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CABLE DYNAMICS—A FINITE SEGMENT APPROACH†‡

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Abstract—The paper presents and discusses a nonlinear, three-dimensional, finite-segment, dynamic model of a cable or chain. The model consists of a series of links connected to each other by ball-and-socket joints. The size, shape, and mass of the links is arbitrary. Furthermore, these parameters may be distinct for each link. Also, the number of links is arbitrary. The model allows an arbitrary force system to be applied to each link.

The model is used to develop a computer code which consists primarily of subroutines containing algorithms to develop the kinematics, force systems, and governing dynamical equations. Although the integration of the equations is performed with a Runge–Kutta algorithm, the code is developed so that any other suitable integration technique or algorithm may be substituted. The input for the code requires the following: the number of links; the mass, centroidal inertia matrix, mass-center position, connection point, and external forces on each link; and the time history of the specified variables. The output consists of the time history of each variable, the position, velocity, and the acceleration of the mass-center of each link, and the unknown forces and moments.

An example problem is presented which describes the motion of a sphere drug through water by a partially submerged cable suspended from a rotating surface crane. Viscous forces of the water are included. Although the example simulates a typical nautical rig, its inclusion in the paper is introduced primarily to illustrate the capability of the model.

INTRODUCTION

The advent of the digital computer and the subsequent introduction of finite-element and finite-segment modelling has fundamentally changed the mode of analysis of structural systems. Recently (that is, in the last five years) this change has been incorporated into the analysis of cable dynamics. This paper is a presentation of a recently developed finite-segment model of nonlinear cable and chain dynamics.

Two years ago Choo and Casarella[1] published an excellent survey of the literature on analytical methods for cable dynamics. They indicate that of four methods for studying cable dynamics (the method of characteristics, the finite-element method, the linearization method, and the equivalent lumped mass method), the finite-element method is the most versatile. Indeed, they indicate that the finite-element or finite-segment method offers the greatest hope for a simple method that can solve nonlinear, unsteady state problems with good accuracy and yet require only a moderate amount of computation time.

References[2–11] provide a summary of recent finite-element approaches to cable dynamics. The approach in this paper is to make a finite-segment model of the cable in the form of a linear chain. The dynamics of this model is then a specialization of a recently developed dynamic analysis of general chain systems[12]. (A general chain system is a set of rigid bodies arbitrarily assembled and connected provided that adjoining bodies have at least one common point and that no closed loops are formed. Examples are manipulators, chain links, teleoperators and human-body models.)

There have been a number of recent attempts to develop efficient methods for obtaining equations of motion for chain systems[13–20]. These efforts generally proceed by first modelling or replacing the given dynamical system by a discrete system or chain or interconnected rigid bodies. Dynamical equations of motion are then written for the chain system. In the derivation of these equations, some use Newton's Laws, and some use geometrical theories. But each has the objective of efficient derivation of computer-oriented equations. The relative advantages and disadvantages of these various methods depends upon: (a) the particular dynamical principle which is used and (b) the method of accounting and organizing the complex geometry. The difficulties encountered in these approaches usually include some of the following: (a) the introduction of "non-working" constraint forces between adjoining bodies (Newton's Laws); the tedious, often unwieldly, calculation of derivatives (Lagrange's Equations); the complex geometrical description of the system; and (c) the solution of the developed equations.

Huston and Passerello[12] have developed a new method of obtaining equations of motion which systematically avoids each of these difficulties. The method uses Lagrange's form of d'Alembert's principle[21–24] which provides for the automatic elimination of the "non-working" internal constraint forces without introducing tedious differentiation or other calculations. The method also uses a geometrical organization and accounting procedure as developed by Kane[23] and Huston and Passerello[15, 16]. The method allows the chain system to be in any general force field and either the moments or the orientation between adjoining bodies may be specified or left unknown. Finally, the method is developed in a form so that all the computation may be systematically and efficiently performed on a digital computer.

The analysis of this paper closely follows this method. It is divided into six parts with the first part providing the necessary preliminary background material. The second part outlines the kinematics and the third part contains the governing dynamical equations of motion. This is

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followed by a brief description of the computer code. An example problem of a semi-submerged towing cable and a general discussion of the method and model are presented in the remaining parts.

