Thu 7 Feb

Lecture 2

Recall:

$$A := \{(a_1, a_2) : a_1, a_2 \in \mathbb{R}\}$$
$$= \mathbb{R}^2$$

$$B := \{(b_1, b_2, b_3) \in \mathbb{R}^3 : b_1 - b_2 + b_3 = 0\}$$

Shorthand: $\mathbb{R}^n := \{(c_1, \cdots, c_n) : c_i \in \mathbb{R}, \ i = 1 \ldots n\}$

$$C := \{\text{polynomials (with real coefficients) of degree} \leq 4\}$$

Addition operation in C:

$$\left(a_4 x^4 + a_3 x^3 + \cdots + a_1 x + a_0\right)$$
$$+\left(b_4 x^4 + b_3 x^3 + \cdots + b_1 x + b_0\right)$$

$$:= (a_4 + b_4) x^4 + (a_3 + b_3) x^3 + \cdots + (a_1 + b_1) x$$
$$+ (a_0 + b_0)$$

Also, A, B and C have a "zero element"

In A, $\vec{0} = (0,0)$

In B, $\vec{0} = (0,0,0)$

In C, $\vec{0} = 0x^4 + 0x^3 + \cdots + 0x + 0$
$= "0"$

and a "scalar multiplication" operation:

In A,

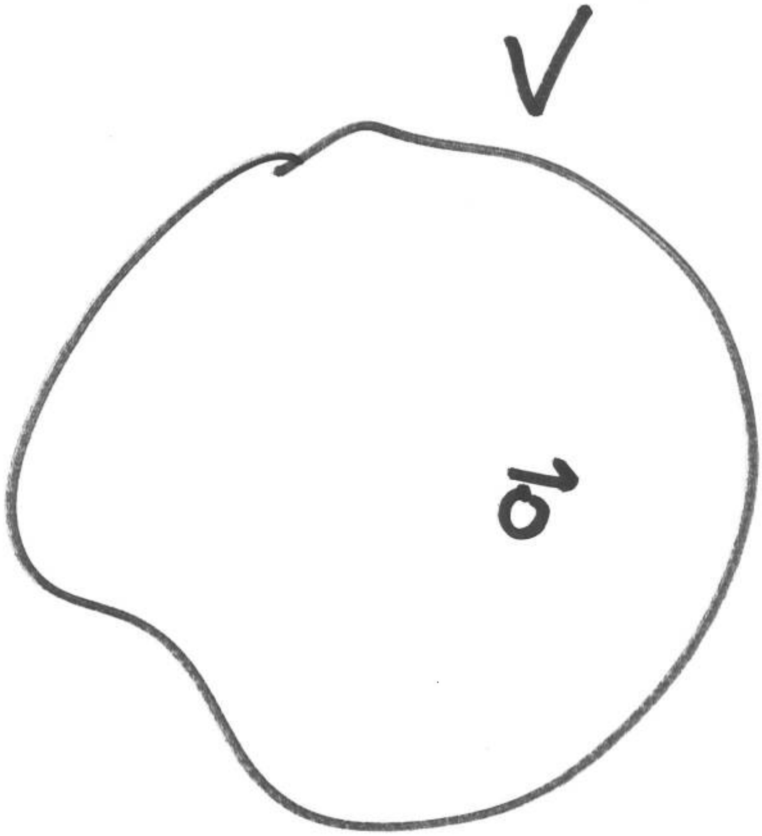
$$\underbrace{k}_{\in \mathbb{R}} \cdot \underbrace{(a_1, a_2)}_{\in A} := \underbrace{(ka_1, ka_2)}_{\in A}$$

In B,

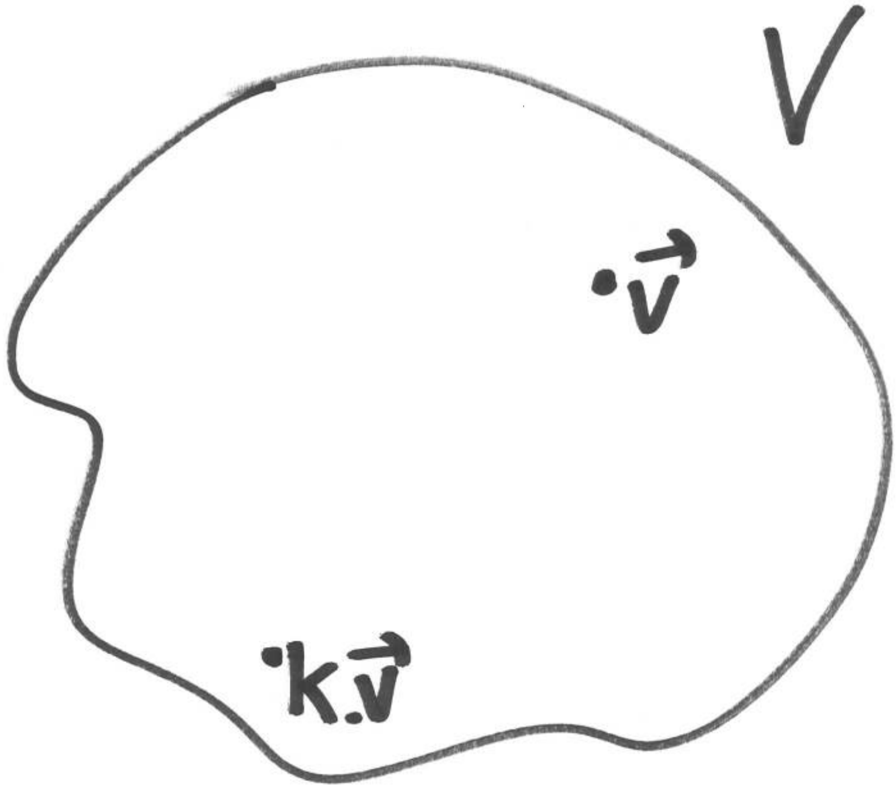
$$k.(b_1, b_2, b_3) := \underbrace{(kb_1, kb_2, kb_3)}_{\in B}$$

