EE 200: Solution Set 7

1. Determine the discrete-time Fourier series representation of a periodic pulse sequence $\tilde{x}[n]$ whose one period is given by:

$$\begin{cases} 1, & 0 \leq n \leq 3 \\ 0, & 4 \leq n \leq 7 \end{cases}$$

Solution 1: The fundamental period of the periodic sequence, $N_0 = 8$. Thus, the fundamental angular frequency,

$$\omega_0 = \frac{2\pi}{8} = \frac{\pi}{4}.$$

The Fourier coefficients in the exponential form of the DFS are

$$\begin{aligned}
c_k &= \frac{1}{8}\sum_{k=0}^{3} e^{-jk(\pi/4)n} \\
&= \frac{1}{8}\left(\frac{1-e^{-jk(\pi)}}{1-e^{-jk(\pi/4)}}\right) \\
&= \frac{1}{8}\frac{e^{-jk(\pi/2)}\left(e^{jk(\pi/2)}-e^{-jk(\pi/2)}\right)}{e^{-jk(\pi/8)}\left(e^{jk(\pi/8)}-e^{-jk(\pi/8)}\right)} \\
&= \frac{e^{-j3k(\pi/8)}}{8}\frac{\sin(k\pi/2)}{\sin(k\pi/8)}
\end{aligned}$$

2. Determine the DTFT of each of the following sequences:

(a) $x_1[n] = \alpha^n \mu[n-1], \; |\alpha| < 1$
(b) $x_2[n] = \alpha^n \mu[n+1], \; |\alpha| < 1$
(c) $x_3[n] = n\alpha^n \mu[n], \; |\alpha| < 1$
(d) $x_4[n] = n\alpha^n \mu[n+1], \; |\alpha| < 1$

Solution 2: Let $x[n] = \alpha^n \mu[n]$ with $|\alpha| < 1$. Then,

$$X(e^{j\omega}) = \frac{1}{1 - \alpha e^{-j\omega}}$$

(a) $x_1[n] = \alpha^n \mu[n-1],\ |\alpha| < 1$

$$\begin{aligned} X_1(e^{j\omega}) &= \sum_{n=1}^{\infty} \alpha^n e^{-j\omega n} = \sum_{n=1}^{\infty} (\alpha e^{-j\omega})^n = \sum_{n=0}^{\infty} (\alpha e^{-j\omega})^n - 1 \\ &= \frac{1}{1-\alpha e^{-j\omega}} - 1 = \frac{\alpha e^{-j\omega}}{1-\alpha e^{-j\omega}} \end{aligned}$$

(b) $x_2[n] = \alpha^n \mu[n+1],\ |\alpha| < 1$

$$\begin{aligned} X_2(e^{j\omega}) &= \sum_{n=-1}^{\infty} \alpha e^{-j\omega n} = \alpha^{-1} e^{j\omega} + \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n} \\ &= \alpha^{-1} e^{j\omega} + \frac{1}{1-\alpha e^{-j\omega}} = \frac{1}{\alpha}\left(\frac{e^{j\omega}}{1-\alpha e^{-j\omega}}\right) \end{aligned}$$

(c) $x_3[n] = n\alpha^n \mu[n],\ |\alpha| < 1$. Note that $x_3[n] = nx[n]$. Therefore, using the differentiation-in-frequency property, we get

$$X_3(e^{j\omega}) = j\frac{dX(e^{j\omega})}{d\omega} = j\frac{d}{d\omega}\left(\frac{1}{1-\alpha e^{-j\omega}}\right) = \frac{\alpha e^{-j\omega}}{(1-\alpha e^{-j\omega})^2}$$

(d) $x_4[n] = n\alpha^n \mu[n+1],\ |\alpha| < 1$

$$\begin{aligned} X_4(e^{j\omega}) &= \sum_{n=-1}^{\infty} n\alpha^n e^{-j\omega n} \\ &= \sum_{n=0}^{\infty} n\alpha^n e^{-j\omega n} - \alpha^{-1} e^{j\omega} \end{aligned}$$

Thus, using the result of part (c), we get

$$X_4(e^{j\omega}) = \frac{\alpha e^{-j\omega}}{(1-\alpha e^{-j\omega})^2} - \alpha^{-1} e^{j\omega}$$

