# MSO 201A: Probability and Statistics

## 2021 (2nd Semester)

## Assignment-IV

1. Let

$$F(x,y) = \begin{cases} 1, & \text{if } x+2y \geq 1 \\ 0, & \text{if } x+2y < 1 \end{cases}.$$

   Does $F(\cdot,\cdot)$ define a d.f.?

2. Let

$$F(x,y) = \begin{cases} 0, & \text{if } x < 0 \text{ or } x+y < 1 \text{ or } y < 0 \\ 1, & \text{otherwise} \end{cases}.$$

   Does $F(\cdot)$ define a d.f.?

3. Let $\underline{X} = (X_1, X_2)$ be a bivariate random vector having the d.f.

$$F(x,y) = \begin{cases} 0, & \text{if } x < 0 \text{ or } y < 0 \\ \frac{1+xy}{2}, & \text{if } 0 \leq x < 1, 0 \leq y < 1 \\ \frac{1+x}{2}, & \text{if } 0 \leq x < 1, y \geq 1 \\ \frac{1+y}{2}, & \text{if } x \geq 1, 0 \leq y < 1 \\ 1 & \text{if } x \geq 1, y \geq 1 \end{cases}.$$

   (a) Verify that $F$ is a d.f.; (b) Determine whether $\underline{X}$ is a discrete or a continuous random vector; (c) Find the marginal distribution functions of $X_1$ and $X_2$; (d) Find $P(\frac{1}{2} \leq X_1 \leq 1, \frac{1}{4} < X_2 < \frac{1}{2}), P(X_1 = 1)$ and $P(X_1 \geq \frac{3}{2}, X_2 < \frac{1}{4})$; (e) Are $X_1$ and $X_2$ independent?

4. Let $\underline{X} = (X_1, X_2)$ be a bivariate random vector having the d.f.

$$F(x,y) = \begin{cases} 0, & \text{if } x < 0 \text{ or } y < 1 \\ \frac{y^2-1}{6}, & \text{if } 0 \leq x < 1, 1 \leq y < 2 \\ \frac{1}{2}, & \text{if } 0 \leq x < 1, y \geq 2 \\ \frac{y^2-1}{3}, & \text{if } x \geq 1, 1 \leq y < 2 \\ 1 & \text{if } x \geq 1, y \geq 2 \end{cases}.$$

   (a) Verify that $F$ is a d.f.; (b) Determine whether $\underline{X}$ is a discrete or a continuous r.v.; (c) Find the marginal distribution functions of $X_1$ and $X_2$; (d) Find $P(\frac{1}{2} \leq X_1 \leq 1, \frac{5}{4} < X_2 < \frac{3}{2}), P(X_1 = 1)$ and $P(X_1 \geq \frac{3}{2}, X_2 < \frac{5}{4})$; (e) Are $X_1$ and $X_2$ independent?

5. Let the r.v. $\underline{X} = (X_1, X_2)'$ have the joint p.m.f.

$$f_{\underline{X}}(x_1, x_2) = \begin{cases} c(x_1 + 2x_2), & \text{if } x_1 = 1, 2,\ x_2 = 1, 2 \\ 0, & \text{otherwise} \end{cases},$$

where $c$ is a real constant. (a) Find the constant $c$; (b) Find marginal p.m.f.s of $X_1$ and $X_2$; (c) Find conditional variance of $X_2$ given $X_1 = x_1$, $x_1 = 1, 2$; (d) Find $P(X_1 < \frac{X_2}{3}), P(X_1 = X_2), P(X_1 \geq \frac{X_2}{2})$ and $P(X_1 + X_2 \leq 3)$; (e) Find $\rho(X_1, X_2)$; (f) Are $X_1$ and $X_2$ independent?

6. Let the r.v. $\underline{X} = (X_1, X_2)'$ have the joint p.m.f.

$$f_{\underline{X}}(x_1, x_2) = \begin{cases} cx_1x_2, & \text{if } x_1 = 1, 2,\ x_2 = 1, 2,\ x_1 \leq x_2 \\ 0, & \text{otherwise} \end{cases},$$

where $c$ is a real constant. (a) Find the constant $c$; (b) Find marginal p.m.f.s of $X_1$ and $X_2$; (c) Find conditional variance of $X_2$ given $X_1 = 1$; (d) Find $P(X_1 > X_2), P(X_1 = X_2), P(X_1 < \frac{2}{3}X_2)$ and $P(X_1 + X_2 \geq 3)$; (e) Find $\rho(X_1, X_2)$; (f) Are $X_1$ and $X_2$ independent?

