EE 250: Control Systems Analysis
Module III: s-plane analysis
Lecture 15: Time Response

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3 Physical systems vs Simulators

All physical devices are causal by nature. An aircraft, an automobile, and a DC servo motor are examples of physical devices. By definition, an input u at time $t = \tau$, can have influence on the response of the system y only at $t \geq \tau$. All these systems will have transfer functions $G(s) = \frac{N(s)}{D(s)}$, where the denominator order is greater than or equal to numerator order.

Any system whether causal or non-causal can be simulated using analogue circuits or digital circuits - respectively called as analogue computer or digital computer. In the class, we have shown how to simulate following transfer functions:

$$G(s) = \frac{s+2}{s^3 + 2s^2 + 3s + 2} \quad (4)$$

$$G(s) = \frac{s^2 + 2s + 3}{s + 1} \quad (5)$$

First order response

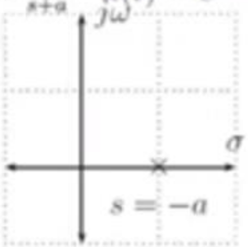
$H(s) = \frac{1}{s+a}$ $h(t) = e^{-at}$

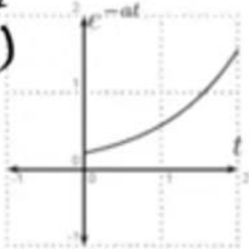
unstable

$H(s) = \frac{1}{s+a}$

$h(t) = e^{-at}$

$(1-\kappa e^{-at})$



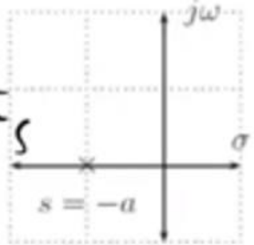


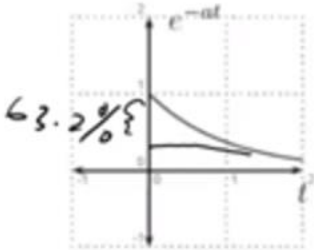
$\underline{a < 0}$

$-a$ is positive

$H(s) = \frac{1/a}{1+\tau s} = \frac{\tau}{1+\tau s}$

$\tau = 1/a$





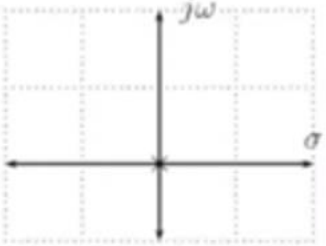
$a > 0$

$\tau = 1/a$ is called time cons

$-a$ is negative

$H(s) = 1/s$

$h(t) = 1$



$a = 0$

$a = 0$

$\tau = \frac{1}{a}$ is the **time constant**, the time when the response is 63.2% of the initial value.

Response of a Standard 2nd order System

The transfer function of a second order system is often given as follows:

$K = \omega_n^2$

$a = 2\zeta\omega_n$

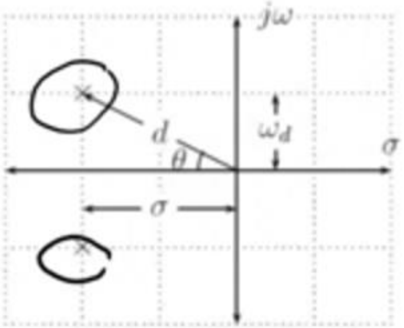
$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{K}{s^2 + as + K}$$

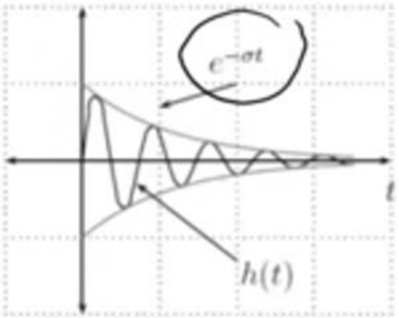
It has two poles, $s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$. Here ω_n is termed as natural frequency while ζ is known as damping ratio or damping coefficient. Taking inverse Laplace transform, we get the following *impulse* response:

$$h(t) = \mathscr{L}^{-1}H(s) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\sigma t} sin(\omega_d t)u(t)$$

$$\cos\theta = \frac{\sigma}{d} = \frac{\zeta\omega_n}{\omega_n}$$

where $\sigma = \zeta\omega_n$ and $\omega_d = \omega_n\sqrt{1-\zeta^2}$. ω_d is also called as frequency of transient oscillation.





$$\begin{aligned} d &= \sqrt{\zeta^2\omega_n^2 + \omega_n^2(1-\zeta^2)} \\ &= \sqrt{\omega_n^2} = \omega_n \\ \cos\theta &= \frac{\zeta\omega_n}{\omega_n} = \zeta \\ \theta &= \cos^{-1}\zeta \end{aligned}$$

