DMA Hand-in, week 5

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Part 1 Euclids algorithm

1. By using the formula $m = q \cdot n + r$ or $a = q \cdot b + r$ for $0 <= r < n$ we calculate the GCD of the following:

   - GCD(3,2) = 1, steps = 2:
   $$\begin{aligned} 3 &= 1 \cdot 2 + 1 \\ 2 &= 2 \cdot 1 + 0 \end{aligned} \tag{1}$$
   - GCD(5,3) = 1, steps = 3
   $$\begin{aligned} 5 &= 1 \cdot 3 + 2 \\ 3 &= 1 \cdot 2 + 1 \\ 2 &= 2 \cdot 1 + 0 \end{aligned} \tag{2}$$
   - GCD(8,5) = 1, steps = 4
   $$\begin{aligned} 8 &= 1 \cdot 5 + 3 \\ 5 &= 1 \cdot 3 + 2 \\ 3 &= 1 \cdot 2 + 1 \\ 2 &= 2 \cdot 1 + 0 \end{aligned} \tag{3}$$
   - GCD(13,8) = 1, steps = 5
   $$\begin{aligned} 13 &= 1 \cdot 8 + 5 \\ 8 &= 1 \cdot 5 + 3 \\ 5 &= 1 \cdot 3 + 2 \\ 3 &= 1 \cdot 2 + 1 \\ 2 &= 2 \cdot 1 + 0 \end{aligned} \tag{4}$$

2. If t_n is the max (worst-case) amount of steps to calculate GCD(a,b) for $n >= a >= b > 0$ then for $n = 1..15$ we use figure 1 to see the following:

   - $t_1 = 1$ by GCD(1,1)
   - $t_2 = 1$ by GCD(2,1)
   - $t_3 = 2$ by GCD(3,2)
   - $t_4 = 2$ by GCD(3,2)
   - $t_5 = 3$ by GCD(5,3)
   - $t_6 = 3$ by GCD(5,3)
   - $t_7 = 3$ by GCD(5,3)
   - $t_8 = 4$ by GCD(8,5)
   - $t_9 = 4$ by GCD(8,5)
   - $t_{10} = 4$ by GCD(8,5)
   - $t_{11} = 4$ by GCD(8,5)
   - $t_{12} = 4$ by GCD(8,5)
   - $t_{13} = 4$ by GCD(8,5)

- $t_{14} = 4$ by GCD(8,5)
- $t_{15} = 4$ by GCD(8,5)

3. for $K = 2, 3, 4, 5, 6$, we have found pair (a_k, b_k) such that GCD(a_k, b_k) has k divisions and max(a_k, b_k) becomes as small as possible. Using figure 1, we find the pairs

   - for k = 2: (3,2)
   - for k = 3: (5,3)
   - for k = 4: (8,5)
   - for k = 5: (13,8)

4. We recognize that the pattern of increasing number of steps follows the fibonacci sequence. With this knowledge we can easily determine (a_7, b_7) and (a_8, b_8) as the next numbers in the sequence by using $(a_6, b_6) = (21, 13)$. (a_7, b_7) = ((21+(21+13)),(21+13)) = (55,34) and (a_8, b_8) = ((55+(55+34),(55+34) = (144,89)

5. The number of steps grows in relation to n and there can never be more steps than n. Therefore t_n must be upperbounded by n, O(n)

6. As we saw in part 1.2, for any increasing set of following fibonacci numbers t_n increases by one. As there are infinite fibonacci numbers, t_n cannot have a constant amount of steps and cannot be O(1)

Part 2

1. We determine P(n) based on the description of the assignment:

   P(n) : $4 \mid 3^n + 6n - 1$ for n > 0

2. We test P(n) for 1-4:

   P(1) : 4 | 3^1 + 6 · 1 - 1 = 4 | 8 : TRUE
   P(2) : 4 | 3^2 + 6 · 2 - 1 = 4 | 20 : TRUE
   P(3) : 4 | 3^3 + 6 · 3 - 1 = 4 | 44 : TRUE
   P(4) : 4 | 3^4 + 6 · 4 - 1 = 4 | 104 : TRUE

