MAE 598 Analysis and Modeling of Fluid Flows

Final Project Report

Instructor: Professor Jeonglae Kim

"Finite-Time Lyapunov Exponent analysis of Unsteady Gyre Flow"

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1 Abstract

This project investigates the dynamic behavior of fluid flows using Finite-Time Lyapunov Exponents (FTLE), a method pivotal in identifying Lagrangian Coherent Structures (LCS) in unsteady gyre flows. The aim is to utilize MATLAB, to develop a simulation to compute FTLE fields based on the parameters defining the double-gyre flow model, specifically focusing on the effects of amplitude and frequency of the gyre oscillations on particle trajectories and mixing patterns. The integration was performed using a fourth-order Runge-Kutta method, ensuring accurate and efficient computation of particle trajectories over a set integration time. The FTLE fields were visualized to reveal the presence of LCS, which are crucial for understanding material transport and mixing in fluid dynamics. The results indicate significant correlations between the gyre flow parameters and the complexity of the LCS, offering insights into how variations in these parameters impact the behavior of fluid flows. This study not only enhances the understanding of fluid dynamics in theoretical and applied contexts but also demonstrates the robust capabilities of numerical simulations in studying complex systems.

2 Introduction

Understanding the behavior of fluid flows is fundamental in various fields such as oceanography, meteorology, and engineering. The ability to predict and analyze fluid motion is crucial for applications ranging from weather forecasting and environmental science to designing efficient hydraulic systems and improving aerodynamic performance. Among the various tools available for analyzing fluid dynamics, Finite-Time Lyapunov Exponents (FTLE) have emerged as a powerful method to identify and visualize the dynamic structures within fluid flows, known as Lagrangian Coherent Structures (LCS).

LCS are the backbone of fluid flow dynamics, often acting as barriers or facilitators to the transport of mass, energy, and momentum within a fluid system. By delineating regions of the flow where nearby particle trajectories diverge significantly over a finite time interval, FTLE provides a quantitative measure of flow separation and mixing. This is particularly useful in unsteady or time-dependent flows, where traditional Eulerian methods may fall short.

The gyre model, a paradigmatic example in fluid dynamics, exhibits complex behavior typical of larger, more chaotic systems. This project employs a double-gyre flow model to simulate unsteady behavior typical of oceanic and atmospheric flows. By investigating

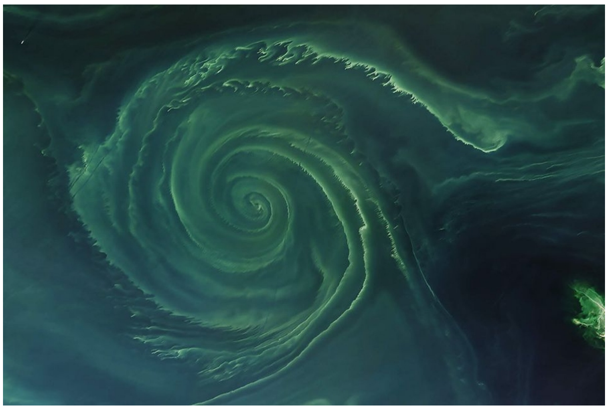


Figure 1: Oceanic Gyres oberved from satellite

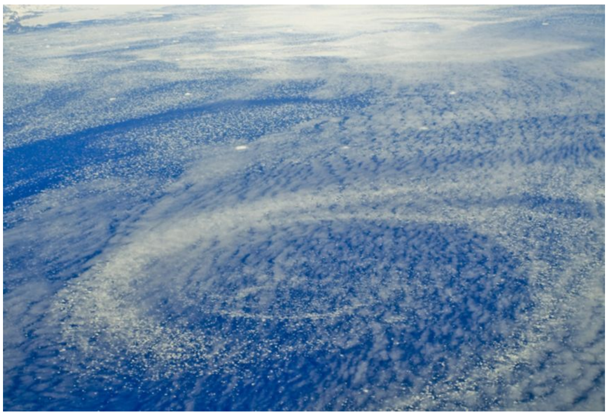


Figure 2: Gyres in the atmosphere

this model using FTLE, the project aims to uncover the underlying mechanisms of particle separation and highlight the impact of flow parameters—specifically the amplitude and frequency of gyre oscillations—on the formation and evolution of LCS.

This report documents the methodology employed to simulate the double-gyre flow, the use of FTLE for analyzing the flow, and the interpretation of results obtained from MATLAB simulations. Through this study, we seek not only to advance understanding of gyre dynamics but also to demonstrate the applicability of FTLE in broader fluid dynamics research.