PRELIMINARY CONSIDERATIONS

Consider a cable as being modelled by an arbitrary number, N of rigid links connected by spherical (ball-and-socket) joints as shown in Fig. 1. All the links need not have the same mass. Next, let this system be subjected to a general force field (modelling an arbitrary force field on the cable) such that each link has a force system entered on it which may be represented as a single force passing through its mass center together with a couple.

To organize the geometrical accounting of this system, select one link (say the first link) of the system as the reference link, and then label or number the other links in ascending progression away from this link as shown in Fig. 1. The configuration and kinematics of each body of the system may be developed relative to the reference link which, in turn, has its configuration and kinematics defined relative to an inertial reference frame† R.

Consider a typical pair of adjoining links such as L_j and L_k as shown in Fig. 2. Let $j < k$, e.g. $j = k - 1$. Then the general orientation of L_k relative to L_j, may be defined in terms of the relative inclination of the dextral orthogonal unit vector sets, $\mathbf{n}_{ji}$ and $\mathbf{n}_{ki}$ $(i = 1, 2, 3)$ fixed in L_j and L_k, as shown in Fig. 2. Specifically, let L_j and L_k be oriented so that $\mathbf{n}_{ji}$ and $\mathbf{n}_{ki}$ are respectively parallel. Then L_k may be brought into any given orientation relative to L_j by three successive dextral rotations about axes parallel to

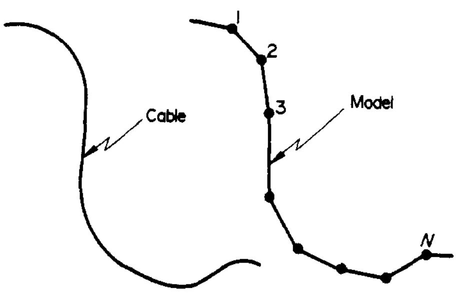

Fig. 1. Cable model.

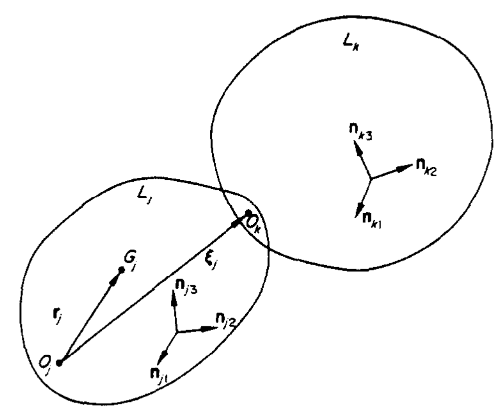

Fig. 2. Two typical adjoining links.

†Alternative approach is to refer each link independently to R, but this is found to be inconvenient when describing the configuration of actual systems.

$\mathbf{n}_{k1}$, $\mathbf{n}_{k2}$, and $\mathbf{n}_{k3}$ through the angles α_{jk}, β_{jk}, and γ_{jk}. $\mathbf{n}_{ji}$ and $\mathbf{n}_{ki}$ are then related to each other as:

$$\mathbf{n}_{ji} = SJK_{im}\mathbf{n}_{km} \tag{1}$$

where SJK is a 3×3 orthogonal tranformation matrix ("shifter") defined as[25]

$$SJK_{im} = \mathbf{n}_{ij} \cdot \mathbf{n}_{km}. \tag{2}$$

(Regarding notation, repeated subscripts, such as m in the right side of eqn (1) represent a sum over the range $(1, \ldots, 3)$ of that index.)

SJK may be written as the product of three orthogonal matrices as

$$SJK = \alpha JK \beta JK \gamma JK \tag{3}$$

$$\alpha JK = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\alpha_{jk} & -s\alpha_{jk} \\ 0 & s\alpha_{jk} & c\alpha_{jk} \end{bmatrix}$$

$$\beta JK = \begin{bmatrix} c\beta_{jk} & 0 & s\beta_{jk} \\ 0 & 1 & 0 \\ -s\beta_{jk} & 0 & c\beta_{jk} \end{bmatrix} \tag{4}$$

$$\gamma JK = \begin{bmatrix} c\gamma_{jk} & -s\gamma_{jk} & 0 \\ s\gamma_{jk} & c\gamma_{jk} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where $S\alpha_{jk}$ and $c\alpha_{jk}$ represent the sine and cosine of α_{jk}.

