Lecture Note: 4
Root Locus

1 Introduction

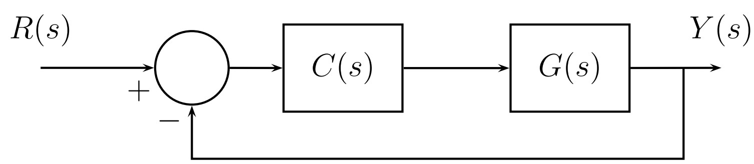

Figure 1: A closed loop system

A generic control system is shown in figure 1 where $G(s)$ is plant transfer function and $C(s)$ is the compensator. The closed loop transfer function is given as:

$$\frac{Y(s)}{R(s)} = \frac{C(s)G(s)}{1 + C(s)G(s)} \tag{1}$$

Let

$$C(s) = K \ \ and \ \ G(s) = \frac{1}{(s+1)(s+2)(s+3)}$$

Then,

$$\frac{Y(s)}{R(s)} = \frac{K}{s^3 + 6s^2 + 11s + 6 + K} \tag{2}$$

The open loop system has poles in LH of s-plane. However, as K varies, the poles of closed loop system varies. For example, when K=50, poles are $-5.77 \ and \ -0.1128 \pm j3.112$. Thus, the system is stable. When K= 150, poles are $-7.376 \ and \ 0.688 \pm j4.547$. In this case, the system is unstable as two of the poles are in right half of s-plane.

Lets take another example where the system is open loop unstable:

$$G(s) \ = \ \frac{(s+1)}{s(s-2)}, \ C(s) \ = \ K \tag{3}$$

Here, the open loop system is unstable as one of the poles is at s=2. The closed loop system is:

$$\frac{Y(s)}{R(s)} = \frac{K}{s^2 - 2s + K(s+1)} \tag{4}$$

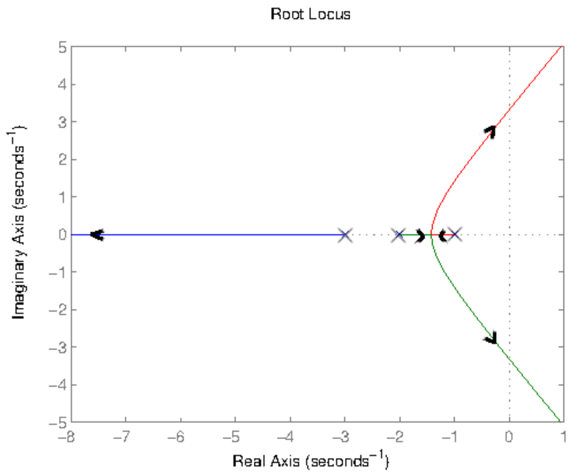

Figure 2: Positive Root Locus

For K=10, the closed loop system has poles at -6.4495 and -1.55 which are roots of $s^2 + 8s + 10$. Here, the closed loop system is stable although open loop system is unstable.
Lets go back to the example in eq(2)

$$\frac{Y(s)}{R(s)} = \frac{K}{s^3 + 6s^2 + 11s + 6 + K}$$

The characteristic polynomial $s^3 + 6s^2 + 11s + 6 + K$ has three roots. As K varies, these three poles give rise to three root locus branches. In Matlab, roots([1 6 11 6+K]) will give the pole locations as K is varied. The plot is as given in figure 2. This is called positive root locus as $0 < K < \infty$.

1.1 Observations:

Looking at figure 2, we can make the following observations:

- Because, we have a third order system, there are three separate plots(3 root locus branches).
- The plot is symmetric about real axis. This is because, complex roots occur in conjugate pairs.
- Each branch starts from open loop poles and ends at ∞.
- Asymptotes of each branch as they tend to ∞, share a common centre and their angles are evenly spread out over 360^o. In this case, these are $60^o, 180^o$ and 300^o.
- Two branches cross imaginary axis $\Rightarrow$ for higer K, the system becomes unstable.
- All parts of real axis to the left of odd numbers of poles are part of one of the root locus branches.
- When two branches meet, they break off, two poles become complex conjugate thereafter.

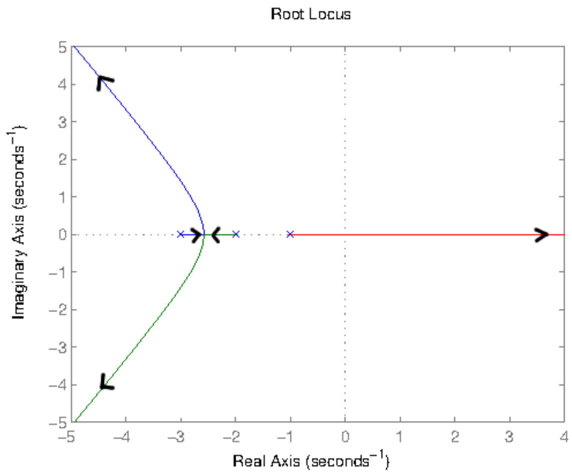

Figure 3: Negative Root Locus

- A root locus plot also exists for negative K ($0 \geq K > -\infty$) as shown in figure 3.

1.2 Concept of Root Locus:

If C(s)= K, then

$$\frac{Y(s)}{R(s)} = \frac{C(s)G(s)}{1 + C(s)G(s)} = \frac{KG(s)}{1 + KG(s)}$$

Roots are found when

$$1 + KG(s) = 0 \tag{5}$$

$$|KG(s)| = 1 \tag{6}$$

From, equation (5),

$$G(s) = -\frac{1}{K} \tag{7}$$

$$\angle G(s) = -180^o \; or \; 180 \pm k(360) \; where \; k = 1, 2, 3, ... \tag{8}$$

Equation (6) is called magnitude criterion ;and equation(8) is called angle criterion.

The root locus is the path of the roots of the characteristic equation due to K as K is varied.

Lets analyze the characteristic equation further,

$$1 + KG(s) = 0 \tag{9}$$

$$1 + K\frac{\prod_i^m (s + z_i)}{\prod_i^n (s + p_i)} = 0 \tag{10}$$

$$\prod_{i}^{n}(s+p_i)+K\prod_{i}^{m}(s+z_i)=0 \tag{11}$$

When K=0, poles of characteristic equation are same as poles of G(s). When K is ∞, equation (11) can be written as,

$$\frac{1}{K}\prod_{i}^{n}(s+p_i)+\prod_{i}^{m}(s+z_i)=0 \tag{12}$$

When K=∞, equation (12) shows that the roots of the characteristic equation as given in equation (10) are open loop zeros of G(s). This implies that root locus branch will start from the open loop poles and will converge to open loop zeros or at infinity.

2 How to plot root locus?

Given $L(s) = KG(s)$, one can plot root locus using some standard rules given in any text book. Root locus is known as positive root locus if K is positive and is varied from 0 to ∞. Root locus is known as negative if K is negative and $|K|$ is varied from 0 to ∞. Here follows some examples.

Example 1 (Motor Dynamics)**.** Consider the following open loop transfer function of a unity feedback system

$$G(s)=\frac{K}{s(s+1)}$$

Plot the root locus as K varies.

Solution

Root locus polynomial is given as $L(s) = \frac{1}{s(s+1)}$. The characteristic polynomial is given as:

$$1+KL(s)=s^2+s+k=0$$

Since this equation has two roots, root locus will consist of two branches. For each value of K, roots are given $s_{1,2} = \frac{-1\pm\sqrt{1-4K}}{2}$. The plot of root locus is shown in figure 4.

When

K=0 $s_{1,2}$=-1,0

K=1 $s_{1,2} = -0.5 \pm j\sqrt{2}$

K=2 $s_{1,2} = -0.5 \pm j\sqrt{7}$

K=10 $s_{1,2} = -0.5 \pm j\sqrt{39}$

It should be noted that the order of characteristic polynomial is same as the number of poles of transfer function $L(s)$. Thus number of root locus branches are always equal to number of poles in $L(s)$.

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Example 2. Consider the loop transfer function

$$L(s)=\frac{K}{s(s+1)(s+4)}$$

The corresponding root locus is given in Fig. 5.

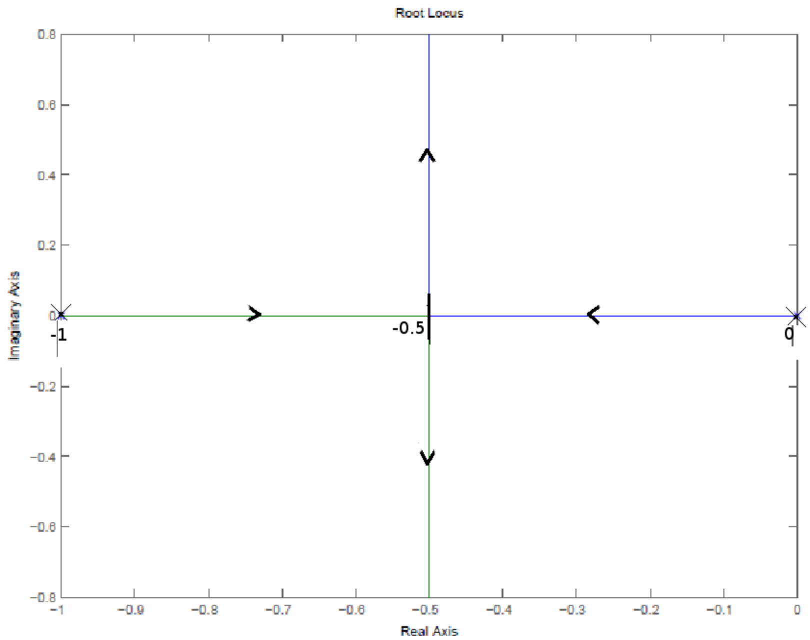

Figure 4: Root locus of example 1

1. It has three root locus branches.

2. Path between the poles at $s = 0$ and $s = -1$ on the real axis lie on root locus, so also the path on the real axis between the poles at $s = -4$ and $s = -\infty$.

