Lecture 10

Last time:

$$\mathrm{Dim}(\mathbb{R}^n) = n$$

<u>Example</u> (Dimension of Poly_n)

Standard basis:

$$p_0 = 1, \; p_1 = x, \; p_2 = x^2, \; \dots, \; p_n = x^n$$

- <u>Spans Poly_n</u>? Let $q \in \mathrm{Poly}_n$,

i.e.
$$q = q_0 + q_1 x + q_2 x^2 + \dots + q_n x^n$$

We can write

$$q = \boxed{q_0} p_0 + \boxed{q_1} p_1 + \boxed{q_2} p_2$$

$$+ \cdots + \boxed{q_n} p_n$$

$\therefore$ list $\{p_0, \cdots, p_n\}$ spans $Poly_n$.

<u>lin. ind.</u> Suppose

$$a_0 p_0 + a_1 p_1 + \cdots + a_n p_n = 0$$

$$\therefore a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0$$
$$\forall x \in \mathbb{R}$$

Fundamental theorem of Algebra: Every polynomial of order n has at most n roots.

$\therefore$ only the trivial soln $a_0 = 0, \cdots, a_n = 0$

$\therefore \{p_0, \dots, p_n\}$ is lin. ind.

$\therefore$ a basis for $Poly_n$

$\therefore Dim(Poly_n) = n+1$

<u>Example</u> (Dimension of $Trig_n$)

Standard basis :

$T_0 = 1$, $T_1 = \cos x$, $T_2 = \sin x$,

$T_3 = \cos 2x$, $T_4 = \sin 2x$,

$\dots, T_{2n-1} = \cos nx$, $T_{2n} = \sin nx$

- spans $Trig_n$ (by defn)
- lin ind. (won't do) $\therefore$ a basis

$\therefore Dim(Trig_n) = 2n+1$

Example (Dimension of $\mathrm{Mat}_{n,m}$)

$$\mathrm{Mat}_{n,m} = \left\{ \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \\ a_{n1} & \cdots & a_{nm} \end{bmatrix} \right\}$$

Standard basis:

$$E^{11} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & & \\ 0 & & \cdots & 0 \end{bmatrix}, E^{12} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & & & 0 \\ \vdots & & & \\ 0 & \cdots & & 0 \end{bmatrix}$$

$$\dots E^{21} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & & & \\ 0 & & \ddots & \\ \vdots & & & \\ 0 & & & 0 \end{bmatrix}, \dots, E^{nm} = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & 0 \\ 0 & \cdots & 1 \end{bmatrix}$$

· <u>Spans $Mat_{n,m}$?</u>

Yes:

$$\begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \\ a_{n1} & \cdots & a_{nm} \end{bmatrix} = a_{11}E^{11} + a_{12}E^{12}$$

$$+ \; a_{13}E^{13} + \cdots + a_{nm}E^{nm}$$

· <u>Lin ind</u>? Suppose

$$a_{11}E^{11} + a_{12}E^{12} + \cdots + a_{nm}E^{nm} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \\ a_{n1} & \cdots & a_{nm} \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$$

$\therefore a_{11} = 0, \; a_{12} = 0,$

$\cdots, a_{nm} = 0$

$\therefore$ list is lin. ind.

