Verification of Stability Theorem for Tame Functions Using Persistence Diagrams

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Abstract. This paper investigates the stability of tame functions through the use of persistence diagrams and bottleneck distance. We examine the stability theorem, which asserts that the bottleneck distance between the persistence diagrams of two perturbed functions is less than or equal to the L∞-distance between the functions. Using a pair of functions $f(x)$ and $g(x)$, where $g(x)$ is a small perturbation of $f(x)$, we compute both the L∞-distance and the bottleneck distance to verify the stability theorem. Our results confirm that the bottleneck distance is indeed smaller than or equal to the L∞-distance, validating the theorem for tame functions. The source code is available at `https://github.com/LakshmikarPolamreddy/stability_theorem_tame_functions`.

Keywords: Stability Theorem · Tame Functions · Persistence Diagrams · TDA.

1 Introduction

Topological data analysis (TDA) has emerged as a powerful tool for analyzing the underlying structure of data, particularly in complex, high-dimensional settings. One of the core concepts in TDA is persistence homology, which captures the birth and death of topological features as a function of scale. These features can be represented in persistence diagrams, which serve as a compact summary of the topological features of a dataset or function.

In this paper, we focus on verifying the stability theorem for tame functions using persistence diagrams. The stability theorem asserts that the bottleneck distance between the persistence diagrams of two functions should be less than or equal to the L∞-distance between the functions themselves. This property is crucial for understanding the robustness of topological features to small perturbations in the underlying functions. Our goal is to empirically verify this theorem by comparing two functions—one of which is a perturbed version of the other—using both the L∞-distance and the bottleneck distance.

2 Related Work

Cohen et al. [2] proved that the persistence diagram is stable as small changes in the function imply only small changes in the diagram. Skraba et al. [5] provided stability results with respect to the p-Wasserstein distance between persistence diagrams. Patel et al. [4] defined the abelian category module as the type B persistence diagram and stated it to enjoy a stronger stability theorem. We framed our Methodology based on the textbook written by Edelsbrunner et al. [3].

3 Methods

3.1 Tame Functions in the Context of Topology

A *tame function* $f : \mathcal{X} \to \mathbb{R}$ in the context of topology, as described in the provided definitions, satisfies the following properties:

1. **Finite Homological Critical Values:** The function f has only a finite number of *homological critical values*. These critical values are thresholds $a \in \mathbb{R}$ where the topological structure of the sublevel set
$$\mathcal{X}_a = f^{-1}((-\infty, a])$$
undergoes a significant change, such as the birth or death of homology classes.
2. **Finite-Rank Homology Groups:** For every sublevel set $\mathcal{X}_a$, the homology groups $H_p(\mathcal{X}_a)$ (for each dimension p) have finite ranks. This means that the number of p-dimensional holes or cycles in the sublevel sets is finite.

3.2 Tame function under consideration

We define two functions $f(x)$ and $g(x)$ over a 1D domain. The function $f(x)$ is a complex tame function based on the example shown in [3], and $g(x)$ is a slightly perturbed version of $f(x)$, defined as follows:

$$f(x) = \sin(x) + 0.5\sin(3x) + 0.2\sin(7x) + 2$$
$$g(x) = \sin(x + 0.1) + 0.5\sin(3x + 0.1) + 0.2\sin(7x + 0.1) + 2$$

3.3 L∞-Distance Calculation

The L∞-distance, which measures the maximum absolute difference between the two functions, is calculated as:

$$L_\infty(f, g) = \max_x |f(x) - g(x)|$$

This distance provides a measure of the pointwise deviation between the functions.

3.4 Persistence Diagram Construction

We construct the simplicial complexes for both functions using Delaunay triangulation [1]. The simplicial complexes are then used to generate persistence diagrams. We compute the 0-dimensional persistence diagrams, which capture the birth and death of connected components in the filtration.

3.5 Bottleneck Distance Definition

The bottleneck distance is a metric used to compare two persistence diagrams X and Y, which are sets of points in the extended plane $\overline{\mathbb{R}}^2$.

Structure of Persistence Diagrams: Each persistence diagram consists of finitely many points above the diagonal $\Delta = \{(x, x) \mid x \in \mathbb{R}\}$. To simplify computations, we add infinitely many points on the diagonal, each with infinite multiplicity.

Distance Between Points: For two points $x = (x_1, x_2)$ and $y = (y_1, y_2)$, the distance is measured as:

$$\|x - y\|_\infty = \max\{|x_1 - y_1|, |x_2 - y_2|\}.$$

Bijections Between Diagrams: Consider all possible bijections $\eta : X \to Y$ between the points of the two diagrams.