7. Let $(X, Y)$ be a random vector such that the p.d.f. of $X$ is

$$f_X(x) = \begin{cases} 4x(1 - x^2), & \text{if } 0 < x < 1 \\ 0, & \text{otherwise} \end{cases},$$

and, for fixed $x \in (0, 1)$, the conditional p.d.f. of $Y$ given $X = x$ is

$$f_{Y|X}(y|x) = \begin{cases} c(x)y, & \text{if } x < y < 1 \\ 0, & \text{otherwise} \end{cases},$$

where $c : (0, 1) \to \mathbb{R}$ is a given function. (a) Determine $c(x), 0 < x < 1$; (b) Find marginal p.d.f. of $Y$; (c) Find the conditional variance of $X$ given $Y = y,\ y \in (0, 1)$; (d) Find $P(X < \frac{Y}{2}), P(X + Y \geq \frac{3}{4})$ and $P(X = 2Y)$; (e) Find $\rho(X, Y)$; (f) Are $X$ and $Y$ independent?

8. Let $\underline{X} = (X_1, X_2, X_3)$ be a random vector with joint p.d.f.

$$f_{\underline{X}}(\underline{x}) = \begin{cases} \frac{c}{x_1x_2}, & \text{if } 0 < x_3 < x_2 < x_1 < 1 \\ 0, & \text{otherwise} \end{cases},$$

where $c$ is a real constant. (a) Find the value of constant $c$; (b) Find marginal p.d.f. of $X_2$; (c) Find the conditional variance of $X_2$ given $(X_1, X_3) = (x, y), 0 < y <$

$x < 1$; (d) Find $P(X_2 < \frac{X_1}{2})$ and $P(X_3 = 2X_2 > \frac{X_1}{2})$; (e) Find $\rho(X_1, X_2)$; (f) Are $X_1, X_2, X_3$ independent?.

9. Let $\underline{X} = (X_1, X_2, X_3)'$ be a random vector with p.m.f.

$$f_{X_1,X_2,X_3}(x_1, x_2, x_3) = \begin{cases} \frac{1}{4}, & \text{if } (x_1, x_2, x_3) \in A \\ 0, & \text{otherwise} \end{cases},$$

where $A = \{(1,0,0), (0,1,0), (0,0,1), (1,1,1)\}$. (a) Are $X_1, X_2, X_3$ independent?; (b) Are $X_1, X_2, X_3$ pairwise independent?; (c) Are $X_1 + X_2$ and $X_3$ independent?

10. Let $\underline{X} = (X_1, X_2, X_3)'$ be a random vector with joint p.d.f.

$$f_{\underline{X}}(x_1, x_2, x_3) = \frac{1}{(2\pi)^{\frac{3}{2}}} e^{-\frac{1}{2}(x_1^2+x_2^2+x_3^2)} \left(1 + x_1 x_2 x_3 e^{-\frac{1}{2}(x_1^2+x_2^2+x_3^2)}\right), \quad -\infty < x_i < \infty,$$

$i = 1, 2, 3$. (a) Are $X_1, X_2, X_3$ independent?; (b) Are $X_1, X_2, X_3$ pairwise independent?; (c) Find the marginal p.d.f.s of $(X_1, X_2)$, $(X_1, X_3)$ and $(X_2, X_3)$.

11. Let $(X, Y, Z)$ have the joint p.m.f. as follows:

| $(x, y, z)$ | $(1,1,0)$ | $(1,2,1)$ | $(1,3,0)$ | $(2,1,1)$ | $(2,2,0)$ | $(2,3,1)$ |
|---|---|---|---|---|---|---|
| $f_{X,Y,Z}(x, y, z)$ | $\frac{2}{15}$ | $\frac{4}{15}$ | $\frac{3}{15}$ | $\frac{1}{15}$ | $\frac{1}{15}$ | $\frac{4}{15}$ |

and $f_{X,Y}(x, y) = 0$, elsewhere. (a) Are $X + Y$ and $Z$ independent?; (b) Find $\rho =$ Corr(X+Y,Z).

12. Let $\underline{X} = (X_1, X_2, X_3)'$ be a random vector with p.d.f.

$$f_{X_1,X_2,X_3}(x_1, x_2, x_3) = \begin{cases} 2e^{-(x_2+2x_3)}, & \text{if } 0 < x_1 < 1, x_2 > 0, x_3 > 0 \\ 0, & \text{otherwise} \end{cases}.$$

(a) Are $X_1, X_2, X_3$ independent?; (b) Are $X_1 + X_2$ and $X_3$ independent?; (c) Find marginal p.d.f.s of $X_1, X_2$ and $X_3$; (d) Find conditional p.d.f. of $X_1$ given $X_2 = 2$.

