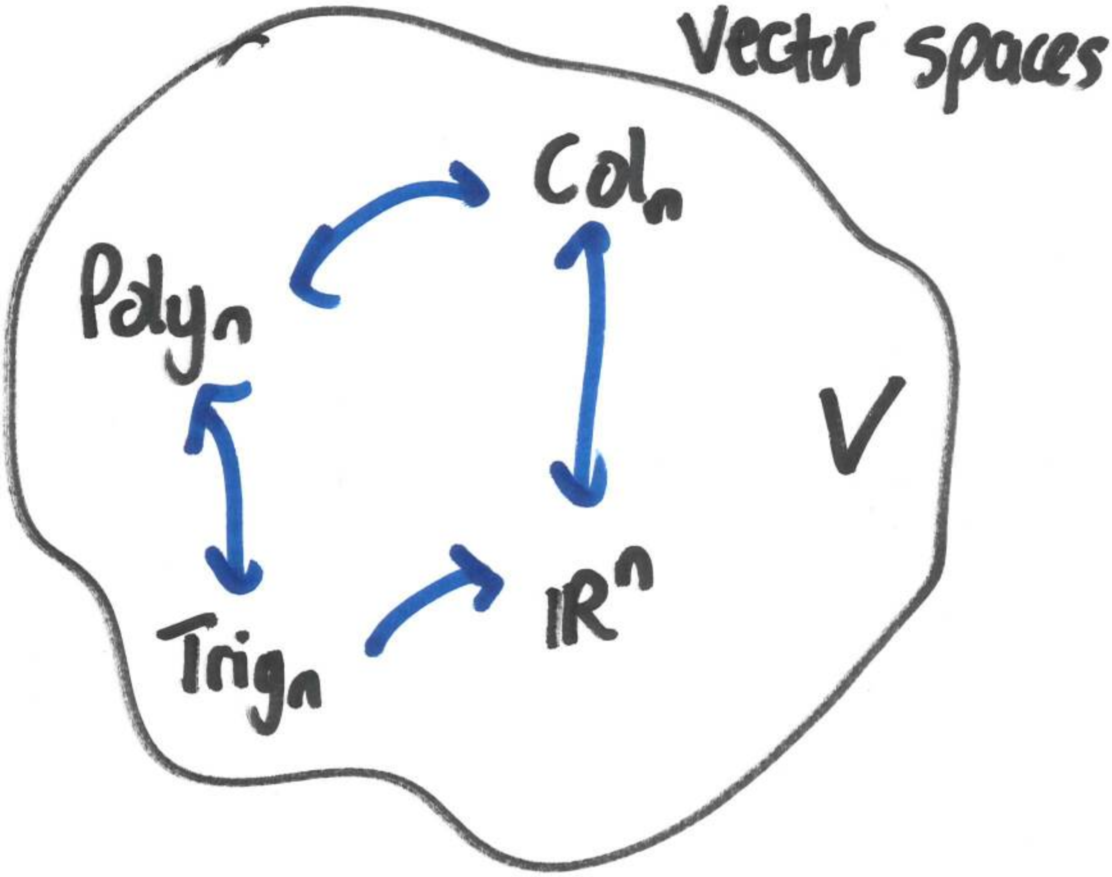
Lecture 16

3. Linear Maps



Linear algebra is the study of <u>linear maps</u> between vector spaces.

