Course outline

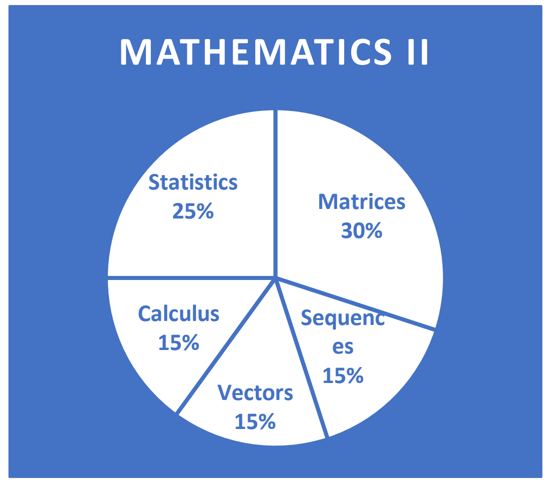
Basic Statistics (25 marks)

5.1 Random Variables (2 marks)

5.1.1 Introduction

5.1.2 Types

5.1.3 Probability



5.2 Discrete Random Variables (15 marks)

5.2.1 Introduction (7 marks)

5.2.2 Binomial Distribution (4 marks)

5.2.3 Poisson Distribution (4 marks)

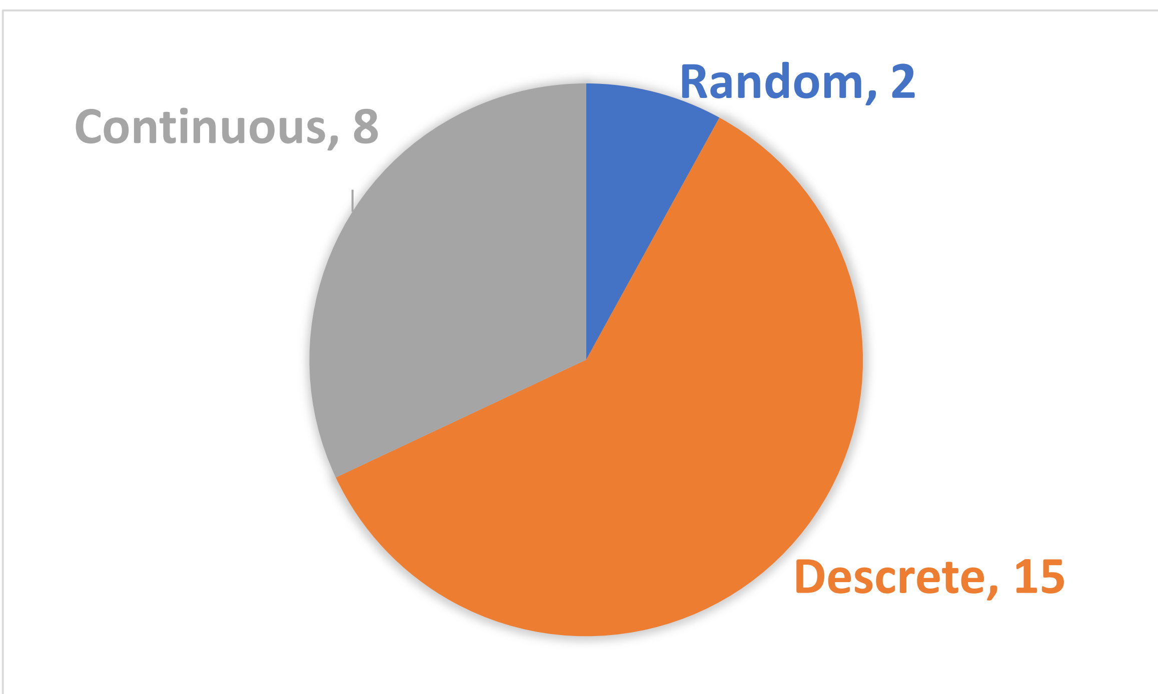
5.3 Continuous Random Variables (8 marks)

5.3.1 Introduction

5.3.2 Uniform Distribution

5.3.3 Exponential Distribution

5.3.4 Normal Distribution



Random variables

Introduction

Unlike usual variables, the value of a random variable is uncertain. Two differentiate random variables from other variables we denote them using Capital English Letters.

Consider an experiment of randomly picking a group of 4 girls. Let X be the number of girls, who are taller than 150 cm and Let Y be the number of boys in the selected group. Observe that that the value of X is not certain (X can be 0,1,2,3 or 4). In other word we cannot say the value of X exactly, without performing the experiment. Therefore, X is a random variable. In contrast, the value of Y is certain (Y=0). Therefore, Y is not a random variable.

Q01: There are 10 balls in a box. 6 of them are red. Rest of the balls are green. A person randomly picks two balls without replacement. Which of the following are random variables?

a) Total number of balls in the box

b) Number of red balls picked

c) Number of balls picked

d) Color of the first ball picked

e) Time taken to picks the two balls

(Ans: b, d and e)

Although, the value of a random variable is not known exactly, it is required to know the possible values it can take. The set of all possible outcomes that a random variable can take is known as the sample space. Sample space of a random variable X is demoted by $S(X)$.

Consider the following example. An unbiased coin in tossed until a head is flipped. Let X be the number of times the coin is flipped. Then $S(X) = \{1,2,3, \dots\}$. Let Y be the side obtain in the first attempt. Then $S(Y) = \{H, T\}$.

Q02: An ordinary die is rolled twice. Find the sample space of each of the following random variables.

a) X = Number obtained in the first attempt

b) Y = Sum of the two numbers obtained

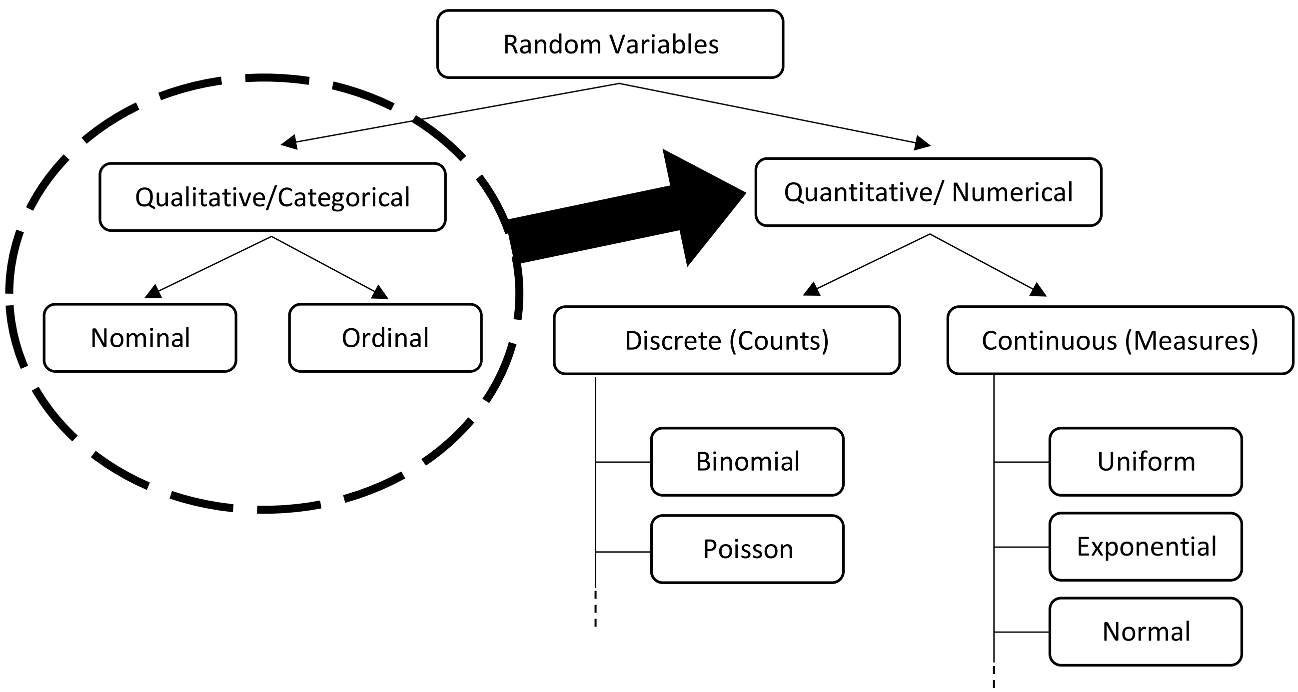
c) Z = Number of sixes obtained

(Ans: $S(X) = \{1,2,3,4,5,6\}$, $S(Y) = \{2,3,4,5,6,7,8,9,10,11,12\}$ and $S(Z) = \{0,1,2\}$)

Types of Random variables

Random variables can be categorized based on the nature of the sample space. Sample space of some random variables consists of labels. These labels represent some quality. Hence, we name such random variables as qualitative (aka categorical) random variable.

Labels can either be name or numbers. When they are numbers, we can order the elements of the sample space either in ascending order or in descending order. Hence, we name such random variables as ordinal random variable. When the labels are just names, such random variables are known as nominal random variable.



