POWER SERIES AND TAYLOR SERIES

(ADDITIONAL NOTES TO STEWART §11.8 – §11.10)

Theorem A

1. If $\sum_{n=1}^{\infty} c_n x^n$ converges at $a \neq 0$, then $\sum_{n=1}^{\infty} c_n x^n$ converges absolutely for all x such that $|x| < |a|$.

2. If $\sum_{n=1}^{\infty} c_n x^n$ diverges at $b \neq 0$, then $\sum_{n=1}^{\infty} c_n x^n$ diverges for all x such that $|x| > |b|$.

Proof.

1. Suppose that $\sum_{n=1}^{\infty} c_n a^n$ converges. Then $\lim_{n\to\infty} (c_n a^n) = 0$, so that $|c_n a^n| \leq 1$ for all $n \geq N$, say. It follows that $0 \leq |c_n x^n| = |c_n a^n| \left|\frac{x}{a}\right|^n \leq \left|\frac{x}{a}\right|^n$. If $|x| < |a|$, then $\left|\frac{x}{a}\right| < 1$, so that $\sum_{n=1}^{\infty} \left|\frac{x}{a}\right|^n < \infty$. Hence, using the comparison test, it follows that $\sum_{n=1}^{\infty} |c_n x^n|$ converges.

2. Suppose that $\sum_{n=1}^{\infty} c_n b^n$ diverges. Also suppose that there exists an x such that $|x| > |b|$ and $\sum_{n=1}^{\infty} c_n x^n$ converges. Then it follows from (1) that $\sum_{n=1}^{\infty} c_n b^n$ converges absolutely — contradiction. Hence the result follows. $\square$

Theorem B (Theorem 4, p.789 in Stewart)

For a power series $\sum_{n=1}^{\infty} c_n (x-a)^n$ there are (only) three possibilities:

1. The series converges only for $x = a$.

2. The series converges for all $x \in \mathbb{R}$.

3. There exists an $R > 0$ such that the series converges for all x such that $|x-a| < R$ and diverges for all x such that $|x - a| > R$.

Proof. We only prove the case where $a = 0$. Suppose that (1) and (2) do not hold. Then there exist non-zero numbers b and d such that $\sum_{n=1}^{\infty} c_n b^n$ converges and $\sum_{n=1}^{\infty} c_n d^n$ diverges. Hence the set $S = \{|x| : \sum_{n=1}^{\infty} c_n x^n < \infty\}$ is not empty. By Theorem A (2) $\sum_{n=1}^{\infty} c_n x^n$ diverges for all x such that $|x| > |d|$. Hence it follows that if $|x| \in S$, then $|x| \leq |d|$, so that $|d|$ is an upper bound for S. It follows from the Completeness Property of $\mathbb{R}$ that $R = \sup S$ exists. If $|x| > R$, then $|x| \notin S$, so that $\sum_{n=1}^{\infty} c_n x^n$ diverges. If $|x| < R$, then $|x|$ is not an upper bound for S, so that there exists an element

$|a| \in S$ for which we have that $|a| > |x|$. Since $|a| \in S$, it follows that $\sum_{n=1}^{\infty} c_n a^n$ converges, so that, by Theorem A (1), $\sum_{n=1}^{\infty} c_n x^n$ converges. $\square$

The number R is called the *radius of convergence* of the series, and the interval I consisting of all x where the series converges is called the *interval of convergence* of the series. In (1) $R = 0$ and in (2) $R = \infty$.

Theorem C

Suppose that $\sum_{n=1}^{\infty} c_n (x-a)^n$ is a power series with $c_n \neq 0$ for all n and radius of convergence R. Let

$$L = \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right|.$$

1. If $L \in \mathbb{R}\backslash\{0\}$, then $R = \frac{1}{L}$.
2. If $L = 0$, then $R = \infty$.
3. If $L = \infty$, then $R = 0$.

