Introduction to Dynamical Systems Modeling

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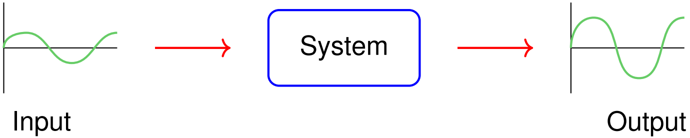
September 15, 2019

Signals and Systems

A system transforms an input signal into an output signal according to well defined operations.



Signals and Systems

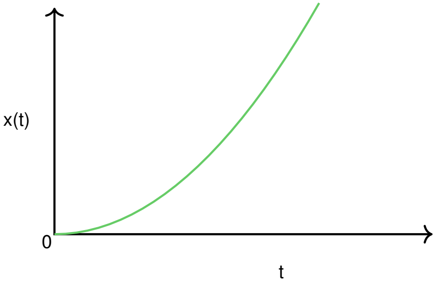
- **Static system**. The output of the system at a given time instant depends only on the input at that instant. Example: the voltage drop across a resistor
$$v(t) = Ri(t)$$
- **Dynamic system**. The output depends on current and past inputs; it is usually described with differential equations. Example: the voltage drop across a capacitor
$$\frac{dv(t)}{dt} = \frac{1}{C}i(t)$$

Differential Equations

Differential equations describe the evolution of a system signals along time. We use the symbol $\dot{x}$ to represent the first derivative of the variable x with respect to time:

$$\dot{x} = \frac{dx}{dt}$$

For example, the differential equation $\dot{x} = 0.5t$ for the initial condition $x(0) = 0$, describes the following behavior



Solving Differential Equations

Different methods exist to solve differential equations. The simplest method consist in the separation of variables and integration of the equation.

Consider, for example, the differential equation $\dot{x} = sin(t)$. Separating the variables x and t, we get

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$$\Rightarrow x(t) = -cos(t) + C$$

Initial Conditions

The trajectory that is actually followed by the variable $x(t)$ along time depends on the integration constant C. The value of this constant can be obtained if we know the initial value of $x(t)$ when $t = 0$, that is, $x(0)$.

In our previous example, substitute $t = 0$ in the general solution $x(t) = -cos(t) + C$, to obtain

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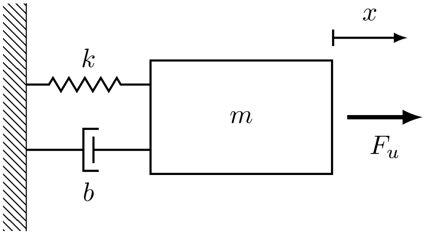
The final trajectory of $x(t)$ is described by the solution

$$x(t) = -cos(t) + x(0) + 1$$

Example: Mass-Spring-Damper System



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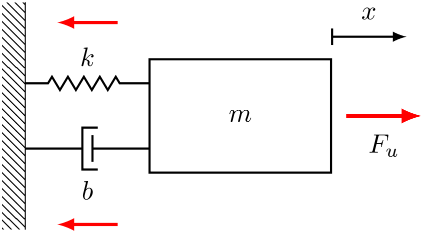

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Example: Mass-Spring-Damper System

- The force produced by the spring is proportional to the position x. Thus, $F_s = kx$.
- The force generated by the damper is proportional to the speed of the mass, $\dot{x}$. Thus, $F_d = b\dot{x}$.
- By the second Newton's law, the total force over a mass is given by $F = ma = m\ddot{x}$. Then we get

$$m\ddot{x} = F_u - F_s - F_d$$

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$$m\ddot{x} = F_u - F_s - F_d = F_u - kx - b\dot{x}$$

$$\Rightarrow m\ddot{x} + b\dot{x} + kx = F_u$$

Laplace Transform

The Laplace transform is an operation that can be used to solve differential equations. It transforms many functions, like sine or exponential functions, into algebraic expressions of a complex variable s.

The Laplace transform of a function $f(t)$ is defined as

$$F(s) = \int_{-\infty}^{\infty} e^{-st} f(t) dt$$

Laplace Transform

In systems theory, the Laplace transform allows a detailed analysis of certain dynamical models.

Here, when the complex variable s takes imaginary values ($s = j\omega$), the variable ω represents the frequency of a sinusoidal signal.

