PQIC :-

ASSIGNMENT - 1 :-

1A) Given

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Eigen values are the solutions of the equation in λ

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = 0 \Rightarrow -\lambda(\lambda^2 - 1) - 1(-\lambda) = 0$$

$$-\lambda^3 + \lambda + \lambda = 0$$

$$-\lambda^3 + 2\lambda = 0$$

$$\lambda(2 - \lambda^2) = 0$$

$\lambda = 0$ and $\pm\sqrt{2}$ are the ~~eq~~ eigen values of A

Let the eigen vectors of A be denoted as $|a_1\rangle$, $|a_2\rangle$, $|a_3\rangle$

$|a_1\rangle$ corresponding to $\lambda_1 = 0$ is the solution of

$$\langle A - \lambda I | a_1 \rangle = 0$$

Let $a_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \Rightarrow \begin{matrix} y = 0 \\ x + z = 0 \end{matrix}$$

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$\Rightarrow \quad -z = -x$

$\therefore \quad |a_1\rangle = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

normalized form

$|a_1\rangle = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix}$

Similarly if $|a_2\rangle = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $\quad \lambda_2 = \sqrt{2}$

$$\begin{pmatrix} -\sqrt{2} & 1 & 0 \\ 1 & -\sqrt{2} & 1 \\ 0 & 1 & -\sqrt{2} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$\sqrt{2}x + y = 0 \Rightarrow y = \sqrt{2}x$

$y - \sqrt{2}z = 0 \Rightarrow y = \sqrt{2}z$

$\Rightarrow x = z$

$\therefore \quad |a_2\rangle = \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}$

normalized form $|a_2\rangle = \begin{pmatrix} 1/2 \\ 1/\sqrt{2} \\ 1/2 \end{pmatrix}$

if $|a_3\rangle = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $\quad \lambda_3 = -\sqrt{2}$

$$\begin{pmatrix} \sqrt{2} & 1 & 0 \\ 1 & \sqrt{2} & 1 \\ 0 & 1 & \sqrt{2} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$\sqrt{2}x + y = 0 \Rightarrow y = -\sqrt{2}x$

$y + \sqrt{2}z = 0 \Rightarrow y = -\sqrt{2}z$

$\Rightarrow z = x$

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$$|a_2\rangle = \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$

normalized form $|a_3\rangle = \begin{pmatrix} 1/2 \\ -1/\sqrt{2} \\ 1/2 \end{pmatrix}$

$$\langle a_1|a_2\rangle = \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/2 \\ 1/\sqrt{2} \\ 1/2 \end{pmatrix} = 0 = \delta_{12}$$

$$\langle a_1|a_3\rangle = \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/2 \\ -1/\sqrt{2} \\ 1/2 \end{pmatrix} = 0 = \delta_{13}$$

$$\langle a_2|a_3\rangle = \begin{pmatrix} 1/2 & 1/\sqrt{2} & 1/2 \end{pmatrix} \begin{pmatrix} 1/2 \\ -1/\sqrt{2} \\ 1/2 \end{pmatrix} = 0 = \delta_{23}$$

$\langle a_2|a_1\rangle = \overline{\langle a_1|a_2\rangle} = 0 = \delta_{21}$

Similarly $\langle a_3|a_1\rangle = \delta_{31}$, $\langle a_3|a_2\rangle = \delta_{32}$

Since all are normalized $\langle a_1|a_1\rangle = 1$ $\langle a_2|a_2\rangle = 1$

$\langle a_3|a_3\rangle = 1$

$$\therefore \boxed{\langle a_j|a_k\rangle = \delta_{jk}}$$

$$|a_1\rangle\langle a_1| = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1/2 & 0 & -1/2 \\ 0 & 0 & 0 \\ -1/2 & 0 & 1/2 \end{pmatrix}$$

