MSO 201a: Probability & Statistics

Assignment - II

Solutions

Problem No. 1 **Claim I** Each D_n $(n=1,2,\dots)$ is finite.

On contrary suppose that some D_{n_0} is infinite. Then D_{n_0} contains a countably infinite set (every infinite set has a countably infinite subset), say $E=\{a_1, a_2, \dots\}$ (every countably infinite set can be represented by a sequence $\{a_n\}_{n\geq 1}$).

For $x \in E \ (\subseteq D_{n_0})$, $F(x)-F(x-) \geq \frac{1}{n_0}$,

$$P(X\in D_{n_0}) \geq P(X\in E) = \sum_{x\in E} P(X=x) = \sum_{x\in E}[F(x)-F(x-)] \geq \sum_{x\in E}\frac{1}{n_0} = \infty$$

$\rightarrow$ Contradiction

Claim II D (= set of discontinuity points of F) is countable.

$$D=\{x\in\mathbb{R}: F(x)-F(x-)>0\} = \bigcup_{n=1}^{\infty}\{x\in\mathbb{R}: F(x)-F(x-)\geq \tfrac{1}{n}\} = \bigcup_{n=1}^{\infty} D_n$$

$\rightarrow$ countable (countable union of countable sets is countable).

Problem No. 2 (i) $F_1(\frac{1}{2}+)=1 \neq \frac{1}{2} = F_1(\frac{1}{2}) \Rightarrow F_1$ is not a d.f.

(ii) F_2 is non-decreasing, $F_2(-\infty)=0$, $F_2(+\infty)=1$ and F_2 is right continuous (in fact continuous) $\Rightarrow F_2$ is a d.f.

(iii) Using arguments of (ii) F_3 is a d.f.

Problem No. 3 (i) $F(3)=F(3+) \Rightarrow \dfrac{4c^2-9c+6}{4}=1 \Rightarrow c=\frac{1}{4},2$

$F(1-)\leq F(1) \Rightarrow \frac{2}{3} \leq \dfrac{7-6c}{6} \Rightarrow c\leq\frac{1}{2}$

$\Rightarrow c=\frac{1}{4}$

(ii) $P(1<X<2)$ $F(2-)-F(1) = \frac{11}{12}-\frac{11}{12}=0$

$P(2\leq X<3) = F(3-)-F(2-) = 1-\frac{11}{12}=\frac{1}{12}$

$P(0<X\leq 1) = F(1)-F(0) = \frac{11}{12}-\frac{2}{3}=\frac{1}{4}$

$P(1\leq X\leq 2) = F(2)-F(1-) = 1-\frac{2}{3}=\frac{1}{3}$

$P(X\geq 3) = 1-F(3-) = 1-1=0$

$P(X=\frac{5}{2}) = F(\frac{5}{2})-F(\frac{5}{2}-)=0$

$P(X=2) = F(2)-F(2-) = 1-\frac{11}{12}=\frac{1}{12}$

(iii) $P(X=1 \mid 1 \le X \le 2) = \dfrac{P(X=1,\ 1 \le X \le 2)}{P(1 \le X \le 2)}$

$$= \frac{P(X=1)}{P(1 \le X \le 2)} = \frac{F_X(1) - F_X(1-)}{F_X(2) - F_X(1-)}$$

$$= \frac{\frac{11}{12} - \frac{2}{3}}{1 - \frac{2}{3}} = \frac{3}{4}$$

$P(1 \le X < 2 \mid X > 1) = \dfrac{P(1 \le X < 2,\ X > 1)}{P(X > 1)}$

$$= \frac{P(1 < X < 2)}{P(X > 1)} = \frac{F_X(2-) - F_X(1)}{1 - F_X(1)}$$

$$= \frac{\frac{11}{12} - \frac{11}{12}}{1 - \frac{11}{12}} = 0.$$

(iv) $D =$ set of discontinuity points of $F = \{0, 1, 2\}$

Sum of jumps $= [F_X(0) - F_X(0-)] + [F_X(1) - F_X(1-)] + [F_X(2) - F_X(2-)]$

$$= \left(\frac{2}{3} - 0\right) + \left(\frac{11}{12} - \frac{2}{3}\right) + \left(1 - \frac{11}{12}\right)$$

$$= 1$$

$\Rightarrow$ X is of discrete type.

