Assigment 6 DMA

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Part 1 The statement "f(x) is O(g(x))" can be expressed using logical operators as the following:

$$\exists c > 0 \; \exists x_0 \in \mathbb{R}^+ \; \forall x \geq x_0 \; f(x) \leq cg(x) \tag{1}$$

(a) The negation of this expression can simply be expressed as:

$$\sim [\exists c > 0 \; \exists x_0 \in \mathbb{R}^+ \; \forall x \geq x_0 \; f(x) \leq cg(x)] \tag{2}$$

I will now use theorem 3 from KBR chapter 2.2 in order to simplyfy it. Now the equation will look like this:

$$\forall c > 0 \; \forall x_0 \in \mathbb{R}^+ \; \exists x \geq x_0 \; \sim [f(x) \leq cg(x)] \tag{3}$$

now all thats left is to remove the $\sim$ notation, this is done as follows:

$$\forall c > 0 \; \forall x_0 \in \mathbb{R}^+ \; \exists x \geq x_0 \; f(x) > cg(x) \tag{4}$$

and now we have the negation of the former expression.

(b) To express this very shortly and directly in danish one could say: "For enhver konstant $c > 0$ og for ethvert x_0 i mængden $\mathbb{R}^+$, findes der et $x \geq x_0$, sådan at $f(x) > cg(x)$.

Part 2 The statement "If r is an irrational number, then $r^{1/5}$ is an irrational number" can be proven by proving the contrapositive statement.

Impliation $p \Rightarrow q$: If r is an irrational number, then $r^{1/5}$ is an irrational number.
Contrapositive $\sim q \Rightarrow \sim p$: If $r^{1/5}$ is a rational number, then r is a rational number.

Now if we assume that $r^{1/5}$ is a rational number, then it means that we can write it as a/b so $r^{1/5} = a/b$ for some intergers a as a numerator and b as a non-zero denominator. So from this we can also conclude that:

$$(r^{1/5})^5 = (a/b)^5 \leftrightarrow r = a^5/b^5 \tag{5}$$

Now from this we get an expression for r, which is a^5/b^5 where both a and b are intergers meaning that a^5 and b^5 is also intergers and therefore we might aswell express it as $r = p/q$, where $p = a^5$ and $q = b^5$ meaning that r is rational number and we have therefor proved the contrapositive of our original statement.

Part 3 By the definition of Big-O then 4^n is not $O(2^n)$.
This can be proven by doing a proof of contradiction.

The definition of big-O is as follows: "We say that $f(x)$ is $O(g(x))$ if there exists a constant $c > 0$ and x_0 such that $f(x) \leq cg(x)$ for all $x \geq x_0$.

Now because this is a proof of contradiction, then we want to try to proof the negation of our original statement and meet a contradiction for this, which we can use to conclude that our original statement is true since the negation then is false. Therefore we wish to prove the negation of "4^n *is not* $O(2^n)$" which is "4^n *is* $O(2^n)$" meaning that we can find a constant $c > 0$ and n_0 such that $4^n \leq c * 2^n$ for all $n \geq n_0$. So we have:

$$4^n \leq c * 2^n \tag{6}$$

By dividing each side of the equation with 2^n then we get

$$2^n \leq c \tag{7}$$

Now this expression has to count for all $n \geq n_0$, which is a contradiction because no matter which c we chose, there will eventually be an $n \geq n_0$ that will overcome this expression so that $2^n \geq c$ will count instead and therefor 4^n is not O(2^n)

Part 4 (a) An equivalent expression for (∼P) using ⊙ is showed in the following table:

P	Q	P ⊙ Q	(∼P)	P ⊙ P
T	T	F	F	F
T	F	T	F	F
F	T	T	T	T
F	F	T	T	T

$\odot$ can be considered the following way to support the result of the truth table above:
T $\odot$ T = F
T $\odot$ F = T
F $\odot$ T = T
F $\odot$ F = T

(b) An equivalent expression for each of the two expressions can be expressed as following:

(i) $P \vee Q \leftrightarrow (\sim P) \odot (\sim Q)$

P	Q	P $\vee$ Q	P $\odot$ Q	($\sim$P)	($\sim$Q)	($\sim$P) $\odot$ ($\sim$Q)
T	T	T	F	F	F	T
T	F	T	T	F	T	T
F	T	T	T	T	F	T
F	F	F	T	T	T	F

(ii) $P \texttt{ xor } Q \leftrightarrow \sim ((P \odot Q) \odot ((\sim P) \odot (\sim Q)))$

P	Q	P `xor` Q	$\sim ((P \odot Q) \odot ((\sim P) \odot (\sim Q)))$
T	T	F	F
T	F	T	T
F	T	T	T
F	F	F	F

The former truth tables can be used to see that this expression holds.