13. Let $X_1, \ldots, X_n$ be $n$ r.v.s with $E(X_i) = \mu_i$, $\text{Var}(X_i) = \sigma_i^2$ and $\rho_{ij} = \text{Corr}(X_i, X_j)$, $i, j = 1, \ldots, n$, $i \neq j$. For real numbers $a_i$, $b_i$, $i = 1, \ldots, n$, define $Y = \sum_{i=1}^n a_i X_i$ and $Z = \sum_{i=1}^n b_i X_i$. Find $\text{Cov}(Y, Z)$.

14. Let $X$ and $Y$ be jointly distributed random variables with $E(X) = E(Y) = 0$, $E(X^2) = E(Y^2) = 2$ and $\text{Corr}(X, Y) = 1/3$. Find $\text{Corr}(\frac{X}{3} + \frac{2Y}{3}, \frac{2X}{3} + \frac{Y}{3})$.

15. Let $X_1, \ldots, X_n$ be random variables and let $p_1, \ldots, p_n$ be positive real numbers with $\sum_{i=1}^n p_i = 1$. Prove that: (a) $\sqrt{\text{Var}(\sum_{i=1}^n p_i X_i)} \leq \sum_{i=1}^n p_i \sqrt{\text{Var}(X_i)} \leq \sqrt{\sum_{i=1}^n p_i \text{Var}(X_i)}$; (b) $\text{Var}(\frac{\sum_{i=1}^n X_i}{n}) \leq \frac{1}{n} \sum_{i=1}^n \text{Var}(X_i)$.

16. Let $(x_i, y_i) \in \mathbb{R}^2, i = 1, \ldots, n$ be such that $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i = 0$. Using a statistical argument show that

$$\left(\sum_{i=1}^n x_i y_i\right)^2 \leq \left(\sum_{i=1}^n x_i^2\right)\left(\sum_{i=1}^n y_i^2\right).$$

17. Let $(X, Y)$ have the joint p.m.f. as follows:

| $(x,y)$ | $(1,1)$ | $(1,2)$ | $(1,3)$ | $(2,1)$ | $(2,2)$ | $(2,3)$ |
|---|---|---|---|---|---|---|
| $f_{X,Y}(x,y)$ | $\frac{2}{15}$ | $\frac{4}{15}$ | $\frac{3}{15}$ | $\frac{1}{15}$ | $\frac{1}{15}$ | $\frac{4}{15}$ |

and $f_{X,Y}(x, y) = 0$, elsewhere. Find $\rho = \text{Corr(X,Y)}$.

18. Let the joint m.g.f. of $(Y, Z)$ be $M_{Y,Z}(t_1, t_2) = \frac{e^{\frac{t_1^2}{1-2t_2}}}{1-2t_2}$, $t_2 < \frac{1}{2}$. (a) Find $\text{Corr}(Y, Z)$; (b) Are $Y$ and $Z$ independent?; (c) Find m.g.f. of $Y + Z$.

19. Let the joint m.g.f. of $(Y, Z)$ be $M_{Y,Z}(t_1, t_2) = e^{\frac{t_1^2+t_2^2+t_1t_2}{2}}$, $(t_1, t_2) \in \mathbb{R}^2$. (a) Find $\text{Corr}(Y, Z)$; (b) Are $Y$ and $Z$ independent?; (c) Find m.g.f. of $Y - Z$.

20. Let $\underline{X} = (X_1, X_2)$ have the joint p.m.f.

$$f_{\underline{X}}(x_1, x_2) = \begin{cases} (\frac{2}{3})^{x_1+x_2}(\frac{1}{3})^{2-x_1-x_2}, & \text{if } (x_1, x_2) = (0,0), (0,1), (1,0), (1,1) \\ 0, & \text{otherwise} \end{cases}.$$

(a) Find the joint p.m.f. of $Y_1 = X_1 - X_2$ and $Y_2 = X_1 + X_2$; (b) Find the marginal p.m.f.s of $Y_1$ and $Y_2$; (c) Find $\text{Var}(Y_2)$ and $\text{Cov}(Y_1, Y_2)$; (d) Are $Y_1$ and $Y_2$ independent?

