Intro to Quantum Information and Computation

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1 Finite-dimensional Hilbert space

A d-dimensional Hilbert space ($1 \leq d \leq \infty$) is defined to be a complex vector space equipped with an inner product. We use the notation $|\psi\rangle$ to denote a vector in $\mathcal{H}$. An inner product is a function $\langle\cdot|\cdot\rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ that satisfies the following properties :

- Non-negativity: $\langle\psi|\psi\rangle \geq 0$ for all $|\psi\rangle \in \mathcal{H}$, and $\langle\psi|\psi\rangle = 0$ if and only if $|\psi\rangle = 0$.
- Conjugate bi linearity: For all $|\psi_1\rangle, |\psi_2\rangle, |\phi_1\rangle, |\phi_2\rangle \in \mathcal{H}$ and $\alpha_1, \beta_1, \alpha_2, \beta_2 \in \mathbb{C}$,

  $$\langle\alpha_1\psi_1 + \beta_1\phi_1|\alpha_2\psi_2 + \beta_2\phi_2\rangle = \bar{\alpha_1}\alpha_2\langle\psi_1|\psi_2\rangle + \bar{\alpha_1}\beta_2\langle\psi_1|\phi_2\rangle + \bar{\beta_1}\alpha_2\langle\phi_1|\psi_2\rangle + \bar{\beta_1}\beta_2\langle\phi_1|\phi_2\rangle .$$

- Conjugate symmetry : $\langle\psi|\phi\rangle = \overline{\langle\phi|\psi\rangle}$ for all $|\psi\rangle, |\phi\rangle \in \mathcal{H}$

2 Postulates

2.1 Postulate 1:The complete description of physical system is given by its state represented by $|\psi\rangle\langle\psi|$ such that$\langle\psi|\psi\rangle = 1$

The state of a quantum system is represented by $|\psi\rangle$ in a Hilbert space. The state $|\psi\rangle$ contains all the information that we can obtain about the system. We work with the normalized states $\langle\psi|\psi\rangle = 1$ called the state vectors. The above statement implies that the sum of the squares of the coordinates must be 1.

2.2 Observables are given by Hermitian operators, which take only real eigenvalues.

In quantum theory, dynamic variables like position, momentum, angular momentum and energy are called Observables.
Let the dimension of a Hilbert space $\mathcal{H}$ be d.
The Orthonormal basis of the considered Hilbert space be $\{|\alpha_1\rangle, |\alpha_2\rangle, \ldots, |\alpha_d\rangle\}$

Let $\mathcal{X} = \sum_{i=1}^{d} c_i |\alpha_i\rangle$ a vector in the hilbert space.
For $\mathcal{X}$ to be a state vector, the condition $\langle\mathcal{X}|\mathcal{X}\rangle = 1$ must be satisfied. Therefore from postulate 1, it implies that

$$\therefore \sum_i |c_i^2| = 1$$

Note: The Standard normal basis of Hilbert space is represented as $|0\rangle, |1\rangle, \ldots, |d-1\rangle$. $|\psi\rangle$ denotes a vector in the Hilbert space. $\langle\psi|$ denotes a vector in the dual of Hilbert space. The inner product of two vectors is given by $\langle\mathcal{X}|\mathcal{X}\rangle = \bar{\mathcal{X}}^{\mathcal{T}}\mathcal{X}$

Linear operators:
Linear operators in quantum theory are for describing the states of a quantum system and the physical evolution of states. Physical evolutions involve measurements and unitary evolutions. Linear operators can be assumed as a transformation mapping from a Hilbert space to another Hilbert space. $\hat{\mathcal{O}} : \mathcal{H}_1 \to \mathcal{H}_2$
Linear operators must be Hermitian ie, $\hat{O} = \hat{O}^\dagger$

Properties $\langle\mathcal{X}, \hat{\mathcal{O}}\mathcal{Y}\rangle = \langle\hat{\mathcal{O}}^\dagger\mathcal{X}, \mathcal{Y}\rangle$ $\mathcal{Y}$ is in $\mathcal{H}_1$ and $\mathcal{X}$ is in $\mathcal{H}_2$ If $\hat{\mathcal{O}}$ is hermitian $\hat{\mathcal{O}} = \hat{\mathcal{O}}^\dagger$ $\implies \langle\mathcal{X}, \hat{\mathcal{O}}\mathcal{Y}\rangle = \langle\hat{\mathcal{O}}\mathcal{X}, \mathcal{Y}\rangle$

2.3 Postulate 3: Measurement $\hat{\mathcal{M}}$ corresponding to an observable $\hat{\mathcal{O}}$ given when for any state $|\psi\rangle$, then $\hat{\mathcal{M}} |\psi\rangle = |a_i\rangle$

The possible results of measurement of an operable are its eigenvalues. Using spectral decomposition, O can be written in terms of its eigenvalues and corresponding projection operators.

