Lecture 17

Addition to 1.6 (Subspaces)

Example (Polynomials in multiple variables)

A monomial in two variables x, y is an expression of the form

$$x^m y^n \quad (m, n \text{ non negative integers})$$

The degree of the monomial is $m+n$.

eg.

$$\underbrace{x^3 y^2}_{\text{degree } 5}, \quad \underbrace{y^7}_{\text{deg. } 7}$$

A __polynomial__ in x and y is a linear comb. of monomials.

The __degree__ of the polynomial is the highest degree of the monomials occuring in the lin. comb.

eg. $p = 2x^3y^2 - y^7 + 3xy$

has degree 7.

We write

$$\mathrm{Poly}_n[x,y] := \left\{ \begin{array}{l} \text{all polynomials} \\ \text{in } x, y \text{ of} \\ \text{degree} \leq n \end{array} \right\}$$

We can regard $p \in \mathrm{Poly}_n[x,y]$ as a function

$$p : \mathbb{R}^2 \longrightarrow \mathbb{R}$$
$$(x,y) \longmapsto p(x,y)$$

So,

$$\mathrm{Poly}_n[x,y] \overset{\text{subspace}}{\subseteq} \mathrm{Fun}(\mathbb{R}^2, \mathbb{R})$$

and so $\mathrm{Poly}_n[x,y]$ is a vector space.

Also eg. $\mathrm{Poly}_n[x,y,z]$, etc.

Note

$$\mathrm{Poly}_n[x] = \mathrm{Poly}_n$$

<u>Example</u> (Vector space of polynomial vector fields)

We write

$$\mathrm{Vect}_n(\mathbb{R}^2) := \left\{ \begin{array}{l} \text{all vector fields} \\ \vec{V} = (p(x,y), q(x,y)) \\ \text{with} \\ p, q \in \mathrm{Poly}_n[x,y] \end{array} \right.$$

eg.

$$\vec{V} = (2x^2y - y, \; x^3y)$$

$$\in \mathrm{Vect}_4(\mathbb{R}^2)$$

Back to 3.1

Example (Gradient as a linear map)

The operation 'take the gradient' can be thought of as a linear map

$$\nabla : \mathrm{Poly}_n[x,y] \longrightarrow \mathrm{Vect}_{n-1}(\mathbb{R}^2)$$

$$f \longmapsto \nabla f$$

eg. $x + xy \longmapsto (1 + y, x)$

Linear? $\nabla(f+g) = \nabla f + \nabla g$

$\nabla(kf) = k\nabla(f)$

Example (Double integral as a linear map)

Let $D \subseteq \mathbb{R}^2$ be a region in the plane. Have linear map

$$I : Poly_2[x,y] \longrightarrow \mathbb{R}$$
$$f \longmapsto \iint_D f\,dA$$

Linear?

① $\iint_D (f+g)\,dA = \iint_D f\,dA + \iint_D g\,dA$

Example (Identity linear map)

Let V be a vector space. Have linear map

$$\begin{aligned} \mathrm{id} : V &\longrightarrow V \\ \vec{v} &\longmapsto \vec{v} \end{aligned}$$

<u>Example</u> (Shift) Define

$$S : \mathrm{Poly}_n \longrightarrow \mathrm{Poly}_n$$
$$p \longmapsto S(p)$$

where $S(p)(x) := p(x-1)$

eg. $p(x) = 3x^2 + 4x - 2$

$$S(p)(x) = p(x-1)$$
$$= 3(x-1)^2 + 4(x-1) - 2$$
$$= 3x^2 - 2x - 3$$

Lemma Suppose $T : V \to W$ is a linear map. Then:

1. $T(\vec{0}_V) = \vec{0}_W$

2. $T(-\vec{v}) = -T(\vec{v})$

Proof (1). $T(\vec{0}_V) = T(0.\vec{0}_V)$ [R8]

$= 0.T(\vec{0}_V)$ [T is linear]

$= \vec{0}_W$ [R8]