Step response: The output of the system with above transfer function is given by

$$Y(s) = H(s)\frac{1}{s} = \frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)}$$

$$y(t) = 1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta\omega_n}{\omega_d}\sin\omega_d t\right)$$

$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t+\theta)$$

$$where \quad \theta = \cos^{-1}\zeta$$

rise time: t_r

peak time $= t_p$

Peak overshoot $= M_p$

t_s = settling time

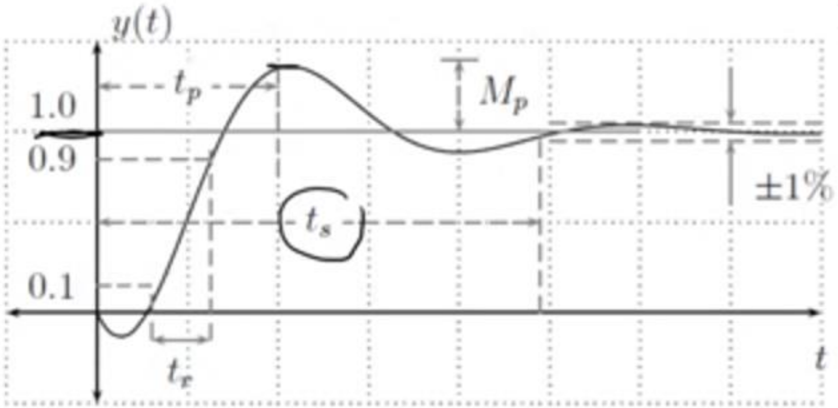
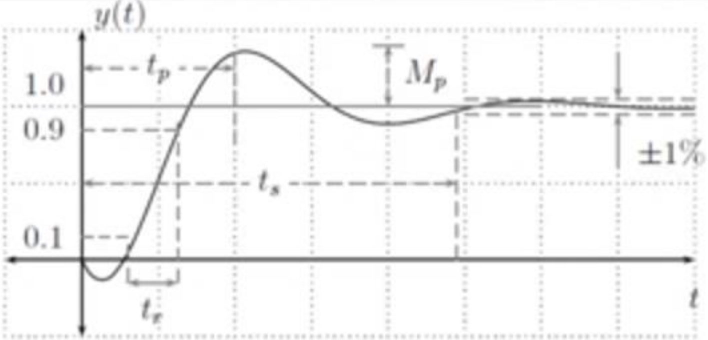


Figure 1: Time response of a second order system



Rise time (t_r): It is the time required by the output to go from 10% to 90% of the final value. The time it takes the response to reach the value 1 can be considered to be upper bound of t_r. This value can be obtained by making the sin term of the response vanish (zero).

$$1 - e^{-\zeta\omega_n t_r}(\cos \omega_d t_r + \frac{\zeta\omega_n}{\omega_d} \sin \omega_d t_r) = 1$$

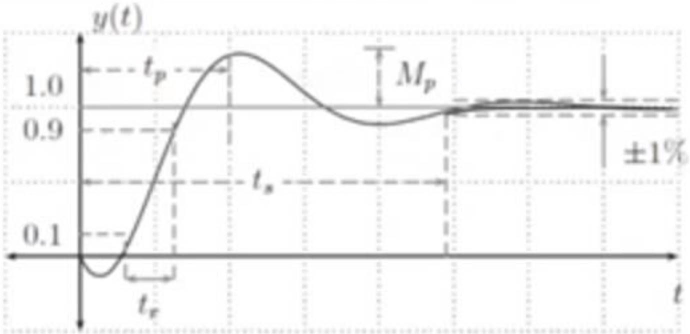
$$\sin(\omega_n\sqrt{1-\zeta^2}t_r + \theta) = \sin\pi = 0$$

$$\omega_n\sqrt{1-\zeta^2}t_r + \theta = \pi$$

$\omega_d t + \theta = \pi$

$$t_r = \frac{\pi - \cos^{-1}\zeta}{\omega_n\sqrt{1-\zeta^2}} \quad (maximum \quad value)$$

This is an upperbound. The actual value is obtained through empirical observation. For a system with $\zeta = 0.5$, t_r can be approximated as $t_r = \frac{1.8}{\omega_n}$.



$$y(t) = 1 - e^{-\sigma t}\sqrt{1 + \frac{\sigma^2}{\omega_d^2}}cos(\omega_d t - \beta)$$

$\dot{y} = 0$

$t = t_p$

Settling time (t_s): It is the time taken by the output to enter ±1% band around the final value.

$$\begin{aligned} t_s &= \frac{4.6}{\zeta\omega_n} \quad \text{for } \pm 1\% \text{ band} \\ &= \frac{4}{\zeta\omega_n} \quad \text{for } \pm 2\% \text{ band} \end{aligned}$$