What are projective operators?
An operator that satisfies the following properties can be called a projective operator:

$$\mathbb{P}_i^2 = \mathbb{P}_i$$

$$\mathcal{P}_i\mathcal{P}_j = \delta_{xy}\mathbb{P}_x$$

Here, The operable $\hat{\mathcal{O}}$ can be decomposed as (spectral decomposition)
$\hat{\mathcal{O}} = \sum \lambda_i |a_i\rangle \langle a_i| = \sum_i \lambda_i \mathbb{P}_i$
where $\lambda_i \to$ eigenvalues $a_i \to$ eigen vectors.

$\therefore$ $\mathbb{P}_i$=$|a_i\rangle\langle a_i|$ is called a projection operation.
where $\mathbb{P}_i^2 = \mathbb{P}_i$

$$\mathbb{P}_i^2 = |a_i\rangle\langle a_i| \cdot |a_i\rangle\langle a_i|$$

$$\implies \mathbb{P}_i^2 = |a_i\rangle \cdot \langle a_i|a_i\rangle \cdot \langle a_i|$$

$$|a_i\rangle\langle a_i| = \mathbb{P}_i$$

2.4 Postulate 4: Evolution of quantum states is given by Unitary transformation

The evolution of a state in a quantum system is described by a quantum channel, which is linear, completely positive, trace-preserving map acting on the state of the system.

Let's say the initial state is $|\psi\rangle$ at time 0. After an evolution (say at time t) let state become $|\psi'\rangle$

Then there exists a Linear Transformation such that:

$$A|\psi\rangle = |\psi'\rangle$$

Also, On applying $\dagger$:

$$\langle\psi'| = (|\psi'\rangle)^\dagger = (A|\psi\rangle)^\dagger = \langle\psi|A^\dagger$$

$|\psi'\rangle$ is also a quantum state then from 1^{st} postulate, we must have
$\langle\psi'|\psi'\rangle = 1$.
$\langle\psi'|\psi'\rangle = \langle\psi|A^\dagger A|\psi\rangle$
$\langle\psi|A^\dagger A|\psi\rangle = \mathcal{I}$.iff $A^\dagger A = \mathcal{I}$
$\therefore$ A must be a unitary operator.
The whole process is reversible,

$$A^\dagger A = AA^\dagger = \mathcal{I}$$

3 Tensor product

In quantum mechanics, we usually work with multiparticle states. Mathematically to understand the multiparticle system, we need to construct a Hilbert space that is a composite of the independent quantum states of the two particles. The method we follow does this is Tensor product. Let the resultant Hilbert space be H H is given by $\mathcal{H}_A \otimes \mathcal{H}_B$

The product follows the stack and multiply procedure

$$\dim(\mathcal{H}_A \otimes \mathcal{H}_B) = \dim(\mathcal{H}_A) \cdot \dim(\mathcal{H}_B)$$

Let $|\psi\rangle \in \mathcal{H}_A$ $\quad |\phi\rangle \in \mathcal{H}_B$
$|\psi\rangle = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad |\phi\rangle = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$

$\mathcal{H}_A \otimes \mathcal{H}_B$ is the set of vectors obtained by $|\psi\rangle \otimes |\phi\rangle, \forall |\psi\rangle \in \mathcal{H}_A$ and $|\phi\rangle \in \mathcal{H}_B$ by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes \begin{bmatrix} p & q \\ r & s \end{bmatrix} =$$

$$\begin{bmatrix} a\begin{bmatrix} p & q \\ r & s \end{bmatrix} b\begin{bmatrix} p & q \\ r & s \end{bmatrix} \\ c\begin{bmatrix} p & q \\ r & s \end{bmatrix} d\begin{bmatrix} p & q \\ r & s \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} ap & aq & bp & bq \\ ar & as & br & bs \\ cp & cq & dp & dq \\ cr & cs & cr & cs \end{bmatrix}$$

Operators act on the tensor product as follows: let $|\phi\rangle \in H_1$ and $|X\rangle \in H_2$,
Let H be the Hilbert space formed by the composition of $H_1 \& H_2$
Now,

$$(A \otimes B)|\psi\rangle = (A \otimes B)(|\phi\rangle \otimes |X\rangle) = (A|\phi\rangle \otimes B|X\rangle)$$

If A and B are hermitian operators, then $A \otimes B$ is also a hermitian operator.
If A and B are unitary operators, then $A \otimes B$ is also a unitary operator.
In general, $|0\rangle \otimes |0\rangle$ is denoted as $|00\rangle$

4 Noisy quantum theory

The expectation value of an observable respective to a quantum state $|\psi\rangle$ Is represented and obtained as

$$\langle \hat{A} \rangle_\psi = \langle \psi | \hat{A} | \psi \rangle$$

Now consider, $e^{i\phi}|\psi\rangle$,

$$\begin{aligned} \langle A \rangle_{e^{i\phi}|\psi\rangle} &= \langle \psi | e^{-i\phi} \hat{A} \cdot e^{i\phi} | \psi \rangle \\ &= \langle \psi | \hat{A} | \psi \rangle \\ &= \langle A \rangle_\phi \end{aligned}$$

$\therefore$ The same expectation value infers that the two quantum states are similar.