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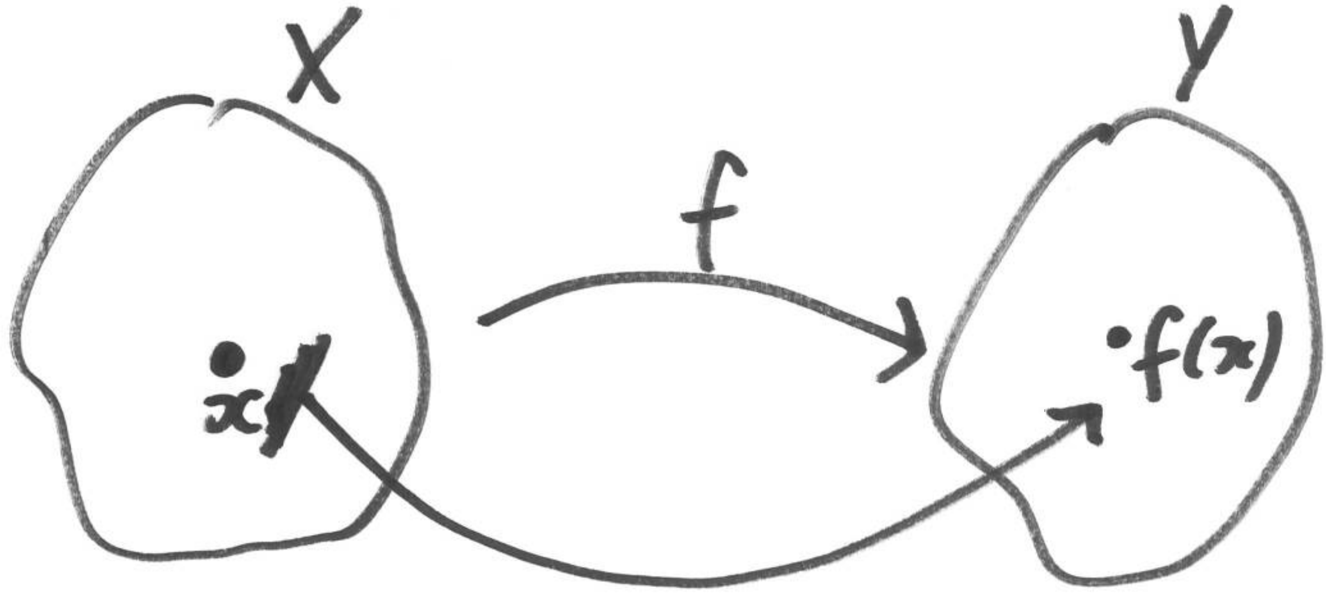
Recall that a function

$$f: X \longrightarrow Y$$

from a set X to a set Y is a rule which assigns to each element $x \in X$ an element $f(x) \in Y$.



We write

$$x \longmapsto f(x)$$

to indicate that $x \in X$ "maps to" $f(x) \in Y$

Two functions

$$f, g : X \longrightarrow Y$$

are <u>equal</u> if for each $x \in X$,

$$f(x) = g(x).$$

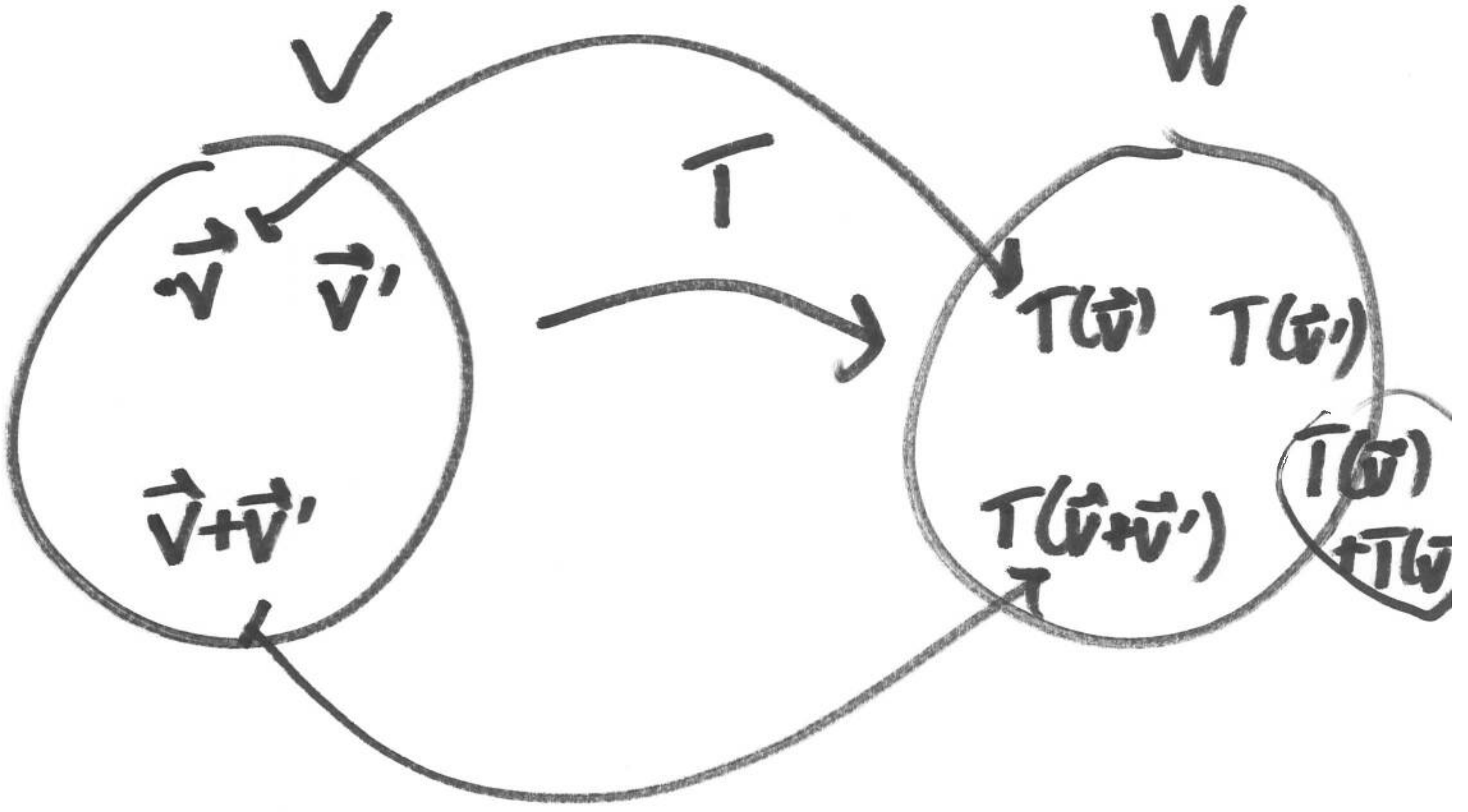
<u>Defn</u> Let V and W be vector spaces. A <u>linear map</u> from V to W is a function

