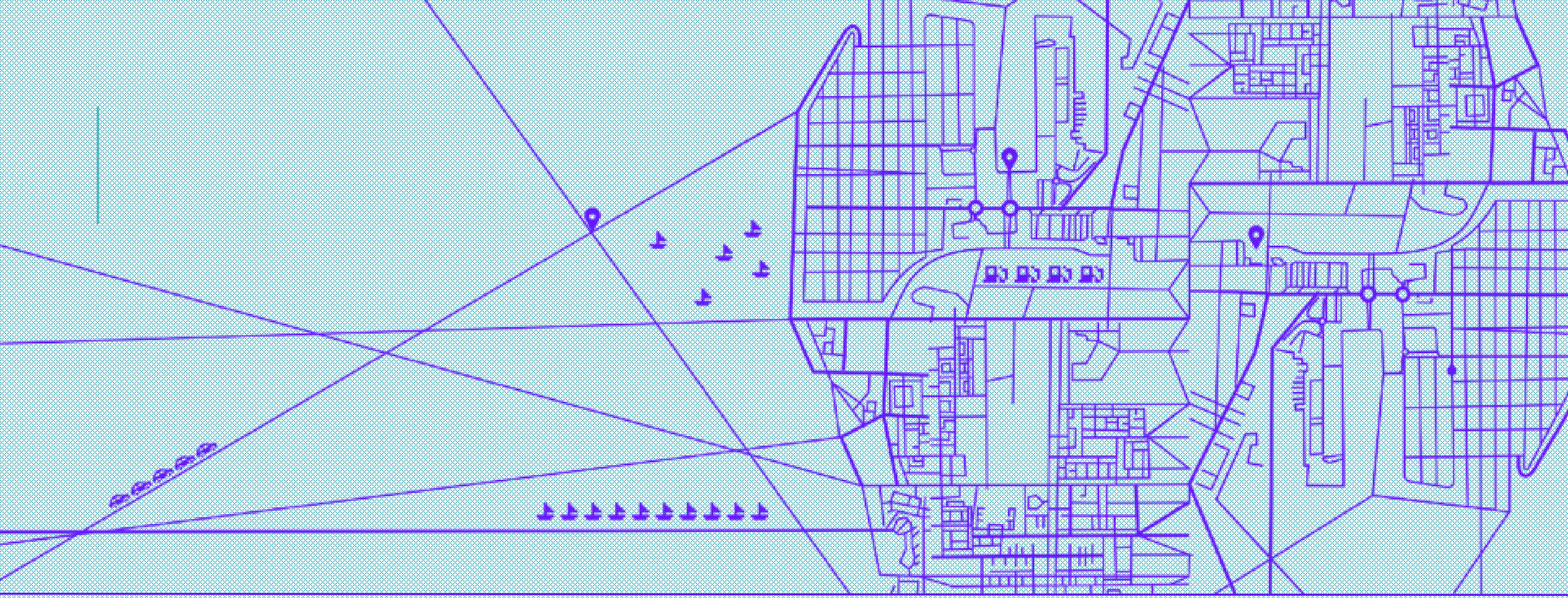


TOPIC 5: VECTORS

OUTLINE

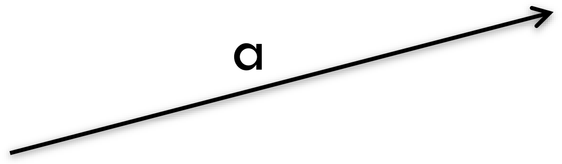
- Addition
- Subtraction
- Multiplication
- Unit Vector
- Dot Product
- Cross Product

VECTOR

A vector is something that has both magnitude and direction. Examples of vectors are Force and Velocity. Both Force and Velocity are in a particular direction. The magnitude of the vector would indicate the strength of the Force or the speed associated with the Velocity.

On diagrams, they are denoted by an arrow, where the length tells us the magnitude and the arrow tells us the direction.

Vectors are often split into two parts, which we call components: an *x* component, which moves left or right, and *y* component, which moves up or down.



