MATHEMATICS - II F/2023

1. Choose the correct

1. $A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}_{1\times3}$, $B = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ $A \times B = ?$

(b) [4]

$[1 \times 2 + 2 \times 1 + 3 \times 0] = [4]$

2. $y^2 = 4ax$ represents

(a) Parabola

3. Area between curve $y = f(x)$ and x axis from $x = a$ to $x = b$

(a) $\int_a^b y \, dx$

4. vector $\vec{a}$ and $\vec{b}$ will be perpendicular to each other if

(c) $\vec{a} \cdot \vec{b} = 0$

5. Solution of differential equation $2x dx + 2y dy = 0$ is

Q2 (a) If $\begin{bmatrix} x & 2 \\ 1 & 3 \end{bmatrix} + 2\begin{bmatrix} 1 & y \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 7 & 5 \end{bmatrix}$

find x and y

Solve

$$\begin{bmatrix} x & 2 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 2y \\ 6 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 7 & 5 \end{bmatrix}$$

$$\begin{bmatrix} x+2 & 2+2y \\ 1+6 & 3+2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 7 & 5 \end{bmatrix}$$

Since corresponding elements of equal matrix are equal

$$x + 2 = 0$$

$$x = -2$$

and

$$2+2y = 1$$

$$2y = 1-2$$

$$y = -1$$

$$y = \frac{-1}{2}$$

Ans The value of x and y is -2 and $\frac{-1}{2}$ respectively

(b) Prove that

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$$

Solve Taking LHS

$R_1 \rightarrow R_1 - R_2$

$R_2 \rightarrow R_2 - R_3$

$$\begin{vmatrix} 1-1 & a-b & a^2-b^2 \\ 1-1 & b-c & b^2-c^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$\begin{vmatrix} 0 & (a-b) & (a-b)(a+b) \\ 0 & (b-c) & (b-c)(b+c) \\ 1 & c & c^2 \end{vmatrix}$$

$$(a-b)(b-c)\begin{vmatrix} 0 & 1 & (a+b) \\ 0 & 1 & (b+c) \\ 1 & c & c^2 \end{vmatrix}$$

Expanding in C_1

$(a-b)(b-c)\left[1(b+c) - 1(a+b)\right]$

$(a-b)(b-c)\ (b+c-a-b)$

$(a-b)(b-c)\ (c-a) = RHS$

Hence proved

(c) Solve the given system of linear equations by Cramer rule

$$2x + y - z = 3$$

$$x + y + z = 1$$

$$x - 2y - 3z = 4$$

Solve

$$\Delta = \begin{vmatrix} 2 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -2 & -3 \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & 1 \\ -2 & -3 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} + (-1)\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix}$$

$$= 2[1(-3) - 1(-2)] - 1[1(3) - 1(1)] - 1[1(-2) - 1(1)]$$

$$= 2[-3+2] - 1[3-1] - 1[-2-1]$$

$$= 2(-1) - 1(2) - 1(-3)$$

$$= -2 - 2 + 3$$

$$= -1$$

Since $\Delta \neq 0$ Crammer's rule can be applied

$$\Delta x = \begin{vmatrix} 3 & 1 & -1 \\ 1 & 1 & 1 \\ 4 & -2 & -3 \end{vmatrix}$$

$$\Delta x = 3\begin{vmatrix} 1 & 1 \\ -2 & -3 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 4 & -3 \end{vmatrix} + (-1)\begin{vmatrix} 1 & 1 \\ 4 & -2 \end{vmatrix}$$

$$\Delta x = 3[1(-3) - 1(-2)] - 1[1(-3) - 1(4)]$$
$$- 1[1(-2) - 1(4)]$$

$$\Delta x = 3[-3 + 2] - 1[-3 - 4] - 1[-2 - 4]$$

$$\Delta x = 3(-1) - 1(-7) - 1(-6)$$

$$\Delta x = -3 + 7 + 6$$

$$\Delta x = 10$$

$$\Delta y = \begin{vmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 1 & 4 & -3 \end{vmatrix}$$

$$\Delta y = 2\begin{vmatrix} 1 & 1 \\ 4 & -3 \end{vmatrix} - 3\begin{vmatrix} 1 & 1 \\ 1 & -3 \end{vmatrix} + (-1)\begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix}$$

