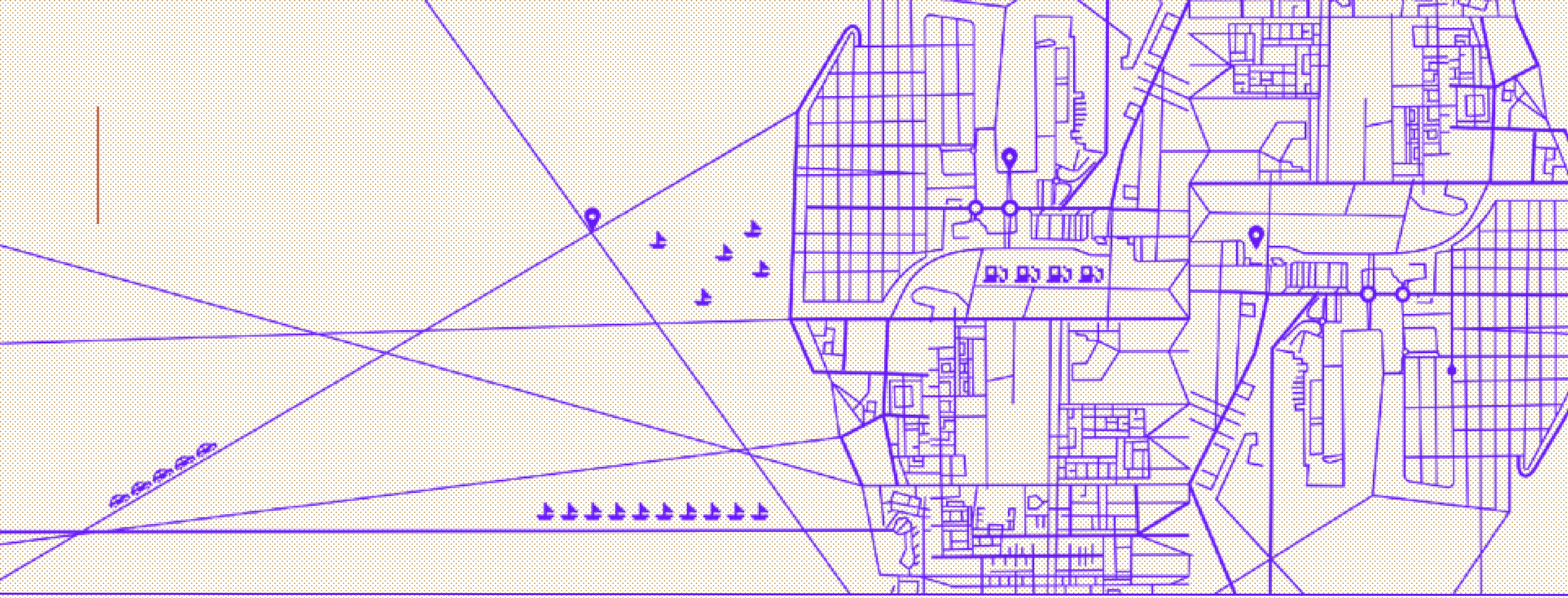

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TOPIC 3: PROBABILITY

OUTLINE

- Basic Theoretical Probability
- Probability using sample spaces: Independent Probability
- Basic set operations
- Dependent Probability
- Conditional probability
- Permutations
- Combinations
- Combinatorics and Probability

BASIC THEORETICAL PROBABILITY

Probability is simply how likely some outcome is to happen. Probability of an event can only be between 0 and 1, and can be written as a percentage.

Whenever we are unsure about the outcome of an event, we can talk about the probabilities of certain outcomes. The analysis of events governed by probability is called **statistics.**

Probability of an event = (number of outcomes that satisfy constraint)/(total number of possible outcomes)

Example 1: Flipping a fair coin. What is the probability of the coin landing on heads?

Solution: P(Head) = ½ = 50%

Example 2: Rolling a fair dice. What is the probability of rolling a 1? What is the probability of rolling an even number? What is the probability of rolling a 2 or 5?

