Lecture 12

Sifting algorithm

Apply to a list of vectors

$\{ \vec{v_1}, \cdots, \vec{v_n} \}$

in a vecter space.

1. Is $\vec{v_1} = \vec{0}$?

   Yes : remove it, and carry on.

   No : carry on.

2. Is $\vec{v_2} = k\vec{v_1}$?

   Yes: remove it

   No: carry on.

3. Is $\vec{v_3}$ a l.c. of $\vec{v_1}, \vec{v_2}$?

Yes: remove it

No: carry on

$\vdots$

<u>Example</u> Sift the following list of vectors in $\mathbb{R}^3$:

$\vec{v_1} = (1, 2, -1)$ $\quad$ $\vec{v_2} = (0, 0, 0)$

$\vec{v_3} = (3, 6, -3)$ $\quad$ $\vec{v_4} = (1, 0, 5)$

$\vec{v_5} = (5, 4, 13)$ $\quad$ $\vec{v_6} = (1, 1, 0)$

• Is $\vec{v_1} = \vec{0}$?

No. Leave it.

current list : $\{\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}, \vec{v_5}, \vec{v_6}\}$

here.

• Is $\vec{v_2} = k\vec{v_1}$?

Yes. $\vec{v_2} = 0.\vec{v_1}$.

Remove it.

current list : $\{\vec{v_1}, \vec{v_3}, \vec{v_4}, \vec{v_5}, \vec{v_6}\}$

• Is $\vec{v_3} = k\vec{v_1}$?

Yes. $\vec{v_3} = 3\vec{v_1}$.

Remove it.

current list : $\{\vec{v_1}, \vec{v_4}, \vec{v_5}, \vec{v_6}\}$

- Is $\vec{v_4} = k\vec{v_1}$?

No. Leave it.

Current list : $\{\vec{V_1}, \vec{V_4}, \vec{V_5}, \vec{V_6}\}$

- Is $\vec{v_5} = a\vec{v_1} + b\vec{v_4}$?

$$(5,4,13) = a(1,2,-1) + b(1,0,5)$$
$$= (a+b,\ 2a,\ -a+5b)$$

$$2a = 4 \Rightarrow a = 2$$
$$a+b = 5 \Rightarrow b = 3$$
$$-a+5b = 13 \ \checkmark$$

Yes: $\vec{v_5} = 2\vec{v_1} + 3\vec{v_4}$.

Remove it.

Current list : $\{\vec{v_1}, \vec{v_4}, \vec{v_6}\}$

- Is $\vec{v_6} = a\vec{v_1} + b\vec{v_4}$?

No. Leave it.

Final sifted list:

$$\{\vec{v_1}, \vec{v_4}, \vec{v_6}\}.$$

Lemma If

$$B = \{\vec{v_1}, \cdots, \vec{v_n}\}$$

is a list of vectors in a vector space, then sifting B produces a linearly independent list B'.

Proof This follows from the "Linear Combination of Preceding Vectors" ~~criterion~~ formulation of linear dependence.

By construction, the first vector in B' is not $\vec{0}$, and no vector in B' is a linear comb. of the prec. vectors.

$\therefore$ B' is linearly independent. $\square$

Lemma If

$$\mathcal{B} = \{ \vec{v_1}, \cdots, \vec{v_n} \}$$

spans V, then the sifted list $\mathcal{B}'$ *also* spans V.

Proof If you remove a vector $\vec{v_i}$ from a list of vectors, which is a l.comb. of the *other* vectors in the list, the *span* of the vectors remains the same (prev. exercise).

$\square$

<u>Corollary</u> Any linearly independent list of vectors $A = \{\vec{v}_1, \cdots, \vec{v}_n\}$ in a finite-dim vector space V can be <u>extended</u> to a basis for V.

<u>Proof</u> Since V is finite-dim, it has a basis

$$\mathcal{B} = \{\vec{e}_1, \cdots, \vec{e}_m\}$$

Now form the big list

$$\cancel{C = \{\vec{e}_1, \cdots, \vec{e}_m, \vec{v}_1, \cdots, \vec{v}_n\}}$$

$$C = \{\vec{v}_1, \cdots, \vec{v}_n, \vec{e}_1, \cdots, \vec{e}_m\}$$

Sift C to arrive at a list C'.