$$\Delta y = 2[1(-3) - 1(4)] - 3[1(-3) - 1(1)] - 1[1(4) - 1(1)]$$

$$\Delta y = 2[-3-4] - 3[-3-1] - 1[4-1]$$

$$\Delta y = 2(-7) - 3(-4) - 1(3)$$

$$\Delta y = -14 + 12 - 3$$

$$\Delta y = -5$$

$$\Delta z = \begin{vmatrix} 2 & 1 & 3 \\ 1 & 1 & 1 \\ 1 & -2 & 4 \end{vmatrix}$$

$$\Delta z = 2\begin{vmatrix} 1 & 1 \\ -2 & 4 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix} + 3\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix}$$

$$\Delta z = 2[1(4) - (-2)1] - 1[1(4) - 1(1)]$$
$$+3[1(-2) - 1(1)]$$

$$\Delta z = 2[4+2] - 1[4-1] + 3[-2-1]$$

$$\Delta z = 2(6) - 1(3) + 3(-3)$$

$$\Delta z = 12 - 3 - 9$$

$$\Delta z = 0$$

$$x = \frac{\Delta x}{\Delta} = \frac{10}{-1} = -10$$

$$y = \frac{\Delta y}{\Delta} = \frac{-5}{-1} = 5$$

$$z = \frac{\Delta z}{\Delta} = \frac{0}{-1} = 0$$

Q 3 (a) Find centre and radius of circle
$x^2+y^2+2x+4y+1=0$

Solve Comparing given equation with standard equation of circle

$$x^2+y^2+2gx+2fy+c=0$$

$$2gx = 2x$$

$$2g = 2$$

$$g = \frac{2}{2} = 1$$

$$2fy = 4y$$

$$2f = 4$$

$$f = \frac{4}{2} = 2$$

and $C = 1$

Centre of circle $= (-g, -f)$

$= (-1, -2)$

Radius of circle

$r = \sqrt{g^2 + f^2 - C}$

$r = \sqrt{1^2 + 2^2 - 1}$

$r = \sqrt{1 + 4 - 1}$

$r = \sqrt{4}$

$r = 2$

Ans The centre and radius of ~~circle~~ given equation of circle is $(-1, -2)$ and 2 units

(b) Find the length of Perpendicular drawn from point $(0, 1)$ to line $4x + 3y + 7 = 0$

Solve

Length of perpendicular from a point to line

$$= \left| \frac{Ax_1 + By_1 + C}{\sqrt{A^2 + B^2}} \right|$$

$$= \left| \frac{4(0) + 3(1) + 7}{\sqrt{4^2 + 3^2}} \right|$$

$$= \left| \frac{0+10}{\sqrt{16+9}} \right|$$

$$= \left| \frac{10}{\sqrt{25}} \right|$$

$$= \left| \frac{\cancel{10}^{2}}{\cancel{5}} \right|$$

$$= |2|$$

$$= 2 \text{ units}$$

Answer Length of perpendicular drawn from point $(0, 1)$ to $4x + 3y + 7 = 0$ is 2 units

(C) Find the Foci, vertices, eccentricity, length of latus rectum and equation of directrices of given hyperbola

$$\frac{x^2}{9} - \frac{y^2}{25} = 1$$

Solve Given

$$\frac{x^2}{9} - \frac{y^2}{25} = 1$$

so. $\frac{x^2}{3^2} - \frac{y^2}{5^2} = 1$

Comparing this equation of Hyperbola with standard equation.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$a = 3$ and $b = 5$

using relation

$$a^2 + b^2 = c^2$$

$$3^2 + 5^2 = c^2$$

$$9 + 25 = c^2$$

$$c = \sqrt{34}$$

Foci $= (-c, 0)\ (c, 0)$

$= (-\sqrt{34}, 0), (\sqrt{34}, 0)$

vertices $= (-a, 0), (a, 0)$

$= (-3, 0), (3, 0)$

eccentricity $= \frac{c}{a} = \frac{\sqrt{34}}{3}$

length of latus rectum $= \frac{2b^2}{a}$

$$= \frac{2(5^2)}{3}$$

$$= \frac{50}{3}$$

Equation of directrices of hyperbola

$$x = \frac{a}{e} \quad , \quad x = \frac{-a}{e}$$

$$x = \frac{3}{\frac{\sqrt{34}}{3}} \quad , \quad x = \frac{-3}{\frac{\sqrt{34}}{3}}$$

