Probability Refresher

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- Probability of event A is denoted by $\mathbb{P}(A)$.

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Probability measure $\mathbb{P}$ is a **set function**, i.e. it acts on sets and measures the probability of such sets.

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- Is it also closed under the formation of countable intersections?

When Ω is countable and finite, we will consider power-set $\mathcal{P}(\Omega)$ as the domain.

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- The trio $(\Omega, \mathcal{F}, \mathbb{P})$ is called as a probability space.

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- Bayes rule: $P(B/A) = \frac{P(A/B)P(B)}{P(A)}$.

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- If A and B are mutually exclusive, then they are not independent (and vice versa).

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- P_X is called as an induced probability measure on Ω'.

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- Notation: Random variables denoted by capital letters like X, Y, Z etc. anf their realizations by small letters x, y, z..

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- For $Y = aX + b$, what is $E[Y]$? $E[Y] = aE[X] + b$. (Linearity of expectation)

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- Mean of binomial is np so p should decrease while n increases.

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$$\frac{dF_X(x)}{dx} = f_X(x) \text{ or } P_X(x < X \leq x+h) \simeq f_X(x)h.$$

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- $Var[X] = E[g(X)]$ where $g(x) = (x - E[X])^2$.
- For $Y = aX + b$, $E[Y] = aE[X] + b$.

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Also revise conditioning of variables.

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HW: What if X and Y are not independent?

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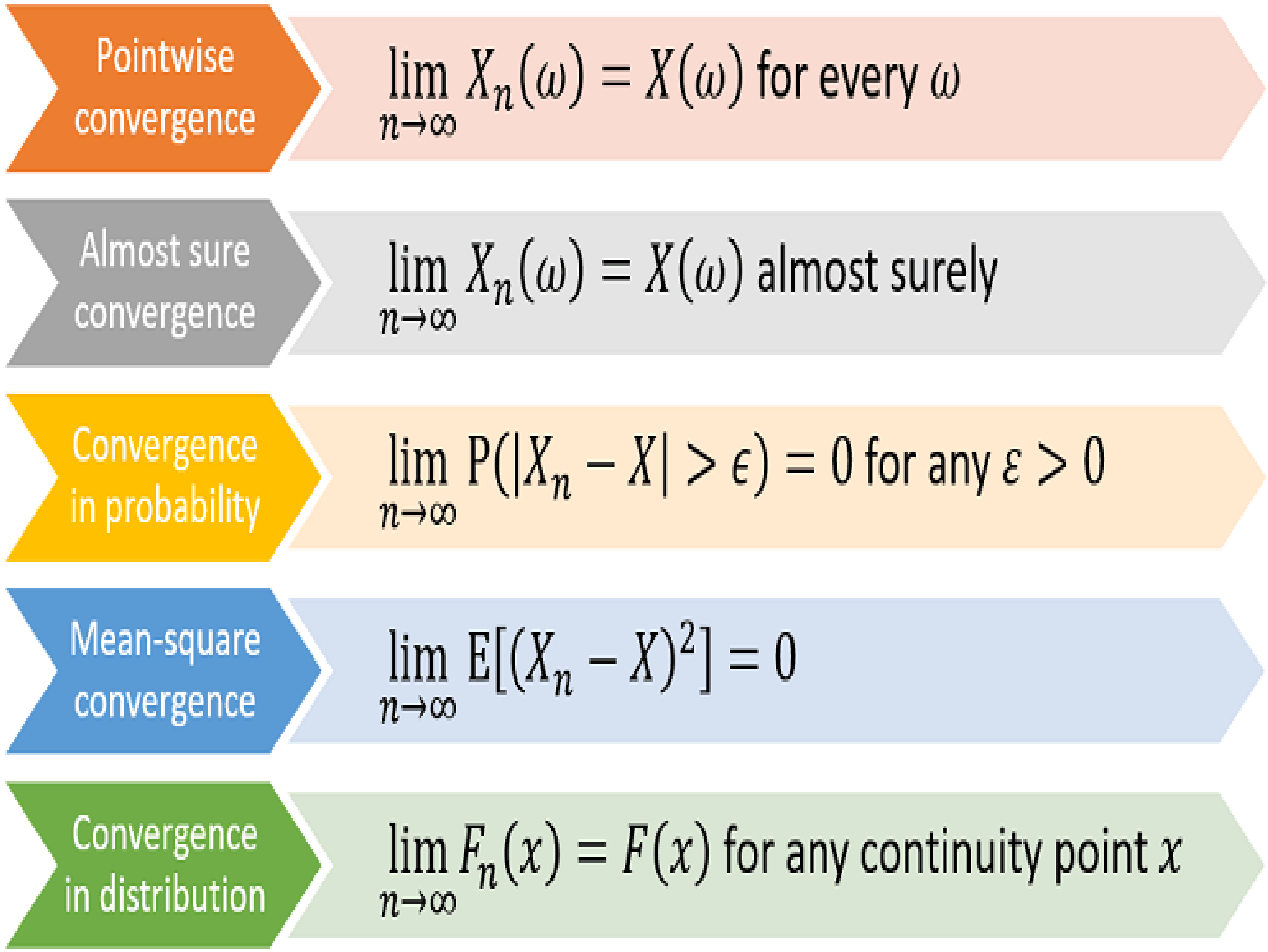
Convergence of Random Variables

