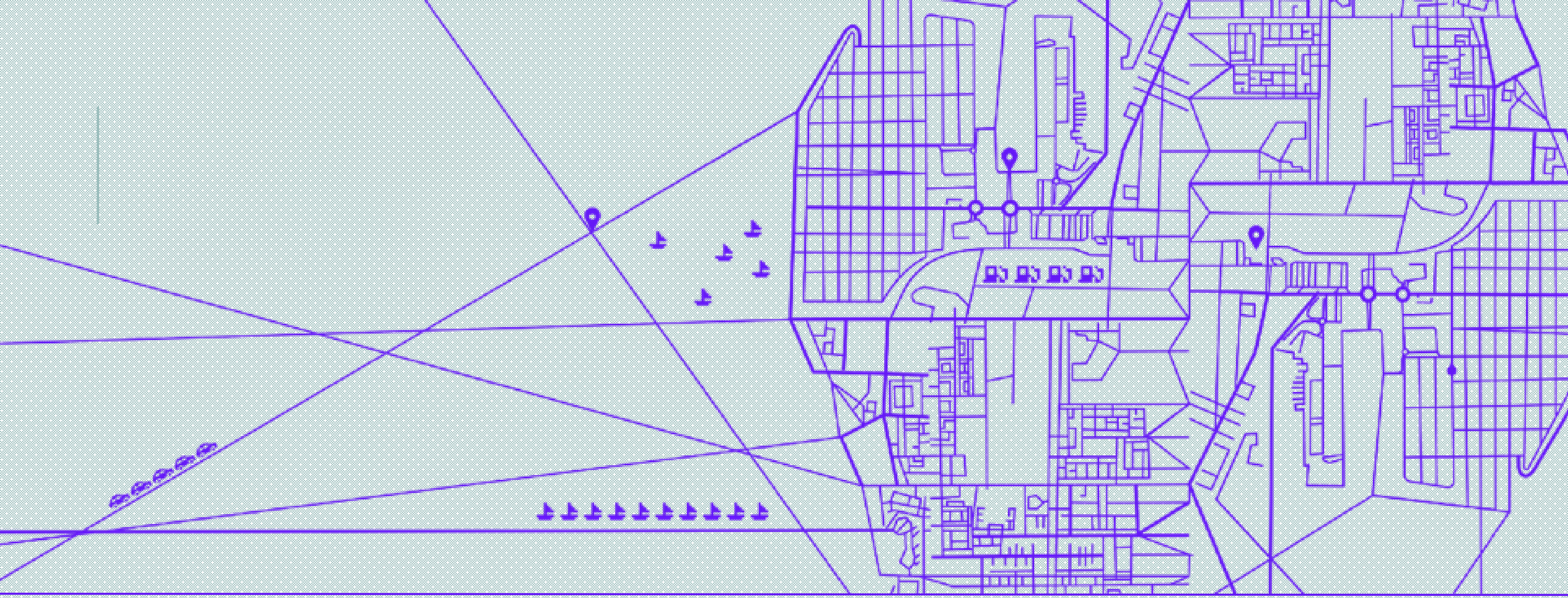


TOPIC 6: MATRICES

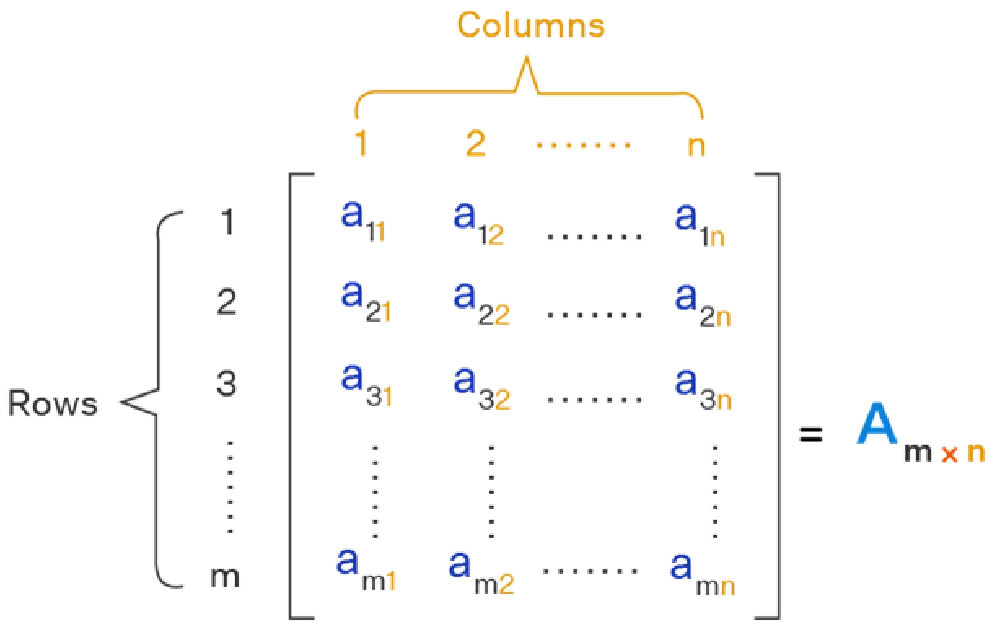
OUTLINE

- Addition
- Subtraction
- Multiplication
- Transpose of a Matrix
- Determinant of a Matrix
- Inverse of a Matrix
- Cryptography

MATRIX

Matrices, the plural form of a matrix, are the arrangements of numbers, variables, symbols, or expressions in the rectangular table which contains various numbers of rows and columns. Matrices are rectangular shaped arrays, for which different operations like addition, subtraction, multiplication are defined. The numbers or entries in the matrix are known as its elements. Horizontal entries for matrices are called rows and vertical entries are known as columns.

The size of a matrix is the number of rows and columns in the matrix.



ADDITION OF MATRICES

Matrices addition can only be possible if the number of rows and columns of both the matrices are the same. In adding two matrices, we add the elements in each row and column to the respective elements in the row and column of the next matrix. Hence $(A + B) = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$, where i and j are the number of rows and columns respecttively.

Example 1: If $A = \begin{bmatrix} 2 & -1 \\ 0 & 5 \end{bmatrix}$ and $\mathrm{B} = \begin{bmatrix} 0 & 2 \\ 1 & -2 \end{bmatrix}$, Find A+B?

Solution: $\mathrm{A} + \mathrm{B} = \begin{bmatrix} 2 & -1 \\ 0 & 5 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 2+0 & -1+2 \\ 0+1 & 5+(-2) \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$

SUBTRACTION OF MATRICES

