EE 200: Solution to Problem Set 2

1. Define the energy and average power of the following analog signals:

   (a) $y_1(t) = \begin{cases} 0.5, & -1 \le t \le 1 \\ 0, & \text{otherwise} \end{cases}$

   (b) $y_2(t) = \begin{cases} t, & t \ge 0 \\ 0, & \text{otherwise} \end{cases}$

   (c) $y_3(t) = e^{-2t}u(t)$

   (d) $y_4(t) = 3\sin(0.5\pi t + 0.4)$

   Solution 1:
   (a): $E_1 = \int_{-\infty}^{\infty} |y_1(t)|^2 dt = \frac{1}{4}\int_{-1}^{1} dt = \frac{1}{4} \times 2 = \frac{1}{2}$
   (b):

$$
\begin{aligned}
E_2 &= \lim_{T\to\infty} \int_{-T/2}^{T/2} |y_2(t)|^2 dt \\
&= \lim_{T\to\infty} \int_{0}^{T/2} t^2 dt = \lim_{T\to\infty} \frac{(T/2)^3}{3} = \infty
\end{aligned}
$$

$$
\begin{aligned}
P_2 &= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |y_2(t)|^2 dt \\
&= \lim_{T\to\infty} \frac{1}{T} \int_{0}^{T/2} t^2 dt = \lim_{T\to\infty} \frac{1}{T}\frac{(T/2)^3}{3} = \infty
\end{aligned}
$$

   Both the energy and average power of $y_2(t)$ are infinite.

(c):

$$\begin{aligned} E_3 &= \int_{-\infty}^{\infty} |y_3(t)|^2 dt \\ &= \int_0^{\infty} e^{-4t} dt = \frac{e^{-4t}}{-4}\bigg|_0^{\infty} = \frac{1}{4} = 0.25 \end{aligned}$$

(d): y_4 is periodic sinusoidal signal with a fundamental period $T_4 = 2\pi/0.5\pi = 4$

$$\begin{aligned} P_4 &= \frac{1}{T_4}\int_0^{T_4} [y_4(t)]^2 dt \\ &= \frac{1}{4}\int_0^4 9\sin^2(0.5\pi t + 0.4) dt \\ &= \frac{9}{4}\int_0^4 \frac{1}{2}[1 - \cos(\pi t + 0.8)] dt \\ &= \frac{9}{4} \times \frac{1}{2}[t - \frac{1}{\pi}\sin(\pi t + 0.8)]_0^4 \\ &= \frac{9}{2} \end{aligned}$$

2. Determine the average power of the analog signal:
$x(t) = A_1 \sin(\Omega_1 t) + A_2 \sin(\Omega_2 t), \Omega_1 \neq \Omega_2$
Solution 2:

$$\begin{aligned} P_x &= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x^2(\tau) d\tau \\ &= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} [A_1 \sin(\Omega_1 \tau) + A_2 \sin(\Omega_2 \tau)]^2 d\tau \\ &= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} A_1^2 \sin^2(\Omega_1 \tau) d\tau + \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} A_2^2 \sin^2(\Omega_2 \tau) d\tau \\ &\quad + \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} 2A_1 A_2 \sin(\Omega_1 \tau)\sin(\Omega_2 \tau) d\tau \end{aligned}$$

Now,

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\sin^2(\Omega\tau)d\tau=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\frac{1}{2}[1-\cos(2\Omega\tau)]d\tau=\frac{1}{2}$$

Hence,

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}A_1^2\sin^2(\Omega_1\tau)d\tau=\frac{A_1^2}{2}$$
$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}A_2^2\sin^2(\Omega_2\tau)d\tau=\frac{A_2^2}{2}$$

Next, we note

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}2\sin(\Omega_1\tau)\sin(\Omega_2\tau)d\tau$$
$$=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\cos((\Omega_1-\Omega_2)\tau)d\tau-\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\cos((\Omega_1+\Omega_2)\tau)d\tau$$
$$=0\ \ (\text{because}\Omega_1\neq\Omega_2)$$

Therefore, $P_x=\frac{A_1^2}{2}+\frac{A_2^2}{2}$

3. Show that
   (a) $\int_{-\infty}^{\infty}(t-T)\delta(t-T)dt=0$
   (b) $\int_{-\infty}^{\infty}\cos(t)\delta(t+\pi)dt=-\int_{-\infty}^{\infty}\delta(t+\pi)dt$
   (c) $\int_{-\infty}^{\infty}\cos(t)\delta(t+\pi/2)dt=0$

   Solution 3:
   We use the property of the unit impulse function:

$$\int_{-\infty}^{\infty}f(t)\delta(t-t_0)dt=\int_{-\infty}^{\infty}f(t_0)\delta(t-t_0)dt=f(t_0)$$

(a):

$$\int_{-\infty}^{\infty}(t-T)\delta(t-T)dt=(t-T)|_{t=T}=0$$

(b):

$$\int_{-\infty}^{\infty}\cos(t)\delta(t+\pi)dt=\int_{-\infty}^{\infty}\cos(-\pi)\delta(t+\pi)dt=-\int_{-\infty}^{\infty}\delta(t+\pi)dt$$