3 Methodology

This section outlines the comprehensive methods utilized to analyze the dynamics of fluid flows using Finite-Time Lyapunov Exponents (FTLE). It includes the description of the physical model used, the numerical methods for solving the model, and the implementation of these methods in MATLAB.

3.1 Description of the Physical Model

The study utilizes the double-gyre flow model, a well-established model in fluid dynamics used to represent idealized, large-scale oceanic and atmospheric circulations. This model is characterized by two counter-rotating gyres, which are typical features in geophysical fluid dynamics. The flow is described by the streamfunction:

$$\psi(x, y, t) = A \sin(\pi f(x, t)) \sin(\pi y)$$

where

$$f(x, t) = \epsilon \sin(\omega t)x^2 + (1 - 2\epsilon \sin(\omega t))x$$

Here, A represents the amplitude of the gyre, set to 0.1; ϵ, controlling the nonlinearity of the flow, is 0.25; and ω, the frequency of gyre oscillations, is $2\pi/10$. These parameters are crucial as they directly influence the dynamics and complexity of the flow, affecting the formation and behavior of Lagrangian Coherent Structures (LCS) within the fluid.

3.2 Numerical Methods

3.2.1 Integration Method

The trajectories of fluid particles are computed using the fourth-order Runge-Kutta (RK4) integration method. This method provides a good balance between computational efficiency and accuracy, making it suitable for solving the ordinary differential equations derived from the velocity field of the double-gyre model. The RK4 method updates particle positions using a weighted average of four increments, where each increment considers the velocity at a different point in the interval:

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = \Delta t \cdot f(t_n, y_n), \quad k_2 = \Delta t \cdot f\left(t_n + \frac{\Delta t}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_3 = \Delta t \cdot f\left(t_n + \frac{\Delta t}{2}, y_n + \frac{k_2}{2}\right), \quad k_4 = \Delta t \cdot f(t_n + \Delta t, y_n + k_3)$$

3.2.2 Finite-Time Lyapunov Exponents (FTLE)

FTLE fields are calculated to identify and visualize the LCS in the fluid flow. This involves computing the largest eigenvalue of the Cauchy-Green strain tensor, derived from the flow map gradient, over a finite time interval. The tensor quantifies the deformation of the fluid particles, with higher eigenvalues indicating regions of significant stretching and hence potential LCS locations.

3.3 Implementation

The simulation was implemented in MATLAB, leveraging its robust matrix operations and visualization capabilities. The main steps in the implementation are:

1. Initialization: Define the grid and initial conditions for the particle positions.

2. Vector Field Calculation: Use the `doublegyre` function to compute the velocity field at each timestep.

3. Trajectory Computation: Integrate particle trajectories using the `runge_kutta` function, updating positions based on the velocity field.

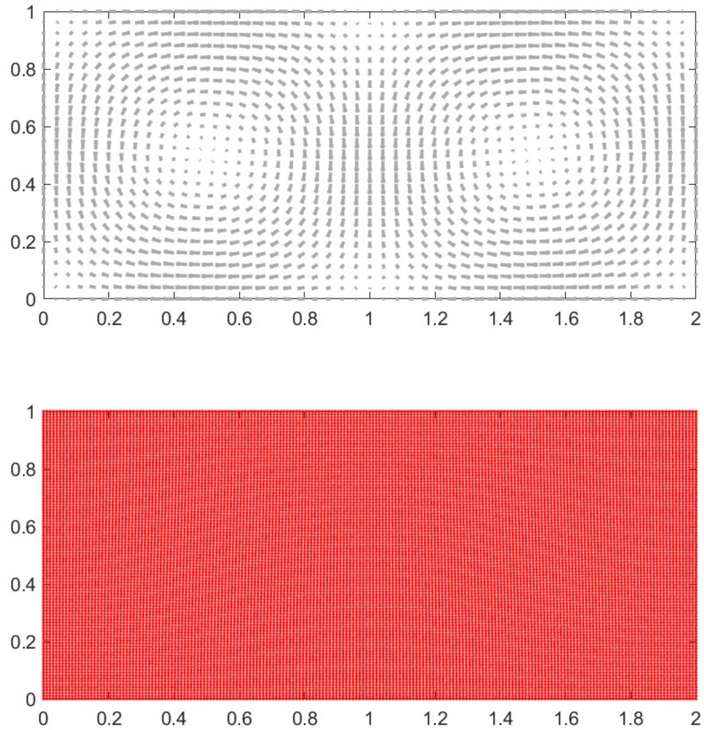


Figure 3: Grid Initiation

4. FTLE Calculation: After computing the trajectories, reshape the output and calculate the gradient of the flow map to determine the FTLE values.