Thus, we use this tool to study the *frequency response* of a system.

Transfer function

Consider a differential equation of the form

$$a_0\ddot{y} + a_1\dot{y} + a_2 y = b_0\ddot{u} + b_1\dot{u} + b_2 u$$

Assume all initial conditions equal to zero. In this case, we obtain the Laplace transform of this equation by using the fact that the transform of $y^{(n)}(t)$ is $s^n Y(s)$:

$$a_0 s^2 Y(s) + a_1 s Y(s) + a_2 Y(s) = b_0 s^2 U(s) + b_1 s U(s) + b_2 U(s)$$

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$$\Rightarrow \left(a_0 s^2 + a_1 s + a_2\right) Y(s) = \left(b_0 s^2 + b_1 s + b_2\right) U(s)$$

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$$\Rightarrow \left(a_0 s^2 + a_1 s + a_2\right) Y(s) = \left(b_0 s^2 + b_1 s + b_2\right) U(s)$$

$$\Rightarrow \frac{Y(s)}{U(s)} = \frac{b_0 s^2 + b_1 s + b_2}{a_0 s^2 + a_1 s + a_2}$$

Transfer Function

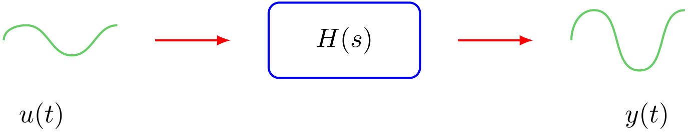
- The complex function

$$H(s) = \frac{Y(s)}{U(s)}$$

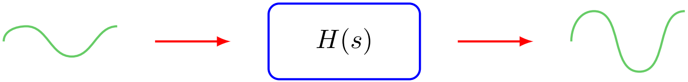
  is regarded as the transfer function of the system.

- The expression $H(s)$ is the mathematical model of a dynamical system in the frequency domain.

- It characterizes the input-output relation of the system.



Transfer Function



- What are the properties of the output signal? How does it relate to the input?
- Maybe the signal gets attenuated at some frequencies. At other frequencies, the signal gets amplified.
- The transfer function is only useful to model systems with linear differential equations and it always assumes initial conditions equal to zero.

Transfer Function of the Spring-Damper-Mass System

In our differential equation for the spring-damper-mass system, define the output as the position, $y(t) = x(t)$ and the input as the force applied to the mass, $u(t) = F_u$. Then, we have

$$m\ddot{y} + b\dot{y} + ky = u$$

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The transfer function is

$$H(s) = \frac{Y(s)}{U(s)} = \frac{1}{ms^2 + bs + k}$$

State Space Modeling

We are interested in a model in time domain, that can represent non-linear systems and that describes trajectories from arbitrary initial conditions.

Define the **state variables** as the smallest set of variables that are necessary to describe the behavior of a dynamical system.

If the differential equation is of order n, then the system has n states. The derivative of highest order is not a state, and neither are the inputs to the system.

State Space Model of the Spring-Damper-Mass System

Recall the differential equation

$$m\ddot{x} + b\dot{x} + kx = F_u$$

This is a second order equation and, therefore, has two states. Notice that the states are the position $x(t)$ and the velocity $\dot{x}$ of the mass.

Rename the states of the system as

$$x_1 = x$$

$$x_2 = \dot{x}$$

State Space Model of the Spring-Damper-Mass System

The state space model of a system is a first order differential equation.

The derivative of our state variables are

$$\dot{x}_1 = \dot{x}$$

$$\dot{x}_2 = \ddot{x}$$

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$$\dot{x}_1 = \dot{x} = x_2$$

$$\dot{x}_2 = \ddot{x} = -\frac{b}{m}\dot{x} - \frac{k}{m}x + \frac{1}{m}F_u$$

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The state space model of a system is a first order differential equation.