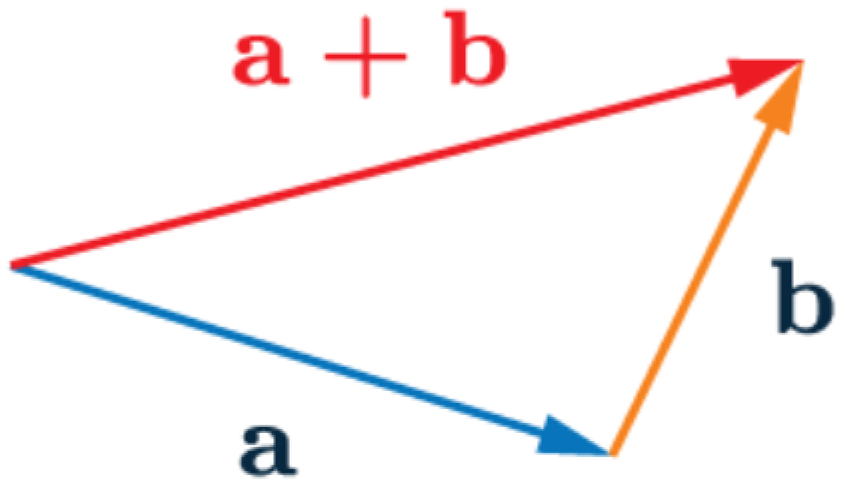
VECTOR ADDITION AND SUBTRACTION

To add two vectors, we need to add the corresponding components.

The sum of two vectors $\vec{a}$ and $\vec{b}$ is

$\vec{a} = (x_1, y_1), \vec{b} = (x_2, y_2)$

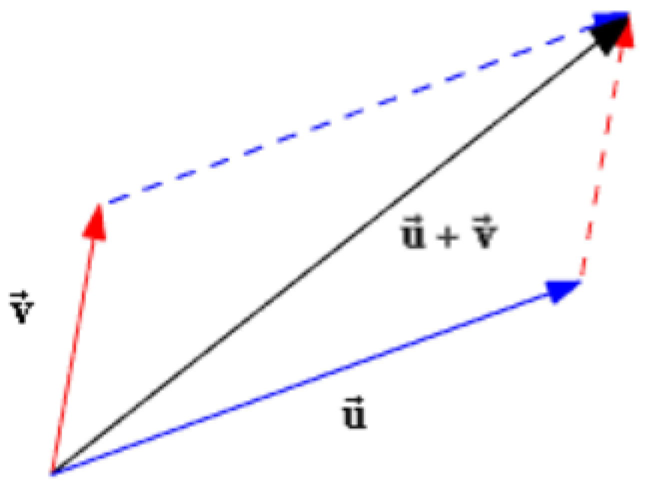
$\vec{a} + \vec{b} = (x_1 + x_2), (y_1 + y_2)$



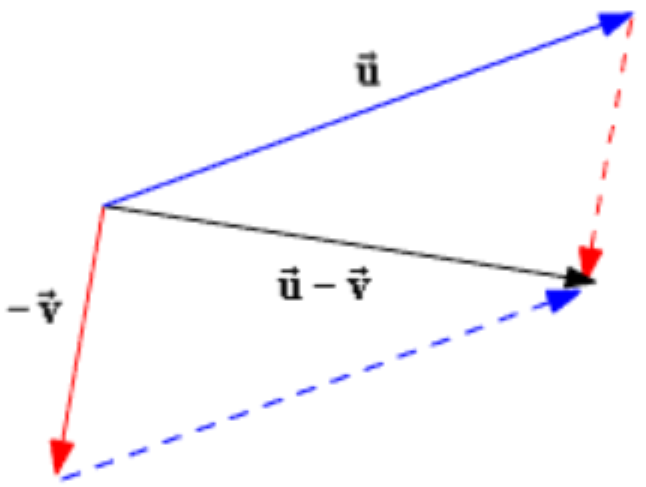
The sum of two or more vectors is called the resultant. The resultant of two vectors can be found using either the *parallelogram method* or the *triangle method*.

PARALLELOGRAM METHOD

Draw the vectors so that their initial points coincide. Then draw lines to form a complete parallelogram. The diagonal from the initial point to the opposite vertex of the parallelogram is the resultant.