Matrices subtraction is also possible only if the number of rows and columns of both the matrices are the same. In subtracting two matrices, we subtract the elements in each row and column from the respective elements in the row and column of the previous matrix. Hence, (A - B) = [a_{ij}] - [b_{ij}] = [a_{ij} - b_{ij}], where i and j are the number of rows and columns respectively.

Example 2: If $A = \begin{bmatrix} 2 & -1 \\ 0 & 5 \end{bmatrix}$ and $\mathrm{B} = \begin{bmatrix} 0 & 2 \\ 1 & -2 \end{bmatrix}$, Find A-B?

Solution: $\mathrm{A} - \mathrm{B} = \begin{bmatrix} 2 & -1 \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} 0 & 2 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 2-0 & -1-2 \\ 0-1 & 5-(-2) \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -1 & 7 \end{bmatrix}$

SCALAR MULTIPLICATION OF MATRICES

The product of a matrix A with any number 'c' is obtained by multiplying every entry of the matrix A by c, i.e., $(cA)_{ij} = c(A_{ij})$

Properties of scalar multiplication in matrices

The different properties of matrices for scalar multiplication of any scalars K and l, with matrices A and B are given as,

$K(A + B) = KA + KB$

$(K + l)A = KA + lA$

$(Kl)A = K(lA) = l(KA)$

$(-K)A = -(KA) = K(-A)$

$1 \cdot A = A$

$(-1)A = -A$

MULTIPLICATION OF MATRICES

Matrices multiplication is defined only if the number of columns in the first matrix and rows in the second matrix are equal. To understand how matrices are multiplied, let us first consider a row vector and a column vector which are both of order *n*.