The p.m.f. of X is

$$f_X(x) = P(X=x) = F_X(x) - F_X(x-)$$

$$= \begin{cases} \frac{2}{3}, & \text{if } x = 0 \\ \frac{1}{4}, & \text{if } x = 1 \\ \frac{1}{12}, & \text{if } x = 2 \\ 0, & \text{otherwise} \end{cases}$$

$\boxed{2/II}$

Problem No. 4 (i) D = set of discontinuity points of F_X

$= \{-2, 0, 5, 6\}$

sum of jumps $= \sum_{x \in D} [F(x) - F(x-)]$

$= \frac{1}{3} + \frac{1}{6} + \frac{1}{4} + \frac{1}{4} = 1$

$\Rightarrow$ X is of discrete type with p.m.f.

$$f_X(x) = P(X=x) = F(x) - F(x-) = \begin{cases} \frac{1}{3}, & x=-2 \\ \frac{1}{6}, & x=0 \\ \frac{1}{4}, & x=5, 6 \\ 0 & \text{otherwise} \end{cases}$$

(ii) Clearly F_X is continuous everywhere and differentiable everywhere except at $x=0$.

$$F'(x) = \begin{cases} 0, & x<0 \\ e^{-x}, & x>0 \end{cases}$$

$$\int_{-\infty}^{\infty} F'(x)\,dx = \int_0^{\infty} e^{-x}\,dx = 1$$

$\Rightarrow$ X is of ~~absolutely~~ continuous type with a p.d.f.

$$f(x) = \begin{cases} e^{-x}, & \text{if } x>0 \\ 0, & \text{otherwise.} \end{cases}$$

Problem No. 5 (i) D = set of discontinuity points of F_X

$= \{1, 2\} \neq \phi$

$\Rightarrow$ X is not of continuous type

sum of jumps $= [F(1) - F(1-)] + [F(2) - F(2-)]$

$= \frac{1}{3} + \frac{1}{3} = \frac{2}{3} \neq 1$

$\Rightarrow$ X is not of discrete type.

$\boxed{3/5}$

(ii) $P(X=1) = F(1) - F(1-) = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$;

$P(X=2) = F(2) - F(2-) = 1 - \frac{2}{3} = \frac{1}{3}$;

$P(X=1.5) = F(1.5) - F(1.5-) = 0$ ($x=1.5$ is a continuity point of F_X)

$P(1<X<2) = F(2-) - F(1) = \frac{2}{3} - \frac{2}{3} = 0$.

(iii) $$P(1 \le X < 2 \mid 1 \le X \le 2) = \frac{P(1 \le X < 2, 1 \le X \le 2)}{P(1 \le X \le 2)}$$

$$= \frac{P(1 \le X < 2)}{P(1 \le X \le 2)} = \frac{F(2-) - F(1-)}{F(2) - F(1-)}$$

$$= \frac{\frac{2}{3} - \frac{1}{3}}{1 - \frac{1}{3}} = \frac{1}{2}.$$

$\boxed{\text{Problem No. 6}}$ (i) $F(20) = F(20+) \Rightarrow 16k^2 - 16k + 3 = 0$

$\Rightarrow k = \frac{1}{4}, \frac{3}{4}$ ..... (I)

$F(5-) \le F(5) \Rightarrow k \le \frac{1}{2}$. .... (II)