5. Visualization: Generate plots to visualize the vector field, particle trajectories, and FTLE fields using built-in MATLAB functions.

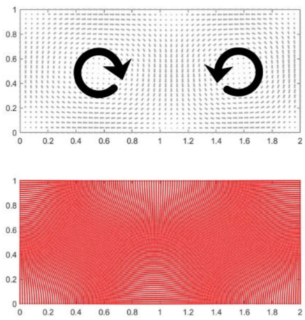
This methodology section aims to provide a detailed description of the tools and techniques used in the analysis, ensuring that the study can be replicated and verified by others in the field.

4 Results

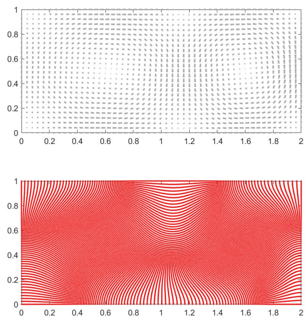
The results from the simulation provide insightful visualization and quantification of the dynamics within the double-gyre flow system, applying both forward and backward integration techniques. The analysis was conducted using MATLAB, which facilitated the computation and visualization of the velocity fields, particle trajectories, and FTLE fields. This section discusses the key findings and visualizations produced during the study.

4.1 Visualization of Vector Fields and Particle Trajectories

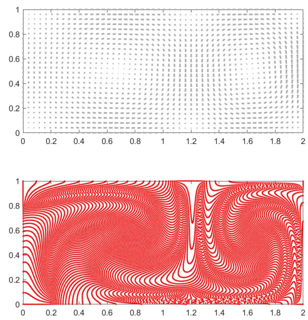
The initial setup of the vector field displays a uniform grid of vectors, which represent the velocity field at the start of the simulation (t=0). The direction and magnitude of



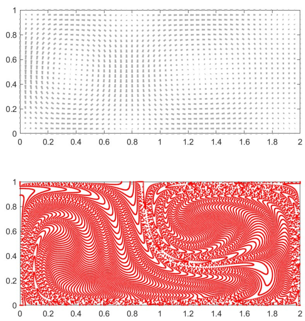
(a) At time t = 0



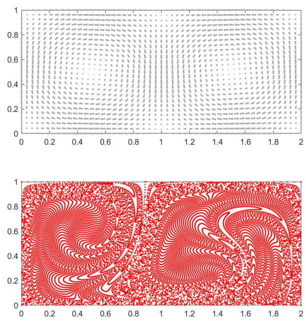
(b) At time t = 0.2t



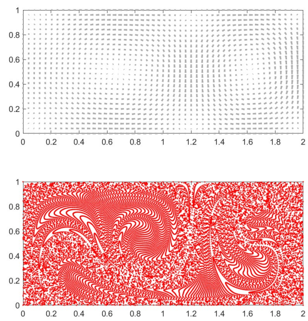
(c) At time t = 0.4t



(d) At time t = 0.6t



(e) At time t = 0.75t



(f) At time t = 0.9t

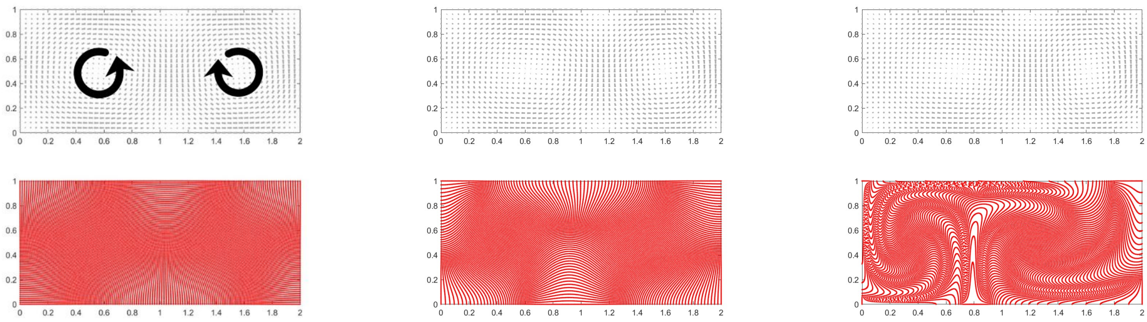
Figure 4: Grid Simulation using Forward Integration

these vectors are indicative of the gyre movements influenced by the specified parameters of amplitude and frequency. Subsequent figures show the evolution of these vectors over time, demonstrating the dynamic nature of the gyre flow.