$R = \begin{bmatrix} r_1 & \cdots & r_n \end{bmatrix}$ and $C = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

The product of *RC* can be defined as

$$RC = \begin{bmatrix} r_1 & \cdots & r_n \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = [r_1 c_1 + \cdots + r_n c_n]$$

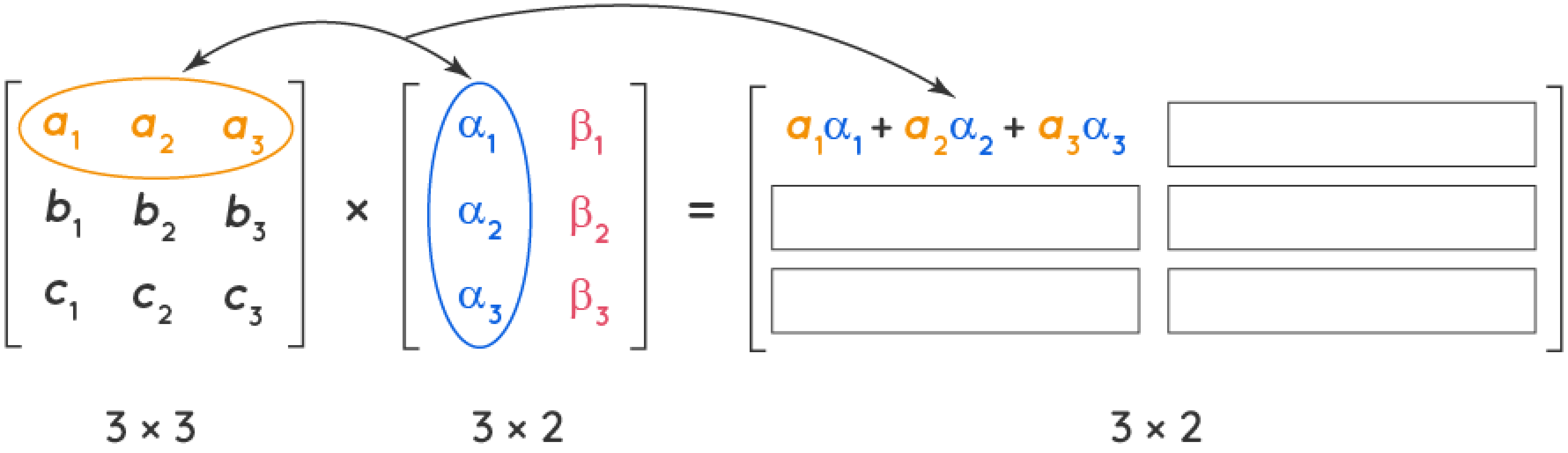
Therefore, *RC* is a scalar quantity.

Example 3: $\begin{bmatrix} 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} = \left[(1 \times 2) + \big(3 \times (-1)\big) + (2 \times 4)\right] = [2 - 3 + 8] = [7]$

Let A be of order $m \times n$ and B be of order $n \times p$. The matrix AB will be of order $m \times p$ and will be obtained by multiplying each row vector of A successively with column vectors in B.

Example 4: $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$ and $B = \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{bmatrix}$. Find AB?

Solution: To obtain the element a_{11} of AB, we multiply R_1 of A with C_1 of B :



To obtain the element a_{12} of AB, we multiply R_1 of A with C_2 of B:

$$\underset{3 \times 3}{\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}} \times \underset{3 \times 2}{\begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{bmatrix}} = \underset{3 \times 2}{\begin{bmatrix} \square & a_1\beta_1 + a_2\beta_2 + a_3\beta_3 \\ \square & \square \\ \square & \square \end{bmatrix}}$$

To obtain the element a_{21} of AB, we multiply R_2 of A with C_1 of B:

$$\underset{3 \times 3}{\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}} \times \underset{3 \times 2}{\begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{bmatrix}} = \underset{3 \times 2}{\begin{bmatrix} \square & \square \\ b_1\alpha_1 + b_2\alpha_2 + b_3\alpha_3 & \square \\ \square & \square \end{bmatrix}}$$

Proceeding in a similar way, we can obtain all the elements of AB.

Properties of Matrix Multiplication

There are different properties associated with the multiplication of matrices. The important properties are listed below,

- $AB \neq BA$, given matrices A and B.
- $A(BC) = (AB)C$, given matrices A, B and C.
- $A(B + C) = AB + AC$, given matrices A, B and C.
- $(A + B)C = AC + BC$, given matrices A, B and C.