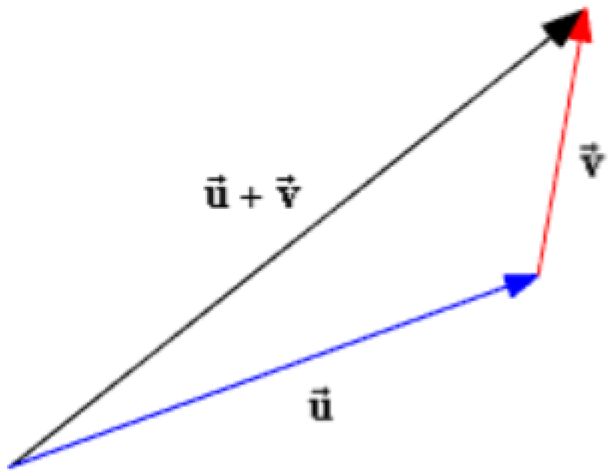
Vector Addition



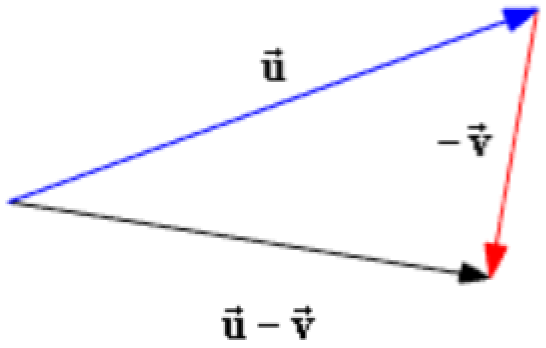
Vector Subtraction

TRIANGLE METHOD

Draw the vectors one after another, placing the initial point of each successive vector at the terminal point of the previous vector. Then draw the resultant from the initial point of the first vector to the terminal point of the last vector. This method is also called the *head-to-tail method*.



Vector Addition



Vector Subtraction

VECTOR NOTATION

A vector from point X to point Y is shown as $\overrightarrow{XY}$.

From the diagram we can see that:

$\overrightarrow{XY} = a$

$\overrightarrow{YZ} = b$

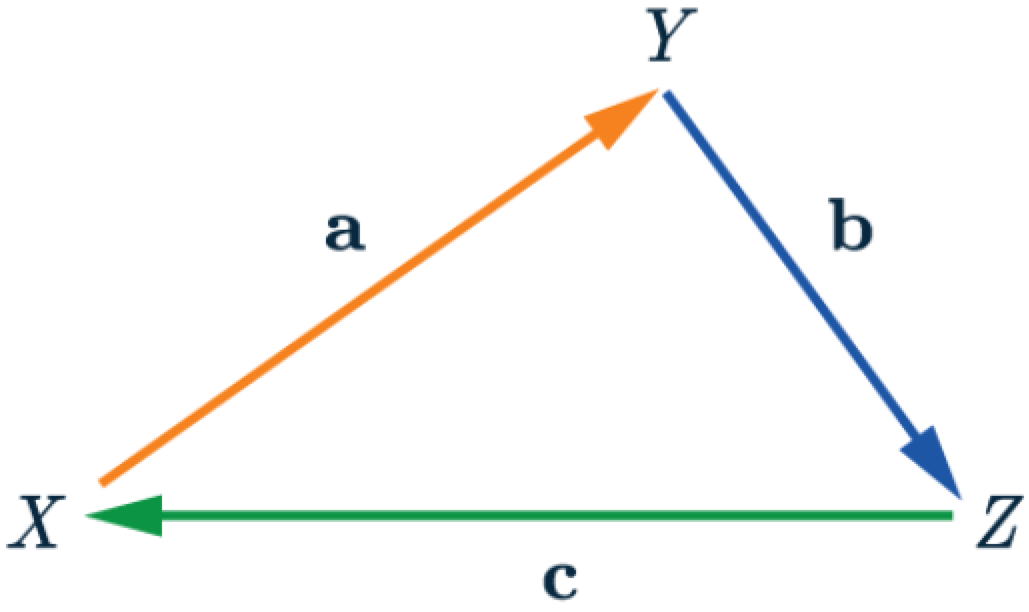
$\overrightarrow{ZX} = c$

If we go against the arrow, the vector becomes negative:

$\overrightarrow{YX} = -a$

We can also combine vectors:

$\overrightarrow{XZ} = a + b = -c$



Example 1: Given $\vec{u} = \langle 3,4 \rangle$ and $\vec{v} = \langle 5, -1 \rangle$. Find (a) $\vec{u} + \vec{v}$ (b) $\vec{u} - \vec{v}$

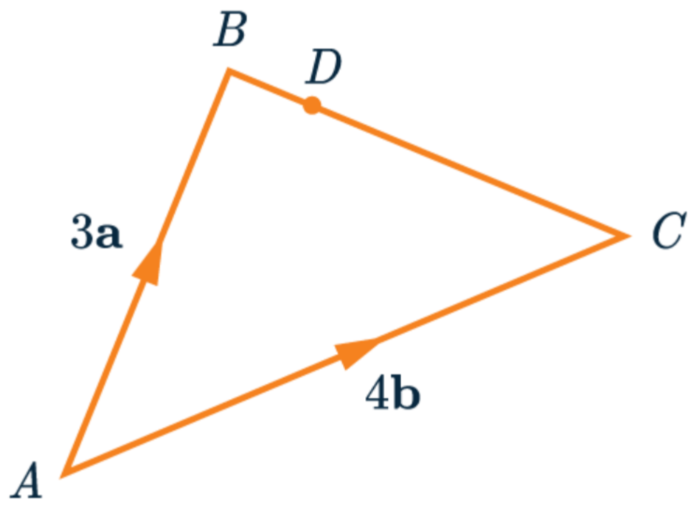
Solution:

(a) $\vec{u} + \vec{v} = \langle 3 + 5, 4 + (-1) \rangle = \langle 8,3 \rangle$

(b) $\vec{u} - \vec{v} = \langle 3 - 5, 4 - (-1) \rangle = \langle -2,5 \rangle$

Example 2:

In the diagram below, we have vectors $\overrightarrow{AB} = 3\boldsymbol{a}$ and $\overrightarrow{AC} = 4\boldsymbol{b}$. Point *D* lies on the line *BC* such that *BD*:*DC* = 1:3. Write the vector $\overrightarrow{AD}$ in terms of ***a*** and ***b***.



$\overrightarrow{BC} = \overrightarrow{AC} - \overrightarrow{AB} = 4b - 3a$

$\overrightarrow{BD} = \frac{1}{4}\overrightarrow{BC} = \frac{1}{4}(4b - 3a) = b - \frac{3}{4}a$

Therefore,

$$\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BD} = 3a + \left(b - \frac{3}{4}a\right) = \frac{9}{4}a + b$$

Magnitude of a line segment is given by:

$$|\overrightarrow{uv}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Magnitude of a position vector is given by:

$$|\overrightarrow{uv}| = \sqrt{x^2 + y^2 + z^2}$$

Example 3: Given a line segment from (-9, 2) to (4, -1). Find its magnitude?

Solution:

Vector for this line segment: $\vec{v} = (4 - (-9), -1 - 2) = (13, -3)$

Magnitude: $|\vec{v}| = \sqrt{(13)^2 + (-3)^2} = \sqrt{178}$

Or

Magnitude: $|\vec{v}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(4 - (-9))^2 + (-1 - 2)^2} = \sqrt{178}$

MULTIPLICATION

We can multiply a vector by a scalar (number).

For example: $3(\bar{a} + \bar{b}) = 3\bar{a} + 3\bar{b}$

Scalar multiples are all parallel to each other.

For example: $3\bar{a} + 3\bar{b}$ is parallel to $\bar{a} + \bar{b}$

Example 4: Given $\vec{a}$=⟨8,5⟩ and $\vec{b}$=⟨−3,6⟩, compute each of the following.

(a) $6\vec{a}$, (b) $7\vec{b}$-$2\vec{a}$, (c) $|10\vec{a} + 3\vec{b}|$

Solution: (a) $6\vec{a} = 6\langle 8,5\rangle = \langle 48,30\rangle$

(b) $7\vec{b}$-$2\vec{a} = 7\langle -3,6\rangle - 2\langle 8,5\rangle = \langle -21,42\rangle - \langle 16,10\rangle = \langle -37,32\rangle$

(c) $10\vec{a} + 3\vec{b} = 10\langle 8,5\rangle + 3\langle -3,6\rangle = \langle 80,50\rangle + \langle -9,18\rangle = \langle 71,68\rangle$

$|10\vec{a} + 3\vec{b}| = \sqrt{(71^2) + (68^2)} = \sqrt{9665}$

UNIT VECTOR

The unit vector in the direction of the *x-axis* is *i*, the unit vector in the direction of the *y-axis* is *j* and the unit vector in the direction of the *z-axis* is *k*.

