Autonomous Robots

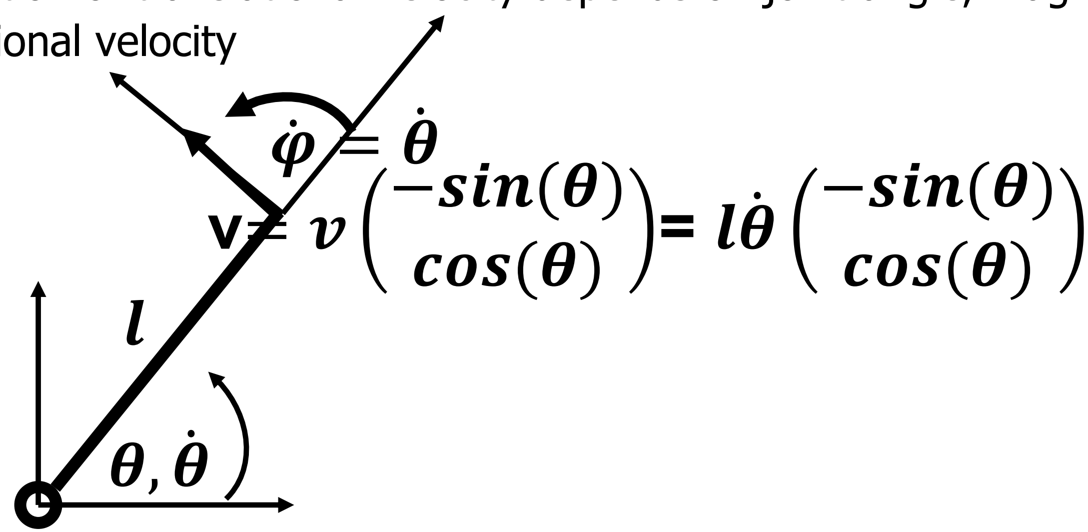
Manipulator Kinematics
Jacobians

Kinematics and Velocities

- Forward kinematics deals with the transformation of internal robot configurations to locations in space of the end-effector
  - Note: Forward kinematics can also be derived for other points on the manipulator but is most often used for the end-effector
- There are situations where we are interested in how the end-effector is movine (i.e. its velocity)
  - When playing a ball game, the velocity at which a robot hits the ball is important
  - When performing interaction tasks (such as painting), velocity of the interaction is important

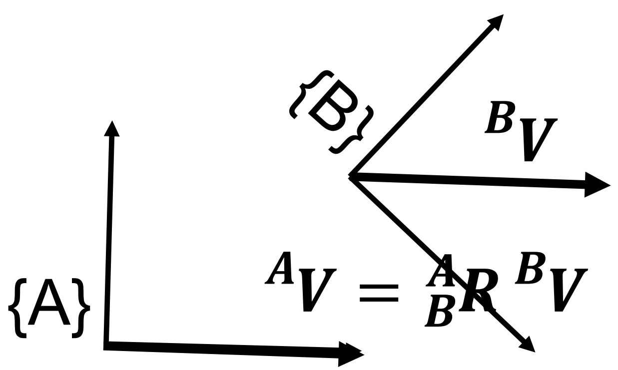
Velocities

- Joint rotations produce translational and rotational velocities in the subsequent frames/links
  - The rotational velocity of a revolute joint produces both translational and rotational velocities at the end of the link
    - Direction of translational velocity depends on joint angle, magnitude on rotational velocity



Velocities

- Translational velocities in a frame can be transformed into another frame in the same way as positions as long as the frames do not rotate or move
  - Velocities are vectors and thus only have magnitude and direction but no position (can draw them at frame origin)