$$T : V \longrightarrow W$$

satisfying:

1. $T(\vec{v}+\vec{v}') = T(\vec{v}) + T(\vec{v}')$
2. $T(k\vec{v}) = kT(\vec{v})$

for all $\vec{v}, \vec{v}' \in V$ and scalars $k \in \mathbb{R}$.



V
W
T
$\vec{v}$
$\vec{v}'$
$\vec{v}+\vec{v}'$
$T(\vec{v})$
$T(\vec{v}')$
$T(\vec{v}+\vec{v}')$
$T(\vec{v})$
$+T(\vec{v}$

Example (Matrices give rise to linear maps)

Let A be an $n \times m$ matrix.

Then have a linear map

$$T_A : \mathrm{Col}_m \longrightarrow \mathrm{Col}_n$$
$$\underline{v} \longmapsto A\underline{v}$$

$\underline{v} \in \mathrm{Col}_m$

eg. $A = \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix}$

$$T_A : \mathrm{Col}_2 \longrightarrow \mathrm{Col}_2$$
$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$
$$= \begin{bmatrix} v_1 + 2v_2 \\ v_1 - 2v_2 \end{bmatrix}$$

T_A Linear ?

$$T_A(\underline{v}+\underline{v}') = T_A(\underline{v}) + T_A(\underline{v}') \ ?$$

$$\begin{aligned} LHS &= A(\underline{v}+\underline{v}') \\ &= A\underline{v} + A\underline{v}' \\ &= T_A(\underline{v}) + T_A(\underline{v}') \\ &= RHS \end{aligned}$$

$$T_A(k\underline{v}) = k\,T_A(\underline{v}) \ ?$$

$$\begin{aligned} LHS &= A(k\underline{v}) \\ &= k(A\underline{v}) \\ &= k\,T_A(\underline{v}) \\ &= RHS. \end{aligned}$$

Example (cross product with a fixed vector)

Fix a vector $\vec{w} \in \mathbb{R}^3$. The function

$$C : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$
$$\vec{v} \longmapsto \vec{w} \times \vec{v}$$

is a linear map:

1. $$C(\vec{v}+\vec{v}\,') = \vec{w} \times (\vec{v}+\vec{v}\,')$$
$$= \vec{w}\times\vec{v} + \vec{w}\times\vec{v}\,'$$
$$= C(\vec{v}) + C(\vec{v}\,')$$