From eqn (1), it is easily seen that with three links L_j, L_k and L_l, the shifter transformation matrix obeys the following "chain" and identity rules

$$SJL = SJK\ SKL \tag{5}$$

and

$$SJJ = I = SJK\ SKJ = SJK\ SJK^{-1} \tag{6}$$

where I is the identity matrix.

These expressions allow for the transformation of components of vectors referred to one link of the system into components referred to any other link of the system, and, in particular to the inertial reference frame R.

Since these transformation matrices play a central role throughout the analysis it is helpful to also have an algorithm for their derivative, especially the derivative of SOK (where 0 refers to R). Using eqn (1) and noting that $\mathbf{n}_{oi}$ are fixed, and therefore constant in R, the following is obtained:

$$\mathrm{d}(SOK_{ij})/\mathrm{d}t = \mathbf{n}_{oi} \cdot {}^{R}d\mathbf{n}_{kj}/\mathrm{d}t \tag{7}$$

where the superscript R indicates that the derivative is computed in R. However, since $\mathbf{n}_{kj}$ are fixed in L_k, their derivative may be written as $\boldsymbol{\omega}^k x \mathbf{n}_{ki}$ where $\boldsymbol{\omega}^k$ is the angular velocity of L_k in R. Equation (7) may then be rewritten as

$$\mathrm{d}(SOK_{ij})/\mathrm{d}t = -e_{imn}\omega_n{}^k \mathbf{n}_{om} \cdot \mathbf{n}_{kj} \tag{8}$$

or as

$$\mathrm{d}(SOK)/\mathrm{d}t = WOK\ SOK \tag{9}$$

where WOK is a matrix defined as

$$WOK_{im} = -e_{imn}\omega_n{}^k \tag{10}$$

where ω_n^k are the $\mathbf{n}_{on}$ components of $\boldsymbol{\omega}^k$ and e_{imn} is the standard permutation symbol[25, 26]. WOK is simply the matrix whose dual vector[26] is $\boldsymbol{\omega}^k$. Equation (9) thus shows that the derivative may be computed by a simple matrix multiplication.

Finally, Fig. 2 contains symbols not yet defined, but which are also needed in the sequel. O_j $(j = 1, \ldots, N)$ is the reference point of L_j and it is the common point of L_j and its adjacent lower numbered link. G_j $(j = 1, \ldots, N)$ represents the mass center of L_j. $\mathbf{r}_j$ $(j = 1, \ldots, N)$ is the position vector of G_j relative to O_j, and $\boldsymbol{\xi}_k$ is the position vector of O_k relative to O_j. $\mathbf{r}_j$ and $\boldsymbol{\xi}_k$ are thus fixed in L_j.

KINEMATICS

The system shown in Fig. 1 will in general, have $3N + 3$ degrees of freedom. These may be defined in terms of generalized coordinates, x_j $(j = 1, \ldots, 3N + 3)$. X_1, X_2, and X_3 represent the position coordinates of O_1 in R. The succeeding triplets of coordinates represent dextral rotations of the links relative to the adjacent lower links. For example, X_{3k+1}, X_{3k+2}, and X_{3k+3} define the orientation of L_k relative to L_j (see Fig. 2).

Angular velocity

The angular velocity of L_k in R is readily obtained from the familiar addition formula[23]

$$\boldsymbol{\omega}^k = \boldsymbol{\omega}^1 + {}^1\boldsymbol{\omega}^2 + \cdots + {}^j\boldsymbol{\omega}^k \tag{11}$$

where ${}^j\boldsymbol{\omega}^k$ is the angular velocity of L_k relative to L_j. By using the transformation properties of the shifters, ${}^j\boldsymbol{\omega}^k$ may be written as