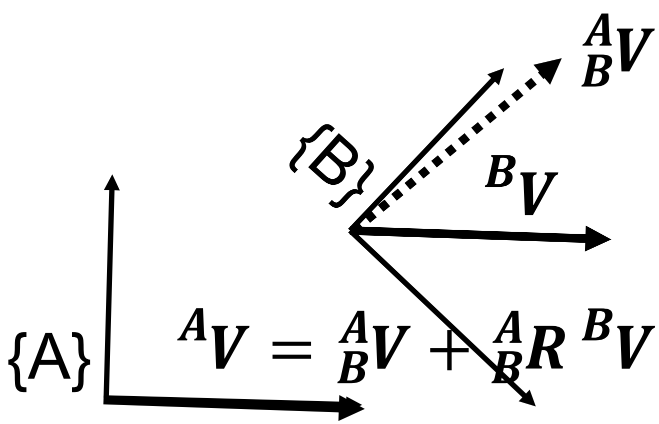


$$ {}^{A}V = {}^{A}_{B}R\, {}^{B}V $$

- In homogeneous transformations this would be achieved by making the 4th element in the velocity vector (the one we added) equal to 0
  - Remember, adding a 1 means it is a position, adding a 0 means it is a vector

Velocities

- Translational velocities between frames simply add to the velocity of the point



Velocities

- Rotational velocities of frames are more complex as they affect translational and rotational velocities
  - We can express the rotational velocity of a frame using a vector of rotational velocities around all the axes

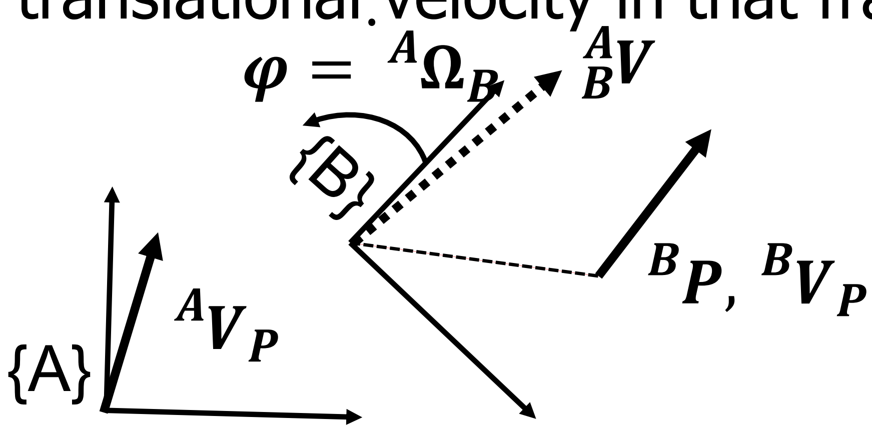
$$ {}^{A}\boldsymbol{\Omega}_{B} = \begin{pmatrix} \omega_X \\ \omega_Y \\ \omega_Z \end{pmatrix} $$

  - Effect of frame rotational velocity on translational velocity is more complex and depends on the location of the moving point in the second frame (point at the origin of B will not move due to rotation of frame B)

$$ {}^{A}\boldsymbol{V}_{P} = {}^{A}\boldsymbol{\Omega}_{B} \times {}^{A}_{B}\boldsymbol{R}\,{}^{B}\boldsymbol{P} $$

Velocities

- Using both translational and rotational velocities we can express the velocity as seen from frame A due to the rotational and translational velocities of a frame B and the translational velocity in that frame



$$ {}^{A}V_{P} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{P} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}P $$

  - Represents the effect of the movement of one proximal joint (in A) on the end-effector velocity (in B)

Velocities

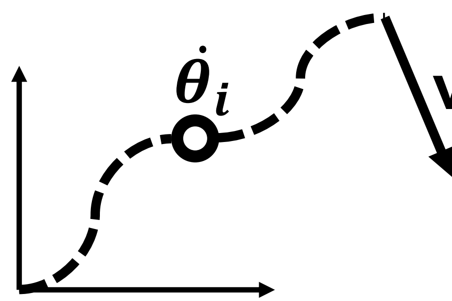
- Using the previous result, we could perform something akin to outward inward iteration
  - Starting from the base frame, use the kinematic relations to derive the rotational and translational effects form one joint frame to the next (${}^{A}_{B}\boldsymbol{V},\ {}^{A}\boldsymbol{\Omega}_{B}$)
  - Starting from the tool frame (${}^{T}\boldsymbol{P}_{TORIG} = \mathbf{0},\ {}^{T}\boldsymbol{V}_{TORIG} = \mathbf{0}$)we can now work backward towards the base and compute the resulting velocities
    - We also have to estimate the location of the end-effector for each frame (as in the forward kinematics)

