Introduction to Quantum Information and Quantum Computation (CS312.9), Spring 2023, IIIT Hyderabad

Assignment

February 13, 2023

Total Points: 25 (+5 Bonus points) Due date: 23rd February, 2023 (**Hard deadline**)

General Instructions: Submit handwritten or typed **PDFs**.

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1. [3 points] **(a)** Consider a trit with the initial probability distribution

$$\mathcal{P} = \left\{\frac{1}{6}, \frac{2}{6}, \frac{3}{6}\right\}.$$

What is the minimum heat cost, determined by the Clausius inequality, due to resetting this trit?

(b) Show that the quantum states $(U \otimes I)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ are mutually orthogonal. Here $U = \{I, \hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z\}$.

2. [4 points] **Pauli rotations**

(a) Let A be matrix such that $A^2 = I$ and x be a real number. Prove that $e^{iAx} = \cos(x)I + i\sin(x)A$.

(b) Use the result of (a) to prove that

$$\exp(-i\theta\hat{\sigma}_x) = \cos(\theta)I - i\sin(\theta)\hat{\sigma}_x$$

(c) Consider the three component spin vector $\vec{\sigma} = (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$. If $\widehat{n} = (n_x, n_y, n_z)$ is a real unit vector in three dimensions, then prove that

$$\exp(-i\theta\widehat{n}\cdot\vec{\sigma}) = \cos(\theta)I - i\sin(\theta)\widehat{n}\cdot\vec{\sigma}$$

3. [5 points] **Evolution of a quantum state and Quantum Zeno effect**

Suppose you are given the Hamiltonian $\hat{H} = \hbar\omega\hat{\sigma}_x$ and that we are working in units where $\hbar\omega = 1$. Assume that the quantum system is initialized in the state $|\psi(0)\rangle = |0\rangle$.

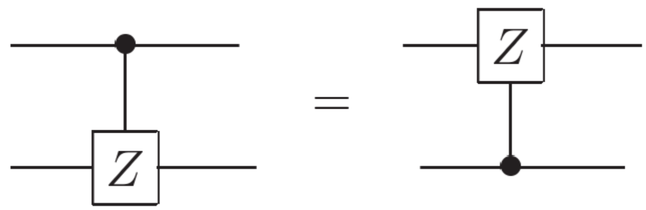
(a) Find the state of the system after time t.

(b) After what time T is the system in the state $|\psi(T)\rangle = |1\rangle$?

(c) Imagine that you make n projective measurements in the $\{|0\rangle, |1\rangle\}$ basis at time intervals $\delta T = T/n$. What is the probability of obtaining $|0\rangle$ in each of the n times when $n = 5$ and when $n = \infty$? Interpret the results.

4. [5 points] **Quantum Gates**

(a) Show that



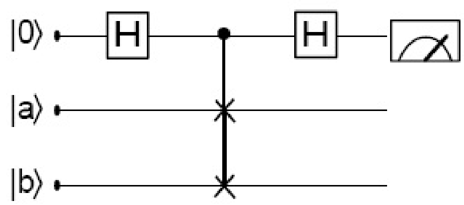
(b) Write the unitary corresponding to the controlled-Z quantum gate shown here.

(c) What would be the output state of the given quantum circuit if the input state is $|+\rangle\,|0\rangle$?

(d) Construct a Controlled SWAP gate using only Toffoli gates.

5. [4 points] **Swap test**

Consider the quantum circuit shown here. The two-qubit gate is a controlled-SWAP gate.



(a) What is the output state?

(b) What is the probability that the first qubit is in the state $|1\rangle$ after measurement?

6. [4 points] **Randomized algorithm for the Deutsch-Jozsa problem**

In class, we have seen the Deutsch-Jozsa problem. We restate it here for convenience. Suppose we are given a black box for some Boolean function $f : \{0,1\}^n \mapsto \{0,1\}$ with the promise that f is either *constant* or *balanced*. In order to determine which is the case with certainty, we have seen that a classical algorithm requires $2^{n-1}+1$ queries to the black box in the worst case, while a quantum query needs only one query. Now we impose some relaxation to the problem. Now, we want to determine whether f is *constant* or *balanced* with a success probability of $1-\varepsilon$, where $\varepsilon \in (0, 1/2)$. How many queries to the black box are needed by a classical algorithm and the quantum algorithm, respectively?

7. [5 points] **Bonus Question: How does your wavefunction evolve?**

The time-independent Schrödinger equation for a free particle is given by

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}.$$

(a) Verify that a typical solution to this equation is a wavefunction of the form $\psi(x,t) = Ae^{i(px-Et)/\hbar}$, where $E = E(p) = p^2/2m$.

(b) Now, consider a free particle localized in space at $t = 0$. To be more precise, assume that the wavefunction of the particle initially is a Gaussian wavepacket centered around the origin and is given by

$$\psi(x,0) = A\exp\left[-x^2/2d^2\right],$$

where d is the spread of the Gaussian wavepacket. Now, calculate how this wavepacket will evolve in time. Show that $|\psi(x,t)|^2 \propto \exp\left[-\frac{x^2}{d^2(1+\frac{\hbar^2 t^2}{m^2 d^4})}\right]$.

(c) What happens to the wavepacket with time? Does the probability of observing the particle in space spreads out or becomes more localized? How fast does it localize or delocalize?

(d) To the result obtained in (c), substitute (i) the mass of the particle by the mass of the electron ($\approx 10^{-27}$ g), the value of d for an electron is $\approx 10^{-8}$cm. (ii) the mass of the particle by your own mass in grams and assume $d \approx 1$cm. For both cases, consider $\hbar \approx 10^{-27}$g cm$^2/s$. How fast does the wavefunction of the electron localize or delocalize? What happens to your wavefunction?