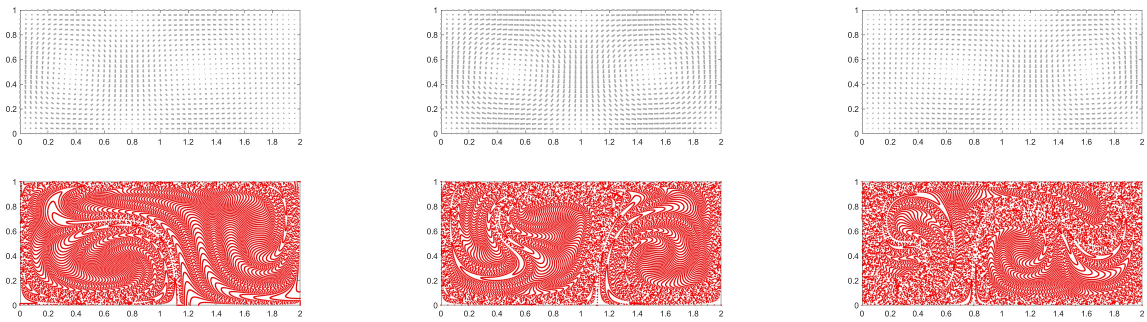
For both forward and backward integration methods, the particle trajectories reveal distinct behaviors:

- **Forward Integration:** Particle trajectories highlight the stretching and folding processes typical of chaotic systems. Over time, particles initially placed in a structured grid pattern increasingly distort, aligning along the ridges of high FTLE values. These trajectories help identify the regions where fluid particles diverge significantly, indicating sensitivity to initial conditions.

- **Backward Integration:** This technique often highlights converging regions, where particles from different initial locations come together. The trajectories tend to converge into the troughs between high FTLE ridges, suggesting areas of relative stability in the flow field.



(a) At time t = 0 (b) At time t = 0.2t (c) At time t = 0.4t



(d) At time t = 0.6t (e) At time t = 0.75t (f) At time t = 0.9t

Figure 5: Grid Simulation using Backward Integration

4.2 Finite-Time Lyapunov Exponent (FTLE) Fields

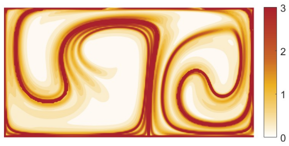
The FTLE fields calculated from both integration methods effectively illustrate the locations and structures of Lagrangian Coherent Structures (LCS):

- **Forward Integration FTLE Fields:** The high values of FTLE, visualized in warm colors (red to yellow), delineate the regions of the flow where there is significant divergence of particle trajectories. These areas represent barriers to transport within the fluid, where material lines stretch extensively.
- **Backward Integration FTLE Fields:** Conversely, areas of lower FTLE values, shown in cool colors (blue shades), correspond to regions of convergence. These are often associated with attracting material lines where particles come closer together.

The contrasting results from the two integration methods provide a comprehensive understanding of the flow dynamics. Forward integration highlights barriers that inhibit particle crossing, essential for understanding mixing and boundary generation in fluid flows. Backward integration, on the other hand, helps in identifying attracting regions which are crucial for predicting areas of particle accumulation.

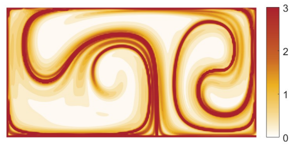
4.3 Interpretation of FTLE Results

The results from the FTLE analysis using both integration directions provide a dual perspective on the flow behavior:



(a) FTLE field at 0.2t

(b) FTLE field at 0.4t



(c) FTLE field at 0.6t

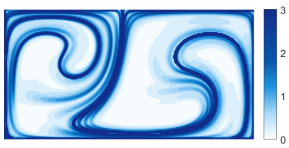
(d) FTLE field at 0.8t

(e) FTLE field at time 0.9t

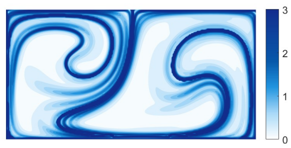
(f) FTLE field at time t

Figure 6: FTLE Field Simulation using Forward Integration

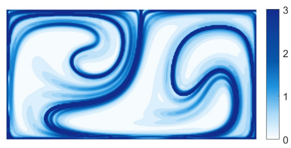
- The forward integration results are crucial for applications where barrier detection is necessary, such as environmental containment and pollution control.