Consider the following experiment. John is one of the 5 competitors of a race. Let X be the place of John in the race. Then $S(X) = \{1,2,3,4,5\}$. Observe that X is just a label assign to each player as they finished the race. Let Y be the gender of the winner. Then $S(Y) = \{Male, Female\}$. Y is also a label assigned to the winner at birth. Therefore, both X and Y are categorical random variables. In contrast with Y, values of X can be sorted. Therefore, X is an ordinal random variable. However, there is no particular order that female must come after male or vice versa. They are just names. Hence, Y is a nominal random variable.

Q03: Complete the following past paper. 2016_21 (Ans: d), 2017_19 (Ans: b, c and d), 2018_20 (Ans: a), 2019_20 (Ans: b)

Some random variables represent a quantity rather than a quality. We name such random variables as quantitative (aka numerical) random variables. These quantities can either be counts or measurements. When they are counts, we call them as discrete random variables. When they are measures, we call them as continuous random variables.

Not like continuous random variables, discrete random variables can only take some values in a range.

Consider the following experiment. Temperature of a particular city is recorded in 7 consecutive days. Let X be the temperature of the 5th day. Assume that temperature of the city varies from 5 to 45 degrees in Celsius. Then $S(X) = \{any\ value\ in\ between\ 5\ and\ 45\} = \{x: 5 < x < 45\} = (5, 45)$. Let Y be the number of days which have temperature above 30 degrees in Celsius. Then $S(Y) = \{0,1,2,3,4,5,6,7\}$. Therefore, both X and Y are numerical random variables. Clearly X is a measurement. In contrast Y is a count. Therefore, it is a discrete random variable. You can further verify this by looking at the fact that Y can only take certain values in the range [0,7]. For example, Y can never be 4.7.

Q04: Five students are facing an MCQ paper consist of 40 questions. For each correctly answered question is given 1/4 marks. Maximum time allocated for the paper is 2 hours. Write the sample space of each of the following random variables. Hence obtain the type of the random variable.

a) Grade of a particular student

b) Number of correct answers a particular student marked

c) Time taken to finish the paper by a particular student

d) Home town of the student who takes the highest marks

e) Marks of the student who obtained the lowest marks

(Ans:

a) {A+, A, A-, B+, B, B-, C+, C, C-, D+, D, D-, E}-Ordinal

b) {0,1, 2, …, 38,39,40}-Discrete

c) (0,2]- Continuous

d) {Galle, Colombo, Gampaha, …}- Nominal

e) {0,1/4,1/2,3/4,1, 1 ¼, 1 ½, …, 9 ¾, 10}-Discrete)

When a sample space consists only integers, it may be confusing to distinguish discrete random variables from ordinal random variables.

Consider the following example. Bob, Sam and Tom sat for an exam and obtain average of 90%,80% and 50% respectively. Let X be the rank of a randomly selected student amount them. Then $S(X) = \{1,2,3\} \subseteq Z$. There ranks just indicate an order of their performance but it is not an indicator of the level of their performance. Tom is significantly weaker when compared to the other two students. Thus, X is not quantitative. Therefore, X is an ordinal random variable.

Q05: Complete the following past paper.

2010_21 (Ans: a), 2011_19 (Ans: a and c), 2012_20 (Ans: c), **2014_19 (Given Ans: b and c, Cor. Ans: c)**, 2015_21 (Ans: c and e)

Probability

Let X be a random variable. Define a function $P: S(X) \to (0,1)$ such that $\sum_{x \in S(X)} P(x) = 1$ representing the likelihood of $X = x$. This definition of probability requires two conditions to be established by a probability distribution.

1) $P: S(X) \to (0,1)$
2) $\sum_{x \in S(X)} P(x) = 1$

Q06: Complete the following past paper.

2009_20 (Ans: a, c, d and e), 2019_25 (Ans: c), 2013_19 (Typo $S(X) = \{1,2\}$)

Discrete Random Variables

In statistics, we will not use categorical random variables for computations. If a random variable is categorical, we can convert it into a numerical variable by using special methods which is out of the scope of this discussion. Therefore, eventually all random variable we encounter in statistics are numerical.

In this part, we will identify and calculate the following five values for discrete random variables.

- Mode
- Expected Value
- Mean
- Variance
- Standard Deviation

Then, we will calculate the above values for two special kind of discrete random variables called binomial and Poisson.

Introduction

Mode

Recall that the value of a random variable is uncertain. However, by looking at probability function, we can predict the value of X, that is most likely to happen. Let X be a random variable and P be the probability function of X. If $\max\{P(X)\} = P(X = x)$, the x is said to be the mode of X.

Consider the following example. An ordinary tetrahedron is rolled twice. Let X be the difference of the values obtained in the two attempts. Then $S(X) = \{0,1,2,3\}$. Observe that the probabilities of X being 0,1,2 and 3 are 0.25, 0.375, 0.250 and 0.125 respectively. Since maximum value of probability is 0.375 and $P(X = 1) = 0.375$, we can conclude that the mode of X is 1.

Q07: Following table represents the probability function of a random variable Y. Which of the following statement are true?

Y	1	2	2.5	3
$P(Y)$	0.2	0.1	0.5	a

a) Value of a is 0.2.

b) Mode of Y is 0.5.

c) Mode of Y is 2.5.

d) Mode of Y is ‘a’. e) Mode of Y is 3.

(Ans: a and c)

Expected Value

Observe that the mode of a random variable is always a member of the sample space. The main problem of using the mode to represent the distribution is that it completely neglects the minority and only focus on the majority.

Consider the following historical data. 60% of a student of a certain college needs 2m for a uniform cloth. Others need 3m for it. Suppose that the college administration has decided to distribute a common uniform cloth for all the students. Now if we consider the mode as a common value 40% the students will be in need of 1m. This is because we completely neglect the requirement of the minority. However, if we select something like 2.5 m, then both of the parties will be benefited. Since the majority (60%) needs 2m, something like 2.4 m would be economical. This is where expected value come into the play.

Expected value is defined as follows: Let P be the probability function of a random variable X and g be another function of X. Then the expected value of g, denoted by $E\big(g(X)\big)$ is the sum of products of g and P.

$$\boxed{E\big(g(X)\big) = \sum g(X) \times P(X)}$$

In the above example, let X be the amount of uniform cloth needed by a randomly selected student. Then $S(X) = \{2,3\}$. Define $g(X) = X$. Then $E\big(g(X)\big) = 2(0.6) + 3(0.4) = 2.4$ as we thought would be realistic.

Q08: Complete the following past paper.

2013_20 (Ans: c), 2010_22 (Ans: c), 2018_3, 2016_3, 2015_3, 2012_3

Q09: Days taken to show symptoms of a particular virus are as follows:

Days	Less than 7	7	8	9	10	More than 10
Probability	x	60%	20%	15%	5%	0%

Note that an infected person will start spreading the virus to 3 people per day, commencing from the 5th day.

a) Find the value of x.

b) What is the expected no of weeks that an infected person takes to show symptoms?

c) What is the expected no of days that an infected person takes to show symptoms?

d) What is the expected no of people that will get infected from an infected person?

e) Verify that $E(3 \times Days - 12) = 3E(Days) - 12$.

(Ans: 0, 1.0929, 7.65, 10.95, $E(3 \times Days - 12) = 10.95 = 3E(Days) - 12$)

It is useful to remember the following relationship.

Now consider $E(aX^2 + bX + c)$.By definition $E(aX^2 + bX + c) = \sum(aX^2 + bX + c)P(X) = \sum aX^2P(X) + \sum bXP(X) + \sum cP(X) = a\sum X^2P(X) + b\sum XP(X) + c\sum P(X) = aE(X^2) + bE(X) + c$.

$$\boxed{E(aX^2 + bX + c) = aE(X^2) + bE(X) + c}$$

Q10: Complete the following past paper.

2018_21(Ans: c), 2012_21 (Ans: d), 2008_25 (Ans: c)

Mean

Consider the following experiment. 5 questions were asked from each of the students of an undergraduate class. Following table shows how many questions they answered correctly.

Reg No	R2001	R2002	R2003	R2004	R2005	R2006
No of correctly answered questions	5	3	3	4	5	5

Let X be the no of questions that a randomly selected student of that class had correctly answered. Then $S(X) = \{3,4,5\}$. The mean (aka average) number of questions that a randomly selected student of that class had correctly answered is $\frac{5+3+3+4+5+5}{6} = 4.17$. This fact is denoted by $\mu(X) = 4.17$. Observe that $\mu(X) = \frac{5+3+3+4+5+5}{6} = \frac{3(2)+4(1)+5(3)}{6} = \frac{3(2)}{6} + \frac{4(1)}{6} + \frac{5(3)}{6} = 3\left(\frac{2}{6}\right) + 4\left(\frac{1}{6}\right) + 5\left(\frac{3}{6}\right) = \sum XP(X) = E(X)$.

Therefore, mean of a discrete random variable can be found just by calculating its expected value.

$$\boxed{\mu(X) = E(X)}$$

Q11: Two lotteries A and B has Rs 20, Rs 100 and Rs 5000 prizes. Tickets from lottery A and lottery B are sold for Rs 20 and Rs 30 respectively. Following table shows the probability of winning a prize from the lotteries.

