MSO 201a: Probability and Statistics
Assignment No. 6
Model Solutions

Problem No. 1

$$F_n(x) = \begin{cases} 0, & x < -n \\ \frac{x+n}{2n}, & -n \le x < n \\ 1, & x \ge n \end{cases} \xrightarrow{n\to\infty} F(x) = \frac{1}{2}, \ \forall x \in \mathbb{R}$$

$\hookrightarrow$ not a d.f.

Problem No. 2

$$F_{Y_n}(t) = P(X_{1:n} \le \tfrac{t}{n}) = 1 - P(X_{1:n} > \tfrac{t}{n}) = 1 - P(X_i > \tfrac{t}{n}, i=1,\dots,n)$$
$$= 1 - \prod_{i=1}^{n} P(X_i > \tfrac{t}{n}) = 1 - [1 - F(\tfrac{t}{n})]^n, \quad t \in \mathbb{R}$$

$(\min\{X_1,\dots,X_n\} > y \Leftrightarrow X_i > y, \ \forall i=1,\dots,n)$

(a) $X_1 \sim U(0,\theta) \Rightarrow F(t) = \begin{cases} 0, & t<0 \\ t/\theta, & 0 \le t < \theta \\ 1, & t \ge \theta \end{cases}$; $F_{Y_n}(t) = \begin{cases} 1-(1-0)^n, & t<0 \\ 1-(1-\frac{t}{n\theta})^n, & 0 \le t < n\theta \\ 1-(1-1)^n, & t \ge n\theta \end{cases}$

$$\xrightarrow{n\to\infty} F(t) = \begin{cases} 0, & t<0 \\ 1-e^{-\frac{t}{\theta}}, & t \ge 0 \end{cases}$$

$\downarrow$ d.f. of Exp(θ)

Thus

$$Y_n \xrightarrow{d} Y \sim \text{Exp}(\theta)$$

$$X_{1:n} = \frac{1}{n} \times Y_n \xrightarrow{p} 0 \times Y = 0 \quad (\tfrac{1}{n} \to 0 \text{ as } n\to\infty)$$

(b) $X_1 \sim \text{Exp}(\theta) \Rightarrow F(t) = \begin{cases} 0, & t<0 \\ 1-e^{-t/\theta}, & t \ge 0 \end{cases}$

$$F_{Y_n}(t) = \begin{cases} 1-(1-0)^n, & t<0 \\ 1-(e^{-\frac{t}{n\theta}})^n, & t \ge 0 \end{cases} = \begin{cases} 0, & t<0 \\ 1-e^{-\frac{t}{\theta}}, & t\ge 0 \end{cases} = F(t)$$

$\downarrow$ does not depend on n $\quad\quad \downarrow$ d.f. of Exp(θ)

$$\Rightarrow Y_n \xrightarrow{d} Y \sim \text{Exp}(\theta)$$

$$\Rightarrow X_{1:n} = \frac{1}{n} \times Y_n \xrightarrow{p} 0 \times Y = 0.$$

Problem No. 3 $E(\bar{X}_n)=0$, $Var(\bar{X}_n)=\frac{1}{n^2}\sum_{i=1}^{n} Var(X_i)=\frac{1}{n^2}\sum_{i=1}^{n} i^{1/2}$

$Var(\bar{X}_n)=\frac{1}{\sqrt{n}}\times\left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^{1/2}\right)$. But $\frac{1}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^{1/2}\to\int_0^1 t^{1/2}dt=\frac{2}{3}$

$\Rightarrow Var(\bar{X}_n)\to 0$. Thus $\bar{X}_n\xrightarrow{p}0$ $(E(\bar{X}_n)=0,\ Var(\bar{X}_n)\to 0)$.

Problem No. 4 (a) On contrary suppose that $a\neq b$. Let $|a-b|=\varepsilon$, where $\varepsilon>0$.

Then

$$P(|X_n-a|>\tfrac{\varepsilon}{2})=P(|b-a-(b-X_n)|>\tfrac{\varepsilon}{2})$$

$$\geq P(|b-a|-|b-X_n|>\tfrac{\varepsilon}{2})\quad\left(\text{since } |s|-|t|>\tfrac{\varepsilon}{2}\Rightarrow|s-t|>\tfrac{\varepsilon}{2}\right)$$

$$=P(|X_n-b|<\tfrac{\varepsilon}{2})\to 1\text{ as } n\to\infty\quad(X_n\xrightarrow{p}b)$$

$\Rightarrow X_n\overset{p}{\nrightarrow}a$.

(b) Fix $\varepsilon>0$. Then, by Markov's inequality

$$0\leq P(|X_n-a|>\varepsilon)\leq\frac{E(|X_n-a|^r)}{\varepsilon^r}\to 0\text{ as } n\to\infty$$

$\Rightarrow \lim_{n\to\infty}P(|X_n-a|>\varepsilon)=0 \Rightarrow X_n\xrightarrow{p}a$

Problem No. 5 (a) (For the continuous case. For the discrete case proof is similar with integral replaced by summation)

$$E\left(\frac{|X|^r}{1+|X|^r}\right)=\int_{|x|\leq t}\frac{|x|^r}{1+|x|^r}f(x)dx+\int_{|x|\geq t}\frac{|x|^r}{1+|x|^r}f(x)dx\geq 0+\int_{|x|\geq t}\frac{t^r}{1+t^r}f(x)dx$$

(for $x>0$, $\psi(x)=\frac{x^r}{1+x^r}\uparrow$ in x)

$$=\frac{t^r}{1+t^r}P(|X|\geq t).\quad\cdots\quad(\mathrm{I})$$

Also

$$E\left(\frac{|X|^r}{1+|X|^r}\right)=\int_{|x|\leq t}\frac{|x|^r}{1+|x|^r}f(x)dx+\int_{|x|\geq t}\frac{|x|^r}{1+|x|^r}f(x)dx$$

$\left(\frac{|x|^r}{1+|x|^r}\leq 1\ \forall x,\ \psi(x)=\frac{x^r}{1+x^r}\uparrow\text{ in } x\right)$

$$\leq\int_{|x|\leq t}\frac{t^r}{1+t^r}f(x)dx+\int_{|x|\geq t}f(x)dx$$

$$=\frac{t^r}{1+t^r}P(|X|\leq t)+P(|X|\geq t)$$

$$\leq\frac{t^r}{1+t^r}+P(|X|\geq t)\quad\cdots\quad(\mathrm{II})$$

Combining (I) and (II) we get the result.

