CS 3.307: Intro to Stochastic Processes

Tejas Bodas

Assistant Professor, IIIT Hyderabad

Recap

Recap

Introduction to Stochastic processes

Introduction to Stochastic processes

- Stochastic process $\{X(t), t \in T\}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a collection of random variables defined such that for every $t \in T$ we have $X(t) : \Omega \to \mathcal{S}$.
- T is the parameter space (often resembles time) and $\mathcal{S}$ is the state space.
- Random variable $X(t)$ is often denoted by $X(\omega, t)$.
- When t is fixed and ω is the only variable, we have a random variable $X(\cdot, t)$. When ω is fixed and t is the variable, we have a $X(\omega, \cdot)$ as a function of time. This is also called as a realization or sampe path of a stochastic process.

Introduction to Stochastic processes

- When T is countable, we have a discrete time process.
- If T is a subset of real line, we have a continuous time process.
- State space could be integers or real numbers
- State space could be $\mathbb{R}^n$ or $\mathbb{Z}^n$ valued

Elementary Examples

- The process of rolling a dice 6 times.
- You bank balance over a week.
- Temperature fluctuations in a 1hr window.
- Number of customers in IKEA every day.

Introduction to Stochastic processes

A c.t.s.p. is called an *independent increment process* if for any choice of parameters $t_0 < t_1 < \ldots < t_n$, the n increment random variables $X(t_1) - X(t_0), X(t_2) - X(t_1), \ldots, X(t_n) - X(t_{n-1})$ are independent.

The c.t.m.p. is said to have *stationary increments* if in addition $X(t_2 + s) - X(t_1 + s)$ has the same distribution as $X(t_2) - X(t_1)$ for all $t_1, t_2 \in T$ and any $s > 0$.

Examples

- Sequence of i.i.d random variables.

- General random walk: If $X_1, X_2, \ldots$ is a sequence i.i.d of random variables, then $S_n = \sum_{i=1}^{n} X_i$ is a random walk.

- Weiner process: $\{X(t), t \geq 0\}$ is a Weiner process if
  1. $X(0) = 0$
  2. $\{X(t), t \geq 0\}$ has stationary and independent increments
  3. for every $t > 0$, $X(t)$ is normally distributed with mean 0 and variance t.

- $\{X(t), t \geq 0\}$ is a Markov process if for $t_1 < t_2 < \ldots t_n < t$ we have
  $P(X(t) \leq x | X(t_1) = x_1, \ldots, X(t_n) = x_n) = P(X(t) \leq x | X(t_n) = x_n)$

- Random walk and Weiner process are examples of Markov processes.

Bernoulli/Binomial process

- Bernoulli(p) random variable

- Bernoulli process is a sequence of independent r.v.'s $\{X_i, i = 1, 2, \ldots\}$ where each X_i is a Bernoulli(p) random variable.

- Binomial random variable S_n counts the sum of n independent Bernoulli(p) variables

- let X_i denote the associated Bernoulli variable for toss i, $i = 1, \ldots, n$. Then $S_n = \sum_{i=1}^{n} X_i$ denotes the number of heads/event and $P(S_n = k) = \binom{n}{k} p^k (1-p)^{n-k}$.

- $E[S_n]$? $\mathrm{Var}(S_n)$?

Bernoulli/Binomial process

- $\{S_n = \sum_{i=1}^{n} X_i, n = 1, 2, \ldots\}$ is called as a Binomial process.
- Let $T := \{\text{smallest } n : S_n > 0.\}$.
- T is a geometric random variable with parameter p, i.e., $P(T = n_1) = p(1-p)^{(n_1-1)}$.
- Memoryless property: $P(T > m + n / T > n) = P(T > m)$.

Counting process

Stochastic process $\{N(t), t \geq 0\}$ is a counting process if it represents the total number of events upto time t.

It satisfies the following

- $N(t) \geq 0$ and is integer valued
- For $s \leq t$, we have $N(s) \leq N(t)$. $N(t) - N(s)$ denotes the number of events in the interval (t, s)
- $N(t)$ can have independent increments
- $N(t)$ can have stationary increments

Poisson process

A Poisson process with rate $\lambda, \lambda \geq 0$ is a counting process $\{N(t), t \geq 0\}$ with the following properties

- $N(0) = 0$
- $N(t)$ has independent and stationary increments
- Number of events in an interval of length t is a Poisson distribution with mean λt. (Hence stationary increments)
- $E[N(t+s) - N(t)] = \lambda s$

Condition 3 is difficult to verify ! Hence ...

Poisson process – Alternative definition

A function f is said to be $o(h)$ if $\lim_{h \to 0} \frac{f(h)}{h} = 0$.

A Poisson process with rate $\lambda, \lambda \geq 0$ is a counting process $\{N(t), t \geq 0\}$ with the following properties

- $N(0) = 0$
- $N(t)$ has independent and stationary increments
- $P\{N(h) = 1\} = \lambda h + o(h)$
- $P\{N(h) \geq 2\} = o(h)$

Poisson process

Lemma

Definition 1 $\implies$ *Definition 2*

Proof on board.

Lemma

Definition 2 $\implies$ *Definition 1*

Self Study: Refer Sheldon Ross, Stochastic processes, Theorem 2.1.1

Poisson Processes Definition 3

A ctsp $\{N(t), t \geq 0\}$ is a Poisson process with rate $\lambda > 0$ if

- $N(0) = 0$
- $N(t)$ is a counting process with stationary and independent increments
- X_i, the time interval between $i - 1$th and ith event is exponentially distributed with parameter λ.