$\square$

Proposition (Sufficient to Define a Linear Map on a Basis) Suppose

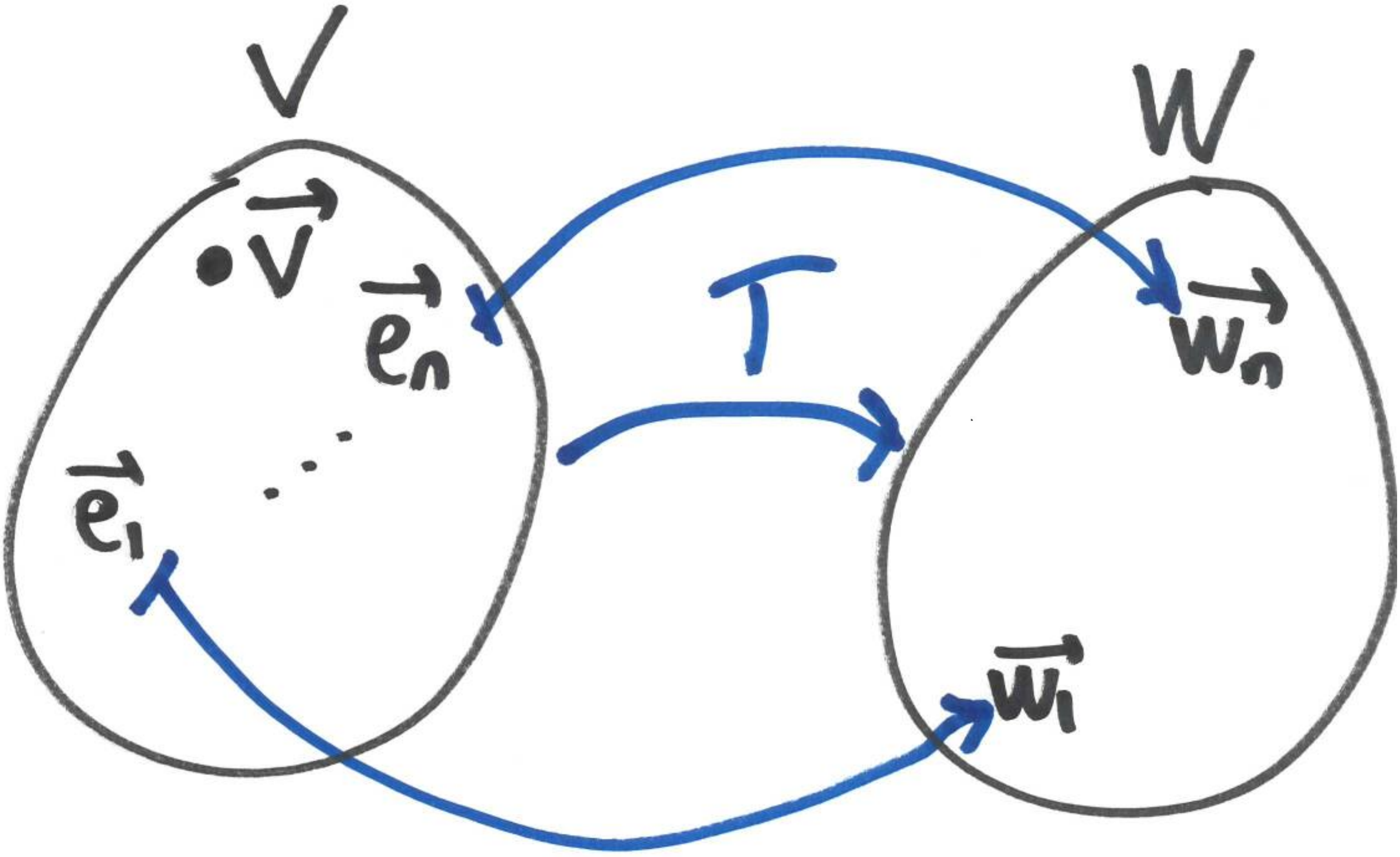
$\mathcal{B} = \{ \vec{e}_1, \cdots, \vec{e}_n \}$ is a basis for V, and let W be any vector space, with

$$\vec{w}_1, \cdots, \vec{w}_n \in W.$$

Then there exists a unique linear map

$$T : V \longrightarrow W$$

such that $\vec{e}_i \longmapsto \vec{w}_i$



Proof Existence

Define T by

$$T(\vec{v}) := \sum_{i=1}^{n} [\vec{v}]_{B,i} \, \vec{w_i}$$

$$= [\vec{v}]_{B,1} \vec{w_1} + \cdots + [\vec{v}]_{B,n} \vec{w_n}$$

This is a well-defined function

$$T : V \longrightarrow W$$

Must check it satisfies the rules to be a linear map.