More on Transience and Recurrence

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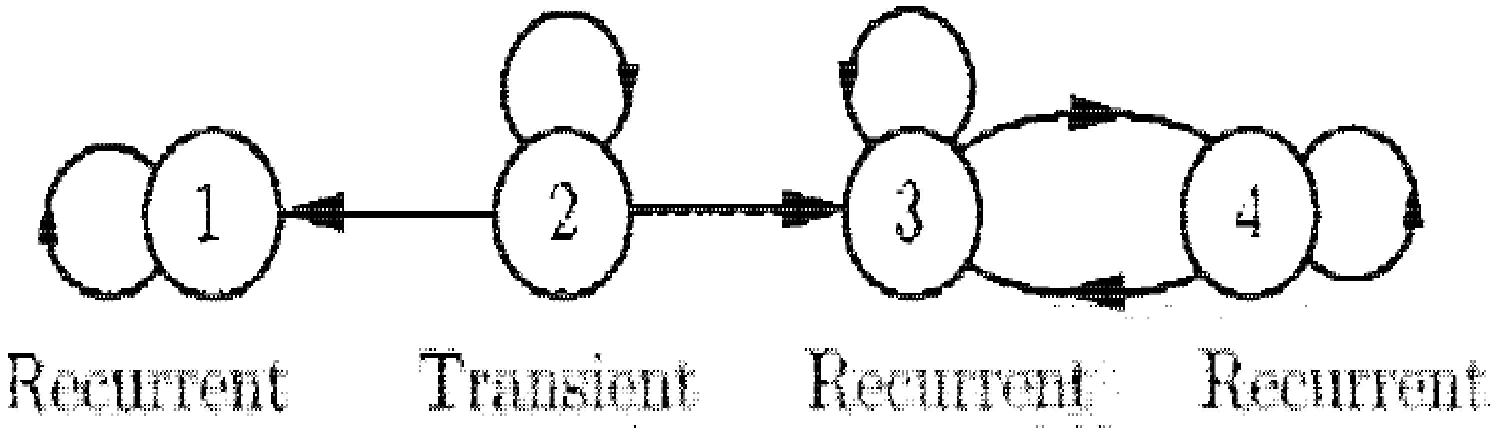
- If $i \leftrightarrow j$ and i is recurrent, then j is recurrent.



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- Let T_{ij} denote the first passage time from i to j.
- T_{ij} has the probability mass function $\{f_{ij}^{n}, n \geq 0\}$. In other words, $P(T_{ij} = k) = f_{ij}^{k}$.

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- For any $i \in \mathcal{M}$, the first return time T_{ii} has the probability mass function $\{f_{ii}^{n}, n \geq 0\}$.

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- For eg: f_{ij}^{t} has a natural interpretation. We wont go further into this.

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- Convergence or divergence of this sum also defines transient or recurrent states.

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- P(exactly infinite visits to i) $= 1$.
- Mean total number of visits is also infinite and hence $\sum_{n=1}^{\infty} p_{ii}^{n}$ diverges.

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- Mean total number of visits to state i is $\frac{F_{ii}}{1-F_{ii}}$ which is finite.
- Hence for transient state $\sum_{n=1}^{\infty} p_{ii}^{n}$ must converge.