Prize (Rs)	20	100	5000
Lottery A	35%	10%	5%
Lottery B	80%	16%	4%

Let X and Y be the profit that one can get by buying a ticket from this lottery A and lottery B respectively.

a) What is the probability of not winning a price from lottery A? (Ans: 50%)

b) Find $S(X)$. (Ans: $\{-20, 0, 80, 4980\}$)

c) Find $S(Y)$. (Ans: $\{-10, \ 70, 4970\}$)

d) What is the average profit of a buyer of lottery A? (Ans: Rs 247)

e) What is the average profit of a buyer of lottery B? (Ans: Rs 202)

f) Which lottery ticked would you like to buy if you have Rs. 30 in your hand? Justify your answer. (Ans: Lottery A. We can expect a higher profit from A than B)

Variance

For most of the incidences, mean can be taken as a representative for the data set. What if we encounter two data sets having the same mean? Then we have to take a look at the spread of data points. One such simple representative is range. Range of a random variable X is the difference between the minimum value of X and the maximum value of X.

But range is also suffering from the same problem of Mode. Range also taken only two values (minimum and maximum) in to account when determining the spread of a distribution. Let's define a better representative by considering all the data points. This new representative is known as Variance.

Variance is the expected value of the squares of differences with respect to mean.

$$\boxed{V(X) = E\{(X-\mu)^2\}}$$

For the ease of calculation, we can write $V(X) = E\{(X-\mu)^2\} = E\{X^2 - 2\mu X + \mu^2\} = E(X^2) - 2\mu E(X) + \mu^2$. Replacing μ by $E(X)$, we can write,

$$\boxed{V(X) = E(X^2) - [E(X)]^2}$$

Salary Sums as an example

Q12: Complete the following past paper.

2009_3

Now consider $V(aX+b)$.By definition $V(aX+b) = E[(aX+b)^2] - [E(aX+b)]^2 = E[a^2X^2 + 2abX + b^2] - [aE(X)+b]^2 = a^2E(X^2) + 2abE(X) + b^2 - \{a^2[E(X)]^2 + 2abE(X) + b^2\} = a^2\{E(X^2) - [E(X)]^2\} = a^2V(X)$.

$$\boxed{V(aX+b) = a^2V(X)}$$

Q13: Complete the following past paper.

2019_21 (Ans: b)

Standard Deviation

Variance is a good measure in all aspects with respect to range except that it is not having the square units. We can overcome this this just by taking the square root of variance. This new measure is called standard deviation.

Standard deviation of a random variable X, denoted by $\sigma(X)$. Thus $\sigma(X) = \sqrt{\sigma^2} = \sqrt{V(X)}$

$$\sigma(X) = \sqrt{V(X)}$$

Q14: Complete the following past paper.

2011_3

Q15: Let X be a random variable having the mean of 0.64. The Probability function of X is given below.

X	b	-1	0	$2b+1$
$P(X)$	a^2	0.5	$0.3-0.8a$	0.35

Answer the following question.

a) Find the value of a. (Ans: 0.3)

b) Find the probability of X being 0. (Ans: 0.06)

c) What is the mode of X? (Ans: -1)

d) Find the expected value of $2X-1$. (Ans: 0.28)

e) Find the value of b. (Ans: 1)

f) Verify that $P(X>-1)=1-P(X\leq -1)$. (Ans: $LHS=0.5=RHS$)

g) Find the expected value of X^2. (Ans: 3.74)

h) Find the variance of X. (Ans: 3.3304)

i) Find the variance of $10X-3$. (Ans: 333.04)

j) Find the standard deviation of X. (Ans: 1.82)

Binomial Distribution

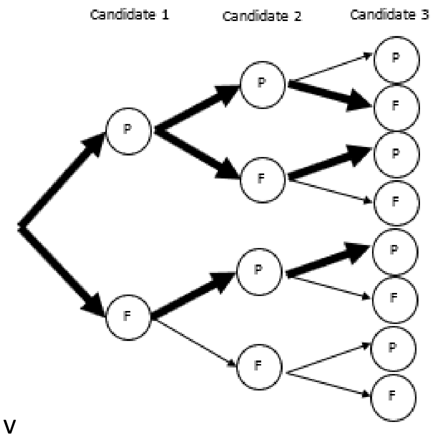
A discrete random variable X is called binomial random variable if it represents the number of successful trials from a set of n independent trials each having the same probability of success. This fact is denoted by, $X\sim Bin(n,p)$ and $S(X)=\{0,1,\dots,n\}$.

E.g.: Flipping a coin 10 times to see 6 heads, Number of patients who get cured from 350 infected, etc.

Q16: Complete the following past papers.

2018_24, 2016_24, 2010_18, 2008_27

Consider the following experiment. Probability of passing a particular examination is 0.7. Three persons sat for this examination. Let X be the number of students who passed the examination. Then $X\sim Bin(3{,}0.7)$ and $S(X)=\{0,1{,}2{,}3\}$. Let's draw a tree diagram to find out the probability of $X=2$.



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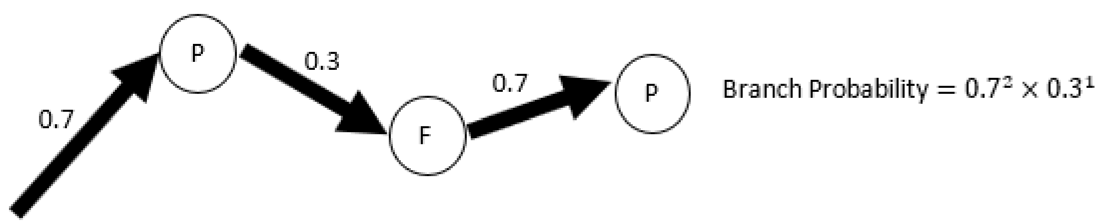
Observe that in each path bolded in the above tree diagram corresponds to $X = 2$. We have $\binom{3}{2} = \frac{3!}{1!2!} = 3$ such branches. Now let us consider one such branch.



Therefore, $P(X = 2) = \begin{matrix} Number \\ of \\ Banches \end{matrix} \times \begin{matrix} Branch \\ Probability \end{matrix} = 6 \times 0.7^2 \times 0.3^1$.

Q17: Your friend suggests you two games. To win the first game you need to roll an ordinary die and get a perfect square (Perfect square is a number in which the square root is an integer). To win the second game you need to get a head after flipping an unbiased coin 3 times. What do you prefer? Justify your answer.

(Ans: Game 2, because it has a higher probability of wining, which is 0.375)

In general, for $X \sim Bin(n, p)$, we can write,

$$\boxed{P(X = x) = \binom{n}{x} p^x (1 - p)^{n-x}}$$

Q18: A coin in tossed 2 times. If the probability of getting 1 head is 0.48, show that the coin is biased.

(Ans: p= 0.6 or p=.04 thus $p \neq 0.5$)

Q19: Complete the following past papers.

2015_24, 2013_21, 2012_22, 2009_18

Mean and Variance of Binomial Distribution

Let $Y_1, Y_2, \dots, Y_n \sim Bin(1, p)$ and $X \sim Bin(n, p)$ be a binomial random variables such that $X = Y_1 + Y_2 + \dots + Y_n$. Then $P(Y_i = 0) = 1 - p$ and $P(Y_i = 1) = p$where $i = 1,2, \dots, n$. Now $E(Y_i) = 0(1 - p) + 1(p) = p$.

Therefore, $\mu = E(X) = E(Y_1 + Y_2 + \dots + Y_n) = E(Y_1) + E(Y_2) + \dots + E(Y_n) = p + p + \dots + p = np$

$$\boxed{\mu = np}$$

Q20: Complete the following past papers.

2018_22, 2010_19, 2008_28

Further, observe that $E\left({Y_i}^2\right) = 0^2(1 - p) + 1^2(p) = p$ and $V(Y_i) = E\left({Y_i}^2\right) - [E(Y_i)]^2 = p - p^2 = p(1 - p)$.

Therefore, $\sigma^2 = V(X) = V(Y_1 + Y_2 + \dots + Y_n) = V(Y_1) + V(Y_2) + \dots + V(Y_n) = p(1 - p) + p(1 - p) + \dots + p(1 - p) = np(1 - p)$.

$$\boxed{\sigma^2 = np(1 - p)}$$

Q21: Complete the following past papers.

2019_22, 2017_21, 2016_22, 2015_22, 2014_20, 2011_20

Poisson Distribution

Poisson distribution is actually a special case of binomial distribution, studied separately due to its wide application in day to day life. Recall that in binomial distribution, we need to know to probability of success (p) beforehand, in order to study the probability distribution. However, in most of the day to day activities, we do not know the probability of success (p). But we know that p is very small. (rare event)

Consider the following experiment. John is looking up '3' times in a selected hour, wishing that he could see an areophane in the sky. Let us declare a binomial random variable X as the no of times he could see an areophane in an hour. Then $X \sim Bin(5, p)$ where p is the probability of success. Further $P(X = x) = \binom{3}{x} p^x (1 - p)^{3-x}$. But the problem is that we are not aware about the exact probability of seeing an aero plane once he looked up.