(I) + (II) $\Rightarrow k = \frac{1}{4}$

$$F(x) = \begin{cases} 0, & x < 2 \\ \frac{2}{3}, & 2 \le x < 5 \\ \frac{11}{12}, & 5 \le x < 9 \\ \frac{91}{96}, & 9 \le x < 14 \\ 1, & x \ge 14 \end{cases}$$

(ii) D = set of discontinuity points of $F = \{2, 5, 9, 14\}$

sum of jumps $= \sum_{x \in D} [F(x) - F(x-)]$

$= (\frac{2}{3} - 0) + (\frac{11}{12} - \frac{2}{3}) + (\frac{91}{96} - \frac{11}{12}) + (1 - \frac{91}{96})$

$= 1$

$\Rightarrow X$ is of discrete type with support $S_X = D_X = \{2, 5, 9, 14\}$.

(iii) The p.m.f. of X is

$$f(x) = P(X=x) = F(x) - F(x-) = \begin{cases} \frac{2}{3}, & x=2 \\ \frac{1}{4}, & x=5 \\ \frac{1}{32}, & x=9 \\ \frac{5}{96}, & x=14 \\ 0, & \text{otherwise} \end{cases}$$

$\boxed{\text{Problem No.7}}$ (i) $S_X =$ support of $X = \{1, 2, 3, \ldots\}$

$$\sum_{x \in S_X} f_X(x) = 1$$

$$\Rightarrow \sum_{i=1}^{\infty} \frac{c}{(2i-1)(2i+1)} = 1$$

$$\Rightarrow \lim_{n\to\infty} \sum_{i=1}^{n} \frac{c}{(2i-1)(2i+1)} = 1$$

$$\Rightarrow \frac{c}{2} \lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{1}{2i-1} - \frac{1}{2i+1}\right] = 1$$

$$\Rightarrow \frac{c}{2} \lim_{n\to\infty} \left[\sum_{i=1}^{n} \frac{1}{2i-1} - \sum_{i=1}^{n} \frac{1}{2i+1}\right] = 1$$

$$\Rightarrow \frac{c}{2} \lim_{n\to\infty} \left[1 - \frac{1}{2n+1}\right] = 1 \quad \Rightarrow c = 2.$$

(ii) $P(X < m+1) = P(X \le m)$

$$= \sum_{i=1}^{m} \frac{2}{(2i-1)(2i+1)}$$

$$= \sum_{i=1}^{m} \left[\frac{1}{2i-1} - \frac{1}{2i+1}\right] = 1 - \frac{1}{2m+1} = \frac{2m}{2m+1} \quad \ldots\ldots (A)$$

$$P(X \ge m) = 1 - P(X < m)$$

$$= 1 - \frac{2(m-1)}{2(m-1)+1} \quad (\text{from } (A))$$

$$= \frac{1}{2m-1} \quad \ldots\ldots (A_1)$$

$$P(m \le X < n) = P(X < n) - P(X < m)$$

$$= \frac{2(n-1)}{2n-1} - \frac{2(m-1)}{2m-1} \quad (\text{using } (A))$$

$$= \frac{2(n-m)}{(2n-1)(2m-1)} \quad \text{......} \quad (B)$$

$$P(m < X \le n) = P(m+1 \le X < n+1)$$

$$= \frac{2(n-m)}{(2n+1)(2m+1)} \quad (\text{using } (B)).$$

(iii): $$P(X > 1 \mid 1 \le X < 4) = \frac{P(X > 1,\ 1 \le X < 4)}{P(1 \le X < 4)}$$

$$= \frac{P(1 < X < 4)}{P(1 \le X < 4)} = \frac{P(2 \le X < 4)}{P(1 \le X < 4)}$$

$$= \frac{\frac{2(4-2)}{7 \times 3}}{\frac{2(4-1)}{7 \times 1}} = \frac{2}{9} \quad \left(\begin{matrix}\text{using} \\ (B)\end{matrix}\right)$$

