing at that node, contributed by the different sources

- ***Example***: To find V_1 appearing across R_3 using Superposition Principle

Circuit has two independent sources, V_S and I_S

First consider V_S and null I_S (i.e., open circuit it)





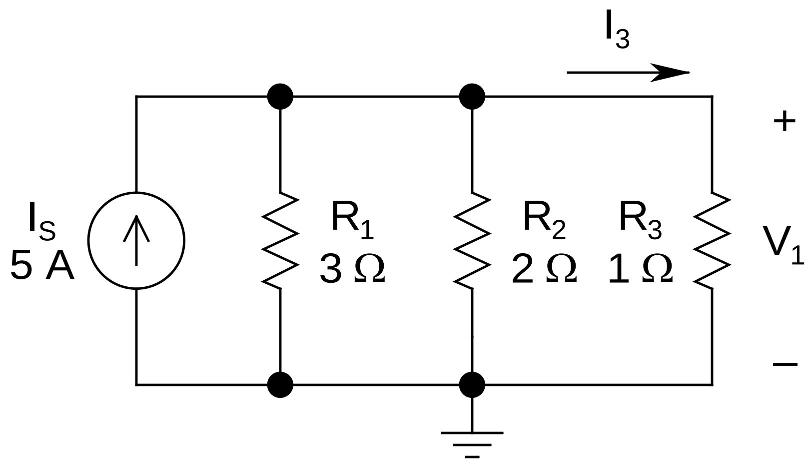
KCL at Node A :

$$I_1 = I_2 + I_3 \quad \Rightarrow \quad \frac{V_S - V_A}{R_2} = \frac{V_A}{R_1} + \frac{V_A}{R_3}$$

$$\Rightarrow V_A = V_1 = \frac{V_S}{R_2\left(1/R_1 + 1/R_2 + 1/R_3\right)}$$

$$= \frac{5}{2\times\left(1/3 + 1/2 + 1\right)} = 1.364\ \text{V}$$

Next, consider I_S and null V_S (i.e., short-circuit it)



Note : $G_{net} = G_1 + G_2 + G_3$

where $G = 1/R$

Hence, $V_1 = I_S / G_{net} = I_3 / G_3$

$$\Rightarrow \; I_3 = \frac{G_3}{G_{net}} I_S = \frac{1/R_3}{1/R_1 + 1/R_2 + 1/R_3} I_S = \frac{1}{1/3 + 1/2 + 1} \times 5 = 2.727 \text{ A}$$

and $V_1 = I_3 R_3 = 2.727$ V

Hence, the net voltage appearing across R_3 would be a superposition of the two results :

$$\Rightarrow \; V_1(\text{net}) = V_1(\text{due to } V_S \text{ alone}) + V_1(\text{due to } I_S \text{ alone})$$