$$x = \frac{\cancel{3}}{\cancel{3}\sqrt{34}} \quad , \quad x = \frac{-\cancel{3}}{\cancel{3}\sqrt{34}}$$

$$x = \frac{1}{\sqrt{34}} \quad , \quad x = \frac{-1}{\sqrt{34}}$$

Q4(a) Find

$$\int \frac{dx}{(x+1)(x+2)}$$

Solve

Let $I = \int \frac{dx}{(x+1)(x+2)}$

Let, $\frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{(x+2)}$

$$\frac{1}{(x+1)(x+2)} = \frac{A(x+2) + B(x+1)}{(x+1)(x+2)}$$

$$1 = A(x+2) + B(x+1) \quad -(2)$$

putting $x = -1$ in eq 2

$$1 = A(-1+2) + B(-1+1)$$

$$1 = A(1) + B(0)$$

$$1 = A$$

$$A = 1$$

putting $x = -2$ in eq 2

$$1 = A(-2+2) + B(-2+1)$$

$$1 = 0 - B$$

$B = -1$

Hence our equation became

$$\int \frac{dx}{(x+1)(x+2)} = \int \frac{1\,dx}{(x+1)} + \int \frac{-1}{(x+2)}.dx$$

$$= \int \frac{dx}{(x+1)} - \int \frac{dx}{(x+2)}$$

$$= \log(x+1) - \log(x+2) + C$$

$$= \log\left|\frac{x+1}{x+2}\right| + C$$

(b) Find $\int x^2 \sin x . dx$

Solve

Integrating by parts taking x^2 as first function

$$\int [f(x).g(x)].dx = f(x)\int g(x).dx - \int\left[\frac{d}{dx}f(x).\int g(x)dx\right].dx$$

$$= x^2\int \sin x.dx - \int\left[\frac{d}{dx}x^2 . \int \sin x.dx\right].dx$$

$$= x^2(-\cos x) - \int\left[2x(-\cos x)\right].dx$$

$$= -x^2 \cos x + 2\int x \cos x \cdot dx$$

Again integrating by parts taking x as first function

$$= -x^2 \cos x + 2\left[x\int \cos x \cdot dx - \int\left[\frac{d(x)}{dx}\int \cos x . dx\right].dx\right]$$

$$= -x^2 \cos x + 2\left[x \sin x - \int \sin x \cdot dx\right]$$

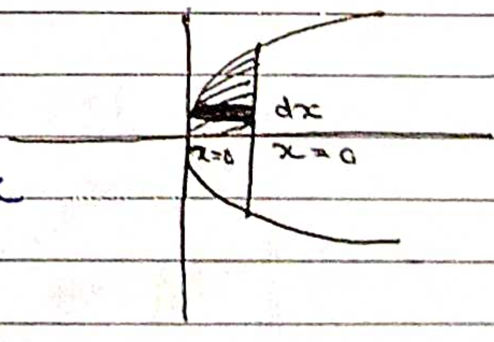
$$= -x^2 \cos x + 2\left[x \sin x + \cos x\right] + C$$

Ans $= -x^2 \cos x + 2x \sin x + 2\cos x + C$

(c) Find the area between curves $y^2 = 4ax$, x axis, $x = 0$, $x = a$

Solve

$$A = \int_a^b y \cdot dx$$



Given equation $y^2 = 4ax$

$$y = \sqrt{4ax}$$

$$A = \int_0^a \sqrt{4ax} \cdot dx$$

$$A = 2\int_0^a \sqrt{ax} \cdot dx$$

$$A = 2\int_0^a \sqrt{a}\, x^{\frac{1}{2}} \cdot dx$$

$$A = 2\sqrt{a} \int_0^a x^{\frac{1}{2}} \cdot dx$$

$$= 2\sqrt{a} \left[\frac{2x^{\frac{3}{2}}}{3}\right]_0^a$$

$$= \frac{4}{3}\sqrt{a} \left[x^{\frac{3}{2}}\right]_0^a$$

$$= \frac{4}{3}\sqrt{a} \left[a^{\frac{3}{2}} - 0^{\frac{3}{2}}\right]$$

$$= \frac{4}{3}\sqrt{a} \left[a\sqrt{a}\right]$$

$$= \frac{4}{3} a^2 \text{ sq.units}$$

Q5(a) If $\vec{a} = \hat{i} + 2\hat{j} - 3\hat{k}$
$\vec{b} = 3\hat{i} - \hat{j} + 2\hat{k}$

then prove that $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ are perpendicular to each other