3. Obviously two root locus branches will start from $s = 0$ and $s = -1$ and move towards each other until breakaway point is reached and then it will move along the asymptotes. The third branch will start from $s = -4$ and move towards $s = -\infty$ along the third asymptote.

4. Angle of asymptotes

$$\phi_l = \frac{180 + 360(l-1)}{n-m}, \quad l = 1, 2, \ldots, n-m$$

   In this case, $n = 3$ and $m = 0$. Thus $\phi_{1,2,3} = 60,\ 180,\ 300$. The centroid (the point of intersection of asymptotes is given by

$$\alpha = \frac{\sum p_i - \sum z_i}{n-m}$$

   In this case, centroid $\alpha = -5/3 = -1.667$.

5. Breakaway points (point of multiple roots):
   Rewriting characteristic equation as

$$1 + L(s) = 1 + K\frac{b(s)}{a(s)} = 0$$

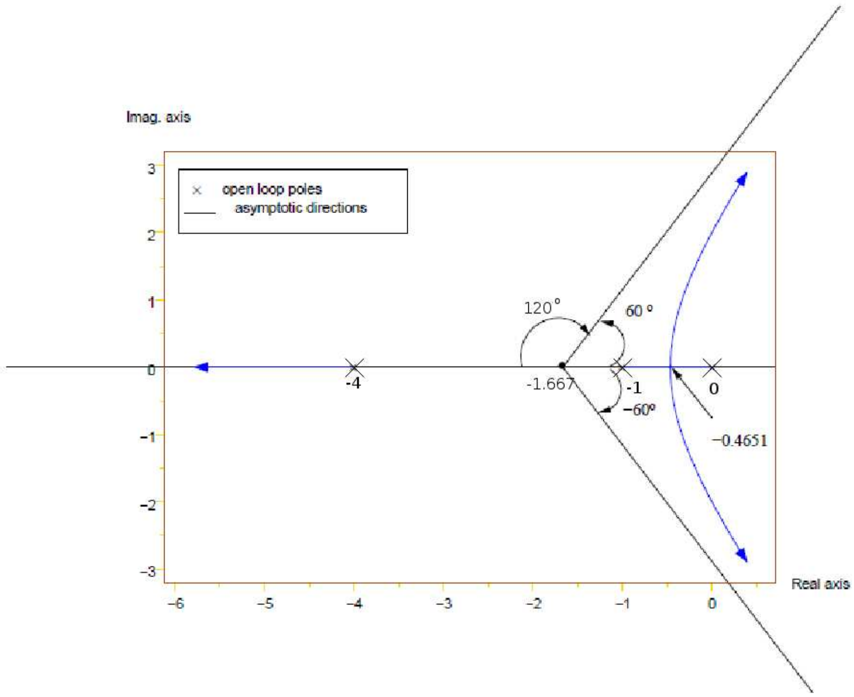

Figure 5: Root Locus

the breakpoints are defined as the points where

$$\frac{dK}{ds} = 0 \quad or, \quad b\frac{da}{ds} - a\frac{db}{ds} = 0$$

Here, $b(s) = 1$, and $a = s^3 + 5s^2 + 4s$. Solving above equation, we get two breakaway points,

$$s_{1,2} = -0.4651, \ -2.8689$$

Obviously, the first solution is right since it lies on root locus.

6. The root locus crosses the $j\omega$ axis at points where Routh criterion shows a transition from roots in the left half plane to roots in right half plane or vice versa. The characteristic equation is rewritten as

$$1 + \frac{K}{s^3 + 5s^2 + 4s} = 0$$

The characteristic polynomial is $d(s) = s^3 + 5s^2 + 4s + K$. Computing the Routh's array, we have

$$\begin{array}{ccc} s^3 & 1 & 4 \\ s^2 & 5 & K \\ s^1 & \frac{20-K}{5} & 0 \\ s^0 & K & \end{array}$$

It is required that $\frac{20-K}{5} = 0$. This gives $K = 20$. The auxilliary equation is given by $5s^2 + 20 = 0$ which gives $s_{1,2} = \pm j2$. Thus the root locus crosses the $j\omega$ axis at $\pm j2$ when $K = 20$.

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Example 3. Let us consider another example that consists of conjugate poles,

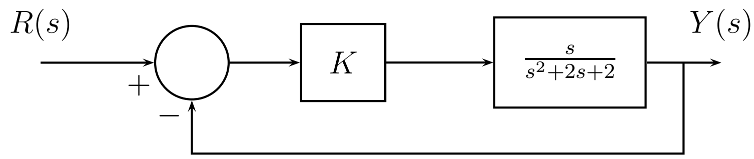


Figure 6: A closed loop system

$$
\begin{aligned}
C(s) = K \; and \; G(s) &= \frac{s}{s^2+2s+2} \\
Then, \; \frac{Y(s)}{R(s)} &= \frac{Ks}{s^2+(2+K)s+2}
\end{aligned}
$$

It has a zero at s=0 and poles at $s_{1,2} = -1 \pm j$

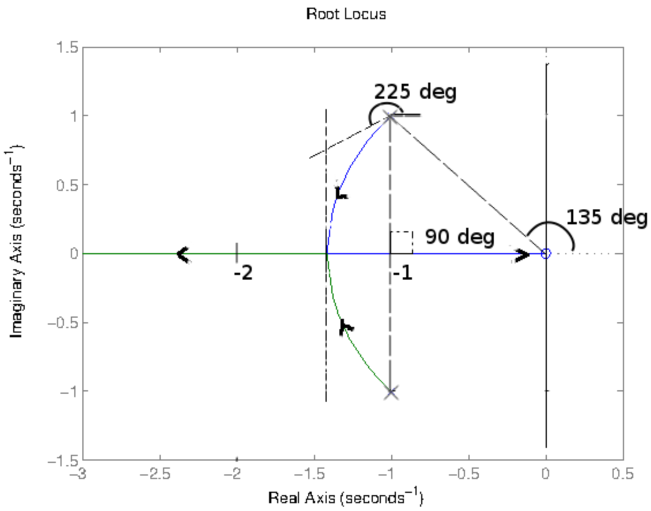
The entire LH s-plane is part of root locus branches. One branch will converge to zero at s=0 while other branch will converge to ∞. This implies that one asymptote angle is -180^o.
The centroid

$$\sigma_A = \frac{-1+j-1-j-0}{1} = -2$$

Breakaway point:

$$
\begin{aligned}
G(s) &= \frac{s}{s^2+2s+2} \\
1 &+ KG(s) = 0 \\
\Rightarrow K &= -\frac{1}{G(s)} \\
\frac{dK}{ds} &= \frac{d}{ds}\left(-\frac{s^2+2s+2}{s}\right) \\
&= \frac{-s(2s+2)+(s^2+2s+2)}{s^2} \\
&= -\frac{s^2-2}{s^2} = 0 \\
\Rightarrow s^2-2 &= 0 \Rightarrow \quad s = \pm\sqrt{2}
\end{aligned}
\tag{13}
$$

Since, $+\sqrt{2}$ does not lie on the root locus, $-\sqrt{2}$ is the breakaway point.



The value of K at breakaway point is,

$$
\begin{aligned}
K &= -\frac{1}{G(s)}_{|s=-\sqrt{2}} \\
&= -\frac{s^2+2s+2}{s}_{|s=-\sqrt{2}} \\
&= -\frac{2-2\sqrt{2}+2}{-\sqrt{2}} \\
&= \frac{4-2\sqrt{2}}{\sqrt{2}} = 2\sqrt{2}-2 = 0.828
\end{aligned}
$$

Angle of Departure/Arrival: Angle of departure at complex pole, $\phi_p = 180 + \phi^o$ where, ϕ is the net angle contribution at this pole
Angle of arrival at complex zero, $\phi_z = 180 - \phi^o$where, ϕ is the net angle contribution at zero
Here, in this example $\phi_{s=-1+j}$= $180 + (135 - 90)^o = 225^o$
Note that angle of departure is calculated only for complex poles and angle of arrival is calculated only for complex zeros.

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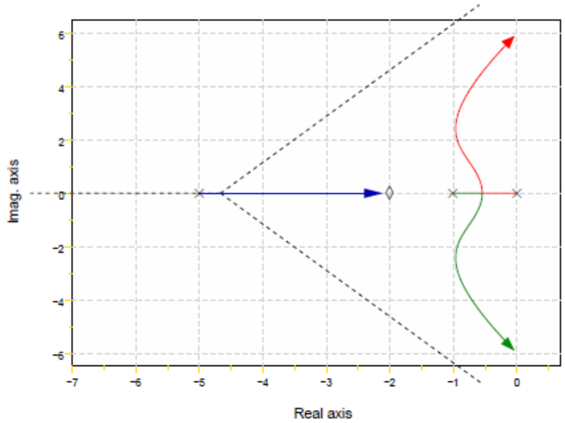
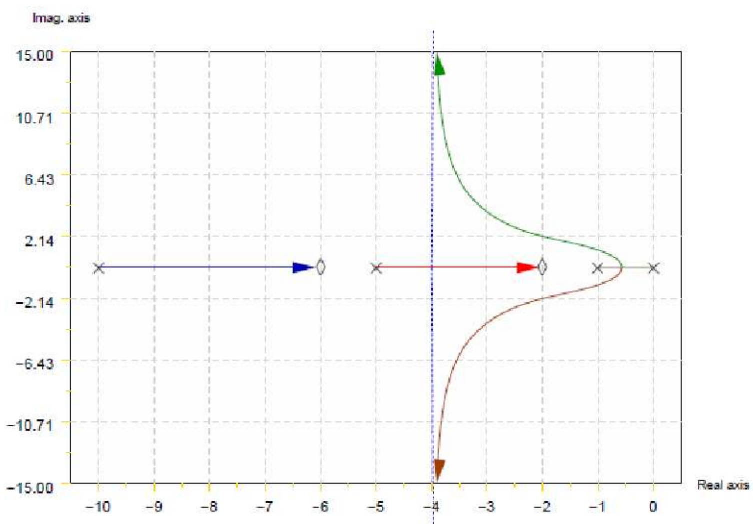
2.1 Practice Problems: Positive root locus

