Lec 18

Prop. (Sufficient to define a linear map on basis)

Proof (cont.)

Existence

$$T(\vec{v}) := \sum_{i=1}^{n} [\vec{v}]_{\beta,i} \vec{w_i}$$

eg.

$$\underbrace{T(2\vec{e_1}+3\vec{e_2})}_{T(\vec{v})} = 2\vec{w_1} + 3\vec{w_2}$$

Linear ?

1. $T(\vec{v}+\vec{v}')$

$$= \sum_{i=1}^{n} [\vec{v}+\vec{v}']_{B,i} \vec{w_i} \quad \left(\text{defn of } T(\vec{v}+\vec{v}')\right)$$

use prev Lemma:

- $[\vec{v}+\vec{v}']_B = [\vec{v}]_B + [\vec{v}']_B$
- $[k\vec{v}]_B = k[\vec{v}]_B$

$$= \sum_{i=1}^{n} \left([\vec{v}]_{B,i} + [\vec{v}']_{B,i}\right) \vec{w_i}$$

$$= \sum_{i=1}^{n} [\vec{v}]_{B,i} \vec{w_i} + \sum_{i=1}^{n} [\vec{v}']_{B,i} \vec{w_i}$$

$$= T(\vec{v}) + T(\vec{v}')$$

2. $T(k\vec{v})$

$$= \sum_{i=1}^{n} [k\vec{v}]_{B,i} \vec{w_i}$$

$$= \sum_{i=1}^{n} k[\vec{v}]_{B,i} \vec{w_i}$$

$$= k \sum_{i=1}^{n} [\vec{v}]_{B,i} \vec{w_i}$$

$$= k T(\vec{v})$$

$\therefore$ T exists !

<u>Uniqueness</u> Suppose

$$T : V \longrightarrow W$$

is an <u>arbitrary</u> linear map s.t.

$$T(\vec{e_i}) = \vec{w_i}$$

Then, for all $\vec{v} \in V$,

$$T(\vec{v}) = T\left(\sum_{i=1}^{n} [\vec{v}]_{B,i}\, \vec{e_i}\right) \quad \left(\begin{array}{c}\text{Expand}\\ \vec{v}\\ \text{w.r.t. } B\end{array}\right)$$

$$= \sum_{i=1}^{n} [\vec{v}]_{B,i}\, T(\vec{e_i}) \quad \left(\begin{array}{c}T \text{ is}\\ \text{linear}\end{array}\right)$$

$$= \sum_{i=1}^{n} [\vec{v}]_{B,i}\, \vec{w_i} \quad \left(\begin{array}{c}T(\vec{e_i})\\ = \vec{w_i}\end{array}\right)$$

So, if

$$S : V \longrightarrow W$$

is <u>another</u> linear map s.t.

$$S(\vec{e_i}) = \vec{w_i},$$

then also necessarily

$$S(\vec{v}) = \sum_{i=1}^{n} [\vec{v}]_{B,i} \vec{w_i} \quad \forall \vec{v} \in V.$$

$\therefore S = T.$ $\square$

<u>Example</u> Define a linear map

$$T : \mathrm{Col}_2 \longrightarrow \mathrm{Fun}(\mathbb{R})$$

by using

$$\mathcal{B} = \left\{ \underset{\vec{e}_1}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}, \underset{\vec{e}_2}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} \right\}$$

