Gamma functions and Beta functions

Definition 1
The gamma function $\Gamma : (0, \infty) \to \mathbb{R}$ is defined by

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx.$$

Theorem 2

(1) $\Gamma(n+1) = n\Gamma(n)$ for each $n \in (0, \infty)$.

(2) $\Gamma(n+1) = n!$ for each $n \in \mathbb{N}$.

(3) $\displaystyle\int_0^\infty e^{-ax} x^{n-1}\, dx = \frac{\Gamma(n)}{a^n}$ for each $n \in (0, \infty)$ and for each $a \in (0, \infty)$.

See examples on pp. B5–B6.

Homework
p. B14 nr. 1, 3(a),(b),(c)

Definition 3
The beta function $\beta : (0, \infty) \times (0, \infty) \to \mathbb{R}$ is defined by

$$\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx.$$

Theorem 3
For each $(m, n) \in (0, \infty) \times (0, \infty)$ we have

(1) $\beta(m, n) = \beta(n, m)$.

(3) $\beta(m, n) = 2 \int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$.

(6) $\beta(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$.

(7) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

See examples on pp. B11–B14: (10), (11), (13), (15).

Homework
p. B14 nr. 3(e),(g),(k),(l)