Solution: P(1) = 1/6

P(even) = 3/6 = ½ = 50%

P(2 or 5) = 2/6 = 1/3

Example 3: A bag has 1 yellow, 2 red, 2 green and 3 blue marbles in it. Find the probability of pulling a blue marble, yellow marble, red or green marble and a non-blue marble from the bag.

Solution: possible outcomes {Y, R, R, G, G, B, B, B}

P(blue) = 3/8

P(yellow) = 1/8

P(red or green) = 4/8 = 1/2

P(non-blue) = 5/8

Question 1: If a number is randomly chosen from [3, 4, 5, 6, 7, 8, 9], what is the probability that the number is a multiple of 2?

Solution: P(multiple of 2) = 3/7

Question 2: The circumference of circle is 36π. Contained within that circle is a smaller circle with area 16π. A point is selected at random from inside the larger circle. What is the probability that the point also lies in the smaller circle?

PROBABILITY USING SAMPLE SPACES

A **sample space** is the set of all possible outcomes of a statistical experiment and **outcomes** are observations of the experiment. An **event** is a subset of a sample space.

How to find sample space?

Three possible ways to find a sample space are:

- To list all the possible outcomes.
- Use a Venn Diagram.
- Create a Tree Diagram.

Example 4:

Q. Find the probability of flipping exactly two heads on three coins?

A. Possible outcomes = 8, Outcomes = 3; P(two heads)= outcomes/possible outcomes = 3/8.

Q. Find the probability of rolling doubles on two six-sided dice numbered 1 to 6.

A. Possible outcomes = 36, Outcomes = 6; P(doubles)= outcomes/possible outcomes = 1/6.

Example 5: Let's suppose we flip a coin and roll a die. List all the possible outcomes?

When we flip a coin, there are only two possible outcomes {heads or tails}, and when we roll a die, there are six possible outcomes {1,2,3,4,5,6}.

That means we have two events:

One event consists of "heads and tails."

The other event consists of {1,2,3,4,5,6}.

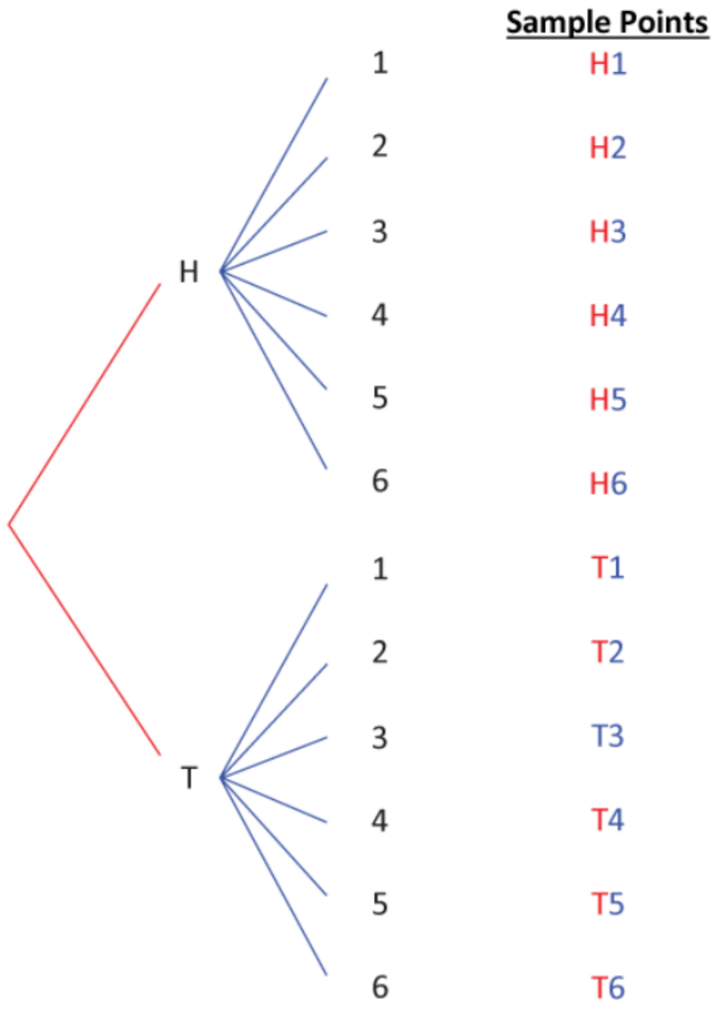
List of **possible outcomes**:

If we flip and roll, then we can get any of the following scenarios:

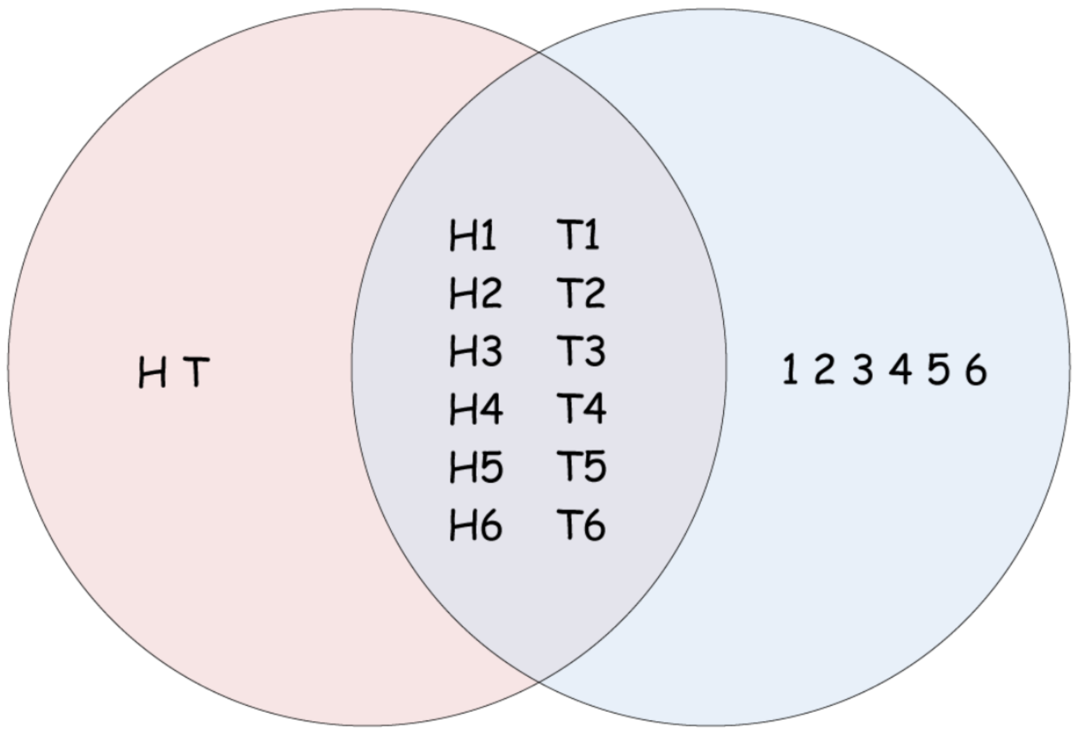
Heads and 1	Tails and 1
Heads and 2	Tails and 2
Heads and 3	Tails and 3
Heads and 4	Tails and 4
Heads and 5	Tails and 5
Heads and 6	Tails and 6

Or more simply stated in a sample space {H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6}.

Tree Diagram:



Venn Diagram:



BASIC SET OPERATIONS

Finite vs Infinite: countable vs uncountable.

Intersection: elements that are common in all events.

Union: complete list of elements that are in all events.

Complement: elements that are not part of an event.

Mutually Exclusive: events that have no outcomes in common and can never occur simultaneously.

Example 6:

1. $\varepsilon = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{1, 3, 5, 7, 9\}$, $B = \{2, 4, 6, 8\}$

Solution:

$A \cap B = \emptyset$ (disjoint set)

$A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

$A' = \{2, 4, 6, 8, 10\}$

$B' = \{1, 3, 5, 7, 9, 10\}$

$A' \cap B = \{2, 4, 6, 8\}$

$A \cap B' = \{1, 3, 5, 7, 9\}$

$A' \cap B' = \{10\}$

$A' \cup B' = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

Question 3:

A = {-5, 0, 3, 7, 13}, B = {0, 5, 9, 13, 17}, C = {3, 5, 9, 17}

Find: (BUC)' ∩ (B∩C)' U (A∩C)

Solution:

Universal Set = {-5, 0, 3, 5, 7, 9, 13, 17}

(BUC) = {0, 3, 5, 9, 13, 17}

(BUC)' = {-5, 7}

(B∩C) = {5, 9, 17}

(B∩C)' = {-5, 0, 3, 7, 13}

(A∩C) = {3}

(BUC)' ∩ (B∩C)' U (A∩C) = {-5, 7, 3}

DEPENDENT PROBABILITY

Example 7:

A bag contains 3 green marbles and 2 blue marbles. What is the probability that a green marble is picked twice without replacement.

