Lec 19

Continuing proof:

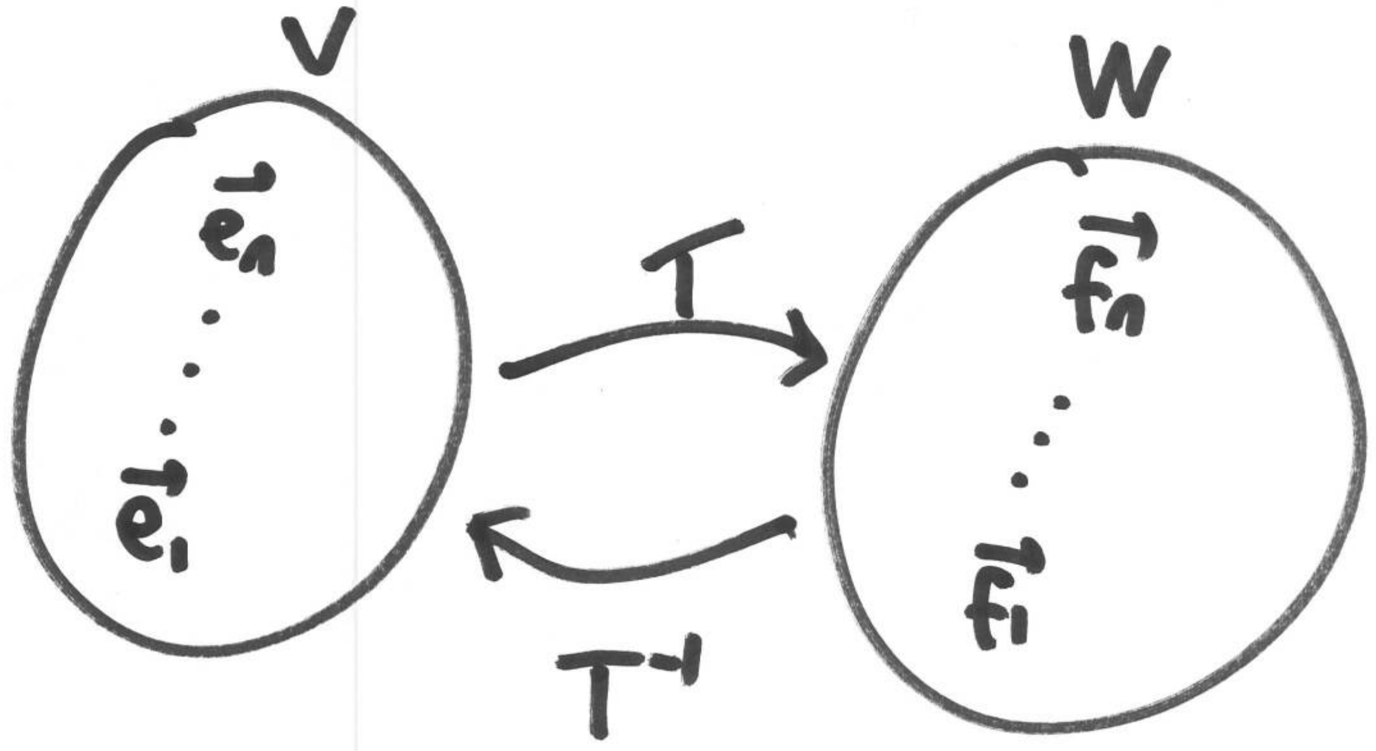
$\Leftarrow$ Let $\{\vec{e_1}, \cdots, \vec{e_n}\}$ be a basis for V.

Let $\{\vec{f_1}, \cdots, \vec{f_n}\}$ be a basis for W.

