# Lecture 13

## 2.3. Basis and Dimension (cont.)

### Dimension of vector space of solutions to a homogenous ODE

**Theorem** (Existence and uniqueness of solns to linear ODE's).

Let

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y^{(1)} + a_0(x)y = 0 \quad (*)$$

be a linear homogenous ODE on an interval $I$, where $a_0(x), \cdots, a_{n-1}(x)$ are continuous on $I$.

Let $x_0 \in I$, and:

initial conditions
$$\begin{cases} y(x_0) = c_0 \\ y^{(1)}(x_0) = c_1 \\ \vdots \\ y^{(n-1)}(x_0) = c_{n-1} \end{cases} \quad (**)$$

where $c_0, \cdots, c_{n-1}$ are arbitrary constants. Then there <u>exists</u> a <u>unique</u> soln to the ODE $(*)$ satisfying the initial conditions $(**)$.

Proof - won't do.

**Corollary** Let $V$ be the vector space of solns to an $n^{th}$ order homogenous linear ODE on an interval $I$,

$$\boxed{y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y^{(1)} + a_0(x)y = 0}$$

where $a_1(x), \cdots, a_{n-1}(x)$ are continuous on $I$. Then

$$\mathrm{Dim}(V) = n.$$

<u>Proof</u> Choose $x_0 \in I$. By the <u>excistence</u> part of prev. Theorem, we know that there exist

$$y_0, \cdots, y_{n-1} \in V$$

Satisfying:

$$y_i^{(j)}(x_0) = \delta_{ij} \qquad i,j = 1 \ldots n-1$$

| $y_0$ | $y_1$ | | $y_{n-1}$ |
|---|---|---|---|
| $y_0 \in V$ | $y_1 \in V$ | $\cdots$ | $y_{n-1} \in V$ |
| $y_0(x_0) = 1$ | $y_1(x_0) = 0$ | | $y_{n-1}(x_0) = 0$ |
| $y_0'(x_0) = 0$ | $y_1^{(1)}(x_0) = 1$ | | $y_{n-1}^{(1)}(x_0) = 0$ |
| $\vdots$ | $\vdots$ | | $\vdots$ |
| $y_0^{(n-1)}(x_0) = 0$ | $y_1^{(n-1)}(x_0) = 0$ | | $y_{n-1}^{(n-1)}(x_0) = 1$ |

<u>Claim</u> $\{y_0, \cdots, y_{n-1}\}$ is a basis for $V$.

<u>$\{y_0, \cdots, y_{n-1}\}$ is linearly independent</u>:

Suppose

$$\begin{cases} ⓪ & k_0 y_0 + \cdots + k_{n-1} y_{n-1} = 0 \\ ① & k_0 y_0' + \cdots + k_{n-1} y_{n-1}' = 0 \\ & \vdots \\ (n-1) & k_0 y_0^{(n-1)} + \cdots + k_{n-1} y_{n-1}^{(n-1)} = 0 \end{cases}$$

Evaluating ⓪ at $x = x_0$:

$$k_0 \underbrace{y_0(x_0)}_{1} + k_1 \underbrace{y_1(x_0)}_{0} + \cdots + k_{n-1} \underbrace{y_{n-1}(x_0)}_{0} = 0$$

$\therefore \; k_0 = 0$

Evaluating ① at $x_0$:

$$k_0 \underbrace{y_0'(x_0)}_{=0} + k_1 \underbrace{y_1'(x_0)}_{=1} + \cdots + k_{n-1} \underbrace{y_{n-1}^{(1)}(x_0)}_{=0} = 0$$

$$\therefore \quad k_1 = 0$$

Similarly, $k_2 = 0, \cdots, k_{n-1} = 0$.

$\therefore \{y_0, \cdots, y_{n-1}\}$ is lin. ind.

## $\{y_0, \cdots, y_{n-1}\}$ spans $V$

Let $y \in V$. Let

$$c_0 := y(x_0)$$
$$c_1 := y^{(1)}(x_0)$$
$$\vdots$$
$$c_{n-1} := y^{(n-1)}(x_0)$$

Let

$$f = c_0 y_0 + c_1 y_1 + \cdots + c_{n-1} y_{n-1}$$

Claim : $f = y$.

We know $f$ solves ODE. And:

$$f^{(0)}(x_0) = c_0 \quad \cdots \quad f^{(n-1)}(x_0) = c_{n-1}$$
$$f^{(1)}(x_0) = c_1$$

$\therefore$ By the uniquness part of prev Thm, we must have

$$f = y.$$

$\therefore \{y_0, \cdots, y_{n-1}\}$ spans $V$.