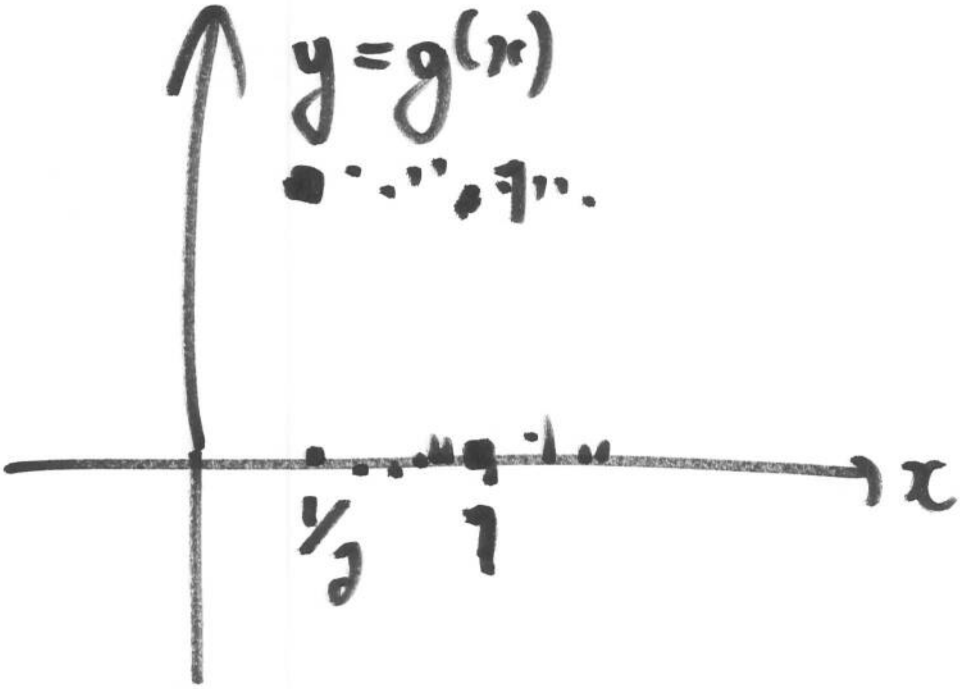
basis for Col_2

Set

$$T(\vec{e}_1) = f$$

$$T(\vec{e}_2) = g$$

where $f(x) = |x|$, $g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$



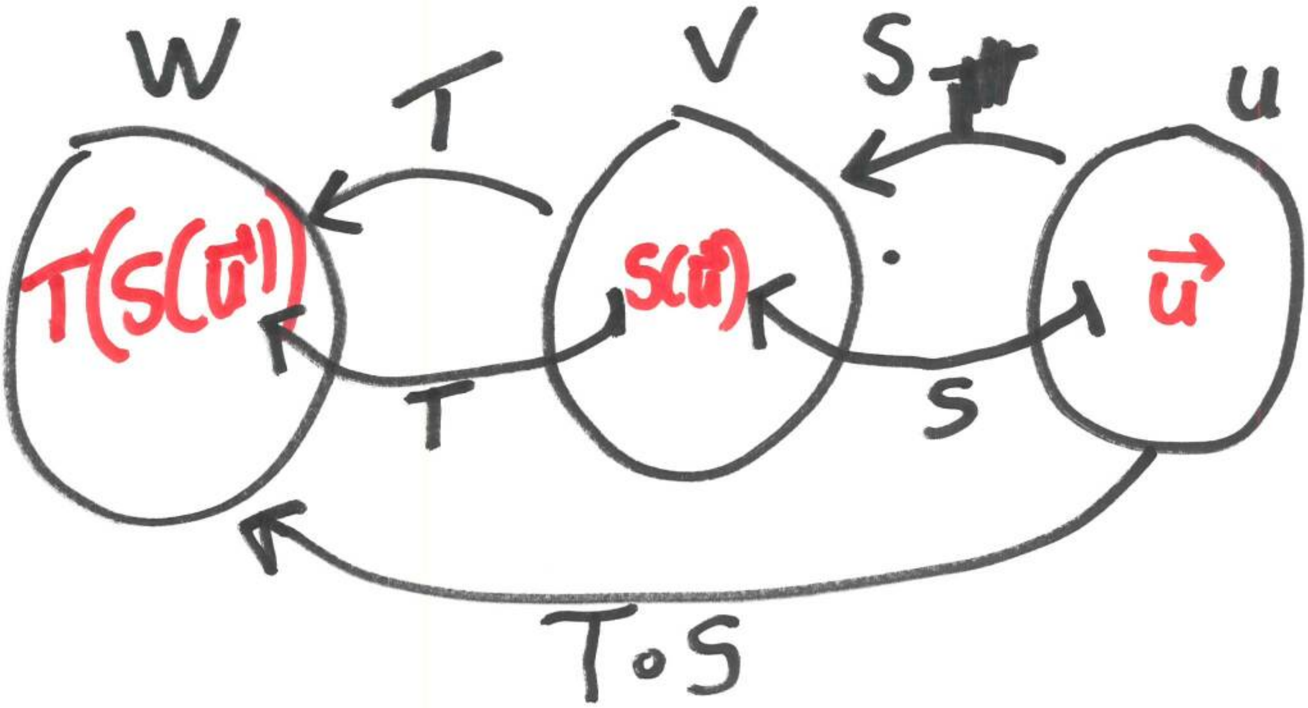
y = g(x)
1/2
1
x

3.2. Composition of linear maps

Defn Let $S : U \to V$ and $T : V \to W$ be linear maps. The <u>composite</u> $T \circ S$ is defined by

"T after S" $T \circ S : U \longrightarrow W$

$$\vec{u} \longmapsto T(S(\vec{u})).$$



Proposition $T \circ S$ is a linear map.

Proof

1. $(T \circ S)(\vec{u} + \vec{u}')$

$= T(S(\vec{u} + \vec{u}'))$ [defn of $T \circ S$]

$= T(S(\vec{u}) + S(\vec{u}'))$ [S is linear]

$= T(S(\vec{u})) + T(S(\vec{u}'))$ [T is linear]

$= (T \circ S)(\vec{u}) + (T \circ S)(\vec{u}')$

[defn of $T \circ S$]

2. $(T \circ S)(k\vec{u}) = k(T \circ S)(\vec{u})$

similar.

