Performance modeling of CS

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Renewal Processes

A Renewal process is a counting process for which the inter-arrival times are i.i.d with an arbitrary distribution.

Renewal process is a generalization of Poisson process where the inter-arrival times were i.i.d exponential.

Renewal Processes - Notations

- $\{X_n, n \geq 0\}$ denote the sequence of inter-arrival times of a renewal process.
- X_n is the time between $n-1$th and nth renewal.
- $\{X_n, n \geq 0\}$ are non negative iid random variables with law F.
- Let $S_0 = 0$ and $S_n = \sum_{i=1}^{n} X_i$

> The term renewal process refers to any of the following:
> 1) The sequence $\{X_n, n \geq 0\}$ of inter-arrival times
> 2) The sequence $\{S_n, n \geq 0\}$ of arrival times
> 3) The associated counting process $\{N(t), t \geq 0\}$.

Renewal Process Examples

- Bernoulli/Binomial process.
- Poisson process.
- Successive times between your water bottle gets empty.
- Time instants when number of customers in Ikea is exactly 10.
- Time between successive visits to a particular state of a Markov Chain.

Relation between S_n and $N(t)$

- Define $N(t) = sup\{n : S_n \leq t\}$. $N(t)$ signifies the number of renewals until time t.

$$N(t) \geq n \Leftrightarrow S_n \leq t$$

- $P\{N(t) \geq n\} = P\{S_n \leq t\}$
- $P\{N(t) = n\} = P\{N(t) \geq n\} - P\{N(t) \geq n+1\}$.
- $P\{N(t) = n\} = P\{S_n \leq t\} - P\{S_{n+1} \leq t\}$.
- $P\{N(t) = n\} = F_n(t) - F_{n+1}(t)$.
- How do you obtain F_n from F ?

Convolution basics

- Convolution of two functions functions:
  $(f * g)(t) := \int_{-\infty}^{\infty} f(t-u)g(u)du$
- Convolution of two positive functions:
  $(f * g)(t) := \int_{0}^{t} f(t-u)g(u)du.$
- Convolution of a function w.r.t a distribution:
  $(f * G)(t) := \int_{-\infty}^{\infty} f(t-u)dG(u) = \int_{-\infty}^{\infty} f(t-u)g(u)du$

Convolution basics

- Convolution of distirbutions is also a distribution.
- If F and G are two distributions then $(F * G)(t) := \int_{-\infty}^{\infty} F(t-u)dG(u)$. $F^{(*2)}(t) = (F * F)(t)$.
- $P(S_n \leq t) := F_n(t)$. We will now express $F_n(t)$ as a convolution!
- $P(S_1 \leq t) = P(X_1 \leq t) = F(t)$ where $F(t)$ is the cdf of the interarrival time X_1.
- $P(S_2 \leq t) = P(X_1 + X_2 \leq t) = \int_0^t F(t-u)f(u)du$
- $= \int_0^t F(t-u)dF(u) = (F * F)(t)$.

$$F_n(t) = F^{(*n)}(t)$$

Laplace transform basics

- Bilateral Laplace transform of function $f(\cdot)$ is given by $\bar{f}(s) := \int_{-\infty}^{\infty} e^{-st} f(t) dt$.
- Consider a random variable X with distribution F. Then recall that $M_X(s) = E[e^{sX}] = \int e^{st} dF(t)$.
- The laplace transform of a distribution F is defined as $\bar{F}(s) = \int_{-\infty}^{\infty} e^{-st} dF(t) = E[e^{-sX}]$.Here X is a random variable with distribution F.
- Property: Consider $Z(t) = (f * F)(t)$. Then $\bar{Z}(s) = \bar{f}(s)\bar{F}(s)$
- This implies that if $Z(t) = (F * F)(t)$, then $\bar{Z}(s) = \bar{F}^2(s)$.
- Then by the same logic, $LT\{F^{(*n)}(t)\} = \bar{F}^n(s)$.

Renewal equation $m(t)$

Let $m(t)$ denote the mean number of arrivals by time t, i.e., $m(t) := E[N(t)]$. Then $m(t) = \sum_{n=1}^{\infty} F_n(t)$.

What is $m(t)$ for the Poisson process?

Let $\bar{m}(s), \bar{F}(s)$ and $\bar{F}_n(s)$ denote the Laplace transform of $m(t), F(t)$ and $F_n(t)$ respectively. Then $\bar{m}(s) = \frac{\bar{F}(s)}{1-\bar{F}(s)}$.

- $\bar{m}(s) = \bar{F}(s) + \bar{m}(s)\bar{F}(s)$. Inverse Laplace transform gives
- $m(t) = F(t) + (m * F)(t)$.