None of the $\vec{v_i}$ will have been removed from C, as that would mean that $\vec{v_i}$ is a l.c. of $\vec{v_1}, \dots, \vec{v_{i-1}}$, which contradicts $A = \{\vec{v_1}, \dots, \vec{v_n}\}$ being lin. ind.

So indeed C' is an extension of A. And:

$$\left.\begin{array}{l} \bullet \text{ it spans } V \\ \bullet \text{ it is lin. ind.} \end{array}\right] \text{just proved}$$

$\therefore$ C' is a basis $\square$

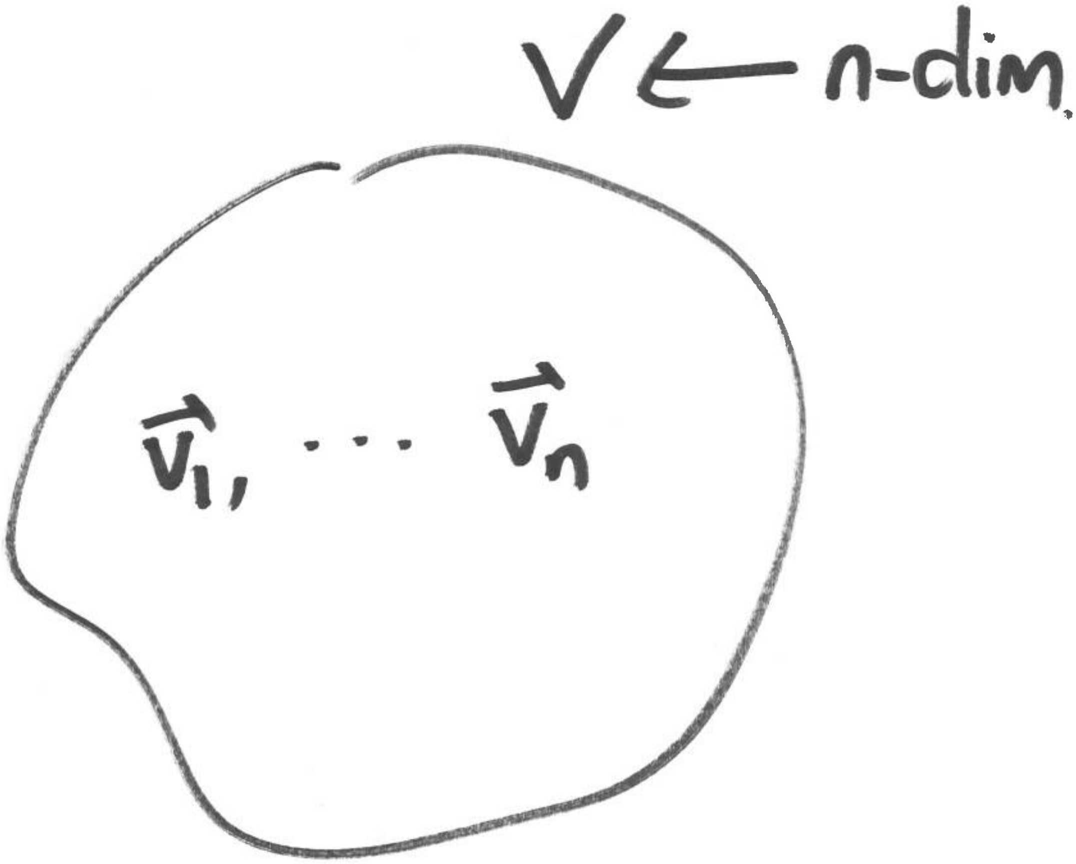
Corollary If

$B = \{ \vec{v}_1, \dots, \vec{v}_n \}$

is a linearly indepent list of n vectors in an n-dimensional vector space V, then B is a basis for V.

Proof By previous corollary, we can extend B to a basis B' for V. But we could not have added any new vectors to B to form B', since

vectors in a lin. ind. list $\leq$ # vectors in a list which span V



(Bumping off prop.)

i.e. # vectors in

vectors in $B' \leq n$

$\therefore$ no vectors were added to B to form B',

i.e. B is a basis for V

$\square$