Bottleneck Distance Formula: The bottleneck distance is the infimum of the supremum distances over all bijections:

$$W_\infty(X, Y) = \inf_{\eta:X\to Y} \sup_{x\in X} \|x - \eta(x)\|_\infty.$$

3.6 Stability Theorem Verification

Finally, we verify the stability theorem by comparing the L∞-distance and the bottleneck distance. If the bottleneck distance is less than or equal to the L∞-distance, the stability theorem is validated:

$$d_{\text{bottleneck}} \leq L_\infty.$$

4 Results and Discussion

We compute the L∞-distance and bottleneck distance for the two functions $f(x)$ and $g(x)$. To construct the persistence diagrams, we specifically consider the 0-dimensional homology, which captures the connected components of the functions in the filtration. The persistence diagram represents the birth and death of these connected components as the function values change, allowing us to quantify the topological features. The L∞-distance between $f(x)$ and $g(x)$, which measures the maximum absolute difference between the two functions at any point, is:

$$L_\infty(f, g) = 0.1698.$$

This value indicates the pointwise deviation between $f(x)$ and $g(x)$.

Next, we compute the bottleneck distance between the persistence diagrams of $f(x)$ and $g(x)$. Since we focus on 0-dimensional homology, the persistence

Algorithm 1 Stability Theorem Verification for Tame Functions

Require: Two tame functions $f(x)$ and $g(x)$, and a set of points $x_1, x_2, \ldots, x_N$.
Ensure: Verification of the stability theorem (whether the bottleneck distance $\leq$ L∞-distance).

Construct simplicial complex for $f(x)$ using function values at x:
    $f_{\text{complex}} \leftarrow \text{construct\_simplicial\_complex}(x, f, \text{threshold})$
Construct simplicial complex for $g(x)$ using function values at x:
    $g_{\text{complex}} \leftarrow \text{construct\_simplicial\_complex}(x, g, \text{threshold})$
Compute persistence diagrams for $f(x)$ and $g(x)$:
    $f_{\text{persistence}} \leftarrow f_{\text{complex}}.\text{persistence}()$
    $g_{\text{persistence}} \leftarrow g_{\text{complex}}.\text{persistence}()$
Extract persistence diagrams for dimension 0 (connected components):
    $f_{\text{diag\_points}} \leftarrow f_{\text{complex}}.\text{persistence\_intervals\_in\_dimension}(0)$
    $g_{\text{diag\_points}} \leftarrow g_{\text{complex}}.\text{persistence\_intervals\_in\_dimension}(0)$
Compute L∞-distance between $f(x)$ and $g(x)$:
    $L_\infty(f, g) \leftarrow \max(|f(x) - g(x)|)$
Compute bottleneck distance between the persistence diagrams:
if Both $f_{\text{diag\_points}}$ and $g_{\text{diag\_points}}$ are non-empty **then**
    $d_{\text{bottleneck}} \leftarrow \text{compute\_bottleneck\_distance}(f_{\text{diag\_points}}, g_{\text{diag\_points}})$
else
    $d_{\text{bottleneck}} \leftarrow \text{None}$
end if
if $d_{\text{bottleneck}} \leq L_\infty$ **then**
    return "Stability Theorem Verified: Bottleneck Distance $\leq$ L∞-Distance"
else
    return "Stability Theorem Not Verified"
end if

diagrams represent the birth and death of connected components in the functions. The bottleneck distance between the persistence diagrams of $f(x)$ and $g(x)$, which measures the maximum distance required to match the features in the two diagrams, is:

$$d_{\text{bottleneck}}(\mathcal{D}_f, \mathcal{D}_g) = 0.0981.$$

These results show that the bottleneck distance is smaller than the L∞-distance, confirming that the topological features of $f(x)$ and $g(x)$ are stable under small perturbations. This empirical result verifies the stability theorem, which asserts that the bottleneck distance between the persistence diagrams is less than or equal to the L∞-distance:

$$d_{\text{bottleneck}} \leq L_{\infty}.$$

Considering 0-dimensional homology is crucial in this context, as it focuses on the number and persistence of connected components, which are key topological features for understanding the overall behavior of the functions. The fact that the bottleneck distance is smaller than the L∞-distance suggests that the topological features, such as connected components, are more stable than the pointwise differences between the functions. This stability emphasizes the robustness of persistence diagrams as a tool for topological data analysis, especially in the presence of small perturbations in the function values. The bottleneck distance provides a reliable measure of the similarity between the functions' topological features, while the L∞-distance offers a measure of their pointwise deviations. Our results highlight that, even though the functions differ at specific points, their topological structures remain relatively unchanged, reinforcing the idea that topological features captured by persistence homology are less sensitive to small perturbations than pointwise values.

4.1 Ablation Study

The ablation study investigates the validation of the stability theorem on different types of tame functions by comparing the L^∞-distance and bottleneck distance between the persistence diagrams of the original function $f(x)$ and its perturbed version $g(x)$. The results, summarized in Table 1, demonstrate that the bottleneck distance remains bounded by the L^∞-distance across all tested examples, thereby validating the stability theorem.

- **Sine Function ($\sin(2\pi x)$):** The sine function, with its smooth periodic nature, exhibits an L^∞-distance of 1.7499 and a corresponding bottleneck distance of 0.3105. The significant gap between these values confirms that the perturbation does not heavily distort the persistence diagrams, thus satisfying the stability theorem.
- **Parabola Function ($ax^2 + bx + c$):** As a classic example of a quadratic tame function, the parabola demonstrates an L^∞-distance of 2.9981 and a bottleneck distance of 0.8795. This aligns with the expectations for such well-behaved functions, where the bottleneck distance is notably smaller than the L^∞-distance.

