EE 250: Control Systems Analysis
Module III: s-plane analysis
Lecture 18: Routh Array

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5 Stability of Linear Time Invariant Systems

$p_i = \sigma + j\omega$

$= \sigma + j\omega$

$$
\begin{aligned}
H(s) &= \frac{Y(s)}{R(s)} \\
&= \frac{b_o s^m + b_1 s^{m-1} + \ldots + b_m}{s^n + a_1 s^{n-1} + \ldots + a_n}, \quad m \le n & (6) \\
&= \frac{k \prod_{i=1}^{m}(s - z_i)}{\prod_{j=1}^{n}(s - p_j)} = \sum_i \frac{K_i}{s - p_i} & (7) \\
\text{or, } h(t) &= \sum_i K_i e^{p_i t} & (8)
\end{aligned}
$$

$Re(p_i) < 0$

The system is **stable** if and only if every term in (8) go to zero as $t \to \infty$, i.e.,

$$e^{p_i t} \to 0 \quad \forall p_i \quad \text{as} \quad t \to \infty$$

This will happen if all the poles of the system are strictly in the left half of the s-plane, i.e., $Re\{p_i\} < 0$. This is called *internal stability*. The system is **unstable** if there exists a pole for which $Re\{p_i\} > 0$.

Neutral/Marginal stability:

The system is said to be marginally stable if it has *non-repeated* roots on $j\omega$ axis. This implies that

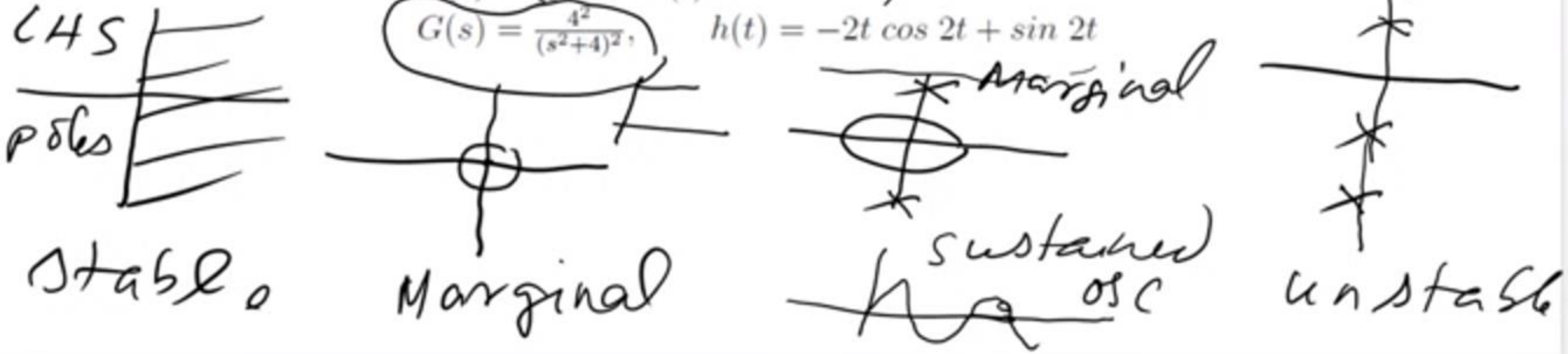
(i) A pole at origin will result in a non-decaying transient, i.e., $\mathscr{L}^{-1}[\frac{1}{s}] = 1$.

(ii) A pair of conjugate pole $\pm j\omega$ will result in oscillatory response with constant magnitude.

However, repeated poles on $j\omega$ axis makes the system unstable. Examples of such unstable systems are

$$G(s) = \frac{1}{s^2}, \quad h(t) = t$$

$$G(s) = \frac{4^2}{(s^2+4)^2}, \quad h(t) = -2t\cos 2t + \sin 2t$$



5.1 Routh's Stability Criterion

The *characteristic polynomial* of a transfer function is represented as

$\frac{Y(s)}{R(s)} = \frac{n(s)}{d(s)}$ where

$$d(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_{n-1} s + a_n \qquad (9)$$

For stability, it is desired that all the roots of $d(s)$ have negative real parts. Without explicitly computing roots of polynomial (9), can we comment about the nature of roots? Routh's stability criterion helps us to do that. This criterion consists of following two conjectures:

1. The **necessary** (but not sufficient) condition for stability is that all the coefficients of the characteristic polynomial $d(s)$ be positive. $a_1, a_2, \ldots, a_n > 0$
2. Sufficient condition for stability:A system is stable if and only if all the elements in the first column of the Routh array are positive.

Routh Array

$$d(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots + a_{n-1} s + a_n$$

For a system of order 'n', the Routh array has $n+1$ rows, where

s^n	1	a_2	a_4	$\dots$
s^{n-1}	a_1	a_3	a_5	$\dots$
s^{n-2}	b_1	b_2	b_3	$\dots$
s^{n-3}	c_1	c_2	c_3	$\dots$
$\vdots$	$d_1,$	d_2	d_3	
s^0	z_1			

$n+1$ rows

$$-\frac{\begin{vmatrix} 1 & a_4 \\ a_1 & a_5 \end{vmatrix}}{a_1}$$

$$a_5 = \frac{a_1 a_6 - a_7}{a_1}$$

$$b_1 = -\frac{\begin{vmatrix} 1 & a_2 \\ a_1 & a_3 \end{vmatrix}}{a_1} = \frac{a_1 a_2 - a_3}{a_1}$$

$$b_2 = -\frac{\begin{vmatrix} 1 & a_4 \\ a_1 & a_5 \end{vmatrix}}{a_1} = \frac{a_1 a_4 - a_5}{a_1}$$

$$\vdots$$

$$c_1 = -\frac{\begin{vmatrix} a_1 & a_3 \\ b_1 & b_2 \end{vmatrix}}{b_1} = \frac{a_3 b_1 - a_1 b_2}{b_1}$$

$$c_2 = -\frac{\begin{vmatrix} a_1 & a_5 \\ b_1 & b_3 \end{vmatrix}}{b_1} = \frac{a_5 b_1 - a_1 b_3}{b_1}$$

$$\vdots$$

Example 6 (Stability). The characteristic polynomial is given as

$$d(s) = s^6 + 4s^5 + 3s^4 + 2s^3 + s^2 + 4s + 4$$

Constructing the Routh table, we have

s^6	1	3	1	4
s^5	4	2	4	0
s^4	2.5	0	4	0
s^3	2	-12/5	0	0
s^2	3	4	0	0
s^1	-76/15	0	0	0
s^0	4	0	0	0

$\frac{12-2}{4} = 2.5$

There are 2 poles in RHS

Since, the elements in the first column corresponding to row s^1 is negative, the system is unstable. The numbe of roots in the right half of the s-plane are given by the number of sign changes in the first column. In this example, the number of roots in the right half s-plane is 2.