Peak Time (t_p): The time taken by the output to reach the maximum overshoot point.

$$t_p = \frac{\pi}{\omega_d}$$

$$\dot{y} = \frac{\zeta\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}sin(\omega_n\sqrt{1-\zeta^2}\,t + \theta) - \frac{e^{-\zeta\omega_n t}}{1-\zeta^2}cos(\omega_n\sqrt{1-\zeta^2}t + \theta)$$

$$= \omega_n\frac{e^{-\zeta\omega_n t}}{1-\zeta^2}(sin(\omega_d t + \theta)cos(\theta) - cos(\omega_d t + \theta)sin(\theta)) = 0$$

sin π

$$\Rightarrow \omega_n\frac{e^{-\zeta\omega_n t}}{1-\zeta^2}(sin(\omega_d t)) = 0$$

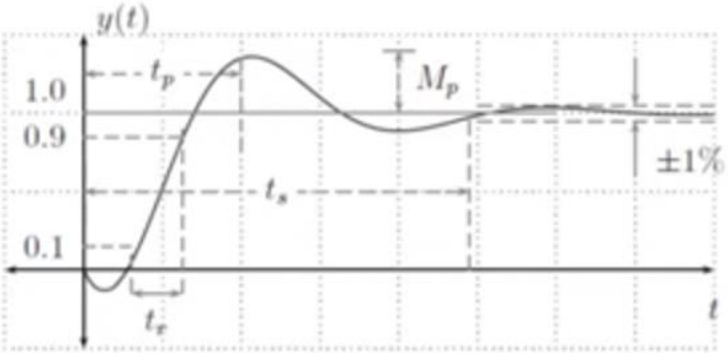
which gives

$$\omega_d t_p = \pi$$

$$\Rightarrow t_p = \frac{\pi}{\omega_d}$$

Peak Overshoot (M_p): It is the ratio of maximum amount of the overshoot of the output over its final value to its final value. The maximum value is given by

$$
\begin{aligned}
y(t_p) &= 1 - e^{-\sigma t}\left(\cos \omega_d t + \frac{\sigma}{\omega_d}\sin \omega_d t\right)\Big|_{t=t_p} \\
&= 1 - e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}\left(\cos\pi + \frac{\sigma}{\omega_d}\sin\pi\right) \\
&= \boxed{1 + e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}}
\end{aligned}
$$



Thus, the maximum overshoot is given as

$$M_p = \frac{y(t_p) - 1}{1} = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}$$

Please note that the peak overshoot is ONLY a function of damping coefficient ζ.

$\zeta = 0 \quad M_p = 1$ undamped

$$y(t) = 1 - e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}\sin(\omega_d t + \theta)$$

$\zeta = 1$

$M_p = 0$ critically damped

2.00

Lecture 13 - PowerPoint (Product Activation Failed)

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FILE HOME INSERT DESIGN TRANSITIONS ANIMATIONS SLIDE SHOW REVIEW VIEW PENS

Pen Highlighter Eraser Lasso Select Select Objects — Write

Pens

Color Thickness

Convert to Shapes — Ink Art

Stop Inking — Close

17 18 19 20 21

First order System

Impulse response e^{-at}

Step response $(1-e^{-at})$

$\tau = 1/a \rightarrow 63.2\%$

2nd order System step response

$t_r \simeq \frac{1.8}{\omega_n}$ $\quad \zeta = 0.5$ $\quad \pm 1\%$

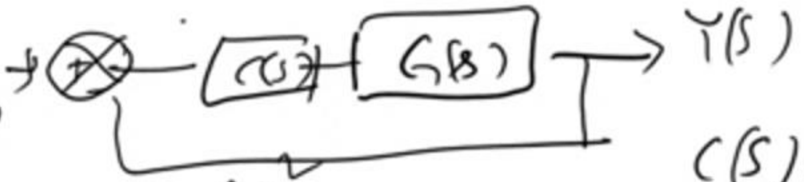
$t_p = \frac{\pi}{\omega_d} = \pi/(\omega_n\sqrt{1-\zeta^2})$ $\quad \pm 2\%$

$M_p = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}$, $\quad t_s = \frac{4.6}{\zeta\omega_n}$ or $\frac{4}{\zeta\omega_n}$

SLIDE 21 OF 21 ENGLISH (INDIA) NOTES COMMENTS 70%

Typically any nth order system after ~~the~~ being integrated with a compensator should behave ~~like~~ like an ideal 2nd order system

$$G(s) = \frac{n(s)}{d(s)}, \quad C(s),$$



$$\frac{\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} \approx \frac{C(s)G(s)}{1 + C(s)G(s)}$$