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- X_i, the time interval between $i - 1$th and ith event is exponentially distributed with parameter λ.

Lemma

Definition 1/2 $\implies$ *Definition 3*

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$$\begin{aligned} P(X_2 > t | X_1 = s) &= P(N(s, t+s] = 0 | X_1 = s) \\ &= P(N(s, t+s] = 0) \text{ (indep. increments)} \\ &= e^{-\lambda t} \text{ (stat. increments)} \end{aligned}$$

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- This implies X_2 is exponential.Repeating the arguments yields the lemma.

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- $f_{S_n}(t) = \lambda \left[\frac{(\lambda t)^{n-1} e^{-\lambda t}}{n-1!} \right]$ and $F_{S_n}(t) = \int_{x=0}^{t} \lambda \left[\frac{(\lambda x)^{n-1} e^{-\lambda x}}{n-1!} \right] dx$

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Lemma

Exponential interarrival times imply $N(t)$ has Poisson distribution with rate λt

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Splitting: If you label each event point of a Poisson(λ) process as type A or type B with probability p or $1 - p$ respectively, then Events of type A form a Poisson ($p\lambda$) process. Similarly Events of type B form a Poisson ($(1-p)\lambda$) process.

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Lemma

Given that 1 event of $P.P.(\lambda)$ *has happened by time* t, *it is equally likely to have happened anywhere in* $[0, t]$ *i.e.,*
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□

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- Example of such systems: Malls, Tourist spots, Gardens, number of active phone calls, etc

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- Then $P(X(t) = k | N(t) = n) = \binom{n}{k} p^k (1-p)^{n-k}$.
- Now unconditioning on $N(t)$, we get

$$
\begin{aligned}
P(X(t) = k) &= \sum_{n=k}^{\infty} P(X(t) = k | N(t) = n) P(N(t) = n) \\
&= e^{-\lambda t p} \frac{(\lambda t p)^j}{j!}
\end{aligned}
$$

where $p = \int_0^t (1 - G(t-x)) \frac{dx}{t}$.