Solve Given
$$\vec{a} = \hat{i} + 2\hat{j} - 3\hat{k}$$
$$\vec{b} = 3\hat{i} - \hat{j} + 2\hat{k}$$

$$\vec{a} + \vec{b} = (\hat{i} + 2\hat{j} - 3\hat{k}) + (3\hat{i} - \hat{j} + 2\hat{k})$$
$$= 4\hat{i} + \hat{j} - \hat{k}$$

$$\vec{a} - \vec{b} = (\hat{i} + 2\hat{j} - 3\hat{k}) - (3\hat{i} - \hat{j} + 2\hat{k})$$

$$\vec{a} - \vec{b} = -2\hat{i} + 3\hat{j} - 5\hat{k}$$

$$(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 4(-2) + 1(3) - 1(-5)$$
$$= -8 + 3 + 5$$
$$= 0$$

Hence $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ are perpendicular to each other.

(b) If $|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}|$, then prove that vectors $\vec{a}$ and $\vec{b}$ are perpendicular to each other

Solve

Given

$$|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}|$$

$$|\vec{a}+\vec{b}|^2 = |\vec{a}-\vec{b}|^2 \quad \boxed{\text{Squaring both sides}}$$

$$(\vec{a}+\vec{b})^2 = (\vec{a}-\vec{b})^2$$

$$\vec{a}^2+\vec{b}^2+2a.b = \vec{a}^2+\vec{b}^2-2a.b$$

$$\cancel{\vec{a}^2}-\cancel{\vec{a}^2}+\cancel{\vec{b}^2}-\cancel{\vec{b}^2}+2a.b+2a.b = 0$$

$$4\,\vec{a}.\vec{b} = 0$$

$$\vec{a}.\vec{b} = 0$$

Since $\vec{a}.\vec{b} = 0$ Therefore $\vec{a}$ and $\vec{b}$ are perpendicular to each other.

(i) If two constant forces $4\hat{i}+\hat{j}-3\hat{k}$ and $3\hat{i}+\hat{j}-\hat{k}$ are applied on a particle and it is displaced from P(1, 2, 3) to Q (5, 4, 1). Then calculate the work done by forces.

Solve

Let $\vec{F}$ the resultant of the three given forces then

$$\vec{F} = (4\hat{i}+\hat{j}-3\hat{k})+(3\hat{i}+\hat{j}-\hat{k})$$

$$= 7\hat{i}+2\hat{j}-4\hat{k}$$

~~Let A~~ Given points

Position vector of P = $\hat{i}+2\hat{j}+3\hat{k}$

position vector of Q = $5\hat{i}+4\hat{j}+\hat{k}$

Displacement $\vec{d} = \vec{AB}$

= position vector of Q − position vector of P

$$= (5\hat{i}+4\hat{j}+\hat{k})-(\hat{i}+2\hat{j}+3\hat{k})$$

$$= 4\hat{i}+2\hat{j}-2\hat{k}$$

Total Work done $= \vec{F} \cdot \vec{d}$

$= (7\hat{i} + 2\hat{j} - 4\hat{k}) \cdot (4\hat{i} + 2\hat{j} - 2\hat{k})$

$= 28 + 4 + 8$

$= 40$ units

Ans. Total work done by forces is 40 units

Q6 Skip

Q7(a) If a line passes from two points (1,2) and (2,-1). Then find the slope of line.