5 Density operator

A Quantum state is represented by a density operator ρ defined on a hilbert space $(\mathcal{H})$ to which the state vector belongs. Mathematically density operator is a linear map on Hilbert space $(\mathcal{H})$ ie,

$$\rho : \mathcal{H} \to \mathcal{H}$$

Any linear map on this Hilbert space obeying the given properties acts as a density operator:

- $\rho \geq 0$ is, positive semi definite
- ρ is hermitian $\rho = \rho^\dagger$
- ρ is a unit trace matrix, that is $Tr[\rho] = 1$

The identical density operator implies equivalent states.

The value $Tr[\rho^2]$ is used to check the purity of a quantum state ρ

- For a pure state ρ
  - $Tr[\rho^2] = 1$
  - Rank of ρ is 1
  - The density operator of a pure state can be obtained as $\rho = |\psi\rangle\langle\psi|$
  - It's a Rank 1 operator as it has only 1 eigenvalue.
- $Tr[\rho^2] < 1 \implies \rho$ is a mixed state

Note:
In general, suppose that there are n possible states. For a state $|\psi\rangle$, from above the density operator is written as $\rho_i = \rho = |\psi\rangle\langle\psi|$. The probability that an observable has been prepared in the state $\rho = |\psi\rangle$ as p_i. Then the density operator for the entire system is given by

$$\rho = \sum_{i=1}^{n} p_i \rho_i = \sum_{i=1}^{n} p_i |\psi_i\rangle\langle\psi_i|$$

5.1 Observables are hermitian operators

- Observables are hermitian operators

The expectation value of an observable $\hat{O}$ for a pure quantum state ρ can be obtained by $Tr[\hat{O}\rho]$.

Note: Never forget the cyclic property of trace ie,

$$Tr[ABC] = Tr[BCA] = Tr[CAB].$$

$\therefore$Tr[$\hat{O}|\psi\rangle\langle\psi|$]=Tr[$\langle\psi|\hat{O}|\psi\rangle$] (Cyclic property of trace)

An identity matrix can be decomposed into the sum of operators ie,

$$\mathbb{I} = \sum |i\rangle\langle i|$$

Here $|i\rangle$ denote the Orthonormal Basis of d dimension.

5.2 Measurement operators can be represented by positive operator valued measures(POVM's) Λ^x

Let the POVM be denoted as Λ^x, which obey the following properties:

- $\Lambda^x \geq 0, \forall x$ x$\rightarrow$outcome
- $\sum_x \Lambda^x = \mathbb{I}$

For a state ρ,probability of outcome 'x' when ${\Lambda^x}_x$ is performed is

$$P(x) = Tr[\Lambda^x \rho]$$

6 Projective Measurement

An observable can be decomposed as (spectral decomposition)

$$\hat{O} = \sum_i \lambda_i |v_i\rangle\langle v_i|$$

where $\lambda_i \rightarrow$ eigenvalues $a_i \rightarrow$ eigen vectors.
$\sum_i |v_i\rangle\langle v_i| = \mathcal{I}$
Also, $|v_i\rangle\langle v_i| \geq 0$ so, we can see $|v_i\rangle\langle v_i|$ as Λ^i

Conversion of a normal operator into a density operator:
Let '$\mathcal{O}$' be an operator

$\dfrac{e^{\mathcal{O}}}{Tr[e^{\mathcal{O}}]}$ is a valid density operator. Since all the entries were made real (makes the matrix positive semi-definite), and trace was made 1 by dividing with $Tr[e^{\mathcal{O}}]$.

Using this set of POVM's, we can find the final state ie,
The final or resultant state =

$$\frac{P_i \rho P_i}{Tr[P_i \rho]}$$

7 Quantum Channel

A transformation that changes the state of a quantum system is called a quantum channel.