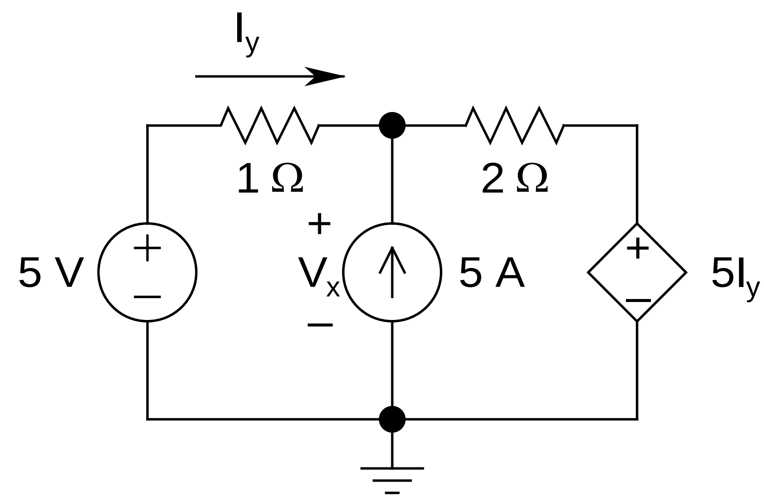
$$= 1.364 + 2.727 = 4.091 \text{ V}$$

- ***Example***: Superposition Principle for Circuits Having Dependent Sources

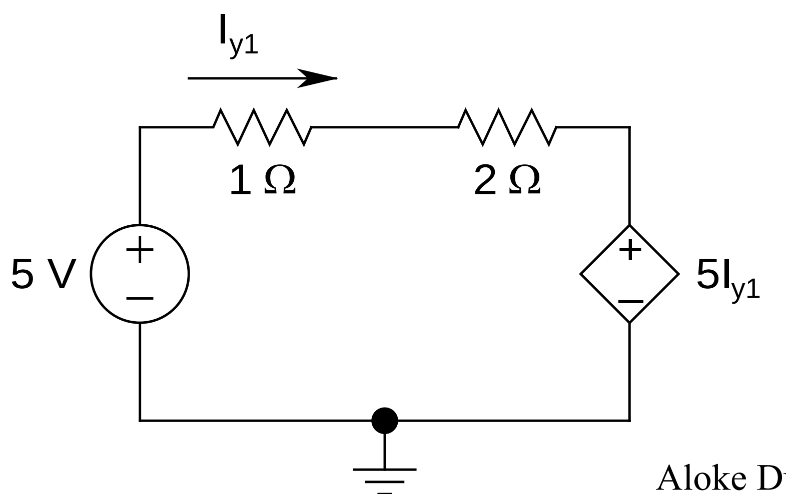
3 sources : Two independent (5 V and 5 A) and one dependent ($5I_y$)

Recall: ***Dependent sources are never nulled***!

First, consider the 5 V source and null (open) the 5 A source



To determine V_x and I_y



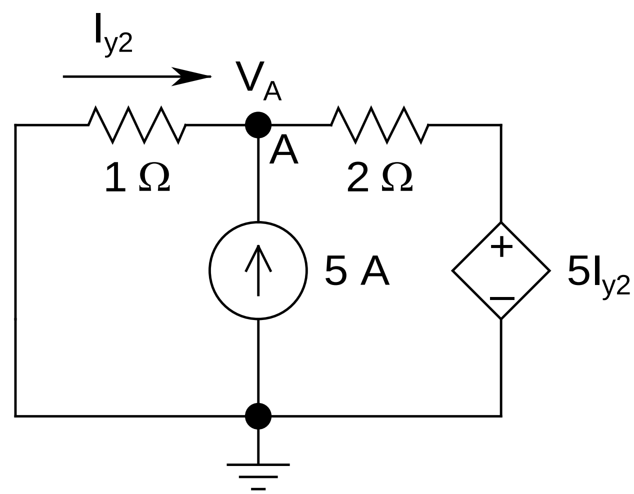
$$5 = I_{y1} + 2I_{y1} + 5I_{y1} = 8I_{y1}$$

$$\Rightarrow I_{y1} = 0.625 \text{ A}$$

Next, consider the 5 A source

and short the 5 V source

KCL at node A:



$$I_{y2} + 5 = \frac{V_A - 5I_{y2}}{2}$$

with $V_A = -I_{y2} \Rightarrow I_{y2} = -1.25$ A

Negative sign implies that I_{y2} is actually flowing opposite to the direction shown in the figure

Therefore, the net current $I_y = I_{y1} + I_{y2} = -0.625$ A

Opposite to the direction shown

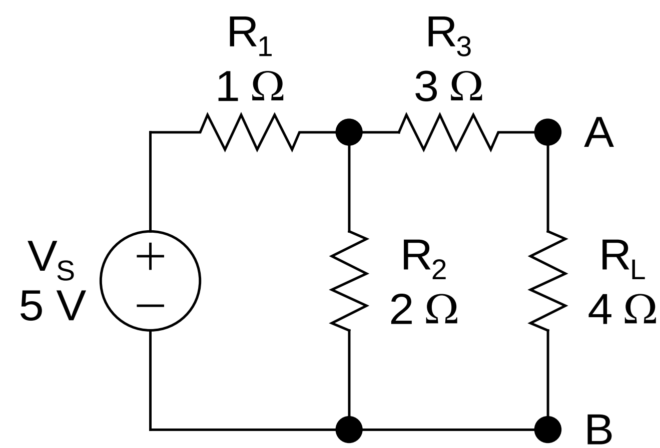
and $V_x = 5 - I_y = 5 - (-0.625) = 5.625$ V

