Revision Notes

Class 11 Mathematics

Chapter 2 - Relation & Function-I

1. INTRODUCTION:

- In this chapter, we'll learn how to link pairs of objects from two sets to form a relation between them.
- We'll see how a relation can be classified as a function.
- Finally, we'll look at several types of functions, as well as some standard functions.

2. RELATIONS:

2.1 **Cartesian product of sets**

Definition:

- Given two non-empty sets P and Q .
- The Cartesian product $P \times Q$ is the set of all ordered pairs of elements from P and Q that is
- $P \times Q = \{(p, q) \; ; \; p \in P \; ; \; q \in Q\}$

2.2 **Relation:**

2.2.1 **Definition:**

- Let A and B be two non-empty sets.
- Then any subset 'R' of $A \times B$ is a relation from A and B.
- If $(a, b) \in R$, then we can write it as $a \; R \; b$ which is read as a is related to b 'by the relation R', 'b' is also called image of 'a' under R.

2.2.2 **Domain and range of a relation:**

- If R is a relation from A to B, then the set of first elements in R is known as **domain** & the set of second elements in R is called **range** of R symbolically.
- Domain of $R = \{ x : (x, y) \in R\}$

- Range of $R = \{ y:(x, y) \in R\}$
- The set B is considered as **co-domain of relation** R.
- Note that,

range $\subset$ co-domain

- **Note :**

Total number of relations that can be defined from a set A to a set B is the number of possible subsets of $A \times B$.
If $n(A) = p$ and $n(B) = q$, then
$n(A \times B) = pq$ and total number of relations is 2^{pq}.

2.2.3 **Inverse of a Relation**:

- Let A, B be two sets and let R be a relation from a set A to set B. Then the inverse of R denoted as R^{-1}, is a relation from B to A and is defined by

$R^{-1} = \{(b, a):(a, b) \in R\}$

- Clearly

$(a, b) \in R \Leftrightarrow (b, a) \in R^{-1}$

- Also,

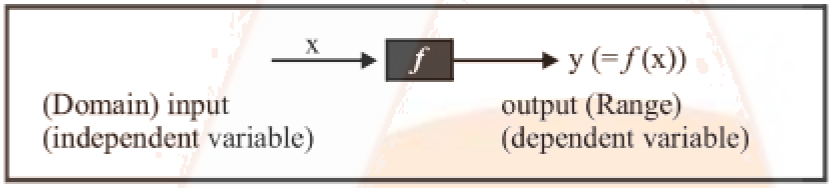
$\text{Dom}(R) = \text{Range}(R^{-1})$ and
$\text{Range}(R) = \text{Dom}(R^{-1})$

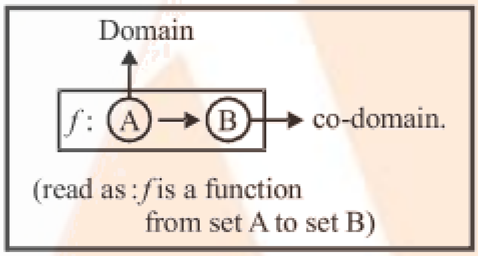
3. FUNCTIONS:

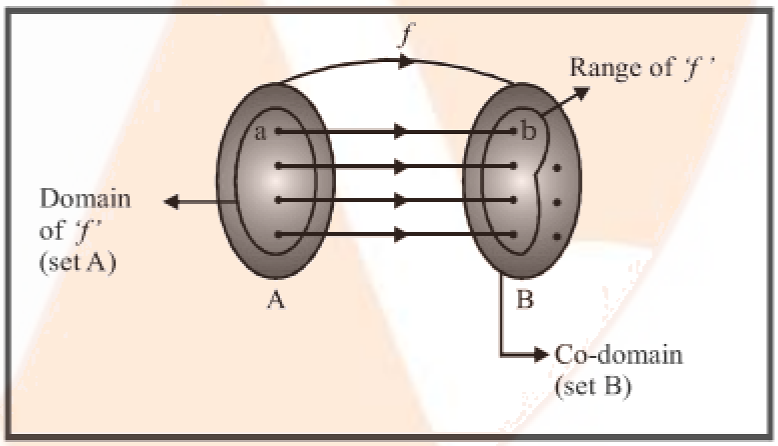
3.1 **Definition:**

A relation 'f' from a set A to set B is said to be a function if every element of set A has **one and only one image** in set B.