In C,

$$k.(a_4x^4 + \cdots + a_0) := ka_4x^4 + \cdots + ka_0$$



$k \in \mathbb{R}$



These operations satisfy rules, eg.

$$\vec{a} + \vec{b} = \vec{b} + \vec{a}$$

$$\vec{0} + \vec{a} = \vec{a}$$

<u>Definition</u> A <u>vector space</u> is a set V equipped with the following data:

D1. <u>an addition operation</u>: for every $\vec{u}, \vec{v} \in V$, there is defined
$$\vec{u} + \vec{v} \in V$$

D2. a <u>zero vector</u>: a special ~~vector~~ element $\vec{0} \in V$ is "marked out"

D3. a <u>scalar multiplication operation</u>: for every $k \in \mathbb{R}$, ~~there~~ and every $\vec{v} \in V$, there is defined $k.\vec{v} \in V$.

This data must satisfy the following <u>rules</u> for all $\vec{u}, \vec{v}, \vec{w} \in V$ and $k, l \in \mathbb{R}$:

R1. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$

R2. $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$

R3. a) $\vec{0} + \vec{v} = \vec{v}$

b) $\vec{v} + \vec{0} = \vec{v}$

R4. $k.(\vec{v} + \vec{w}) = k.\vec{v} + k.\vec{w}$

R6. $k.(l.\vec{v}) = (kl).\vec{v}$

R5. $(k+l).\vec{v} = k.\vec{v} + l.\vec{v}$

R7. $1.\vec{v} = \vec{v}$

R8. $\underbrace{0.\vec{v}}_{\in V} = \underbrace{\vec{0}}_{\in V}$

In notes: check that B is a vector space.

<u>Non-example</u>

$$V := \{\vec{a}, \vec{b}\}$$

<u>D1</u> Define addition:

$$\vec{a} + \vec{a} := \vec{a} \qquad \vec{b} + \vec{a} := \vec{b}$$

$$\vec{a} + \vec{b} := \vec{b} \qquad \vec{b} + \vec{b} := \vec{c}$$

<u>STOP!</u> $\vec{c} \notin V$ so addition operation not well-defined.

Non-example

$$V := \{\vec{a}, \vec{b}\}$$

Addition :

$\vec{a} + \vec{a} := \vec{a}$

$\vec{a} + \vec{b} := \vec{b}$

$\vec{b} + \vec{a} := \vec{b}$

$\vec{b} + \vec{b} := \vec{a}$

Zero vector

$$\vec{0} := \vec{a}$$

Scalar multiplication

Let $k \in \mathbb{K}$,

$k.\vec{a} := \vec{a}$

$k.\vec{b} := \vec{a}$

well-defined data ✓

<u>R1</u> :

① $\underbrace{\vec{a} + \vec{b}}_{\vec{b}} = \underbrace{\vec{b} + \vec{a}}_{\vec{b}}$? ✓

$\underbrace{\vec{a} + \vec{a}}_{\vec{a}} = \underbrace{\vec{a} + \vec{a}}_{\vec{a}}$ ✓

$\underbrace{\vec{b} + \vec{b}}_{\vec{a}} = \underbrace{\vec{b} + \vec{b}}_{\vec{a}}$ ✓

$\therefore$ R1 satisfied.

R2. $\vec{u} = \vec{a}$, $\vec{v} = \vec{a}$, $\vec{w} = \vec{b}$: $(\vec{a} + \vec{a}) + \vec{b} = \vec{a} + (\vec{a} + \vec{b})$?

$$\text{LHS} = \vec{a} + \vec{b}$$
$$= \vec{b}$$

$$\text{RHS} = \vec{a} + \vec{b}$$
$$= \vec{b}$$

$\therefore$ LHS = RHS

$\vdots$

R2 ✓

R3 ✓

but <u>not</u> R4 !

(Exercise)