- ***Thevenin and Norton Equivalents***:
  - Two very powerful techniques for network analysis, and are actually dual of each other
- ***Thevenin's Theorem***:
  - Any network containing independent and dependent sources and resistors, when viewed from the load, can be represented by an ***equivalent voltage source*** V_T in series with an ***equivalent resistance*** R_T
- ***Norton's Theorem***:
  - Any network containing independent and dependent sources and resistors, when viewed from the load, can be represented by an ***equivalent current source*** I_N in shunt with an ***equivalent resistance*** R_N

- ***Construction of Thevenin's Equivalent***:
  - The Thevenin voltage V_T is also referred to as the ***Open-Circuit Voltage*** (V_{OC})
- ***Example***: To find current through R_L

First, remove R_L (i.e., open-circuit load)

The voltage appearing between terminals A & B is known as the open-circuit voltage (V_{OC})

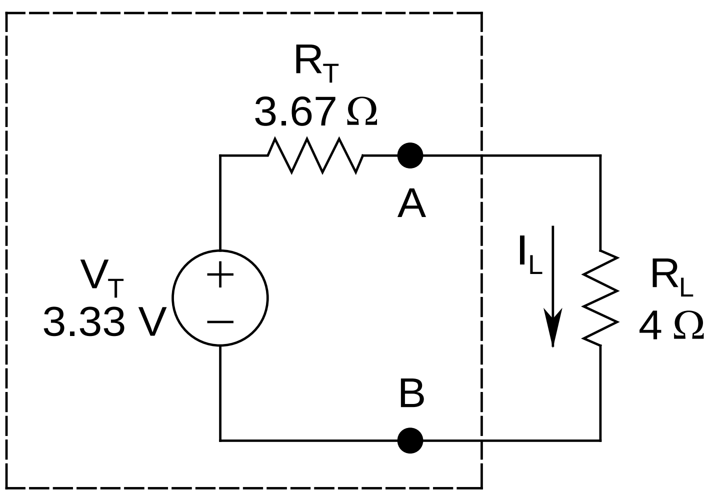


$$V_T = V_{OC} = \frac{R_2}{R_1 + R_2} V_S = \frac{2}{2+1} \times 5 = 3.33 \text{ V} \;\; (\text{no current through } R_3)$$

- Procedure to find the ***Thevenin Resistance*** R_T:
  - R_T is defined as the ***effective resistance*** of the network, looking from the two open-circuited terminals
  - To find R_T, null all independent sources, and find the effective resistance appearing between the two open-circuited terminals
  - For the example, with V_S shorted and looking from terminals A and B, by inspection:

$$R_T = R_3 + \left(R_1 \parallel R_2\right) = 3 + \left(1 \parallel 2\right) = 3.67\ \Omega$$

- ***Complete Thevenin Equivalent***:



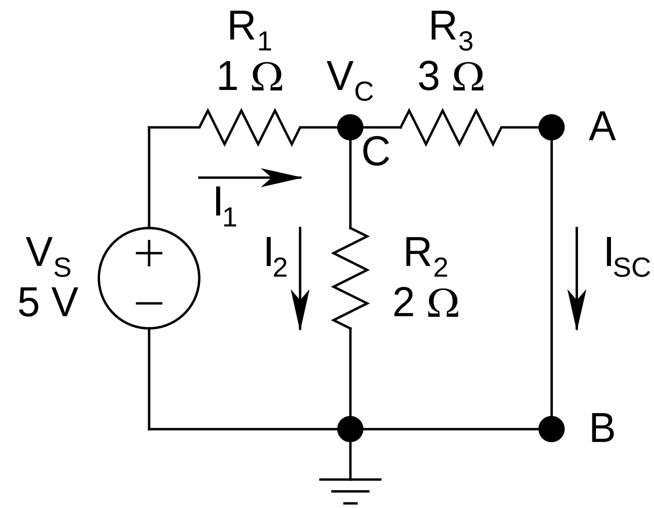
Thevenin Equivalent Representation

$$I_L = \frac{V_T}{R_T + R_L} = \frac{3.33}{3.67 + 4} = 0.434 \text{ A}$$

Note that the Thevenin Equivalent (within the dotted box) remains ***invariant*** for any value of R_L

Thus, computation of I_L for any value of R_L becomes absolutely trivial

- ***Construction of Norton's Equivalent***:
  - The Norton current I_N is also referred to as the ***Short-Circuit Current*** (I_{SC})
- ***Example***: Same as before