3. Determine the DTFT $X(e^{j\omega})$ of the sequence $x[n] = \mu[n+1] - \mu[n-1]$.

Solution 3: We have $x[n] = \mu[n+1] - \mu[n-1]$. Let $M(e^{j\omega})$ denote the DTFT of $\mu[n]$. Using the time-shifting property of the DTFT, we have got

$$X(e^{j\omega}) = (e^{j\omega} - e^{-j\omega})M(e^{j\omega}).$$

Now,

$$M(e^{j\omega}) = \frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k).$$

Therefore,

$$X(e^{j\omega}) = (e^{j\omega} - e^{-j\omega})\left(\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)\right)$$

4. Determine the inverse DTFT of the following functions:

   (a) $X_a(e^{j\omega}) = 2\cos(2\omega)$

   (b) $X_b(e^{j\omega}) = 3\cos(3\omega) + 4\sin(2\omega)$

Solution 4:
(a)

$$X_a(e^{j\omega}) = 2\cos(2\omega) = 2\cos^2(\omega) - 1 = 2\left(\frac{e^{jw} + e^{-jw}}{2}\right)^2 - 1$$
$$= \frac{1}{2}(e^{j2w} + e^{-j2w} + 2) - 1 = \frac{1}{2}e^{j2w} + \frac{1}{2}e^{-j2w}$$

Hence, its inverse DTFT

$$x_a[n] = \left\{\frac{1}{2}, 0, 0, 0, \frac{1}{2}\right\}, \; -2 \leq n \leq 2$$

(b)

$$X_b(e^{j\omega}) = 3\cos(3\omega) + 4\sin(2\omega)$$

$$\begin{aligned}
3\cos(3\omega) &= 3(4\cos^3(\omega) - 3\cos(\omega)) = 12\left(\frac{e^{j\omega}+e^{-j\omega}}{2}\right)^3 - 9\left(\frac{e^{j\omega}+e^{-j\omega}}{2}\right) \\
&= \frac{12}{8}\left(e^{j3\omega} + 3e^{j2\omega}e^{-j\omega} + 3e^{j\omega}e^{-j2\omega} + e^{-j3\omega}\right) - \frac{9}{2}(e^{j\omega}+e^{-j\omega}) \\
&= \frac{3}{2}e^{j3\omega} + \frac{9}{2}e^{j\omega} + \frac{9}{2}e^{-j\omega} + \frac{3}{2}e^{-j3\omega} - \frac{9}{2}e^{j\omega} - \frac{9}{2}e^{-j\omega} \\
&= \frac{3}{2}e^{j3\omega} + \frac{3}{2}e^{-j3\omega}
\end{aligned}$$

and

$$\begin{aligned}
4\sin(2\omega) &= 8\sin(\omega)\cos(\omega) \\
&= 8\left(\frac{e^{j\omega}-e^{-j\omega}}{2j}\right)\left(\frac{e^{j\omega}+e^{-j\omega}}{2}\right) \\
&= \frac{2}{j}(e^{j2\omega}-e^{-j2\omega}) = -j2(e^{j2\omega}-e^{-j2\omega})
\end{aligned}$$

Thus,

$$X_b(e^{j\omega}) = \frac{3}{2}e^{j3\omega} + \frac{3}{2}e^{-j3\omega} - j2(e^{j2\omega}-e^{-j2\omega})$$

Hence, its inverse DTFT is

$$x_b[n] = \left\{3/2, j2, 0, 0, 0, -j2, 3/2\right\}; \ -3 \le n \le 3$$

5. Let $G(e^{j\omega})$ denote the DTFT of the sequence $g[n]$. Determine the inverse DTFT $h[n]$ of the DTFT $H(e^{j\omega}) = G(e^{j4\omega})$ in terms of $g[n]$.
Solution 5: $H(e^{j\omega}) = G(e^{j4\omega})$
Now,

$$G(e^{j\omega}) = \sum_{n=-\infty}^{\infty} g[n]e^{-j\omega n}$$

Hence,

$$\begin{aligned}
H(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} = G((e^{j\omega})^4) \\
&= \sum_{n=-\infty}^{\infty} g[n](e^{-j\omega n})^4 = \sum_{m=-\infty}^{\infty} g[m/4]e^{-j\omega m}
\end{aligned}$$

