2.a.i) $X_1, \ldots, X_n \overset{iid}{\sim} N(0,1)$

$\Rightarrow \bar{X}_n$ is independent of

[Student's theorem] $S_n^2 \ \forall n \geq 1.$

$\Rightarrow \bar{X}_n$ is independent of

$\frac{1}{S_n} \ \forall n \geq 1$

Fix $n=5$.

$$E\left[\frac{\bar{X}_5}{S_5}\right] = E[\bar{X}_5] \cdot E\left[\frac{1}{S_5}\right]$$

$$= 0$$

$\boxed{1 \text{ mark}}$

$$Var\left[\frac{\bar{X}_5}{S_5}\right] = E\left[\frac{\bar{X}_5^2}{S_5^2}\right] - 0$$

$$= E[\bar{X}_5^2] \cdot E\left[\frac{1}{S_5^2}\right]$$

$$E[\bar{X}_5^2] = \frac{1}{5}$$

$\boxed{1 \text{ mark}}$

$$4 S_5^2 \sim \chi_4^2$$

$$= \frac{4}{5} \cdot E\left[\frac{1}{4S_5^2}\right]$$

$$= \frac{4}{5} \cdot \frac{2^{2/2} \cdot \Gamma(2/2)}{2^{4/2} \cdot \Gamma(4/2)}$$

$$= \frac{4}{5} \cdot \frac{2 \cdot 1}{4 \cdot 1}$$

So, $\quad \mathrm{Var}\left[\frac{\bar{X}_5}{S_5}\right] = \frac{2}{5}$.

$\boxed{\text{1 mark}}$

2.a.ii) $X_1, \dots, X_n \overset{iid}{\sim} \mathrm{Unif}(0,1)$

<u>Claim</u>: $X_{(1)} \xrightarrow{P} 0$ as $n \to \infty$

$X_{(n)} \xrightarrow{P} 1$ as $n \to \infty$.

<u>Proof</u>: $$E[X_{(1)}^2] = n\int_0^1 y^2(1-y)^{n-1}dy$$

$$= n \cdot B(3,n)$$

$$= n \cdot \frac{2!\,(n-1)!}{(n+2)!}$$

$$= \frac{2}{(n+2)(n+1)} \to 0$$

as $n \to \infty$.

$\Rightarrow X_{(1)} \xrightarrow{L_2} 0$ as $n \to \infty$.

$\boxed{\text{1 mark}}$

Similarly,

$$E[1 - X_{(n)}]^2 = n\int_0^1 y^{n-1}(1-y)^2 dy$$

$$= n \cdot B(n,3)$$

$$\longrightarrow 0 \text{ as } n \to \infty.$$

$$\Rightarrow \quad X_{(n)} \xrightarrow{L_2} 1 \quad \text{as } n \to \infty.$$

1 mark

Combining these two statements, we have

$$X_{(1)} X_{(n)} \xrightarrow{P} 0 \quad \text{as } n \to \infty.$$

1 mark

2.b. $\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \sim N_2\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}\right)$

i) $$\max\{Z_1, Z_2\} = \frac{Z_1 + Z_2 + |Z_1 - Z_2|}{2}$$

$$E[\max\{Z_1, Z_2\}] = \frac{0 + E|Z_1 - Z_2|}{2}$$

1 mark

$$Z_1 - Z_2 \sim N(0,1).$$

So, $E|Z_1 - Z_2| = \sqrt{\frac{2}{\pi}}$

1 mark

$$\Rightarrow E[\max\{Z_1, Z_2\}] = \frac{1}{\sqrt{2\pi}}$$

ii) Recall $Z_2 | Z_1 \sim N\left(\frac{Z_1}{2}, \frac{3}{4}\right)$.

1 mark

$$E[e^{tZ_1Z_2}]$$

$$= E_{Z_1}\left\{E[e^{tZ_1Z_2} | Z_1]\right\}$$

$$= E_{Z_1}\left\{e^{\frac{tZ_1^2}{2} + \frac{3t^2Z_1^2}{8}}\right\}$$

$$= E_{Z_1}\left(e^{sZ_1^2}\right), \quad s = \frac{t}{2} + \frac{3t^2}{8}$$

1 mark

$$= \frac{1}{\sqrt{1-2s}} \quad \text{for} \quad s < \frac{1}{2}$$

$$\left[\because Z_1^2 \sim \chi_1^2\right]$$

$$= \frac{1}{\sqrt{1-\left(t+\frac{3t^2}{4}\right)}} \quad \text{for} \quad -2 < t < \frac{2}{3}.$$

1 mark