ASSIGNMENT - 3

Name :- Ch. Sai. Lakshmanarao
Roll No :- 2021101011

Scanned with OKEN Scanner

1A) Given

$$|\psi_k\rangle = \cos\left(\frac{2\pi k}{5}\right)|0\rangle + \sin\left(\frac{2\pi k}{5}\right)|1\rangle$$

$k = 0, 1, 2, \ldots, 4$

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Scanned with OKEN Scanner

We can assume $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

without loss of generality as if it is some another orthogonal basis then it would add a phase factor of $e^{i\phi}$ to $|\psi_k\rangle$ and

$|\psi_k\rangle\langle\psi_k|$ becomes the same as when taking

$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Scanned with OKEN Scanner

$$|\psi_k\rangle = \begin{bmatrix} \cos\left(\frac{2\pi k}{5}\right) \\ \sin\left(\frac{2\pi k}{5}\right) \end{bmatrix}$$

$$\langle\psi_k| = \begin{bmatrix} \cos\left(\frac{2\pi k}{5}\right) & \sin\left(\frac{2\pi k}{5}\right) \end{bmatrix}$$

$$|\psi_k\rangle\langle\psi_k| = \begin{bmatrix} \cos^2\left(\frac{2\pi k}{5}\right) & \cos\left(\frac{2\pi k}{5}\right)\sin\left(\frac{2\pi k}{5}\right) \\ \sin\left(\frac{2\pi k}{5}\right)\cos\left(\frac{2\pi k}{5}\right) & \sin^2\left(\frac{2\pi k}{5}\right) \end{bmatrix}$$

$$2/5\,|\psi_k\rangle\langle\psi_k| = 2/5 \begin{bmatrix} \cos^2\left(\frac{2\pi k}{5}\right) & \cos\left(\frac{2\pi k}{5}\right)\sin\left(\frac{2\pi k}{5}\right) \\ \sin\left(\frac{2\pi k}{5}\right)\cos\left(\frac{2\pi k}{5}\right) & \sin^2\left(\frac{2\pi k}{5}\right) \end{bmatrix}$$

For these set of measurables to be POVMS, first it should be positive, semidefinite.
i.e eigen values ≥ 0

Scanned with OKEN Scanner

__/__/__

$\frac{2}{5}$ Let $A = \frac{2}{5} |\psi_k\rangle\langle\psi_k|$

eigen values of A are the solutions of the eq $|A - \lambda I| = 0$

$$\begin{vmatrix} \frac{2}{5}\cos^2\left(\frac{2\pi k}{5}\right) - \lambda & \frac{2}{5}\cos\left(\frac{2\pi k}{5}\right)\sin\left(\frac{2\pi k}{5}\right) \\ \frac{2}{5}\cos\left(\frac{2\pi k}{5}\right)\sin\left(\frac{2\pi k}{5}\right) & \frac{2}{5}\sin^2\left(\frac{2\pi k}{5}\right) - \lambda \end{vmatrix} = 0$$

on finding eigen values $= 0, \frac{2}{5}$ which is ≥ 0

$\therefore \left\{\frac{2}{5}|\psi_k\rangle\langle\psi_k|\right\}$ is semi positive definite.

For it to be POVM another condition is the completeness relation

i.e

$$\sum_{k=0}^{4} \frac{2}{5} |\psi_k\rangle\langle\psi_k| = I_{2\times 2}$$

Scanned with OKEN Scanner

If $A = \frac{2}{5} \sum_{k=0}^{4} |\psi_k\rangle\langle\psi_k|$

then $A_{11} = \frac{2}{5}\left(\cos^2(0) + \cos^2\left(\frac{2\pi}{5}\right) + \cos^2\left(\frac{4\pi}{5}\right) + \cos^2\left(\frac{6\pi}{5}\right) + \cos^2\left(\frac{8\pi}{5}\right)\right)$

$= \frac{2}{5}\left(1 + 2\cos^2\left(\frac{2\pi}{5}\right) + 2\cos^2\left(\frac{4\pi}{5}\right)\right)$

$= \frac{2}{5}\left(1 + 2\cos^2\left(\frac{2\pi}{5}\right) + 2\cos^2\left(\frac{\pi}{5}\right)\right)$

$\cos^2\left(\frac{\pi}{5}\right) + \cos^2\left(\frac{2\pi}{5}\right) = 1 - \sin^2\left(\frac{\pi}{5}\right) + \cos^2\left(\frac{2\pi}{5}\right)$