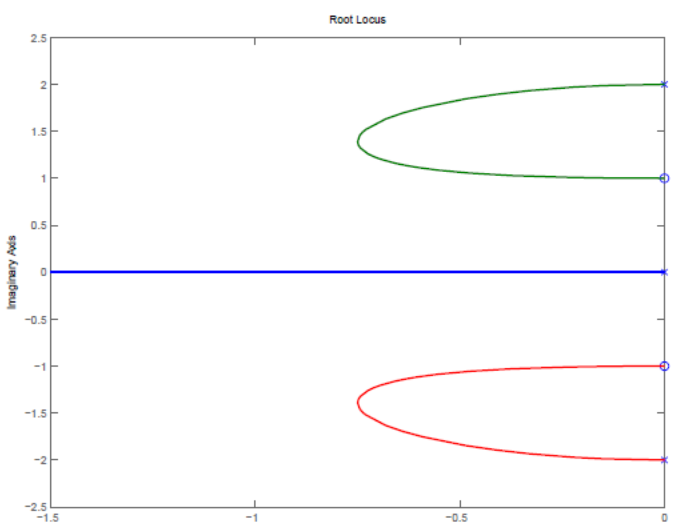
1. 

$$L(s) = \frac{s+2}{s(s+1)(s+5)(s+10)}$$

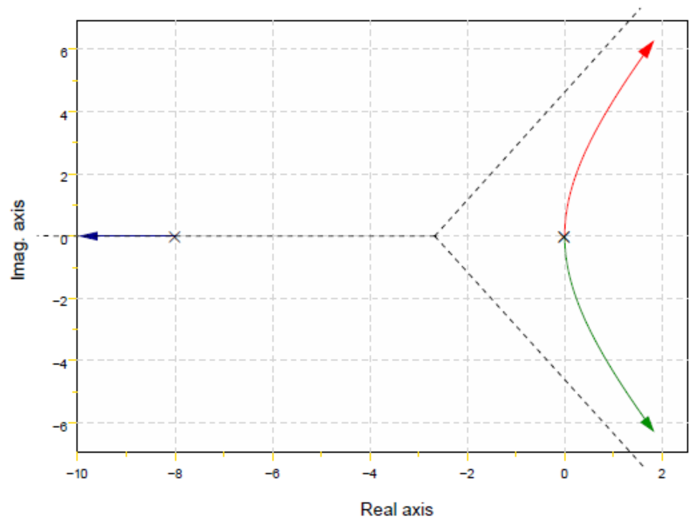
Root locus crosses the $j\omega$ axis at $\pm j6$ when $K \approx 516$.

2. 

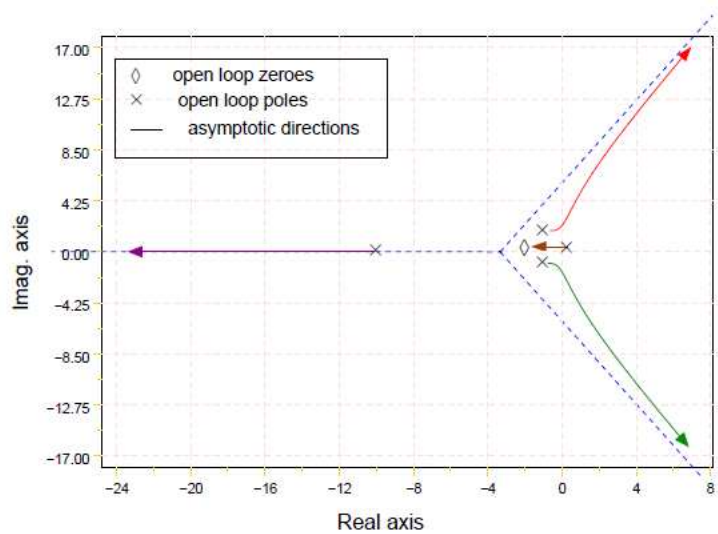
$$L(s) = \frac{(s+2)(s+6)}{s(s+1)(s+5)(s+10)}$$

3. 

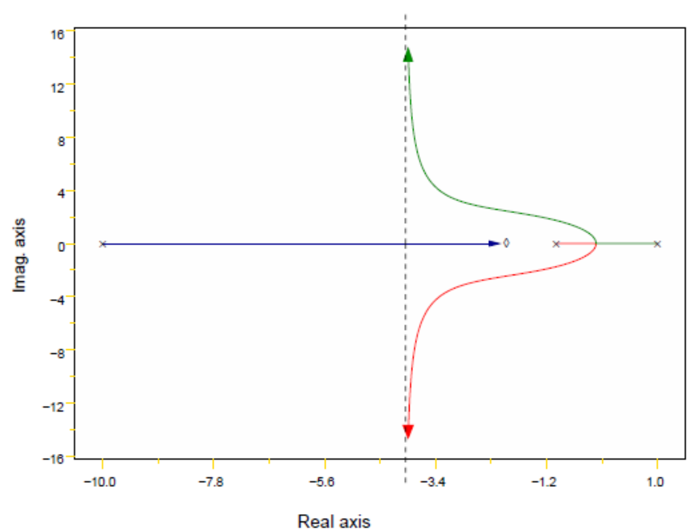
$$L(s) = \frac{s^2+1}{s(s^2+4)}$$

4. 

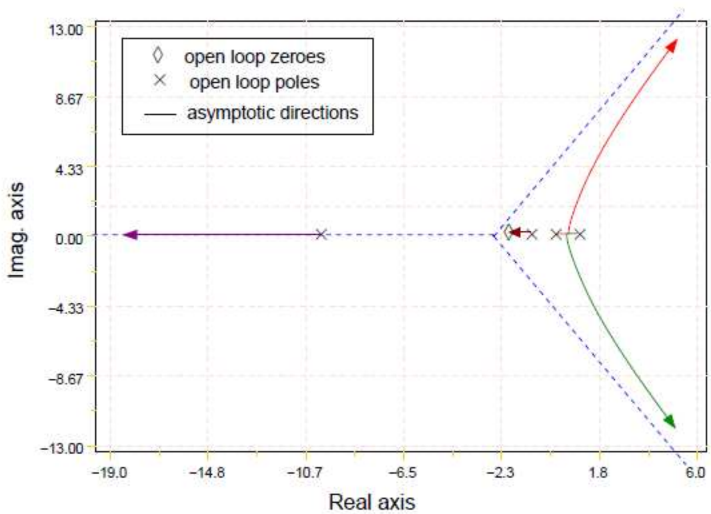
$$L(s) = \frac{1}{s^2(s+8)}$$

5. 

$$L(s) = \frac{s+2}{s(s+10)(s^2+2s+2)}$$

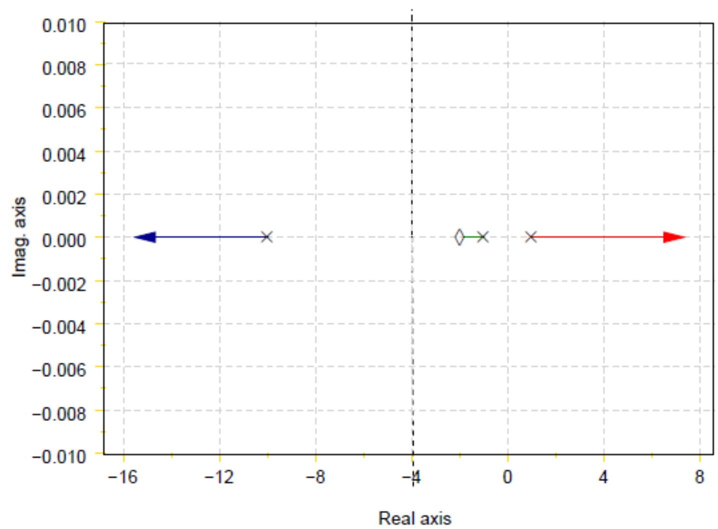
6. 

$$L(s) = \frac{s+2}{(s+10)(s^2-1)}$$

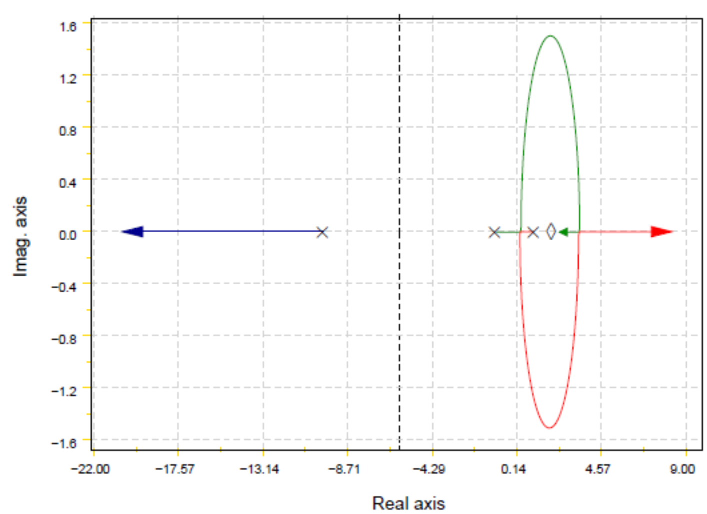
7. 