Notations:







3.2 **Domain, Co-domain and Range of a function:**

Domain:

The domain is believed to be the biggest set of x - values for which the formula provides real y - values when $y = f(x)$ is defined using a formula and the domain is not indicated explicitly.

The **domain** of $y = f(x)$ is the set of all real x for which $f(x)$ is defined (real).

Rules for finding Domain:

1. Even roots (square root, fourth root, etc.) should have non–negative expressions.
2. Denominator $\neq 0$
3. $\log_a x$ is defined when $x > 0$, $a > 0$ and $a \neq 1$
4. If domain $y = f(x)$ and $y = g(x)$ are D_1 and D_2 respectively then the domain of $f(x) \pm g(x)$ or $f(x).g(x)$ is $D_1 \cap D_2$.

While domain of $\dfrac{f(x)}{g(x)}$ is $D_1 \cap D_2 - \{x : g(x) = 0\}$

Range:

The set of all f - **images of elements** of A is known as the range of f and can be denoted as $f(A)$.

$\text{Range} = f(A) = \{f(x) : x \in A\}$

$f(A) \subseteq B$ {Range $\subseteq$ Co-domain}

Rule for finding range:

First of all find the domain of $y = f(x)$

i. If domain $\in$ finite number of points $\Rightarrow$ range $\in$ set of corresponding $f(x)$ values.

ii. If domain $\in R$ or $R -$ {Some finite points}

Put $y = f(x)$

Then express x in terms of y. From this find y for x to be defined. (i.e., find the values of y for which x exists).

iii. If domain $\in$ a finite interval, find the least and greater value for range using monotonicity.

Note:

1. Question of format:

$$\left(y = \frac{Q}{Q}; y = \frac{L}{Q}; y = \frac{Q}{L} \right)$$

$Q \rightarrow$ Quadratic

$L \rightarrow$ Linear

Range is found out by **cross-multiplying** & **creating a quadratic** in 'x' & making $D \geq 0$ (as $x \in R$)

2. Questions to determine the range of values in which the given expression $y = f(x)$ can be converted into x (or some function of $x =$ expression in ' y ' .

Do this & apply method (ii) .

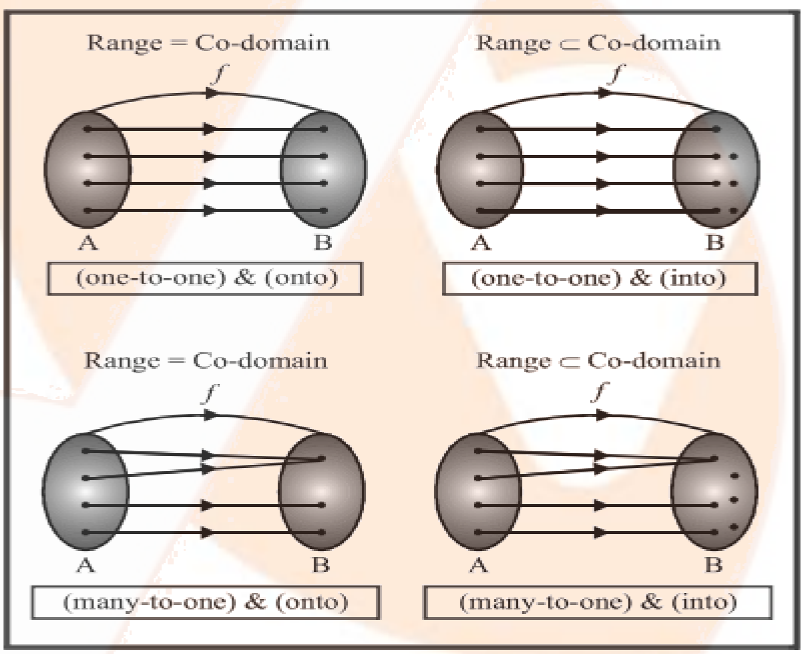
3. Two functions f & g are said to be equal if

a. Domain of $f =$ Domain of g

b. Co-domain of $f =$ Co-domain of g

c. $f(x) = g(x) \ \forall x \in$ Domain

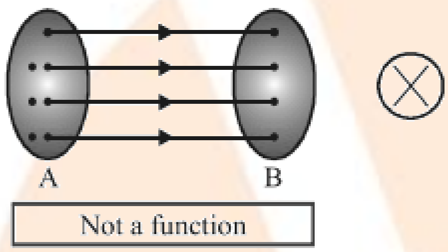
3.3 **Kinds of functions:**



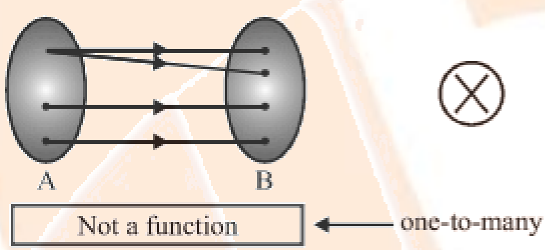
Note:

- **Injective functions** are called as **one-to-one functions**.
- **Surjective functions** are also known as **onto functions**.
- **Bijective functions** are also known as (**one-to-one**) and (**onto**) **functions.**

Relations which cannot be categorized as a function:



As not all elements of set A are associated with some elements of set B . (Violation of– point (i) – definition 2.1)



An element of set A is not associated with a unique element of set B, (violation of point (ii) definition 2.1)

Methods to check one-one mapping:

1. Theoretically:

$f(x_1) = f(x_2)$

$\Rightarrow x_1 = x_2$

then $f(x)$ is **one-one.**

2. Graphically:

A function is one-one, if no line parallel to $x-\text{axis}$ meets the graph of function at more than one point.

3. By Calculus:

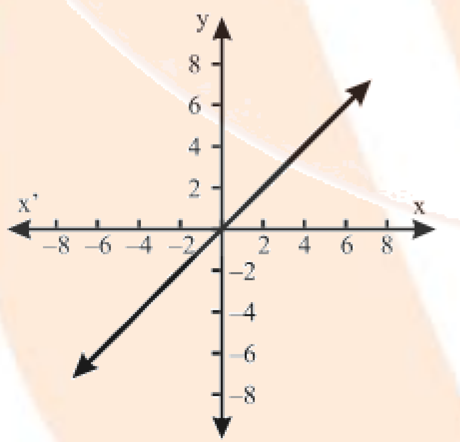
For checking whether $f(x)$ is One-One, find whether function is only increasing or only decreasing in their domain. If yes, then function is one-one, that is if $f'(x) \geq 0, \forall x \in$ domain or, if $f'(x) \geq 0, \forall x \in$ domain, then function is one-one.

3.4 Some standard real functions & their graphs:

3.4.1 Identity Function:

The function $f: R \to R$ defined by

$y = f(x) = x \forall x \in R$ is called **identity function.**

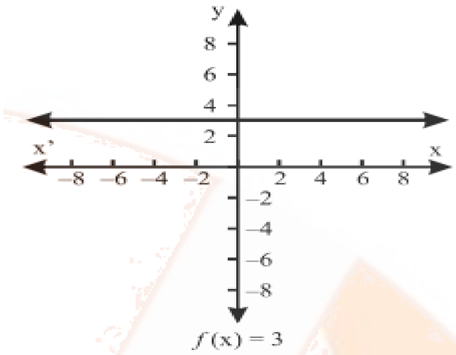


$f(x) = x$

3.4.2 Constant function:

The function $f: R \to R$ defined by

$y = f(x) = c, \forall x \in R$



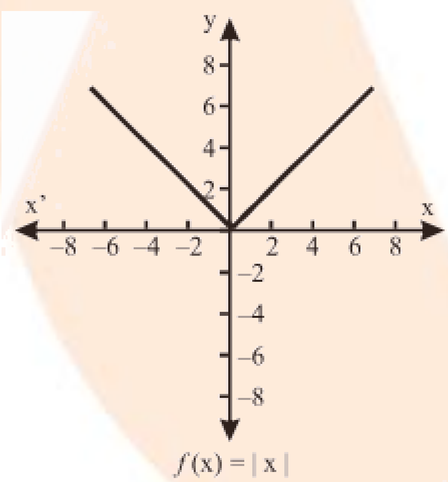
3.4.3 **Modulus function:**

The function $f : R \to R$ defined by

$$f(x) = \begin{cases} x; & x \geq 0 \\ -x; & x < 0 \end{cases}$$

is called **modulus function**. It is denoted by

$y = f(x) = |x|$



It is also known as "**Absolute value function**".