Example 4: Given $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$. Find AB and BA?

Solution: $A \times B = \begin{bmatrix} 1(3) + 2(1) & 1(2) + 2(4) \\ 3(3) + 4(1) & 3(2) + 4(4) \end{bmatrix} = \begin{bmatrix} 5 & 10 \\ 13 & 22 \end{bmatrix}$

B$\times A = \begin{bmatrix} 3(1) + 2(3) & 3(2) + 2(4) \\ 1(1) + 4(3) & 1(2) + 4(4) \end{bmatrix} = \begin{bmatrix} 9 & 14 \\ 13 & 18 \end{bmatrix}$

Question 1: Given $A = \begin{bmatrix} 12 & 8 & 4 \\ 3 & 17 & 14 \\ 9 & 8 & 10 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 19 & 3 \\ 6 & 15 & 9 \\ 7 & 8 & 16 \end{bmatrix}$. Find AB?

Check solution to Question 1:

$$AB = \begin{bmatrix} 136 & 380 & 172 \\ 215 & 424 & 386 \\ 163 & 371 & 259 \end{bmatrix}$$

TRANSPOSE OF A MATRIX

The transpose of a matrix is done when we replace the rows of a matrix to the columns and columns to the rows. Interchanging of rows and columns is known as the transpose of matrices.

$$A_{2\times 3} = \begin{array}{c} \begin{bmatrix} 2 & -3 & -4 \\ -1 & 7 & -7 \end{bmatrix} \end{array} \begin{array}{l} \leftarrow \text{Row 1} \\ \leftarrow \text{Row 2} \end{array}$$

$$A^{T}_{3\times 2} = \begin{array}{c} \begin{array}{cc} \text{Column 1} & \text{Column 2} \\ \downarrow & \downarrow \end{array} \\ \begin{bmatrix} 2 & -1 \\ -3 & 7 \\ -4 & -7 \end{bmatrix} \end{array}$$

DETERMINANT OF A MATRIX

The determinant of a matrix is a number defined only for square matrices. It is used in the analysis of linear equations and their solution. Determinant of a matrix A is denoted as $|A|$.

Determinant Formula of matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is

$$|A| = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Example 5: Find the determinant of Matrix $D = \begin{bmatrix} 6 & 1 & 1 \\ 4 & -2 & 5 \\ 2 & 8 & 7 \end{bmatrix}$?

Solution:

$$|D| = 6[(-2 \times 7) - (5 \times 8)] - 1[(4 \times 7) - (5 \times 2)] + 1[(4 \times 8) - (-2 \times 2)]$$

$$= 6(-14 - 40) - 1(28 - 10) + 1(32 + 4) = 6(-54) - 1(18) + 1(36) = -306$$

INVERSE OF A MATRIX

The inverse of any matrix is denoted as the matrix raised to the power (-1), i.e. for any matrix "*A*", the inverse matrix is denoted as A^{-1}. There is a possibility that sometimes the inverse of a matrix does not exist if the determinant of the matrix is equal to zero ($|A| = 0$).

The first step is to (1) calculate the determinant of 3×3 matrix, (2) find its **minors**, (3) **cofactors**, and (4) **transpose** of the cofactor matrix (adjugate), and then include the results in the below given inverse matrix formula.

Inverse of a Matrix, $A^{-1} = (\frac{1}{|A|})(Adj\ A)$

where

$|A|$ is the determinant of the matrix *A* and $|A| \neq 0$.

Adj A is the adjugate of the given matrix *A*.

(2) **Minor** of matrix for a particular element in the matrix is defined as the matrix obtained when the row and column of the matrix in which that particular element lies are deleted, and the minor of the element a_{ij} is denoted as M_{ij}.

For example, for the given matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, minor a_{12} is:

$$M_{12} = \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}$$

Similarly, we can find all the minors of the matrix and will get a minor matrix M of the given matrix A as:

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}$$

(3) Cofactor of the matrix A is obtained when the minor M_{ij} of the matrix is multiplied with $(-1)^{i+j}$. The cofactor of a matrix is denoted as C_{ij}. If the minor of a matrix is M_{ij}, then the cofactor of the matrix would be:

$C_{ij}=(-1)^{i+j}M_{ij}$

Or simply multiply the Minor Matrix M by $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$ to obtain $C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$

(4) Transpose of cofactor matrix gives the adjugate matrix.

$$Adj\ A = C^T = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

Inverse of Matrix, $A^{-1} = (\frac{1}{|A|})(Adj\ A)$

Example 6: Find the inverse of Matrix $D = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 5 & 2 \\ 1 & -1 & -2 \end{bmatrix}$?

