Lecture 6

Chapter 2 Finite-dimensional vector spaces

Goal: Prove that every finite-dimensional vector space is isomorphic to $\mathbb{R}^n$ for some $n \geq 0$.

"the same as"

2.1. Linear combinations and span

Definition A linear combination of a finite collection $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ of vectors in a vector space V is a vector of the form

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n$$

where $a_1, \cdots, a_n \in \mathbb{R}$.

From now on [$\geq$ Chap 2], we will write

$k\vec{v}$ for $k.\vec{v}$

eg. In $\mathbb{R}^3$, $(6,2,-14)$ is a linear combination of

$\vec{v_1} = (-3,1,2)$, $\vec{v_2} = (-2,0,3)$

because

$$(6,2,-14) = \boxed{2}(-3,1,2) + \boxed{-6}(-2,0,3)$$

check! RHS $= \Big(2.(-3) + (-6)(-2), 2.1 + (-6).0$
$, 2.2 + (-6).3\Big)$

$= (6,2,-14).$ $=$ LHS.

<u>Example</u> Consider $f, f_1, f_2 \in \text{Fun}(\mathbb{R})$:

$f(x) = \cos^3 x$, $f_1(x) = \cos x$, $f_2(x) = \cos 3x$

Is f a linear combination of f_1, f_2 ?

Let's see. Suppose

$$f = a_1 f_1 + a_2 f_2$$

$$\therefore \quad f(x) = a_1 f_1(x) + a_2 f_2(x) \quad \text{for all } x \in \mathbb{R}$$

$$\cos^3 x = a_1 \cos x + a_2 \cos 3x \quad \text{for all } x \in \mathbb{R}$$

Use:

$$\cos^3 x = \frac{1}{4}\left(3\cos x + \cos(3x)\right)$$

$\therefore$

$$\boxed{f = \frac{3}{4} f_1 + \frac{1}{4} f_2.}$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B \quad \cdots ①$$

$$\cos(A-B) = \cos A \cos B + \sin A \sin B \quad \cdots ②$$

$$\therefore \cos(3A)$$

$$= \cos(2A + A)$$

$$= \cos(2A)\cos(A) - \sin(2A)\sin A$$

$$= (2\cos^2 A - 1)\cos(A) - 2\sin A \cos A \sin$$

$$= (2\cos^2 A - 1)\cos A - 2(1-\cos^2 A)\cos$$

$$= 4\cos^3 A - 3\cos A$$

$$\therefore \cos^3 A = \frac{3\cos A + \cos(3A)}{4}$$

eg. In $\mathbb{R}^4$, $\vec{v} = (2, -1, 3, 0)$ is

not a linear combination of

$\vec{v_1} = (1, 3, 2, 0)$, $\vec{v_2} = (5, 1, 2, 4)$,

$\vec{v_3} = (-1, 0, 2, 1)$

Suppose $\vec{v}$ is a linear combination

of $\vec{v_1}, \vec{v_2}, \vec{v_3}$.

$\therefore$ there exist scalars $a_1, a_2, a_3 \in \mathbb{R}$

such that

$$\vec{v} = a_1 \vec{v_1} + a_2 \vec{v_2} + a_3 \vec{v_3}$$

$$\therefore (2,-1,3,0) = a_1 (1,3,2,0) + a_2 (5,1,2,4) + a_3 (-1,0,2,1)$$

$$\therefore (2,-1,3,0) = (a_1 + 5a_2 - a_3,\ 3a_1 + a_2,\ 2a_1 + 2a_2 + 2a_3,\ 4a_2 + a_3)$$

$$\therefore \begin{cases} a_1 + 5a_2 - a_3 = 2 \\ 3a_1 + a_2 = -1 \\ 2a_1 + 2a_2 + 2a_3 = 3 \\ 4a_2 + a_3 = 0. \end{cases}$$

<u>Exercise</u>: Check that these eq^{n}s have <u>no</u> solution.

$\therefore \vec{v}$ is <u>not</u> a liner combination of $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

<u>Defn</u> Let $\vec{v}_1, \cdots, \vec{v}_n$ be a list of vectors in a vector space V.