$$\begin{aligned} {}^j\boldsymbol{\omega}^k = SOJ_{im}(&\dot{\alpha}_{jk}\delta_{ml} + \dot{\beta}_{jk}\alpha JK_{m2} \\ &+ \dot{\gamma}_{jk}\alpha JK_{m1}\beta JK_{l3})\mathbf{n}_{oi} \end{aligned} \tag{12}$$

where δ_{mn} is Kronecker's delta symbol or the identity tensor[25, 26]. Hence, by repeatedly substituting eqn (12) into (11) $\boldsymbol{\omega}^k$ takes the form

$$\boldsymbol{\omega}^k = \omega_{k1m}\dot{X}_L\mathbf{N}_{om} \tag{13}$$

where there is a sum from 1 to $3N + 3$ on 1 and from 1 to 3 on m. From eqn (12), it is seen that the non-zero ω_{k1m} take one of the three forms

$$\omega_{klm} = \begin{matrix} SOJ_{m1} \\ SOJ_{mn}\alpha JK_{n2} \\ SOJ_{mn}\alpha JK_{np}\beta JK_{p3} \end{matrix} \tag{14}$$

depending upon whether x_l is the first, second, or third dextral angle defining the orientation of L_k relative to its adjacent lower link.

Angular acceleration

The angular acceleration of L_k in R may be obtained by differentiating eqn (13). Noting that the n_{om} are constant, this becomes

$$\boldsymbol{\alpha}^k = (\omega_{klm}\ddot{X}_l + \dot{\omega}_{klm}\dot{X}_l)\mathbf{n}_{om} \tag{15}$$

where from eqn (14) the non-zero $\dot{\omega}_{klm}$ take one of the three forms

$$\dot{\omega}_{klm} = \begin{matrix} S\dot{O}J_{m1} \\ S\dot{O}J_{mn}\alpha JK_{n2} + SOJ_{mn}\alpha\dot{J}K_{n2} \\ S\dot{O}J_{mn}\alpha JK_{np}\beta JK_{p3} + SOJ_{mn}\alpha\dot{J}K_{np}\beta JK_{p3} \\ + SOJ_{mn}\alpha JK_{np}\beta\dot{J}K_{p3} \end{matrix} \tag{16}$$

where $S\dot{O}J$ is given by eqn (9) and where $\alpha\dot{J}K$ and $\beta\dot{J}K$ are obtained by differentiating the expressions in eqn (4).

Velocity

From Fig. 1 the position vector of G_j $(j = 1, \ldots, N)$ relative to a fixed point in R is

$$\mathbf{P}^j = \left(X_k + SOJ_{kl}r_l^j + \sum_{M=1}^{j-1} SOM_{kl}\xi_l^M\right)\mathbf{n}_{ok} \tag{17}$$

where, as before, there is a sum over k and 1 from 1 to 3.

The velocity of G_j in R is now obtained by differentiating $\mathbf{p}^j$ in eqn (13) the result may be written as

$$\mathbf{V}^j = V_{jlk}\dot{X}_l\mathbf{n}_{ok} \tag{18}$$

where by eqns (17), (9) and (10) the non-zero V_{jlk} are given by

$$V_{jlk} = \delta_{lk} \quad (j = 1, \ldots, N; l,\ k = 1, \ldots, 3) \tag{19}$$

and

$$V_{jlk} = WSJ_{lkp}r_p^j + \sum_{M=1}^{j-1} WSM_{lkp}\xi_p^M \tag{20}$$

where by eqns (10) and (13) WSM is defined as

$$\begin{aligned} WSM_{lkp} &= -e_{kqn}\,{}_{mln}SOM_{qp} \\ (l = 1, \ldots, 3N + 3;k,\quad & p = 1, \ldots, 3). \end{aligned} \tag{21}$$

Acceleration

The acceleration of G_j in R may be obtained by differentiating eqn (18). This becomes

$$\mathbf{a}^k = (V_{jlk}\ddot{X}_l + \dot{V}_{jlk}\dot{X}_l)\mathbf{n}_{ok} \tag{22}$$

where by eqns (19) and (20), the non-zero $\dot{V}_{jlk}$ are given by

$$\dot{V}_{jlk} = W\dot{S}J_{lkp}r_p^j + \sum_{M=1}^{j-1} W\dot{S}M_{lkp}\xi_p^M \tag{23}$$

where by eqn (21) $W\dot{S}M$ is given by

$$W\dot{S}M_{lkp} = -e_{kqn}(\dot{\omega}_{mln}SOM_{qp} + \omega_{mln}S\dot{O}M_{qp}). \tag{24}$$

