Lec 20 Last time

$$\left[T(\vec{v})\right]_C = \left[T\right]_{C\leftarrow B}\left[\vec{v}\right]_B$$

Example Recall:

$$T : \mathrm{Poly}_2 \longrightarrow \mathrm{Poly}_3$$

$$T(p)(x) = x\,p(x)$$

$$\mathcal{B} = \{\underbrace{1+x}_{p_1},\ \underbrace{1-x}_{p_2},\ \underbrace{1+x+x^2}_{p_3}\}$$

$$C = \{\underbrace{1}_{q_1},\ \underbrace{1+x}_{q_2},\ \underbrace{1+x+x^2}_{q_3},\ \underbrace{x^3}_{q_4}\}$$

$$[T]_{C \leftarrow B} = \begin{bmatrix} -1 & -1 & -1 \\ 0 & 2 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T(p_3) = x(1+x+x^2)$$
$$= x + x^2 + x^3$$
$$= -q_1 + q_3 + q_4$$

Pick $\vec{v} \in Poly_2$ to be x

$$x = \frac{1}{2}(p_1 - p_2)$$
$$\therefore [x]_B = \begin{bmatrix} 1/2 \\ -1/2 \\ 0 \end{bmatrix}$$

$$T(x) = x^2.$$

$$= -q_2 + q_3$$

$$[T(x)]_C = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

Check the theorem (for $\vec{v} = x$)

$$\text{LHS} = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\text{RHS} = \begin{bmatrix} -1 & -1 & -1 \\ 0 & 2 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 \\ -1/2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} = \text{LHS}!$$

Another fact about matrices.

<u>Lemma</u> Let

$$\underline{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \underline{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, \underline{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Let A be a matrix with n ~~[illegible]~~ columns.

Then:

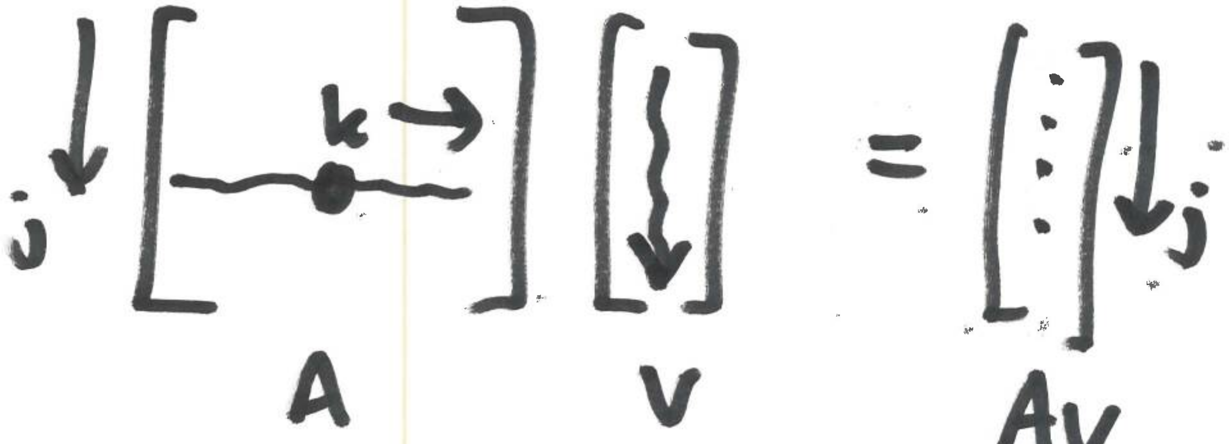
$$\begin{pmatrix} i^{th} \text{ column} \\ \text{of } A \end{pmatrix} = A e_i$$

<u>Proof</u> $\begin{pmatrix} i^{th} \text{ column} \\ \text{of } A \end{pmatrix}_j = A_{ji}$

$$(Ae_i)_j = \sum_k A_{jk} (e_i)_k$$

$$= \sum_k A_{jk}\, \delta_{ik}$$

$$= A_{ji}\,.$$



$$(Av)_j = \sum_k A_{jk}\, v_k$$

<u>Theorem</u> (Functoriality of the Matrix of a Linear Map)

