ASSIGNMENT - 1

1A) (a) (a) Given probability distribution

$$P = \{1/6, 2/6, 3/6\}$$

W.K.T Shannon entropy $= \sum P_x \log(1/P_x)$

$$= \frac{1}{6}\log 6 + \frac{2}{6}\log 3 + \frac{3}{6}\log 2$$

$$= \frac{\log 6}{6} + \frac{\log 3}{3} + \frac{\log 2}{2}$$

After assorting it will collapse to a state

$\therefore$ entropy $= 1 \cdot \log 1 = 0$

$$\Delta S = -\left(\frac{\log 6}{6} + \frac{\log 3}{3} + \frac{\log 2}{2}\right)$$

From Clausius inequality:-

$$\frac{\Delta Q}{T} \geq K_B \Delta S$$

$$\Delta Q \geq K_B \Delta S\, T$$

$$\geq -K_B T \left(\frac{-\log 6}{6} + \frac{\log 3}{3} + \frac{\log 2}{2}\right)$$

$\therefore$ The minimum heat cost

$$= K_B T \left(\frac{\log 2}{2} + \frac{\log 3}{3} + \frac{\log 6}{6}\right)$$

where K_B = boltzmann constant

T = temperature

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(b)

<u>R-T.P:</u> The quantum states

$$(U \otimes I)\ \tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

are mutually orthogonal.

where $U = (I, \hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$

<u>Proof:</u>

Consider $x_1, x_2 \in U$ and $x_1 \neq x_2$

and $|\psi_1\rangle = (x_1 \otimes I)\ \tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

$|\psi_2\rangle = (x_2 \otimes I)\ \tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

$\langle\psi_2| = \tfrac{1}{\sqrt{2}}(\langle 00| + \langle 11|)\ (x_2 \otimes I)^{+}$

$\langle\psi_2|\psi_1\rangle = \tfrac{1}{\sqrt{2}}(\langle 00| + \langle 11|)\ (x_2 \otimes I)^{+}\ (x_1 \otimes I)\ \tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

$= \tfrac{1}{2}(\langle 00| + \langle 11|)\ (x_2^{+} \otimes I^{+})\ (x_1 \otimes I)\ (|00\rangle + |11\rangle)$

$= 0$ Since $x_2^{+}, x_1 = 0$

and $(x_1 \otimes I)(x_2 \otimes I) = (x_1 x_2) \otimes (I \otimes I)$

$\langle \psi_i | \psi_i \rangle = 0$

$\therefore$ The state are mutually orthogonally

2A) (a)

Given $A^2 = I$

$$e^{iAx} = \cos(x) I + i \sin x (A)$$

W.K.T

$$e^{i\theta} = \cos\theta + i \sin\theta$$

$$\therefore e^{iAx} = \cos(Ax) + i \sin x (A) (Ax)$$

W.K.T :

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \text{———}$$

$$\cos Ax = I - \frac{(Ax)^2}{2!} + \frac{(Ax)^4}{4!} + \text{———}$$

$$= I - A^2 \frac{x^2}{2!} + A^4 \frac{x^4}{4!} + \text{———}$$

Given $A^2 = I$

$$= I - \frac{x^2}{2!} I + I \frac{x^4}{4!} + \text{———}$$

$$= I \cos x$$

$$\therefore e^{iAx} = \cos(x) I + i \sin x (A) (Ax)$$

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(b)

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

$$\sin Ax = Ax - \frac{A^3x^3}{3!} + \frac{A^5x^5}{5!} \cdots$$

$$= A\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)$$

$$= A\sin x$$

$$\therefore \boxed{e^{iAx} = \cos x\, I + i\sin x\, A} \quad \text{—(1)}$$

(b) W.K.T

for pauli matrices A

$$A^2 = I$$

$\therefore$ From (1)

$$e^{i\sigma_x x} = \cos x\, I + i\sin x\, A$$

Substituting x with $-x$

$$e^{-i\sigma_x x} = \cos x\, I - i\sin x\, A \quad \text{—(2)}$$

(c) consider $\hat{n}\cdot\hat{\sigma}$

$\hat{n} = (n_x, n_y, n_z)$ $\quad \bar{\sigma} = (\bar{\sigma}_x, \bar{\sigma}_y, \bar{\sigma}_z)$