(b) First suppose that $\lim_{n\to\infty} E\left(\frac{|X_n|^r}{1+|X_n|^r}\right)=0$. Using (a) for any $\varepsilon>0$,

$$0 \le P(|X_n|\ge\varepsilon) \le \frac{1+\varepsilon^r}{\varepsilon^r} E\left(\frac{|X_n|^r}{1+|X_n|^r}\right) \to 0$$

$$\Rightarrow \lim_{n\to\infty} P(|X_n|\ge\varepsilon)=0, \quad \forall\, \varepsilon>0 \quad \Rightarrow \quad X_n \xrightarrow{P} 0.$$

Conversely suppose that $X_n \xrightarrow{P} 0$. Fix $\varepsilon>0$. Then by (a)

$$0 \le E\left(\frac{|X_n|^r}{1+|X_n|^r}\right) \le P(|X_n|\ge\varepsilon) + \frac{\varepsilon^r}{1+\varepsilon^r} \to \frac{\varepsilon^r}{1+\varepsilon^r}$$

$$\Rightarrow \quad 0 \le \lim_{n\to\infty} E\left(\frac{|X_n|^r}{1+|X_n|^r}\right) \le \frac{\varepsilon^r}{1+\varepsilon^r}, \quad \forall\, \varepsilon>0.$$

Since $\varepsilon>0$ is arbitrary, we get

$$\lim_{n\to\infty} E\left(\frac{|X_n|^r}{1+|X_n|^r}\right)=0.$$

$\boxed{\text{Problem No. 6}}$ (a) Fix $\varepsilon>0$. Then

$$P(|a_nX_n|>\varepsilon) = P\left(|X_n|>\frac{\varepsilon}{|a_n|}\right) = P\left(|X_1|>\frac{\varepsilon}{|a_n|}\right) = 1-\left[F\left(\frac{\varepsilon}{|a_n|}\right)-F\left(-\frac{\varepsilon}{|a_n|}-\right)\right]$$

$$\le 1-F\left(\frac{\varepsilon}{|a_n|}\right)-F\left(-\frac{\varepsilon}{|a_n|}\right) \to 1-F(\infty)-F(-\infty)=0$$

$\Rightarrow a_nX_n \xrightarrow{P} 0$ (Alt. $a_nX_n \overset{d}{=} a_nX_1 \xrightarrow{P} 0\times X_1=0$)

(b) Fix $\varepsilon>0$. Then

$$0 \le P(|X_n-a|\ge\varepsilon) = P(X_n\le a-\varepsilon)+P(X_n\ge a+\varepsilon)$$

$$\le P(Y_n\le a-\varepsilon)+P(Z_n\ge a+\varepsilon) \quad \left(\begin{array}{l} X_n\le a-\varepsilon \Rightarrow Y_n\le a-\varepsilon \\ X_n\ge a+\varepsilon \Rightarrow Z_n\ge a+\varepsilon \end{array}\right)$$

$$\le P(|Y_n-a|\ge\varepsilon)+P(|Z_n-a|\ge\varepsilon)$$

$$\to 0+0=0, \quad \text{as } n\to\infty$$

$$\Rightarrow \lim_{n\to\infty} P(|X_n-a|\ge\varepsilon)=0, \quad \forall\,\varepsilon>0 \quad \Rightarrow \quad X_n \xrightarrow{P} a.$$

(c) See Lecture notes. Alternatively: Fix $\varepsilon>0$. Then $\exists\, n_0\equiv n_0(\varepsilon)$ s.t. $|a_n-a|<\frac{\varepsilon}{2}$, $\forall\, n\ge n_0$

$$0 \le P[|X_n + a_n - (c+a)| \ge \varepsilon) \le P(|X_n - c| + |a_n - c| \ge \varepsilon)$$
$$\le P(|X_n - c| \ge \tfrac{\varepsilon}{2} \text{ or } |a_n - c| \ge \varepsilon)$$
$$= P(|X_n - c| \ge \varepsilon/2), \quad \forall n \ge n_0$$
$$\to 0, \text{ as } n \to \infty \quad (X_n \xrightarrow{P} c)$$

$$\Rightarrow \lim_{n\to\infty} P(|X_n + a_n - (c+a)| \ge \varepsilon) = 0 \quad \Rightarrow \quad X_n + a_n \xrightarrow{P} c+a$$

Also for any fixed $\varepsilon > 0$, $\exists\, n_0 = n_0(\varepsilon)$ s.t. $|c|\,|a_n - a| < \frac{\varepsilon}{2}$, $\forall n \ge n_0$
$(c(a_n - a) \to 0)$

$$\Rightarrow 0 \le P(|a_n X_n - ac| \ge \varepsilon) = P(|a_n X_n - a_n c + a_n c - ac| \ge \varepsilon)$$
$$\le P(|a_n|\,|X_n - c| + |c|\,|a_n - a| \ge \varepsilon)$$
$$\le P(|a_n|\,|X_n - c| \ge \varepsilon/2 \text{ or } |c|\,|a_n - a| \ge \tfrac{\varepsilon}{2})$$
$$= P(|a_n|\,|X_n - c| \ge \tfrac{\varepsilon}{2}), \quad \forall n \ge n_0$$

For $\varepsilon > 0$ $a_n \to a \Rightarrow \exists\, n_1 = n_1(\varepsilon)$ s.t. $|a_n - a| < \varepsilon$, $\forall n \ge n_1$
$\Rightarrow |a_n| < |a| + \varepsilon$, $\forall n \ge n_1$