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$$\begin{aligned}\dot{x}_2 = \ddot{x} &= -\frac{b}{m}\dot{x} - \frac{k}{m}x + \frac{1}{m}F_u \\ &= -\frac{b}{m}x_2 - \frac{k}{m}x_1 + \frac{1}{m}F_u\end{aligned}$$

State Space Model of the Spring-Damper-Mass System

Finally,

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{b}{m}x_2 - \frac{k}{m}x_1 + \frac{1}{m}F_u$$

We can express this model in matrix form as

$$\left[\begin{array}{c} \dot{x}_1 \\ \dot{x}_2 \end{array}\right] = \left[\begin{array}{cc} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{array}\right] \left[\begin{array}{c} x_1 \\ x_2 \end{array}\right] + \left[\begin{array}{c} 0 \\ \frac{1}{m} \end{array}\right] F_u$$

General Representation of State Space Models

A linear system with n states is commonly expressed in matrix form as

$$\dot{x} = Ax + Bu$$

where x is an $n \times 1$ state vector, u is an $m \times 1$ input vector, and A and B are matrices of appropriate dimensions.

A general nonlinear system is represented by

$$\dot{x} = f(x, u)$$

or, for some applications,

$$\dot{x} = f(x) + g(x)u$$

Transfer Function vs. State Space

TF	SS
Frequency domain	Time domain
Linear systems	Linear and nonlinear systems
Algebraic polynomials	Complex functions
Zero initial conditions	Arbitrary initial conditions

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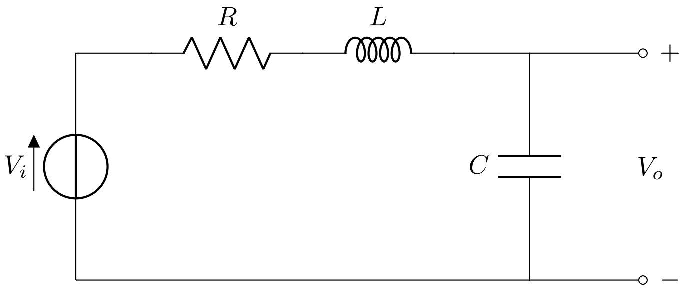
	Kalman (1960)

Example: RLC Circuit



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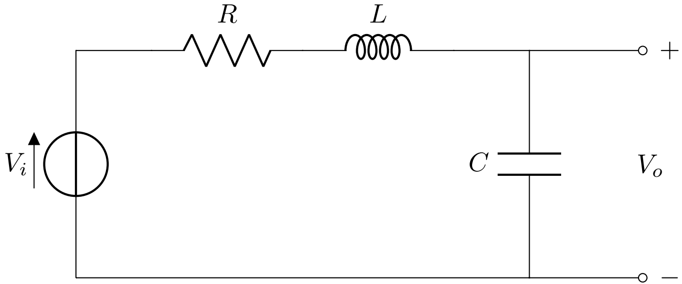

Using Kirchhoff's laws,

$$V_i = Ri_L + L\frac{di_L}{dt} + V_C$$

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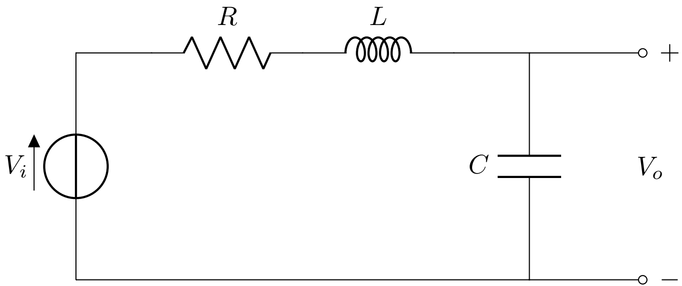


Using Kirchhoff's laws,

$$V_i = Ri_L + L\frac{di_L}{dt} + V_C$$

Also, $i_L = C\frac{dV_C}{dt}$.

Example: RLC Circuit

The states of the system are

$$x_1 = V_C$$

$$x_2 = i_L$$

The final state space model is given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} V_i$$

Example: Model of a Quadrotor

- The translational state variables are $\xi = [x,\ y,\ z]^T$.
- The rotational state variables are $\eta = [\psi,\ \theta,\ \phi]^T$.

Example: Model of a Quadrotor

- The translational state variables are $\xi = [x,\ y,\ z]^T$.
- The rotational state variables are $\eta = [\psi,\ \theta,\ \phi]^T$.
- The Lagrangian model of the system is

$$m\ddot{\xi} = u \begin{bmatrix} -\sin(\theta) \\ \cos(\theta)\sin(\phi) \\ \cos(\theta)\cos(\phi) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix}$$

$$J\ddot{\eta} = -C(\eta, \dot{\eta})\dot{\eta} + \tau$$

where the thrust u and the torque τ are the inputs.