$= 1 + \cos\left(\frac{3\pi}{5}\right)\cos\left(\frac{\pi}{5}\right)$

$= 1 - \sin\left(\frac{\pi}{10}\right)\cos\left(\frac{\pi}{5}\right)$

$= 1 - \frac{4}{16} = \frac{3}{4}$

$\therefore \quad A_{11} = 1$

$A_{12} = A_{21} = \frac{2}{5}\left(\cos(0)\sin(0) + \cos\left(\frac{2\pi}{5}\right)\sin\left(\frac{2\pi}{5}\right) + \cos\left(\frac{4\pi}{5}\right)\sin\left(\frac{4\pi}{5}\right) + \cos\left(\frac{6\pi}{5}\right)\sin\left(\frac{6\pi}{5}\right) + \cos\left(\frac{8\pi}{5}\right)\sin\left(\frac{8\pi}{5}\right)\right)$

since $\sin(2\pi - \theta) = -\theta$

$A_{12} = A_{21} = 0$

Scanned with OKEN Scanner

$$A_{22} = \sum \frac{2}{5} \sin^2\left(\frac{2\pi k}{5}\right)$$

$$= \sum \frac{2}{5}\left(1 - \cos^2\left(\frac{2\pi k}{5}\right)\right)$$

$$= 2 - \sum \cos^2\left(\frac{2\pi k}{5}\right) = 2 - A_{11} = 1$$

$$\therefore A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

$$\therefore \left\{ \frac{2}{5} |\psi_k\rangle\langle\psi_k| \right\}_{k=0}^{4} \text{ is a POVM.}$$

For them to be projective measurement, it should satisfy an additional condition of

$$\cancel{\delta_i \delta_j} = \Pi_i \Pi_j = \delta_{ij} \Pi_i$$

$$\frac{2}{5}\left(|\psi_k\rangle\langle\psi_k|\right)\left(|\psi_l\rangle\langle\psi_l|\right)\frac{2}{5}$$

if $\cancel{k \neq l}$ $k = l$

It should follow that

$$p^2 = p$$

where $p = \frac{2}{5} |\psi_k\rangle\langle\psi_k|$

let $\frac{2\pi k}{5} = \theta$

$$\therefore p^2 = \frac{4}{25}\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$$

$$\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$$

Scanned with OKEN Scanner

$$= \frac{4}{20}\begin{bmatrix} \cos^4\theta + \cos^2\theta\sin^2\theta & \cos^2\theta\sin\theta + \cos\theta\sin^3\theta \\ \cos^2\theta\sin\theta + \sin^3\theta\cos\theta & \cos^2\theta\sin^2\theta + \sin^4\theta \end{bmatrix}$$

$$= \frac{4}{20}\begin{bmatrix} \cos^2\theta(1) & \cos\theta\sin\theta(1) \\ \cos\theta(\sin\theta) & \sin^2\theta(1) \end{bmatrix}$$

$$= \frac{2}{5} \times P$$

$$\therefore P^2 = \frac{2}{5}P \neq P$$

$\therefore$ It is not a projective measurement.

2A) Given,

Walsh - Hadamard transform denoted by H performs the following transformation

$$|0\rangle \rightarrow \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$|1\rangle \rightarrow \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

Regardless of what $|0\rangle$, $|1\rangle$ consider

H to be (H, H_2)

($|0\rangle$, $|1\rangle$ are 2 dimensional)

let $H_1 = \begin{pmatrix} x \\ y \end{pmatrix}$

Then

$$|0\rangle = \frac{x}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) + \frac{y}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

$$x - y = 0$$

$$\frac{x+y}{\sqrt{2}} = 1 \implies x = y = 1/\sqrt{2}$$

$\therefore H_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$ It is already normalized

Similarly let $H_2 = \begin{pmatrix} x \\ y \end{pmatrix}$

Then

$$|\phi\rangle = \frac{x}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) + \frac{y}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

$$x + y = 0$$

$$\frac{x-y}{\sqrt{2}} = 1 \implies x = 1/\sqrt{2} \quad y = -1/\sqrt{2}$$

$\therefore H_2 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$

Scanned with OKEN Scanner

$$\therefore H = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$

(i) The matrix representation of H for the basis $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

is $\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$

(ii) Since H doesn't depend on the values of $|0\rangle$, $|1\rangle$ the matrix representation is the same i.e

$$\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$

(iii) Inverse of this operator H

$$= \frac{\mathrm{Adj}(H)}{|H|}$$

$$= \frac{\begin{pmatrix} +\left(-1/\sqrt{2}\right) & -\left(1/\sqrt{2}\right) \\ -\left(1/\sqrt{2}\right) & +\left(1/\sqrt{2}\right) \end{pmatrix}}{-1/2 - 1/2}$$

Scanned with OKEN Scanner

$$= \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$

3A) Given

$$A = \frac{1}{\sqrt{2}} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & i \\ 0 & -i & 1 \end{bmatrix}$$

~~Since~~

It can easily ~~of~~ be observed that

$$A^* = A$$

$\therefore$ A is a normal matrix and it has a spectral decomposition

eigen values of A are

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2/\sqrt{2} - \lambda & 0 & 0 \\ 0 & 1/\sqrt{2} - \lambda & i/\sqrt{2} \\ 0 & -i/\sqrt{2} & 1/\sqrt{2} - \lambda \end{vmatrix} = 0$$

$$\left(2/\sqrt{2} - \lambda\right)\left[\left(\frac{1}{\sqrt{2}} - \lambda\right)^2 - 1/2\right] = 0$$