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$$|a_2\rangle\langle a_2| = \begin{pmatrix} 1/2 \\ 1/\sqrt{2} \\ 1/2 \end{pmatrix} \begin{pmatrix} 1/2 & 1/\sqrt{2} & 1/2 \end{pmatrix} = \begin{pmatrix} 1/4 & \frac{1}{2\sqrt{2}} & 1/4 \\ \frac{1}{2\sqrt{2}} & 1/2 & \frac{1}{2\sqrt{2}} \\ 1/4 & \frac{1}{2\sqrt{2}} & 1/4 \end{pmatrix}$$

$$|a_3\rangle\langle a_3| = \begin{pmatrix} 1/2 \\ -1/\sqrt{2} \\ 1/2 \end{pmatrix} \begin{pmatrix} 1/2 & -1/\sqrt{2} & 1/2 \end{pmatrix} = \begin{pmatrix} 1/4 & -\frac{1}{2\sqrt{2}} & 1/4 \\ -\frac{1}{2\sqrt{2}} & \frac{1}{2} & -\frac{1}{2\sqrt{2}} \\ 1/4 & -\frac{1}{2\sqrt{2}} & 1/4 \end{pmatrix}$$

$$\sum_{i=1}^{3} |a_i\rangle\langle a_i| = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

Given $|b_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $|b_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $|b_3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Let U be the matrix representation from the basis $\{|a\rangle\}$ to $\{|b\rangle\}$

Let the column vectors of U be U_1, U_2, U_3

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$|a_1\rangle = |b_i\rangle U_1$

Let U_1 be $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$\therefore \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow x = 1/\sqrt{2} \quad y = 0 \quad z = -1/\sqrt{2}$$

Similar $\quad U_1 = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix}$

Similarly $\quad U_2 = \begin{pmatrix} 1/2 \\ 1/\sqrt{2} \\ 1/2 \end{pmatrix} \quad U_3 = \begin{pmatrix} 1/2 \\ -1/\sqrt{2} \\ 1/2 \end{pmatrix}$

$$\therefore U = \begin{pmatrix} 1/\sqrt{2} & 1/2 & 1/2 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ -1/\sqrt{2} & 1/2 & 1/2 \end{pmatrix}$$

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2A) Given

$|\psi\rangle = |\phi_1\rangle + |\phi_2\rangle \quad , \quad |\chi\rangle = |\phi_1\rangle - |\phi_2\rangle$

$\therefore \langle\psi| = \langle\phi_1| + \langle\phi_2| \quad , \quad \langle\chi| = \langle\phi_1| - \langle\phi_2|$

$\cancel{\langle\psi|\psi\rangle = (|\phi_1\rangle + |\phi_2\rangle)}$

$\langle\psi|\psi\rangle = \langle\phi_1|\phi_1\rangle + \langle\phi_1|\phi_2\rangle + \langle\phi_2|\phi_1\rangle + \langle\phi_2|\phi_2\rangle$

$\langle\chi|\chi\rangle = \langle\phi_1|\phi_1\rangle - \langle\phi_1|\phi_2\rangle - \langle\phi_2|\phi_1\rangle + \langle\phi_2|\phi_2\rangle$

$$\boxed{\langle\psi|\psi\rangle + \langle\chi|\chi\rangle = 2\left(\langle\phi_1|\phi_1\rangle + \langle\phi_2|\phi_2\rangle\right)}$$

3A)

(a) If $|\psi\rangle$ is normalized

$\langle\psi|\psi\rangle = 1$

$\langle\psi| = \frac{1}{\sqrt{2}}\langle\phi_1| + \frac{1}{\sqrt{5}}\langle\phi_2| + \frac{1}{\sqrt{10}}\langle\phi_3|$

$\langle\psi|\psi\rangle = \frac{1}{2}\langle\phi_1|\phi_1\rangle + \frac{1}{5}\langle\phi_2|\phi_2\rangle + \frac{1}{10}\langle\phi_3|\phi_3\rangle$

$= \frac{1}{2} + \frac{1}{5} + \frac{1}{10} = \frac{8}{10} = \frac{4}{5}$

It is not normalized

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(b) Expectation value of $\hat{B}$ for the state $|\hat{\psi}\rangle$

$$= \frac{\langle\psi|\hat{B}|\psi\rangle}{\langle\psi|\psi\rangle}$$

$$\hat{B}|\psi\rangle = \frac{1}{\sqrt{2}}\hat{B}|\phi_1\rangle + \frac{1}{\sqrt{5}}\hat{B}|\phi_2\rangle + \frac{1}{\sqrt{10}}\hat{B}|\phi_3\rangle$$

$$= \frac{1}{\sqrt{2}}|\phi_1\rangle + \frac{4}{\sqrt{5}}|\phi_2\rangle + \frac{9}{\sqrt{10}}|\phi_3\rangle$$