First, replace R_L by a short-circuit

The current flowing through this shorted branch is known as the short-circuit current (I_{SC})

KCL at node C:

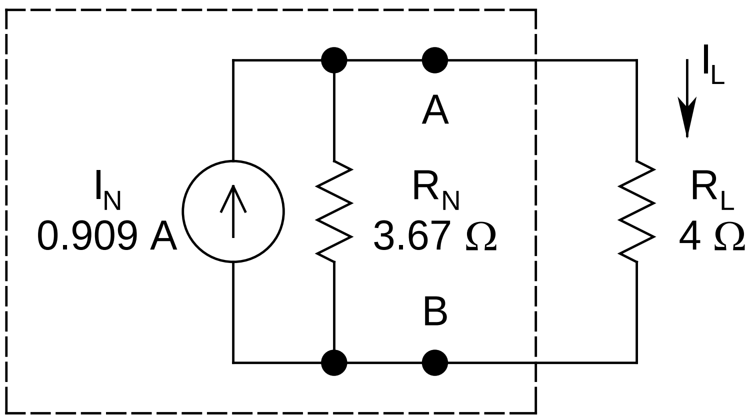
$$I_1 = I_2 + I_{SC} \Rightarrow \frac{V_S - V_C}{R_1} = \frac{V_C}{R_2} + \frac{V_C}{R_3}$$

$$\Rightarrow V_C = \frac{V_S/R_1}{1/R_1 + 1/R_2 + 1/R_3}$$

$$= \frac{5}{1 + 1/2 + 1/3} = 2.727 \text{ V}$$

$$\Rightarrow I_{SC} = I_N = V_C / R_3 = 0.909 \text{ A}$$

- Procedure to find the ***Norton Resistance*** R_N:
  - Identical to that for R_T => $R_N = R_T = 3.67\ \Omega$
- ***Complete Norton Equivalent***:



Norton Equivalent Representation

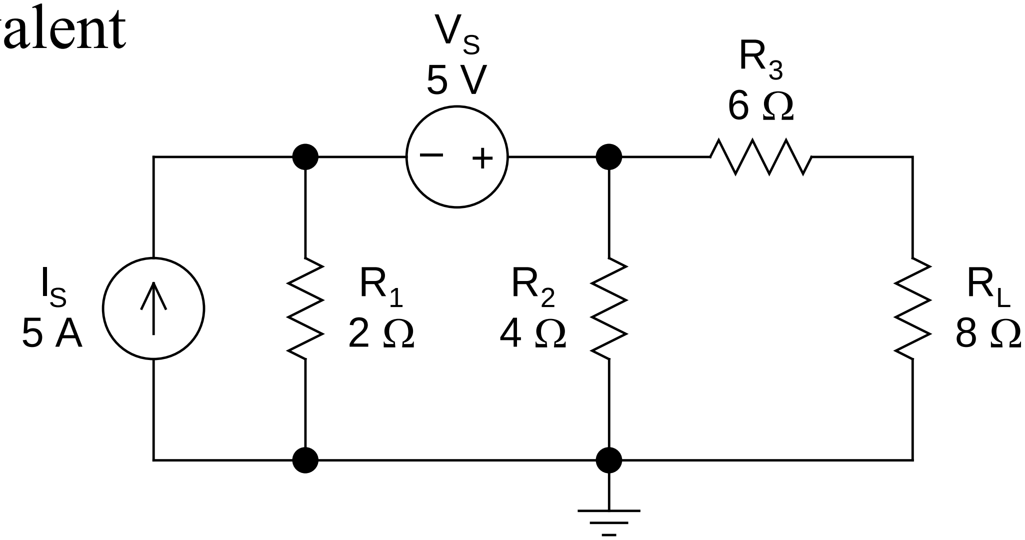
$$I_L = \frac{R_N}{R_N + R_L} I_N = \frac{3.67}{3.67+4} \times 0.909$$