We write

$$\text{span}\{\vec{v}_1, \cdots, \vec{v}_n\} := \left\{ \begin{array}{c} \text{all linear} \\ \text{combinations} \\ \text{of } \vec{v}_1, \cdots, \vec{v}_n \end{array} \right\}$$

We say that $\vec{v}_1, \cdots, \vec{v}_n$ <u>spans</u> V

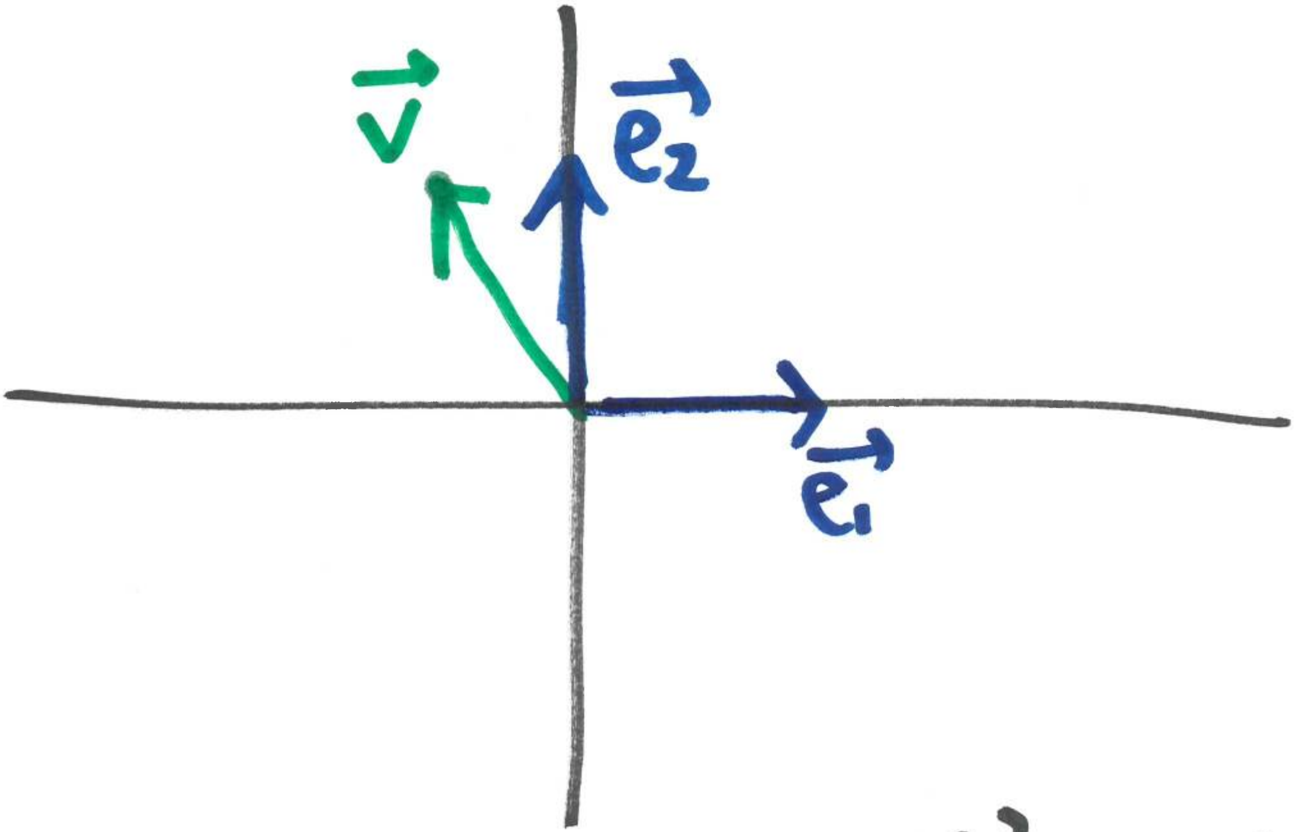
if

$$V = \text{span}\{\vec{v}_1, \cdots, \vec{v}_n\}$$

i.e. every vector in V can be written as a linear comb. of $\vec{v}_1, \cdots, \vec{v}_n$.