Properties of Modulus Function:

The modulus function has the following properties:

1. For any real number x, we have $\sqrt{x^2} = |x|$
2. $|xy| = |x||y|$

3. $|x+y| \le |x| + |y|$ Triangle inequality

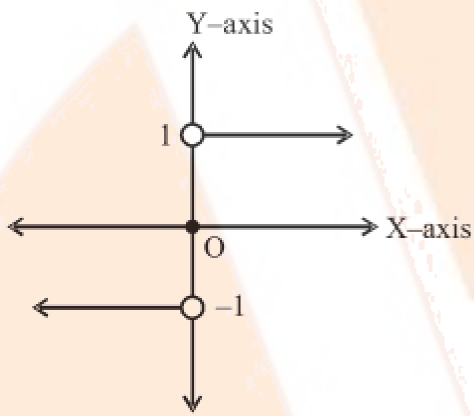
4. $|x-y| \ge ||x| - |y||$ Triangle inequality

3.4.4 **Signum Function:**

The function $f : R \to R$ define by

$$f(x) = \begin{cases} 1 : x > 0 \\ 0 : x = 0 \\ -1 : x < 0 \end{cases}$$ is called **signum** function.

It is usually denoted as $y = f(x) = sgn(x)$



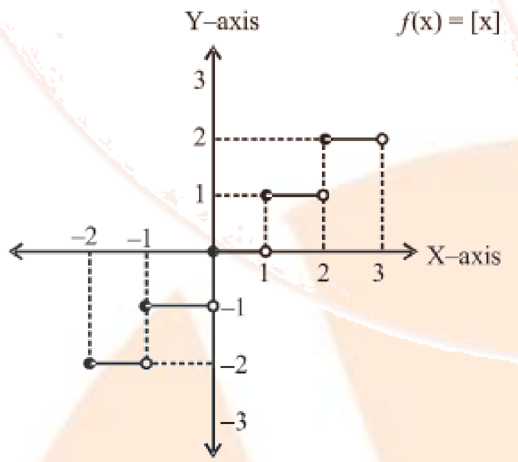
Note:

$$sgn(x) = \begin{cases} \frac{|x|}{x}; & x \ne 0 \\ 0; & x = 0 \end{cases}$$

3.4.5 **Greatest Integer Function:**

The function $f : R \to R$ defined as the greatest integer less than or equal to x.

It is usually denoted as $y = f(x) = [x]$.



Properties of Greatest Integer Function:

If n is an integer and x is any real number between n and $n+1$, then the greatest integer function has the following properties:

1. $[-n]=-[n]$
2. $[x+n]=[x]+n$
3. $[-x]=[x]-1$
4. $[x]+[-x]=\begin{cases}-1, & \text{if } x \notin I \\ 0, & \text{if } x \in I\end{cases}$

Note:

Fractional part of x, denoted by $\{x\}$ is given by $x-[x]$, Hence

$$\{x\}=x-[x]=\begin{cases} x-1; & 1 \le x < 2 \\ x & 0 \le x \le 1 \\ x+1 & -1 \le x < 0 \end{cases}$$

