SEQUENCES

(ADDITIONAL NOTES TO STEWART §11.1)

The Archimedean Property

For each $x > 0$ in $\mathbb{R}$ there exists an $N \in \mathbb{N}$ such that $N > x$.
Hence there also exists for each $x > 0$ an $N \in \mathbb{N}$ such that $\frac{1}{N} < x$.

Theorem A

If the limit of a sequence of real numbers exists, then it is unique.

Proof. Suppose that $a_n \to a$ as $n \to \infty$ and $a_n \to b$ as $n \to \infty$, and also suppose that $a \neq b$. Let $\epsilon = \frac{1}{2}|a - b|$. Then $\epsilon > 0$. Since $a_n \to a$, there exists an $N_1 \in \mathbb{N}$ such that if $n > N_1$, then $|a_n - a| < \epsilon$ and since $a_n \to b$, there exists an $N_2 \in \mathbb{N}$ such that if $n > N_2$, then $|a_n - b| < \epsilon$. Now let $n > \max\{N_1, N_2\}$. Then $|a_n - a| < \epsilon$ and $|a_n - b| < \epsilon$. But then it follows that

$$2\epsilon = |a - b| \leq |a - a_n| + |a_n - b| < 2\epsilon$$

— contradictory. Hence $a = b$. $\square$

Theorem B

A convergent sequence of real numbers is bounded.

Proof. Suppose that $a_n \to a$ as $n \to \infty$. Then (consider $\epsilon = 1$) there exists an $N \in \mathbb{N}$ such that if $n > N$, then $|a_n - a| < 1$, and hence $|a_n| \leq |a_n - a| + |a| < 1 + |a|$ for all $n > N$. Let $r = \max\{|a_1|, |a_2|, \ldots, |a_N|, 1 + |a|\}$. Then $|a_n| \leq r$ for all $n \in \mathbb{N}$, so that $\{a_n\}$ is bounded. $\square$

Theorem C

Suppose that $\{c_n\}$ is a sequence in the domain of a real function f, $\lim\limits_{n\to\infty} c_n = c$ and f is continuous at c. Then $\lim\limits_{n\to\infty} f(c_n) = f(c)$.

Prove this.

The Monotone Sequence Theorem

1. Suppose that $\{a_n\}$ is an increasing sequence. Then $\{a_n\}$ is bounded above if and only if $\{a_n\}$ converges, in which case
$$\lim_{n\to\infty} a_n = \sup\{a_n : n \in \mathbb{N}\}.$$
2. Suppose that $\{a_n\}$ is a decreasing sequence. Then $\{a_n\}$ is bounded below if and only if $\{a_n\}$ converges, in which case
$$\lim_{n\to\infty} a_n = \inf\{a_n : n \in \mathbb{N}\}.$$

Proof.

1. Suppose that $\{a_n\}$ is an increasing sequence which is bounded above. Then $S = \{a_n : n \in \mathbb{N}\}$ is bounded above, so that $L = \sup S$ exists, and $a_n \leq L$ for all $n \in \mathbb{N}$. We show that $\lim_{n\to\infty} a_n = L$:

   Let $\epsilon > 0$. It follows from Theorem 10 (Improper Integrals: Additional Notes) that there exists an $N \in \mathbb{N}$ such that $L - \epsilon < a_N$. Since $\{a_n\}$ is increasing, $a_N \leq a_n$ for all $n \geq N$. Hence it follows that
   $$L - \epsilon < a_N \leq a_n \leq L < L + \epsilon,$$
   so that $|L - a_n| < \epsilon$, for all $n \geq N$. Consequently $\lim_{n\to\infty} a_n = L$.

   Conversely, if $\{a_n\}$ is an increasing sequence which converges, then it follows from Theorem B above that $\{a_n\}$ is bounded, so that $\{a_n\}$ is bounded above.

2. similar. $\square$