$$P(1 < X < 6 \mid X \ge 3) = \frac{P(1 < X < 6,\ X \ge 3)}{P(X \ge 3)}$$

$$= \frac{P(3 \le X < 6)}{P(X \ge 3)} = \frac{\frac{2 \times 3}{11 \times 5}}{\frac{1}{5}} \quad \left(\begin{matrix}\text{using} \\ (A) \text{ and} \\ (B)\end{matrix}\right)$$

$$= \frac{6}{11}$$

(iv) Clearly, for $x < 1$, $F(x) = 0$. For $i \le x < i+1$, $i = 1, 2, 3 \ldots$

$$F(x) = P(X < i+1) = \frac{2i}{2i+1} \quad (\text{using } (A))$$

$$\Rightarrow \quad F(x) = \begin{cases} 0, & x < 1 \\ \frac{2i}{2i+1}, & i \le x < i+1, \quad i = 1, 2, 3 \ldots \end{cases}$$

$\boxed{6/\mathrm{II}}$

$\boxed{\text{Problem No. 8}}$ (i) Clearly F_X is differentiable everywhere except at points 1 and 2.

$$F'(x) = \begin{cases} 0 & x<0 \\ x, & 0<x<1 \\ \frac{1}{2}, & 1<x<2 \\ 0, & x>2 \end{cases}$$

$$\int_{-\infty}^{\infty} F_X'(x)\,dx = \int_0^1 x\,dx + \int_1^2 \frac{1}{2}\,dx = 1$$

$\Rightarrow$ X is of ~~[illegible]~~ continuous type.

(ii) $P(X=x) = F_X(x) - F_X(x-) = 0, \ \forall x \in \mathbb{R}$ (as X is of continuous type)

$\Rightarrow P(X=1) = P(X=2) = 0$

$P(1<X<2) = P(1\le X<2) = P(1<X\le 2) = P(1\le X\le 2)$

$= F(2) - F(1) = 1-\frac{1}{2} = \frac{1}{2}$

$P(X\geq 1) = 1 - F(1) = 1-\frac{1}{2} = \frac{1}{2}.$

(iii) From (i) the p.d.f. of X is

$$f(x) = \begin{cases} x, & 0<x<1 \\ \frac{1}{2}, & 1<x<2 \\ 0, & \text{otherwise} \end{cases}$$

(iv) $F(\xi_{\frac{1}{4}}) = \frac{1}{4}$

$\Rightarrow \xi_{\frac{1}{4}} = \frac{1}{\sqrt{2}}$

$F(\xi_{1/2}) = \frac{1}{2} \Rightarrow \xi_{\frac{1}{2}} = 1$

$F(\xi_{3/4}) = \frac{3}{4} \Rightarrow \xi_{\frac{3}{4}} = \frac{3}{2}$

$\boxed{\text{Problem No. 9}}$ (i) Since f is p.d.f.

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow \int_{-\frac{1}{2}}^{\frac{1}{2}} (k-|x|)\,dx = 1 \Rightarrow k = \frac{5}{4}.$$

(ii) $P(X<0) = P(X\le 0) = \int_{-\infty}^{0} f(x)\,dx = \int_{-\frac{1}{2}}^{0}\left(\frac{5}{4}+x\right)dx = \frac{1}{2}$

$P(0<X\le\frac{1}{4}) = P(0\le X<\frac{1}{4}) = \int_0^{1/4} f(x)\,dx = \int_0^{1/4}\left(\frac{5}{4}-x\right)dx = \frac{9}{32}$

$$P\left(-\frac{1}{8} \le x \le \frac{1}{4}\right) = \int_{-\frac{1}{8}}^{\frac{1}{4}} f(x)\,dx = \int_{-\frac{1}{8}}^{0}\left(\frac{5}{4}+x\right)dx + \int_{0}^{1/4}\left(\frac{5}{4}-x\right)dx = \frac{55}{128}$$