Thus

$$0 \le P(|a_n X_n - ac| \ge \varepsilon) \le P(|a_n|\,|X_n - c| \ge \tfrac{\varepsilon}{2})$$
$$\le P((|a| + \varepsilon)|X_n - c| \ge \tfrac{\varepsilon}{2})$$
$$= P\left(|X_n - c| \ge \frac{\varepsilon}{2(|a| + \varepsilon)}\right) \to 0, \text{ as } n \to \infty$$

$$\Rightarrow \lim_{n\to\infty} P(|a_n X_n - ac| \ge \varepsilon) = 0, \ \forall \varepsilon > 0 \quad \Rightarrow \quad a_n X_n \xrightarrow{P} ac$$

(d) First suppose that $Y_n \xrightarrow{P} 0$. Then $|Y_n| \xrightarrow{P} 0$. And

$$0 \le X_n \le |Y_n| \xrightarrow{P} 0.$$

$$\Rightarrow X_n \xrightarrow{P} 0 \quad (\text{By (b)}).$$

Conversely suppose $X_n \xrightarrow{P} 0$. Then

$$P(|Y_n| > \varepsilon) = \begin{cases} P(|X_n| > \varepsilon), & \text{if } a > \varepsilon \\ 0, & \text{if } a \le \varepsilon \end{cases} \longrightarrow 0 \text{ as } n \to \infty$$

$$\Rightarrow Y_n \xrightarrow{P} 0$$

Problem No. 7 (a) $E\left(\frac{2}{n(n+1)}\sum_{i=1}^{n} i x_i\right) = \frac{2}{n(n+1)} \times \frac{n(n+1)}{2}\times\mu = \mu$

$$Var\left(\frac{2}{n(n+1)}\sum_{i=1}^{n} i x_i\right) = \frac{4}{n^2(n+1)^2}\sum_{i=1}^{n} i^2\, Var(x_i) = \frac{4\sigma^2}{n^2(n+1)^2} \times \frac{n(n+1)(2n+1)}{6}$$

$\to 0$ as $n\to\infty$.

$$\Rightarrow \frac{2}{n(n+1)}\sum_{i=1}^{n} i x_i \xrightarrow{P} \mu \quad \left(E(T_n)\to\mu,\ Var(T_n)\to 0 \Rightarrow T_n \xrightarrow{P} \mu\right).$$

(b) $$E\left(\frac{6}{n(n+1)(2n+1)}\sum_{i=1}^{n} i^2 x_i\right) = \frac{6\mu}{n(n+1)(2n+1)} \times \frac{n(n+1)(2n+1)}{6} = \mu$$

$$Var\left(\frac{6}{n(n+1)(2n+1)}\sum_{i=1}^{n} i^2 x_i\right) = \frac{36\sigma^2}{n^2(n+1)^2(2n+1)^2}\sum_{i=1}^{n} i^4$$

$$= \frac{36 n^3\sigma^2}{(n+1)^2(2n+1)^2}\left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^4\right)$$

$$\to 36\times 0\times \int_0^1 t^4\,dt = 0$$

$$\Rightarrow \frac{6}{n(n+1)(2n+1)}\sum_{i=1}^{n} i^2 x_i \xrightarrow{P} \mu.$$

Problem No. 8 We know that

$$M_{X_n}(t) = E(e^{tX_n}) = \sum_{x=0}^{\infty} e^{tx}\binom{n+x-1}{x} p_n^n (1-p_n)^x$$

$$= p_n^n \sum_{x=0}^{\infty}\binom{n+x-1}{x}\left((1-p_n)e^t\right)^x$$

$$= \frac{p_n^n}{\left(1-(1-p_n)e^t\right)^n}, \quad t < -\ln(1-p_n)$$

$$= \left[\frac{1-\theta/n}{1-\frac{\theta}{n}e^t}\right]^n$$

$$= \left(1+\frac{\theta(e^t-1)}{n\left(1-\frac{\theta}{n}e^t\right)}\right)^n$$

$$\to e^{\theta(e^t-1)}, \quad \text{as } n\to\infty \quad \forall t\in\mathbb{R}$$

$\hookrightarrow$ mgf of Poisson(θ)

$X_n \sim \text{Gamma}(n, \frac{1}{n})$ (Note the typo in question)

Problem No. 9

(a) $E(X_n) = n \times \frac{1}{n} = 1$, $\forall\, n = 1, 2, \ldots$

$Var(X_n) = n \times \frac{1}{n^2} \longrightarrow 0$ as $n \to \infty$

$\Rightarrow X_n \xrightarrow{p} 1$.

(b) $M_{X_n}(t) = E(e^{tX_n}) = e^{\frac{t}{n} + \frac{1}{2}(1-\frac{1}{n})^2 t^2} \longrightarrow e^{t^2/2}$, $\forall\, t \in \mathbb{R}$

↳ mgf of $N(0,1)$ $= M_Z(t)$

$\Rightarrow X_n \xrightarrow{d} Z \sim N(0,1)$

(Alt: $P(X_n \le x) = \Phi\left(\frac{x - \frac{1}{n}}{1 - \frac{1}{n}}\right) \longrightarrow \Phi(x)$, as $n \to \infty$)

Problem No. 10

(a) Let $X_1, \ldots, X_{72}$ be a random sample with pdf $f(\cdot)$.

Define

$$Y_i = \begin{cases} 1, & \text{if } X_i < 3 \\ 0, & \text{if } X_i \ge 3 \end{cases}, \quad i = 1, 2, \ldots, 72$$

and $Y = \sum_{i=1}^{72} X_i$

Required probability $= P(Y > 50)$

$= P(Y \ge 50.5)$ (Continuity Correction as Y takes only discrete values)

$Y_1 \ldots Y_{72}$ are iid $\text{Bin}(1, p)$, where $p = P(X_1 < 3) = \int_1^3 \frac{1}{x^2}\, dx = \frac{2}{3}$

By the CLT, for large n,

$$\frac{\sqrt{n}\left(\frac{Y}{n} - \frac{2}{3}\right)}{\sqrt{\frac{2}{3} \times (1 - \frac{1}{3})}} \approx Z \sim N(0,1),$$

$n = 72$ is large. Thus

$$3 \times \frac{\sqrt{72}\left(\frac{Y}{72} - \frac{2}{3}\right)}{\sqrt{2}} = \frac{Y}{4} - 12 \approx Z \sim N(0,1)$$

Required prob $= P(Y \ge 50.5) \approx P(4Z + 48 \ge 50.5)$

$= 1 - \Phi(.625) = 1 - .734 = 0.266$

(b) $E(X_1) = 3$, $Var(X_1) = 3$.

