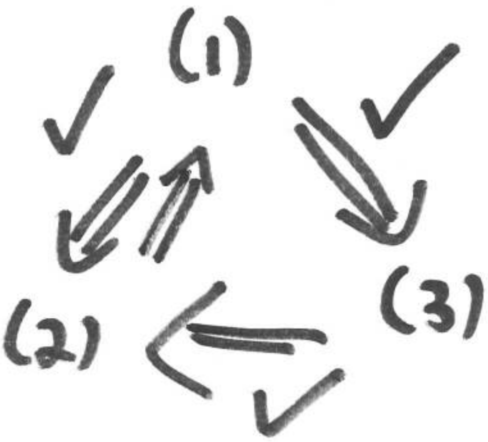
Lecture 8

Finishing 2.2 (Lin. Indep.)

Proof of Lemma from last time:



$(3) \Rightarrow (2)$: Suppose (3) is true.

In other words, either:

Case 1 $\vec{v}_1 = \vec{0}$

$\therefore$ B is lin. dep. (remark from prev. lecture)

$\therefore$ (1) is true.

$\therefore$ (2) is true.

<u>Case 2</u> For some $r \in \{2, \cdots, n\}$,

$\vec{v_r}$ is lin. comb. of $\vec{v_1}, \cdots, \vec{v_{r-1}}$.

$\therefore$ (2) is true.

$\therefore$ (3) $\Rightarrow$ (2) in both cases.

(2) $\Rightarrow$ (1). Suppose (2) is true.

$$\therefore \vec{v_s} = a_1\vec{v_1} + \cdots + a_n\vec{v_n} \quad \left(\begin{matrix}\text{no } \vec{v_s} \\ \text{on RHS}\end{matrix}\right)$$

$$\therefore a_1\vec{v_1} + \cdots + (-1)\vec{v_s} + \cdots + a_n\vec{v_n} = \vec{0}$$

Not all the coefficients are zero ($a_s = -1$).

$\therefore \{\vec{v_1}, \cdots, \vec{v_n}\}$ is lin. dep.

$\square$

Example Show that the list of vectors

$$\vec{f_1} = (-1, 2),\ \vec{f_2} = (1,1),\ \vec{f_3} = (2,-1)$$

in $\mathbb{R}^2$ is linearly dependent.

Soln Use Lemma (2).

Observe:

$$\vec{f_2} = \vec{f_1} + \vec{f_3}$$

$\therefore$ list is linearly dep.

Use Lemma (3).

- Is $\vec{f_1} = \vec{0}$? No.
- Is $\vec{f_2} = k\vec{f_1}$? No.

- Is $\vec{f_3}$ a lin. comb. of $\vec{f_1}, \vec{f_2}$?

Yes: $\vec{f_3} = -\vec{f_1} + \vec{f_2}$

$\therefore \{\vec{f_1}, \vec{f_2}, \vec{f_3}\}$ is lin. dep.

by Lemma (3).

Proposition ("Bumping off") Suppose $\mathcal{L} = \{\vec{l}_1, \cdots, \vec{l}_m\}$ is a **linearly independent** list of vectors in a vector space V, and that $S = \{\vec{s}_1, \cdots, \vec{s}_n\}$ **spans** V. Then $m \leq n$.

Proof Start with the original spanning list

$$S = \{\vec{s}_1, \cdots, \vec{s}_n\}$$

and consider the 'bloated' list

$$S_1 = \{\vec{l}_1, \vec{s}_1, \cdots, \vec{s}_n\}.$$

Since $\vec{l}_1$ is a lin. comb. of the other vectors $\vec{s}_1, \cdots, \vec{s}_n$ (they span V)

, we conclude that S_1 is linearly dependent by prev Lemma (2).

$\therefore$ Part (3) of the Lemma is true.

$\therefore$ either $\vec{\ell}_1 = \vec{0}$ (not possible as $\mathcal{L}$ is lin. ind.)

so some $\vec{s}_r$ is a lin. comb. of the preceding vectors

$$\vec{\ell}_1, \vec{s}_1, \dots, \vec{s}_{r-1}.$$

Hence, we can <u>remove</u> $\vec{s}_r$ from S_1, to arrive a <u>new</u> list

$$S_1 = \{\vec{\ell}_1, \vec{s}_1, \dots, \hat{\vec{s}}_r, \dots, \vec{s}_n\}$$

which still spans V.

We can go on in this way, each time transferring one of the vectors from $\mathcal{L}$ onto the list, and removing one of the vectors from S:

$$\mathcal{L}=\{\vec{l}_1,\cdots,\vec{l}_m\} \quad S=\{\vec{s}_1,\cdots,\vec{s}_n\}$$

$$\mathcal{L}_1=\{\vec{l}_2,\cdots,\vec{l}_m\} \quad S_1=\{\vec{l}_1,\underbrace{\vec{s}_1,\cdots,\vec{s}_n}_{n-1}\}$$

$$\mathcal{L}_2=\{\vec{l}_3,\cdots,\vec{l}_m\} \quad S_2=\{\vec{l}_2,\vec{l}_1,\underbrace{\vec{s}_1,\cdots,\vec{s}_n}_{n-2}\}$$

$$\vdots$$

Suppose $m>n$. Then S_n will consist entirely of vectors from $\mathcal{L}$, but there are more vectors from $\mathcal{L}$ left over!

This is impossible, as it means $\vec{e}_{n+1}$ is a lin. comb. of

$\vec{e}_1, \dots, \vec{e}_n$

which ~~contradicts~~ means $\mathcal{L}$ is lin dep (by Prev Lemma (2)), contradicting our initial assumption.

$\square$