Solve

$\tan\theta = \frac{y_2 - y_1}{x_2 - x_1}$

$\tan\theta = \frac{-1 - 2}{2 - 1}$

$\tan\theta = \frac{-3}{2}$

$\theta = \tan^{-1}\left(\frac{-3}{2}\right)$

(b) Find

$$\int \frac{e^x(x+1)}{\cos^2(x.e^x)} . dx$$

Solve

put $x.e^x = t$
then

$$\frac{d(x.e^x)}{dx} = \frac{dt}{dx}$$

$$\frac{d(x)}{dx} \cancel{x} . e^x + \frac{d}{dx} e^x . x = \frac{dt}{dx}$$

$$e^x + e^x . x = \frac{dt}{dx}$$

$$\boxed{e^x(x+1) . dx = dt}$$

$$\int \frac{dt}{\cos^2 t}$$

$$\int \sec^2 t . dt$$

$$\tan t + C \qquad \left[\int \sec^2 x . dx = \tan x + C\right.$$

Ans $\tan(x.e^x) + C$

(c) $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 3 \\ -1 & 2 & 1 \end{bmatrix}$ then find A^{-1}

Solve

$|A| = \begin{vmatrix} 1 & 0 & 2 \\ 2 & 1 & 3 \\ -1 & 2 & 1 \end{vmatrix}$

$= 1(1-6) + 2(2 \times 2 - (-1)\,1)$

$= 5 + 2(4+1)$

$= 5 + 10 = 15$

Since $|A| \neq 0$ A^{-1} exists

The cofactors of given matrix

$C_{11} = + \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = (1-6) = -5$

$C_{12} = - \begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix} = -(2+3) = -5$

$C_{13} = + \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} = (4+1) = 5$

$C_{21} = - \begin{vmatrix} 0 & 2 \\ 2 & 1 \end{vmatrix} = -(0-4) = 4$

$$C_{22} = +\begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = +(1-2) = -1$$

$$C_{23} = \begin{vmatrix} 1 & 0 \\ -1 & 2 \end{vmatrix} = -(2-0) = -2$$

$$C_{31} = \begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix} = (0-2) = -2$$

$$C_{32} = -\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -(3-4) = 1$$

$$C_{33} = \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = (1-0) = 1$$

$$\text{Adj.}A = \begin{bmatrix} -5 & -5 & 5 \\ 4 & -1 & -2 \\ -2 & 1 & 1 \end{bmatrix}^T$$

$$= \begin{bmatrix} -5 & 4 & -2 \\ -5 & -1 & 1 \\ 5 & -2 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{\text{Adj. } A}{|A|}$$

$$A^{-1} = \begin{bmatrix} -5 & 4 & -2 \\ -5 & -1 & 1 \\ 5 & -2 & 1 \end{bmatrix} \times \frac{1}{15}$$

$$\cancel{A^{-1}}\ A^{-1} = \frac{1}{15} \begin{bmatrix} -5 & 4 & -2 \\ -5 & -1 & 1 \\ 5 & -2 & 1 \end{bmatrix}$$

Q8 a) What is MATLAB ?

(b) Prove that

$$\vec{a} \times (\vec{b} + \vec{c}) + \vec{b} \times (\vec{c} + \vec{a}) + \vec{c} \times (\vec{a} + \vec{b}) = \vec{0}$$

Solve ~~Taking LHS~~

$$\vec{a} \times \vec{b} + \vec{a} \times \vec{c} + \vec{b} \times \vec{c} + \vec{b} \times \vec{a} + \vec{c} \times \vec{a} + \vec{c} \times \vec{b}$$

$$\cancel{\vec{a} \times \vec{b}} + \cancel{\vec{a} \times \vec{c}} + \cancel{\vec{b} \times \vec{c}} - \cancel{\vec{a} \times \vec{b}} - \cancel{\vec{a} \times \vec{c}} - \cancel{\vec{b} \times \vec{c}}$$

$= 0 = \text{RHS}$

Hence proved

(c) If parabola $y^2 = 4ax$ from $x=0$ to $x=a$ is revoluted about x-axis then find the volume of surface generated by revolution.

Solve

Given $y^2 = 4ax$

$$y = \sqrt{4ax}$$

$$y = 2\sqrt{ax}$$

Let $f(x) = 2\sqrt{ax}$

$$V = \pi\int_a^b [f(x)]^2 \cdot dx$$

$$= \pi\int_0^a (2\sqrt{ax})^2 \cdot dx$$

$$= \pi\int_0^a 4ax \cdot dx$$

$$= \pi 4a\int_0^a x \cdot dx$$

$$= \pi 4a \left[\frac{x^2}{2}\right]_0^a$$

$$= \frac{4\pi a}{2}\left[a^2 - 0\right]$$

Answer $= 2\pi a^3$