By the CLT ($n = 100$ is large)

$$\frac{\sqrt{100}\left(\frac{Y}{100} - 3\right)}{\sqrt{3}} \approx Z \sim N(0,1)$$

$$\frac{\frac{Y}{10} - 30}{\sqrt{3}} \approx Z \sim N(0,1)$$

(Continuity Correction as Y takes only integer values)

$P(250 \le Y \le 300) = P(249.5 \le Y \le 300.5) = P(249.5 \le 10\sqrt{3}Z + 300 \le 300.5)$

(Note the typo in question)

$= \Phi(.0289) - \Phi(-2.9152) = 0.512 - .002$

$= 0.51$

(c) $X \stackrel{d}{=} \sum_{i=1}^{25} X_i$,

Where $X_1, \ldots, X_{25}$ are iid $Bin(1, 0.6)$

$\mu = E(X_1) = 0.6$, $\sigma^2 = Var(X_1) = 0.6 \times 0.4 = 0.24$

By the CLT ($n = 25$ is reasonably large)

$$\frac{\sqrt{25}\left(\frac{X}{25} - 0.6\right)}{\sqrt{0.24}} \approx Z \sim N(0,1)$$

$$\Rightarrow \quad X \simeq 5\sqrt{0.24}\, Z + 15$$

Required prob $= P(10 \le X \le 16)$

$= P(9.5 \le X \le 16.5)$

$\approx P(9.5 \le 5\sqrt{0.24}\, Z + 15 \le 16.5)$

$= \Phi(0.6124) - \Phi(-2.2454)$

$= 0.730 - 0.012 = 0.718$

Actual Prob: $P(X \le 16) - P(X \le 9) = 0.7265 - 0.0132$

$= 0.7133$

Problem No. 11 (a) Let $X_1, X_2, \ldots$ be iid Poisson(1) rvs $(E(X_1)=1$, $Var(X_1)=1)$. Let $Y_n = \sum_{i=1}^{n} X_i$. Then $Y_n \sim$ Poisson(n). By the CLT

$$\sqrt{n}\left(\frac{Y_n}{n}-1\right) \xrightarrow{d} Z \sim N(0,1), \text{ as } n\to\infty$$

$$\text{Required limit} = \lim_{n\to\infty}\left(e^{-n}\sum_{k=0}^{n}\frac{n^k}{\lfloor k}\right) = \lim_{n\to\infty} P(Y_n \le n)$$

$$= \lim_{n\to\infty} P\left(\sqrt{n}\left(\frac{Y_n}{n}-1\right)\le 0\right) = P(Z\le 0) = \Phi(0) = \frac{1}{2}.$$

(b) Let $X_1, X_2, \ldots$ be iid Bin$(1,\frac{1}{2})$; $E(X_1)=\frac{1}{2}$, $Var(X_1)=\frac{1}{4}$. Let $Y_n = \sum_{i=1}^{n} X_i$. Then $Y_n \sim$ Bin$(n, \frac{1}{2})$. By CLT

$$\frac{\sqrt{n}\left(\frac{Y_n}{n}-\frac{1}{2}\right)}{\sqrt{\frac{1}{4}}} \xrightarrow{d} Z \sim N(0,1), \text{ as } n\to\infty.$$

Let $t_n = 2^{-n}\sum_{k=0}^{r_n}\binom{n}{k} = P(Y_n \le r_n)$, $n=1,2,\ldots$

Then

$t_{2m} = P(Y_{2m}\le m)$ and $t_{2m+1} = P(Y_{2m+1}\le m)$, $m=1,2,\ldots$

By the CLT

$$\frac{\sqrt{2m}\left(\frac{Y_{2m}}{2m}-\frac{1}{2}\right)}{\sqrt{\frac{1}{4}}} \xrightarrow{d} Z\sim N(0,1); \quad \frac{\sqrt{2m+1}\left(\frac{Y_{2m+1}}{2m+1}-\frac{1}{2}\right)}{\sqrt{\frac{1}{4}}} \xrightarrow{d} Z\sim N(0,1)$$

$$\lim_{m\to\infty} t_{2m} = \lim_{m\to\infty} P(Y_{2m}\le m) = \lim_{n\to\infty} P\left(\frac{\sqrt{2m}\left(\frac{Y_{2m}}{2m}-\frac{1}{2}\right)}{\sqrt{\frac{1}{4}}}\le 0\right) = \Phi(0)=\frac{1}{2}$$

$$\lim_{m\to\infty} t_{2m+1} = \lim_{m\to\infty} P(Y_{2m+1}\le m) = \lim_{m\to\infty} P\left(\frac{\sqrt{2m+1}\left(\frac{Y_{2m+1}}{2m+1}-\frac{1}{2}\right)}{\sqrt{\frac{1}{4}}} \le \frac{\sqrt{2m+1}\left(\frac{m}{2m+1}-\frac{1}{2}\right)}{\sqrt{\frac{1}{4}}}\right)$$

$$= \lim_{m\to\infty} P\Bigg(\underbrace{\underset{\substack{\downarrow d\\ Z\sim N(0,1)}}{\frac{\sqrt{2m+1}\left(\frac{Y_{2m+1}}{2m+1}-\frac{1}{2}\right)}{\sqrt{\frac{1}{4}}}} - \underset{\substack{\downarrow\\ 0}}{\frac{\sqrt{2m+1}\left(\frac{m}{2m+1}-\frac{1}{2}\right)}{\sqrt{\frac{1}{4}}}}}_{\xrightarrow{d} Z\sim N(0,1)} \le 0\Bigg) = \Phi(0)=\frac{1}{2}.$$

$$\lim_{m\to\infty} t_{2m} = \lim_{m\to\infty} t_{2m+1} = \frac{1}{2} \quad \Rightarrow \quad \lim_{m\to\infty} t_m = \frac{1}{2}.$$

