CS 3.307: Intro to Stochastic Processes

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Recap

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Introduction to Stochastic processes

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- Stochastic process $\{X(t), t \in T\}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a collection of random variables defined such that for every $t \in T$ we have $X(t) : \Omega \to \mathcal{S}$.

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- Random variable $X(t)$ is often denoted by $X(\omega, t)$.
- When t is fixed and ω is the only variable, we have a random variable $X(\cdot, t)$. When ω is fixed and t is the variable, we have a $X(\omega, \cdot)$ as a function of time. This is also called as a realization or sampe path of a stochastic process.

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- State space could be $\mathbb{R}^n$ or $\mathbb{Z}^n$ valued

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- Temperature fluctuations in a 1hr window.
- Number of customers in IKEA every day.

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A c.t.s.p. is called an *independent increment process* if for any choice of parameters $t_0 < t_1 < \ldots < t_n$, the n increment random variables $X(t_1) - X(t_0), X(t_2) - X(t_1), \ldots, X(t_n) - X(t_{n-1})$ are independent.

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The c.t.m.p. is said to have *stationary increments* if in addition $X(t_2 + s) - X(t_1 + s)$ has the same distribution as $X(t_2) - X(t_1)$ for all $t_1, t_2 \in T$ and any $s > 0$.

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- Random walk and Weiner process are examples of Markov processes.

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- $E[S_n]$? $\mathrm{Var}(S_n)$?

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- Memoryless property: $P(T > m + n / T > n) = P(T > m)$.

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Condition 3 is difficult to verify ! Hence ...

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- $N(0) = 0$
- $N(t)$ has independent and stationary increments
- $P\{N(h) = 1\} = \lambda h + o(h)$
- $P\{N(h) \geq 2\} = o(h)$

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Lemma

Definition 1 $\implies$ *Definition 2*

Proof on board.

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Self Study: Refer Sheldon Ross, Stochastic processes, Theorem 2.1.1