$\therefore \hat{n}\cdot\hat{A} = n_x\hat{\sigma}_x + n_y\hat{\sigma}_y + n_z\hat{\sigma}_z$

$$\ln\left|\frac{\psi(t)}{\psi(0)}\right| = e^{-(iHt/\hbar)}$$

$$\psi(t) = \psi(0)\, e^{-(iHt/\hbar)}\, \psi(0)$$

$$\boxed{\psi(t) = |0\rangle \cdot e^{-(i\omega_z t)}\, |0\rangle}$$

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(b) $|1\rangle = e^{-i(\omega t)\sigma_x} |0\rangle$

Assuming σ_x to be the Pauli Matrix since $\sigma_x^2 = I$

$$e^{-i(\omega t)\sigma_x} = (\cos(\omega t))\, I - i(\sin \omega t)\, \sigma_x$$

$$|1\rangle = \cos(\omega t)\, |0\rangle - i \sin(\omega t)\, \sigma_x |0\rangle$$

$$|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \cos \omega t \begin{bmatrix} 1 \\ 0 \end{bmatrix} - (i \sin \omega t) \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

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4A) (a)

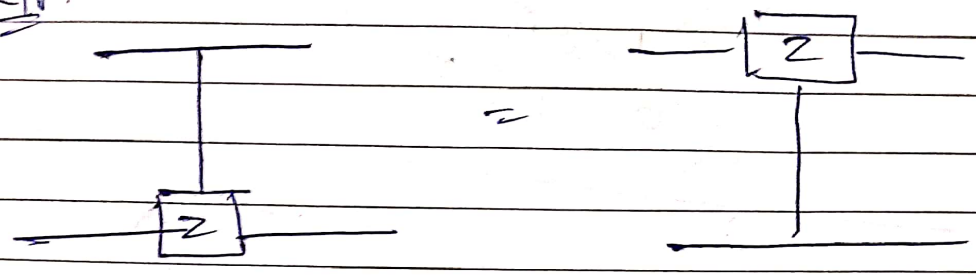
The Z gate corresponds to the pauli-Z

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

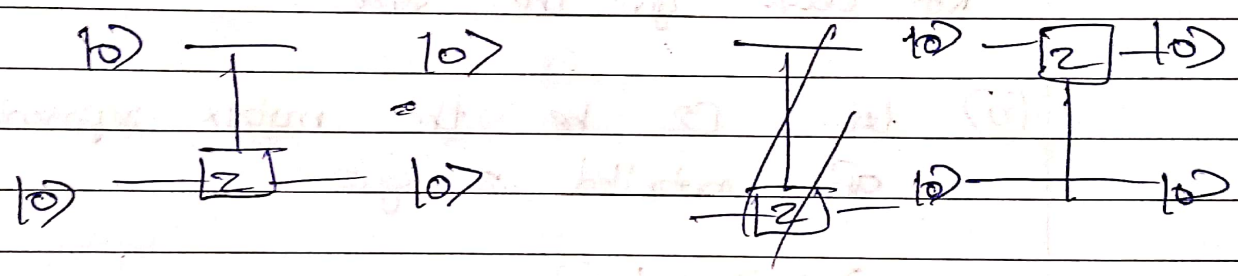
ie $Z|0\rangle = |0\rangle$

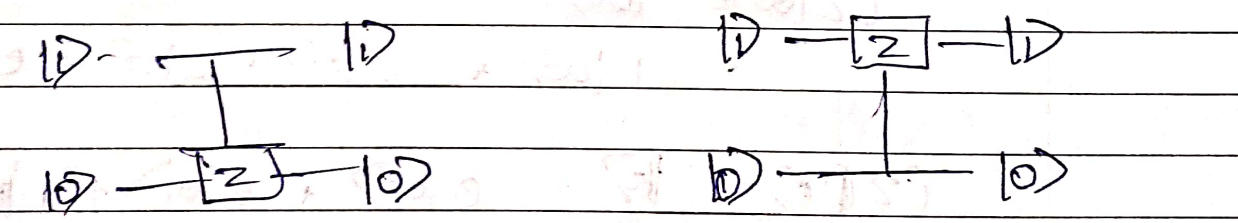