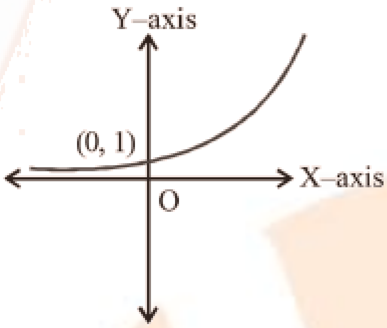
3.4.6 **Exponential Function:**

$f(x)=a^x$,

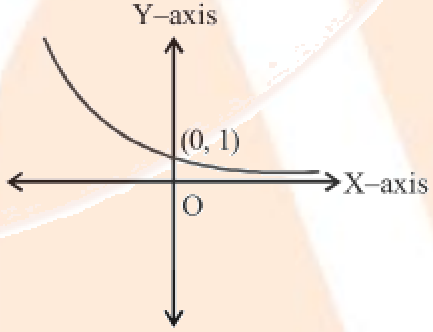
$a>0, a \ne 1$

Domain: $x \in R$

Range: $f(x) \in (0,\infty)$



$f(x) = a^x$, when $a > 1$



$f(x) = a^x$, when $0 < a < 1$

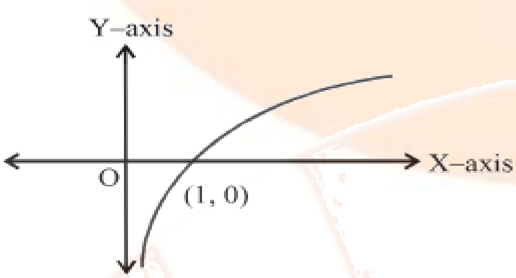
3.4.7 **Logarithm Function:**

$f(x) = \log_a x$,

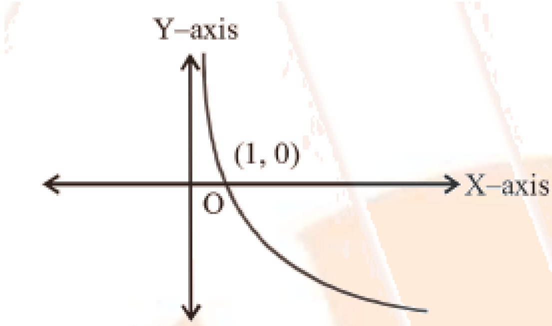
$a > 0, a \neq 1$

Domain: $x \in (0, \infty)$

Range: $y \in R$



$f(x) = \log_a x$, when $a > 1$



$f(x) = \log_a x$, when $0 < a < 1$

a) The Principal Properties of Logarithms:

Let M and N be the **arbitrary positive numbers**, $a > 0,\ a \neq 1,\ b > 0,\ b \neq 1$

1) $\log_b a = a \quad \Rightarrow a = b^c$

2) $\log_a (M.N) = \log_a M + \log_a N$

3) $\log_a \left(\dfrac{M}{N}\right) = \log_a M - \log_a N$

4) $\log_a M^N = N \log_a M$

5) $\log_b a = \dfrac{\log_c a}{\log_c b}, c > 0, c \neq 1$

6) $a^{\log_c b} = b^{\log_c a}, a, b, c > 0, c \neq 1$

Note:

- $\log_a a = 1$
- $\log_b a \ .\ \log_c b\ .\ \log_a c = 1$
- $\log_a 1 = 0$
- $e^{x \ln a} = e^{x \ln a^x} = a^x$

b) Properties of Monotonicity of Logarithm:

1) If $a > 1, \log_a x < \log_a y \quad \Rightarrow 0 < x < y$

2) If $0 < a < 1, \log_a x < \log_a y \quad \Rightarrow x > y > 0$

3) If $a > 1$ then $\log_a x < p \quad \Rightarrow 0 < x < a^p$

4) If $a > 1$ then $\log_a x > p \quad \Rightarrow x > a^p$

5) If $0 < a < 1$ then $\log_a x < p \quad \Rightarrow x > a^p$

6) If $0 < a < 1$ then $\log_a x > p \quad \Rightarrow 0 < x < a^p$

Note:

- The logarithm is **positive** if the **exponent and base** are on the **same side of unity.**
- The logarithm is **negative** if the **exponent and base** are on **opposite sides of unity.**

4. ALGEBRA OF REAL FUNCTION:

We'll learn how to add two real functions, remove one from another, multiply a real function by a scalar (a scalar is a real integer), multiply two real functions, and divide one real function by another in this part.

4.1 **Addition of two real functions:**

Let $f : X \to R$ and $g : X \to R$ by any two real functions, where $x \subset R$. Then, we define $(f+g) : X \to R$ by

$(f+g)(x) = f(x) + g(x)$ for all $x \in X$.

4.2 **Subtraction of a real function from another:**

Let $f : X \to R$ be any two any two real functions, where $x \subset R$.

Then, we define $(f-g) : X \to R$ by

$(f-g)(x) = f(x) - g(x)$ for all $x \in X$.

4.3 **Multiplication by a scalar:**

Let $f : X \to R$ be a real valued function and α be a scalar.

Here by scalar, we mean a real number.

Then the product αf is a function from X to R defined as $(\alpha f)(x) = \alpha f(x), x \in X$.

4.4 **Multiplication of two real functions:**

The product (or multiplication) of two real functions $f: X \to R$ and $g: X \to R$ is a function $fg: X \to R$ defined as

$(fg)(x) = f(x)g(x)$ for all $x \in X$.

This is also known as **pointwise multiplication**.