Suppose that we know that on average 48 aero planes pass per day where John is doing this experiment. This implies that he can expect to see 2 aero planes per hour. However, John is looking up only 3 times. There is a greater probability that he could miss at least one aero plane. Therefore, we cannot expect him to see 2 aero planes.

We can expect him to see two aero planes, if he increases the times, he looks up to see aero planes. Looking at the sky infinitely many times within an hour is equivalent to looking at the sky for an hour continuously. Now we can expect him to see both aero planes. This kind of problems are discussed under Poisson distribution.

A discrete random variable X is said to be a Poisson random variable if it represents the no of times that a rare event is occurring in a given interval. This interval can be a time, distance, length, volume, etc. E.g.: Number of virus attacks per day, number of corona viruses found in a cubic centimeter, number of weeds per square feet, etc. This fact is denoted by $X \sim Poisson(\lambda)$, where λ is the expected value per unit and $S(X) = \{0, 1, 2, \dots\}$

Q22: Complete the following past papers.
2017_23, 2011_21

Before calculating the probability distribution of a Poisson random variable, let us define a constant $e = \frac{19}{7}$ (don't mix up with the well know constant $\pi = \frac{22}{7}$). What is so special about this constant? Surprisingly, its x^{th} power can be represented as

$$e^x = \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n$$

Let X be a Poisson random variable and λ be the expected value per unit. Then $\lambda = \lim_{n\to\infty} \mu_{Binomial} = \lim_{n\to\infty} np = p \times \lim_{n\to\infty} n$. Further, $e^{-\lambda} = \lim_{n\to\infty}\left(1 + \frac{-\lambda}{n}\right)^n = \lim_{n\to\infty}\left(1 - \frac{\lambda}{n}\right)^n$.Now substituting p, we can have

$$\begin{aligned}
P(X = x) &= \lim_{n\to\infty}\left\{\binom{n}{x} p^x (1-p)^{n-x}\right\} = \lim_{n\to\infty}\left\{\binom{n}{x}\left(\frac{\lambda}{n}\right)^x\left(1 - \frac{\lambda}{n}\right)^{n-x}\right\} \\
&= \lim_{n\to\infty}\left\{\frac{n!}{(n-x)!\,x!} \times \frac{\lambda^x}{n^x} \times \left(1 - \frac{\lambda}{n}\right)^{n-x}\right\} = \frac{\lambda^x}{x!} \times \lim_{n\to\infty}\left\{\frac{n!}{(n-x)!} \times \frac{1}{n^x} \times \left(1 - \frac{\lambda}{n}\right)^{n-x}\right\} \\
&= \frac{\lambda^x}{x!} \times \lim_{n\to\infty}\left\{[n \times (n-1) \times \dots \times (n-x+1)] \times \frac{1}{n^x} \times \left(1 - \frac{\lambda}{n}\right)^{n-x}\right\} \\
&= \frac{\lambda^x}{x!} \times \lim_{n\to\infty}\left\{\left[\frac{n}{n} \times \frac{(n-1)}{n} \times \dots \times \frac{(n-x+1)}{n}\right] \times \left(1 - \frac{\lambda}{n}\right)^{n-x}\right\} \\
&= \frac{\lambda^x}{x!} \times \lim_{n\to\infty}\left\{1 \times \left(1 - {}^{1}/_{n}\right) \times \dots \times \left(1 - \frac{x-1}{n}\right) \times \left(1 - \frac{\lambda}{n}\right)^{-x} \times \left(1 - \frac{\lambda}{n}\right)^n\right\} \\
&= \frac{\lambda^x}{x!} \times \lim_{n\to\infty}\left(1 - \frac{\lambda}{n}\right)^n = \frac{\lambda^x}{x!} \times e^{-\lambda}
\end{aligned}$$

$$\boxed{P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}}$$

Q23: Complete the following past papers.
2018_23, 2015_25, 2014_23, 2013_23, 2012_23, 2009_19, 2008_29

Mean and Variance of Poisson Distribution

Mean of Poisson distribution can be deduced from the mean of Binomial distribution as follows:

$$\mu_{Poisson} = \lim_{n\to\infty} \mu_{Binomial} = \lim_{n\to\infty} np = \lambda$$

$$\boxed{\mu = \lambda}$$

> Q24: Probability of not encountering any filling stations on journey of 5km is $\frac{1}{e^2}$ What is the average number of filling stations one can find per 120km?
>
> (Ans 48 stations)

Variance of Poisson distribution can be deduced from the variance of Binomial distribution as follows:

$$\sigma^2{}_{Poisson} = \lim_{n\to\infty} \sigma^2{}_{Binomial} = \lim_{n\to\infty} np(1-p) = \lim_{n\to\infty} np \times \lim_{n\to\infty} (1-p) = \lambda \times \lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right) = \lambda \times 1$$

$$\boxed{\sigma^2 = \lambda}$$

> Q25: Complete the following past papers.
>
> 2019_23, 2017_22, 2016_23, 2015_23, 2014_21, 2013_22, 2008_28

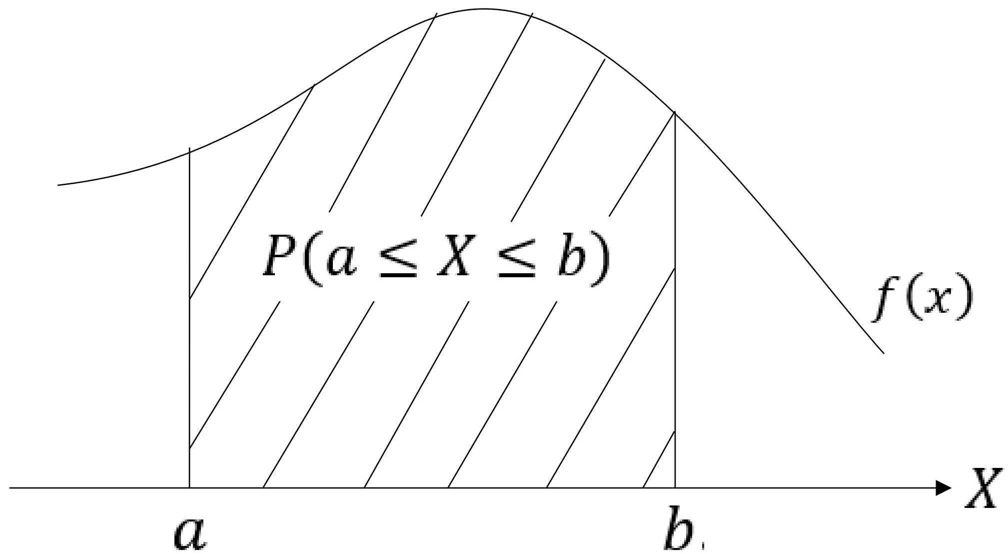
Continuous Random Variables

Recall that if X is a discrete random variable, then the probability mass function P of X need to have the following two conditions.

1) $P: S(X) \to (0,1)$
2) $\sum_{x\in S(X)} P(x) = 1$

Since a continuous random variable can take uncountable number of different values, we cannot add all the probabilities to see whether is satisfy condition 2. Therefore, we need to refine our definition of probability mass function P, by using another function ‘f’.

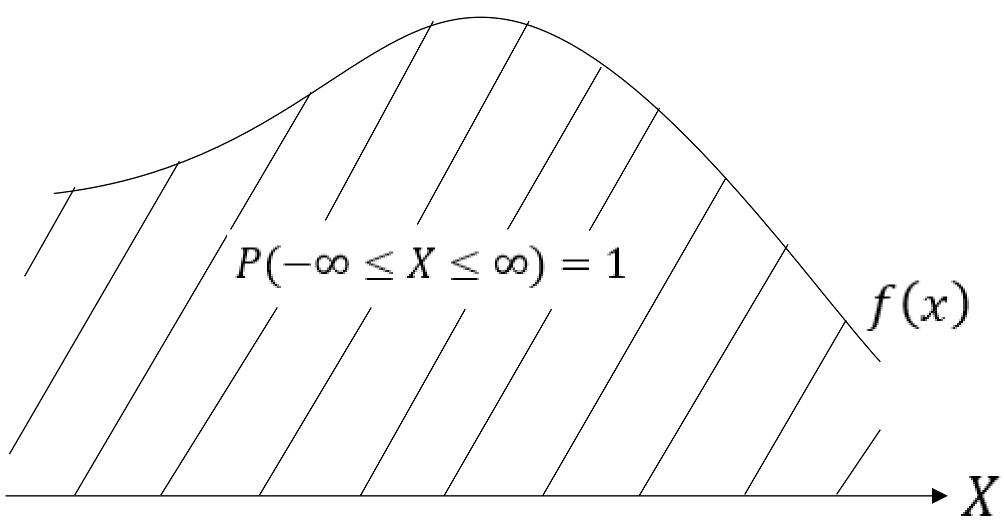
Let f be a function such that $P(a \leq X \leq b) = \int_a^b f(x)\, dx$ where $\int_a^b f(x)\, dx$ denote the area under the curve $f(x)$ within $a \leq X \leq b$. This function is known as the probability density function.