$$
{}^{A}\boldsymbol{V}_{T} = {}^{A}\boldsymbol{V}_{BORG} + {}^{A}_{B}\boldsymbol{R}\,{}^{B}\boldsymbol{V}_{T} + {}^{A}\boldsymbol{\Omega}_{B} \times {}^{A}_{B}\boldsymbol{R}\,{}^{B}\boldsymbol{P}_{T}
$$

$$
{}^{A}\boldsymbol{P}_{T} = {}^{A}\boldsymbol{P}_{BORG} + {}^{A}_{B}\boldsymbol{R}
$$

Manipulator Jacobians

- Another way to derive the end-effector velocities is to realize their relation to the forward kinematics
  - If we move only one joint and keep all others fixed as we change the corresponding joint velocity we can see that
    - The direction of the velocity vector of the end-effector is not affected by the joint speed, only the joint angles
    - The magnitude of the velocity vector of the end-effector scales linearly with the joint angle speed (i.e. twice the joint angle speed yields twice the end-effector speed



$$\mathbf{v} = \dot{\theta}_i J_i(\vec{\theta})$$

Manipulator Jacobians

- This observation yields that the end-effector velocity is linear in terms of the joint velocities
  - It can thus be written (and computed) as:

$$V = J(\vec{\theta})\dot{\vec{\theta}}$$

  - Since we also know that the end-effector velocity is the change of end-effector position over time, we know that it is the derivative of the forward kinematics

$$V = \frac{dP_T}{dt} = \begin{pmatrix} \frac{dx_T}{dt} \\ \frac{dy_T}{dt} \\ \frac{dz_T}{dt} \end{pmatrix}$$

Manipulator Jacobians

- From this we can derive the Jacobian matrix $J(\vec{\theta})$ for the manipulator

$$V = J(\vec{\theta})\dot{\vec{\theta}} = J(\vec{\theta})\begin{pmatrix} \frac{d\theta_1}{dt} \\ \vdots \\ \frac{d\theta_n}{dt} \end{pmatrix} = \begin{pmatrix} \frac{dx_T}{dt} \\ \frac{dy_T}{dt} \\ \frac{dz_T}{dt} \end{pmatrix}$$

  - The Jacobian has as many rows at there are end-effector velocities (usually 3) and as many columns as there are joints and is a matrix of partial derivatives expressing the relation between position and joint changes

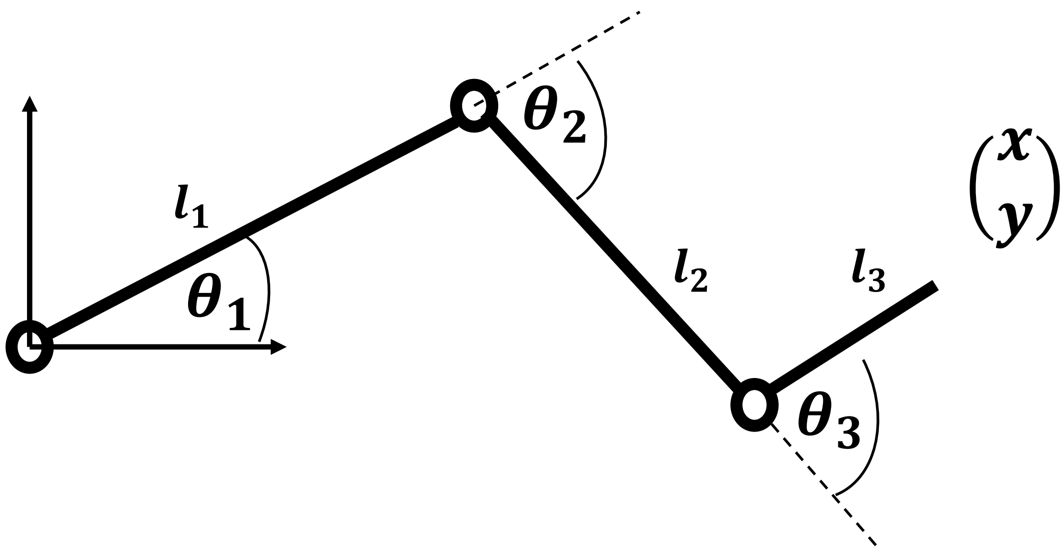
$$J(\vec{\theta}) = \begin{pmatrix} \frac{d\boldsymbol{x_T}}{d\boldsymbol{\theta_1}} & \cdots & \frac{d\boldsymbol{z_T}}{d\boldsymbol{\theta_n}} \\ \frac{d\boldsymbol{y_T}}{d\boldsymbol{\theta_1}} & \cdots & \frac{d\boldsymbol{z_T}}{d\boldsymbol{\theta_n}} \\ \frac{d\boldsymbol{z_T}}{d\boldsymbol{\theta_1}} & \cdots & \frac{d\boldsymbol{z_T}}{d\boldsymbol{\theta_n}} \end{pmatrix}$$

