Lecture 14

2.4. Coordinate Vectors

Proposition (Bases give coordinates)
Let $\mathcal{B} = \{\vec{e}_1, \cdots, \vec{e}_n\}$ be a list of vectors in a vector space V. The following statements are equivalent:

1. $\mathcal{B}$ is a basis for V.
2. For every $\vec{v} \in V$, there exist unique scalars $a_1, \cdots, a_n$ such that

$$\vec{v} = a_1\vec{e}_1 + \cdots + a_n\vec{e}_n$$

Example Find the coordinate vector

of
$$p = 2x^2 - 2x + 3$$

w.r.t. the basis

$$B = \{ \underbrace{1+x}_{p_1} , \underbrace{x^2+x-1}_{p_2} , \underbrace{x^2+x+1}_{p_3} \}$$

of $Poly_2$.

Solution We compute

$$2x^2 - 2x + 3 = \boxed{-4}(1+x)$$
$$+ \boxed{-5/2}(x^2+x-1) + \boxed{9/2}(x^2+x+$$

$$\therefore [p]_B = \begin{bmatrix} -4 \\ -5/2 \\ 9/2 \end{bmatrix}$$

Proof (1) $\Rightarrow$ (2)

Let $B = \{\vec{e_1}, \cdots, \vec{e_n}\}$ be a basis for V. Let $\vec{v} \in V$.

Since B spans V, there exist scalars $a_1, \cdots, a_n$ s.t.

$$\vec{v} = a_1\vec{e_1} + \cdots + a_n\vec{e_n} \quad \cdots (1)$$

To prove uniqueness, suppose that we also have

$$\vec{v} = b_1\vec{e_1} + \cdots + b_n\vec{e_n} \quad \cdots (2)$$

Subtracting (1) from (2):

$$\vec{0} = (b_1 - a_1)\vec{e_1} + \cdots + (b_n - a_n)\vec{e_n}$$

Since $\mathcal{B}$ is lin. ind., we <u>must</u> have

$$(b_1 - a_1) = 0, \cdots, (b_n - a_n) = 0$$

$$\therefore b_1 = a_1, \cdots, b_n = a_n$$

So the scalars are unique.

(2) $\Rightarrow$ (1) Suppose (2) is true.

We need to show that $\mathcal{B} = \{\vec{e}_1, \cdots, \vec{e}_n\}$ is a basis for V.

Clearly $\mathcal{B}$ spans V.

<u>$\mathcal{B}$ is linearly independent</u> Suppose

$$k_1\vec{e}_1 + \cdots + k_n\vec{e}_n = \vec{0} \quad (*)$$

We know <u>one</u> soln to ⊛,
namely

$$\boxed{0}\vec{e}_1 + \boxed{0}\vec{e}_2 + \cdots + \boxed{0}\vec{e}_n = \vec{0}$$

By our assumption (2), this must be the <u>unique</u> soln. In other words, we <u>must</u> have

$$k_1 = 0, \quad \cdots, \quad k_n = 0$$

$\therefore \ \beta = \{\vec{e}_1, \cdots, \vec{e}_n\}$ is lin. ind.

$\therefore \ \beta$ a basis.

$\square$

Definition We call $a_1, \cdots, a_n$ the **coordinates of $\vec{v}$ w.r.t. the basis $\mathcal{B}$**. The column vector

$$[\vec{v}]_{\mathcal{B}} := \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

is called the **coordinate vector of $\vec{v}$ w.r.t. the basis $\mathcal{B}$**.

Example Find the coordinate vector of $\vec{v}, \vec{w} \in \mathbb{R}^2$ w.r.t. the basis $\mathcal{B}$ (Example 2.4.4.)

$$\vec{v} = \boxed{2}\,\vec{b_1} + \boxed{-1}\,\vec{b_2}$$

$$\therefore [\vec{v}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\vec{w} = \boxed{-3}\,\vec{b_1} + \boxed{+2}\,\vec{b_2}$$

$$\therefore [\vec{w}]_{\mathcal{B}} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

<u>Example</u> Find the coordinate vector of

$$f = \sin^2 x - \cos^3 x \in \mathrm{Trig}_3$$

with respect to the standard basis

$$S = \{1, \cos x, \sin x, \cos 2x, \sin 2x, \cos 3x, \sin 3x\}$$

of Trig_3.

<u>Solution</u> We have

$$f = \boxed{1/2}\,1 + \boxed{-3/4}\cos x + \boxed{0}\sin x + \boxed{-1/2}\cos 2x + \boxed{0}\sin 2x + \boxed{-1/4}\cos 3x + \boxed{0}\sin 3x$$

Use:

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\therefore [f]_S = \begin{bmatrix} 1/2 \\ -3/4 \\ 0 \\ -1/2 \\ 0 \\ -1/4 \\ 0 \end{bmatrix}.$$