Proof. Let $a_n = c_n(x-a)^n$. Then

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n} \right| = \lim_{n\to\infty} \left(\left| \frac{c_{n+1}}{c_n} \right| |x-a| \right).$$

It follows from this that if $L = \infty$, then $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$, provided that $x \neq a$, so that the series diverges for all $x \neq a$, i.e. $R = 0$.

It also follows that if $L \in \mathbb{R}$, then $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L|x-a|$. Hence, if $L \neq 0$, then, using the ratio test, it follows that the series converges if $L|x-a| < 1$, i.e. if $|x-a| < \frac{1}{L}$; and diverges if $L|x-a| > 1$, i.e. if $|x-a| > \frac{1}{L}$; so that $R = \frac{1}{L}$. Finally, if $L = 0$, then it follows that $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 0$, so that the series converges for all x, i.e. $R = \infty$. $\square$

Theorem D

Suppose that $\sum_{n=1}^{\infty} c_n (x-a)^n$ is a power series with radius of convergence R. Let $L = \lim_{n\to\infty} \sqrt[n]{|c_n|}$.

1. If $L \in \mathbb{R}\backslash\{0\}$, then $R = \frac{1}{L}$.
2. If $L = 0$, then $R = \infty$.
3. If $L = \infty$, then $R = 0$.

Prove this.

Lemma E
If $\epsilon > 0$, then $|nx^{n-1}| < (|x| + \epsilon)^n$ for all n large enough.

Proof. Let $\epsilon > 0$. Since $\lim\limits_{n\to\infty} |nx^{n-1}|^{\frac{1}{n}} = \lim\limits_{n\to\infty} n^{\frac{1}{n}}|x|^{1-\frac{1}{n}} = |x|$, it follows that there exists an $N \in \mathbb{N}$ such that if $n > N$, then $|x| - \epsilon < |nx^{n-1}|^{\frac{1}{n}} < |x| + \epsilon$. It follows that $|nx^{n-1}| = (|nx^{n-1}|^{\frac{1}{n}})^n < (|x| + \epsilon)^n$ for all $n > N$. $\square$

Theorem F
The power series $\sum\limits_{n=0}^{\infty} a_n(x-a)^n$ converges on $(a-R, a+R)$ if and only if the power series $\sum\limits_{n=1}^{\infty} na_n(x-a)^{n-1}$ converges on $(a-R, a+R)$.

Note that $\sum\limits_{n=1}^{\infty} na_n(x-a)^{n-1} = \sum\limits_{n=0}^{\infty} \left[\frac{d}{dx}(a_n(x-a)^n)\right]$.

Proof of Theorem F. We only prove the case where $a = 0$. So suppose that the series $\sum\limits_{n=0}^{\infty} a_n x^n$ converges on $(-R, R)$. Then it follows from Theorem A that the series is absolutely convergent on this interval. Let $x \in (-R, R)$. Take an $\epsilon > 0$ such that $|x| < |x| + \epsilon < R$ (e.g. $\epsilon = \frac{1}{2}(R - |x|)$). Since $|x| + \epsilon \in (-R, R)$, it follows that $\sum\limits_{n=0}^{\infty} |a_n(|x| + \epsilon)^n|$ converges. It follows from Lemma E that $0 \leq |na_n x^{n-1}| \leq |a_n|(|x| + \epsilon)^n$ for all n large enough, say for $n \geq N$. Using the comparison test it then follows that $\sum\limits_{n=1}^{\infty} |na_n x^{n-1}|$ converges, and hence that $\sum\limits_{n=1}^{\infty} na_n x^{n-1}$ converges.

