Magreekse en Taylorreekse / Power series and Taylor series §11.8–§11.10

Stelling A

1. As $\sum_{n=1}^{\infty} c_n x^n$ konvergeer by $a \neq 0$, dan konvergeer $\sum_{n=1}^{\infty} c_n x^n$ absoluut vir alle x sodat $|x| < |a|$.

2. As $\sum_{n=1}^{\infty} c_n x^n$ divergeer by $b \neq 0$, dan divergeer $\sum_{n=1}^{\infty} c_n x^n$ vir alle x sodat $|x| > |b|$.

Stelling B (Stelling 4, p.789 in Stewart)

Gegee 'n magreeks $\sum_{n=0}^{\infty} c_n (x-a)^n$. Dan geld presies een van die volgende:

1. Die reeks konvergeer slegs vir $x = a$.

2. Die reeks konvergeer vir alle $x \in \mathbb{R}$.

3. Daar bestaan 'n getal $R > 0$ sodat die reeks konvergeer vir $|x - a| < R$ en divergeer vir $|x - a| > R$.

Stelling C

Gestel $\sum_{n=1}^{\infty} c_n(x-a)^n$ is 'n magreeks met $c_n \neq 0$ vir alle n en konvergensiestraal (radius of convergence) R. Laat

$$L = \lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right|.$$

1. As $L \in \mathbb{R}\backslash\{0\}$, dan is $R = \frac{1}{L}$.
2. As $L = 0$, dan is $R = \infty$.
3. As $L = \infty$, dan is $R = 0$.

Stelling D

Gestel $\sum_{n=1}^{\infty} c_n(x-a)^n$ is 'n magreeks met konvergensiestraal R. Laat $L = \lim_{n\to\infty} \sqrt[n]{|c_n|}$.

1. As $L \in \mathbb{R}\backslash\{0\}$, dan is $R = \frac{1}{L}$.
2. As $L = 0$, dan is $R = \infty$.
3. As $L = \infty$, dan is $R = 0$.

Huiswerk

Ex. 11.8 nr. 9, 17, 19, 23, 29

Lemma E
As $\epsilon > 0$, dan is $|nx^{n-1}| < (|x| + \epsilon)^n$ vir alle n groot genoeg.

Stelling F
Die magreeks $\sum_{n=0}^{\infty} a_n(x-a)^n$ konvergeer op $(a-R, a+R)$ as en slegs as die magreeks $\sum_{n=1}^{\infty} na_n(x-a)^{n-1}$ op $(a-R, a+R)$ konvergeer.

$$\sum_{n=1}^{\infty} na_n(x-a)^{n-1} = \sum_{n=0}^{\infty} \frac{d}{dx}\left[a_n(x-a)^n\right]$$

Stelling G (Stelling 2, p.794 in Stewart)

As die magreeks $\sum_{n=0}^{\infty} c_n(x-a)^n$ konvergensiestraal (radius of convergence) $R > 0$ het, dan is die funksie

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$$

differensieerbaar (en dus kontinu, en dus integreerbaar) op $(a-R, a+R)$, en

1.
$$\begin{aligned} f'(x) &= \frac{d}{dx}\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] \\ &= \sum_{n=0}^{\infty}\left[\frac{d}{dx}(c_n(x-a)^n)\right], \end{aligned}$$

   en

2.
$$\begin{aligned} \int f(x)\,dx &= \int\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right]dx \\ &= \sum_{n=0}^{\infty}\left[\int c_n(x-a)^n\,dx\right]. \end{aligned}$$

Beide reekse hierbo het konvergensiestraal R.

Huiswerk

Ex. 11.9 nr. 5, 13, 15, 27, 31

Stelling H (Stellings 5 en 6, p.800 in Stewart)
As f 'n magreeksvoorstelling vanuit a het (power series representation around a), d.w.s.

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \text{ vir } |x-a| < R$$

dan is

$$c_n = \frac{f^{(n)}(a)}{n!},$$

sodat

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 \\ &\quad + \frac{f'''(a)}{3!}(x-a)^3 + \cdots \end{aligned}$$

Taylorreeks van f vanuit a /
Taylor series of f around a

Indien $a = 0$:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \cdots$$

Maclaurinreeks van f / Maclaurin series of f

Stelling I (Stelling 8, p.801 in Stewart)
As $f(x) = T_n(x) + R_n(x)$, waar T_n die n-degraadse Taylorpolinoom van f by a is, en

$$\lim_{n\to\infty} R_n(x) = 0$$

vir $|x-a| < R$, dan is f gelyk aan sy Taylorreeks vir $|x-a| < R$.

Taylor se Stelling
Gestel I is 'n oop interval en f se $(n+1)$-de afgeleide bestaan by elke punt van I. As $a, b \in I$ met $a < b$, dan bestaan daar 'n $d_n \in (a, b)$ sodat

$$\begin{aligned} f(b) &= f(a) + \frac{f'(a)}{1!}(b-a) + \frac{f''(a)}{2!}(b-a)^2 \\ &\quad + \frac{f'''(a)}{3!}(b-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n \\ &\quad + \frac{f^{(n+1)}(d_n)}{(n+1)!}(b-a)^{n+1}. \end{aligned}$$

Gevolg (Taylor se Stelling)
Gestel I is 'n oop interval en f se $(n+1)$-de afgeleide bestaan by elke punt van I. As $a \in I$, dan bestaan daar vir enige $x \neq a$ in dié interval 'n getal d_n eg tussen a en x sodat

$$\begin{aligned} f(x) &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 \\ &\quad + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &\quad + \frac{f^{(n+1)}(d_n)}{(n+1)!}(x-a)^{n+1}. \end{aligned}$$

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$$\begin{aligned} f(x) &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 \\ &\quad + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &\quad + \frac{f^{(n+1)}(d_n)}{(n+1)!}(x-a)^{n+1}. \end{aligned}$$

Dus is

$$R_n(x) = \frac{f^{(n+1)}(d_n)}{(n+1)!}(x-a)^{n+1}.$$

Vir elke reële getal x geld

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0.$$

Huiswerk
Ex. 11.10 nr. 5, 13

Bekende Maclaurinreekse / Well-known Maclaurin series §11.10

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

vir $|x| < 1$

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

vir $|x| \leq 1$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

vir $|x| < 1$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

vir alle $x \in \mathbb{R}$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

vir alle $x \in \mathbb{R}$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

vir alle $x \in \mathbb{R}$

$$\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$

vir alle $x \in \mathbb{R}$

$$\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$$

vir alle $x \in \mathbb{R}$

Huiswerk

Ex. 11.10 nr. 35, 39, 61, 77, 79

Laat die volgende uit: Die binomiaalreeks (pp.806–807); Vermenigvuldiging en deling van magreekse (p.810).