$Z|1\rangle = -|1\rangle$

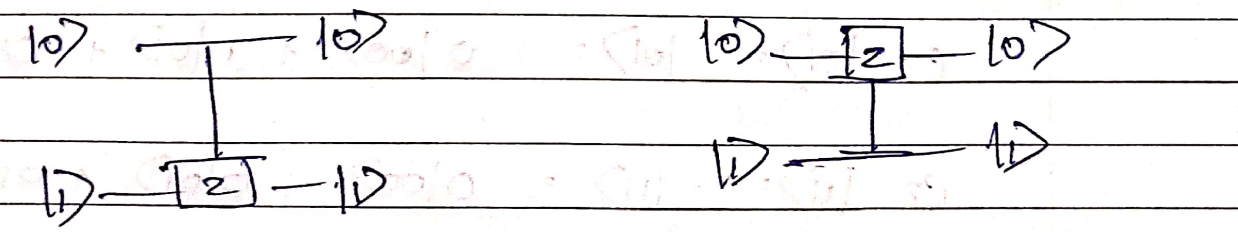
RTP:

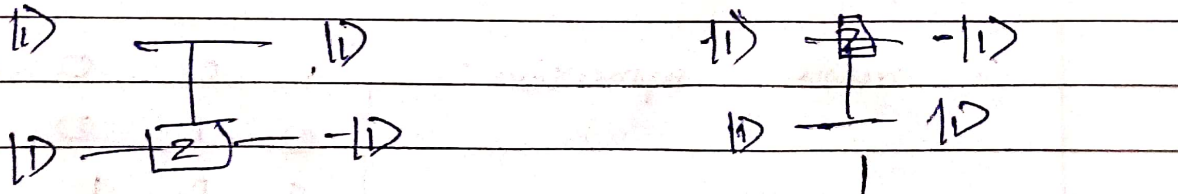


Proof: Consider all possible input (basis)









↳ superposition of $|1\rangle, -|1\rangle$

↳ superposition of $|1\rangle, -|1\rangle$

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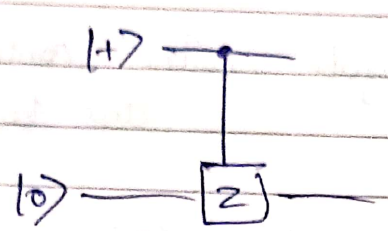
Since it is also unitary, the unitary ~~Representation~~ representation of CZ =

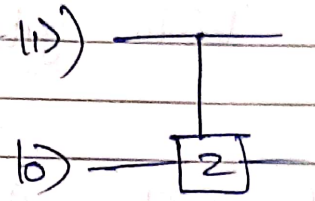
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

(c) If input state = $|+\rangle|0\rangle$

i.e



$$= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$



i.e $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\,|0\rangle$

We need to find $CZ(|+\rangle|0\rangle)$

$$CZ(|+\rangle|0\rangle) = CZ\left(\frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)\right)$$

$$= \frac{1}{\sqrt{2}}\cancel{|00\rangle} + CZ(|00\rangle) + \frac{1}{\sqrt{2}}CZ(|10\rangle)$$

$$= \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|10\rangle$$

$$= \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle) = |+\rangle|0\rangle$$

