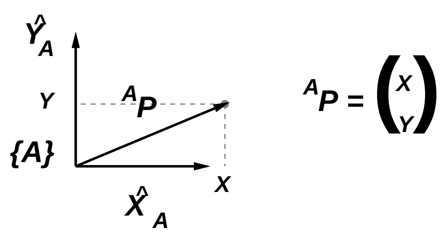
CSE 4360 / CSE 5364 - *Autonomous Robots*

Lecture 2

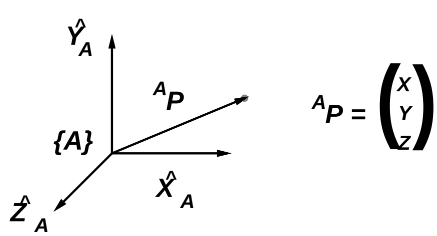
1 Spatial Descriptions of Robot Systems

To assess a situation and to be able to plan future moves of a robot it is necessary to be able to describe its configuration and to be able to relate that to either internal maps of the environment or to other robots.

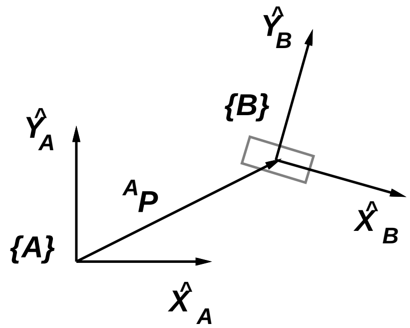
- In the same way as in geometry the position of an object is cescribed by its coordinates in a global coordinate frame $\{A\}$.
  - In 2 dimensions this position in terms of coordinate frame $\{A\}$, ${}^{A}P$, is expressed as a 2×1 vector



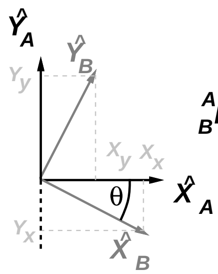
  - Similarly, the position is 3 dimensions is expressed as a 3×1 vector



- The description of the position of a robot is not sufficient to describe its configuration in the world. It is also necessary to indicate its orientation.
  - To represent orientation, a coordinate frame $\{B\}$ is attached to the robot.



– If the origins of the two frames coincide the relative orientation of frame $\{B\}$ in frame $\{A\}$ can be expressed by the rotation matrix ${}^A_BR$



$$ {}^A_BR = \begin{pmatrix} {}^A\hat{X}_B & {}^A\hat{Y}_B \end{pmatrix} = \begin{pmatrix} X_x & X_y \\ Y_x & Y_y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} $$

* The same applies to 3 dimesnions where the rotation matrix takes the form

$$ {}^A_BR = \begin{pmatrix} {}^A\hat{X}_B & {}^A\hat{Y}_B & {}^A\hat{Z}_B \end{pmatrix} $$

  where ${}^A\hat{X}_B$ is the representation of $\hat{X}_B$ in coordinate frame $\{A\}$.

* To obtain the orientation of frame $\{A\}$ in frame $\{B\}$ one of the properies of rotation matrices can be used:

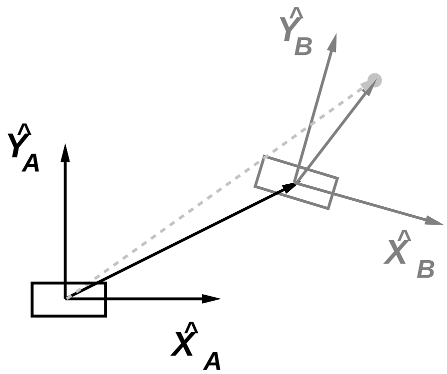
$$ {}^B_AR = {}^A_BR^{-1} = {}^A_BR^T $$

- To represent the configuration of an body in a global coordinate frame $\{A\}$, the frame $\{B\}$ attached to this body is expressed in terms of the position of its origin and by its orientation in global coordinates.

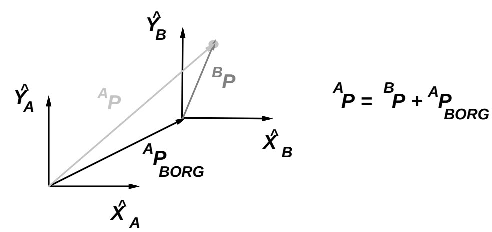
$$ \{B\} = \{{}^A_BR\ ,\ {}^AP_{BORG}\} $$

Mapping between frames

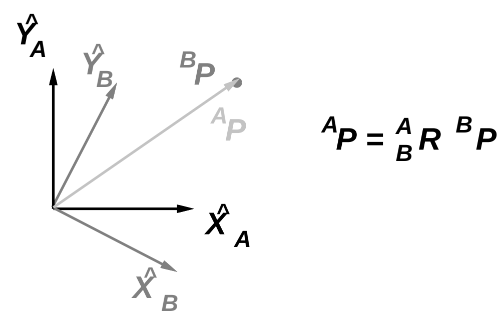
Often it is important to map the location of an object or robot within one frame into a different target frame. Examples for this include the mapping of obstacles form a global model into coordinates relative of the robot or the mapping of a body observed by the robot into a global model or into the frame of a different robot.