Observe that since $P(a \leq X \leq b)$ is non-negative, $\int_a^b f(x)\, dx$ should be non-negative. For that we need $f(x)$ to be non-negative.

$$\boxed{f(x) \geq 0}$$

Further, since $P(-\infty \leq X \leq \infty) = 1$, we have $\int_{-\infty}^{\infty} f(x)\, dx = 1$.

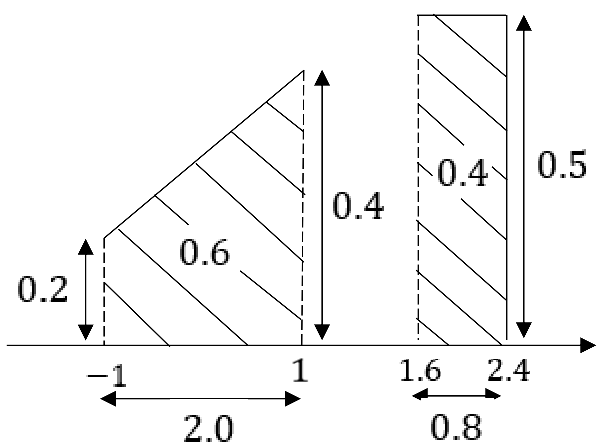


$$\boxed{\int_{-\infty}^{\infty} f(x)\, dx = 1}$$

This definition of probability density function requires two conditions to be established by a probability distribution.

1) $f: S(X) \to [0, \infty)$
2) $\int_{-\infty}^{\infty} f(x)dx = 1$

Consider the function $f = \begin{cases} 0.1x + 0.3 & -1 < x < 1 \\ 0.5 & ; 1.6 < x \leq 2.4 \\ 0 & ; o/w \end{cases}$. You can easily see that $f(x) \geq 0$ for all values of x. As show in the following diagram $\int_{-\infty}^{\infty} f(x)\, dx = \left(\frac{0.2+0.4}{2} \times 2.0\right) + (0.8 \times 0.4) = 0.6 + 0.4 = 1.$



Therefore, f is a probability density function.

Q26: Which of the following are probability density functions.

a) $f(x) = 5$ for $0 < x < \frac{1}{5}$

b) $f(x) = \frac{1}{3}$ for $2 \leq x < 6$

c) $f(x) = \begin{cases} \frac{x}{2} - 5 & ; x \in (10,12) \\ 0 & ; o/w \end{cases}$

d) $f(x) = \begin{cases} 0.5 - x & ; x \in (0,1) \\ 0 & ; o/w \end{cases}$

e) $f(x) = 3x$ for $0 \leq x < 0.6$

(Ans: a and c)

Consider the function $f(x) = \begin{cases} \frac{x^2}{3} & x \in (-1,2) \\ 0 & o/w \end{cases}$. It is clear that $f(x) \geq 0$ for all values of x. To check whether f is a probability density function we need to calculate $\int_{-\infty}^{\infty} f(x)\, dx = \int_{-1}^{2} \frac{x^2}{3} dx$ which is not easy to calculate as in the previous examples. This is where integration come in to the play. Recall that

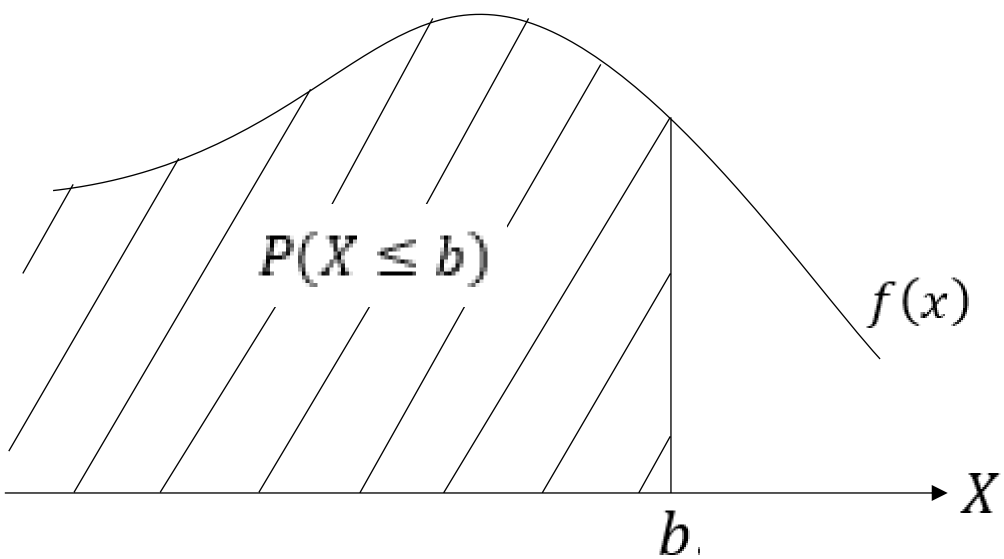
we learn under integration that $\int_a^b f(x)dx = \int f(x)dx|_b - \int f(x)dx|_a = [\int f(x)dx]_a^b$. Therefore, $\int_{-1}^{2} \frac{x^2}{3} dx = \frac{1}{3}\int_{-1}^{2} x^2\, dx = \frac{1}{3}[\int x^2 dx]_{-1}^{2} = \frac{1}{3}\left[\frac{x^3}{3}\right]_{-1}^{2} = \frac{1}{3} \times \left\{\frac{x^3}{3}|_2 - \frac{x^3}{3}|_{-1}\right\} = \frac{1}{3} \times \left\{\frac{2^3}{3} - \frac{(-1)^3}{3}\right\} = 1.$
Therefore, f is probability density function.

Q27: Complete the following past paper. 2009-21

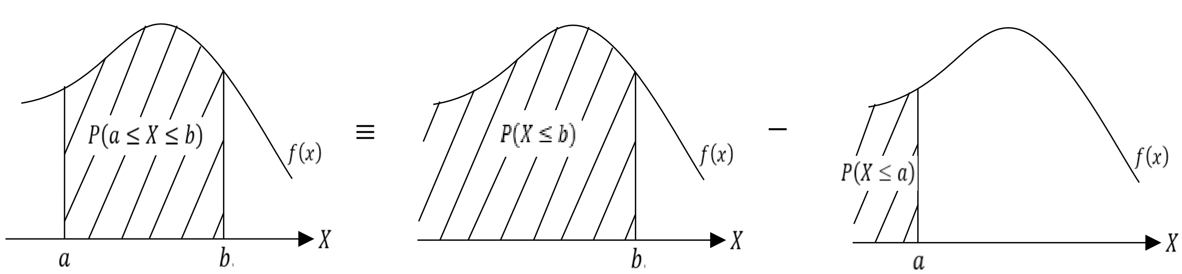
Cumulative density function

Let X be a continues random variable. Then its cumulative density function (denoted by $F(x)$) , is defined as the probability of X being less than or equal to x (i.e. $P(X \le x)$). We can calculate $F(x)$ by using the definition of probability density function as follows.

$$\boxed{F(x) = \int_{-\infty}^{x} f(x)dx}$$



Now we can relate probability mass function(P) to cumulative density function(F) as below:

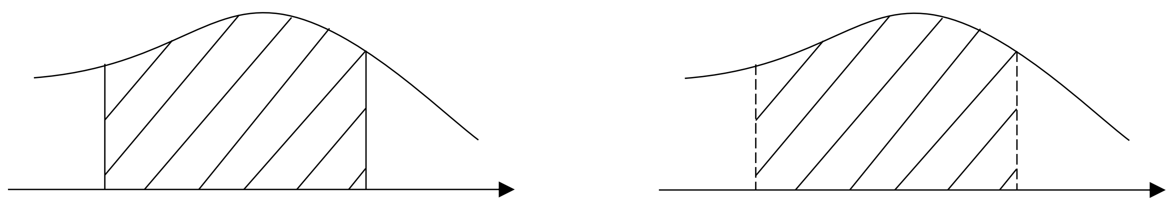


$$\boxed{P(a \le X \le b) = F(b) - F(a)}$$

Observe that $P(X = a) = P(a \le X \le a) = F(a) - F(a) = 0$. This means probability of X being exactly 'a' is always zero irrespective of the value of 'a'.

$$\boxed{P(X = a) = 0}$$

Further, $P(a \le X \le b) = P(X = a) + P(a < X < b) + P(X = b) = 0 + P(a < X < b) + 0 = P(a < X < b)$. Therefore, when considering continuous random variables end points doesn't contribute anything to the probabilities. Therefore, more than is as same as at least and less than is as same as at most. This fact can be visualized graphically as follows.



$$\boxed{P(a \le X \le b) = P(a < X < b)}$$

Q28: Complete the following past paper.

2009-22, 2016-25 (Note that normal distribution is a special kind of continuous random variable.)