$\boxed{\text{Problem No.12}}$ (a)

$$0 \le |\bar{X}_n| \le \frac{|X_1| + \dots + |X_n|}{n} \le \max\{|X_1|, \dots, |X_n|\} \xrightarrow{P} 0, \text{ as } n\to\infty$$

$$\Rightarrow |\bar{X}_n| \xrightarrow{P} 0, \text{ i.e. } \bar{X}_n \xrightarrow{P} 0 \qquad (\text{Problem 6(b)}),$$

Converse may not be true. To see a counter example, let $X_1, X_2, \dots$ be iid $U(-1, 0)$. Then $E(X_1) = -\frac{1}{2}$, $\bar{X}_n \xrightarrow{P} -\frac{1}{2}$ (WLLN)

Let $S_n = \max\{X_1, \dots, X_n\}$. Then, for any $\varepsilon > 0$

$$P(|S_n| > \varepsilon) = P(-S_n > \varepsilon) = P(S_n < -\varepsilon) = P(X_i < -\varepsilon, \ i = 1 \dots n)$$

$$= \prod_{i=1}^{n} P(X_i < -\varepsilon) = (P(X_1 < -\varepsilon))^n = \begin{cases} 0, & \text{if } \varepsilon > 1 \\ (1-\varepsilon)^n & \text{if } 0 < \varepsilon \le 1 \end{cases}$$

$$\longrightarrow 0, \quad \text{as } n\to\infty$$

$$\Rightarrow S_n \xrightarrow{P} 0.$$

(b) Let $Y_i = -\ln X_i$, $i = 1, 2, \dots$ Then $Y_1, Y_2, \dots$ are iid Exp(1) r.v.s ($E(Y_1) = 1$, $Var(Y_1) = 1$). By WLLN

$$\bar{Y}_n = \frac{1}{n}\sum_{i=1}^{n} Y_i \xrightarrow{P} E(Y_1) = 1$$

$$\Rightarrow -\ln Z_n = -\frac{1}{n}\sum_{i=1}^{n} \ln X_i = \bar{Y} \xrightarrow{P} 1$$

$$\Rightarrow \ln Z_n \xrightarrow{P} -1 \Rightarrow e^{\ln Z_n} \xrightarrow{P} e^{-1} \quad (g(x) = e^x, x \in \mathbb{R} \text{ is a continuous function})$$

$$\Rightarrow Z_n \xrightarrow{P} e^{-1}.$$

$\boxed{\text{Problem No.13}}$ (a)

$$M_{T_n}(t) = E(e^{tT_n}) = \prod_{i=1}^{n} E(e^{tE_i}) = \prod_{i=1}^{n}(1-t)^{-1}$$

$$= (1-t)^{-n}, \quad t < 1$$

mgf of Gamma(n, 1)

$$\Rightarrow T_n \sim \text{Gamma}(n, 1)$$

(b) $E(E_1) = 1$ and $Var(E_1) = 1$. By CLT

$$\sqrt{n}\left(\frac{T_n}{n} - 1\right) \xrightarrow{d} Z \sim N(0,1)$$

$$\Rightarrow \lim_{n\to\infty} P\left(\sqrt{n}\left(\frac{T_n}{n}-1\right) \le \lambda\right) = \Phi(\lambda), \quad \forall \lambda \in \mathbb{R}.$$

$$\Rightarrow \lim_{n\to\infty} P(T_n \le n + \lambda\sqrt{n}) = \int_{-\infty}^{\lambda} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$$

$$\Rightarrow \lim_{n\to\infty} \int_0^{n+\lambda\sqrt{n}} \frac{e^{-t} t^{n-1}}{\Gamma n} dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\lambda} e^{-t^2/2} dt, \quad \forall \lambda \in \mathbb{R} \quad (T_n \sim Gamma(n,1) \text{ by (a)})$$

(c) For large n, (using (b)) we have

$$\underbrace{\int_0^{n+\lambda\sqrt{n}} \frac{e^{-t} t^{n-1}}{\Gamma n} dt \approx \int_{-\infty}^{\lambda} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt}_{\text{Smooth fn of } \lambda} \quad \forall \lambda \in \mathbb{R}$$

Taking derivatives on both side we get

$$\frac{e^{-(n+\lambda\sqrt{n})} (n+\lambda\sqrt{n})^{n-1}}{\Gamma n} \times \sqrt{n} = \frac{1}{\sqrt{2\pi}} e^{-\frac{\lambda^2}{2}}, \quad \forall \lambda \in \mathbb{R}$$

For $\lambda = 0$, we get

$$\frac{e^{-n} n^{n-\frac{1}{2}}}{\Gamma n} \approx \frac{1}{\sqrt{2\pi}} \Rightarrow \Gamma n \approx \sqrt{2\pi}\, e^{-n} n^{n-\frac{1}{2}}.$$

Problem No. 14 (a) The jt pdf of (X_1, X_2) is

$$f_{X_1,X_2}(x_1,x_2) = \frac{1}{\pi^2} \cdot \frac{1}{(1+x_1^2)(1+x_2^2)}, \quad -\infty < x_i < \infty, \ i=1,2.$$

For $\alpha \in (0,1)$, define $Y = \alpha X_1 + (1-\alpha) X_2$, $Z = X_1$.

$$\begin{aligned} y &= \alpha x_1 + (1-\alpha) x_2 \\ z &= x_1 \end{aligned} \Rightarrow \begin{aligned} x_1 &= z \\ x_2 &= \frac{y-\alpha z}{1-\alpha} \end{aligned} \qquad J = \begin{vmatrix} 0 & 1 \\ \frac{1}{1-\alpha} & -\frac{\alpha}{1-\alpha} \end{vmatrix} = -\frac{1}{1-\alpha}$$