Unit vector: $\vec{u} = \dfrac{vector}{|magnitude\ of\ vector|}$

Example 5: Given $\vec{a} = 8\vec{\imath} - \vec{\jmath} + 3\vec{k}$ and $\vec{b} = 7\vec{\jmath} - 4\vec{k}$, Compute $|-9\vec{b} - 2\vec{a}|$.

Solution:

$-9\vec{b} - 2\vec{a} = -9\left(7\vec{\jmath} - 4\vec{k}\right) - 2\left(8\vec{\imath} - \vec{\jmath} + 3\vec{k}\right) = \left(-63\vec{\jmath} + 36\vec{k}\right) - \left(16\vec{\imath} - 2\vec{\jmath} + 6\vec{k}\right) =$
$-16\vec{\imath} - 61\vec{\jmath} + 30\vec{k}$

Magnitude: $|-9\vec{b} - 2\vec{a}| = \sqrt{(-16)^2 + (-61)^2 + (30)^2} = \sqrt{4877}$

Example 6: Find a unit vector that points in the same direction as $\vec{q} = \vec{\imath} + 3\vec{\jmath} + 9\vec{k}$.

Solution: $|\vec{q}| = \sqrt{(1)^2 + (3)^2 + (9)^2} = \sqrt{91}$

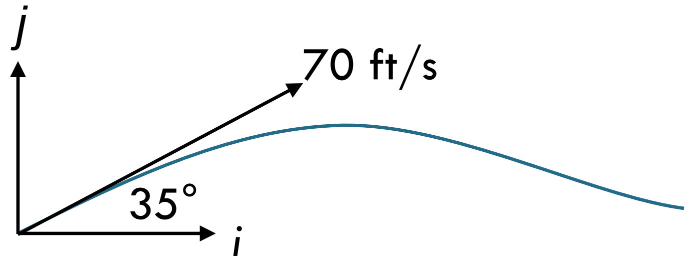
Recall: $\vec{u} = \frac{\vec{\imath}+3\vec{\jmath}+9\vec{k}}{\sqrt{91}} = \frac{1}{\sqrt{91}}\vec{\imath} + \frac{3}{\sqrt{91}}\vec{\jmath} + \frac{9}{\sqrt{91}}\vec{k}$

Check: $|\vec{u}| = \sqrt{(\frac{1}{\sqrt{91}})^2 + (\frac{3}{\sqrt{91}})^2 + (\frac{9}{\sqrt{91}})^2} = \sqrt{\frac{91}{91}} = 1$

WORD PROBLEMS

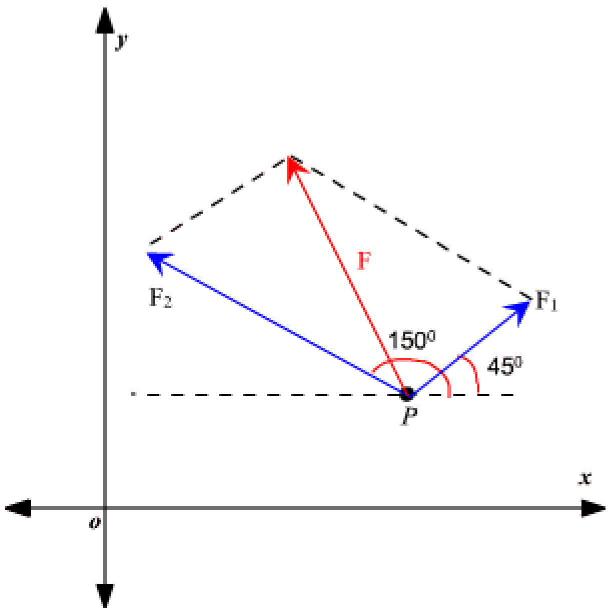
Example 7: A ball is thrown with an initial velocity of 70 ft/s, at an angle of 35° with the horizontal. Find the vertical and horizontal components of the velocity?