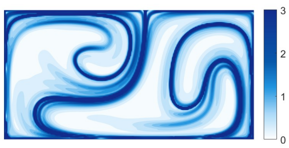
(a) FTLE field at 0.2t



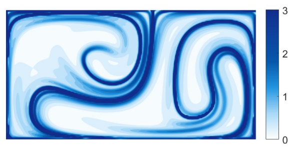
(b) FTLE field at 0.4t



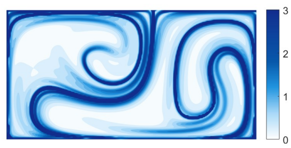
(c) FTLE field at 0.6t



(d) FTLE field at 0.8t



(e) FTLE field at time 0.9t



(f) FTLE field at time t

Figure 7: FTLE Field Simulation using Backward Integration

- The backward integration results are particularly useful for applications involving aggregation phenomena, such as predicting areas where pollutants or biological materials collect.

These findings underscore the utility of FTLE in revealing intricate details of fluid dynamics, which are often not apparent through conventional flow visualization techniques alone. The ability to visualize and quantify the dynamic separation and convergence within the flow enhances our understanding of fluid behavior in natural and industrial applications. The video/gif animations of the said analyses are found here.

4.4 Tracking of Particle Sets in Forward and Backward Integration

The study utilized forward and backward integration techniques to track the evolution of specific sets of particles within the gyre flow, which provided unique insights into the dynamics of particle movement and the influence of integration direction on the interpretation of flow behavior.

4.4.1 Forward Integration Particle Tracking

In forward integration, particles were initialized in a uniform grid and tracked as they moved through the evolving vector field. This approach primarily highlighted the repulsive characteristics of the Lagrangian Coherent Structures (LCS). Particles increasingly diverged along the LCS, delineating areas of high FTLE values. These regions represent zones of significant fluid stretching and are crucial for understanding the barriers that influence fluid mixing and transport. The visualization captured the dynamic separation of particles, providing a vivid depiction of how materials and pollutants might spread in real fluid environments.

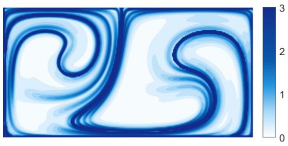
4.4.2 Backward Integration Particle Tracking

Conversely, backward integration provided a contrasting perspective by focusing on the convergence of particles. This method traced particles as they moved towards each other, converging in regions characterized by lower FTLE values. These areas potentially indicate zones of fluid stability where particles are likely to accumulate over time, offering valuable insights for applications concerned with the gathering of materials, such as environmental cleanup and pollutant containment. The backward integration effectively mapped out the attracting LCS, showcasing regions where fluid particles are drawn together, thus aiding in predicting zones of high particle density.

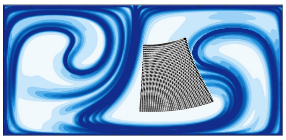
4.4.3 Visualization and Analysis

Both integration methods were visualized using MATLAB, which allowed for the real-time observation of particle trajectories. The figures generated provided a clear and detailed view of the flow dynamics, with distinct color maps enhancing the differentiation between high and low FTLE regions. These visualizations not only supported the quantitative analysis but also facilitated a more intuitive understanding of the flow characteristics.

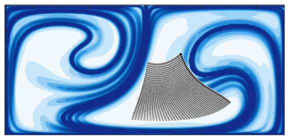
The tracking of particles in both directions proved to be instrumental in offering a comprehensive analysis of the gyre model. It highlighted the dual nature of fluid flows, capturing both the chaotic and orderly phenomena that occur within a dynamic system. This balanced view is essential for applications requiring detailed knowledge of fluid behavior under various conditions.