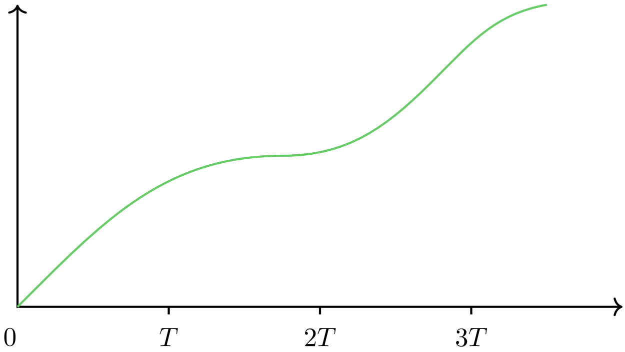
Transform Equivalences

$$sX(s) \rightarrow \frac{dx(t)}{dt}$$

$$\frac{1}{s}X(s) \rightarrow \int x(t)dt$$

Discrete-Time Systems

Computers and microprocessors do not work with exact continuous signals. Instead, they discretize it:



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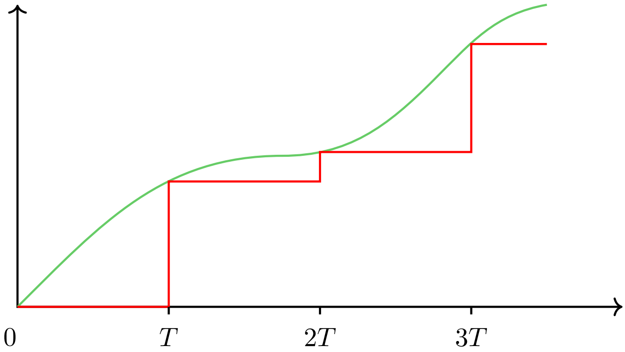

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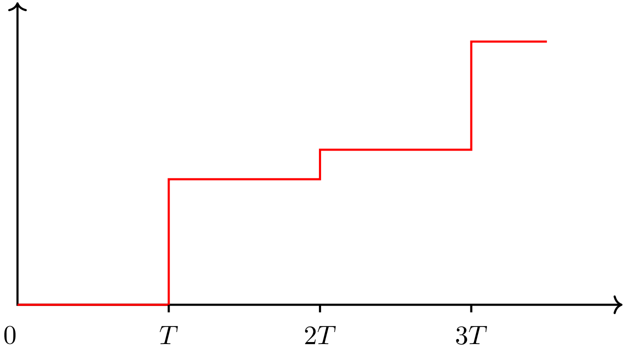

Difference Equations

Instead of differential equations, discrete-time systems are described by difference equations of the form

$$a_0 y(k) + a_1 y(k-1) + \ldots + a_n y(k-n) = \\ b_0 u(k) + b_1 u(k-1) + \ldots + b_n u(k-n)$$

where k is the current time step of a signal.

Using this representation, the current output of the system can be calculated as

$$y(k) = \frac{1}{a_0} \sum_{i=0}^{n} b_i u(k-i) - \frac{1}{a_0} \sum_{i=1}^{n} a_i y(k-i)$$

DT Transfer Function

For discrete-time systems, we define the $\mathcal{Z}$-transform of a signal $x(k)$ as

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

This transform leads to the discrete-time transfer function

$$H(z) = \frac{Y(z)}{U(z)} = \frac{a_n z^{-n} + a_{n-1} z^{-(n-1)} + \ldots + a_1 z^{-1} + a_0}{b_n z^{-n} + b_{n-1} z^{-(n-1)} + \ldots + b_1 z^{-1} + b_0}$$

Discrete State Space Model

- For linear systems, we write

$$x(k+1) = Ax(k) + Bu(k)$$

- A nonlinear system is expressed by

$$x(k+1) = f(x(k), u(k))$$

Transform Equivalences

$$zX(z) \quad \rightarrow x(k+1)$$

$$z^{-1}X(z) \rightarrow x(k-1)$$