Expected value

Now we need to re =dine expected value of a function 'g' in terms of probability density function 'f' as $\int_{-\infty}^{\infty} gf(x)\,dx$.

$$\boxed{E(g) = \int_{-\infty}^{\infty} gf(x)\,dx}$$

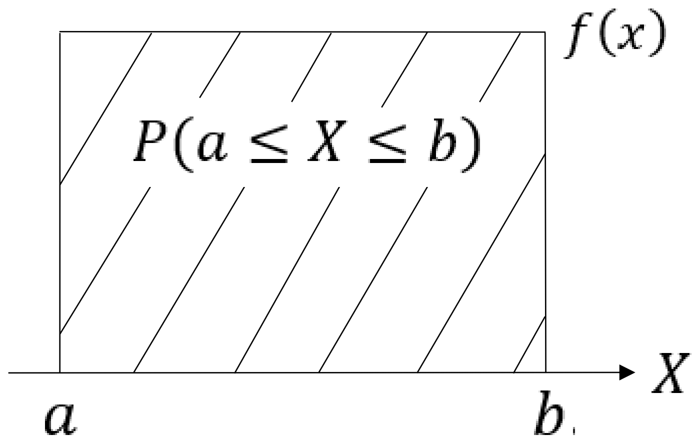
Consider the function $f(x) = 2x + 1$ for $-0.5 < x < 0.5$. Clearly $f(x) \ge 0$ for all values of x. Observe that $\int_{-\infty}^{\infty} f(x)dx = \int_{-0.5}^{0.5}(2x + 1)dx = [x^2 + x]_{-0.5}^{0.5} = [0.5^2 + 0.5] - [(-0.5)^2 + (-0.5)] = 1$. Therefore 'f' is a probability density function. Now $E(5x^3) = \int_{-\infty}^{\infty} 5x^3 f(x)\,dx = \int_{-0.5}^{0.5} 5x^3(2x + 1)\,dx = \int_{-0.5}^{0.5}(10x^4 + 5x^3)\,dx = \left[2x^5 + \frac{5x^4}{4}\right]_{-0.5}^{0.5} = \left[2(0.5)^5 + \frac{5(0.5)^4}{4}\right] - \left[2(-0.5)^5 + \frac{5(-0.5)^4}{4}\right] = 4(0.5)^5 = 0.125$.

Q29: Complete the following past papers.

2008-3, 2019-3

Uniform Distribution

A continuous random variable X is said to be a uniform random variable if its probability density function is a constant for all the values in range $[a, b]$ and 0 otherwise. This fact is denoted by $X \sim U(a, b)$.



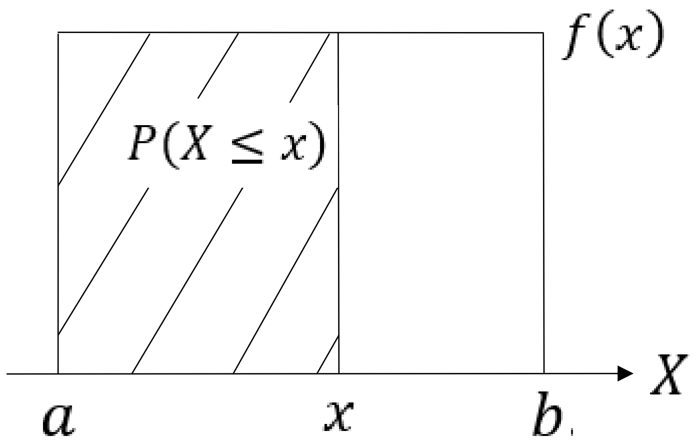
The constant k can be found by considering the total probability. Since $f(x) = \begin{cases} k & ; x \in [a, b] \\ 0 & ; o/w \end{cases}$, we have $\int_{-\infty}^{\infty} f(x)dx = \int_a^b kdx = k \times [x]_a^b = k(b-a) = 1$. Therefore, $k = \frac{1}{b-a}$.

$$\boxed{f(x) = \begin{cases} \dfrac{1}{b-a} & ; x \in [a, b] \\ 0 & ; o/w \end{cases}}$$

Consider the following experiment. Resistance of a 15 Ohms resistor can vary at most 2 Ohms with equal probability throughout the range. Suppose we want to find out the probability of receiving a resistor having a higher true value than its face value. Let X denote the true resistance of a randomly selected resistor. Then $X \sim U(13\Omega, 17\Omega)$ and $f(x) = \begin{cases} \frac{1}{4} & ; x \in [13\Omega, 17\Omega] \\ 0 & ; o/w \end{cases}$. Further, $P(X > 15) = \int_{15}^{\infty} f(x)dx = \int_{15}^{17} f(x)dx + \int_{17}^{\infty} f(x)dx = \int_{15}^{17} \frac{1}{4} dx + \int_{17}^{\infty} 0dx = \frac{1}{4}(17-15) + 0 = \frac{1}{2}$.

> Q30: A Perfect random number generator generates random numbers between 0 and 1 inclusively. What is the probability that it will generate a number between 0.3 and 0.6?
>
> (Ans: 0.3)

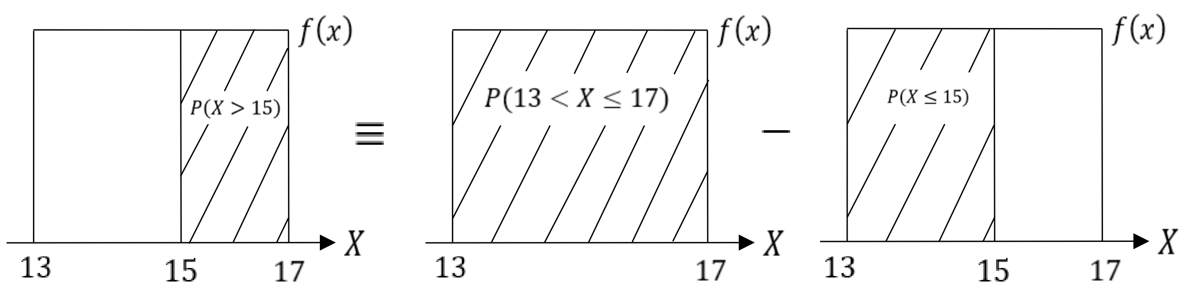
Consider the case $x < a$. Then we have $F(x) = \int_{-\infty}^{x} f(x)dx = \int_{-\infty}^{x} 0dx = 0$. For $a \leq x \leq b$, we have $F(x) = \int_{-\infty}^{x} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_{a}^{x} f(x)dx = \int_{-\infty}^{a} 0dx + \int_{a}^{x} \frac{1}{b-a}dx = 0 + \frac{1}{b-a}[x]_a^x = \frac{1}{b-a} \times (x - a) = \frac{x-a}{b-a}$.



Finally, if $x > b$ we have $F(x) = \int_{-\infty}^{x} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_{a}^{b} f(x)dx + \int_{b}^{x} f(x)dx = \int_{-\infty}^{a} 0dx + \int_{a}^{b} \frac{1}{b-a}dx + \int_{b}^{x} 0dx = 0 + \frac{1}{b-a}[x]_a^b + 0 = \frac{1}{b-a} \times (b-a) = 1.$

$$F(x) = \begin{cases} 0 & ; x < a \\ \left(\frac{x-a}{b-a}\right) & ; x \in [a, b] \\ 1 & ; x > b \end{cases}$$

Reconsider the above experiment. If $X \sim U(13\Omega, 17\Omega)$, then $F(x) = \begin{cases} 0 & ; x < 13 \\ \left(\frac{x-13}{4}\right) & ; x \in [13,17] \\ 1 & ; x > 17 \end{cases}$. Now, $P(X > 15) = 1 - P(X \leq 15) = 1 - F(15) = 1 - \frac{15-13}{4} = \frac{1}{2}$. This can be visualized by the following diagram.



Mean and variance

$$\mu = E(X) = \int_{-\infty}^{\infty} xf(x)dx = \int_{-\infty}^{a} xf(x)dx + \int_{a}^{b} xf(x)dx + \int_{b}^{\infty} xf(x)dx$$
$$= \int_{-\infty}^{a} (x \times 0)dx + \int_{a}^{b} \left(x \times \frac{1}{b-a}\right) dx + \int_{b}^{\infty} (x \times 0)dx = 0 + \int_{a}^{b} \frac{x}{b-a} dx + 0$$
$$= \frac{1}{b-a}\int_{a}^{b} xdx = \frac{1}{b-a} \times \left[\frac{x^2}{2}\right]_a^b = \frac{1}{b-a} \times \left(\frac{b^2}{2} - \frac{a^2}{2}\right) = \frac{1}{b-a} \times \frac{b^2-a^2}{2} = \frac{a+b}{2}$$

$$\boxed{\mu = \frac{a+b}{2}}$$

Q31: Certain African tribe is capable of telling the time by analyzing the sun light. There is no possibility that their predictions exceed more than 1 hour from the correct time. What is the mean of the predicted time if one of the randomly selected tribe members is asked time at 12:45 pm?