Solution:



$$v = 70(cos35°)i + 70(sin35°)j = 57.34i + 40.15j$$

Example 8: Two forces F_1 and F_2 with magnitudes 20 N and 30 N, respectively, act on an object at a point P as shown. Find the magnitude of the resultant force acting at P.



Solution:

$$F_1 = 20(cos45°)i + 20(sin45°)j = 10\sqrt{2}i + 10\sqrt{2}j$$

$$F_2 = -30(cos30°)i + 30(sin30°)j = -15\sqrt{3}i + 15j$$

Resultant Force:

$$F = F_1 + F_2 = \left(10\sqrt{2}i + 10\sqrt{2}j\right) + \left(-15\sqrt{3}i + 15j\right) = \left(10\sqrt{2} - 15\sqrt{3}\right)i + \left(10\sqrt{2} + 15\right)j = -12i + 29j$$

Magnitude:

$$|F| = \sqrt{(-12)^2 + (29)^2} = 31.4\ N$$

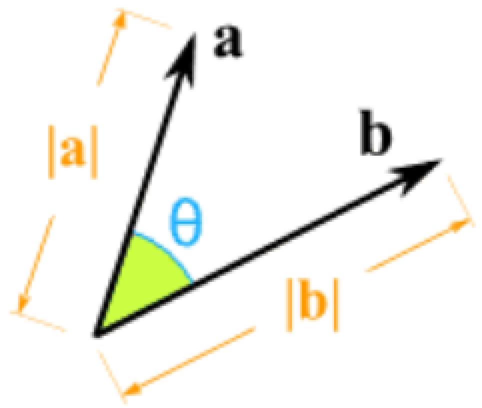
DOT PRODUCT

The dot product of two vectors is equal to the product of the magnitude of the two vectors and the cosine of the angle between the two vectors. The resultant of the dot product of vectors is a scalar quantity. Thus, the dot product is also known as a scalar product.

In vector algebra, if two vectors are given as: $a = [a_1, a_2, a_{3,} \dots \dots a_n]$ and $\mathrm{b} = [b_1, b_2, b_{3,} \dots \dots b_n]$, then their dot product is given by:

$a.b = [a_1 b_1 + a_2 b_2 + a_3 b_3, \dots \dots + a_n b_n]$

$a.b = \sum_{i=1}^{n} a_i b_i$



Let *a* and *b* be two non-zero vectors, and ϑ be the included angle of the vectors. Then the scalar product or dot product is denoted by *a.b*, which is defined as:

$$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|cos\theta$$

The cosine of the angle between two vectors is equal to the sum of the product of the individual constituents of the two vectors, divided by the product of the magnitude of the two vectors.

$$cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}|.|\vec{b}|} = \frac{a_1b_1+a_2b_2+a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}\,.\sqrt{b_1^2+b_2^2+b_3^2}}$$

If the two vectors are expressed in terms of unit vectors, *i*, *j*, *k*, along the *x*, *y*, *z* axes, then the scalar product is obtained as follows:

If $\vec{a} = a_1i + a_2j + a_3k$ and $\vec{b} = b_1i + b_2j + b_3k$, then

$$\vec{a}.\vec{b} = (a_1i + a_2j + a_3k)(b_1i + b_2j + b_3k) = a_1b_1 + a_2b_2 + a_3b_3$$

Note: $\hat{\imath}.\hat{\imath} = \hat{\jmath}.\hat{\jmath} = \hat{k}.\hat{k} = 1$ and $\hat{\imath}.\hat{\jmath} = \hat{\jmath}.\hat{k} = \hat{k}.\hat{\imath} = 0$

Example 9: Find the dot product of two vectors having magnitudes of 6 units and 7 units, and the angle between the vectors is 60°.

Solution:

The magnitudes of the two vectors are $|a| = 6$, $|b| = 7$, and the angle between the vectors is $\theta = 60°$.