7.1 Properties of quantum channel

- A quantum channel is a linear map from a set of density operators to a set of density operators
- A quantum channel is completely positive
- A quantum channel is trace-preserving i.e $Tr[f(\rho)] = Tr[\rho]$
- But it may not be dimension preserving

7.2 What does it mean to be completely positive?

First of all, a positive map is a map that preserves the positivity of the output. i.e

$$\rho \geq 0 \implies f(\rho) \geq 0$$

where ρ is a density operator and f is a map.

Product states

A product state is a state that can be written as a tensor product of two pure states. i.e

$$\rho_{AB} = \rho_A \otimes \rho_B$$

Alternatively given a pure state $|\psi\rangle$ of a composite system AB, it can be expressed in terms of orthogonal states $|i_A\rangle$ for system A and $|i_B\rangle$ for system B as follows from Schmidt Decomposition:

$$|\psi\rangle = \sum_i \lambda_i |i_A\rangle |i_B\rangle$$

where λ_i are the Schmidt coefficients with the property:

$$\sum_i \lambda_i^2 = 1$$

Separable states

A state ρ_{AB} is called separable if it can be written as follows:

$$\rho_{AB} = \sum_x p_x (\rho_A^x \otimes \rho_B^x)$$

where ρ_A^x and ρ_B^x are density operators given an event "x" and p_x are probabilities. It also implies that

$$\sum_x p_x = 1$$

From the above equation, we can see that depending on the event "x", ρ_A and ρ_B change. So, we can say that ρ_A and ρ_B are not independent of each other.

If they are independent i.e they have no correlation, then we can write ρ_{AB} as follows:

$$\rho_{AB} = \rho_A \otimes \rho_B$$

- It is to be noted that determining whether a state is Separable or not is an NP-hard problem.

Entangled States

A state ρ_{AB} is called entangled if it is not separable i.e these states **cannot** be written as

$$\rho_{AB} = \sum_x p_x(\rho_A^x \otimes \rho_B^x)$$

The maximum possible entangled state is the following for a two-qubit system: (Also known as the bell state)

$$\underline{\bar{\phi}}_{AB} = \frac{1}{d}(\sum_{i,j=0}^{d-1,d-1} (|i\rangle_a \otimes |i\rangle_b \langle j|_a \otimes \langle j|_b))$$

where d is the minimum dimension of the two Hilbert spaces.

It can also be written as:

$$\underline{\bar{\phi}}_{AB} = \frac{1}{d}(\sum_{i,j=0}^{d-1,d-1} (|i\rangle_a \langle j|_a \otimes |i\rangle_b \langle j|_b))$$

This is a pure state with ψ as

$$\psi = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle_a \otimes |i\rangle_b$$

It is worth noting that d is the minimum dimension of the two Hilbert spaces. If the dimensions of the two Hilbert spaces are different then we choose d eigenvectors of the two Hilbert spaces in this definition

Completely Positive Maps

Consider a two-state composite system ρ_{AB} and a positive map f. Since it is two state composite we could individually apply a positive map that transforms state A $\rightarrow$ state C to one of the two states and this results in a new composite state $\rho_{CB'}$.

For this map to be completely positive, we need to show that the new composite state $\rho_{CB'}$ is also positive. i.e

$$\rho_{CB'} \geq 0$$

Since $\underline{\overline{\phi}}_{AB}$ is the maximum possible entangled state, checking for this state would be enough to check for all entangled states. Hence if

$$d_B \otimes N_{A\to C}(\underline{\overline{\phi}}_{AB}) \geq 0$$

then the map is completely positive.

The action of any quantum operator Φ on a state ρ is given by:

$$\Phi(\rho) = \sum_k B_k \rho B_k^\dagger$$

where B_k are known as Kraus operators.

These Kraus operators also satisfy $\sum_k B_k^\dagger B_k = I$.

$N_{A\to C}(\underline{\overline{\phi}}_{AB})$ is also known as the Choi operator and checking if the Choi operator is positive is equivalent to checking if the map is completely positive.

Partial Trace Operator

The partial trace operator is defined as follows:

$$Tr_B(X_{AB}) = \sum_i \langle i_B | X_{AB} | i_B \rangle$$

Note that the partial trace operator can output a mixed state i.e $Tr_B(\rho_{AB})$ is not necessarily a pure state although ρ_{AB} is a pure state.

Suppose there exists a two-state composite system with states A, E and a quantum channel N maps the state $A \to B$. Also, let the unitary operator that maps the whole system from $AE \to BE'$ be U

Then N acting on the density operator ρ_A is given by:

$$N(\rho_{A\to B}) = Tr_E(U_{AE\to BE'}(\rho_{AE})U^\dagger_{AE\to BE'})$$

Since U is unitary, we can write:

$$dim(A) * \dim(E) = \dim(B) * \dim(E')$$

Since E' can't be determined, the dimension of B can't be determined. Hence the map is not dimension-preserving.

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