3.a. Consider the following density

$$f(x) = ce^{-x^2/32} \text{ for } x > 0.$$

(a) Find c.
(b) Let X be a random variable with pdf f. Find $M_X(t)$ for $-\infty < t < \infty$.
(c) Use $M_X(t)$ to compute $Var(X)$. [1+2+2]

3.b. Let $Z_1, \ldots, Z_n$ be random variables and $a_1, \ldots, a_n$ be positive numbers. Prove that

$$\sum_{i=1}^{n} a_i \sqrt{Var(Z_i)} \leq \sqrt{\sum_{i=1}^{n} a_i} \sqrt{\sum_{i=1}^{n} a_i Var(Z_i)}. \quad [2]$$

Solution 3. a.

3(a) Given, $f_X(x) = c\,e^{-x^2/32}$, $x > 0$

$$\int_0^\infty f_X(x)\,dx = 1$$

$$\Rightarrow c\int_0^\infty e^{-x^2/32}\,dx = 1$$

$$\Rightarrow c = \frac{1}{4}\sqrt{\frac{2}{\pi}} = \frac{1}{2\sqrt{2\pi}}$$ — $\boxed{\text{1 mark}}$

3(b)
$$M_X(t) = \frac{1}{4}\sqrt{\frac{2}{\pi}}\int_0^\infty e^{tx-\frac{x^2}{32}}\,dx$$

$$= \frac{1}{4}\sqrt{\frac{2}{\pi}}\,e^{\frac{16t^2}{2}}\int_0^\infty e^{-\frac{1}{2}\left(\frac{x^2}{16}-2\cdot\frac{x}{4}\cdot 4t+16t^2\right)}\,dx$$

$$= \frac{1}{4}\sqrt{\frac{2}{\pi}}\,e^{8t^2}\int_0^\infty e^{-\frac{1}{2}\left(\frac{x}{4}-4t\right)^2}\,dx$$

Put $\frac{x}{4} - 4t = z$

$dx = 4\,dz$

$$= \sqrt{\frac{2}{\pi}}\,e^{8t^2}\int_{-4t}^\infty e^{-z^2/2}\,dz$$

$$= 2e^{8t^2}\frac{1}{\sqrt{2\pi}}\int_{-4t}^\infty e^{-z^2/2}\,dz$$

$$= 2e^{8t^2}\left[1-\Phi(-4t)\right]$$

$$= 2e^{8t^2}\Phi(4t),\ t\in\mathbb{R}, \quad \left[\because 1-\Phi(-z)=\Phi(z)\right]$$

— $\boxed{\text{2 marks}}$

3(c): $\psi_X(t) = \log M_X(t) = \log 2 + 8t^2 + \log \Phi(4t)$

$$\frac{\partial}{\partial t}\psi_X(t) = 16t + 4\,\frac{\phi(4t)}{\Phi(4t)}$$

$$\frac{\partial^2}{\partial t^2}\psi_X(t) = 16 + 16\left[\frac{\Phi(4t).\,\phi'(4t) - (\phi(4t))^2}{(\Phi(4t))^2}\right]$$

$$\Rightarrow Var(X) = \frac{\partial^2}{\partial t^2}\psi_X(t)\Big|_{t=0}$$

$$= 16 + 16\left[\frac{\Phi(0).\,\phi'(0) - (\phi(0))^2}{(\Phi(0))^2}\right]$$

$$= 16 + 16\left[\frac{\frac{1}{2}\cdot 0 - \frac{1}{2\pi}}{(\frac{1}{2})^2}\right] \qquad \left[\because \phi'(t) = -t\,\phi(t),\ \phi'(0) = 0\right]$$

$$= 16 + 16\times 4\left(-\frac{1}{2\pi}\right)$$

$$= 16\left(1 - \frac{2}{\pi}\right) \quad \text{———} \quad \boxed{\text{2 marks}}$$

Solution 3.b.

Let $X = z_i$ w.p. p_i for $1 \le i \le n$.

$\sum_{i=1}^{n} p_i = 1$, $p_i > 0$, and $z_i > 0$.

S3

Using Jensen's inequality with $f(x) = \sqrt{x}$,

$$E\left(\sqrt{X}\right) \leq \sqrt{E(X)}$$

$$\Rightarrow \sum_{i=1}^{n} \sqrt{z_i}\, p_i \leq \sqrt{\sum_{i=1}^{n} z_i p_i} \quad — \boxed{1 \text{ mark}}$$

Take $z_i = Var(Z_i)$ and $p_i = \dfrac{a_i}{\sum_{i=1}^{n} a_i}$

$$\Rightarrow \frac{\sum_{i=1}^{n} a_i \sqrt{Var(Z_i)}}{\sum_{i=1}^{n} a_i} \leq \frac{\sqrt{\sum_{i=1}^{n} a_i Var(Z_i)}}{\sqrt{\sum_{i=1}^{n} a_i}}$$

$$\Rightarrow \sum_{i=1}^{n} a_i \sqrt{Var(Z_i)} \leq \sqrt{\sum_{i=1}^{n} a_i} \sqrt{\sum_{i=1}^{n} a_i Var(Z_i)}$$

$— \boxed{1 \text{ mark}}$