MA 6.101
Probability and Statistics

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Surprize Quiz

Just Kidding!

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- What if the alphabets and numbers are not to repeat?

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- Using principles of counting, the number of indistinguishable arrangements are $\frac{6!}{3!2!1!}$

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- $P(C_3/G) = \frac{P(C_3 \cap G|C_1)P(C_1) + P(C_3 \cap G|C_1^c)P(C_1^c)}{P(G)} = 2/3$

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