Lecture 15

2.4. Coordinate Vectors (cont.)

Lemma Let $\mathcal{B} = \{\vec{b_1}, \cdots, \vec{b_n}\}$ be a basis for a vector space V.

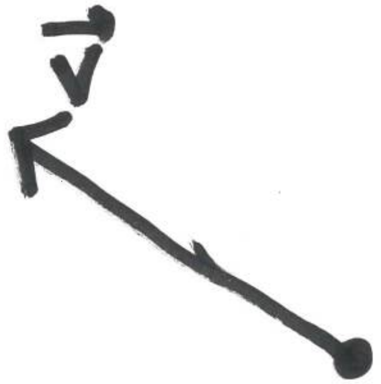
Then:

1. $[\vec{v} + \vec{w}]_{\mathcal{B}} = [\vec{v}]_{\mathcal{B}} + [\vec{w}]_{\mathcal{B}}$
2. $[k\vec{v}]_{\mathcal{B}} = k[v]_{\mathcal{B}}$

for all $\vec{v}, \vec{w} \in V$ and $k \in \mathbb{R}$.

2.5. Change of basis

Suppose we fix a vector $\vec{v} \in \mathbb{R}^2$



Suppose that $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$

and $C = \{\vec{c}_1, \vec{c}_2\}$ are two

different bases of $\mathbb{R}^2$, as shown.

If we know

$$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

how can we calculate

$$[\vec{v}]_C \qquad ?$$

Express each of the B vectors as lin. comb. of the C vectors.

$$\vec{b_1} = \boxed{1}\,\vec{c_1} + \boxed{3}\,\vec{c_2}$$

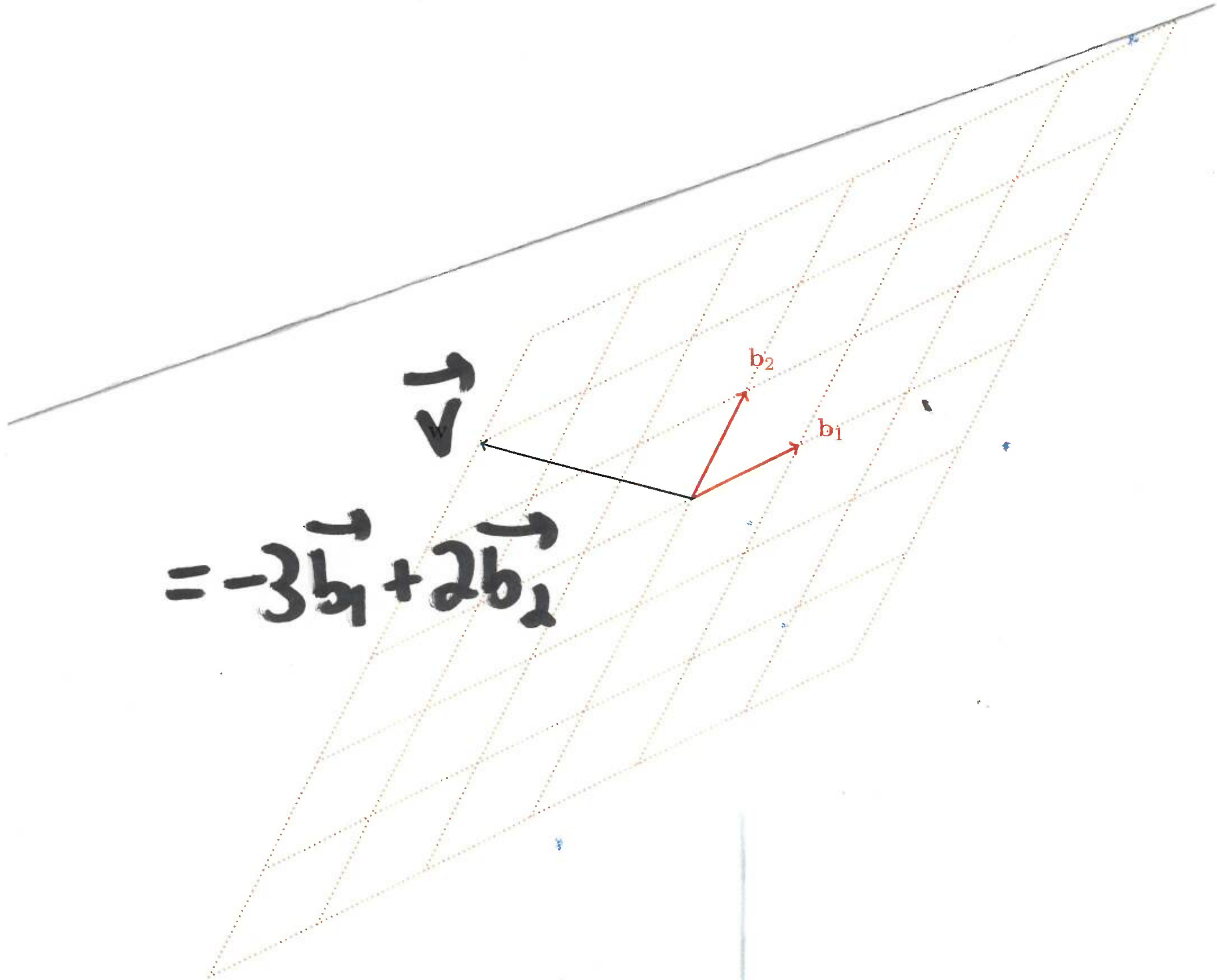
$$b_2 = \boxed{2}\,\vec{c_1} + \boxed{3}\,\vec{c_2}$$