The dot product of the two vectors is:

$$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|cos\theta = (6)(7)\cos(60°) = 21$$

Question 1:

Find the angle between the two vectors $\vec{a} = 2i + 3j + k$ and $\vec{b} = 5i - 2j + 3k$

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Find the angle between the two vectors $\vec{a} = 2i + 3j + k$ and $\vec{b} = 5i - 2j + 3k$

Solution:

$|\vec{a}| = \sqrt{(2)^2 + (3)^2 + (1)^2} = \sqrt{14}$

$|\vec{b}| = \sqrt{(5)^2 + (-2)^2 + (3)^2} = \sqrt{38}$

$\vec{a}.\vec{b} = (2)(5) + (3)(-2) + (1)(3) = 7$

$cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}|.|\vec{b}|} = \frac{7}{\sqrt{14}.\sqrt{38}} = 0.304;\ \theta = cos^{-1}(0.304) = 72.3°$

CROSS PRODUCT

When two vectors are multiplied with each other and the product of the vectors is also a vector quantity, then the resultant vector is called the cross product of two vectors or the vector product. The resultant vector is perpendicular to the plane containing the two given vectors.

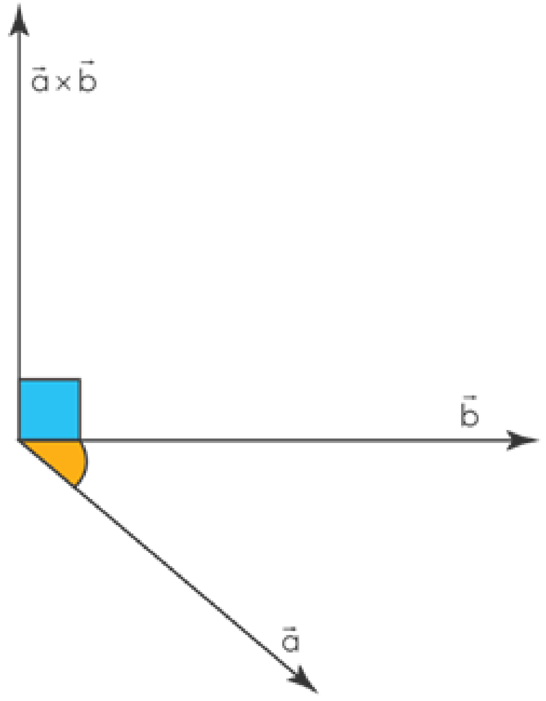
If A and B are two independent vectors, then the result of the cross product of these two vectors (Ax B) is perpendicular to both the vectors and normal to the plane that contains both the vectors. It is represented by:

$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|sin\theta$$

Where

$|\vec{a}|$ is the magnitude of the vector a,

$|\vec{b}|$ is the magnitude of the vector b.



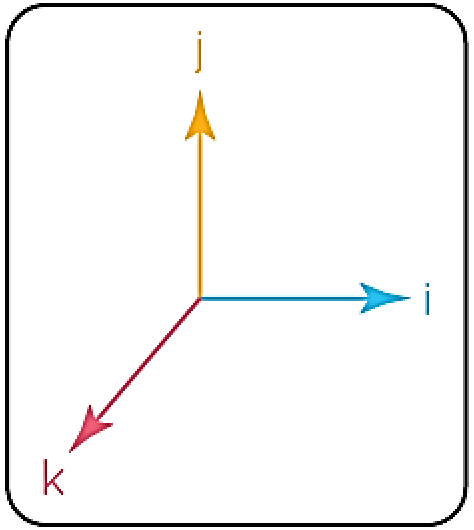
Assume that $\vec{a}$ and $\vec{b}$ are two vectors, such that $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\,\hat{k}$ and $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\,\hat{k}$, then by using determinants, we can find the cross product using matrix notation.

$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k} \qquad \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$

$$\vec{c} = \vec{a} \times \vec{b}$$

$$\vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$\vec{c} = \hat{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \hat{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \hat{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$



$$\boxed{\vec{c} = \hat{i}\,|a_2b_3 - a_3b_2| - \hat{j}\,|a_1b_3 - a_3b_1| + \hat{k}\,|a_1b_2 - a_2b_1|}$$

Note: $\hat{\imath}, \hat{\jmath}$ *and* $\hat{k}$ are the unit vectors in the direction of *x-axis*, *y-axis* and *z-axis* respectively.