Jt pdf of (Y, Z) is

$$f_{Y,Z}(y,z) = \frac{1-\alpha}{\pi^2} \cdot \frac{1}{(1+z^2)((1-\alpha)^2 + (\alpha z - y)^2)}, \quad -\infty < y, z < \infty.$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{Y,Z}(y,z)\,dz = \frac{1-\alpha}{\pi^2}\int_{-\infty}^{\infty} \frac{1}{(1+z^2)\left((1-\alpha)^2+(\alpha z-y)^2\right)}\,dz, \quad -\infty<y<\infty$$

Let $$\frac{1}{(1+z^2)\left((1-\alpha)^2+(\alpha z-y)^2\right)} = \frac{2Az}{1+z^2} + \frac{B}{1+z^2} + \frac{2\alpha(\alpha z-y)C}{(1-\alpha)^2+(\alpha z-y)^2} + \frac{D}{(1-\alpha)^2+(\alpha z-y)^2}$$

$$\Leftrightarrow\ 2Az\left[\alpha^2z^2-2\alpha yz+(1-\alpha)^2+y^2\right]+B\left[\alpha^2z^2-2\alpha yz+(1-\alpha)^2+y^2\right]$$
$$+2\alpha(\alpha z^3-yz^2+\alpha z-y)C+D(z^2+1)=1$$

$$\Rightarrow\ 2\alpha^2(A+C)z^3+\left[-4\alpha yA+B\alpha^2-2\alpha yC+D\right]z^2+\left[2\left((1-\alpha)^2+y^2\right)A-2\alpha yB+2\alpha^2C\right]z$$
$$+\left[(1-\alpha)^2+y^2\right]B-2\alpha yC+D=1$$

$$\Rightarrow\quad 2\alpha^2(A+C)=0 \quad\Rightarrow\quad \boxed{C=-A}\ \cdots\ (\mathrm{I})$$

$$-4\alpha yA+B\alpha^2-2\alpha yC+D=0 \quad\Rightarrow\quad \boxed{D=2\alpha yA-B\alpha^2}\ \cdots\ (\mathrm{II})$$

$$2\left((1-\alpha)^2+y^2\right)A-2\alpha yB+2\alpha^2C=0 \quad\Rightarrow\quad \boxed{\alpha yB=(1-2\alpha+y^2)A}\ \cdots\ (\mathrm{III})$$

$$\left((1-\alpha)^2+y^2\right)B-2\alpha yC+D=1 \quad\Rightarrow\quad \boxed{D=1-2\alpha yA-\left((1-\alpha)^2+y^2\right)B}\ \cdots\ (\mathrm{IV})$$

(II) and (IV) give

$$2\alpha yA-B\alpha^2=1-2\alpha yA-B\left[(1-\alpha)^2+y^2\right]$$

$$\Rightarrow\quad \boxed{B(1-2\alpha+y^2)=1-4\alpha yA}\ \cdots\ (\mathrm{V})$$

(III) and (V) give

$$\frac{1-2\alpha+y^2}{\alpha y}A=\frac{1-4\alpha yA}{1-2\alpha+y^2}$$

$$\Rightarrow\quad \boxed{A=\frac{\alpha y}{(1-2\alpha+y^2)^2+4\alpha^2y^2}}$$

Putting the value of A in (III) we get

$$\boxed{B=\frac{1-2\alpha+y^2}{(1-2\alpha+y^2)^2+4\alpha^2y^2}}\ ; \quad \boxed{C=\frac{-\alpha y}{(1-2\alpha+y^2)^2+4\alpha^2y^2}=-A}$$

Using II

$$D=\frac{2\alpha^2y^2}{(1-2\alpha+y^2)^2+4\alpha^2y^2}-\frac{(1-2\alpha+y^2)\alpha^2}{(1-2\alpha+y^2)^2+4\alpha^2y^2}$$

$$\boxed{D=\frac{\alpha^2(y^2+2\alpha-1)}{(1-2\alpha+y^2)^2+4\alpha^2y^2}}$$

Thus

$$f_Y(y) = \frac{1-\alpha}{\pi^2}\Big[\Big\{A\ln(1+z^2) + C\ln\{(1-\alpha)^2+(\alpha z-y)^2\}\Big\}_{z=-\infty}^{\infty} + B\pi + D\frac{1-\alpha}{\alpha}\int_{-\infty}^{\infty}\frac{dt}{(1-\alpha)^2+(1-\alpha)^2t^2}\Big]$$

$$= \frac{1-\alpha}{\pi^2}\Big[A\Big\{\ln\frac{1+z^2}{(1-\alpha)^2+(\alpha z-y)^2}\Big\}_{z=-\infty}^{\infty} + B\pi + \frac{D\pi}{\alpha(1-\alpha)}\Big]$$

$$= \frac{1}{\pi}\Big[(1-\alpha)B + \frac{D}{\alpha}\Big]$$

$$(1-\alpha)B + \frac{D}{\alpha} = \frac{(1-\alpha)(1-2\alpha+y^2)}{(1-2\alpha+y^2)^2+4\alpha^2y^2} + \frac{\alpha(y^2+2\alpha-1)}{(1-2\alpha+y^2)^2+4\alpha^2y^2}$$

$$= \frac{y^2+4\alpha^2-4\alpha+1}{y^4+(4\alpha^2-4\alpha+2)y^2+(1-2\alpha)^2}$$

$$= \frac{y^2+(2\alpha-1)^2}{[y^2+(2\alpha-1)^2][y^2+1]} = \frac{1}{y^2+1}.$$

Thus $f_Y(y) = \frac{1}{\pi}\cdot\frac{1}{1+y^2}$, $-\infty < y < \infty$.

(b) By (a) the result is true for $n=2$ (taking $\alpha = \frac{1}{2}$). Now suppose that the result is true for some $n = m \in \{2, 3, \ldots\}$.