Conversely, suppose that the series $\sum\limits_{n=1}^{\infty} na_n x^{n-1}$ converges on $(-R, R)$. Then it follows from Theorem A that $\sum\limits_{n=1}^{\infty} |na_n x^{n-1}|$ converges on $(-R, R)$ and hence that $|x| \sum\limits_{n=1}^{\infty} |na_n x^{n-1}| = \sum\limits_{n=1}^{\infty} |na_n x^n|$ converges on $(-R, R)$. Since $0 \leq |a_n x^n| \leq n|a_n x^n| = |na_n x^n|$ for all $n \in \mathbb{N}$, using the comparison test it follows that $\sum\limits_{n=1}^{\infty} |a_n x^n|$ converges on $(-R, R)$, and hence that $\sum\limits_{n=0}^{\infty} a_n x^n$ converges on $(-R, R)$. $\square$

It follows from Theorem F that the radii of convergence of the series

$$\sum_{n=0}^{\infty} a_n(x-a)^n, \quad \sum_{n=0}^{\infty} \frac{d}{dx}(a_n(x-a)^n), \quad \sum_{n=0}^{\infty} \frac{d^2}{dx^2}(a_n(x-a)^n), \quad \ldots$$

are all the same.

Theorem G (Theorem 2, p.794 in Stewart)

If the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ has radius of convergence $R > 0$, then the function

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and hence continuous, and hence integrable) on the interval $(a-R, a+R)$ and

1. $f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$,
2. $\int f(x)\,dx = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$.

Both the power series in (1) and (2) have radius of convergence R.

The equations in (1) and (2) can be rewritten as follows:

$$\frac{d}{dx}\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] = \sum_{n=0}^{\infty}\left[\frac{d}{dx}(c_n(x-a)^n)\right],$$

and

$$\int\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right]dx = \sum_{n=0}^{\infty}\left[\int c_n(x-a)^n\,dx\right].$$

Proof of (2). We only prove the case where $a = 0$. Since $\sum_{n=0}^{\infty} c_n x^n$ converges on $(-R, R)$, it follows from Theorem A that $\sum_{n=0}^{\infty} |c_n x^n|$ converges on $(-R, R)$. If $n \in \mathbb{N}$, then $0 \leq \left|\frac{c_n x^n}{n+1}\right| = \frac{1}{n+1}|c_n x^n| \leq |c_n x^n|$, and hence, using the comparison test, it follows that $\sum_{n=0}^{\infty}\left|\frac{c_n x^n}{n+1}\right|$ converges on $(-R, R)$. Hence $\sum_{n=0}^{\infty}\left|\frac{c_n x^{n+1}}{n+1}\right| = |x|\sum_{n=0}^{\infty}\left|\frac{c_n x^n}{n+1}\right|$ also converges on $(-R, R)$, so that $\sum_{n=0}^{\infty}\frac{c_n x^{n+1}}{n+1}$ converges on $(-R, R)$. It follows from (1) that

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty}\frac{c_n x^{n+1}}{n+1}\right) = \sum_{n=0}^{\infty}\left[\frac{d}{dx}\left(\frac{c_n x^{n+1}}{n+1}\right)\right] = \sum_{n=0}^{\infty} c_n x^n = f(x),$$

so that $\int f(x)\,dx = C + \sum_{n=0}^{\infty}\frac{c_n x^{n+1}}{n+1}$. $\square$

Theorem H (Theorems 5 and 6, p.800 in Stewart)

If f has a power series representation around a, i.e.

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n \text{ for } |x-a| < R$$

then

$$c_n = \frac{f^{(n)}(a)}{n!},$$

so that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots.$$

Prove this. The series above is called the *Taylor series of f around a.*

If $a = 0$, then the series becomes:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots,$$

which is called the *Maclaurin series of f.*

Suppose that $f^{(n)}(a)$ exists for all $n \in \mathbb{N}$. Let

$$T_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Then T_n is called the *n-th degree Taylor polynomial of f* at a.

Theorem I (Theorem 8, p.801 in Stewart)
If $f(x) = T_n(x) + R_n(x)$, where T_n is the n-th degree Taylor polynomial of f at a, and $\lim_{n\to\infty} R_n(x) = 0$ for $|x-a| < R$, then f is equal to its Taylor series for $|x-a| < R$.