3. $$b_n = 3^n + 6 \cdot n - 1$$

   First we insert $n + 1$ instead of n:

   $$b_{n+1} = 3^{n+1} + 6(n + 1) - 1$$
   $$= 3 \cdot 3^n + 6n + 5$$

   Remember that $3^n = b_n - 6n + 1$

   $$b_{n+1} = 3 \cdot (b_n - 6n + 1) + 6n + 5$$
   $$= 3b_n - 18n + 3 + 6n + 5$$
   $$= 3b_n - 12n + 8$$
   $$= 3b_n + 4 \cdot (2 - 3n)$$

4. Assuming that P(n) holds, the expression from 3. is then used to prove that P(n+1) holds as well. Remember that: $b_{n+1} = 3b_n + 4 \cdot (2 - 3n)$.
   In KBR. p.21 theorem 2, we see that if: $4 \mid b_n$ and $4 \mid 4$ then $4 \mid b_n + 4$.
   We already made the assumption that $4 \mid b_n$ and $4 \mid 4$ needs no further explanation, so we can conclude that $4 \mid b_n + 4$.

Since we are working with integer values, we can multiply by whatever we like and still be divisible by 4, so the final conclusion is that:

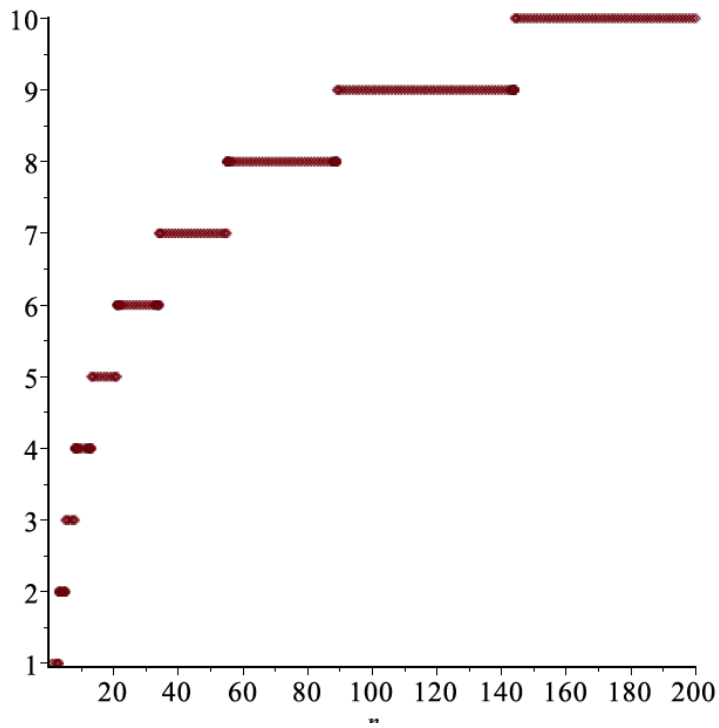
$$4 \mid 3b_n + 4 \cdot (2 - 3n) \qquad is\ TRUE$$

Thus P(n+1) holds.

5. By invoking the principle of mathematical induction we conclude that $P(n): \ 4 \mid 3^n + 6n - 1$ holds for all $n \in \mathbb{Z}^+$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2		1	2	1	2	1	2	1	2	1	2	1	2	1	2
3			1	2	3	1	2	3	1	2	3	1	2	3	1
4				1	2	2	3	1	2	2	3	1	2	2	3
5					1	2	3	4	3	1	2	3	4	3	1
6						1	2	2	2	3	3	1	2	2	2
7							1	2	3	3	4	4	3	1	2
8								1	2	2	4	2	5	3	3
9									1	2	3	2	3	4	3
10										1	2	2	3	3	2
11											1	2	3	4	4
12												1	2	2	2
13													1	2	3
14														1	2
15															1

Figur 1: The number of steps in the calculation of $\mathrm{GCD}(a, b)$



Figur 2: Graph for t_n