Therefore,

$$h[n] = \begin{cases} g[n], & n = 0, \pm 4, \pm 8, \pm 12, \cdots \\ 0, \text{otherwise} \end{cases}$$

$$g[n] \rightarrow \boxed{\uparrow 4} \rightarrow h[n]$$

6. Determine the z-transform and the corresponding ROC of the sequences of Prob.2 and derive their DTFTs from the z-transforms.
Solution 6: Let $x[n] = \alpha^n \mu[n]$; its z-transform is

$$X(z) = \frac{1}{1 - \alpha z^{-1}}, \ |z| > |\alpha|.$$

(a)

$$\begin{aligned} x_1[n] &= \alpha^n \mu[n-1] \\ X_1(z) &= \sum_{n=1}^{\infty} \alpha^n z^{-n} = \sum_{n=1}^{\infty} (\alpha z^{-1})^n \\ &= \sum_{n=0}^{\infty} (\alpha z^{-1})^n - 1 = \frac{1}{1 - \alpha z^{-1}} - 1 \\ &= \frac{\alpha z^{-1}}{1 - \alpha z^{-1}}; \ |z| > |\alpha| \end{aligned}$$

If $|\alpha| < 1$, the unit circle is in the ROC. Hence, the DTFT is obtained by setting $z = e^{j\omega}$:

$$X_1(e^{j\omega}) = X_1(z)\Big|_{z=e^{j\omega}} = \frac{\alpha e^{-j\omega}}{1 + \alpha e^{-j\omega}}, \ |\alpha| < 1$$

(b)

$$\begin{aligned} x_2[n] &= \alpha^n \mu[n+1], \ |\alpha| < 1 \\ X_2(z) &= \sum_{n=-1}^{\infty} \alpha^n z^{-n} = \alpha^{-1} z + \sum_{n=0}^{\infty} \alpha^n z^{-n} \\ &= \alpha^{-1} z + \frac{1}{1 - \alpha z^{-1}} = \frac{1}{\alpha} \left(\frac{z}{1 - \alpha z^{-1}} \right), \ |z| > |\alpha| \end{aligned}$$

(c)

$$x_3[n] = n\alpha^n \mu[n], \ |\alpha| < 1$$

Note that $x_3[n] = nx[n]$. Thus, using the differentiation-in-the z-domain property, we get

$$\begin{aligned} X_3(z) &= -z\frac{dX(z)}{dz} = -z\frac{d}{dz}\left(\frac{1}{1-\alpha z^{-1}}\right) \\ &= \frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}, \ |z| > |\alpha| \end{aligned}$$

(d)

$$x_4[n] = n\alpha^n \mu[n+1], \ |\alpha| < 1$$

$$X_4(z) = \sum_{n=-1}^{\infty} n\alpha^n z^{-n} = \sum_{n=0}^{\infty} n\alpha^n z^{-n} - \alpha^{-1}z$$

From the results of Part (c),

$$\sum_{n=0}^{\infty} n\alpha^n z^{-n} = \frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$$

Hence,

$$X_4(z) = \frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2} - \alpha^{-1}z, \ |z| > |\alpha|$$

7. Determine the z-transform and the corresponding ROC of the following sequences:

   (a) $y_a[n] = \alpha^n \mu[n-3]$

   (b) $y_b[n] = \alpha\mu[-n]$

   Solution 7:
   (a) $y_a[n] = \alpha^n \mu[n-3]$. Note that $y_a[n]$ is a right sided sequence. Hence, the ROC of its z-transform is exterior to a circle.

$$Y_a(z) = \sum_{n=-\infty}^{\infty} \alpha^n \mu[n-3] z^{-n} = \sum_{n=3}^{\infty} \alpha^n z^{-n}$$

Simplifying we get

$$Y_a(z) = \frac{1}{1-\alpha z^{-1}} - 1 - \alpha z^{-1} - \alpha^2 z^{-2} \tag{1}$$

$$= \frac{\alpha^3 z^{-3}}{1-\alpha z^{-1}};\ \text{ROC} : |z| > |\alpha| \tag{2}$$