$$L(s) = \frac{s+2}{s(s+10)(s^2-1)}$$

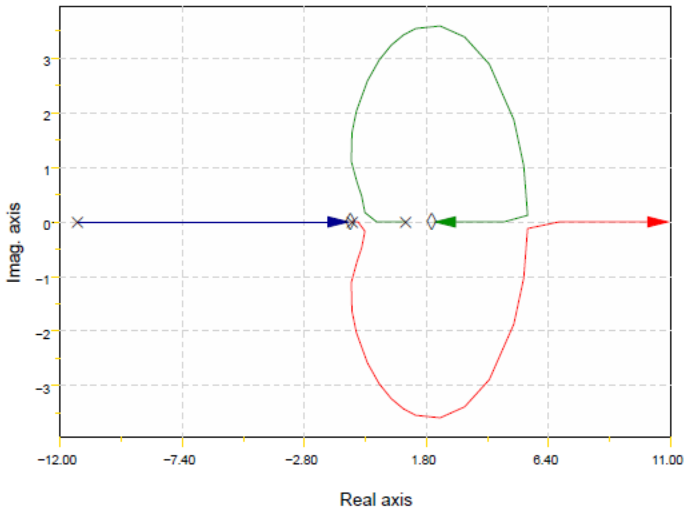
2.2 Practice Problems: Negative root locus

1. 

$$L(s) = -\frac{s+2}{(s+10)(s^2-1)}$$

2. 

$$L(s) = -\frac{s-2}{(s+10)(s^2-1)}$$

3. 

$$L(s) = -\frac{(22s+23)(s-2)}{(35s+34)(s^2-1)}$$

3 Tutorial Problems

1. Plot the root locus when

   (a) $L(s) = \frac{1}{s^2}$

   (b) $L(s) = \frac{s+1}{s+12}\frac{1}{s^2}$

   (c) $L(s) = \frac{s+1}{s+9}\frac{1}{s^2}$

   (d) $L(s) = \frac{s+1}{s+4}\frac{1}{s^2}$

   Comment on the results.

2. Plot the root locus when

   (a) $L(s) = \frac{s-1}{s^2}$

   (b) $L(s) = -\frac{s+2}{s+14}\frac{s-1}{s^2}$

   (c) $L(s) = -\frac{s+0.4}{s+14}\frac{s-1}{s^2}$

   (d) $L(s) = -\frac{s+0.1}{s+14}\frac{s-1}{s^2}$

   Comment on the results.

Solutions:

1. (a)

$$L(s) = \frac{1}{s^2}$$

The corresponding root locus is shown in Fig. 7.
The characteristic polynomial would be $s^2 + K = 0$ and $s_{1,2} = \pm j\sqrt{K}$
The breakaway point can be found by taking $\frac{dK}{ds} = -2s = 0$ which gives $s = 0$.

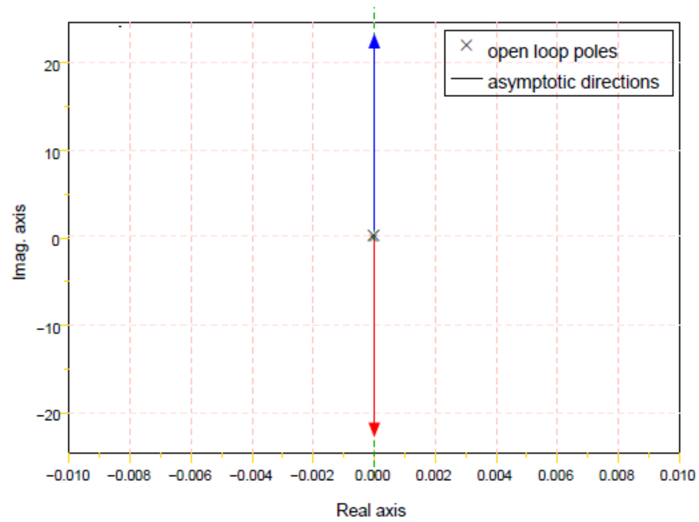


Figure 7: Problem 1(a)

(b)

$$L(s) = \frac{s+1}{s+12} \cdot \frac{1}{s^2}$$

The corresponding root locus is shown in Fig. 8.

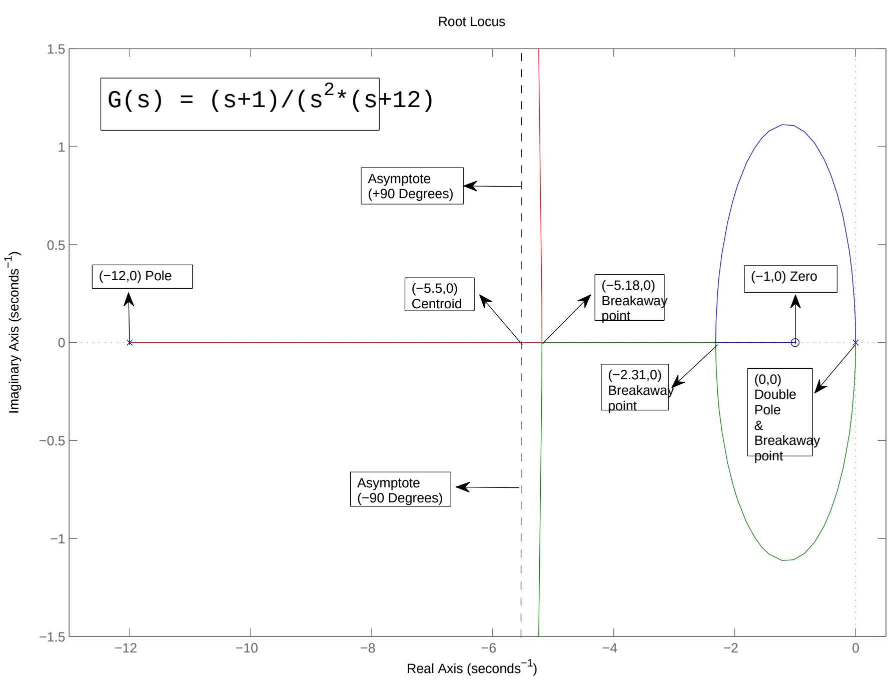


Figure 8: Problem 1(b)

- There are 3 poles and 1 zero $\Rightarrow$ 3 root locus branches exist and $3-1$ branches converge to infinity.
- Centroid is at $C = \frac{\Sigma p_i - \Sigma z_i}{n-m} = \frac{-12-(-1)}{2} = -\frac{11}{2}$
- Routh array confirms that no pole crosses imaginary axis.
- Characteristic equation $s^2(s+12) + K(s+1) = 0 \Rightarrow K = -\frac{s^2(s+12)}{s+1}$
- Break away points (points of multiple roots):

$$\frac{dK}{ds} = 0 \Rightarrow -\frac{(s+1)(3s^2+24s) - s^2(s+12)}{(s+1)^2} = 0 \Rightarrow 2s^3+15s^2+24s = 0 \Rightarrow s[2s^2+15s+24] = \tag{14}$$

(c)

$$L(s) = \frac{s+1}{s+9} \cdot \frac{1}{s^2}$$

The corresponding root locus is shown in Fig. 9.

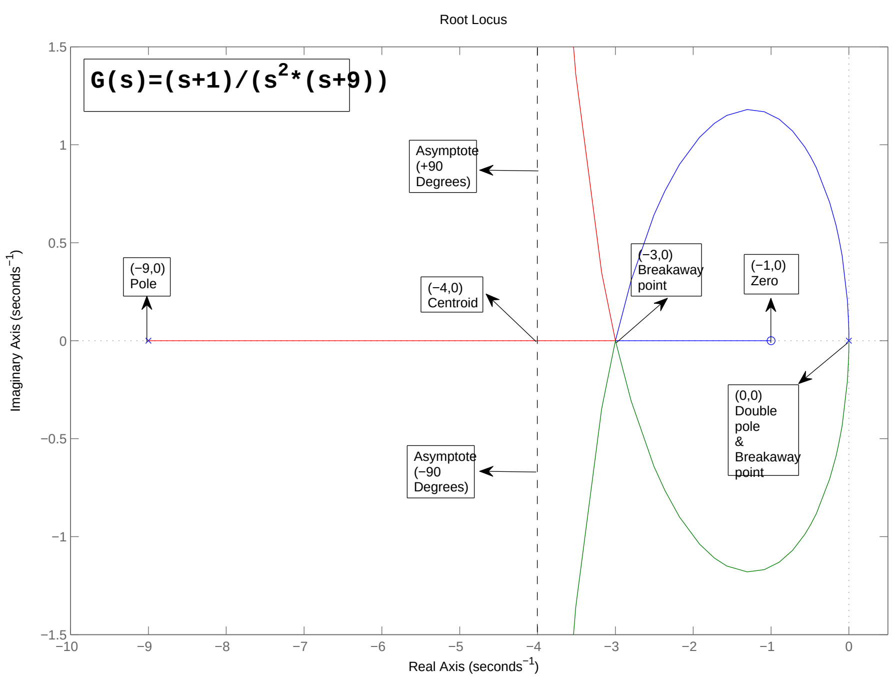


Figure 9: Problem 1(c)

- There are 3 poles and 1 zero $\Rightarrow$ 3 root locus branches exist and $3-1$ branches converge to infinity.
- Asymptote angles would be $+90^o$ and -90^o
- Centroid at $C = \frac{\Sigma p_i - \Sigma z_i}{n-m} = \frac{-9-(-1)}{2} = -\frac{-8}{2} = -4.$
- Root Locus do not cross $j\omega$ axis.
- Characteristic equation would be $s^2(s+9) + K(s+1) = 0 \Rightarrow K = -\frac{s^2(s+9)}{s+1}$

- Break away points :

$$\frac{dK}{ds} = \frac{3s^3 + 18s^2 + 18s - s^3 - 9s^2}{(s+1)^2} = 0 \Rightarrow s_{1,2,3} = 0, -3, -3. \tag{15}$$

Given that

(d)

$$L(s) = \frac{s+1}{s+4} \cdot \frac{1}{s^2}$$

The corresponding root locus is shown in Fig. 10.

