# IMPROPER INTEGRALS

## (ADDITIONAL NOTES TO STEWART §7.8)

### Definition 1

Suppose that $a \in \mathbb{R}$ and $f$ is continuous on $[a, \infty)$. Then

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

provided that this limit exists.
Suppose that $b \in \mathbb{R}$ and $f$ is continuous on $(-\infty, b]$. Then

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

provided that this limit exists.
$\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^b f(x)\,dx$ are called *improper integrals of Type 1.* An improper integral of Type 1 is called *convergent* if the limit involved exists, and it is called *divergent* if the limit involved does not exist.
If both $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ are convergent ($a$ any number), then we define

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx.$$

$\int_{-\infty}^\infty f(x)\,dx$ is also called an improper integral of Type 1.

### Definition 2

Suppose that $f$ is continuous but unbounded on $[a, b)$. Then

$$\int_a^b f(x)\,dx = \lim_{t\to b-} \int_a^t f(x)\,dx$$

provided that this limit exists.
Suppose that $f$ is continuous but unbounded on $(a, b]$. Then

$$\int_a^b f(x)\,dx = \lim_{t\to a+} \int_t^b f(x)\,dx$$

provided that this limit exists.
These integrals are called *improper integrals of Type 2.* An improper integral of Type 2 is called *convergent* if the limit involved exists, and it is called *divergent* if the limit involved does not exist.
Suppose that $a < c < b$, $f$ is discontinuous at $c$ and $f$ is continuous but unbounded on $[a, c)$ and on $(c, b]$. If both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then we define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Such an integral is also called an improper integral of Type 2.

**Definition 3**

Suppose that $a \in \mathbb{R}$, $f$ is continuous on $(a, \infty)$ and for any $c > a$ we have that $f$ is unbounded on $(a, c]$. If the improper integrals $\int_a^c f(x)\,dx$ (of Type 2) and $\int_c^\infty f(x)\,dx$ (of Type 1) converge, then we define

$$\int_a^\infty f(x)\,dx = \int_a^c f(x)\,dx + \int_c^\infty f(x)\,dx.$$

Such an integral is called an *improper integral of Type 3.*

**Lemma 4**

1. Suppose that $a < b$. If $f$ is continuous on $[a, \infty)$, then $\int_a^\infty f(x)\,dx$ is convergent if and only if $\int_b^\infty f(x)\,dx$ is convergent.
2. If both $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ converge and $\alpha$ and $\beta$ are numbers, then $\int_a^\infty [\alpha f(x) + \beta g(x)]\,dx$ converges as well.

Similar results also hold for improper integrals of Type 2.

**Lemma 5**

Suppose that $a < b$. Then $\int_b^\infty \frac{1}{(x-a)^p}\,dx$ is convergent for $p > 1$ and divergent for $p \leq 1$.

**Lemma 6**

Suppose that $a \in \mathbb{R}$. Then $\int_a^\infty e^{-px}\,dx$ is convergent for $p > 0$ and divergent for $p \leq 0$.

**Lemma 7**

Suppose that $a < b$. Then $\int_a^b \frac{1}{(x-a)^p}\,dx$ and $\int_a^b \frac{1}{(x-b)^p}\,dx$ are convergent if $p < 1$ and divergent if $p \geq 1$.

Before we can obtain further results, we must first study the concept of a *smallest upper bound.*

**Definition 8**

1. A set $A$ of real numbers is *bounded above* if there exists a real number $M$ such that $x \leq M$ for all $x \in A$. We call $M$ an *upper bound* for $A$.
2. A set $A$ of real numbers is *bounded below* if there exists a real number $m$ such that $x \geq m$ for all $x \in A$. We call $m$ a *lower bound* for $A$.
3. A set $A \subset \mathbb{R}$ is *bounded* if $A$ is both bounded below and bounded above.

4. If $A \subset \mathbb{R}$ is not bounded, we say that $A$ is *unbounded.*

**Definition 9**

If a set $A \subset \mathbb{R}$ has a smallest upper bound $S$, then we call $S$ the *supremum* or *sup* of $A$ and write $S = \sup A$. This means:

1. $S$ is an upper bound for $A$.