(Ans: 1:15pm)

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx = \int_{-\infty}^{a} x^2 f(x)dx + \int_{a}^{b} x^2 f(x)dx + \int_{b}^{\infty} x^2 f(x)dx$$
$$= \int_{-\infty}^{a} (x^2 \times 0)dx + \int_{a}^{b} \left(x^2 \times \frac{1}{b-a}\right) dx + \int_{b}^{\infty} (x^2 \times 0)dx = 0 + \int_{a}^{b} \frac{x^2}{b-a} dx + 0$$
$$= \frac{1}{b-a}\int_{a}^{b} x^2 dx = \frac{1}{b-a} \times \left[\frac{x^3}{3}\right]_a^b = \frac{1}{b-a} \times \left(\frac{b^3}{3} - \frac{a^3}{3}\right) = \frac{1}{b-a} \times \frac{b^3-a^3}{3}$$
$$= \frac{1}{b-a} \times \frac{(b-a)(b^2+ab+a^2)}{3} = \frac{a^2+ab+b^2}{3}$$

$$\sigma^2 = V(X) = E(X^2) - [E(X)]^2 = \frac{a^2+ab+b^2}{3} - \left(\frac{a+b}{2}\right)^2 = \frac{a^2+ab+b^2}{3} - \frac{a^2+2ab+b^2}{4}$$
$$= \frac{a^2-2ab+b^2}{12} = \frac{(a-b)^2}{12}$$

$$\boxed{\sigma^2 = \frac{(a-b)^2}{12}}$$

Q32: Web site of Smart Cabs show that the arrival time of their vehicles uniformly distributed with a mean of 5 minutes and a variance of 3 minutes. Is it suitable to have a coffee immediately after booking a vehicle from this cab service, if it takes 2 minutes and 30 seconds for a coffee? Justify your answer.

(Ans: No. Since a < 2 minutes and 30 seconds)

Exponential Distribution

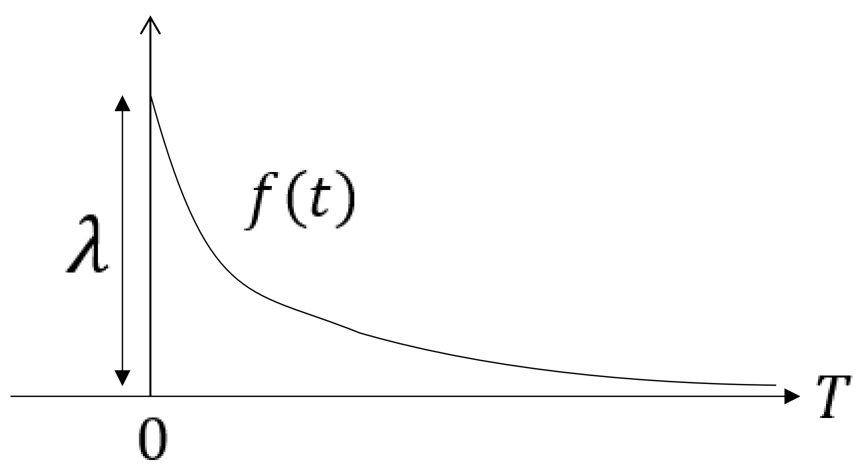
A continuous random variable X is said to be an exponential random variable if its probability density function represents the interval between two Poisson events with parameter λ. This fact is denoted by $T \sim Expo(\lambda)$.

Consider an event occurring λ times for a unit interval. Then for an interval t we can expect λt events to occur. Let X be the number of events occur during this interval. Then $X \sim Pois(\lambda t)$ and $P(X = 0) = \frac{e^{-\lambda t}(\lambda t)^0}{0!} = e^{-\lambda t}$. Now the next event can occur at $t + 1$ or $t + 2$ or at any time after t. Let T be the interval to the next event. Then $P(T > t) = P(X = 0) = e^{-\lambda t}$. Further $F(t) = P(T \leq t) = 1 - P(T > t) = 1 - e^{-\lambda t}$.

Now recall that $F(t) = \int_{-\infty}^{t} f(t)dt$. By the fundamental theorem of calculus, we have $f(t) = \frac{d}{dt}[F(t)] = \frac{d}{dt}\left[1 - e^{-\lambda t}\right] = \lambda e^{-\lambda t}$.

$$\boxed{f(t) = \begin{cases} \lambda e^{-\lambda t} & ; t \geq 0 \\ 0 & ; o/w \end{cases}}$$

Shape of the exponential distribution is as of a decaying function which has its highest value λ at $t = 0$. The intuition comes from the fact that as the waiting interval increases number of expected events also increases making long waiting intervals less probable.



Consider the following experiment. It is known that on average 2 busses pass a certain junction every 30 minutes. Let T be the waiting time in hours for a bus to pass this junction. Then $T \sim Expo(4)$. The probability of waiting at more than 15 minutes for a bus at this junction can be calculated as $P(T > 0.25) = \int_{0.25}^{\infty} f(t)dt = \int_{0.25}^{\infty} 4e^{-4t}dt = [-e^{-4t}]_{0.25}^{\infty} = [-e^{-4\infty}] - \left[-e^{-4(0.25)}\right] = 0 + e^{-1} = \frac{1}{e}$

$$F(x) = \begin{cases} 0 & ; x < 0 \\ 1 - e^{-\lambda x} & ; x \geq 0 \end{cases}$$

$$\mu = E(X) = \int_{-\infty}^{\infty} xf(x)dx = \int_{-\infty}^{0} xf(x)dx + \int_{0}^{\infty} xf(x)dx = \int_{-\infty}^{0} (x \times 0)dx + \int_{0}^{\infty} \left(x \times \lambda e^{-\lambda x}\right)dx$$

$$= 0 + \int_{0}^{\infty} \lambda x e^{-\lambda x} dx = \lambda \int_{0}^{\infty} x e^{-\lambda x} dx = \lambda \times \left\{ \left[\frac{x e^{-\lambda x}}{-\lambda}\right]_{0}^{\infty} - \int_{0}^{\infty} \frac{e^{-\lambda x}}{-\lambda} dx \right\}$$

$$= \lambda \times \left\{ \left[\frac{x e^{-\lambda x}}{-\lambda}\right]_{0}^{\infty} + \frac{1}{\lambda}\int_{0}^{\infty} e^{-\lambda x} dx \right\} = \lambda \times \left\{ \left[\frac{x e^{-\lambda x}}{-\lambda}\right]_{0}^{\infty} + \frac{1}{\lambda}\left[\frac{e^{-\lambda x}}{-\lambda}\right]_{0}^{\infty} \right\}$$

$$= \lambda \times \left\{ [0 - 0] + \frac{1}{\lambda}\left[0 - \frac{1}{-\lambda}\right] \right\} = \lambda \times \frac{1}{\lambda^2} = \frac{1}{\lambda}$$

$$\boxed{\mu = \frac{1}{\lambda}}$$

Q33: Complete the following past paper.

2014-22

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx = \int_{-\infty}^{0} x^2 f(x)dx + \int_{0}^{\infty} x^2 f(x)dx = \int_{-\infty}^{0} (x^2 \times 0)dx + \int_{0}^{\infty} \left(x^2 \times \lambda e^{-\lambda x}\right)dx$$

$$= 0 + \int_{0}^{\infty} \lambda x^2 e^{-\lambda x} dx = \lambda \int_{0}^{\infty} x^2 e^{-\lambda x} dx = \lambda \times \left\{ \left[\frac{x^2 e^{-\lambda x}}{-\lambda}\right]_{0}^{\infty} - \int_{0}^{\infty} \frac{e^{-\lambda x}}{-\lambda} 2x dx \right\}$$

$$= \lambda \times \left\{ \left[\frac{x^2 e^{-\lambda x}}{-\lambda}\right]_{0}^{\infty} + \frac{2}{\lambda}\int_{0}^{\infty} x e^{-\lambda x} dx \right\} = \lambda \times \left\{ \left[\frac{x^2 e^{-\lambda x}}{-\lambda}\right]_{0}^{\infty} + \frac{2}{\lambda}\left(\frac{1}{\lambda^2}\right) \right\} (By\ part\ 1)$$

$$= \lambda \times \left\{ [0 - 0] + \frac{2}{\lambda^3} \right\} = \lambda \times \frac{2}{\lambda^3} = \frac{2}{\lambda^2}$$

$$\sigma^2 = V(X) = E(X^2) - [E(X)]^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

$$\boxed{\sigma^2 = \frac{1}{\lambda^2}}$$

Normal distribution

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Observe that the probability density function is hard to be integrated to get the probability value. Therefore, we will use some special techniques to solve problems of normal distribution.

Let $X \sim N(\mu, \sigma^2)$. Consider the variable $Z = \frac{X-\mu}{\sigma}$. Let us look at the probability density function of Z.

$$P(a < Z < b) = P\left(a < \frac{X-\mu}{\sigma} < b\right) = P(\sigma a + \mu < X < \sigma b + \mu) = \int_{\sigma a+\mu}^{\sigma b+\mu} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$
$$= \int_a^b \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \sigma dz = \int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz = \int_a^b \frac{1}{1 \times \sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{z-0}{1}\right)^2} dz$$

Then $Z \sim N(0,1)$. This normal distribution is known as the standard normal distribution. A random variable which follows standard normal distribution is always denoted by Z and called as Z-score.

Q34: Complete the following past paper.