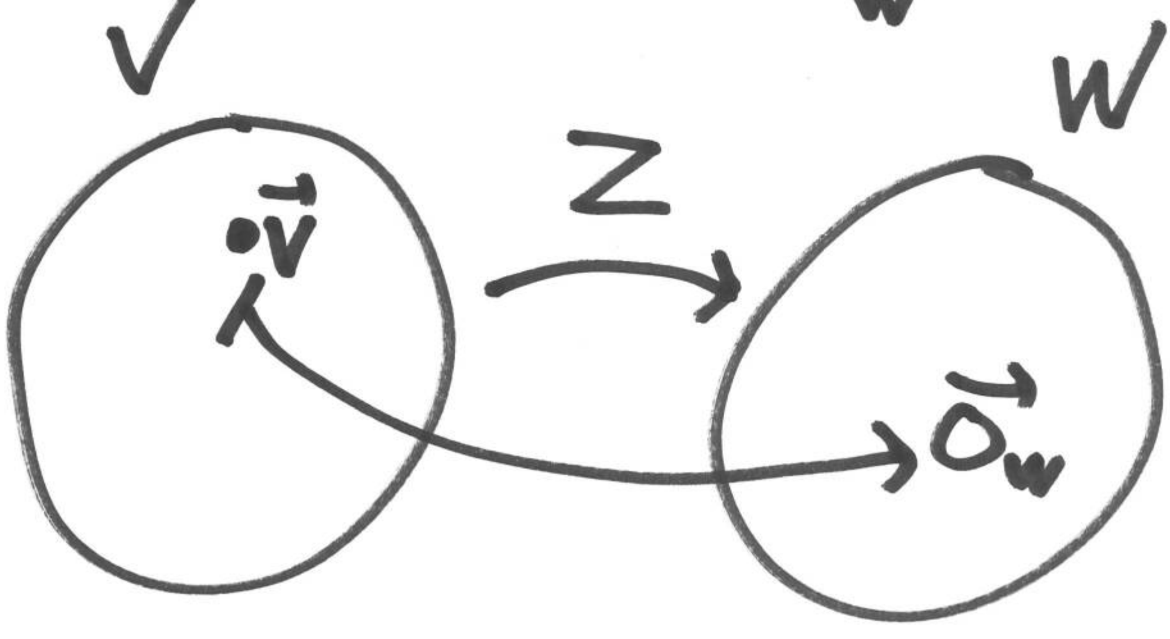
2. $$C(k\vec{v}) = \vec{w} \times (k\vec{v})$$
$$= k\,\vec{w}\times\vec{v}$$
$$= k\,C(\vec{v})$$

<u>Example</u> (The zero map)

V, W vector spaces.

The function

$$Z : V \longrightarrow W$$
$$\vec{v} \longmapsto \vec{0}_W$$



Linear?

1. $Z(\vec{v}+\vec{v}') = \vec{0}$
$= \vec{0} + \vec{0}$
$= Z(\vec{v}) + Z(\vec{v}')$.

<u>Example</u> (Differentiation) The operation 'take the derivative' can be interpreted as a linear map

$$\begin{aligned} D : \mathrm{Poly}_n &\longrightarrow \mathrm{Poly}_{n-1} \\ p &\longmapsto p' \end{aligned}$$

Linear? Yes:

$$(p+q)' = p' + q' \quad \checkmark$$

$$(kp)' = kp' \quad \checkmark$$

<u>Example</u> (Antiderivative) The operation 'compute the unique antiderivative' with zero constant term can be interpreted as a linear map

$$\begin{aligned} A : \mathrm{Poly}_n &\longrightarrow \mathrm{Poly}_{n+1} \\ p &\longmapsto \int_0^x p(t)\,dt \end{aligned}$$

Example (Initial conditions on ODE as a linear map)

Let V be the vector space of all solns to an n^{th} order homogenous linear ODE on an interval I:

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_0(x)y = 0$$

Recall: $\mathrm{Dim}(V) = n$.

Let $b \in I$. Have linear map

$$T_b : \mathbb{R}^n \longrightarrow V$$

$$(x_0, \cdots, x_{n-1}) \longmapsto \text{unique } y \in V \text{ such that } y^{(i)}(b) = x_i.$$

eg.

$$y'' + 4y = 0$$

Basis for $V = \{\cos(2x), \sin(2x)\}$

Pick $b = 0$

$$T : \mathbb{R}^2 \longrightarrow V$$

$$(x_0, x_1) \longmapsto \text{unique soln } y$$

s.t.

$$y(0) = x_0$$
$$y'(0) = x_1$$

i.e. $y = x_0 \cos(2x) + \frac{x_1}{2} \sin(2x)$