CS 3.307
Performance Modeling for Computer Systems

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Markov Process

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- State space in both cases can be integers or general ($\mathbb{R}^d$)
- We will stick with integer or finite state space

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- The row corresponds to present state and the column corresponds to next state.

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- In the ctmc counterpart for these examples, imagine the coin tosses itself/ dice rolls itself after waiting in the state for a random time that is exponentially distributed. (more later)

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- $i, j \in \mathcal{S}$ which is countable and $|\mathcal{S}| \leq \infty$

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- Y_n is geometric random variable.
- What would be the time spent in a state for a continuous time Markov chain ?

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Theorem

$$P(Y_t > u | X(t) = i) := \bar{G}_i(u) = e^{-a_i u}$$

for all $i \in \mathcal{S}$ and $t \geq 0, u \geq 0$ and for some real number $a_i \in [0, \infty]$.

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- $\bar{G}_i(u+v) = \bar{G}_i(u)\bar{G}_i(v)$
- Only CCDF function which satisfies this equation is the exponential distribution. This requires a proof. We will skip this part.