2. If $T$ is any upper bound for $A$, then $S \leq T$.

**Theorem 10**

Let $A \subset \mathbb{R}$. Then $\sup A = S$ if and only if the following two conditions hold:

1. $S$ is an upper bound for $A$.

2. For each $\epsilon > 0$ there exists an $x \in A$ such that $x > S - \epsilon$.

**Proof**

If $\sup A = S$, then (1) holds, since $S$ is an upper bound (the smallest one) for $A$. For any $\epsilon > 0$ we have that $S - \epsilon < S$, and hence $S - \epsilon$ is smaller than the smallest upper bound $S$, so that $S - \epsilon$ is not an upper bound for $A$. Hence there exists at least one $x \in A$ such that $x > S - \epsilon$. So (2) holds. Conversely, suppose that (1) and (2) hold. Then $S$ is an upper bound for $A$. It follows from (2) that each number smaller than $S$ is *not* an upper bound for $A$. Hence $S$ is the smallest upper bound for $A$, i.e. $S = \sup A$. $\square$

**Definition 11**

If a set $A \subset \mathbb{R}$ has a greatest lower bound $I$, then we call $I$ the *infimum* or *inf* of $A$ and write $I = \inf A$. This means:

1. $I$ is a lower bound for $A$.

2. If $J$ is any lower bound for $A$, then $I \geq J$.

**Theorem 12**

Let $A \subset \mathbb{R}$. Then $\inf A = I$ if and only if the following two conditions hold:

1. $I$ is a lower bound for $A$.

2. For each $\epsilon > 0$ there exists an $x \in A$ such that $x < I + \epsilon$.

The proof of Theorem 12 is similar to that of Theorem 10.

Finally, we formulate the following axiom:

**The Completeness Property of $\mathbb{R}$**

Each non-empty set of real numbers which is bounded above has a smallest upper bound.

**Corollary**

Each non-empty set of real numbers which is bounded below has a greatest lower bound.

We are now ready to prove the following important theorem:

**Theorem 13**

Suppose that $f$ is increasing on $[a, \infty)$. Then $\lim_{x\to\infty} f(x)$ exists if and only if $f$ is bounded above on $[a, \infty)$, in which case

$$\lim_{x\to\infty} f(x) = \sup\{f(x) : x \in [a, \infty)\}.$$

**Proof**

Suppose that $f$ is increasing on $[a, \infty)$.
We first prove that if $\lim_{x\to\infty} f(x)$ exists, then $f$ is bounded above on $[a, \infty)$ (so that $\sup\{f(x) : x \in [a, \infty)\}$ exists):
So suppose that $\lim_{x\to\infty} f(x)$ exists, and let $L = \lim_{x\to\infty} f(x)$. Then there exists a number $N \geq a$ so that $|f(x) - L| < 1$ for all $x \geq N$ (take $\epsilon = 1$). Hence $-1 < f(x) - L < 1$, so that $L - 1 < f(x) < L + 1$, for all $x \geq N$. In particular, $f(N) < L + 1$. Since $f$ is increasing on $[a, \infty)$, we have that $f(x) \leq f(N)$, so that $f(x) < L + 1$, for all $x \in [a, N]$. It follows that $f(x) < L + 1$ for all $x \in [a, \infty)$, and hence $f$ is bounded above on $[a, \infty)$.
Next we prove that if $f$ is bounded above on $[a, \infty)$, then $\lim_{x\to\infty} f(x)$ exists and $\lim_{x\to\infty} f(x) = \sup\{f(x) : x \in [a, \infty)\}$:
So suppose that $f$ is bounded above on $[a, \infty)$. Let $S = \sup\{f(x) : x \in [a, \infty)\}$. Then there exists for each $\epsilon > 0$ a $y \in [a, \infty)$ such that $S - \epsilon < f(y)$. Since $f$ is increasing on $[a, \infty)$, we have that $f(y) \leq f(x)$, so that $S - \epsilon < f(x)$, for all $x \geq y$. Also, $f(x) \leq S < S + \epsilon$ for all $x \in [a, \infty)$, so that $S - \epsilon < f(x) < S + \epsilon$, i.e. $|S - f(x)| < \epsilon$, for all $x \geq y$. It follows from this that $\lim_{x\to\infty} f(x)$ exists and $\lim_{x\to\infty} f(x) = S$.
The result follows. $\square$

