Section 17 Limits involving the point at infinity

The complex plane together with the *point at infinity* is called the *extended complex plane*. However, if we refer to a point z, we will mean a point in the *finite plane*, unless specified otherwise. Optional: read the first paragraph on p.50 about the *Riemann sphere* and the *stereographic projection*.

Definition

$\lim_{z \to z_0} f(z) = \infty$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$0 < |z - z_0| < \delta \quad \Rightarrow \quad |f(z)| > \frac{1}{\epsilon}.$$

$\lim_{z \to \infty} f(z) = w_0$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$|z| > \frac{1}{\delta} \quad \Rightarrow \quad |f(z) - w_0| < \epsilon.$$

Theorem p.50

Let $z_0, w_0 \in \mathbb{C}$. Then:

If $\lim\limits_{z \to z_0} \dfrac{1}{f(z)} = 0$, then $\lim\limits_{z \to z_0} f(z) = \infty$.

If $\lim\limits_{z \to 0} f\left(\dfrac{1}{z}\right) = w_0$, then $\lim\limits_{z \to \infty} f(z) = w_0$.

Example

Determine $\lim\limits_{z \to -1} \dfrac{iz+3}{z+1}$ and $\lim\limits_{z \to \infty} \dfrac{2z+i}{z+1}$.

Section 18 Continuity

Definition

A complex function f is *continuous at a point* $z_0 \in \mathbb{C}$ if $\lim_{z \to z_0} f(z) = f(z_0)$. f is *continuous on a set* R if f is continuous at each point of R.

Theorem 1, p.52

A composition of continuous functions is continuous.

Theorem 2, p.53

If a function f is continuous and nonzero at z_0, then $f(z) \neq 0$ for all z in some neighbourhood of z_0.

Theorem 3, p.53

Suppose that $f(z) = u(x, y) + iv(x, y)$ with $z = x + iy$. Then f is continuous at $z_0 = x_0 + iy_0$ if and only if both u and v are continuous at (x_0, y_0).

Theorem 4, p.54
If f is continuous on a closed, bounded set R, then there exists $M \geq 0$ such that $|f(z)| \leq M$ for all $z \in R$ (and equality holds for at least one z).

We say that f is *bounded* on R.

Tutorial: pp.54–55 nr. 1, 4–5, 7–9, 10(a),(b).

Section 19 Derivatives

Definition
Suppose f is defined in a neighbourhood of z_0. Then the *derivative* of f at z_0 is

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to 0} \frac{f(z_0 + z) - f(z_0)}{z}.$$

f is said to be *differentiable* if $f'(z_0)$ exists.

Examples
Investigate the differentiability of $f(z) = \frac{1}{z}$ and $f(z) = |z|^2$.

Note:

- It is possible for a function to be differentiable at a point but nowhere else in any neighbourhood of that point.

- It is possible that the real and imaginary components of a function f have continuous partial derivatives of all orders at some point, while f is not differentiable at that point.

- f may be continuous but not differentiable.

- If f is differentiable, then f is continuous.

Section 20 Differentiation Rules

Similar as for real functions —– see pp.59–60.
We have, e.g., the sum rule, product rule, quotient rule and chain rule.
(Optional: Proof of chain rule, p.60.)

Tutorial: pp.61–62 nr. 2(d), 6(a), 7, 9.

Sections 21–22 Cauchy-Riemann equations

Theorem p.64
Suppose that $f(z) = u(x, y) + iv(x, y)$ and f is differentiable at $z_0 = x_0 + iy_0$. Then the first order partial derivatives of u and v exist at (x_0, y_0) and satisfy the *Cauchy-Riemann equations*

$$u_x = v_y, \quad u_y = -v_x$$

at (x_0, y_0). Also

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0).$$

Examples
Use the Cauchy-Riemann equations to investigate the differentiability of the following functions:

1. $f(z) = z^2$
2. $f(z) = |z|^2$
3. $f(z) = \begin{cases} \frac{\overline{z}^2}{z} & \text{if} \quad z \neq 0 \\ 0 & \text{if} \quad z = 0. \end{cases}$