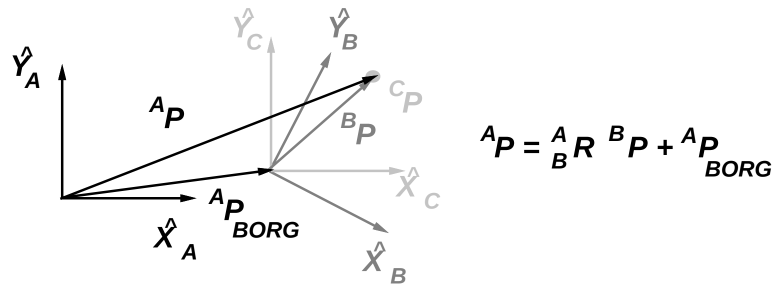
- If both frames have the same orientation this transformation is achieved by adding the position of the object and the relative position of the corresponding coordinate frame with respect to the target frame (i.e. the frame into which the coordinates are to be mapped).



- In the case where the origins of both frames coincide, the mapping is performed by rotating the position vector of the object into the target coordinate frame.



- For general frames the mapping can be achieved by first transfering the coordinates into an intermediate frame $\{C\}$ which has the same origin as the original frame but the orintation of the target frame, $\{C\} = \{0, {}^AP_{BORG}\}$. This corresponds to a pure change in orientation and thus ${}^CP = {}^C_BR\ {}^BP = {}^A_BR\ {}^BP$. This coordinate is then tranferred into the target frame, ${}^AP = {}^CP + {}^AP_{CORG} = {}^A_BR\ {}^BP + {}^AP_{BORG}$.



Homogeneous Transformations

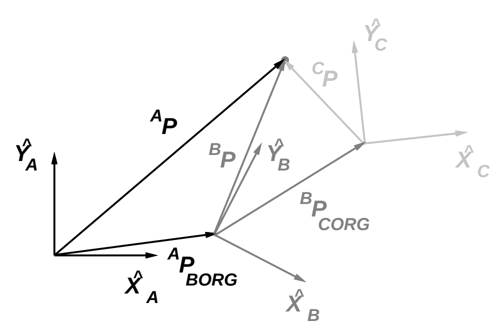
Treating orintation and position separately makes the chaining of multiple frames cumbersome. To solve this problem homogeneous transformations can be used.

- A homogeneous transformation incorporates rotation and translation between two frames in a single 4×4 matrix.
$$ {}^{A}_{B}T = \left(\begin{array}{c|c} {}^{A}_{B}R & {}^{A}P_{BORG} \\ \hline 0 \;\; 0 \;\; 0 & 1 \end{array} \right) $$
- The transformation of a coordinate ${}^{B}P$ into frame $\{A\}$ can then be expressed using a single matrix multiplication.
$$ \begin{pmatrix} {}^{A}P \\ 1 \end{pmatrix} = {}^{A}_{B}T \begin{pmatrix} {}^{B}P \\ 1 \end{pmatrix} $$
- NOTATION: For simplification ${}^{A}P$ is often used to already represent a 4×1 position vector in homogeneous coordinates (with the 4^{th} row being a 1).

Chaining Frames and Compound Transformations

In many situations multiple frames are present but only pairwise relations are available. For example in the case of multiple cooperating mobile robots, the relative location of both frames with respect to the world frame might be available but not the relative location of one robot with respect to the other robot. To solve for this relation it is necessary to form a chain of multiple frames.

- In the case where the position of an object is known within frame $\{C\}$ and also the relative positions and orientations of frame $\{C\}$ with respect to frame $\{B\}$ and of frame $\{B\}$ with respect to frame $\{A\}$ is available the location of the object within frame $\{A\}$ can be computed.



$$
\begin{aligned}
{}^{B}P &= {}^{B}_{C}T \ {}^{C}P \\
{}^{A}P &= {}^{A}_{B}T \ {}^{B}P = {}^{A}_{B}T \ {}^{B}_{C}T \ {}^{C}P = {}^{A}_{C}T \ {}^{C}P
\end{aligned}
$$

- The relation between frame $\{C\}$ and frame $\{A\}$ can be expressed by the compound transformation ${}^{A}_{C}T = {}^{A}_{B}T \ {}^{B}_{C}T$.