P(GG) = (3/5)(2/4) = 3/10 = 30%

Question 4:

You have 8 coins in a bag, 3 of them are unfair in that they have a 60% chance of being a heads when flipped. The rest are fair coins. You randomly choose one coin from the bag and flip it 2 times. What is the probability of getting 2 heads?

Fair coin (5/8), P(HH) = (0.5)(0.5)(5/8) = 0.15625

Unfair coin (3/8), P(HH) = (0.6)(0.6)(3/8) = 0.135

Probability P(HH) = 0.15625 + 0.135 = 0.29125 = 29.125%

CONDITIONAL PROBABILITY: MULTIPLICATION RULE

Bayes' Theorem: For any two events we can say that

$P(A \cap B) = P(A|B) \,.\, P(B)$ or $P(A \cap B) = P(B|A) \,.\, P(A)$ or $P(A|B) \,.\, P(B) = P(B|A) \,.\, P(A)$

$P(A|B) = \frac{P(A \cap B)}{P(B)}$ or $P(B|A) = \frac{P(A \cap B)}{P(A)}$

The vertical bar in $P(A|B)$ and $P(B|A)$ means "given". This could also be read as "the probability that "A occurs *given* that B has occurred" and "B occurs *given* that A has occurred", respectively.

This formula means that we can multiply the probabilities of two events, but we need to take the first event into account when considering the probability of the second event. This is true for dependent events.

If the events are independent, one happening does not impact the probability of the other, and in that case, $P(B|A) = P(B)$

Therefore, for independent events, $P(A \cap B) = P(A) \,.\, P(B)$

Example 8:

John's two favorite foods are sandwich and rice. Let A represent the event that he eats a sandwich for breakfast and B represents the event that he eats rice for lunch.

On a randomly selected day, the probability that John will eat a sandwich for breakfast, $P(A)$, is equal to 0.6, the probability that he will eat rice for lunch, $P(B)$, is equal to 0.5, and the conditional probability that he eats rice for lunch given that he eats a sandwich for breakfast, $P(B|A)$, is equal to 0.7.

Based on this information, what is $P(A|B)$, the conditional probability that John eats a sandwich for breakfast, given that he eats rice for lunch?

Solution:

P(A) = 0.6, P(B) = 0.5, P(B | A) = 0.7

$P(A \cap B) = P(B|A) . P(A)$

$P(A \cap B) = (0.7) . (0.6) = 0.42$

Also, $P(A \cap B) = P(A|B) . P(B)$

$0.42 = P(A|B) . (0.5)$

$P(A|B) = (0.42/0.5) = 0.84$

OR

$$P(A|B) = \frac{P(B|A).P(A)}{P(B)} = \frac{(0.7)(0.6)}{(0.5)} = 0.84$$

TREE DIAGRAM

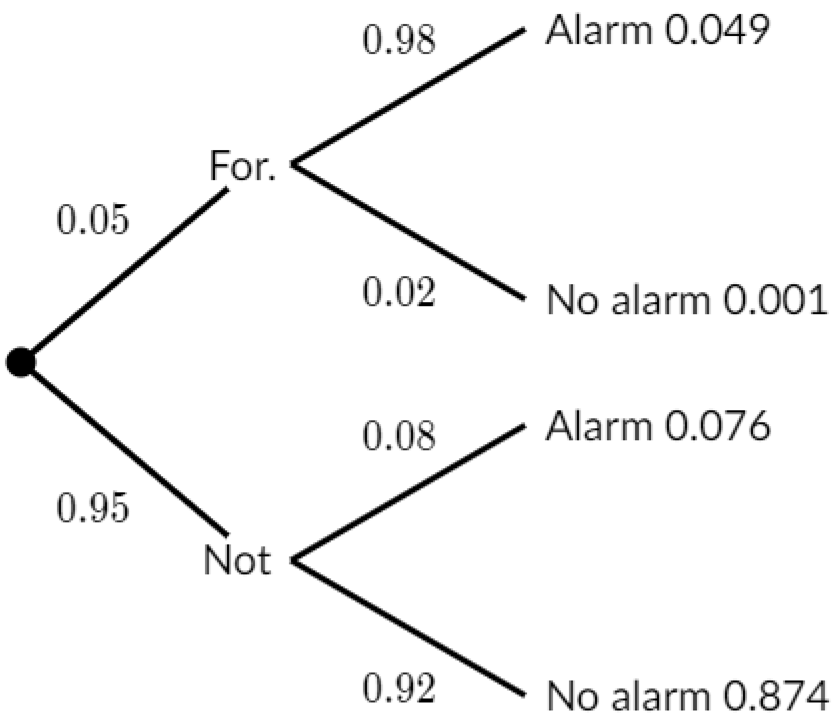
Example 9:

An airport screens bags for forbidden items, and an alarm is supposed to be triggered when a forbidden item is detected.

- Suppose that 5% of bags contain forbidden items.
- If a bag contains a forbidden item, there is a 98% chance that it triggers the alarm.
- If a bag doesn't contain a forbidden item, there is an 8% chance that it triggers the alarm.

Given a randomly chosen bag triggers the alarm, what is the probability that it contains a forbidden item?

Solution:



$$P(F|A) = \frac{P(F \cap A)}{P(A)} = \frac{(0.05)(0.98)}{(0.05)(0.98)+(0.95)(0.08)} = \frac{0.049}{0.049+0.076} = 0.392$$

Question 5:

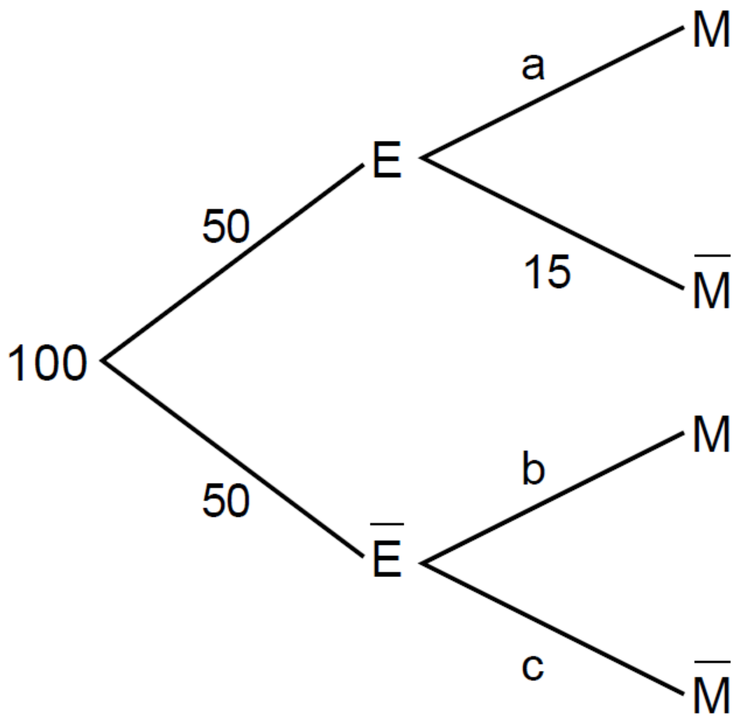
100 cars are selected for a road worthiness test which is in two parts, electrical and mechanical. A car passes only if it passes both parts. Half the cars fail the electrical test and 62 pass the mechanical test. 15 pass the electrical test but fail the mechanical test.

Find the probability that a car chosen at random:

a) passes overall

b) fails on one test only

c) given that it has failed, it has failed the mechanical test only.


100
50
50
E
$\overline{E}$
a
15
b
c
M
$\overline{M}$
M
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PERMUTATION

A permutation is a mathematical calculation of the number of ways a particular set can be arranged, where the order of the arrangement matters.