The cross product of the unit vectors is:

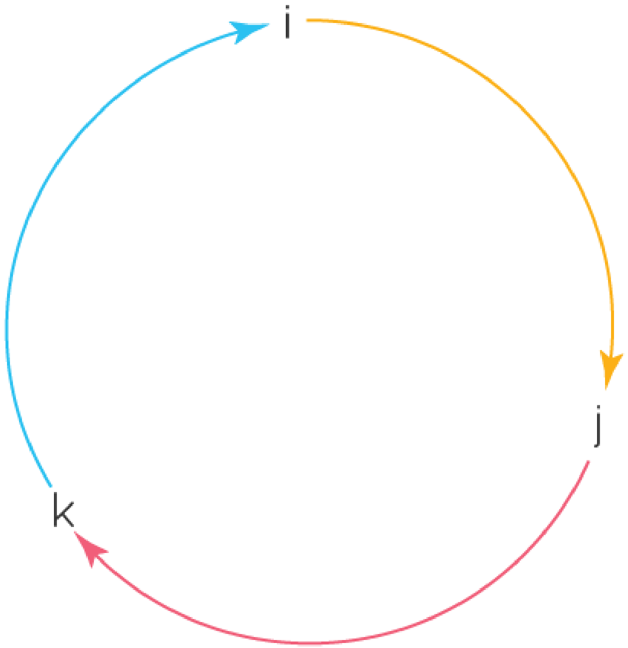
- $\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = 0$
- $\hat{\imath} \times \hat{\jmath} = \hat{k}$

  $\hat{\jmath} \times \hat{k} = \hat{\imath}$

  $\hat{k} \times \hat{\imath} = \hat{\jmath}$
- $\hat{\jmath} \times \hat{\imath} = -\hat{k}$

  $\hat{k} \times \hat{\jmath} = -\hat{\imath}$

  $\hat{\imath} \times \hat{k} = -\hat{\jmath}$



Example 10: Find the cross product of two vectors $\vec{a} = (3, 4, 5)$ and $\vec{b} = (7, 8, 9)$.

Solution:

The cross product is given as:

$$\vec{a} \times \vec{b} = \begin{matrix} i & j & k \\ 3 & 4 & 5 \\ 7 & 8 & 9 \end{matrix} = [(4 \times 9) - (5 \times 8)]\hat{\imath} - [(3 \times 9) - (5 \times 7)]\hat{\jmath} + [(3 \times 8) - (4 \times 7)]\hat{k}$$

$$= (36 - 40)\hat{\imath} - (27 - 35)\hat{\jmath} + (24 - 28)\hat{k} = -4\hat{\imath} + 8\hat{\jmath} - 4\hat{k}$$

Question 2: If $\vec{a} = (2, -4, 4)$ and $\vec{b} = (4, 0, 3)$, find the angle between the two vectors using the cross product?

Question 2: If $\vec{a} = (2, -4, 4)$ and $\vec{b} = (4, 0, 3)$, find the angle between the two vectors using the cross product?

Solution:

$$|\vec{a}| = \sqrt{(2)^2 + (-4)^2 + (4)^2} = 6$$

$$|\vec{b}| = \sqrt{(4)^2 + (0)^2 + (3)^2} = 5$$

$$\vec{a} \times \vec{b} = \begin{bmatrix} i & j & k \\ 2 & -4 & 4 \\ 4 & 0 & 3 \end{bmatrix} = [(-4 \times 3) - (0)]\hat{\imath} - [(2 \times 3) - (4 \times 4)]\hat{\jmath} + [(0) - (-4 \times 4)]\hat{k}$$

$$= (-12)\hat{\imath} - (6 - 16)\hat{\jmath} + (16)\hat{k} = -12\hat{\imath} + 10\hat{\jmath} + 16\hat{k}$$

$$\left|\vec{a} \times \vec{b}\right| = \sqrt{(-12)^2 + (10)^2 + (16)^2} = \sqrt{500} = 10\sqrt{5}$$

$$sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|} = \frac{10\sqrt{5}}{(6 \times 5)} = 0.74$$

$$\theta = sin^{-1}(0.74) = 48°$$

Resources:

https://natureofcode.com/book/chapter-1-vectors/

https://www.haroldserrano.com/blog/vectors-in-computer-graphics

https://blog.terresquall.com/2020/01/vector-math-basics-for-games-programming/

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