Define linear maps (using the Proposition that it is sufficient to define a linear map on a basis)

$$T : V \longrightarrow W$$
$$\vec{e_i} \longmapsto \vec{f_i}$$

$$T^{-1} : W \longrightarrow V$$
$$\vec{f_i} \longmapsto \vec{e_i}$$


V
W
T
T⁻¹

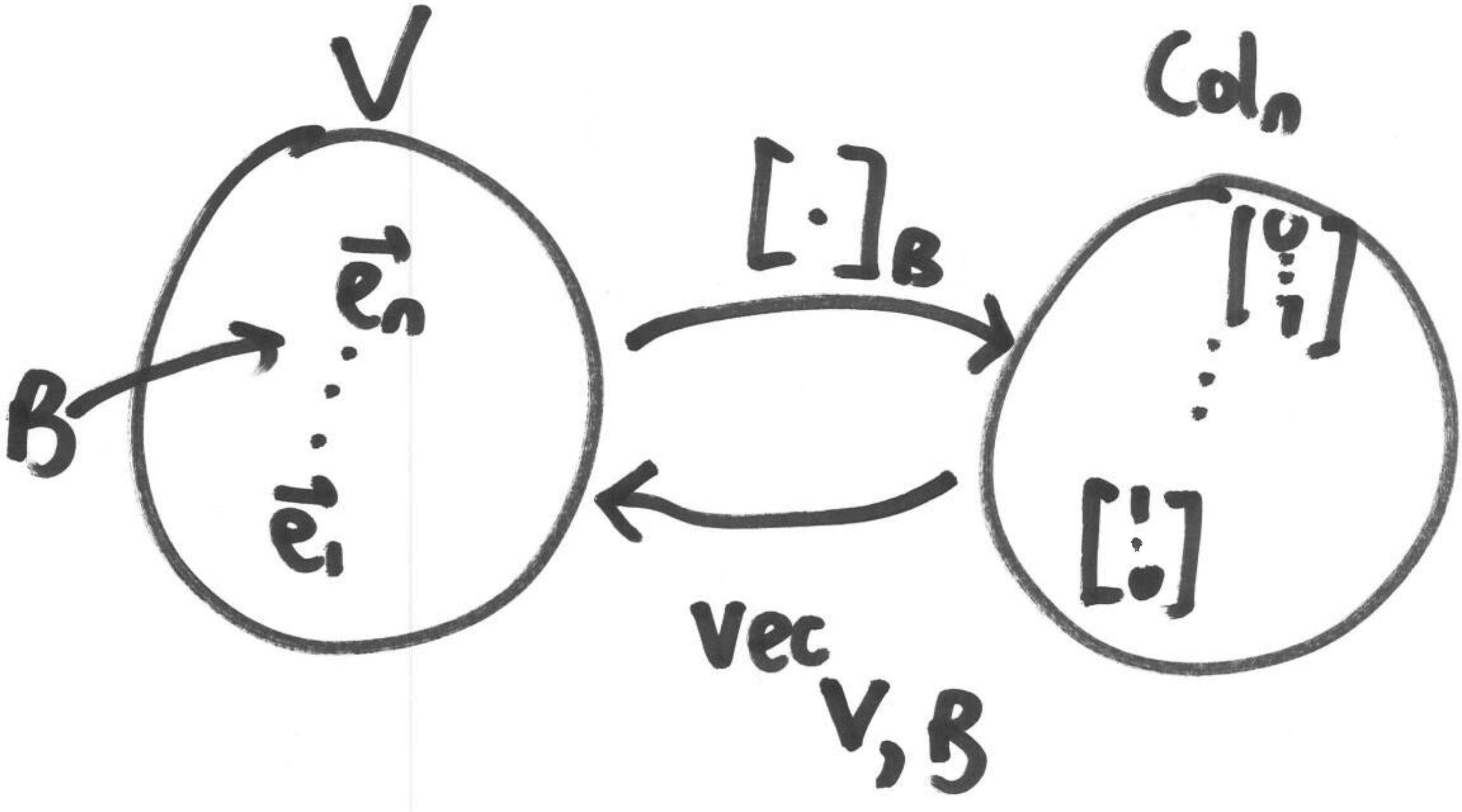
<u>Theorem</u> Let V be a vector space with basis $\mathcal{B}=\{\vec{e}_1,\cdots,\vec{e}_n\}$

The linear map

$$
\begin{aligned}
[\cdot]_{\mathcal{B}} : V &\longrightarrow \mathrm{Col}_n \\
\vec{v} &\longmapsto [\vec{v}]_{\mathcal{B}}
\end{aligned}
$$

is an isomorphism, with explicit inverse

$$
\begin{aligned}
\mathrm{Vec}_{V,\mathcal{B}} : \mathrm{Col}_n &\longrightarrow V \\
\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} &\longmapsto a_1\vec{e}_1 + a_2\vec{e}_2 + \cdots + a_n\vec{e}_n
\end{aligned}
$$


V
Col$_n$
$[\cdot]_B$
$Vec_{V,B}$
B
$\vec{e}_n$
$\vec{e}_1$

Clearly,

$$T^{-1} \circ T = id_V \quad , \quad T \circ T^{-1} = id_W$$

$\therefore$ T is an isomorphism.