Formula: $_{n}P_{k} = \frac{n!}{(n-k)!}$

Example 10:

Given: Colors blue, green, red, yellow, orange and white.

How many 4 color combinations can be made if the colors cannot be repeated?

Solution:

$\underline{6}\ \underline{5}\ \underline{4}\ \underline{3} = 6\times5\times4\times3 = 6!/2! = 360$

Question 5: A club of 9 people wants to choose a board of three officers: President, Vice President and Secretary. How many ways are there to choose the board from the 9 people?

Solution:

$\underline{9}\ \underline{8}\ \underline{7} = 9\times8\times7 = 9!/6! = 504$

Example 11: In how many different ways can 3 pool balls be arranged out of 16 balls?

Solution:

$16 \times 15 \times 14 = 3360$ ways

Or

$${}_{16}P_3 = \frac{16!}{(16-3)!} = \frac{16!}{13!} = 3360$$

COMBINATIONS

A combination is a mathematical calculation that determines the number of possible arrangements in a collection of items where the order of the selection does not matter. In combinations, you can select the items in any order.

Formula: $_{n}C_{k} = \frac{n!}{k!(n-k)!}$

Example 12:

$$_{6}C_{4} = \frac{6!}{4!(6-4)!} = \frac{6\times5\times4\times3\times2\times1}{(4\times3\times2\times1)(2\times1)} = \frac{30}{2} = \mathbf{15}$$

Question 6:

$$_{9}C_{3} = \frac{9!}{3!(9-3)!} = \frac{9\times8\times7\times6\times5\times4\times3\times2\times1}{(3\times2\times1)(6\times5\times4\times3\times2\times1)} = \frac{504}{6} = \mathbf{84}$$

Example 13: In how many ways a committee consisting of 5 men and 3 women, can be chosen from 9 men and 12 women?

Solution:

Choose 5 men out of 9 men = ${}_{9}C_{5}$ ways = 126 ways

Choose 3 women out of 12 women = ${}_{12}C_{3}$ ways = 220 ways

The committee can be chosen in = 126×220 =27720 ways.

Question 7: Given numbers 1, 2 and 3. In how many ways can these numbers be arranged?

Solution:

If order matters: ${}_{3}P_{3} = \frac{3!}{(3-3)!} = ?$

If order doesn't matter: ${}_{3}C_{3} = \frac{3!}{(3-3)!3!} = ?$

Example 14: In how many ways can 3 pool balls be arranged out of 16 balls?

Solution:

$${}_{16}C_{3} = \frac{16!}{3!(16-3)!} = \frac{16!}{3!13!} = 560$$

COMBINATORICS AND PROBABILITY

Example 15:

There are 10 balls in a bag, 6 red and 4 green. If 3 are picked out at random, what is the probability of 1 red and 2 greens?

Solution:

Total no of ways of picking 3 balls from 10 (regardless of color) in which the order doesn't matter

$$ {}_{10}C_3 = \frac{10!}{7!3!} = 120 $$

Outcomes: 3 reds, 2 reds 1 green, 1 red 2 greens, 3 greens

The outcome we seek, 1 red and 2 greens, contains any 1 of the 6 reds and any 2 of the 4 greens. The number of ways it can come about is (since order doesn't matter)

$$ {}_6C_1 \times {}_4C_2 = \frac{6!}{5!1!}\frac{4!}{2!2!} = 36 $$

Thus probability $= \frac{36}{120} = \frac{3}{10}$

We can check the number of ways that the other outcomes can come about. (The total number, according to our answer, should be 120)

3 reds: ${}_6C_3 = \frac{6!}{3!3!} = 20$

2 reds, 1 green: ${}_6C_2 \times {}_4C_1 = \frac{6!}{4!2!}\frac{4!}{3!1!} = 60$

3 greens: ${}_4C_3 = \frac{4!}{3!1!} = 4$

Total = 36+20+60+4 = 120

Resources:

https://www.datasciencecentral.com/understanding-the-applications-of-probability-in-machine-learning/

https://www.researchgate.net/publication/346655414_Probability_and_it's_use_in_Artificial_Intelligence

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