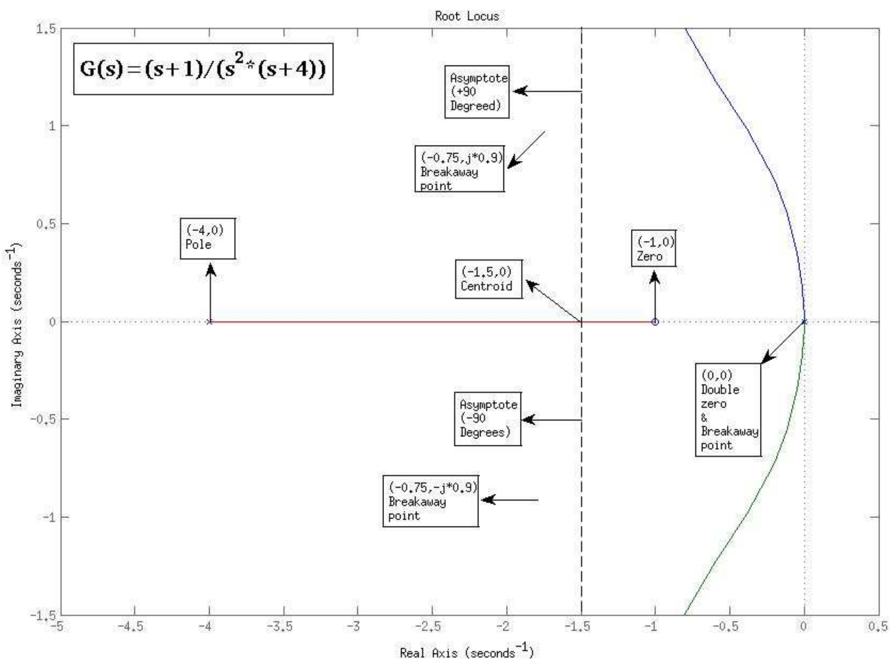


Figure 10: Problem 1(d)

- Centroid $= \frac{\Sigma p_i - \Sigma z_i}{n-m} = \frac{-4-(-1)}{2} = -\frac{3}{2} = -1.5.$
- Root Locus do not cross $j\omega$ axis.
- Break away points (points of multiple roots):
  $K = -\frac{s^2(s+4)}{s+1}$
  $\frac{dK}{ds} = 0 \Rightarrow (s+1)(3s^2+8s) - s^2(s+4) = 0 \Rightarrow 2s^3 + 7s^2 + 8s = 0$ $s_{1,2,3} = 0, -1.75 \pm j0.9$. s_2 and s_3 do not lie on root locus.

Comments: $\frac{s+1}{s+12}$, $\frac{s+1}{s+9}$ and $\frac{s+1}{s+4}$ are valid compensators. A specific compensator is finally selected based on desired specifications.

2. (a)

$$L(s) = \frac{s-1}{s^2}$$

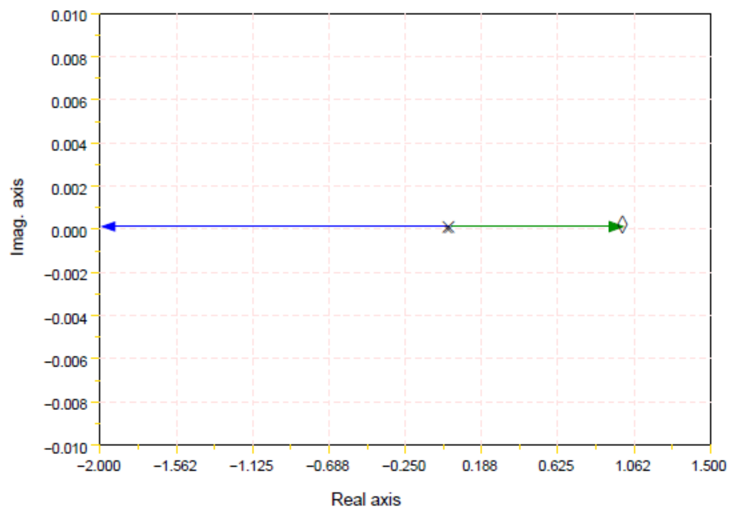


Figure 11: Problem 2(a)

The corresponding root locus is shown in Fig. 11. This is a non-minimum phase system. The closed loop system is always unstable since zero in RHP will always drag one of the double poles at origin.

(b)

$$L(s) = -\frac{s+2}{s+14} \cdot \frac{s-1}{s^2}$$

The corresponding root locus is shown in Fig. 12. The negative sign is introduced since

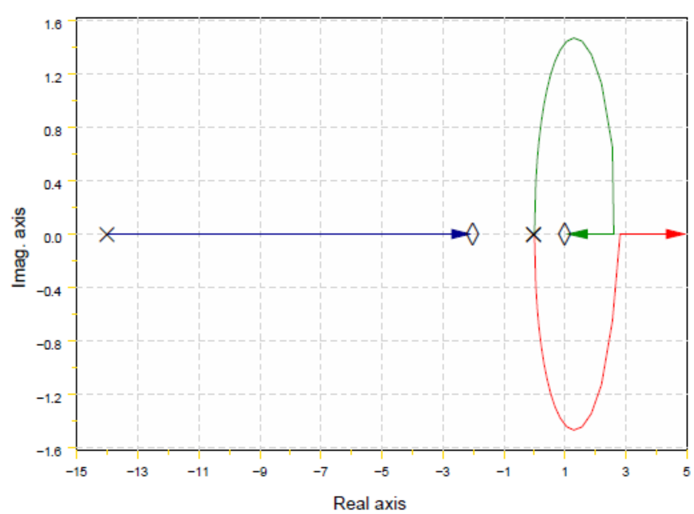


Figure 12: Problem 2(b)

this system can be made stable in the negative root locus plot. In this case, the system is probably marginally stable for a small range of K.

(c)

$$L(s) = -\frac{s+0.4}{s+14} \cdot \frac{s-1}{s^2}$$

The corresponding root locus is shown in Fig. 13.

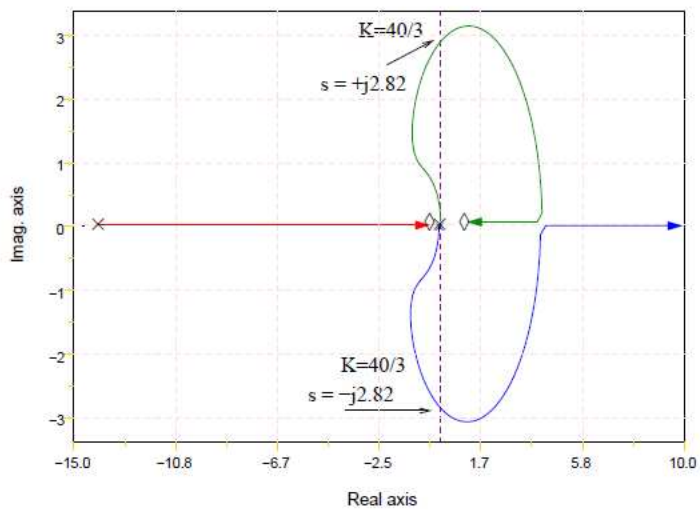


Figure 13: Problem 2(c)

$$\begin{aligned} 1 + KL(s) &= 0 \\ 1 - K\frac{(s+0.4)(s-1)}{s^2(s+14)} &= 0 \\ d(s) &= s^3 + (14-K)s^2 + (K-0.4)s + 0.4K \end{aligned}$$

Routh array

$$\begin{array}{lll} s^3 & 1 & 0.6K \\ s^2 & 14-K & 0.4K \\ s^1 & \frac{(14-K)0.6K-0.4K}{14-K} & 0 \end{array}$$

Since, root locus crosses the $j\omega$ axis, we have $(14-K)0.6K - 0.4K = 0$, this gives $K = \frac{40}{3}$. Root locus crosses the $j\omega$ axis at $s = \pm j2.82$.

(d)

$$L(s) = -\frac{s+0.1}{s+14} \cdot \frac{s-1}{s^2}$$

The corresponding root locus is shown in Fig. 14. Using Routh array for $d(s) = s^3 + (14-K)s^2 + 0.9Ks + 0.1K$, we find that the root locus crosses the $j\omega$ axis at $s_{1,2} = \pm j3.5$ for a gain $K = \frac{125}{9}$.

Comments: For non-minimum phase systems, it is generally very difficult to design a compensator which can be stable over large range of gain K. In this specific case, compensator $-\frac{s+0.4}{s+14}$ can provide stability over some range of K $(0 < K < \frac{40}{3})$.