$\square$

<u>Example</u> $\mathbb{R}^n$ is isomorphic to $Poly_{n-1}$

because: $Dim(\mathbb{R}^n) = n$

$Dim(Poly_{n-1}) = n$

An explicit isomorphism is given by:

$$(1,0,\cdots,0) \longmapsto 1$$

$$(0,1,0,\cdots,0) \longmapsto x$$

$$\vdots$$

$$(0,\cdots,0,1) \longmapsto x^{n-1}$$

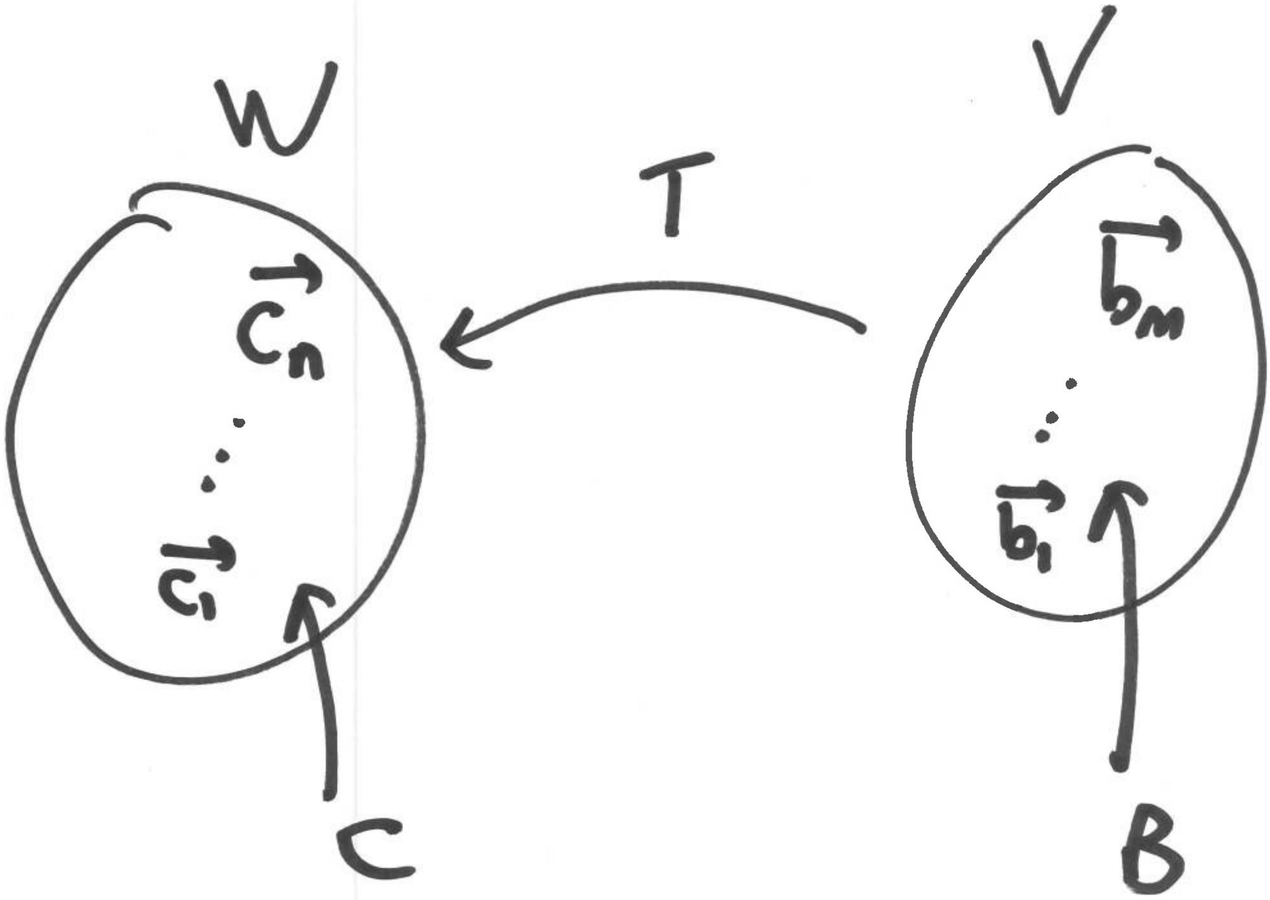
3.4 Linear Maps and Matrices

<u>Defn</u> Let $T: V \to W$ be a linear map. Let

$$B = \{\vec{b}_1, \cdots, \vec{b}_m\}$$
$$C = \{\vec{c}_1, \cdots, \vec{c}_n\}$$

be bases for V and W. The <u>matrix of T relative to the bases B and C</u> is:

$$[T]_{C \leftarrow B} := n \underbrace{\begin{bmatrix} \begin{bmatrix} T(\vec{b}_1) \\ \\ \\ \end{bmatrix}_C & \cdots & \begin{bmatrix} T(\vec{b}_m) \\ \\ \\ \end{bmatrix}_C \end{bmatrix}}_{m}$$


W
V
T
$\vec{c}_n$
$\vec{c}_1$
$\vec{b}_m$
$\vec{b}_1$
C
B

Example

$$T : \mathrm{Poly}_2 \longrightarrow \mathrm{Poly}_3$$

$$T(p)(x) := x\, p(x)$$

Let

$$B = \{ \underbrace{1+x}_{p_1}, \underbrace{1-x}_{p_2}, \underbrace{1+x+x^2}_{p_3}$$

be a basis for Poly_2,

$$C = \{ \underbrace{1}_{q_1}, \underbrace{1+x}_{q_2}, \underbrace{1+x+x^2}_{q_3},$$

$$q_4 = x^3 \}$$

be a basis for Poly_3.

$$T(p_1) = x(1+x)$$

$$= x + x^2$$

$$= -q_1 + q_3$$

$$\therefore [T(p_1)]_C = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$T(p_2) = x(1-x)$$

$$= x - x^2$$

$$= -q_1 + 2q_2 - q_3$$

$$\therefore [T(p_2)]_C = \begin{bmatrix} -1 \\ 2 \\ -1 \\ 0 \end{bmatrix}$$