Renewal equation

- Renewal equation is an integral equation for $m(t)$ that is obtained by conditioning on time for first renewal.
- Suppose $X_1 = x$. Since this is the time interval between 0th and 1st arrival, $S_1 = x$ and the first arrival has happened at x.
- $m(t) = E[N(t)] = E_F[E[N(t)/X_1]]$.
- Therefore $m(t) = \int_0^\infty E[N(t)/X_1 = x] dF(x)$
- What if $t < x$? Then $E[N(t)/X_1 = x] = 0$.
- What happens when $t \geq x$?
- $E[N(t)/X_1 = x] = 1 + m(t - x)$.
- This gives us the renewal equation $m(t) = \int_0^t (1 + m(t - x)) dF(x)$.

Stopping times

- Let $X_1, X_2, \ldots$ be a sequence of independent random variables.
- An integer valued positive random variable N is said to be a stopping time for this sequence if the event $\{N = n\}$ is independent of $X_{n+1}, X_{n+2}, \ldots$ for $n = 1, 2, \ldots$.
- N is not independent of the entire sequence $\{X_i\}$.
- Think as if we are seeing $X_n's$ one at a time and stop after a stopping criteria is met.
- So if we stop after seeing $X_1, X_2, \ldots, X_n$, then $N = n$.
- Suppose $P(X_n = 1) = P(X_n = -1) = 0.5$. Then $N = min\{n : X_1 + \ldots + X_n = 1\}$ is a stopping time.
- Stop one roll before you see 6. Is this a stopping time ?No.

Stopping times for Renewal process

- Is $N(t)$ a stopping time for the sequence of interarrivals X_i?
- Suppose $N(t) = n$, i.e., by time t there have been only n arrivals. Then what we know is that $S_n \leq t$ and that $S_{n+1} > t$.
- Therefore $N(t) = n$ depends on X_{n+1}. For it to be a stopping time, it should have been independent of X_{n+1}.
- Therefore $N(t)$ is not a stopping time.
- However $N(t) + 1$ is a stopping time. This is because $N(t) + 1 = n$ implies $N(t) = n - 1$ for which $S_{n-1} \leq t$ and that $S_n > t$.
- $N(t) + 1 = n$ depends on $X_1, \ldots X_n$ and is independent of $X_{n+1}, X_{n+2}, \ldots$

Wald's Equation

Theorem

If $X_1, X_2, \ldots,$ are independent and identically distributed random variables having finite expectations, and if N is a stopping time for $X_1, X_2, \ldots$ such that $E[N] < \infty$, then

$$E[\sum_{i=1}^{N} X_i] = ENEX$$

Proof on board. Also Refer Sheldon Ross, Thm 3.3.2.

Corollary

$E[S_{N(t)+1}] = E[X](m(t)+1)$

Time average versus Ensemble average

$$\bar{X}^{time-avg} = \lim_{t\to\infty} \frac{\int_0^t X(u,\omega)du}{t}$$

$$\bar{X}^{ensemble} = \lim_{t\to\infty} E(X(t))$$

For an ergodic process, $\bar{X}^{time-avg} = \bar{X}^{ensemble}$

- Consider a Markov coin (with unknown transition probabilities) and given a budget of 10,00,000 (10 lakh) tosses, how will you find the stationary probability of head?
- Exhaust all at once (time average)
- Perform 100 runs each of length 10000 and average across the last toss in each run! (ensemble average)

Renewal theorem

Lemma

- *With probability 1,* $\frac{N(t)}{t} \to \frac{1}{E[X_1]}$ *as* $t \to \infty$.
  (Proof hint:- $S_{N(t)} \leq t \leq S_{N(t)+1}$*)*

- $\frac{m(t)}{t} \to \frac{1}{E[X_1]}$ *as* $t \to \infty$.

See Sheldon Ross (Stochastic Processes, 2nd edition) Proposition 3.3.1 and Thm 3.3.4 for proof.
NOTE: $S_{N(t)+1} > t$. Taking Expectations on both sides, invoking Wald's lemma, and rearranging gives us $\liminf_{t\to\infty} \frac{m(t)}{t} \geq \frac{1}{E[X_1]}$

Renewal Reward theorem

- Consider a renewal process with interarrival times $X_i, i = 1, 2, \ldots$. Suppose a random reward Y_i is earned at the time of the ith arrival. While Y_i may depend on X_i, the pairs (X_i, Y_i) are independent and identically distributed.

- Let $Y(t)$ denote the total reward accrued till time t. Then $Y(t) = \sum_{i=1}^{N(t)} Y_i$.

Lemma

- *With probability 1,* $\frac{Y(t)}{t} \to \frac{E[Y]}{E[X]}$ *as* $t \to \infty$.

- $\frac{E[Y(t)]}{t} \to \frac{E[Y]}{E[X]}$ *as* $t \to \infty$.

See Sheldon Ross Theorem 3.6.1 for proof.