$$= 0.434\ \text{A}$$

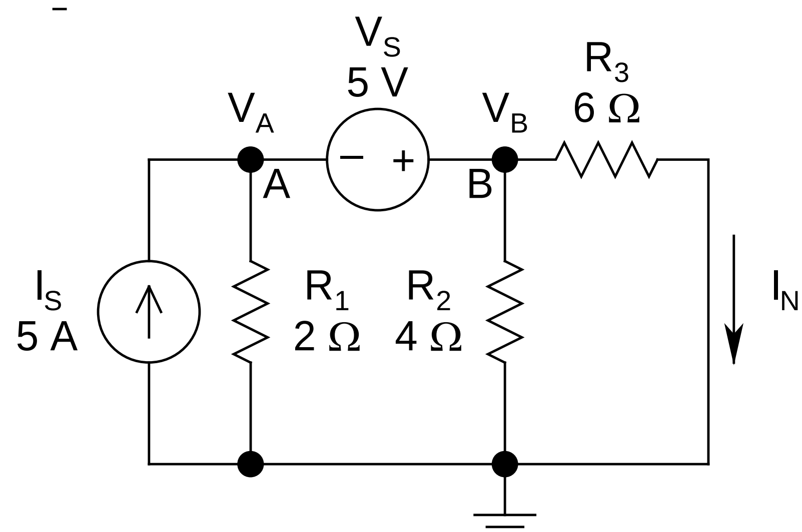
Identical to that computed earlier using Thevenin equivalent, which should be the case since it is the same circuit!

- ***Source Transformation Technique***:
  - Thevenin and Norton equivalents are ***dual*** of each other (can extract one from the other)
  - Note that $V_T = I_N R_T$
  - A voltage source V_1 in series with a resistance R_1 can be replaced by a current source I_1 ($= V_1/R_1$) in shunt with a resistance R_1, or vice versa
    - Known as the ***source transformation technique***
    - Extremely powerful in analysis of electrical networks
  - Also, note that $R_T = R_N = V_T/I_N = V_{OC}/I_{SC}$
    - Thevenin (or equivalently, Norton) resistance is also defined as the ***ratio of open-circuit voltage and short-circuit current***

- ***Example***: To find the current through R_L using Norton's equivalent



Remove R_L and short the two terminals:
I_N is the short-circuit current (I_{SC})



Note that nodes A and B can be combined to a supernode, with $V_B = V_A + 5$

KCL at node A:

$$I_S = \frac{V_A}{R_1} + \frac{V_B}{R_2} + \frac{V_B}{R_3} \quad \Rightarrow \quad 5 = \frac{V_B - 5}{2} + \frac{V_B}{4} + \frac{V_B}{6}$$

$\Rightarrow\ V_B = 8.182\ \text{V, and}\ I_N = V_B / R_3 = 8.182/6 = 1.364\ \text{A}$

For R_N, open-circuit R_L and look across its two terminals, open-circuit all independent current sources, and short-circuit all independent voltage sources

$$\Rightarrow R_N = R_3 + \left(R_1 \,\|\, R_2\right) = 6 + \left(2 \,\|\, 4\right) = 7.33\ \Omega$$

Note: $V_T = R_N I_N = 10\ \text{V, and}\ R_T = R_N = 7.33\ \Omega$

Alternate Method:

Note also that for a supernode, the total current entering is equal to the total current leaving

Total current entering node A = $I_S - \frac{V_A}{R_1}$

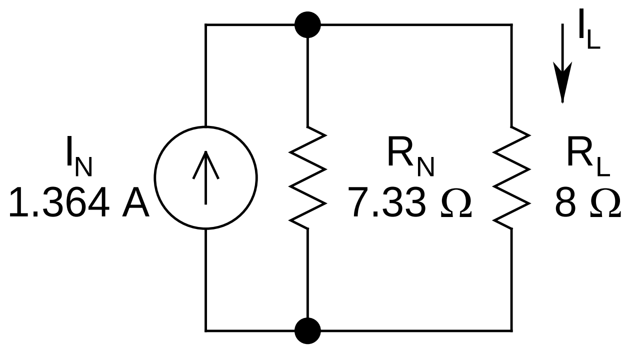
Total current leaving node B = $\frac{V_B}{R_2} + \frac{V_B}{R_3}$