$$\langle\psi|\hat{B}|\psi\rangle = \frac{1}{\sqrt{2}}\langle\phi_1|\times\frac{1}{\sqrt{2}}|\phi_1\rangle + \frac{4}{\sqrt{5}}\times\frac{1}{\sqrt{5}}\langle\phi_2|\phi_2\rangle + \frac{9}{\sqrt{10}}\times\frac{1}{\sqrt{10}}\langle\phi_3|\phi_3\rangle$$

$$= \frac{1}{2} + \frac{4}{5} + \frac{9}{10} = \frac{22}{10}$$

$$\langle\psi|\psi\rangle = \frac{8}{10}$$

Expectation : $\frac{22}{8} = \frac{11}{4}$

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(C) Expectation value of $\hat{B}^2$

$$= \frac{\langle \psi | \hat{B}^2 | \psi \rangle}{\langle \psi | \psi \rangle}$$

$$\hat{B}^2 |\psi\rangle = B|B|\psi\rangle$$

$$= \cancel{B|n^2|\phi}$$

$$= B\left| \left(\frac{1}{\sqrt{2}} B|\phi_1\rangle + \frac{1}{\sqrt{5}} B|\phi_2\rangle + \frac{1}{\sqrt{10}} B|\phi_3\rangle \right) \right.$$

$$= B\left| \left(\frac{1}{\sqrt{2}} |\phi_1\rangle + \frac{4}{\sqrt{5}} |\phi_2\rangle + \frac{9}{\sqrt{10}} |\phi_3\rangle \right) \right.$$

$$= \frac{1}{\sqrt{2}} |\phi_1\rangle + \frac{16}{\sqrt{5}} |\phi_2\rangle + \frac{81}{\sqrt{10}} |\phi_3\rangle$$

$$\langle \psi | \hat{B}^2 | \psi \rangle = \frac{1}{2} + \frac{16}{5} + \frac{81}{10}$$

$$= \frac{5 + 32 + 81}{10} = \frac{118}{10}$$

$$\langle \psi | \psi \rangle = \frac{8}{10}$$

Expected value $= \frac{118}{8} = \frac{59}{4}$

4A) Given A is a hermitian operator

RTP: all the eigen values are real and the eigen vectors corresponding to diff eigen values are orthogonal

PROOF:

Since A is a hermitian operator

$$A^{+} = A$$

The eigen value equation for A is

$$A|\psi\rangle = \lambda|\psi\rangle$$

where $|\psi\rangle$ is the eigen state vector and λ be the eigen value

$$(A|\psi\rangle)^{+} = (\lambda|\psi\rangle)^{+}$$

$$\langle\psi| A^{+} = \bar{\lambda}\langle\psi|$$

$$\langle\psi| A = \bar{\lambda}\langle\psi|$$

$$\langle\psi| A|\psi\rangle = \bar{\lambda}\langle\psi|\psi\rangle$$

$$\langle\psi| \lambda|\psi\rangle = \bar{\lambda}\langle\psi|\psi\rangle$$

$$\lambda\langle\psi|\psi\rangle = \bar{\lambda}\langle\psi|\psi\rangle$$

since $\langle\psi|\psi\rangle \neq 0$

$$\lambda = \bar{\lambda}.$$

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i.e λ is a real value

Hence eigen values are real.

Consider two eigen vectors corresponding to different eigen values λ_1, λ_2 are $|\psi_1\rangle$ and $|\psi_2\rangle$

$$\therefore \quad A|\psi_1\rangle = \lambda_1|\psi_1\rangle$$

$$\text{and} \quad A|\psi_2\rangle = \lambda_2|\psi_2\rangle$$

$$(A|\psi_2\rangle)^\dagger = (\lambda_2|\psi_2\rangle)^\dagger$$

$$\langle\psi_2|A^\dagger = \bar{\lambda}_2\langle\psi_2|$$

$$\langle\psi_2|A = \lambda_2\langle\psi_2|$$

$$\langle\psi_2|A|\psi_1\rangle = \lambda_2\langle\psi_2|\psi_1\rangle$$

$$\langle\psi_2|\lambda_1|\psi_1\rangle = \lambda_2\langle\psi_2|\psi_1\rangle$$

$$\lambda_1\langle\psi_2|\psi_1\rangle = \lambda_2\langle\psi_2|\psi_1\rangle$$

$$(\lambda_1 - \lambda_2)\langle\psi_2|\psi_1\rangle = 0$$

Since $\lambda_1 \neq \lambda_2$

$$\langle\psi_2|\psi_1\rangle = 0$$

i.e The vectors are orthogonal