Lecture 4

1.4 (cont.)

Example ($n \times m$ matrices)

The set

$$\mathrm{Mat}_{n,m} := \left\{ \underbrace{\begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \\ a_{n1} & \cdots & a_{nm} \end{bmatrix}}_{m} \updownarrow n \right\}$$

is a vector space, where:

D1. $\begin{bmatrix} a_{11} & a_{12} & \cdots \\ & \ddots & \\ & & a_{nm} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots \\ & \ddots & \\ & & b_{nm} \end{bmatrix}$

$$:= \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots \\ & \ddots & \\ & & a_{nm}+b_{nm} \end{bmatrix}$$

D2. $$\vec{0} := \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & & & & \\ & & \ddots & & \\ 0 & & & & 0 \end{bmatrix}$$

D3. $$k. \begin{bmatrix} a_{11} & a_{12} & \\ & \ddots & \\ & & a_{nm} \end{bmatrix} := \begin{bmatrix} ka_{11} & ka_{12} & \\ & \ddots & \\ & & ka_{nm} \end{bmatrix}$$

Special case (n-dim column vectors)

$$\mathrm{Col}_n := \mathrm{Mat}_{n,1}$$

$$= \left\{ \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} : a_i \in \mathbb{R},\ i = 1 \ldots n \right\}$$

1.5. Some results about abstract vector spaces

Lemma Suppose V is a vector space with zero vector $\vec{0}$. If $\vec{0}' \in V$ satisfies

$$(*) \quad \vec{0}' + \vec{v} = \vec{v} \quad \forall \vec{v} \in V$$

then $\vec{0}' = \vec{0}$.

Proof

$$\vec{0}' = \vec{0}' + \vec{0} \quad \left[\text{using R3b) for } \vec{v} = \vec{0}'\right]$$

$$= \vec{0} \quad \left[\text{using } (*) \text{ for } \vec{v} = \vec{0}\right] \quad \square$$

Recall

Rules of vector space:

For all $\vec{u}, \vec{v}, \vec{w} \in V$ and $k, l \in \mathbb{R}$:

R1. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$

R2. $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$

R3. a) $\vec{0} + \vec{v} = \vec{v}$

b) $\vec{v} + \vec{0} = \vec{v}$

R4. $k.(\vec{v} + \vec{w}) = k.\vec{v} + k.\vec{w}$

R5. $(k+l).\vec{v} = k.\vec{v} + l.\vec{v}$

R6. $k.(l.\vec{v}) = (kl).\vec{v}$

R7. $1.\vec{v} = \vec{v}$

R8. $0.\vec{v} = \vec{0}$

Definition Let V be a vector space, and $\vec{v} \in V$. We define the **additive inverse** of $\vec{v}$ as

$$\underbrace{-\vec{v}}_{\text{additive inverse of } \vec{v}} := (-1) \cdot \vec{v}$$

Lemma Let V be a vector space. Then for all $\vec{v} \in V$,

1. $\vec{v} + (-\vec{v}) = \vec{0}$

and

2. $(-\vec{v}) + \vec{v} = \vec{0}$

__Proof__

1. $\vec{v} + (-\vec{v}) = \vec{v} + (-1).\vec{v}$ [def of a.i.]

$= 1.\vec{v} + (-1).\vec{v}$ [R7]

$= (1 + (-1)).\vec{v}$ [R5]

$= 0.\vec{v}$ [$1+(-1)=0$]

$= \vec{0}$ [R8].

2. $(-\vec{v}) + \vec{v} = \vec{v} + (-\vec{v})$ [R1]

$= \vec{0}$ [by 1] $\square$

Lemma Suppose that two vectors $\vec{w}$ and $\vec{v}$ in a vector space satisfy

$$\vec{w} + \vec{v} = \vec{0}.$$

$$\boxed{\text{Then} \quad \vec{w} = -\vec{v}}$$

Proof

$$
\begin{aligned}
\vec{w} &= \vec{w} + \vec{0} && [\text{R3b}] \\
&= \vec{w} + \left(\vec{v} + (-\vec{v})\right) && [\text{prev Lemma}] \\
&= (\vec{w} + \vec{v}) + (-\vec{v}) && [\text{R2}] \\
&= \vec{0} + (-\vec{v}) && [\text{by assumption}] \\
&= -\vec{v} && [\text{R3a}]
\end{aligned}
$$

$\blacksquare$

Example Let $\vec{v}, \vec{x}, \vec{w} \in \underbrace{V}_{\text{v. space}}$.

Solve for $\vec{x}$, using the rules of a v. space.

$$\vec{v} + 7.\vec{x} = \vec{w}$$

Solution

$$\vec{v} + 7.\vec{x} = \vec{w}$$

$$\therefore\ -\vec{v} + (\vec{v} + 7.\vec{x}) = -\vec{v} + \vec{w}$$

$$\therefore\ (-\vec{v} + \vec{v}) + 7.\vec{x} = -\vec{v} + \vec{w} \quad \left[\begin{matrix}\text{R2} \\ \text{on LHS}\end{matrix}\right]$$

$$\therefore\ \vec{0} + 7.\vec{x} = -\vec{v} + \vec{w} \quad \left[\begin{matrix}\text{by} \\ \text{Lemma}\end{matrix}\right]$$

$$\therefore\ 7.\vec{x} = -\vec{v} + \vec{w} \quad [\text{R3a}]$$

$$\therefore \frac{1}{7} \cdot (7 . x) = \frac{1}{7} \cdot (-\vec{v} + \vec{w})$$

$$\therefore \left(\frac{1}{7} 7\right) . \vec{x} = \frac{1}{7} \cdot (-\vec{v} + \vec{w}) \; [R6]$$

$$\therefore 1 . \vec{x} = \frac{1}{7} (-\vec{v} + \vec{w}) \quad \left[\frac{1}{7} 7 = 1\right]$$

$$\therefore \vec{x} = \frac{1}{7} . (-\vec{v} + \vec{w}) \; [R7].$$