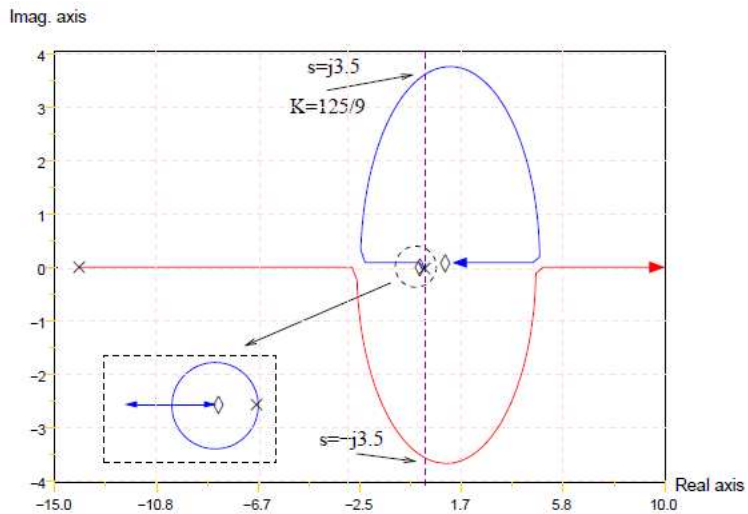


Figure 14: Problem 2(d)

4 Compensator Design Using Root-Locus Technique

Design Problem 1. Given $G(s) = \frac{K}{s^2}$, Design a lead compensator so that the settling time $t_s \leq 4$ *sec* and $M_p \leq 20\%$.

Solution: $M_p \leq 20\%$ implies $\zeta \geq 0.45$. $t_s \leq 4$ together with this value of ζ implies that $\zeta\omega_n \geq 1$. Thus, the desired dominant poles are

$$\begin{aligned} s_d &= -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} \\ &= -1 \pm j2 \end{aligned}$$

Angle contributed by all the poles of $G(s)$ at $s_d = -2 \times 116 = -232$. Thus compensator must

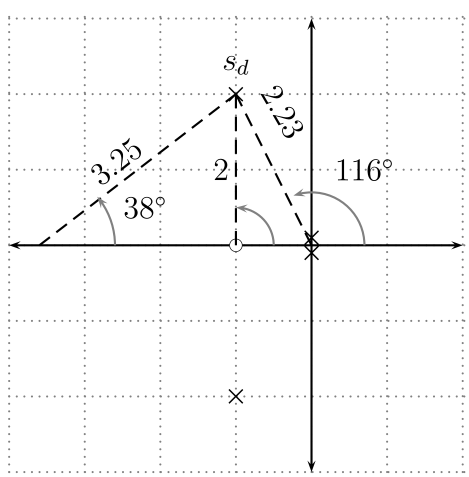


Figure 15: Root Locus Design

contribute an angle of $-180 + 232 = 52°$ so that s_d becomes a part of the root locus thus satisfying the angle criterion. For a lead compensator, first fix the zero at any suitable position, say, at -1. Angle contributed by it at s_d is $+90°$. Thus, angle contributed by the pole of compensator must be $-(90-52) = -38°$. Hence, we have following compensator

$$G_c(s) = \frac{s+1}{s+3.6}$$

The loop transfer function may be written as

$$L(s) = C(s)G(s) = \frac{K(s+1)}{s^2(s+3.6)}$$

From magnitude criterion, we have

$$K = \frac{1}{|L(s)|}$$

where

$$|L(s)| = \left|\frac{s+1}{s^2(s+3.6)}\right|_{s=-1+j2}$$

This gives $K = 8.2$. $|L(s)|$ may be computed graphically as follows

$$|L(s)| = \frac{\prod \text{distance from zeros to } s_d}{\prod \text{distance from poles to } s_d} = \frac{(2.23)^2 \times 3.25}{2}$$

Therefore, the final lead compensator is given as

$$C(s) = \frac{8.2(s+1)}{s+3.6}$$

□

Design Problem 2 (Compensators for speed control of DC motors)**.** Consider a separately excited DC motor with following parameters:

$$\begin{aligned} &J = 0.01; \\ &b = 0.1; \\ &Kt = Kb = 0.01; \\ &R = 1; \\ &L = 0.5; \end{aligned}$$

1. Design a PI (Lag) compensator so that with a 1 rad/sec step reference, the design criteria are:
   - Settling time less than equal to 2 seconds
   - Overshoot less than 5%
   - Steady-state error less than 5%

   (a) Plot the system step response of the compensated system showing that design specifications are met.

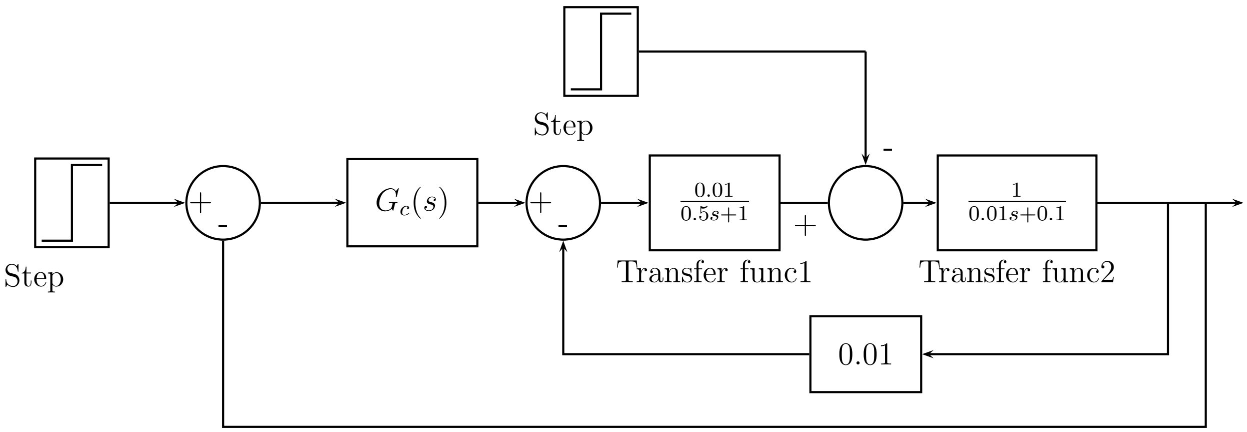


Figure 16: DC motor

(b) Can you design a PD (Lead) compensator to meet the above specifications? Explain why?

2. Design a PID (Lag-lead compensator) so that settling time is approximately 1 sec while other specifications remain same as that for PI compensator. Plot the system step response with the compensated system showing if the desired specifications are met.

Solution: The transfer function is given as:

$$G(s) = \frac{\omega(s)}{V(s)} = \frac{0.01}{(0.5s+1)(0.01s+0.1)+0.0001} = \frac{2}{(s+2)(s+10)}$$

From $t_s = 2\ sec$ and $M_p = 0.05$, we obtain $\zeta = 0.6901$, $\omega_n = 3.3328$ using equations

$$0.05 = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \quad \text{and} \quad t_s = \frac{4.6}{\zeta\omega_n}$$

The dominant pole pairs are found to be $s_{1,2} = -2.3 \pm j2.4120$.

1. **Design of PI (Lag) Compensator:**

The root locus of the uncompensated system is shown in Fig. 17, where two open loop poles are located at -2 and -10. The desired objective is to design a PI compensator of the form $G_c(s) = K\frac{s+\alpha}{s+\beta}$ such that the new pole is very near to origin and the zero is located between two original poles. It requires trial and error method to exactly fix the required dominant poles to lie on the root locus of the compensated system. The pole was fixed at $s = 0.1$. Using angle criteria, the zero was found to be at $s = 3.3$. Using gain criteria, the gain K' at $s = -2.3 + 2.4j$ is found to be 24.6. Since the transfer function $G(s)$ has a gain 2, the extra gain that must be mutliplied with the compensator transfer function is $K = 12.3$. Thus compensator transfer function is $G_c(s) = 12.3\frac{s+3.3}{s+0.1}$. The root locus of the compensated system $\frac{1}{K}G_c(s)G(s)$ is shown in the Fig. 18. The response of the closed loop system to a step input of magnitude 10 is shown in Fig. 19. As can be seen from the figure

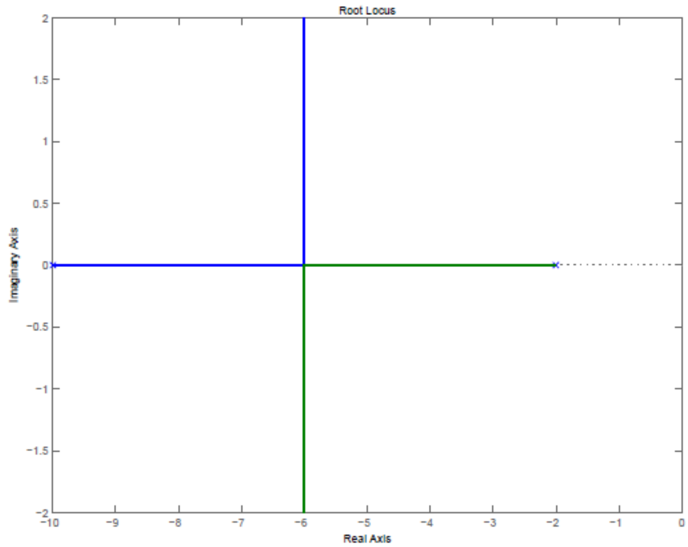


Figure 17: Root Locus of $G(s)$

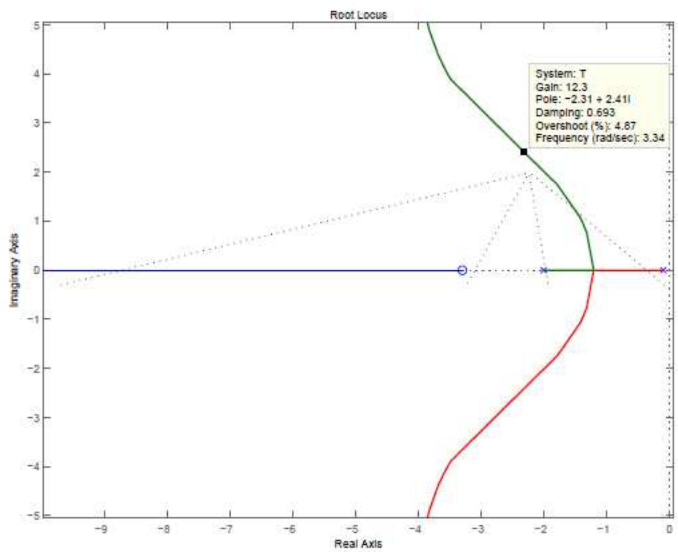


Figure 18: Root Locus of $\frac{1}{K}G_c(s)G(s)$

that the settling time is around 2 sec, peak overshoot is 7% and a steady state error is 2.4%. The steady state error can be eliminated if one takes a pure integrator design, i.e, $\beta = 0$. So the PI design has almost met the desired specifications.

