Magreekse en Taylorreekse / Power series and Taylor series §11.8–§11.10

Theorem A

1. If $\sum_{n=1}^{\infty} c_n x^n$ converges at $a \neq 0$, then $\sum_{n=1}^{\infty} c_n x^n$ converges absolutely for all x such that $|x| < |a|$.

2. If $\sum_{n=1}^{\infty} c_n x^n$ diverges at $b \neq 0$, then $\sum_{n=1}^{\infty} c_n x^n$ diverges for all x such that $|x| > |b|$.

Theorem B (Theorem 3, p.789 in Stewart)

Given a power series $\sum_{n=0}^{\infty} c_n (x-a)^n$, then precisely one of the following holds:

1. The series converges only for $x = a$.

2. The series converges for all $x \in \mathbb{R}$.

3. There exists a number $R > 0$ such that the series converges for $|x-a| < R$ and diverges for $|x-a| > R$.

Theorem C

Suppose $\sum_{n=1}^{\infty} c_n(x-a)^n$ is a power series with $c_n \neq 0$ for all n and radius of convergence R. Let

$$L = \lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right|.$$

1. If $L \in \mathbb{R}\backslash\{0\}$, then $R = \frac{1}{L}$.
2. If $L = 0$, then $R = \infty$.
3. If $L = \infty$, then $R = 0$.

Theorem D

Suppose $\sum_{n=1}^{\infty} c_n(x-a)^n$ is a power series with radius of convergence R. Let $L = \lim_{n\to\infty} \sqrt[n]{|c_n|}$.

1. If $L \in \mathbb{R}\backslash\{0\}$, then $R = \frac{1}{L}$.
2. If $L = 0$, then $R = \infty$.
3. If $L = \infty$, then $R = 0$.

Homework

Ex. 11.8 nr. 9, 17, 19, 23, 29

Lemma E

If $\epsilon > 0$, then $|nx^{n-1}| < (|x| + \epsilon)^n$ for all n large enough.

Theorem F

The power series $\sum_{n=0}^{\infty} a_n(x-a)^n$ converges on $(a-R, a+R)$ if and only if the power series $\sum_{n=1}^{\infty} na_n(x-a)^{n-1}$ converges on $(a-R, a+R)$.

$$\sum_{n=1}^{\infty} na_n(x-a)^{n-1} = \sum_{n=0}^{\infty} \frac{d}{dx}\left[a_n(x-a)^n\right]$$

Theorem G (Theorem 2, p.794 in Stewart)

If the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ has radius of convergence $R > 0$, then the function

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and hence continuous, and hence integrable) on $(a-R, a+R)$, and

1.
$$\begin{aligned} f'(x) &= \frac{d}{dx}\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] \\ &= \sum_{n=0}^{\infty}\left[\frac{d}{dx}(c_n(x-a)^n)\right], \end{aligned}$$

and

2.
$$\begin{aligned} \int f(x)\,dx &= \int\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] dx \\ &= \sum_{n=0}^{\infty}\left[\int c_n(x-a)^n\,dx\right]. \end{aligned}$$

Both series above have radius of convergence R.

Homework

Ex. 11.9 nr. 5, 13, 15, 27, 31

Theorem H (Theorems 5 and 6, p.800 in Stewart)

If f has a power series representation around a, i.e.

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \text{ for } |x-a| < R$$

then

$$c_n = \frac{f^{(n)}(a)}{n!},$$

so that

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 \\ &\quad + \frac{f'''(a)}{3!}(x-a)^3 + \cdots \end{aligned}$$

Taylorreeks van f vanuit a /
Taylor series of f around a

If $a = 0$:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

Maclaurinreeks van f / Maclaurin series of f

Theorem I (Theorem 8, p.801 in Stewart)
If $f(x) = T_n(x) + R_n(x)$, where T_n is the n-th degree Taylor polynomial of f at a, and

$$\lim_{n\to\infty} R_n(x) = 0$$

for $|x-a| < R$, then f is equal to its Taylor series for $|x-a| < R$.

Taylor's Theorem
Suppose that I is an open interval and the $(n+1)$-th derivative of f exists at each point of I. If $a, b \in I$ with $a < b$, then there exists $d_n \in (a, b)$ such that

$$\begin{aligned} f(b) &= f(a) + \frac{f'(a)}{1!}(b-a) + \frac{f''(a)}{2!}(b-a)^2 \\ &\quad + \frac{f'''(a)}{3!}(b-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n \\ &\quad + \frac{f^{(n+1)}(d_n)}{(n+1)!}(b-a)^{n+1}. \end{aligned}$$

Corollary (Taylor's Theorem)
Suppose that I is an open interval and the $(n+1)$-th derivative of f exists at each point of I. If $a \in I$, then there exists for any $x \neq a$ in this interval a number d_n properly between a and x such that

$$\begin{aligned} f(x) &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 \\ &\quad + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &\quad + \frac{f^{(n+1)}(d_n)}{(n+1)!}(x-a)^{n+1}. \end{aligned}$$

Corollary (Taylor's Theorem)
Suppose that I is an open interval and the $(n+1)$-th derivative of f exists at each point of I. If $a \in I$, then there exists for any $x \neq a$ in this interval a number d_n properly between a and x such that

$$\begin{aligned} f(x) &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 \\ &\quad + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &\quad + \frac{f^{(n+1)}(d_n)}{(n+1)!}(x-a)^{n+1}. \end{aligned}$$

Hence

$$R_n(x) = \frac{f^{(n+1)}(d_n)}{(n+1)!}(x-a)^{n+1}.$$

For each real number x we have

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0.$$

Homework
Ex. 11.10 nr. 5, 13

Bekende Maclaurinreekse / Well-known Maclaurin series §11.10

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

for $|x| < 1$

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

for $|x| \leq 1$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

for $|x| < 1$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

for all $x \in \mathbb{R}$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

for all $x \in \mathbb{R}$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

for all $x \in \mathbb{R}$

$$\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$

for all $x \in \mathbb{R}$

$$\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$$

for all $x \in \mathbb{R}$

Homework

Ex. 11.10 nr. 35, 39, 61, 77, 79

Leave out the following: The binomial series (pp.806–807); Multiplication and division of power series (p.810).