Let $S: U \to V$ and $T: V \to W$ be linear maps between f. dim. vector spaces. Let $\mathcal{B}, \mathcal{C}, \mathcal{D}$ be bases for U, V, W resp. Then

$$[T \circ S]_{\mathcal{D} \leftarrow \mathcal{B}} = [T]_{\mathcal{D} \leftarrow \mathcal{C}} [S]_{\mathcal{C} \leftarrow \mathcal{B}}$$

$$\begin{tikzcd}
\underset{D=\{d_1,\dots,d_r\}}{W} & \underset{C=\{c_1,\dots,c_q\}}{V} \arrow[l, "T"'] & \underset{B=\{\vec{b}_1,\dots,\vec{b}_p\}}{U} \arrow[l, "S"'] \arrow[ll, bend left, "T\circ S"]
\end{tikzcd}$$

<u>Proof</u>

i^{th} column of LHS

$= \left[(T\circ S)(\vec{b}_i)\right]_D$ $\left[\text{defn of } [T\circ S]_{D\leftarrow B}\right]$

$= \left[T\left(S(\vec{b}_i)\right)\right]_D$ $\left(\text{defn of } T\circ S\right)$

$= [T]_{D\leftarrow C}\left[S(\vec{b}_i)\right]_C$ (Theorem from last time)

$$= [T]_{D \leftarrow C} [S]_{C \leftarrow B} [\vec{b}_i]_B \quad (\text{"})$$

$$= \left([T]_{D \leftarrow C} [S]_{C \leftarrow B} \right) \underline{e}_i \leftarrow \underline{e}_i = \begin{array}{c} \\ \\ i \\ \\ \end{array} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

$= i^{th}$ column of

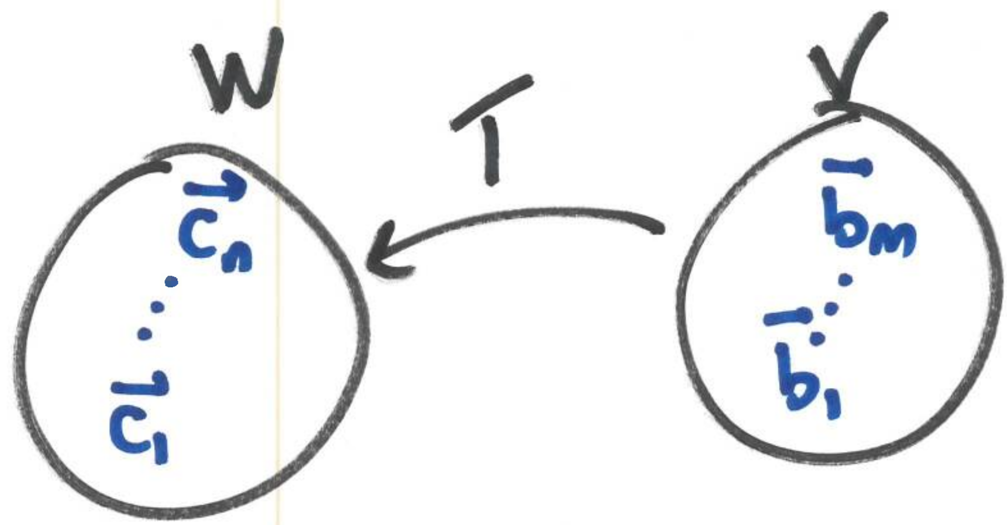
$$[T]_{D \leftarrow C} [S]_{C \leftarrow B}.$$

(fact about matrices)

$\square$

Let's give a different, index-based proof!

But first:



$$T(\vec{b}_i) = \sum_{j=1}^{n} \left([T]_{C \leftarrow B} \right)_{ji} \vec{c}_j$$