2. **Design of lead-lag Compensator:**

Since the required settling time t_s is now reduced to 1 sec, a lead compensator is introduced to make the response faster. When $t_s = 1$, and $M_p = 0.05$, one can find that $\zeta = 0.69$ and $\omega_n = 6.66$. Thus dominat pole pair must be located at $s_{1,2} = -4.6 \pm 4.8j$. The lead-lag compensator has the following form: $G_c(s) = KG_{c1}(s)G_{c2}s$ where $G_{c1}(s) = \frac{s+z_1}{s+p_1}$ is the lag compensator and

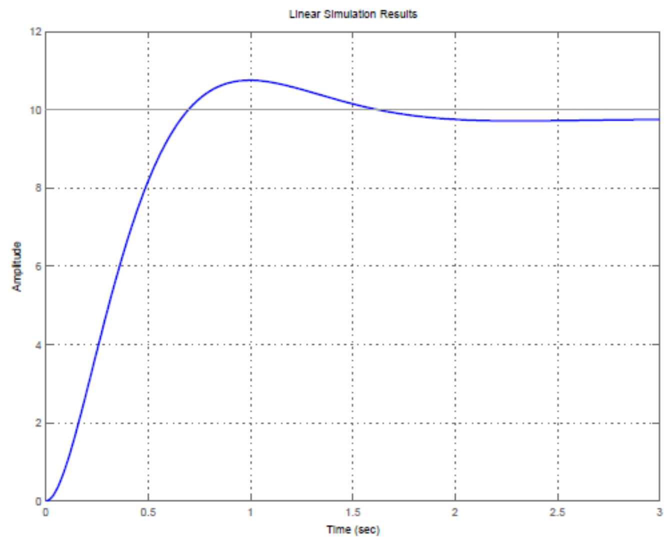


Figure 19: Step Response of $\frac{G_c(s)G(s)}{1+G_c(s)G(s)}$

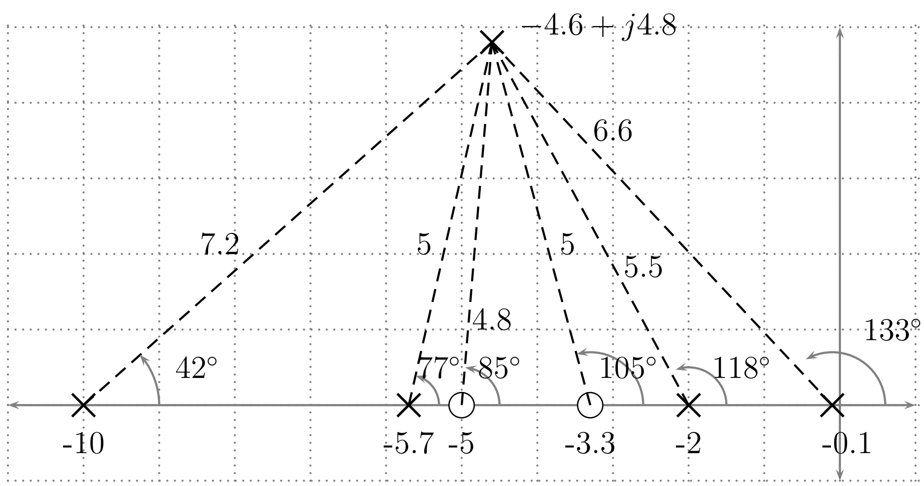


Figure 20: Root Locus Design

$G_{c2}(s) = \frac{s+z_2}{s+p_2}$ is a lead compensator. One can retain the same lag compensator, i.e. $G_{c1}(s) = \frac{s+3.3}{s+0.1}$. The zero of the lead compensator is fixed at $z_2 = 5.0$. Using angle criteria (please refer figure 20), the pole of the lead compensator is found to be at $p_2 = 5.7$. Using gain criteria, the gain was found to be $K' = \frac{7.2\times6.6\times5.5\times5}{5\times4.8} = 54.5$. Since the actual system $G(s)$ has a agin 2, the gain that must be pultiplied with the compensator is $K = 27.25$. The complete expression of the compensator is thus $G_c(s) = 27.25\frac{s+3.3}{s+0.1}\frac{s+5}{s+5.7}$. The root locus of the compensator is shown in figure 21. It is seen from the root locus that the gain is 26.7 at the desired location $s = -4.6 + 4.8j$. This value is very close to the gain value 27.25 computed graphically. The closed loop transfer function of the compensated system is

$$G_f(s) = \frac{54.4s^2 + 452.4s + 899.3}{s^4 + 17.8s^3 + 144.7s^2 + 575.2s + 910.6s}$$

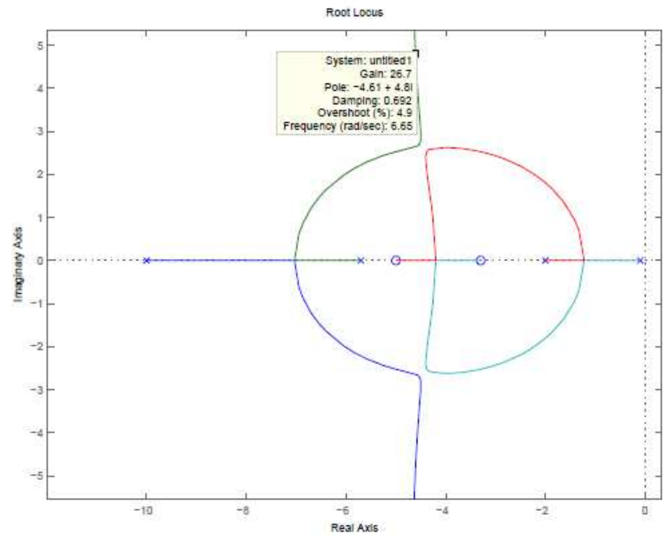


Figure 21: Root Locus of DC motor with Lead-lag compensator

The step response of this feedback system is shown in Fig. 22. As can be seen that settling time has reduced to 1 sec and steady state error is now within 2%. However, the overshoot is 13% which is slightly more than required.

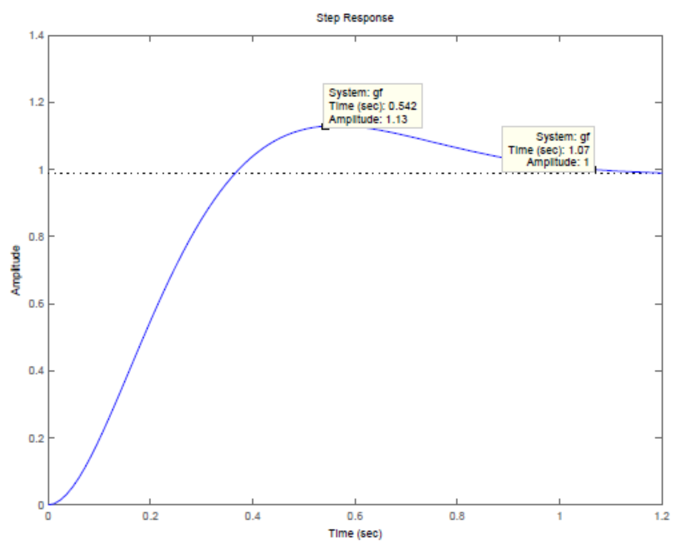


Figure 22: Step response of DC motor with Lead-lag compensator

□

5 Summary

Root locus based design is all about how to place a pole or a zero. Following points may be kept in mind while designing:

1. The primary objective of root locus based design is that over all closed loop compensated dynamics should approximately behave as an ideal second order system $G_f(s) = \frac{{\omega_n}^2}{s^2+2\zeta\omega_n s+{\omega_n}^2}$. Thus design is primarily based on placing desired dominant poles on the root locus. This assumption makes root locus based design as somewhat unattractive.

2. If steady state error performance has to be improved, then a pole is placed very close to the origin in the left half of the s-plane.

3. If the uncompensated closed loop system is unstable, a zero is suitably placed in the left half of the s-plane so that root locus branches are drawn towards left half of the s-plane.

4. A PI compensator is also known as lag compensator because its phase angle contribution at the desired dominant pole is negative.

5. A PD compensator is also known as lead compensator because its phase angle contribution at the desired dominant pole is positive.

6. A PI compensator makes system sluggish while a PD compensator actuates fast response.

7. If steady state error is required to be reduced as well as response has to be faster, i.e. settling time t_s has to be reduced, then a lead-lag compensator is designed.