$\therefore$ a basis

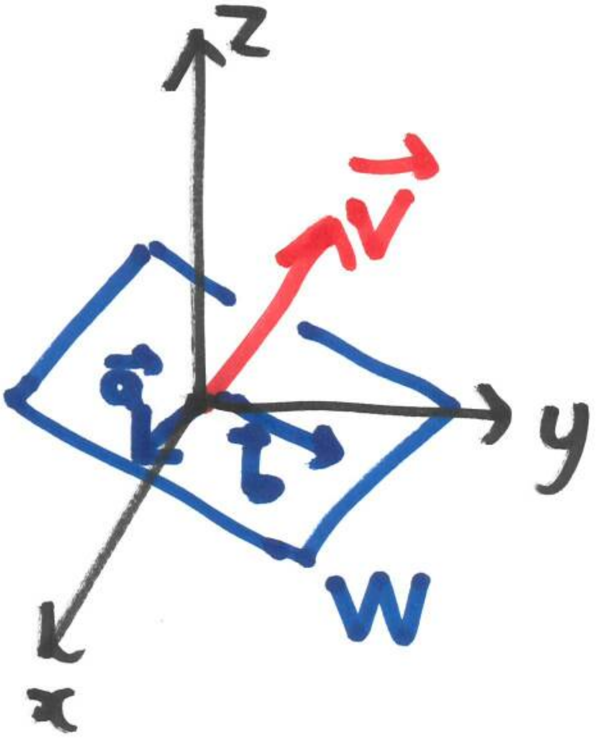
$\therefore \mathrm{Dim}(\mathrm{Mat}_{n,m}) = nm$

<u>Example</u> (hyperplanes)

Fix $\vec{v} \in \mathbb{R}^n$

$$W := \left\{ \vec{w} \in \mathbb{R}^n : \vec{w} \cdot \vec{v} = 0 \right\}$$

eg. $\vec{V} = (1,1,2) \in \mathbb{R}^3$



$$W = \left\{ \vec{w} \in \mathbb{R}^3 : \vec{w} \cdot \vec{v} = 0 \right\}$$

$\mathrm{Dim}(W) = 2.$

In our example, the equation is

$\vec{w} = (w_1, w_2, w_3)$

$\vec{v} = (1,1,2)$

$\vec{w} \cdot \vec{v} = 0 \iff W_1 + W_2 + 2w_3 = 0$

$\vec{a} = (1, 0, -1/2) \in W$

$\vec{b} = (0, 1, -1/2) \in W$

or

$\vec{u} = (2, 3, -5/2)$

$\vec{v} = (-10, 6, 2)$

Check $\{\vec{a}, \vec{b}\}$ is a basis for W.

Lin. ind. Suppose

$$k_1 \vec{a} + k_2 \vec{b} = \vec{0}$$

$$\therefore k_1 (1, 0, -1/2) + k_2 (0, 1, -1/2)$$
$$= (0, 0, 0)$$

$$\therefore \left(k_1, k_2, -1/2 k_1 - 1/2 k_2\right) = (0, 0, 0)$$

$\therefore$ $k_1 = 0$

$k_2 = 0$ $\therefore \dim W = 2.$

$-1/2 k_1 - 1/2 k_2 = 0$

$\therefore$ $k_1 = 0$ and $k_2 = 0$

$\therefore$ list is l. ind.

- spans W ? Exercise!

In general, the dimension of a hyperplane $W \subseteq \mathbb{R}^n$ is

$$\mathrm{Dim}(W) = n-1.$$

<u>Example</u>

$$\mathrm{Col}_n = \left\{ \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \right\}$$

$$= \mathrm{Mat}_{n,1}$$

$$\therefore \mathrm{Dim}(\mathrm{Col}_n) = n.$$

Proposition Let W be a subspace of a finite-dim vector space V. Then W is finite-dimensional, and

$$\mathrm{Dim}(W) \leq \mathrm{Dim}(V).$$

Proof Let $n = \mathrm{Dim}(V)$.

If $W = \{\vec{0}\}$, then $\mathrm{Dim} W = 0$

$$\therefore \ \mathrm{Dim} W \leq n.$$

So, assume $W \neq \{\vec{0}\}$.

$\therefore$ there exists a nonzero vector $\vec{e_1} \in W$.

Let $B_1 = \{\vec{e_1}\}$

Note: B_1 is lin. ind.

If $\mathcal{B}_1$ spans W, we are done.

If not, there exists $\vec{e_2} \in W$ which is not a multiple of $\vec{e_1}$.

Consider

$$B_2 = \{\vec{e_1}, \vec{e_2}\}$$

Note: B_2 is l. ind.

If B_2 spans W, we are done.

If not, w carry on adding vectors to the list.