Using the above theorem we can extend the theory of improper integrals further:

**The comparison test for improper integrals of Type 1**

Suppose that $a \in \mathbb{R}$ and $f$ and $g$ are continuous functions on $[a, \infty)$ such that $0 \leq g(x) \leq f(x)$ for all $x \geq a$.

1. If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is convergent.
2. If $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is divergent.

**Proof**

1. Let $G(t) = \int_a^t g(x)\,dx$ and $F(t) = \int_a^t f(x)\,dx$. Since $g(x) \leq f(x)$ for all $x \geq a$, it follows that $G(t) \leq F(t)$ for all $t \geq a$. The fact that $f(x) \geq 0$ and $g(x) \geq 0$ for all $x \geq a$ implies that the functions $F$ and $G$ are increasing on $[a, \infty)$.

   Now suppose that $\int_a^\infty f(x)\,dx$ is convergent. Then $\lim_{t\to\infty} F(t)$ exists (with $\lim_{t\to\infty} F(t) = \int_a^\infty f(x)\,dx$). Since $F$ is increasing on $[a, \infty)$ and $\lim_{t\to\infty} F(t)$ exists, it follows from Theorem 13 that $\{F(t) : t \in [a, \infty)\}$ is bounded above. Since $G(t) \leq F(t)$ for all $t \geq a$, we have that $\{G(t) : t \in [a, \infty)\}$ is bounded above as well. Together with the fact that $G$ is increasing on $[a, \infty)$, it follows from Theorem 13 that $\lim_{t\to\infty} G(t)$ exists, i.e. that $\int_a^\infty g(x)\,dx$ converges.
2. This is the contrapositive of (1).

**The comparison test for improper integrals of Type 2**

Suppose that $a < b$, $f$ and $g$ are continuous but unbounded on $(a, b]$ and $0 \leq g(x) \leq f(x)$ for all $x \in (a, b]$.

1. If $\int_a^b f(x)\,dx$ is convergent, then $\int_a^b g(x)\,dx$ is convergent.
2. If $\int_a^b g(x)\,dx$ is divergent, then $\int_a^b f(x)\,dx$ is divergent.

The above theorem also holds if the interval $(a, b]$ is replaced by $[a, b)$. The proofs are similar to the proof of the comparison test for improper integrals of Type 1.

**The quotient test for improper integrals of Type 1**

Suppose that $a \in \mathbb{R}$ and $f$ and $g$ are continuous functions on $[a, \infty)$ such that $f(x) \geq 0$ and $g(x) \geq 0$ for all $x \geq a$.

1. If $\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$ and $\int_a^\infty g(x)\,dx$ is convergent, then $\int_a^\infty f(x)\,dx$ is convergent.
2. If $\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)} = A > 0$, then $\int_a^\infty g(x)\,dx$ is convergent if and only if $\int_a^\infty f(x)\,dx$ is convergent.
3. If $\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$ and $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is divergent.

**Proof**

1. Suppose that $\lim_{x\to\infty} \dfrac{f(x)}{g(x)} = 0$ and $\int_a^\infty g(x)\,dx$ is convergent. Since $\lim_{x\to\infty} \dfrac{f(x)}{g(x)} = 0$, there exists a number $N$ — and we may take $N \geq a$ — such that $\left|\frac{f(x)}{g(x)} - 0\right| < 1$ (take $\epsilon = 1$), and hence $\frac{f(x)}{g(x)} < 1$, or $0 \leq f(x) < g(x)$, for all $x \geq N$. By Lemma 4 $\int_N^\infty g(x)\,dx$ is convergent, and hence it follows from the comparison test that $\int_N^\infty f(x)\,dx$ is convergent. Hence it follows from Lemma 4 that $\int_a^\infty f(x)\,dx$ is convergent.