Summary

The kinetical description of the system is defined by eqns (13), (15), (18) and (22), and specifically by the four $N \times 3N + 3 \times 3$ block matrices ω_{jlk}, $\dot{\omega}_{jlk}$, V_{jlk} and $\dot{V}_{jlk}$. From eqns (14), (16), (20) and (23) it is seen that each of these matrices may be computed by vector and matrix multiplications which are easily developed into computer algorithms. These matrices play a central role in the development of the equations of motion in the next part.

EQUATIONS OF MOTION

Consider again the chain system of Fig. 1. Let the externally applied force system on each link L_j be replaced by an equivalent force system consisting of a single force $\mathbf{F}_j$, passing through G_j together with a couple with torque $\mathbf{M}_j$. Then Lagranges form of d'Alembert's principle states that the governing dynamical equations of motion for the chain system are

$$F_l + F_l* = 0 \quad l = 1, \ldots, 3N + 3 \tag{25}$$

F_l $(l = 1, \ldots, 3N + 3)$ is called the generalized active force and is given by

$$F_l = V_{jlk}F_{jk} + \omega_{jlk}M_{jk} \tag{26}$$

where there is a sum from 1 to N an j and 1–3 on k, and where F_{jk} and M_{jk} are the components of $\mathbf{F}_j$ and $\mathbf{M}_j$ with respect to $\mathbf{n}_{ok}$. F_l^* $(l = 1,.0.,3N+3)$ is called the generalised inertia force and is given by

$$F_l^* = V_{jlk}F_{jk} + \omega_{jlk}M^*_{jk} \tag{27}$$

where the indices follow the same rules as in eqn (26) and where F^*_{jk} and M^*_{jk} and $\mathbf{n}_{ok}$ components of the inertia forces $\mathbf{F}_j^*$ and inertia torques M_j^* which are given by [23]

$$\mathbf{F}_j^* = -m_j\mathbf{a}^j \quad \text{(no sum)} \tag{28}$$

and

$$\mathbf{M}_j^* = -\mathbf{I}_j \cdot \boldsymbol{\alpha}^j - \boldsymbol{\omega}^j \times (\mathbf{I}_j \cdot \boldsymbol{\omega}^j) \quad \text{(no sum)} \tag{29}$$

where m_j is the mass of L_j and $\mathbf{I}_j$ is the inertia dyadic of L_j relative to G_j $(j = 1, \ldots, N)$.

The forces exerted by the surrounding fluid on the cable links is modelled as follows: If L_j is a typical link then the contribution to the active force system by the fluid forces is modelled by the single $\mathbf{F}_{wj}$ as shown in Fig. 3. $\mathbf{F}_{wj}$ passes through G_j and is perpendicular to the axis of L_j. (The axis is the line joining the reference points of L_j and L_{j+1}.) Analytically, it is expressed as

$$\mathbf{F}_{wj} = -\rho A_j C_D |\mathbf{V}_{j\perp}|\mathbf{V}_{j\perp} \tag{30}$$

where ρ is the mass density of the fluid, A_j is the projected link area on the plane of the axis of L_j, C_D is the drag coefficient, and $\mathbf{V}_j$ is the component of the relative velocity of the link mass center and the fluid velocity perpendicular to the axis of L_j. $\mathbf{V}_j$ is given by

$$\mathbf{V}_j = \boldsymbol{\lambda}_j X[(\mathbf{V}_{Bj} - \mathbf{V}_w) X \boldsymbol{\lambda}_j] \tag{31}$$

where $\mathbf{V}_w$ is the ambient fluid velocity and $\boldsymbol{\lambda}_j$ is a unit vector along the axis of L_j (see Fig. 3).