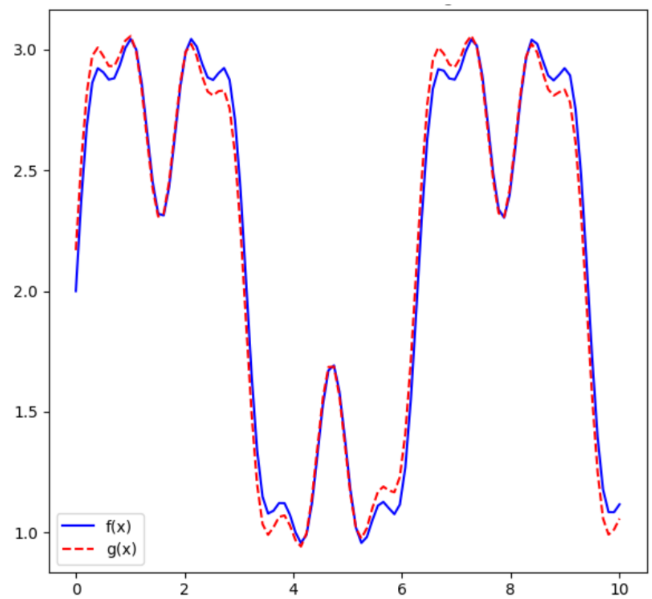

Fig. 1: Tame functions: $f(x) = \sin(x) + 0.5\sin(3x) + 0.2\sin(7x) + 2$
$g(x) = \sin(x + 0.1) + 0.5\sin(3x + 0.1) + 0.2\sin(7x + 0.1) + 2$

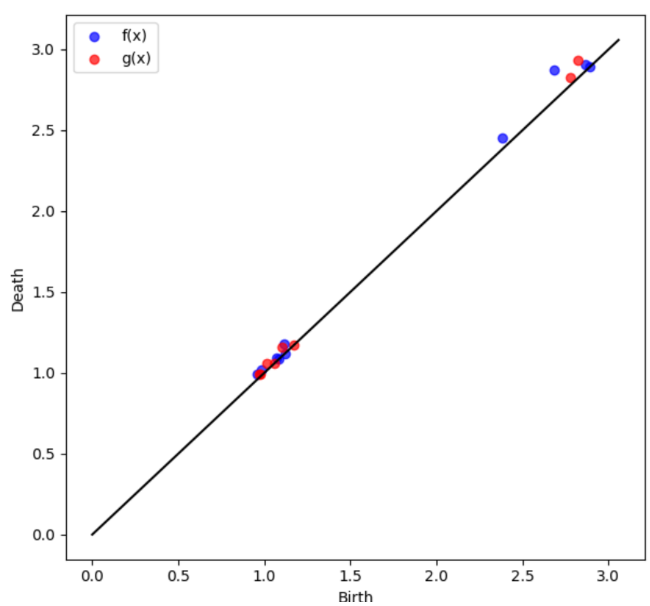

Fig. 2: 0-dimensional (H_0) persistence diagrams of $f(x)$ and $g(x)$

– **Complex Function ($\sin(x) + 0.5\sin(3x) + 0.2\sin(7x) + 2$):** This more intricate tame function exhibits smaller distances, with an L^{∞}-distance of 0.1698 and a bottleneck distance of 0.0981. The close proximity of these values underscores the sensitivity of persistence diagrams to the function's structure, while still upholding the stability theorem.

The study reveals that the bottleneck distance between persistence diagrams is consistently smaller than the L^{∞}-distance for all considered examples, irrespective of the function's complexity. This finding underscores the robustness of the stability theorem and its applicability to a diverse range of tame functions, including smooth, polynomial, and composite oscillatory functions. These results further affirm that persistence diagrams serve as reliable topological summaries under perturbations.

Table 1: Stability theorem validation on different tame functions.

Examples	$f(x)$	L∞	Bottleneck	Validated
Sine function	$\sin(2\pi x)$	1.7499	0.3105	Yes
Parabola function	$ax^2 + bx + c$	2.9981	0.8795	Yes
Complex function	$\sin(x) + 0.5\sin(3x) + 0.2\sin(7x) + 2$	0.1698	0.0981	Yes

5 Conclusion

In this paper, we successfully verified the stability theorem for tame functions using persistence diagrams. By computing both the L∞-distance and the bottleneck distance between two perturbed functions, we showed that the bottleneck distance is indeed less than or equal to the L∞-distance, thereby validating the stability theorem. Our results demonstrate the effectiveness of persistence diagrams in capturing the topological stability of functions, and we conclude that persistence homology is a valuable tool for analyzing and comparing functions, particularly in the context of small perturbations. This work contributes to the growing body of research in topological data analysis, offering further evidence for the utility of persistence homology in understanding the structure and stability of mathematical functions.

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