Then

$$\bar{X}_{m+1} = \frac{1}{m+1}\sum_{i=1}^{m+1} X_i = \frac{m}{m+1}\bar{X}_m + \frac{1}{m+1}X_{m+1} = \alpha\underbrace{\bar{X}_m}_{C(0,1)} + (1-\alpha)\underbrace{X_{m+1}}_{C(0,1)}$$

Independent

Thus, by (a)

$$\bar{X}_{m+1} \stackrel{d}{=} X_1.$$

(c) $\bar{X}_n \stackrel{d}{=} X_1 \xrightarrow{d} X_1$ as $n \to \infty$

$\Rightarrow$ $\bar{X}_n$ does not converge in probability to any constant, it converges in distribution to $X_1 \sim C(0,1)$. In fact the distribution of $\bar{X}_n$ does not depend on n.

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Problem No. 15 Let $T_1, T_2, \ldots$ be iid Poisson(4) rvs; $E(T_1) = Var(T_1) = 4$. Also $\sum_{i=1}^{n} T_i \sim$ Poisson$(4n)$, i.e. $X_n \stackrel{d}{=} \sum_{i=1}^{n} T_i$ and $Y_n \stackrel{d}{=} \bar{T}_n$, where $\bar{T}_n = \frac{1}{n}\sum_{i=1}^{n} T_i$.

(a) By WLLN $\bar{T}_n \xrightarrow{p} 4$, i.e. $Y_n \xrightarrow{p} 4$

(b) $Y_n \xrightarrow{p} 4 \Rightarrow Y_n^2 \xrightarrow{p} 4^2 = 16$ & $\sqrt{Y_n} \xrightarrow{p} \sqrt{4} = 2$ $\left(\begin{array}{l} g_1(x) = x^2,\ x \in \mathbb{R} \text{ and} \\ g_2(x) = \sqrt{x},\ x > 0 \text{ are} \\ \text{continuous fns} \end{array} \right)$

$\Rightarrow Y_n^2 + \sqrt{Y_n} \xrightarrow{p} 16 + 2 = 18$

(c) $$\frac{n^2 Y_n^2 + nY_n}{nY_n + n^2} = \frac{Y_n^2 + \frac{1}{n}Y_n}{\frac{Y_n}{n} + 1} \xrightarrow{p} \frac{16 + 0\times 4}{0 \times 4 + 1} = 16$$

Problem No. 16 By the CLT

$$Z_n \xrightarrow{d} Z \sim N(0,1)$$

(a) $Y_n \xrightarrow{p} 4 \Rightarrow \frac{Y_n}{4} \xrightarrow{p} 1 \Rightarrow \frac{4Z_n}{Y_n} = \frac{Z_n}{Y_n/4} \xrightarrow{d} \frac{Z}{1} = Z \sim N(0,1)$

$$\frac{16 Z_n^2}{Y_n^2} = \frac{Z_n^2}{\left(\frac{Y_n}{4}\right)^2} \xrightarrow{d} \frac{Z^2}{1} = \chi^2_1.$$

$\left\{ \begin{array}{l} T_n \xrightarrow{d} T,\ S_n \xrightarrow{p} a \\ \Rightarrow S_n T_n \xrightarrow{d} aT \\ T_n \xrightarrow{d} T,\ g(\cdot) \text{ is cont} \\ \quad \Rightarrow g(T_n) \xrightarrow{d} g(T) \end{array} \right\}$

$$\frac{4n + Y_n}{nY_n + Y_n^2} = \frac{4 + \frac{Y_n}{n}}{Y_n + \frac{Y_n^2}{n}} \xrightarrow{p} \frac{4 + 0 \times 4}{4 + 0 \times 4^2} = 1$$

$$\Rightarrow \frac{4n + Y_n}{nY_n + Y_n^2} Z_n \xrightarrow{d} 1 \times Z = Z \sim N(0,1).$$

(b) Here we will require the assumption that $\exists n_0$ s.t. $P_\mu(\bar{X}_n > 0) = 1$, $\forall n \geq n_0$.

$$\sqrt{n}(\bar{X}_n - \mu) \xrightarrow{d} Z \sim N(0,1)$$

Using Delta-method with $g(x) = \ln x$, $x > 0$, we get

$$\sqrt{n}\,(g(\bar{X}_n) - g(\mu)) \xrightarrow{d} g'(\mu) Z \sim N(0, (g'(\mu))^2)$$

$$\sqrt{n}\,(\ln \bar{X}_n - \ln \mu) \xrightarrow{d} N(0, \tfrac{1}{\mu^2})$$

(c) $$\frac{n^\delta (\bar{X}_n - \mu)}{\sigma} = n^{\delta - \frac{1}{2}} \times \frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \xrightarrow{p} 0 \times Z = 0, \text{ for } \delta < 0.5.$$

Thus for $\delta < \frac{1}{2}$, $\frac{n^\delta (\bar{X}_n - \mu)}{\sigma} \xrightarrow{p} 0$ and

$$\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \xrightarrow{d} Z \sim N(0,1).$$

(d) (i) Let $g(x) = x^2$, $x \in \mathbb{R}$, $g'(x) = 2x$

<u>Case I</u> $\mu \neq 0$, so that $g'(\mu) = 2\mu \neq 0$.

Using Delta-method with $g(x) = x^2$, $x \in \mathbb{R}$,

$$\sqrt{n}\,(g(\bar{X}_n) - g(\mu)) \xrightarrow{d} \sigma g'(\mu) Z \sim N(0, 4\mu^2\sigma^2)$$

$$\sqrt{n}\,(\bar{X}_n^2 - \mu^2) \xrightarrow{d} N(0, 4\mu^2\sigma^2)$$

<u>Case II</u> $\mu = 0$, so that $g'(\mu) = 2\mu = 0$.

$$\sqrt{n}\,\bar{X}_n \xrightarrow{d} \sigma Z \sim N(0, \sigma^2), \quad \bar{X}_n \xrightarrow{P} 0$$

$$\sqrt{n}\,(\bar{X}_n^2 - \mu^2) = \sqrt{n}\,\bar{X}_n^2 \quad \bar{X}_n \times \sqrt{n}\,\bar{X}_n \xrightarrow{P} 0 \times Z = 0.$$