$$
\begin{aligned}
T(p_3) &= x(1+x+x^2) \\
&= x + x^2 + x^3 \\
&= -q_1 + q_3 \qquad + q_4
\end{aligned}
$$

easy!

$$
\therefore [\cancel{A}T]_{C \leftarrow B} = \begin{bmatrix} -1 & -1 & -1 \\ 0 & 2 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & \cancel{0}1 \end{bmatrix}
$$

Theorem (Linear Maps and Matrix Multiplication Of Coordinate vectors)

For all vectors $\vec{v} \in V$,

$$[T(\vec{v})]_C = [T]_{C \leftarrow B} [\vec{v}]_B$$

Proof Expand $\vec{v}$ w.r.t. the basis B:

$$\vec{v} = a_1 \vec{b_1} + \cdots + a_m \vec{b_m}$$

i.e. $$[\vec{v}]_B = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix}$$

Then:

$$[T(\vec{v})]_C = [T(a_1\vec{b_1} + \cdots + a_m\vec{b_m})]_C$$

$$= [a_1 T(\vec{b_1}) + \cdots + a_m T(\vec{b_m})]_C$$

[T is linear]

$$= a_1 [T(\vec{b_1})]_C + \cdots + a_m [T(\vec{b_m})]$$

$$\left[[\vec{w} + \vec{w}']_C = [\vec{w}]_C + [\vec{w}']_C \right]$$

$$= \begin{bmatrix} [T(\vec{b_1})]_C & \cdots & [T(\vec{b_m})]_C \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix} \quad (*)$$

$$a_1 \begin{bmatrix} c_1 \end{bmatrix} + \cdots + a_m \begin{bmatrix} c_m \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} c_1 \end{bmatrix} & \begin{bmatrix} c_2 \end{bmatrix} & \cdots & \begin{bmatrix} c_m \end{bmatrix} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix}$$

(*)

$$= [T]_{C \leftarrow B} [\vec{v}]_B .$$

$\square$

3 blue 1 brown