21. Let $X_1, \ldots, X_n$ be a random sample of continuous random variables and let $X_{1:n} < X_{2:n} \cdots < X_{n:n}$ be the corresponding order statistics. If the expectation of $X_1$ is finite and the distribution of $X_1$ is symmetric about $\mu \in (-\infty, \infty)$, show that: (a) $X_{r:n} - \mu \stackrel{d}{=} \mu - X_{n-r+1:n}$, $r = 1, \ldots, n$; (b) $E(X_{r:n} + X_{n-r+1:n}) = 2\mu$; (c) $E(X_{\frac{n+1}{2}:n}) = \mu$, if $n$ is odd; (d) $P(X_{\frac{n+1}{2}:n} > \mu) = 0.5$, if $n$ is odd.

22. (a) Let $X_1, \ldots, X_n$ denote a random sample, where $P(X_1 > 0) = 1$. Show that

$$E\left(\frac{X_1 + X_2 + \cdots + X_k}{X_1 + X_2 + \cdots + X_n}\right) = \frac{k}{n}, \; k = 1, 2, \ldots, n.$$

(b) Let $X_1, \ldots, X_n$ be a random sample and let $E(X_1)$ be finite. Find the conditional expectation $E(X_1 | X_1 + \cdots + X_n = t)$, where $t \in \mathbb{R}$ is such that the conditional expectation is defined.

(c) Let $X_1, \ldots, X_n$ be a random sample of random variables. Find $P(X_1 < X_2 < \cdots < X_r),\ r = 2, 3, \ldots, n$.

23. Let $X_1$ and $X_2$ be independent and identically distributed random variables with common p.m.f.

$$f(x) = \begin{cases} \theta(1-\theta)^{x-1}, & \text{if } x = 1, 2, \ldots \\ 0, & \text{otherwise} \end{cases},$$

where $\theta \in (0,1)$. Let $Y_1 = \min\{X_1, X_2\}$ and $Y_2 = \max\{X_1, X_2\} - \min\{X_1, X_2\}$. (a) Find the marginal p.m.f. of $Y_1$ without finding the joint p.m.f. of $\underline{Y} = (Y_1, Y_2)$; (b) Find the marginal p.m.f. of $Y_2$ without finding the joint p.m.f. of $\underline{Y} = (Y_1, Y_2)$; (c) Find the joint p.m.f. of $\underline{Y} = (Y_1, Y_2)$; (d) Are $Y_1$ and $Y_2$ independent; (e) Using (c), find the marginal p.m.f.s of $Y_1$ and $Y_2$.

24. Let $\underline{X} = (X_1, X_2, X_3)'$ be a random vector with p.m.f.

$$f_{X_1,X_2,X_3}(x_1, x_2, x_3) = \begin{cases} \frac{2}{9}, & \text{if } (x_1, x_2, x_3) = (1,1,0), (1,0,1), (0,1,1) \\ \frac{1}{3}, & \text{if } (x_1, x_2, x_3) = (1,1,1) \\ 0, & \text{otherwise} \end{cases}.$$

Define $Y_1 = X_1 + X_2$ and $Y_2 = X_2 + X_3$. (a) Find the marginal p.m.f. of $Y_1$ without finding the joint p.m.f. of $\underline{Y} = (Y_1, Y_2)$; (b) Find the marginal p.m.f. of $Y_2$ without finding the joint p.m.f. of $\underline{Y} = (Y_1, Y_2)$; (c) Find the joint p.m.f. of $\underline{Y} = (Y_1, Y_2)$; (d) Are $Y_1$ and $Y_2$ independent; (e) Using (c), find the marginal p.m.f.s of $Y_1$ and $Y_2$.

25. Let $X_1$ and $X_2$ be independent random variables with p.d.f.s

$$f_1(x) = \begin{cases} 1, & \text{if } 0 < x < 1 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad f_2(x) = \begin{cases} 1, & \text{if } 1 < x < 2 \\ 0, & \text{otherwise} \end{cases},$$

respectively. Let $Y = X_1 + X_2$ and $Z = X_1 - X_2$. (a) Find the d.f. of $Y$ and hence find its p.d.f.; (b) Find the joint p.d.f. of $(Y, Z)$ and hence find the marginal p.d.f.s of $Y$ and $Z$; (c) Are $Y$ and $Z$ independent?

26. Let $X_1$ and $X_2$ be i.i.d. random variables with common p.d.f.

$$f(x) = \begin{cases} \frac{1}{2}, & \text{if } -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}.$$

Let $Y = |X_1| + X_2$ and $Z = X_2$. (a) Find the d.f. of $Y$ and hence find its p.d.f.; (b) Find the joint p.d.f. of $(Y, Z)$ and hence find the marginal p.d.f.s of $Y$ and $Z$; (c) Are $Y$ and $Z$ independent?