2. Suppose that $\lim_{x\to\infty} \frac{f(x)}{g(x)} = A > 0$. Let $m$ and $M$ be positive numbers such that $m < A < M$. Since $\lim_{x\to\infty} \dfrac{f(x)}{g(x)} = A$, it follows that there exists a number $N \geq a$ such that $m \leq \frac{f(x)}{g(x)} \leq M$ for all $x \geq N$. Hence $0 \leq mg(x) \leq f(x) \leq Mg(x)$ for all $x \geq N$. Now suppose that $\int_a^\infty g(x)\,dx$ is convergent. Then $\int_N^\infty g(x)\,dx$ is convergent, so that $\int_N^\infty Mg(x)\,dx$ is convergent, by Lemma 4. By the comparison test it follows that $\int_N^\infty f(x)\,dx$ is convergent, so that $\int_a^\infty f(x)\,dx$ is convergent. Conversely, suppose that $\int_a^\infty f(x)\,dx$ is convergent. Then $\int_N^\infty f(x)\,dx$ is convergent, so that, by the comparison test, $\int_N^\infty mg(x)\,dx$ is convergent. Hence it follows from Lemma 4 that $\int_a^\infty g(x)\,dx$ is convergent.

3. Suppose that $\lim_{x\to\infty} \dfrac{f(x)}{g(x)} = \infty$ and $\int_a^\infty g(x)\,dx$ is divergent. Since $\lim_{x\to\infty} \dfrac{f(x)}{g(x)} = \infty$, there exists a number $N$ such that $\frac{f(x)}{g(x)} > 1$, i.e. $0 \leq g(x) < f(x)$, for all $x \geq N$. It follows from Lemma 4 that $\int_N^\infty g(x)\,dx$ is divergent and hence, by the comparison test, that $\int_N^\infty f(x)\,dx$ is divergent. Hence $\int_a^\infty f(x)\,dx$ is divergent. $\square$

**The quotient test for improper integrals of Type 2**

Suppose that $a < b$, $f$ and $g$ are continuous but unbounded on $(a, b]$ and $f(x) \geq 0$ and $g(x) \geq 0$ for all $x \in (a, b]$.

1. If $\lim_{x\to a+} \dfrac{f(x)}{g(x)} = 0$ and $\int_a^b g(x)\,dx$ is convergent, then $\int_a^b f(x)\,dx$ is convergent.

2. If $\lim_{x\to a+} \dfrac{f(x)}{g(x)} = A > 0$, then $\int_a^b g(x)\,dx$ is convergent if and only if $\int_a^b f(x)\,dx$ is convergent.

3. If $\lim_{x\to a+} \dfrac{f(x)}{g(x)} = \infty$ and $\int_a^b g(x)\,dx$ is divergent, then $\int_a^b f(x)\,dx$ is divergent.

A similar theorem as the above also holds if the interval $(a, b]$ is replaced by $[a, b)$. The proofs are similar to the proof of the quotient test for improper integrals of Type 1.

We conclude with an important concept:

**Definition 14**

Suppose that $f$ is continuous on $[a, \infty)$. Then $\int_a^\infty f(x)\,dx$ is *absolutely convergent* if $\int_a^\infty |f(x)|\,dx$ is

convergent.
$\int_a^\infty f(x)\,dx$ is *conditionally convergent* if $\int_a^\infty f(x)\,dx$ is convergent and $\int_a^\infty |f(x)|\,dx$ is divergent.

Similar definitions also hold for improper integrals of Type 2.

**Theorem 15**

If $\int_a^\infty f(x)\,dx$ is absolutely convergent, then $\int_a^\infty f(x)\,dx$ is convergent.

**Proof**

For $x \geq a$ we have that $0 \leq f(x)+|f(x)| \leq 2|f(x)|$. Since $2\int_a^\infty |f(x)|\,dx$ is convergent, it follows using the comparison test that $\int_a^\infty [f(x)+|f(x)|]\,dx$ is convergent. Since $f(x) = [f(x)+|f(x)|] - |f(x)|$, the result follows from Lemma 4. $\square$