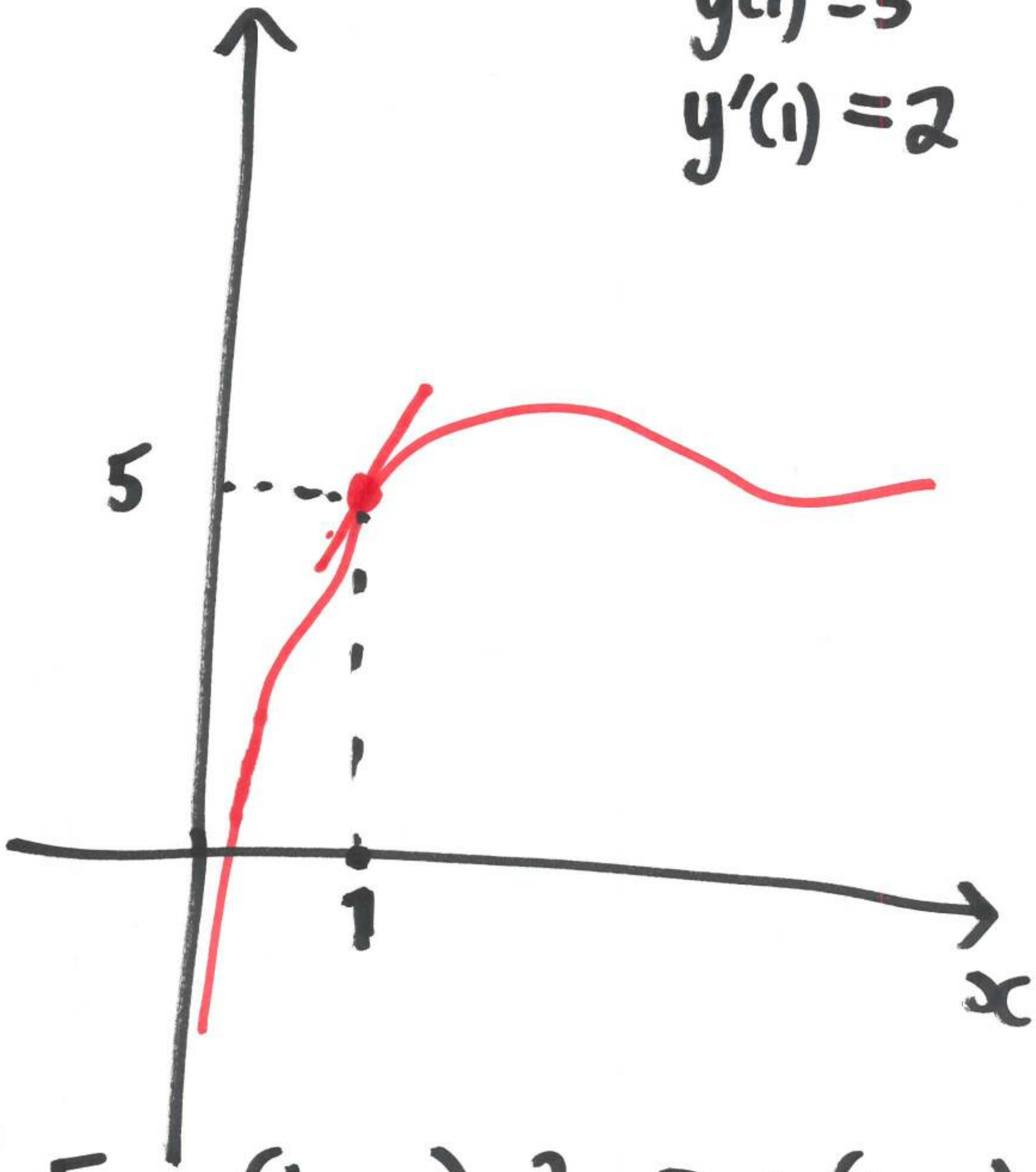
$\therefore$ a basis

$\therefore \operatorname{Dim}(V) = n.$

$\square$

$$y'' - \frac{3}{x} y' + \frac{5}{x^2} y = 0$$

$x \in (0,\infty)$




$y(1) = 5$

$y'(1) = 2$

$$y = 5\cos(\log x)x^2 - 3\sin(\log x)x^2$$

$y_0(1) = 1$

$y_0'(1) = 0$

$y_0 = x^2\left(\cos(\log x) - 2\sin(\log x)\right)$

$y_1(1) = 0$

$y_1'(1) = 1$

$y_1 = x^2 \sin(\log x).$

$$\boxed{y'' - \frac{3}{x} y' + \frac{5}{x^2} y = 0}$$