Example $\mathbb{R}^2$ is spanned by

$$\vec{e}_1 = (1,0), \quad \vec{e}_2 = (0,1).$$

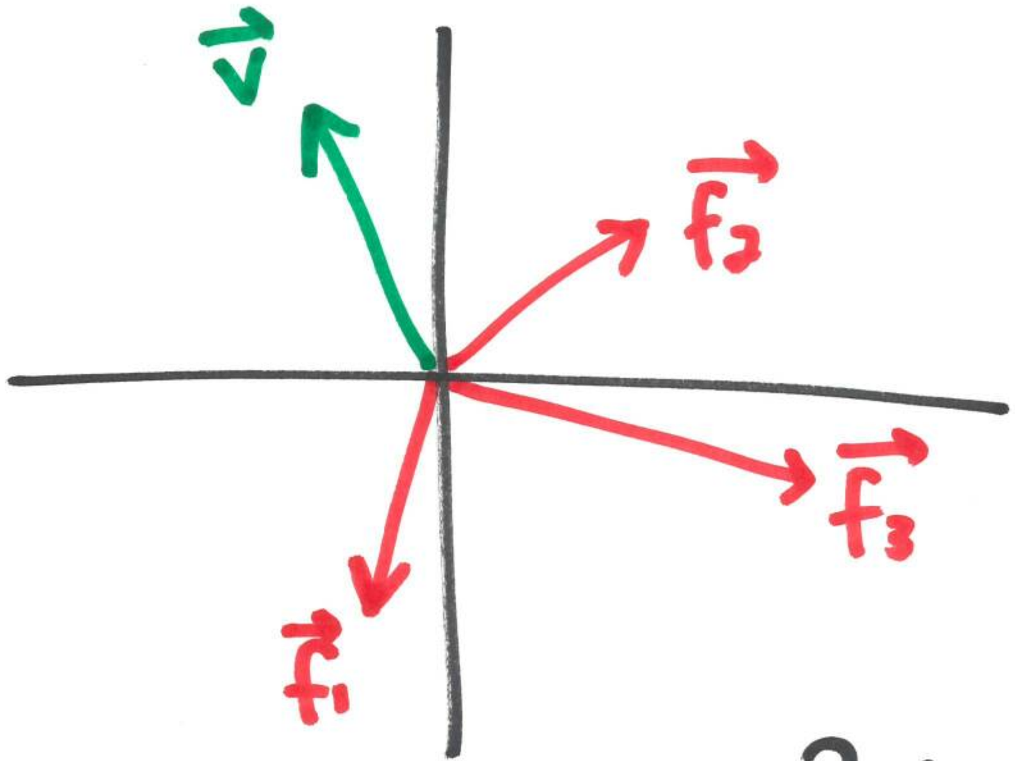


General vector $\vec{v} \in \mathbb{R}^2$ can be written as

$$\vec{v} = (x, y)$$
$$= \boxed{x}\,\vec{e}_1 + \boxed{y}\,\vec{e}_2$$

$\mathbb{R}^2$ is <u>also</u> spanned by

$$\vec{f_1} = (-1, 2), \quad \vec{f_2} = (1,1), \quad \vec{f_3} = (2,-1)$$



General vector $\vec{v}$ in $\mathbb{R}^2$ is

$$\vec{v} = (x, y)$$

So we need to solve:

$$\vec{v} = a_1\vec{f_1} + a_2\vec{f_2} + a_3\vec{f_3}$$

i.e.

$$(x,y) = a_1(-1,2) + a_2(1,1) + a_3(2,-1)$$
$$= (-a_1 + a_2 + 2a_3,\ 2a_1 + a_2 - a_3)$$

$\therefore$ trying to solve (for fixed x,y)

$$\begin{cases} -a_1 + a_2 + 2a_3 = x \\ 2a_1 + a_2 - a_3 = y \end{cases}$$