Solution:

(1) Determinant, $|A|$ =-33

(2) Minor Matrix, $M = \begin{bmatrix} -8 & -2 & -5 \\ 5 & -7 & -1 \\ -17 & 4 & 10 \end{bmatrix}$

(3) Cofactor Matrix, $C = \begin{bmatrix} -8 & 2 & -5 \\ -5 & -7 & 1 \\ -17 & -4 & 10 \end{bmatrix}$

(4) Transpose of C, $C^T = Adj\ A = \begin{bmatrix} -8 & -5 & -17 \\ 2 & -7 & -4 \\ -5 & 1 & 10 \end{bmatrix}$

(5) Inverse of Matrix A, $A^{-1} = \left(\frac{1}{|A|}\right)(Adj\ A) = \left(\frac{1}{-33}\right)\begin{bmatrix} -8 & -5 & -17 \\ 2 & -7 & -4 \\ -5 & 1 & 10 \end{bmatrix} = \begin{bmatrix} 0.24 & 0.15 & 0.51 \\ -0.06 & 0.21 & 0.12 \\ 0.15 & -0.03 & -0.39 \end{bmatrix}$

Example 7: Solve the following set of two equations: $x + y = 8$, $2x + 3y$ =10 ?

Solution:

1) arrange the set of equations in the Matrix form:

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, B = \begin{bmatrix} 8 \\ 10 \end{bmatrix}$$

To solve the equations, we need to find matrix *X*. It can be found by multiplying the inverse of matrix *A* with *B*, which is given as $X = (A^{-1})B$. To find the inverse of *A*, we will need the determinant and adjugate of matrix *A*.

2) $|A| = (1 \times 3) - (1 \times 2) = 1$

3) Minor, $\mathrm{M} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$

4) Cofactor Matrix, $\mathrm{C} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} + & - \\ - & + \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$

5) Transpose, $C^T = Adj\ A = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$

6) Inverse, $A^{-1} = \left(\frac{1}{|A|}\right)(Adj\ A) = (\frac{1}{1}) \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$

7) Now to find the matrix X, we'll multiply A^{-1} and *B*.

$$X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 8 \\ 10 \end{bmatrix} = \begin{bmatrix} (3 \times 8) + (-1 \times 10) \\ (-2 \times 8) + (1 \times 10) \end{bmatrix} = \begin{bmatrix} 14 \\ -6 \end{bmatrix}$$, therefore, $x = 14, y = -6$

ALTERNATE METHOD – CRAMER'S RULE

Cramer's Rule

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

$$\text{Let } D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\text{If } D \neq 0 \text{ then}$$

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{D} \qquad y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{D} \qquad z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{D}$$

Question 2: Solve the following set of two equations: $2x - y = 6$, $x + 2y = 8$?

CRYPTOGRAPHY

Cryptography is the study of secure communications techniques that allow only the sender and intended recipient of a message to view its contents. The term is derived from the Greek word ‘kryptos’, which means hidden.

Encryption is now widely used in communication and to secure private data; it is no longer limited to military applications. Modern encryption systems are increasingly complex, incorporating numerous ways to encrypt data, making it more secure and difficult to decrypt. Some modern methods make use of matrices in the encryption and decryption process.

The approach for encoding secret messages using matrices is as follows.

To encode a message:

1. Divide the letters of the message into groups of two or three (depending on the size of the square matrix used to encode the message).

2. Convert each group into a string of numbers by assigning a number to each letter of the message. Remember to assign numbers to blank spaces.

3. Convert each group of numbers into column matrices.

4. Convert these column matrices into a new set of column matrices by multiplying them with a compatible square matrix of your choice that has an inverse. This new set of numbers or matrices represents the coded message.

We'll use the following correspondence in this section: letters A to Z correspond to numbers 1 to 26, a space is represented by number 27, and punctuation is ignored.