Caution: Keep track of the sign of the currents

Thus,

$$I_S - \frac{V_A}{R_1} = \frac{V_B}{R_2} + \frac{V_B}{R_3}$$

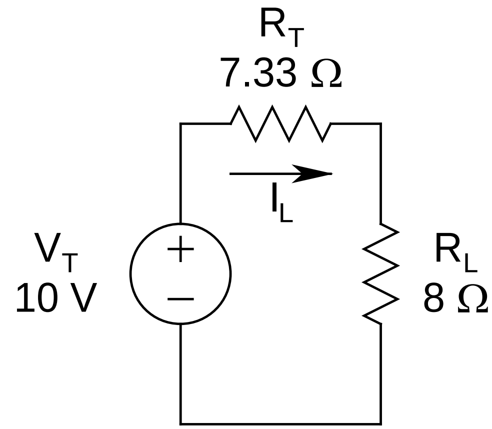
which leads to the same result as before



Norton Representation

$\Leftrightarrow$

Dual



Thevenin Representation

From Norton Equivalent:

$$I_L = R_N I_N / (R_N + R_L) = 7.33 \times 1.364 / (7.33 + 8) = 0.652 \text{ A}$$

From Thevenin Equivalent:

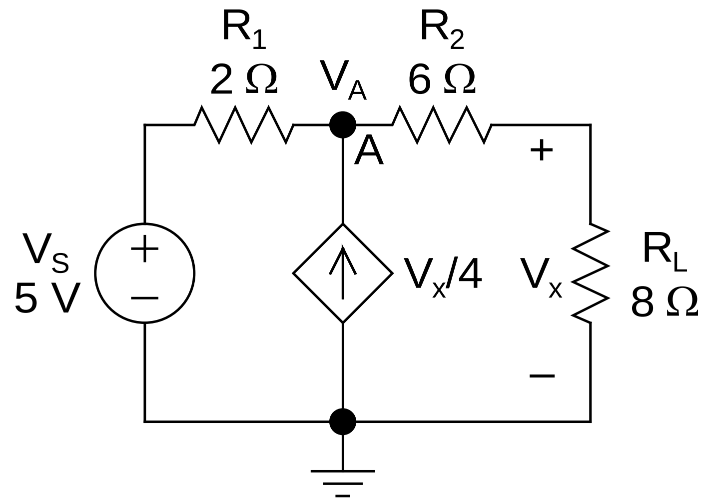
$$I_L = V_T / (R_T + R_L) = 10 / (7.33 + 8) = 0.652 \text{ A}$$

They are same, as expected!

- ***Example***: Application of Thevenin and Norton Techniques for Circuits having Dependent Sources:

To find current through R_L :

First, remove R_L : Voltage appearing across the two terminals is the Thevenin voltage V_T



$$\text{Note: no current through } R_2 \Rightarrow V_T = V_{OC} = V_A = V_X$$

KCL at node A:

$$\frac{V_S - V_A}{R_1} + \frac{V_X}{4} = 0 \Rightarrow \frac{5 - V_{OC}}{2} + \frac{V_{OC}}{4} = 0 \Rightarrow V_{OC} = V_T = 10\ \text{V}$$

To find R_T:

For circuits having dependent sources, it can't be obtained by inspection

Instead, apply the ***short-circuit current technique***

Algorithm:

* Replace the load by a short-circuit
* Find the short-circuit current I_{SC}
* Open-circuit the load
* Find the open-circuit voltage V_{OC}
* Then, obtain R_T from the relation: $R_T = V_{OC} / I_{SC}$