Properties of Transformations

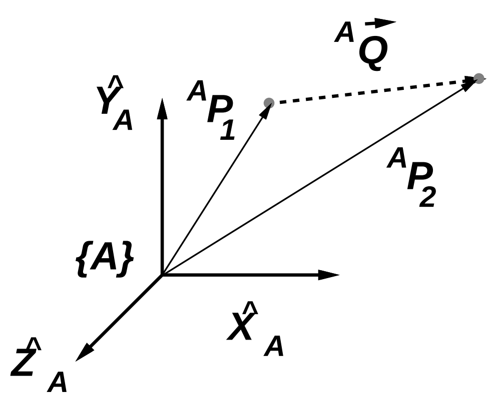
Transformations between two coordinate frames can easily be inverted. This is important, for example, in the case of two mobile robots where each one has to know where the other robot is with respect to its own local coordinate frame. Given the transformation representing robot B's position with respect to robot A, robot A's position with repsect to robot A can be computed by inverting the former transformation.

$$
{}^{A}_{B}T = {}^{B}_{A}T^{-1}
$$

Transformations and Movement

Homogenenous transformations can also be used as operators to describe the movement of an object in a coordinate frame.

- If an object at location ${}^{A}P_1$ moves distance q in direction $\hat{Q}$, then the movement can be expressed using a translation matrix $D_{\hat{Q}}(q)$ and the new location of the object, ${}^{A}P_2$ can be computed.

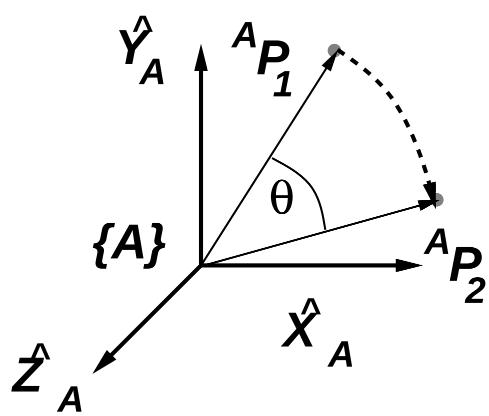


$$
\begin{aligned}
D_{\hat{Q}}(q) &= \begin{pmatrix} 1 & 0 & 0 & q_x \\ 0 & 1 & 0 & q_y \\ 0 & 0 & 1 & q_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
{}^{A}P_2 &= D_{\hat{Q}}(q) \ {}^{A}P_1
\end{aligned}
$$

- Similarly, if the object rotates around the origin of the coordinate frame, its movement can be captured using rotation matrices, $R_Q(\theta)$, where Q represents the rotation axis.

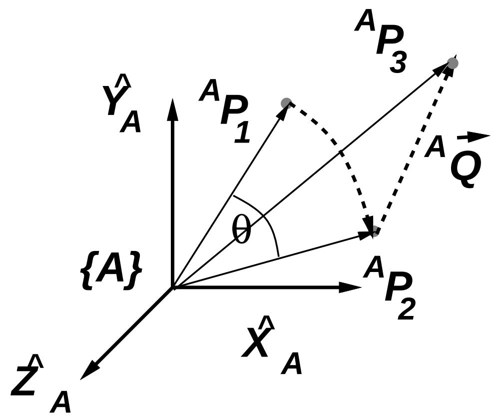
$$R_X(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & cos(\theta) & -sin(\theta) & 0 \\ 0 & sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$R_Y(\theta) = \begin{pmatrix} cos(\theta) & 0 & sin(\theta) & 0 \\ 0 & 1 & 0 & 0 \\ -sin(\theta) & 0 & cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$R_Z(\theta) = \begin{pmatrix} cos(\theta) & -sin(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ sin(\theta) & cos(\theta) & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

For example, if an object at located at position ${}^AP_1$ and moves along an arc around the Z axis by an angle θ, then the new location ${}^AP_2$ can be computed as:



$${}^AP_2 = R_Z(\theta)\ {}^AP_1$$

- To discribe general displacements and motions, motion segments can be combined by multiplying the corresponding transformation matrices. Note that the operations are performed from right to left, i.e. the rightmost transformation represents the first motion segment. For example, if the object first moves along an arc around the Z axis by an angle θ and then moves in direction $\hat{Q}$ for a distance q, the final position ${}^AP_3$ can be computed as:



$$^A P_3 = D_{\hat{Q}}(q) R_Z(\theta)\ ^A P_1$$

The product of a translation matrix and a rotation matrix, $D_{\hat{Q}}(q)R_W\theta$ can be written as one transformation matrix where the rotational component is the one of the rotation matrix and the position vector (in the fourth column) is the one of the translation matrix. E.g.:

$$D_{\hat{Q}}(q)R_Z(\theta) = \begin{pmatrix} cos(\theta) & -sin(\theta) & 0 & q_x \\ sin(\theta) & cos(\theta) & 0 & q_y \\ 0 & 0 & 1 & q_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Note: The order in which operations are performed is important. Changing the order will change the final result.