Manipulator Jacobians

- Manipulator Jacobian expresses the relation between joint movements and end-effector movements
  - If an entry is 0 then the corresponding joint can not cause any movement in this direction
  - The larger a value, the more velocity the corresponding joint can produce in this direction
- The Manipulator Jacobian can be derived directly from the Forward Kinematic function
  - All entries are derivatives of the kinematics with respect to a single joint angle

Manipulator Jacobian

- To derive the Jacobian we first derive the Forward Kinematics



x= $l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2) + l_3 \cos(\theta_1 + \theta_2 + \theta_3)$

y= $l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2) + l_3 \sin(\theta_1 + \theta_2 + \theta_3)$

Manipulator Jacobian

x= $l_1 \cos(\theta_1) + l_2\cos(\theta_1 + \theta_2) + l_3\cos(\theta_1 + \theta_2 + \theta_3)$
y= $l_1 \sin(\theta_1) + l_2\sin(\theta_1 + \theta_2) + l_3\sin(\theta_1 + \theta_2 + \theta_3)$

- Compute the partial derivatives of the spatial dimensions with respect to each of the joint angles

$$dx/d\theta_1 = -l_1 \sin(\theta_1) - l_2 \sin(\theta_1 + \theta_2) - l_3\sin(\theta_1 + \theta_2 + \theta_3)$$
$$dx/d\theta_2 = -l_2 \sin(\theta_1 + \theta_2) - l_3\sin(\theta_1 + \theta_2 + \theta_3)$$
$$dx/d\theta_3 = -l_3\sin(\theta_1 + \theta_2 + \theta_3)$$

$$dy/d\theta_1 = l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2) + l_3 cos(\theta_1 + \theta_2 + \theta_3)$$
$$dy/d\theta_2 = l_2 \cos(\theta_1 + \theta_2) + l_3\cos(\theta_1 + \theta_2 + \theta_3)$$
$$dy/d\theta_3 = l_3\cos(\theta_1 + \theta_2 + \theta_3)$$

  - These are the entries in the Manipulator Jacobian

Manipulator Jacobians

- Besides calculating end-effector velocity (or velocities of other point with a forward kinematic function), the Manipulator Jacobian is also used in other ways
  - Incrementally moving towards a position
    - Jacobian provides approximate joint velocities to move in a given direction

$$\dot{\vec{\theta}} \approx J(\vec{\theta})^{\boldsymbol{T}} \boldsymbol{V_T}$$

      Note: While $J(\vec{\theta})^{-\mathbf{1}}$ would be the precise answer, we can not invert the Jacobian in general since it is not a square matrix
  - Compute relation between joint torque $\vec{\boldsymbol{\tau}}$ and static end-effector force $\vec{F}$ (forces not due to impact and movement)
    - Intuitively, the faster something can move, the less hard it can push

$$\vec{\boldsymbol{\tau}} \approx J(\vec{\theta})^{\boldsymbol{T}} \vec{\boldsymbol{F}}$$