(c):

$$\int_{-\infty}^{\infty} \cos(t)\delta(t+\pi/2)dt = \int_{-\infty}^{\infty} \cos(-\pi/2)\delta(t+\pi/2)dt = 0$$

4. Evaluate the following definite integrals:
   (a) $\int_{-\infty}^{t} \sin(\tau)\delta(\tau)d\tau$
   (b) $\int_{-\infty}^{t} \sin(\tau)\mu(\tau)d\tau$
   (c) $\int_{-\infty}^{\infty} \sin(\tau)\delta(\tau)\mu(\tau-2)d\tau$
   (d) $\int_{-\infty}^{\infty} \tau\cos(\tau/2)\delta(\tau-\pi)d\tau$

   Solution 4:
   (a): For $t < 0$,

$$\int_{-\infty}^{t} \sin(\tau)\delta(\tau)d\tau = 0$$

   as $\delta(t) = 0$.
   For $t > 0$,

$$\int_{-\infty}^{t} \sin(\tau)\delta(\tau)d\tau = \int_{-\infty}^{t} \sin(0)\delta(\tau)d\tau = 0.$$

   b):

$$\int_{-\infty}^{t} \sin(\tau)\mu(\tau)d\tau = \int_{0}^{t} \sin(\tau)d\tau = -\cos(\tau)\bigg|_{0}^{t} = 1-\cos(t)$$

   (c):

$$\int_{-\infty}^{\infty} \sin(\tau)\delta(\tau)\mu(\tau-2)d\tau = \sin(0)\mu(-2) = 0.$$

   (d):

$$\int_{-\infty}^{\infty} \tau\cos(\tau/2)\delta(\tau-\pi)d\tau = \pi\cos(\pi/2) = 0.$$

5. Develop a differential equation representation relating the analog signals $y(t)$ and $x(t)$ of the following equation:

$$3y(t) = 2\int_{-\infty}^{t} x(\tau)d\tau - 5x(t) + 9\int_{-\infty}^{t} y(\tau)d\tau$$

Solution 5: Differentiating both sides of the equation, we get

$$3\frac{dy(t)}{dt} = 2x(t) - 5\frac{dx(t)}{dt} + 9y(t)$$

whihc can be written as

$$3\frac{dy(t)}{dt} - 9y(t) = -5\frac{dx(t)}{dt} + 2x(t)$$

6. Develop a differential equation representation relating the analog signals $y(t)$ and $x(t)$ of the following two equations:

$$2w(t) = 8x(t) - 7\int_{-\infty}^{t} x(\tau)d\tau + 3\int_{-\infty}^{t}\left[\int_{-\infty}^{\tau} x(\xi)d\xi\right]d\tau,$$

$$5y(t) = 4w(t) + 6\int_{-\infty}^{t} y(\tau)d\tau - 10\int_{-\infty}^{t}\left[\int_{-\infty}^{\tau} y(\xi)d\xi\right]d\tau,$$

Solution 6: Differentiating both sides of the two equations twice, we get

$$2\frac{d^2w(t)}{dt^2} = 8\frac{d^2x(t)}{dt^2} - 7\frac{dx(t)}{dt} + 3x(t)$$

$$5\frac{d^2y(t)}{dt^2} = 4\frac{d^2w(t)}{dt^2} + 6\frac{dy(t)}{dt} - 10y(t)$$

Substituting first equation into second, we get

$$5\frac{d^2y(t)}{dt^2} - 6\frac{dy(t)}{dt} + 10y(t) = 16\frac{d^2x(t)}{dt^2} - 14\frac{dx(t)}{dt} + 6x(t)$$

7. Write a MATLAB program to generate and plot the sinusoidal signal

$$\tilde{y}(t) = 7.5\cos(0.6\pi t + \frac{\pi}{3})$$

Solution 7: MATLAB Code:
$t = -5 : 0.01 : 10$; % Time range
$y = 7.5 * \cos(0.6 * pi * t + pi/3)$ % Generate the signal
plot(t, y) % Plot the signal
xlabel('Time t');
ylabel('Amplitude');

8. Write a MATLAB program to generate and plot the exponential signals of slide 31, Ch3-1 (Notes).

Solution 8: MATLAB COde:

$t = 0 : 0.1 : 10;$ %Time range
$y1 = \exp(-0.1 * pi * t);$ % Generate the signal for $alpha < 0$
$y2 = \exp(0.1 * pi * t);$ % Generate the signal for $alpha > 0$
subplot$(2, 2, 1)$
plot$(t, y1)$ %Plot the signal for $alpha < 0$
xlabel('Time t');
ylabel('Amplitude');
title(' \ alpha < 0');
subplot$(2, 2, 2)$
plot$(t, y2)$ %Plot the signal for $alpha > 0$
xlabel('Time t');
ylabel('Amplitude');
title(' \ alpha > 0');