(iii) $$P\left(x > \frac{1}{4} \,\middle|\, |x| > \frac{2}{5}\right) = \frac{P\left(x > \frac{1}{4},\ |x| > \frac{2}{5}\right)}{P\left(|x| > \frac{2}{5}\right)} = \frac{P\left(x > \frac{2}{5}\right)}{P\left(|x| > \frac{2}{5}\right)}$$

$$P\left(x > \frac{2}{5}\right) = \int_{2/5}^{1/2}\left(\frac{5}{4}-x\right)dx = \frac{2}{25}$$

$$P\left(|x| > \frac{2}{5}\right) = P\left(x < -\frac{2}{5} \text{ or } x > \frac{2}{5}\right) = P\left(x < -\frac{2}{5}\right) + P\left(x > \frac{2}{5}\right) = \int_{-\frac{1}{2}}^{-2/5}\left(\frac{5}{4}+x\right)dx + \frac{2}{25} = \frac{4}{25}$$

$$\Rightarrow P\left(x > \frac{1}{4} \,\middle|\, |x| > \frac{2}{5}\right) = \frac{\frac{2}{25}}{\frac{4}{25}} = \frac{1}{2}$$

$$P\left(\frac{1}{8} < x < \frac{2}{5} \,\middle|\, \frac{1}{10} < x < \frac{1}{5}\right) = \frac{P\left(\frac{1}{8} < x < \frac{2}{5},\ \frac{1}{10} < x < \frac{1}{5}\right)}{P\left(\frac{1}{10} < x < \frac{1}{5}\right)} = \frac{P\left(\frac{1}{8} < x < \frac{1}{5}\right)}{P\left(\frac{1}{10} < x < \frac{1}{5}\right)}$$

$$= \frac{\int_{\frac{1}{8}}^{\frac{1}{5}}\left(\frac{5}{4}-x\right)dx}{\int_{\frac{1}{10}}^{\frac{1}{5}}\left(\frac{5}{4}-x\right)dx} = \frac{261}{352}$$

(iv) $$F(x) = \int_{-\infty}^{x} f(t)\,dt = \begin{cases} 0, & x < -\frac{1}{2} \\ \int_{-\frac{1}{2}}^{x}\left(\frac{5}{4}+t\right)dt, & -\frac{1}{2} \le x < 0 \\ \int_{-\frac{1}{2}}^{0}\left(\frac{5}{4}+t\right)dt + \int_{0}^{x}\left(\frac{5}{4}-t\right)dt, & 0 \le x < \frac{1}{2} \\ 1, & x \ge \frac{1}{2} \end{cases}$$

$$= \begin{cases} 0, & x < -\frac{1}{2} \\ \frac{x^2}{2}+\frac{5}{4}x+\frac{1}{2}, & -\frac{1}{2} \le x < 0 \\ -\frac{x^2}{2}+\frac{5}{4}x+\frac{1}{2}, & 0 \le x < \frac{1}{2} \\ 1, & x \ge \frac{1}{2} \end{cases}$$

(v)
$F(\xi_{1/4}) = \frac{1}{4}$
$\Rightarrow \frac{\xi_{1/4}^2}{2} + \frac{5}{4}\xi_{1/4} + \frac{1}{2} = \frac{1}{4}$
$2\xi_{1/4}^2 + 5\xi_{1/4} + 1 = 0$
$\xi_{1/4} = \frac{-5+\sqrt{17}}{4}$
$F(\xi_{1/2}) = \frac{1}{2}$
$\Rightarrow \xi_{1/2} = 0$

$\boxed{8/\text{II}}$

$F(\xi_{3/4}) = \frac{3}{4}$
$\Rightarrow -\frac{\xi_{3/4}^2}{2} + \frac{5}{4}\xi_{3/4} + \frac{1}{2} = \frac{3}{4}$
$\Rightarrow 2\xi_{3/4}^2 - 5\xi_{3/4} + 1 = 0$
$\xi_{3/4} = \frac{5-\sqrt{17}}{4}$