$\therefore$ The output state is $|+\rangle|0\rangle$

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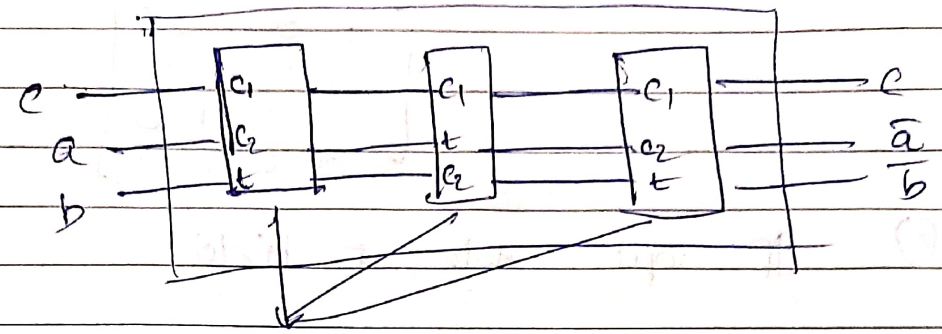
(d)

Construct the circuit as follows



Tofolli gate with controls c_1, c_2 and target t.

If $c=0$ ie control is zero Tofolli gate would'nt be set and $\therefore$ $\bar{a}=a$, $\bar{b}=b$

If $c=1$, $a=0$, $b=0$

after 1st Tofolli gate : 1, 0, 0

2nd : 1, 0, 0

3rd : 1, 0, 0

$\therefore$ $\bar{a} = b = 0$

$\bar{b} = a = 0$

If $c=1$, $a=0$, $b=1$

1st Tofolli gate : 1, 0, 1

2nd : 1, 1, 1

3rd : 1, 1, 0

$\bar{a} = b = 1$, $\bar{b} = a = 0$

If $c=1$, $a=1$, $b=0$

1st Tofolli gate : 1, 1, 1

2nd Tofolli gate : 1, 0, 1

3rd : 1, 0, 1

$\bar{a} = b = 0$, $\bar{b} = a = 1$

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If $c=1$, $a=1$, $b=1$

1st Tofolli gate : 1, 1, 0

2nd Tofolli gate : 1, 1, 0

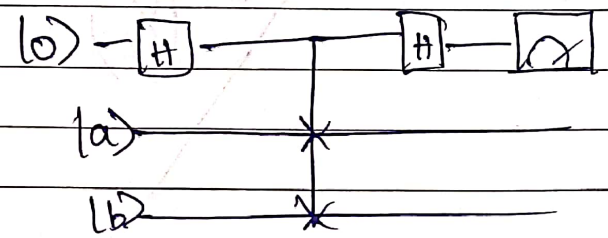
3rd Tofolli gate : 1, 1, 1

$a=b=1$, $b=a=1$

$\therefore$ The given circuit is a C-SWAP

5A) b

(a)



$$|0ab\rangle \xrightarrow{H\otimes I\otimes I} \left|\tfrac{1}{\sqrt{2}}(|0\rangle+|1\rangle)\,ab\right\rangle = \tfrac{1}{\sqrt{2}}\left(|0ab\rangle + |1ab\rangle\right)$$

$$\xrightarrow{\text{CSWAP}} \tfrac{1}{\sqrt{2}}\left(|0ab\rangle + |1ba\rangle\right)$$

$$\xrightarrow{H\otimes I\otimes I} \tfrac{1}{\sqrt{2}}\left(\tfrac{1}{\sqrt{2}}(|0\rangle+|1\rangle)|ab\rangle + \tfrac{1}{\sqrt{2}}(|0\rangle-|1\rangle)|ba\rangle\right)$$

$$= \tfrac{1}{2}\left(|0ab\rangle + |1ab\rangle + |0ba\rangle - |1ba\rangle\right)$$

Output state

$$= \tfrac{1}{2}\left[|0ab\rangle + |1ab\rangle + |0ba\rangle - |1ba\rangle\right]$$