4.5 **Quotient of two real functions:**

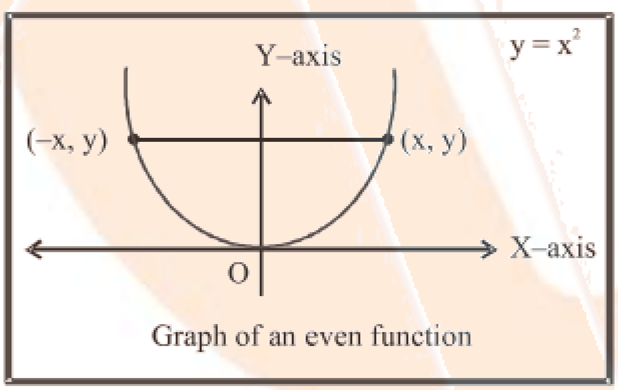
Let f and g be two **real** functions defined from $X \to R$ where $X \subset R$.

The quotient of f by g denoted by $\frac{f}{g}$ a is a function defined as $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$

Provided $g(x) \neq 0, x \in X$.

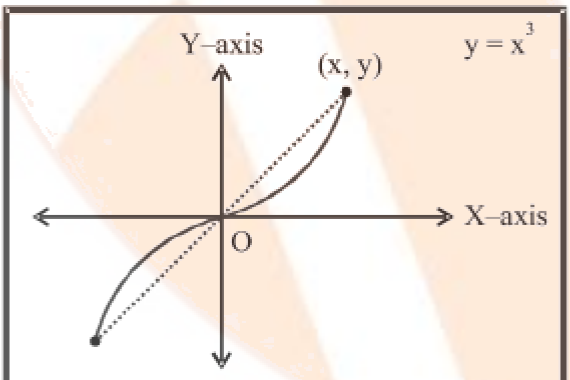
4.6 **Even and Odd Functions**

- **Even function:**
  - $f(-x) = f(x), \forall x \in$ Domain
  - The graph of an even function $y = f(x)$ is **symmetric** about the $y-$ axis, that is (x, y) lies on the graph $\Leftrightarrow (-x, y)$ lies on the graph.



Graph of an even function

- **Odd Function:**
  - $f(x) = -f(x), \forall x \in$ Domain
  - The graph of an odd function $y = f(x)$ is **symmetric about origin** that is if point (x, y) is on the graph of an odd function, then $(-x, -y)$ will also lie on the graph.



(–x, –y) Graph of an odd function

5. PERIODIC FUNCTION

- **Definition:**

A function $f(x)$ is said to be periodic function, if there exists a positive real number T , such that

$f(x+T) = f(x), \forall x \in R$

Then, $f(x)$ is a **periodic function** where **least positive value of T is called fundamental period**.

- **Graphically:**

The function is said to be periodic if the graph repeats at a set interval, and its period is the width of that interval.

Some standard results on periodic functions:

	Functions	Periods
i	$\sin^n x, \cos^n x, \sec^n x, \text{cosec}^n x$	π ; if n is even 2π ;(if n is odd or fraction)
ii	$\tan^n x, \cot^n x$	π ; n is even or odd
iii	$\lvert\sin x\rvert, \lvert\cos x\rvert, \lvert\tan x\rvert,$ $\lvert\cot x\rvert, \lvert\sec x\rvert, \lvert\text{cosec} x\rvert$	π
iv	$x-[x],[.]$ represents greatest integer function	1
v	Algebraic functions for example $\sqrt{x}, x^2, x^3+5$, ... etc.	Period does not exist

Properties of Periodic Function:

i. If $f(x)$ is periodic with period T, then

a) $c.f(x)$ is periodic with period T

b) $f(x \pm c)$ is periodic with period T

c) $f(x) \pm c$ is periodic with period T

Where c is any constant

i. If $f(x)$ is periodic with period T, then

$kf(cx+d)$ has period $\frac{T}{|c|}$

That is **Period** can be only affected by coefficient of x where $k, c, d \in$ constant.

ii. If $f_1(x), f_2(x)$ are periodic functions with periods T_1, T_2 respectively,

Then we have,

$h(x) = af_1(x) \pm bf_2(x)$ has period as, LCM of $\{T_1, T_2\}$

Note:

a. $\text{LCM of}\left(\frac{a}{b},\frac{c}{d},\frac{e}{f}\right)=\frac{\text{LCM of }(a,c,e)}{\text{HCF of }(b,d,f)}$

b. LCM **of rational and rational always exists.**
LCM **of irrational and irrational sometime exists.**
But LCM **of rational and irrational never exists.**
For example, LCM of $(2\pi, 1, 6\pi)$ is not possible because $2\pi, 6\pi \in$ **irrational** and $1 \in$ **rational.**