By substituting eqns (15) and (22) into eqns (28) and (29) and ultimately into eqn (25), the equations of motion may be written in the form

$$a_{lp}\ddot{X}_p = f_l \quad (l = 1, \ldots, 3N+3) \tag{32}$$

where there is a sum from 1 to $3N+3$ on p, and where a_{lp} and f_l are given by

$$a_{lp} = m_j V_{jpk} V_{jlk} + I_{jkn}\omega_{jpn}\omega_{jlk} \tag{33}$$

and

$$f_l = -(F_l + m_j V_{jlk}\dot{V}_{jqk}\dot{X}_q + I_{jkn}\omega_{jln}\dot{\omega}_{jqk}\dot{X}_q + e_{mnk}\omega_{jqn}\omega_{jsr}\omega_{jlk}I_{jmr}\dot{X}_q\dot{X}_s)$$

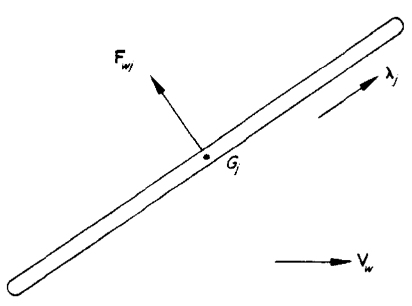

Fig. 3. Fluid force model.

where there is a sum from 1 to N on j, q, and s and a sum from 1 to 3 on the other repeated indices.

Equations (32) form a set of $3N+3$ simultaneous ordinary, nonlinear differential equations determining the $3N+3$ generalized coordinates X_j of the cable or chain system. If some (or all) of the x_j are specified then the differential equations become algebraic equations for the unknown forces or moments associated with the specified x_j. Since the coefficients, a_{lp} and f_l, of these equations are algebraic functions of the physical parameters, and the four block arrays, V_{jlk}, $\dot{V}_{jlk}$, ω_{jlk} and $\dot{\omega}_{jlk}$, the equations may be generated by a computer. Further, once they are developed they may also be solved numerically by the computer by using one of the standard numerical integration routines. The following part presents a brief description of the input–output parameters of a computer code developed for these computations.

COMPUTER CODE

Numerical algorithms to evaluate the above parameters and expressions have been developed, written and compiled into a user-oriented computer code. As input, the code requires: the number of links; the masses; the centroidal principal inertia matrices; the mass center positions; the connection (or reference) point positions; the motion profile for those links with specified motion; the applied forces and moments; the initial configuration; the ambient fluid velocity; the surface height; the fluid mass density; the fluid kinematic viscosity; the projected link areas; the link diameters; and the mass densities of the links.

The code provides for the evaluation of the drag coefficient (see eqn 30) from the Reynolds number using an algorithm which models Hoerner's [27] drag coefficient curve.

The governing equations of motion (eqns 32) are numerically integrated using a fourth order Runge–Kutta technique. However, the computer code is developed so that any other similar integration technique may be substituted.

The output of the code includes: the values of all variables and their first derivatives; the connection (or reference) point positions; the mass center positions; the mass center velocities and accelerations; and the moments and forces associated with the specified variables. All the output data is given at arbitrarily spaced time intervals.

EXAMPLE CONFIGURATION

A sample cable configuration is described in this part of the paper. And although this configuration simulates an off-shore oil rig or a ship crane, it is intended simply as an illustration of the kind of problems that can be studied using the above method. Hence, the results do not describe any specific physical situation, but they are simply qualitative estimates of physical behavior.

The configuration is a rotating surface crane with a 25 ft boom dragging a 50 ft cable attached to a submerged 1 ft dia. sphere. The cable is modelled by 10 links each 5 ft long. The cable diameter is 1 in. and each link has a mass of 0.4025 slug. The sphere mass is 7.727 slug. The water is still and its surface is 10 ft below the boom.

The boom makes a 90° turn in 5 sec. The angular acceleration is given by the graph in Fig. 5. The resultant motion of the sphere is shown in Figs. 6 and 7. The effect of the viscous force is seen by making the same run while

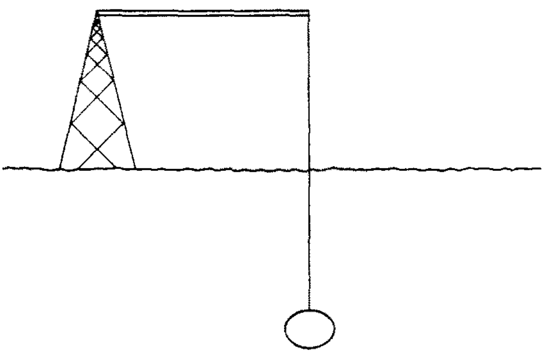

Fig. 4. Boom, cable and submerged sphere.