8. If the plant is non-minimum phase system, then compensator may need to be designed in negative root locus.

9. In general, root locus makes us aware that there are systems for which increase of gain K leads to unstability. An example is $L(s) = \frac{K}{s(s+2)(s+4)}$. Similarly there are systems for which increase of gain leads to stability. Example: $L(s) = \frac{K(s-1)}{(s-6)(s^2-9)}$

6 Problems

1. Plot the root locus when:

   (a) $L(s) = \frac{s+2}{s(s+1)(s+5)(s+10)}$

   (b) $L(s) = \frac{(s+2)(s+6)}{s(s+1)(s+5)(s+10)}$

   (c) $L(s) = \frac{(s+2)(s+4)}{s(s+1)(s+5)(s+10)}$

   (d) $L(s) = \frac{s^2+1}{s(s^2+4)}$

   (e) $L(s) = \frac{s^2+4}{s(s^2+1)}$

   (f) $L(s) = \frac{1}{s^2(s+8)}$

   (a) $L(s) = \frac{s+2}{s(s+10)(s^2+2s+2)}$

   (b) $L(s) = \frac{1}{s^2-1}$

   (c) $L(s) = \frac{s+2}{s+10}$

   (d) $L(s) = \frac{s+2}{s(s+10)(s^2-1)}$

   (e) $L(s) = \frac{s-1}{s^2}$

2. Set up the following characteristic equations in the form suited to Evan's root-locus method. Give $L(s)$, $a(s)$, and $b(s)$ and the parameter, K, in terms of the original parameters in each case. Be sure to select K so that $a(s)$ and $b(s)$ are monic in each case and the degree of $b(s)$ is not greater than that of $a(s)$.

   (a) $s + (1/\tau) = 0$ versus parameter τ.

   (b) $s^2 + cs + c + 1 = 0$ versus parameter c.

   (c) $(s + c)^3 + A(Ts + 1) = 0$

      i. versus parameter A,

      ii. versus parameter T,

      iii. versus parameter c, if possible. Say why you can or cannot. Can a plot of the roots be drawn versus c for given constant values of A and T by any means at all?

   (d) $1 + \left[k_p + \frac{k_I}{s} + \frac{k_D s}{\tau s+1}\right] G(s) = 0$
   Assume that $G(s) = A\frac{c(s)}{d(s)}$, where $c(s)$ and $d(s)$ are monic polynomials with the degree of $d(s)$ greater than that of $c(s)$.

      i. versus k_p,

      ii. versus k_I,

      iii. versus k_D,

      iv. versus τ.

3. A simplified model ofthe longitudinal motion of a certain helicopter near hover has the transfer function

$$G(s) = \frac{9.8(s^2 - 0.5s + 6.3)}{(s + 0.66)(s^2 - 0.24s + 0.15)}$$

   and the characteristic equation is $1 + D(s)G(s) = 0$. Let $D(s) = k_p$ at first.

   (a) Compute the departure and arrival angles at the complex poles and zeros.

   (b) Sketch the root locus for this system for parameter $K = 9.8k_p$. Use axes $-4 \leq x \leq 4$, $-3 \leq y \leq 3$;

   (c) Verify your answer using MATLAB. Use the command axes ([-4 4 -3 3]) to get right scales.

   (d) Suggest a practical (at least as many poles or zeros) alternative compensation $D(s)$ which will at least result in a stable system.

4. For the feedback configuration of Fig. 23, use asymptotes, center of asymptotes, angles of departure and arrival, and the Routh array to sketch root loci for the characteristic equations of the following feedback control systems versus the parameter K. Use MATLAB to verify your results.

   (a) $G(s) = \frac{1}{s(s+1+3j)(s+1-3j)}$, $\quad H(s) = \frac{s+2}{s+8}$

   (b) $G(s) = \frac{1}{s^2}$, $\quad H(s) = \frac{s+1}{s+3}$

   (c) $G(s) = \frac{s+5}{s+1}$, $\quad H(s) = \frac{s+7}{s+7}$

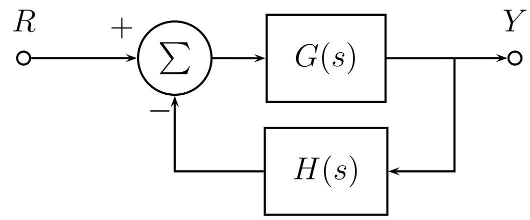


Figure 23: Problem 4

(d) $G(s) = \frac{(s+3+4j)(s+3-4j)}{s(s+1+2j)(s+1-2j)}$, $\qquad H(s) = 1 + 3s$

5. Consider the system in Fig. 24.

   (a) Using Routh's stability criterion, determine all values of K for which the system is stable.

   (b) Sketch the root locus of the characteristic equation versus K. Include angles of departure and arrival, and find the values for K and s at all breakaway points, break-in points, and imaginary-axis crossings.

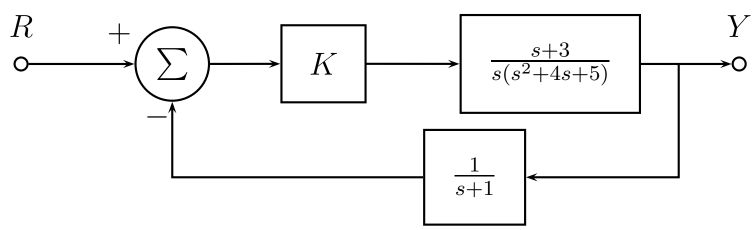


Figure 24: Problem 5

6. Suppose you are given the plant

$$L(s) = \frac{1}{s^2 + (1+\alpha)s + (1+\alpha)},$$

   where α is a system parameter that is subject to variations. Use both positive and negative root-locus methods to determine what variations in α can be tolerated before instability occurs.

7. For the system in Fig. 25

   (a) Find the locus of closed-loop roots with respect to K.

   (b) Is there a value of K that will cause all roots to have a damping ratio greater than 0.5.

   (c) Find the values of K that yield closed-loop poles with the damping ratio $\zeta = 0.707$.

   (d) use MATLAB to plot the response of the resulting design to a reference step.

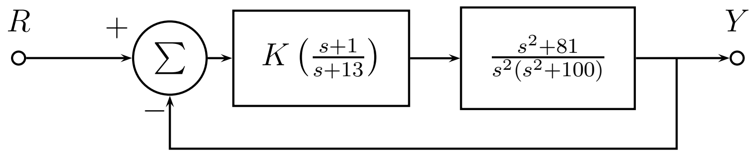


Figure 25: Problem 7

8. Suppose that in Fig. 26

$$G(s) = \frac{1}{s(s^2+2s+2)} \quad \text{and} \quad D(s) = \frac{K}{s+2}$$

Sketch the root-locus with respect to K of the characteristic equation for the closed-loop system, paying particular attention to points that generate multiple roots if $KL(s) = D(s)G(s)$.

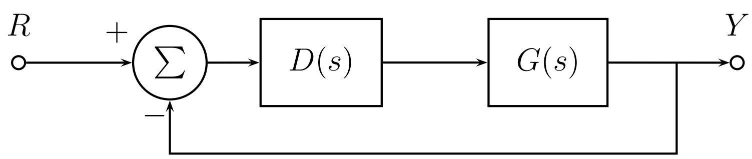


Figure 26: Problem 8

9. Assume that the unity feedback system of Fig. 26 has the open-loop plant

$$G(s) = \frac{1}{s(s+3)(s+6)}$$

Design a lag compensation to meet the following specifications:

- The step response settling time is to be less than 5 sec.
- The step response overshoot is to be less than 17%.
- The steady-state error to a unit ramp input must not exceed 10%.

10. Assume the closed-loop system of Fig. 26 has a feed-forward transfer function $G(s)$ given by

$$G(s) = \frac{1}{s+2}$$

Design a lag compensation so that the dominant poles of the closed-loop system are located at $s = -1 \pm j$ and the steady-state error to a unit ramp input is less than 0.2.

11. Consider the system in Fig. 27.

(a) Use Routh's criterion to determine the regions in the $(K_1,\ K_2)$ plane for which the system is stable.

(b) Use root-locus methods to verify your answer to part (a).

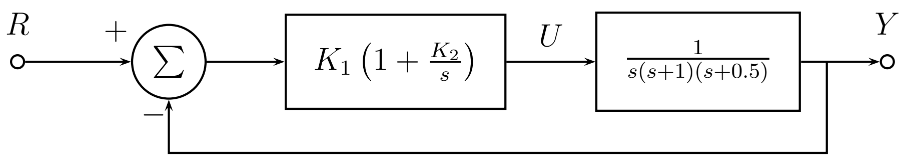


Figure 27: Problem 11

12. Consider the third-order system shown in Fig. 28.

(a) Sketch the root locus for this system with respect to K, showing your calculations for the asymptote angles, departure angles and so on.

(b) Using graphical techniques, locate carefully the point at which the locus crosses the imaginary axis. What is the value of K at that point?

(c) Assume that, due to some unknown mechanism, the amplifier output is given by the following saturation non linearity (instead of by a proportional gain K):

$$u = \begin{cases} e, & |e| \leq 1; \\ 1, & e > 1; \\ -1, & e < -1 \end{cases}$$

Qualitatively describe how you would expect the system to respond to a unit step input.

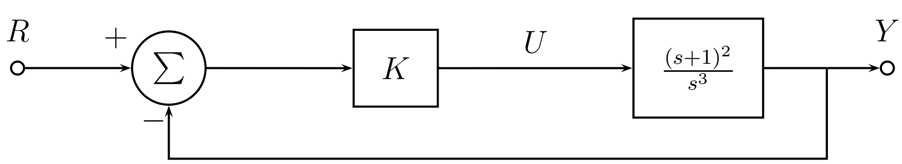


Figure 28: Problem 12

13. Sketch the root locus with respect to K for the system in Fig. 29. What is the range of values of K for which the system is unstable?

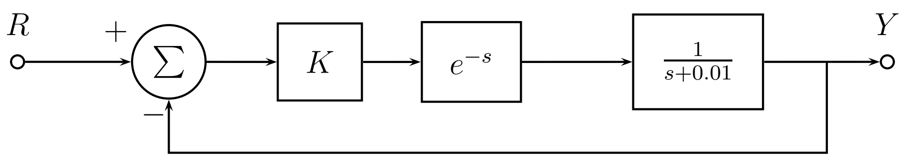


Figure 29: Problem 13