$\square$

3.3. Isomorphisms

<u>Defn</u> A linear map

$$T : V \longrightarrow W$$

is called an <u>isomorphism</u> if there exists a linear map

$$T^{-1} : W \to V$$

such that

$$T^{-1} \circ T = id_V \, , \quad T \circ T^{-1} = id_W$$

We say that V and W are <u>isomorphic</u> if there exists an isomorphism $T : V \to W$.

Theorem Let V and W be f. dim. vector spaces. Then

V and W are isomorphic $\Longleftrightarrow \operatorname{Dim}(V) = \operatorname{Dim}(W)$

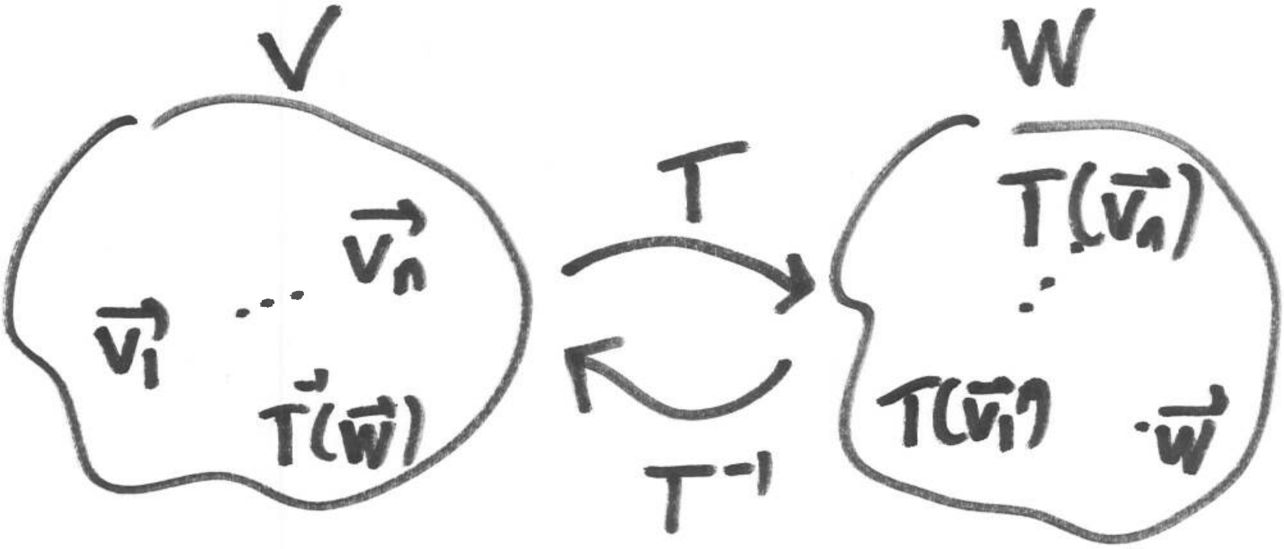
Proof $\Longrightarrow$ Suppose V and W are isomorphic via an isomorphism

$$T : V \longrightarrow W$$

Let $\mathcal{B} = \{\vec{v_1}, \dots, \vec{v_n}\}$ be a basis for V. I claim that

$$C = \{T(\vec{v_1}), \dots, T(\vec{v_n})\}$$

is a basis for W.



① C spans W :

Let $\vec{w} \in W$

$\therefore T^{-1}(\vec{w}) \in V$

$\therefore T^{-1}(\vec{w}) = a_1\vec{v_1} + \cdots + a_n\vec{v_n}$

(B spans V)

$\therefore T(T^{-1}(\vec{w})) = T(a_1\vec{v_1} + \cdots + a_n\vec{v_n})$

$\therefore \underbrace{id_W(\vec{w})}_{\vec{w}} = a_1 T(\vec{v_1}) + \cdots + a_n T(\vec{v_n})$

$\therefore$ C spans W.

(2) <u>C is lin. ind.</u>

Suppose

$$k_1 T(\vec{v_1}) + \cdots + k_n T(\vec{v_n}) = \vec{0}_W$$

$$\therefore T^{-1}\left(k_1 T(\vec{v_1}) + \cdots + k_n T(\vec{v_n})\right) = T^{-1}(\vec{0}_W)$$

$$\therefore k_1 T^{-1} T(\vec{v_1}) + \cdots + k_n T^{-1} T(\vec{v_n}) = \vec{0}_V$$

$$\therefore k_1 \vec{v_1} + \cdots + k_n \vec{v_n} = \vec{0}_V$$

$$\therefore k_1 = 0, \ \ldots, \ k_n = 0$$

[B is lin. ind.]

$\square$