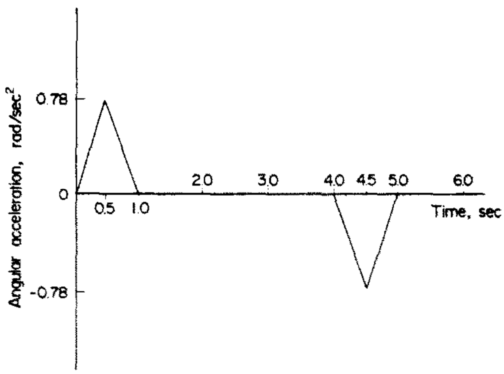

Fig. 5. Angular acceleration of the boom.

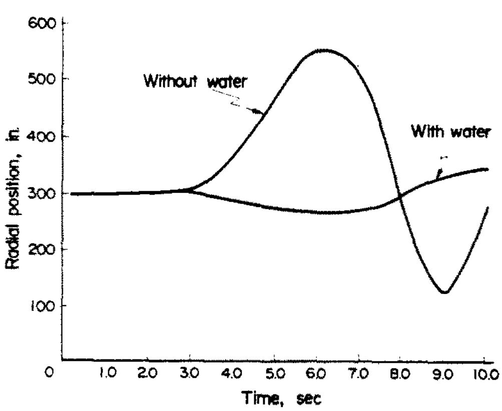

Fig. 6. Radial position of the sphere.

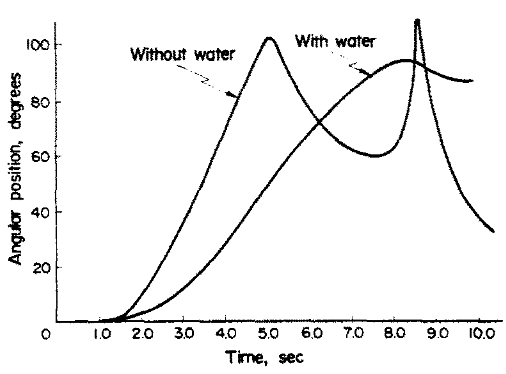

Fig. 7. Angular position of the space.

neglecting the water. It is seen (as one would expect) that the viscous forces tend to counteract the inertia forces.

RESULTS AND DISCUSSION

A nonholonomic, three-dimensional, finite-segment, dynamic model of a cable or chain has been developed. The model which consists of a series of physically arbitrary links connected by ball-and-socket joints is developed so that if the applied force system is known, the motion (position, velocity, acceleration, angular velocity and angular acceleration) of each link is known. Conversely, if the motion is specified, the required driving forces are determined. Also, a combination of the specified forces and motion will allow the determination of the remaining unknown motion and forces.

The model uses "local" as opposed to "global" coordinates. That is, the generalized coordinates of the system are (except for translation of the first link) relative orientation angles measured between the respective links as opposed to absolute orientation angles of the links in space. (This allows for a more convenient specification of initial conditions, constraints, and interpretation of results.) The kinematics is developed following the techniques of vector algebra which provide for the computation of derivatives using vector cross products. This, in turn, provides a means for writing efficient computer algorithms to calculate these derivatives since the basic operations are reduced to addition and multiplication. The development of the governing dynamical equations of motion then follows immediately after the development of the kinematic analysis through using Lagrange's form of d'Alembert's principle. The governing equations are, in general, a set of $3N+3$ second order, quasi-linear ordinary differential equations, where N is the number of links of the cable model. For a given specific configuration, these equations may be integrated numerically using a fourth-order Runge–Kutta technique or another similiar method.

Finally, since the model is primarily oriented toward cable dynamics with finite motion (as opposed to cable statics), its application will be primarily in the analysis of the motion associated heavy chains or long towing cables.

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