$\lambda = 2/\sqrt{2}$ (or) 0 i.e $\lambda = \sqrt{2}, 0$

Scanned with OKEN Scanner

Let Eigen vector corresponding to $\lambda = \sqrt{2}$ be $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

So $AX = \lambda X$

$$\begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1/\sqrt{2} & i/\sqrt{2} \\ 0 & -i/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} \sqrt{2}x \\ \frac{y+iz}{\sqrt{2}} \\ \frac{-yi+z}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \lambda x \\ \lambda y \\ \lambda z \end{pmatrix}$$

For $\lambda = \sqrt{2}$ $\quad y + iz = 2y$, $\quad -iy + z = 2z$ $\Rightarrow y = iz$

$\therefore$ eigen vector corresponding to $\lambda = \sqrt{2}$ is the basis of $\begin{pmatrix} x \\ iz \\ z \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + z\begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix}$

$\therefore$ normalized eigen vectors $= \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ i/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \right]$

Similarly

For $\lambda = 0$

normalized eigen vectors $= \begin{pmatrix} 0 \\ -i/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$

Scanned with OKEN Scanner

(i) $\therefore$ The P_m corresponding to eigen vector with value 0

$$= \begin{pmatrix} 0 \\ -i/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 0 & i/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1/2 & -i/2 \\ 0 & i/2 & 1/2 \end{pmatrix}$$

$$p(m) = \langle \psi | P_m | \psi \rangle$$

$$|\psi\rangle = \begin{pmatrix} 1/6 \\ 0 \\ 2/3 \end{pmatrix} \times \frac{6}{\sqrt{17}} \quad \text{(Normalizing factor)}$$

$$P_m|\psi\rangle = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1/2 & -i/2 \\ 0 & i/2 & 1/2 \end{pmatrix} \begin{pmatrix} 1/6 \\ 0 \\ 2/3 \end{pmatrix} \times \frac{6}{\sqrt{17}}$$

$$= \begin{pmatrix} 0 \\ -i/3 \\ 1/3 \end{pmatrix} \times \frac{6}{\sqrt{17}}$$

$$\langle \psi | P_m | \psi \rangle = \begin{pmatrix} 1/6 & 0 & 2/3 \end{pmatrix} \begin{pmatrix} 0 \\ -i/3 \\ 1/3 \end{pmatrix} \times \frac{36}{17} = 2/9 \times \frac{36}{17}$$

The probability that value is 0 $= 2/9 \times \frac{36}{17} = \frac{8}{17}$

Scanned with OKEN Scanner

(ii). First we should calculate the new state $|\psi\rangle$

new state : $\dfrac{P_m|\psi\rangle}{\sqrt{p(m)}}$

$$= \frac{\begin{pmatrix} 0 \\ -i/3 \\ 1/3 \end{pmatrix} \times \sqrt{\frac{8}{17}}}{\sqrt{2}/3 \times \sqrt{\frac{8}{17}}} = \begin{pmatrix} 0 \\ -i/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

We have to find the probability of getting value $= 1$

Let eigen vector of B corresponding to eigen value 1 $= \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$\therefore \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} x \\ -iz \\ iy \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\therefore y = -iz$$

$\therefore$ corresponding to 1 eigen vector(s) = basis of $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$= x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ -i \\ 1 \end{pmatrix}$$

Scanned with OKEN Scanner

eigen vectors = $\begin{pmatrix}1\\0\\0\end{pmatrix}$, $\begin{pmatrix}0\\-i/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}$

~~P~~ Let $p_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$ $p_2 = \begin{pmatrix}0\\-i/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}$

$\therefore$ $P(\text{getting } 1) = \langle\psi|P_1|\psi\rangle + \langle\psi|P_2|\psi\rangle$

~~$P_1|\psi$~~ where $P_1 = p_1 p_1^*$

$= \begin{pmatrix}1\\0\\0\end{pmatrix}\begin{pmatrix}1 & 0 & 0\end{pmatrix}$

$= \begin{pmatrix}1 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{pmatrix}$

$P_2 = \begin{pmatrix}0\\-i/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}\begin{pmatrix}0 & i/\sqrt{2} & 1/\sqrt{2}\end{pmatrix} = \begin{pmatrix}0 & 0 & 0\\0 & 1/2 & -i/2\\0 & i/2 & 1/2\end{pmatrix}$

Since clearly one of the eigen vector $p_2 = \begin{pmatrix}0\\-i/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}$

current

is the ~~new~~ state, it would mean that we are applying the same measurement

$\therefore$ $P(\text{getting } 1) = 1$