Lemma

Definition 1/2 $\implies$ *Definition 3*

Proof:

- What is $P(X_1 > t) =$?

$$P(X_1 > t) = P(N(0,t) = 0) = e^{-\lambda t}$$

- This implies $F_{X_1}(t) = P(X_1 \leq t) = 1 - e^{-\lambda t}$ and hence X_1 has exponential distribution.
- What is $P(X_2 > t | X_1 = s)$?

$$\begin{aligned} P(X_2 > t | X_1 = s) &= P(N(s, t+s] = 0 | X_1 = s) \\ &= P(N(s, t+s] = 0) \text{ (indep. increments)} \\ &= e^{-\lambda t} \text{ (stat. increments)} \end{aligned}$$

- This implies X_2 is exponential.Repeating the arguments yields the lemma.

Definition 3 $\implies$ Definition 1

Lemma

i.i.d exponential interarrival time implies $N(0,t)$ has Poisson distribution with rate λt.

- Let $S_0 = 0$ and $S_n = \sum_{i=1}^{n} X_i$
- If $S_n = t$, we say that the nth renewal happened at time t.
- $f_{S_n}(t) = \lambda \left[\frac{(\lambda t)^{n-1} e^{-\lambda t}}{n-1!}\right]$ and $F_{S_n}(t) = \int_{x=0}^{t} \lambda \left[\frac{(\lambda x)^{n-1} e^{-\lambda x}}{n-1!}\right] dx$

More on $F_{S_n}(t)$

- $F_{S_n}(t) = \int_0^t \lambda \left[\frac{(\lambda x)^{n-1} e^{-\lambda x}}{n-1!}\right] dx$
- Integration by parts $(u(x) = e^{-\lambda x},\ v'(x) = \lambda \left[\frac{(\lambda x)^{n-1}}{n-1!}\right])$

$$\int_a^b u(x)v'(x)dx = [u(x)v(x)]_a^b - \int_a^b u'(x)v(x)dx$$

- $F_{S_n}(t) = \left[\frac{(\lambda x)^n e^{-\lambda x}}{n!}\right]_0^t - \int_0^t \left[\frac{-\lambda e^{-\lambda x}(\lambda x)^n}{n!}\right] dx$
- $F_{S_n}(t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!} + F_{S_{n+1}}(t)$

$$F_{S_n}(t) - F_{S_{n+1}}(t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}$$

Relation between S_n and $N(t)$

$$N(t) = sup\{n : S_n \leq t\}$$

$$N(t) \geq n \Leftrightarrow S_n \leq t$$

- $P\{N(t) \geq n\} = P\{S_n \leq t\}$
- $P\{N(t) = n\} = P\{N(t) \geq n\} - P\{N(t) \geq n+1\}.$
- $P\{N(t) = n\} = P\{S_n \leq t\} - P\{S_{n+1} \leq t\}.$
- $P\{N(t) = n\} = Poisson(\lambda t).$

Lemma

Exponential interarrival times imply $N(t)$ has Poisson distribution with rate λt

Properties of Poisson Process (Self Study)

Merging: Merging two independent Poisson processes with rate λ_1 and λ_2 leads to a Poisson process with rate $\lambda_1 + \lambda_2$.

Splitting: If you label each event point of a Poisson(λ) process as type A or type B with probability p or $1 - p$ respectively, then Events of type A form a Poisson ($p\lambda$) process. Similarly Events of type B form a Poisson ($(1-p)\lambda$) process.

Conditional distribution of Arrival times

Lemma

Given that 1 event of $P.P.(\lambda)$ has happened by time t, it is equally likely to have happened anywhere in $[0, t]$ i.e.,
$P\{X_1 < s|N(t) = 1\} = \frac{s}{t}$.

Proof.

$$
\begin{aligned}
P\{X_1 < s|N(t) = 1\} &= \frac{P\{X_1 < s, N(t) = 1\}}{P(N(t) = 1)} \\
&= \frac{P\{N[0, s) = 1, N[s, t] = 0\}}{P(N(t) = 1)} \\
&= \frac{P\{N[0, s) = 1\}P\{N[s, t] = 0\}}{P(N(t) = 1)} \\
&= \frac{\lambda s e^{-\lambda s} e^{-\lambda(t-s)}}{\lambda t e^{-\lambda t}} = \frac{s}{t}
\end{aligned}
$$

$\square$

First Queueing Example: Infinite server Queues

- Imagine a system with infinite servers and jobs arrive to this system according to $PP(\lambda)$.
- Every arriving job has a independent service requirement with distribution G and is immediately assigned a server for service.
- When the job receives service, he leaves the system.
- Let $N(t)$ denote the number of arrivals till time t.
- Let $X(t)$ denote the number of customers present in this system at time t.
- Example of such systems: Malls, Tourist spots, Gardens, number of active phone calls, etc

First Queueing Example: Infinite server Queues

- What is the pmf of $X(t)$, i.e., $P(X(t)=k)$?
- First condition on $N(t)$. What is $P(X(t)=k|N(t)=n)$?
- Of the n jobs that arrived (uniformly placed in the interval $[0,t]$), k are yet to complete service.
- Let p denote the probability that an arbitrary of these customers is still receiving service at time t.
- Then $P(X(t)=k|N(t)=n) = \binom{n}{k}p^k(1-p)^{n-k}$.
- Now unconditioning on $N(t)$, we get

$$
\begin{aligned}
P(X(t)=k) &= \sum_{n=k}^{\infty} P(X(t)=k|N(t)=n)P(N(t)=n) \\
&= e^{-\lambda tp}\frac{(\lambda tp)^j}{j!}
\end{aligned}
$$

where $p = \int_0^t (1-G(t-x))\frac{dx}{t}$.