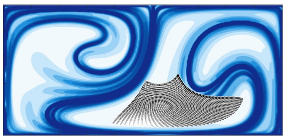
(a) Partciles at time = 0



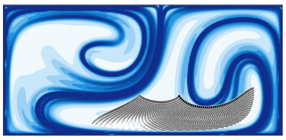
(b) Particles at 0.2t



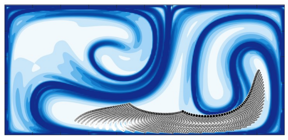
(c) Particles at 0.4t



(d) Particles at 0.6t



(e) Particles at time 0.8t



(f) Particles at time t

Figure 8: FTLE Field Particle-set Tracking

5 Discussion

This section evaluates the implications of the results obtained from the Finite-Time Lyapunov Exponent (FTLE) analysis of the unsteady gyre flow using both forward and backward integration techniques. The interpretation of these results, the limitations of the current study, and potential future research directions are discussed.

5.1 Interpretation of Results

The FTLE fields generated from the MATLAB simulations have revealed significant insights into the behavior of the double-gyre flow. The forward integration primarily illuminated the repulsive Lagrangian Coherent Structures (LCS), which act as barriers to fluid transport. These structures are critical in understanding the mechanisms of mixing and barrier formation in fluid dynamics, particularly in oceanographic and atmospheric contexts where such phenomena can influence large-scale environmental processes.

Conversely, backward integration highlighted the attractive LCS, which are indicative of regions where fluid particles converge. These areas could be critical in applications such as environmental science, where understanding the accumulation points of pollutants or biological materials is necessary for effective management and remediation strategies.

The dual analysis approach, utilizing both forward and backward integrations, offers a comprehensive view of the flow dynamics, providing a more complete picture of the fluid behavior than would be possible through a single method of analysis.

5.2 Limitations

While the study has produced valuable insights, several limitations are noteworthy:

- **Resolution Dependency:** The resolution of the particle grid significantly impacts the accuracy of the FTLE calculations. Higher resolutions can provide more detailed insights but at the cost of increased computational demand.

- **Model Simplifications:** The double-gyre model, while useful in simulating general gyre behavior, is an idealization and does not account for numerous real-world factors such as variable topography, inhomogeneous fluid properties, or external forcings.

- **Numerical Errors:** The Runge-Kutta integration method, though robust, introduces numerical errors, particularly in areas of the flow with high divergence or convergence, which can affect the accuracy of the FTLE calculations.

5.3 Future Work

To build upon the current study and address its limitations, several avenues for future work are suggested:

- **Model Enhancement:** Incorporating more complex physical factors into the model, such as non-uniform depth and external forces, could provide a more realistic simulation of environmental fluid flows.
- **Algorithm Optimization:** Exploring advanced numerical integration techniques that reduce errors and computational costs could enhance the feasibility of conducting high-resolution studies.
- **Application to Real-World Data:** Applying the FTLE analysis framework to real-world fluid flow data, such as ocean current data from satellite measurements or atmospheric data from weather models, could validate the utility of this method in practical scenarios.
- **Interdisciplinary Studies:** Collaborating with environmental scientists or engineers to apply FTLE analysis to specific challenges, such as pollution dispersion or ecosystem modeling, could broaden the impact of this research.

By addressing these limitations and exploring these suggested future directions, the utility of FTLE analysis in understanding and predicting fluid dynamics can be significantly enhanced.

6 Conclusion

The study conducted has significantly contributed to the analysis and understanding of unsteady gyre flows through the application of Finite-Time Lyapunov Exponent (FTLE) methodology. The dual approach of utilizing both forward and backward integration techniques in MATLAB has effectively highlighted the intricate structures within the flow—identifying both repulsive and attractive Lagrangian Coherent Structures (LCS).

From the forward integration, it is clear that LCS act as dynamic barriers that can influence the mixing and boundary formations in fluid flows. This knowledge is pivotal for the prediction and control of transport phenomena in various fluid dynamic applications. The backward integration complements this by mapping the convergence zones, areas critical to understanding where particles or substances may accumulate over time.

The study underscores the robust capability of numerical simulation tools like MATLAB in providing valuable insights into complex dynamical systems. It also illustrates the importance of Lagrangian methods in fluid mechanics, where Eulerian approaches may not adequately capture the essence of fluid transport.

While the model and methods employed in this study have limitations—chief among them being the simplification of the gyre model and the resolution constraints—the insights gained offer a solid foundation for future investigations. It is anticipated that subsequent research will build upon these findings, incorporating more complex physical phenomena and utilizing more sophisticated computational techniques.

In conclusion, this project not only furthers our comprehension of gyre dynamics and FTLE analysis but also opens avenues for future interdisciplinary research. The methodologies refined and the results obtained will serve as a valuable reference for studies aimed at solving practical problems in environmental science, engineering, and beyond.

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