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(b) From the output state $|\psi\rangle$

$$\frac{1}{2}\left[|0ab\rangle + |0ba\rangle + |1ab\rangle - |1ba\rangle\right]$$

$$= \frac{1}{2}\left[|0\rangle\left[|ab\rangle + |ba\rangle\right] + |1\rangle\left[|ab\rangle - |ba\rangle\right]\right]$$

~~norm of $\left(\psi' = |1\rangle\left[|ab\rangle - |ba\rangle\right]\right)$~~

~~$\langle ba|$~~

norm of $\left(\psi' = |1\rangle\left[|ab\rangle - |ba\rangle\right]\right)$

$$= \langle\psi'|\psi'\rangle$$

$$\langle\psi'| = \langle ba| - \langle ab|$$

$$\langle\psi'|\psi'\rangle = \left[\left(\langle ba|\right) - \left(\langle ab|\right)\right]\left[|ab\rangle - |ba\rangle\right]$$

$$= 1 - \langle ba|ba\rangle - \langle ab|ab\rangle + 1$$

$$= 2 - \langle ba|ba\rangle - \langle ab|ab\rangle$$

If $\langle ba|ba\rangle = \langle ab|ab\rangle = 1$

i.e $|ab\rangle = |ba\rangle$ ~~then~~ where

i.e $a = b$

~~$\langle\psi'|\psi'\rangle = 0$~~

~~since $a, b \in \{0,1\}$ if~~

$a, b \in \{0,1\}$

if $a \neq b$ $\langle ba|ba\rangle = \langle ab|ab\rangle = 0$

as $\langle a|b\rangle = \langle 1|0\rangle$ or $\langle 0|1\rangle = 0$

$\therefore$ if $a \neq b$ $\langle \psi'|\psi'\rangle = 2$

$$\therefore |\psi\rangle = \frac{1}{2}\Big[|0\rangle \big[|ab\rangle + |ba\rangle\big] + |1\rangle \big[|ab\rangle - |ba\rangle\big]$$

can be rewritten as

if $(a \neq b)$

$$|\psi\rangle = \tfrac{1}{2}\big(|0ab\rangle + |0ba\rangle\big) + \tfrac{1}{2} \times \sqrt{2}\,|\psi'\rangle$$

where $|\psi'\rangle$ is normalized

$\therefore$ probability (of getting 1 in the first wire)

$$= (\text{amplitude})^2$$

$$= \left(\tfrac{1}{2} \times \sqrt{2}\right)^2 = \tfrac{1}{2}$$

if $a = b$,

then

$$|\psi\rangle = \tfrac{1}{2}\big(|0ab\rangle + |0ba\rangle\big)$$

$\therefore$ probability (getting 1 in the first wire) $= 0$

$$\therefore \quad p = \begin{cases} 0 & \text{if } a = b \\ 1/2 & a \neq b \end{cases}$$

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(c)

W.K.T

$$|\psi(t)\rangle = e^{-(i\omega_x t)}|0\rangle \quad \text{— (1)}$$

$$(\theta = \omega t)$$

$$e^{-i\omega_x t} = \cos\theta\, I - i\sin\theta\, \sigma_x$$

$$(\sigma_x|0\rangle = |1\rangle)$$

$$|\psi(t)\rangle = \cos\theta\,|0\rangle - i\sin\theta\,|1\rangle$$

The probability of getting $|0\rangle$ ~~is~~ the when a projective measurement is made in the basis of $\{|0\rangle, |1\rangle\}$:

$$|\langle 0|\psi(t)\rangle|^2$$

$$= |\cos\theta\,\langle 0|0\rangle - i\sin\theta\,\langle 0|1\rangle|^2$$

$$= \cos^2\theta$$

$$(\theta = \omega t)$$

$$= \cos^2 \omega t$$

The new state after this measurement would be $|0\rangle$.

Again the system starts changing according to eqn (1) i.e it "resets" if $|0\rangle$ is observed.