2010-20

Observe that a standard normal distribution has no parameters. Therefore, it is unique. Which means that its probability values only depend on the values of Z-score. Therefore, cumulative probability values of the standard normal distribution ($F(z)$) are available as a ready to use reference table.

Q35: Find each of the following for a standard normal distribution.

a) $F(1)$

b) $F(-2.65)$

c) $P(Z \geq 2)$

d) $P(-1 \leq Z \leq 2)$

e) $F(z) = 22.66\%$

(Ans: 0.8413, 0.0040, 0.0228, 0.8185, -0.75)

If the required value is not in the table, we can use the interpolation to find out the required values.

$$\frac{F - F_L}{z - z_H} = \frac{F_H - F_L}{z_H - z_L}$$

Suppose we need to find $F(0.43)$ of a standard normal distribution. But only $F(0.40) = 0.6554$ and $F(0.45) = 0.6736$ is available in the given table. Now substituting these values in the interpolation equation, we can get $\frac{F-0.6554}{0.43-0.40} = \frac{0.6736-0.6554}{0.45-0.40}$. By simplifying this, we get $F(0.43) = (0.43 - 0.40) \times \frac{0.6736-0.6554}{0.45-0.40} + 0.6554 = 0.6663$.

Q36: Find each of the following for a standard normal distribution.

a) $F(-1.285)$

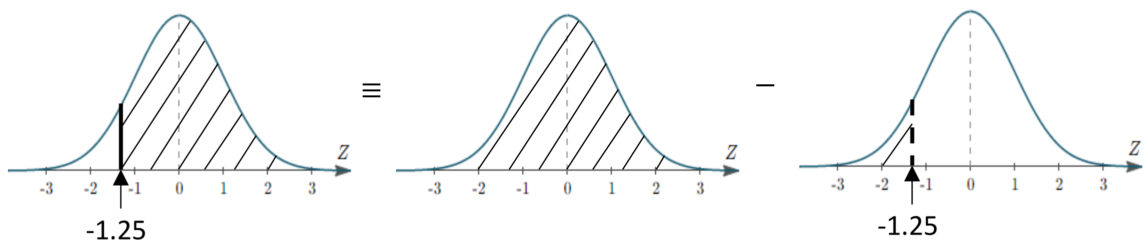
d) $P(0 \leq Z \leq 1.44)$

c) $F(z) = 80\%$

d) $P(Z \geq z) = 0.3000$

(Ans: 0.0994, 0.4250, 0.84, 0.52)

Consider the following example. Birth weight for babies follows a normal distribution with mean 3.5 kg and a standard deviation of 1.2 kg. Let W be the birth weight of a randomly selected baby. Then $W \sim N(3.5, 1.44)$. Suppose we want to find $P(W \geq 2)$. Now $P(W \geq 2) = P\left(\frac{W-\mu}{\sigma} \geq \frac{2-3.5}{1.2}\right) = P(Z \geq -1.25) = 1 - P(Z < -1.25) = 1 - F(-1.25) = 1 - 0.1056 = 0.8944 = 89.44\%$.



Q37: Complete the following past papers.

2017-3, 2015-3, 2014-3, 2013-3, 2008-3, 2019-24, 2018-25, 2017-24, 2012-24, 2011-22,2008-30

Normal Approximation to Binomial Distribution

Let $X \sim Bin(n, p)$. Then $P(a \leq X \leq b)$ can be approximated as $P(a - 0.5 \leq Y \leq b + 0.5)$ considering $Y \sim Nor\big(np, np(1-p)\big)$ whenever σ^2 of X is at least 10.

Q38: Which of the following distributions can be approximated as a normal distribution?

a) $X \sim Bin(3, 0.7)$

b) $X \sim Bin(100, 0.9)$

c) $X \sim Bin(50, 0.4)$

(Ans: c)

Consider the following example. Let $X \sim Bin(42, 0.6)$. Clearly $\sigma^2 = np(1-p) = 10.08 \geq 10$. Therefore, this can be approximated by as $Y \sim Nor(25.2, 10.08)$. Further $P(X = 22) = P(22 \leq X \leq 22) \approx P(21.5 \leq Y \leq 22.5) = P\left(\frac{21.5-25.2}{\sqrt{10.08}} \leq \frac{Y-\mu}{\sigma} \leq \frac{22.5-25.2}{\sqrt{10.08}}\right) = P(-1.17 \leq Z \leq -0.85) = F(-0.85) - F(-1.17) = 0.1977 - 0.1211 = 0.0766$. This is satisfactorily close to the actual value of $P(X = 22)$ which is 0.0744.

Q39: Consider the following experiment. 60 students sat for an examination which has the pass percentage of 75%. Let X be the number of students who will pass the examination. Answer the following equations based on the above description.

a) What distribution does X follows?

b) Find the average of X.

c) Find the variance of X.

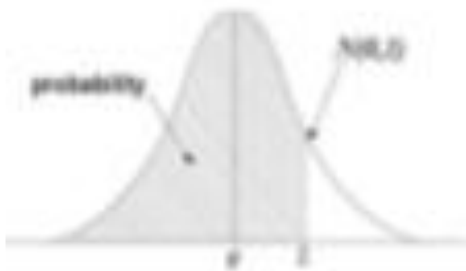
d) What is the normal distribution that can be used to approximate X?

e) What is the probability that at least 40 students will pass the examination?

f) What is the minimum no of students need to be sat for the examination, in order to make an approximation using a normal distribution?

(Ans: $X \sim Bin(60,0.75)$, 45, 11.25, $Y \sim Nor(45,11.25)$, 0.9494, 54)

The Standard Normal Distribution Table



The distribution tabulated is that of the normal distribution with mean **zero** and standard deviation **1**. For each value of $\boldsymbol{Z}$, the standardized normal deviate, (the proportion $\boldsymbol{P}$, of the distribution less than $\boldsymbol{Z}$) is given. For a normal distribution with mean μ and variance σ^2 the proportion of the distribution less than some particular value $\boldsymbol{X}$ is obtained by calculating $\boldsymbol{Z} = (X - \mu)/\sigma$ and reading the proportion corresponding to this value of $\boldsymbol{Z}$.

Z	*P*	*Z*	*P*	*Z*	*P*
-4.00	0.00003	-1.00	0.1587	1.05	0.8531
-3.50	0.00023	-0.95	0.1711	1.10	0.8643
-3.00	0.0014	-0.90	0.1841	1.15	0.8749
-2.95	0.0016	-0.85	0.1977	1.20	0.8849
-2.90	0.0019	-0.80	0.2119	1.25	0.8944
-2.85	0.0022	-0.75	0.2266	1.30	0.9032
-2.80	0.0026	-0.70	0.2420	1.35	0.9115
-2.75	0.0030	-0.65	0.2578	1.40	0.9192
-2.70	0.0035	-0.60	0.2743	1.45	0.9265
-2.65	0.0040	-0.55	0.2912	1.50	0.9332
-2.60	0.0047	-0.50	0.3085	1.55	0.9394
-2.55	0.0054	-0.45	0.3264	1.60	0.9452
-2.50	0.0062	-0.40	0.3446	1.65	0.9505
-2.45	0.0071	-0.35	0.3632	1.70	0.9554
-2.40	0.0082	-0.30	0.3821	1.75	0.9599
-2.35	0.0094	-0.25	0.4013	1.80	0.9641
-2.30	0.0107	-0.20	0.4207	1.85	0.9678
-2.25	0.0122	-0.15	0.4404	1.90	0.9713
-2.20	0.0139	-0.10	0.4602	1.95	0.9744
-2.15	0.0158	-0.05	0.4801	2.00	0.9772
-2.10	0.0179	0.00	0.5000	2.05	0.9798
-2.05	0.0202	0.05	0.5199	2.10	0.9821
-2.00	0.0228	0.10	0.5398	2.15	0.9842
-1.95	0.0256	0.15	0.5596	2.20	0.9861
-1.90	0.0287	0.20	0.5793	2.25	0.9878
-1.85	0.0322	0.25	0.5987	2.30	0.9893
-1.80	0.0359	0.30	0.6179	2.35	0.9906
-1.75	0.0401	0.35	0.6368	2.40	0.9918
-1.70	0.0446	0.40	0.6554	2.45	0.9929
-1.65	0.0495	0.45	0.6736	2.50	0.9938
-1.60	0.0548	0.50	0.6915	2.55	0.9946
-1.55	0.0606	0.55	0.7088	2.60	0.9953
-1.50	0.0668	0.60	0.7257	2.65	0.9960
-1.45	0.0735	0.65	0.7422	2.70	0.9965
-1.40	0.0808	0.70	0.7580	2.75	0.9970
-1.35	0.0885	0.75	0.7734	2.80	0.9974
-1.30	0.0968	0.80	0.7881	2.85	0.9978
-1.25	0.1056	0.85	0.8023	2.90	0.9981
-1.20	0.1151	0.90	0.8159	2.95	0.9984
-1.15	0.1251	0.95	0.8289	3.00	0.9986
-1.10	0.1357	1.00	0.8413	3.50	0.99977
-1.05	0.1469			4.00	0.99997