A	B	C	D	E	F	G	H	I	J	K	L	M
1	2	3	4	5	6	7	8	9	10	11	12	13
N	O	P	Q	R	S	T	U	V	W	X	Y	Z
14	15	16	17	18	19	20	21	22	23	24	25	26

Example 8: Use matrix $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ to encode the message: ATTACK NOW!

Solution:

- Divide the letters of the message into groups of two: AT TA CK –N OW
- We assign the numbers to these letters from the above table, and convert each pair of numbers into 2×1 matrices. In the case where a single letter is left over on the end, a space is added to make it into a pair.

$$\begin{bmatrix} A \\ T \end{bmatrix} = \begin{bmatrix} 1 \\ 20 \end{bmatrix} \quad \begin{bmatrix} T \\ A \end{bmatrix} = \begin{bmatrix} 20 \\ 1 \end{bmatrix} \quad \begin{bmatrix} C \\ K \end{bmatrix} = \begin{bmatrix} 3 \\ 11 \end{bmatrix} \quad \begin{bmatrix} - \\ N \end{bmatrix} = \begin{bmatrix} 27 \\ 14 \end{bmatrix} \quad \begin{bmatrix} O \\ W \end{bmatrix} = \begin{bmatrix} 15 \\ 23 \end{bmatrix}$$

The message expressed as 2×1 matrices is as follows.

$$\begin{bmatrix} 1 \\ 20 \end{bmatrix} \quad \begin{bmatrix} 20 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 3 \\ 11 \end{bmatrix} \quad \begin{bmatrix} 27 \\ 14 \end{bmatrix} \quad \begin{bmatrix} 15 \\ 23 \end{bmatrix}$$

Now to encode, we multiply, on the left, each matrix of our message by the matrix A. For example, the product of A with our first matrix is:

$$\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 20 \end{bmatrix} = \begin{bmatrix} 41 \\ 61 \end{bmatrix}$$

Multiplying each matrix in by matrix A, in turn, gives the desired coded message:

$$\begin{bmatrix} 41 \\ 61 \end{bmatrix} \begin{bmatrix} 22 \\ 23 \end{bmatrix} \begin{bmatrix} 25 \\ 36 \end{bmatrix} \begin{bmatrix} 55 \\ 69 \end{bmatrix} \begin{bmatrix} 61 \\ 84 \end{bmatrix}$$

The approach for decoding secret messages using matrices is as follows.

To decode a message:

1. Take the string of coded numbers and multiply it by the inverse of the matrix that was used to encode the message.

2. Associate the numbers with their corresponding letters.

Question 3: Decode the following message that was encoded using matrix $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$.

$$\begin{bmatrix} 21 \\ 26 \end{bmatrix} \begin{bmatrix} 37 \\ 53 \end{bmatrix} \begin{bmatrix} 45 \\ 54 \end{bmatrix} \begin{bmatrix} 74 \\ 101 \end{bmatrix} \begin{bmatrix} 53 \\ 69 \end{bmatrix}$$

Solution:

Since this message was encoded by multiplying by the matrix A in Example 7, we decode this message by first multiplying each matrix, on the left, by the inverse of matrix A given below.

$$A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$$

For example, the product of A^{-1} with our first matrix is: $\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 21 \\ 26 \end{bmatrix} = \begin{bmatrix} 11 \\ 5 \end{bmatrix}$

By multiplying each of the matrices in by the matrix A^{-1}, we get the following.

$$\begin{bmatrix} 11 \\ 5 \end{bmatrix}\begin{bmatrix} 5 \\ 16 \end{bmatrix}\begin{bmatrix} 27 \\ 9 \end{bmatrix}\begin{bmatrix} 20 \\ 27 \end{bmatrix}\begin{bmatrix} 21 \\ 16 \end{bmatrix}$$

Finally, by associating the numbers with their corresponding letters, we obtain:

$\begin{bmatrix} K \\ E \end{bmatrix}\begin{bmatrix} E \\ P \end{bmatrix}\begin{bmatrix} - \\ I \end{bmatrix}\begin{bmatrix} T \\ - \end{bmatrix}\begin{bmatrix} U \\ P \end{bmatrix}$. The Message reads "KEEP IT UP".

Resources:

https://towardsdatascience.com/coding-a-matrix-5e1d7eb1e6e5

https://www.vedantu.com/maths/application-of-matrices

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