$\therefore$ If we measure at time intervals $\delta T = T/n$

The probability of obtaining ~~zero~~ $|0\rangle$:

$$\cos^2\left(\omega \frac{T}{n}\right)$$

If $n=5$,

probability : $\cos^2(\omega T/5)$

If $n=\infty$,

probability = $\cos^2(\omega T/\infty) = \cos^2 0 = 1$

i.e It can be interpreted as the system is not leaving the initial state $|0\rangle$ when we measure it ~~con~~ frequently as the probability that the measured state is $|0\rangle$ = 1

6A) Consider, a classical algorithm that randomly queries for K values of f(x) and if all the K queries are the same then it outputs the function f(x) is const or else it outputs that f(x) is balanced.

Let the prob of error of this algorithm be e.

Error occurs when f(x) is const but ~~f(x)~~ algo outputs balanced and when f(x) is bal but algo gives f(x) as const.

When algo outputs that f(x) is balanced, that means all the K queries are not the same, hence f(x) will definitely be balanced.

Therefore there would be no error.

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But when algo outputs that $f(x)$ is const, that means all the k queries are the same.

But if $k << n/2$, then $f(x)$ can be balanced.

ie $P\left(\frac{f \text{ is balanced}}{\text{all } k \text{ queries are same}}\right) = \epsilon$

From bias theorem

$$P\left(\frac{f \text{ is bal}}{\text{all } k \text{ queries are same}}\right) = \frac{P\left(\frac{\text{all } k \text{ queries are same}}{f \text{ is bal}}\right)}{P\left(\frac{\text{all } k \text{ queries are same}}{f \text{ is bal}}\right) + P\left(\frac{\text{all } k \text{ queries are same}}{f \text{ is const}}\right)}$$

$$= \frac{2\left(\frac{n/2}{n}\right)^k}{2\left(\frac{n/2}{n}\right)^k + 1}$$

[Since if f is bal, there are $n/2$ 0's and $n/2$ 1's and probability of getting the same elem k times $2\left(\frac{n/2}{n}\right)^k$]

$$= \frac{\frac{2}{2^k}}{\frac{2}{2^k} + 1}$$

$$= \frac{2}{2+2^k}$$

$$\therefore \frac{2}{2+2^k} = \epsilon$$

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$$\frac{2}{\epsilon} = 2 + 2^k$$

$$\frac{2}{\epsilon} - 2 = 2^k$$

$$k = \log_2\left(\frac{2}{\epsilon} - 2\right)$$

$$\therefore \boxed{k = O\left(\log_2\left(\frac{1}{\epsilon}\right)\right)}$$

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While for the quantum counter part we can determine if the function is constant or balanced with zero error in 1 query using the Deutsch-Jozsa Algorithm.

Hence only 1 query is required.

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7A) (a)

Given $\psi(x,t) = Ae^{i(px-Et)/\hbar}$

$$\frac{\partial \psi}{\partial t} = A e^{i/\hbar (px-Et)} \times (-E) \times i/\hbar$$

$$\frac{\partial \psi}{\partial x} = A e^{i/\hbar (px-Et)} \times p \times i/\hbar$$

$$\frac{\partial^2 \psi}{\partial x^2} = A e^{i/\hbar (px-Et)} \times p^2 \times i^2/\hbar^2$$

$$i\hbar \frac{\partial \psi}{\partial t} = i\hbar A e^{i/\hbar (px-Et)} \times (-E) \times i/\hbar$$

$$= i\hbar A e^{i/\hbar (px-Et)} \times \left(-\frac{p^2}{2m}\right) \times i/\hbar$$

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$$i\hbar \frac{\partial \psi}{\partial t} = \left(- i^2 A e^{i/\hbar (px - Et)} \times \frac{p^2}{2m} \right)$$

$$\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} = \frac{\hbar^2}{2m} \times A e^{i/\hbar (px - Et)} \times p^2 \times \frac{i^2}{\hbar^2}$$

$$\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} = i^2 \times A e^{i/\hbar (px - Et)} \times p^2/2m$$

clearly $\boxed{i\hbar \frac{\partial \psi}{\partial t} = \frac{-\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2}}$

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