Prove this. R_n is called the *remainder* of the Taylor series.

Rolle's Theorem
If f is continuous on $[a,b]$ and differentiable on (a,b), and $f(a) = f(b)$, then there exists a $d \in (a,b)$ such that $f'(d) = 0$.

Taylor's Theorem
Suppose that I is an open interval and the $(n+1)$-st derivative of f exists at each point of I. If $a, b \in I$ with $a < b$, then there exists a $d_n \in (a,b)$ such that

$$\begin{aligned} f(b) &= f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \cdots \\ &\qquad \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}(d_n)}{(n+1)!}(b-a)^{n+1}. \end{aligned}$$

Proof. Let $L = f(b) - f(a) - f'(a)(b-a) - \frac{f''(a)}{2!}(b-a)^2 - \cdots - \frac{f^{(n)}(a)}{n!}(b-a)^n$, and define $k_n = \frac{(n+1)!L}{(b-a)^{n+1}}$.
Define $\phi : I \to \mathbb{R}$ by

$$\begin{aligned} \phi(x) &= f(b) - f(x) - f'(x)(b-x) - \frac{f''(x)}{2!}(b-x)^2 - \cdots \\ &\qquad \cdots - \frac{f^{(n)}(x)}{n!}(b-x)^n - \frac{k_n}{(n+1)!}(b-x)^{n+1}. \end{aligned}$$

Then $\phi(a) = L - \frac{k_n(b-a)^{n+1}}{(n+1)!} = 0$ and $\phi(b) = 0$. Furthermore, since $f^{(n+1)}(x)$ exists for each $x \in I$, it follows that each of the functions $f, f', f'', \ldots, f^{(n)}$ is continuous and differentiable on I, so that ϕ is continuous and differentiable on I. In particular, ϕ is continuous on $[a, b]$ and differentiable on (a, b). Hence it follows from Rolle's Theorem that there exists a $d_n \in (a, b)$ such that $\phi'(d_n) = 0$. But

$$\begin{aligned}
\phi'(x) &= -f'(x) + f'(x) - f''(x)(b-x) + f''(x)(b-x) - \frac{f'''(x)}{2!}(b-x)^2 + \cdots \\
&\quad \cdots + \frac{f^{(n)}(x)}{(n-1)!}(b-x)^{n-1} - \frac{f^{(n+1)}(x)}{n!}(b-x)^n + \frac{k_n}{n!}(b-x)^n \\
&= \frac{(b-x)^n}{n!}(k_n - f^{(n+1)}(x)),
\end{aligned}$$

so that

$$\phi'(d_n) = \frac{(b-d_n)^n}{n!}(k_n - f^{(n+1)}(d_n)) = 0.$$

Since $a < d_n < b$, it follows that $b - d_n \neq 0$, and consequently $f^{(n+1)}(d_n) = k_n$. Since $k_n = \frac{(n+1)!L}{(b-a)^{n+1}}$, it follows that $L = \frac{f^{(n+1)}(d_n)}{(n+1)!}(b-a)^{n+1}$, but since $L = f(b) - f(a) - f'(a)(b-a) - \frac{f''(a)}{2!}(b-a)^2 - \cdots - \frac{f^{(n)}(a)}{n!}(b-a)^n$, the result follows. $\square$

Corollary (Taylor's Theorem)
Suppose that I is an open interval and the $(n+1)$-st derivative of f exists at each point of I. If $a \in I$, then there exists for any $x \neq a$ in this interval a number d_n properly between a and x such that

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(d_n)}{(n+1)!}(x-a)^{n+1}.$$

Hence $R_n(x) = \frac{f^{(n+1)}(d_n)}{(n+1)!}(x-a)^{n+1}$. The latter is called the Lagrange Formula for the remainder of the Taylor series of f.