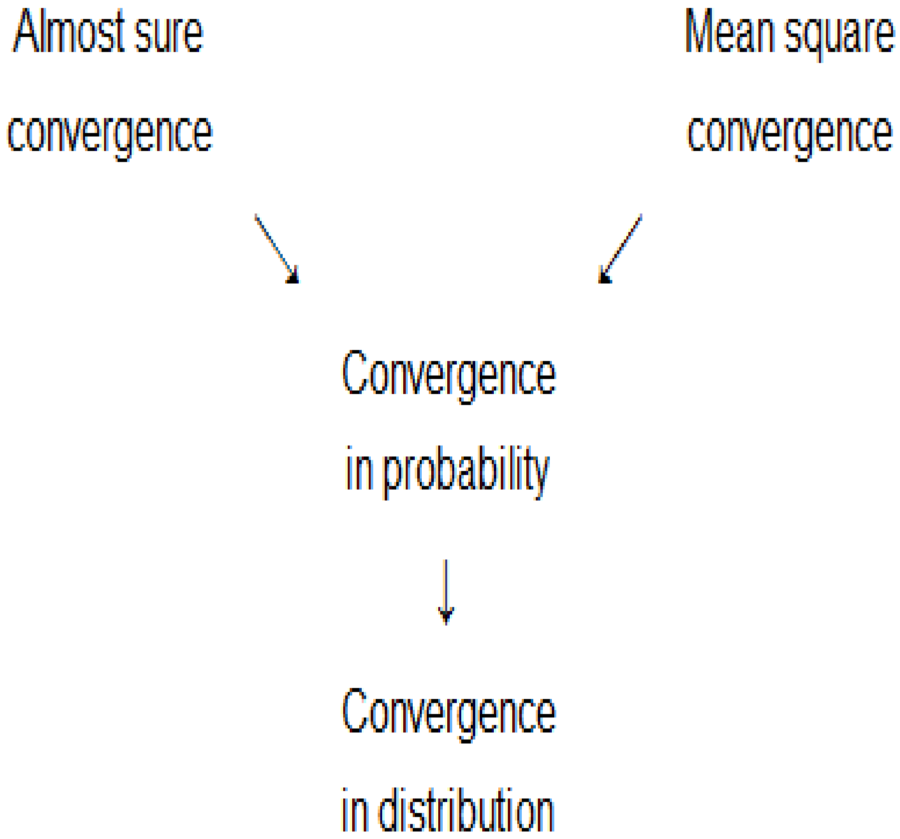
Summary



[1]

[1] Image from probabilitycourse.com

Relation between modes of convergence (no proofs)



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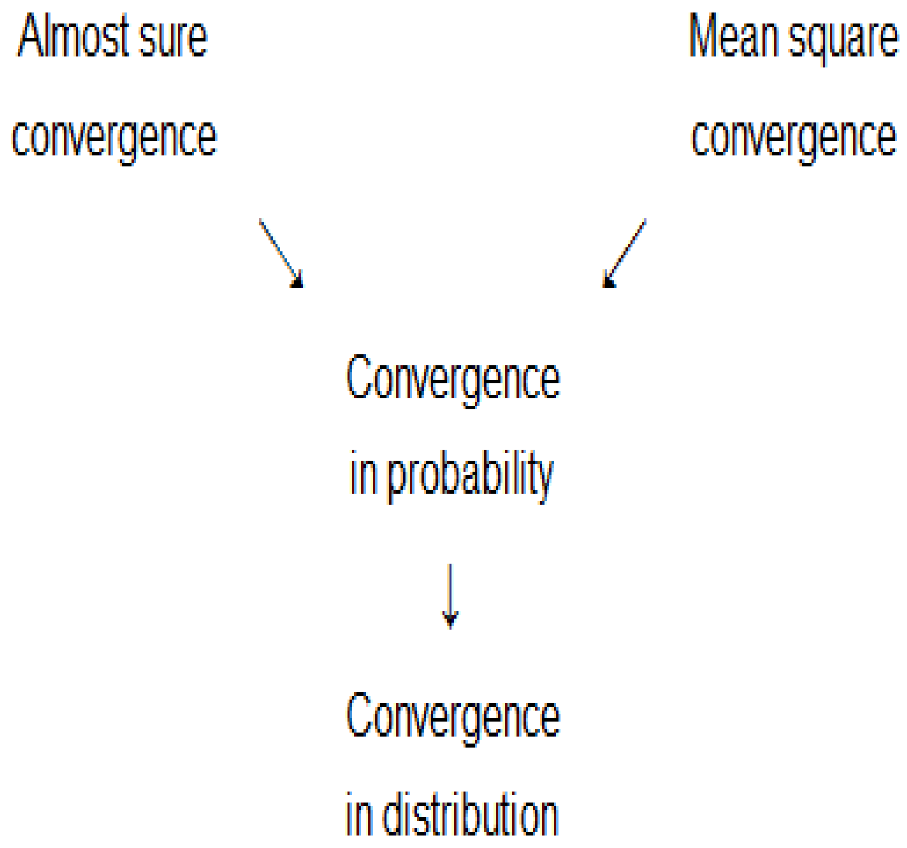

https://en.wikipedia.org/wiki/Proofs_of_convergence_of_random_variables

Introduction to Stochastic processes

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- When t is fixed and ω is the only variable, we have a random variable $X(\cdot, t)$. When ω is fixed and t is the variable, we have a $X(\omega, \cdot)$ as a function of time. This is also called as a realization or sampe path of a stochastic process.

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