Rye van reëele getalle / Sequences of real numbers §11.1

Definition 11
A sequence $\{a_n\}$ is *bounded above / na bo begrens* if there exists a number M such that $a_n \leq M$ for all $n \in \mathbb{N}$, and
bounded below / na onder begrens if there exists a number m such that $m \leq a_n$ for all $n \in \mathbb{N}$.
If $\{a_n\}$ is both bounded above and below, then $\{a_n\}$ is called a *bounded sequence / begrensde ry*.

Hence a sequence is bounded if and only if there exists a number K such that $|a_n| \leq K$ for all $n \in \mathbb{N}$.

Theorem 3
If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ for all natural numbers n, then $\lim_{n\to\infty} a_n = L$.

Theorem 3.4.4, p.234
If $r > 0$, then

$$\lim_{x\to\infty} \frac{1}{x^r} = 0.$$

Equation 4, p.737
If $r > 0$, then

$$\lim_{n\to\infty} \frac{1}{n^r} = 0.$$

Limit theorems for sequences

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then:

$$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} c a_n = c \lim_{n\to\infty} a_n$$

$$\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n . \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \text{ if } \lim_{n\to\infty} b_n \neq 0$$

$$\lim_{n\to\infty} c = c$$

Theorem 6
If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

- Suppose that $c_n \in \mathcal{D}_f$, $\lim_{n\to\infty} c_n = c$ and f is continuous at c. Then $\lim_{n\to\infty} f(c_n) = f(c)$.

The Squeeze Theorem / Die Knyptangstelling
If $a_n \leq b_n \leq c_n$ for all $n \geq n_0$ and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L,$$

then $\lim_{n\to\infty} b_n = L$.

L'Hospital's Rules, p.492

Homework
Ex. 11.1 nr. 25, 35, 37, 42, 43, 47

Theorem 9
The sequence $\{r^n\}$ is convergent if $-1 < r \leq 1$ and divergent for all other values of r. Also,

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if} \quad -1 < r < 1 \\ 1 & \text{if} \quad r = 1. \end{cases}$$

Definition 10

A sequence $\{a_n\}$ is called *increasing / stygend* if $a_n \leq a_{n+1}$ for all $n \geq 1$, and
decreasing / dalend if $a_n \geq a_{n+1}$ for all $n \geq 1$.
The sequence $\{a_n\}$ is called *monotone / monotoon* if it is either increasing or decreasing.

Definition 11

A sequence $\{a_n\}$ is *bounded above* if there exists a number M such that $a_n \leq M$ for all $n \in \mathbb{N}$, and
bounded below if there exists a number m such that $m \leq a_n$ for all $n \in \mathbb{N}$.
If $\{a_n\}$ is both bounded above and below, then $\{a_n\}$ is called a *bounded sequence*.

Theorem 12 (Monotone Sequence Theorem / Monotone Ry-Stelling)

1. Suppose that $\{a_n\}$ is an increasing sequence. Then $\{a_n\}$ is bounded above if and only if $\{a_n\}$ converges, in which case
$$\lim_{n\to\infty} a_n = \sup\{a_n : n \in \mathbb{N}\}.$$

2. Suppose that $\{a_n\}$ is a decreasing sequence. Then $\{a_n\}$ is bounded below if and only if $\{a_n\}$ converges, in which case
$$\lim_{n\to\infty} a_n = \inf\{a_n : n \in \mathbb{N}\}.$$

Theorem 12 (Monotone Sequence Theorem)

1. Suppose that $\{a_n\}$ is an increasing sequence. Then $\{a_n\}$ is bounded above if and only if $\{a_n\}$ converges, in which case

$$\lim_{n\to\infty} a_n = \sup\{a_n : n \in \mathbb{N}\}.$$

The Completeness Property of $\mathbb{R}$ / Die Volledigheidseienskap van $\mathbb{R}$

Each non-empty set of real numbers which is bounded above has a smallest upper bound.

Theorem 10 (notes: improper integrals)

Let $A \subset \mathbb{R}$. Then $\sup A = S$ if and only if the following two conditions hold:

1. S is an upper bound for A.

2. For each $\epsilon > 0$ there exists an $x \in A$ such that $x > S - \epsilon$.

Example

Consider the sequence $\{a_n\}$, where $a_1 = 2$ and $a_{n+1} = \frac{1}{3-a_n}$ for $n \in \mathbb{N}$.

1. Show that $0 < a_n \leq 2$ for all n, and

2. determine $\lim\limits_{n\to\infty} a_n$.

Homework

Ex. 11.1 nr. 69, 73, 77, 79, 81