(b) $y_b[n] = \alpha^n \mu[-n]$. Note that $y_b[n]$ is a left sided sequence. Hence, the ROC of its z-transform is interior to a circle.

$$Y_b(z) = \sum_{n=-\infty}^{\infty} \alpha^n \mu[-n] z^{-n} = \sum_{n=-\infty}^{0} \alpha^n z^{-n}$$

$$= \sum_{m=0}^{\infty} \alpha^{-m} z^m = \sum_{m=0}^{\infty} (z/\alpha)^m$$

Simplifying we get

$$Y_b(z) = \frac{1}{1-(z/\alpha)};\ \text{ROC} : |z| < |\alpha| \tag{3}$$

8. Consider the following z-transforms of causal sequences:

   (a) $X_1(z) = \frac{2+0.4z^{-1}}{1+0.5z^{-1}}$

   (b) $X_2(z) = \frac{3}{1+0.25z^{-2}}$

   (c) $X_3(z) = \frac{1}{1-z^{-4}}$

   Determine their inverse z-transform using the partial-fraction approach.

   Solution 8:

   (a) $X_1(z) = \frac{2+0.4z^{-1}}{1+0.5z^{-1}} = k + \frac{\rho}{1+0.5z^{-1}}$ so, $k = \frac{0.4}{0.5} = 0.8$, and

   $$\rho = X_1(z)(1+0.5z^{-1})\Big|_{z=-0.5} = (2+0.4z^{-1})\Big|_{z=-0.5} = 2+\frac{0.4}{-0.5} = 1.2$$

   Thus, $X_1(z) = 0.8 + \frac{1.2}{1+0.5z^{-1}}$. Its inverse z- transform is

   $$x_1[n] = 0.8\delta[n] + 1.2(0.5)^n \mu[n]$$

   (b)

   $$X_2(z) = \frac{3}{1-0.25z^{-2}} = \frac{3}{(1-0.5z^{-1})(1+0.5z^{-1})}$$

   $$= k + \frac{\rho_1}{1-0.5z^{-1}} + \frac{\rho_2}{1+0.5z^{-1}}$$

So, $k = 0$, and

$$\rho_1 = X_2(z)(1 - 0.5z^{-1})\Big|_{z=0.5} = 1.5$$

$$\rho_2 = X_2(z)(1 + 0.5z^{-1})\Big|_{z=-0.5} = 1.5$$

Thus, $X_2(z) = \frac{1.5}{1-0.5z^{-1}} + \frac{1.5}{1-0.5z^{-1}}$ Its inverse z- transform is

$$x_2[n] = 1.5(0.5)^n \mu[n] + 1.5(-0.5)^n \mu[n]$$

(c)

$$\begin{aligned} X_3(z) &= \frac{1}{1 - z^{-4}} = \frac{1}{(1 - z^{-2})(1 + z^{-2})} \\ &= \frac{\rho_1}{1 + z^{-1}} + \frac{\rho_2}{1 - z^{-1}} + \frac{\rho_3}{1 + jz^{-1}} + \frac{\rho_4}{1 - jz^{-1}} \end{aligned}$$

$$\rho_1 = X_3(z)(1 + z^{-1})\Big|_{z=-1} = \frac{1}{4} = 0.25$$

$$\rho_2 = X_3(z)(1 - z^{-1})\Big|_{z=1} = \frac{1}{4} = 0.25$$

$$\rho_3 = X_3(z)(1 + jz^{-1})\Big|_{z=-j} = \frac{1}{4} = 0.25$$

$$\rho_4 = X_3(z)(1 - jz^{-1})\Big|_{z=j} = \frac{1}{4} = 0.25$$

Thus,

$$X_3(z) = \frac{0.25}{1 + z^{-1}} + \frac{0.25}{1 - z^{-1}} + \frac{0.25}{1 + jz^{-1}} + \frac{0.25}{1 - jz^{-1}}$$

Its inverse z-transform is

$$x_3[n] = 0.25(1 + (-1)^n + (-j)^n + (j)^n)\mu[n]$$