For the example considered, V_{OC} is already known

Now, to find I_{SC}, short R_L

With R_L shorted, V_x vanishes, along with $V_x / 4$

$$\Rightarrow I_{SC} = V_S / (R_1 + R_2)$$

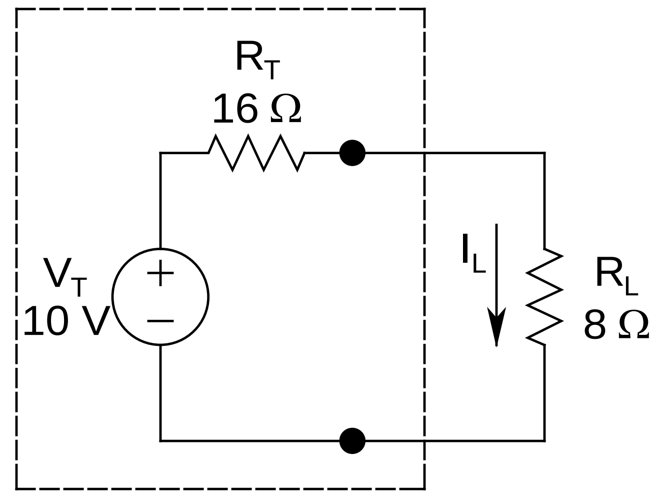
$$= 5/(2 + 6) = 0.625 \text{ A}$$

$$\Rightarrow R_T = V_{OC} / I_{SC} = 10 / 0.625$$

$$= 16\ \Omega$$

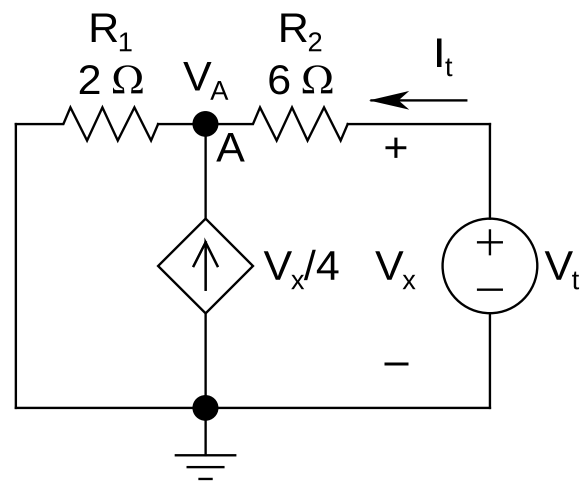
$$\Rightarrow I_L = V_T / (R_T + R_L)$$

$$= 10/(16 + 8) = 0.417 \text{ A}$$



Thevenin Equivalent Representation

- ***Alternate Method to Find R_T***:
- ***Procedure***:
  - Open circuit the load terminals and attach a test voltage source V_t
  - Null all independent sources, keeping dependent source undisturbed
  - Perform a circuit analysis, and find the current I_t drawn from V_t
  - Then, $R_T = V_t/I_t$



V_S shorted, $V_x = V_t$, and

$V_A = V_t - I_t R_2 = V_t - 6I_t$

KCL at node A:

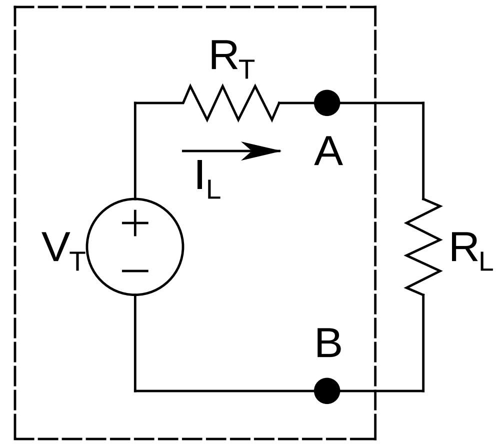
$$I_t + \frac{V_t}{4} = \frac{V_A}{R_1} \Rightarrow I_t + \frac{V_t}{4} = \frac{V_t - 6I_t}{2}$$

$\Rightarrow V_t = 16I_t$, and $R_T = V_t / I_t = 16\ \Omega$

Same as before

- ***Maximum Power Transfer Theorem***:
  - Goal is to ensure that the maximum power is delivered to the load
  - Extremely useful for audio applications:
    - The speaker resistance can be tuned to ensure that the maximum power is transferred to it from the audio amplifier
      - Thus, the maximum possible level of sound is produced
  - One important condition that must be satisfied for this to happen is obtained from the ***maximum power transfer theorem***

Goal: To find the value of R_L that would ensure ***maximum power*** to be transferred to it, and to find this maximum power



Thevenin Equivalent Representation

Power consumed by R_L:

$P_L = I_L^2 R_L$, with $I_L = V_T/(R_T + R_L)$

$$\Rightarrow P_L = \frac{V_T^2 R_L}{(R_T + R_L)^2}$$

V_T and R_T are fixed for a given network

$\Rightarrow P_L$ would show variation only with respect to R_L

To find $P_{L,max}$, put $dP_L / dR_L = 0$

$$\Rightarrow \boldsymbol{R_L = R_T}, \text{ and } P_{L,max} = \frac{V_T^2}{4R_T}$$