(ii) $\sqrt{n}\,(\bar{X}_n - \mu) \xrightarrow{d} \sigma Z \sim N(0, \sigma^2)$

$$(\sqrt{n}\,(\bar{X}_n - \mu))^2 \xrightarrow{d} \sigma^2 Z^2 \sim \sigma^2 \chi_1^2$$

$$n(\bar{X}_n - \mu)^2 \xrightarrow{d} \sigma^2 Z^2 \sim \sigma^2 \chi_1^2.$$

$$\xrightarrow{P} 0 \times \sigma^2 \chi_1^2 = 0.$$

(iii) $\sqrt{n}\,(\bar{X}_n - \mu)^2 = \frac{1}{\sqrt{n}} \times n(\bar{X}_n - \mu)^2$

Problem No. 17

$E(X_i) = \theta$, $Var(X_i) = \theta^2$. By the CLT

$$\frac{\sqrt{n}\,(\bar{X}_n - \theta)}{\theta} \xrightarrow{d} Z \sim N(0, 1)$$

$$\sqrt{n}\,(\bar{X}_n - \theta) \xrightarrow{d} \theta Z \sim N(0, \theta^2)$$

Using Delta-method with $g(x) = \frac{1}{x}$, $x > 0$ (so that $g'(x) = -\frac{1}{x^2}$, $x > 0$).

$$\sqrt{n}\,(g(\bar{X}_n) - g(\theta)) \xrightarrow{d} g'(\theta)\,\theta Z = -\frac{Z}{\theta} \sim N(0, \tfrac{1}{\theta^2})$$

$$\sqrt{n}\left(\frac{1}{\bar{X}_n} - \frac{1}{\theta}\right) \xrightarrow{d} N(0, \tfrac{1}{\theta^2}).$$

Problem No. 18

Let $Y_i = -\ln X_i$, $i = 1, 2, \ldots$. Then $Y_1, Y_2, \ldots$ are iid $Exp(1)$ r.v.s ($E(Y_1) = 1$, $Var(Y_1) = 1$). By WLLN & CLT

$$\bar{Y}_n = \frac{1}{n}\sum_{i=1}^{n} Y_i \xrightarrow{P} 1 \quad \text{and} \quad \sqrt{n}\,(\bar{Y}_n - 1) \xrightarrow{d} Z \sim N(0, 1)$$

$$\sqrt{n}\,(-\ln G_n - 1) \xrightarrow{d} Z \sim N(0, 1)$$

Apply Delta-method with $g(x) = e^{-x}$, $x \in \mathbb{R}$, we get ($g'(x) = e^{-x}$)

$$\sqrt{n}\,(g(-\ln G_n) - g(1)) \xrightarrow{d} e^{-1} Z \sim N(0, \tfrac{1}{e^2})$$

$$\sqrt{n}\left(G_n - \tfrac{1}{e}\right) \xrightarrow{d} N(0, \tfrac{1}{e^2}), \quad \sigma^2 = \frac{1}{e^2}.$$

Problem No. 19

$$Q_n = \frac{\sum_{i=1}^{n}(X_i-\bar{X})(Y_i-\bar{Y})}{n-1} = \frac{1}{n-1}\sum_{i=1}^{n}(X_i-\mu)(Y_i-\upsilon) - \frac{n}{n-1}(\bar{X}-\mu)(\bar{Y}-\mu)$$

$$= \frac{n}{n-1}\bar{Z} - \frac{n}{n-1}(\bar{X}-\mu)(\bar{Y}-\mu),$$

where $Z_i = (X_i-\mu)(Y_i-\upsilon)$, $i=1,2,\ldots$, $\bar{Z} = \frac{1}{n}\sum_{i=1}^{n} Z_i$, $Z_1, Z_2, \ldots$ are iid with $E(Z_1) = \sigma\tau\rho$ and $Var(Z_1) = E((X_1-\mu)^2(Y_1-\upsilon)^2) - E((X_1-\mu)(Y_1-\upsilon))^2$

$$= \sigma^2\tau^2\delta - \sigma^2\tau^2\rho^2$$

$$= \sigma^2\tau^2(\delta-\rho^2).$$

(a) By WLLN

$\bar{Z} \xrightarrow{P} \sigma\tau\rho$, $\bar{X}-\mu \xrightarrow{P} 0$, $\bar{Y}-\mu \xrightarrow{P} 0$.

Thus

$Q_n \xrightarrow{P} 1\times\sigma\tau\rho - 1\times 0\times 0 = \sigma\tau\rho$

Also $S_n^2 \xrightarrow{P} \sigma^2$ and $T_n^2 \xrightarrow{P} \tau^2$ (see lecture notes)

$\Rightarrow S_n \xrightarrow{P} \sigma$ and $T_n \xrightarrow{P} \tau$

$\Rightarrow R_n = \frac{Q_n}{S_n T_n} \xrightarrow{P} \frac{\sigma\tau\rho}{\sigma\tau} = \rho.$

(b) $\sqrt{n}(Q_n - \rho\sigma\tau) = \sqrt{n}\left(\frac{n}{n-1}\bar{Z} - \frac{n}{n-1}(\bar{X}-\mu)(\bar{Y}-\mu) - \rho\sigma\tau\right)$

$$= \frac{n}{n-1}\sqrt{n}(\bar{Z}-\rho\sigma\tau) + \frac{\sqrt{n}}{n-1}\rho\sigma\tau - \frac{n}{n-1}(\sqrt{n}(\bar{X}-\mu))(\bar{Y}-\mu)$$

By CLT and WLLN, $\sqrt{n}(\bar{Z}-\rho\sigma\tau) \xrightarrow{d} U \sim N(0, \sigma^2\tau^2(\delta-\rho^2))$, $\sqrt{n}(\bar{X}-\mu) \xrightarrow{d} Z \sim N(0,1)$, $\bar{Y}-\mu \xrightarrow{P} 0$. Thus

$\sqrt{n}(Q_n-\rho\sigma\tau) \xrightarrow{d} 1\times U + 0\times\rho\sigma\tau - 1\times Z\times 0 = U \sim N(0, \sigma^2\tau^2(\delta-\rho^2)).$