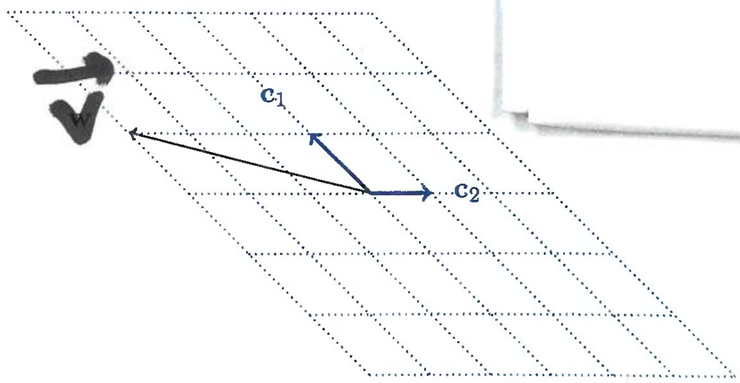
$$\therefore \vec{v} = -3\vec{b_1} + 2\vec{b_2}$$

$$= -3\left(1\vec{c_1} + 3\vec{c_2}\right) + 2\left(2\vec{c_1} + 3\vec{c_2}\right)$$

$$= \vec{c_1} - 3\vec{c_2}$$

$$\therefore [\vec{v}]_C = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$


$\vec{v}$
w
b_2
b_1
$= -3\vec{b_1} + 2\vec{b_2}$


$\vec{v}$
w
c_1
c_2

Definition Let $\mathcal{B} = \{\vec{b}_1, \cdots, \vec{b}_n\}$ and $\mathcal{C} = \{\vec{c}_1, \cdots, \vec{c}_n\}$ be bases for a vector space V. The change-of-basis matrix from $\mathcal{B}$ to $\mathcal{C}$ is the $n \times n$ matrix whose columns are the coordinate vectors $[\vec{b}_1]_{\mathcal{C}}, \cdots, [\vec{b}_n]_{\mathcal{C}}$

$$\underset{C \leftarrow B}{P} = \begin{bmatrix} [\vec{b}_1]_C & [\vec{b}_2]_C & \cdots & [\vec{b}_n]_C \end{bmatrix}$$

Lemma Suppose you collect m column vectors to form a matrix:

$$\begin{bmatrix} \big| & \big| & & \big| \\ c_1 & c_2 & \cdots & c_m \\ \big| & \big| & & \big| \end{bmatrix}$$

Then:

$$\begin{bmatrix} \big| & \big| & & \big| \\ c_1 & c_2 & \cdots & c_m \\ \big| & \big| & & \big| \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix}$$

$$= a_1 \begin{bmatrix} \big| \\ c_1 \\ \big| \end{bmatrix} + a_2 \begin{bmatrix} \big| \\ c_2 \\ \big| \end{bmatrix} + \cdots + a_m \begin{bmatrix} \big| \\ c_m \\ \big| \end{bmatrix}$$

<u>Proof</u>

$$(\mathrm{LHS})_i = \sum_{j=1}^{m} (C_j)_i \, a_j$$

$$\updownarrow$$

$$(\mathrm{RHS})_i = \sum_{j=1}^{m} a_j (C_j)_i$$

Theorem (Change of Basis)

Suppose that $\mathcal{B} = \{\vec{b_1}, \cdots, \vec{b_n}\}$ and $C = \{\vec{c_1}, \cdots, \vec{c_n}\}$ are bases for a v.s. V. Let

$$P_{C \leftarrow \mathcal{B}}$$

be the change-of-basis matrix from $\mathcal{B}$ to C. Then for all vectors $\vec{v} \in V$,

$$[\vec{v}]_C = P_{C \leftarrow \mathcal{B}} [\vec{v}]_{\mathcal{B}}.$$

Proof Let $\vec{v} \in V$.

Expand it in $\mathcal{B}$ basis:

$$\vec{v} = a_1 \vec{b_1} + \cdots + a_n \vec{b_n}$$

i.e. $$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

Then, $$[\vec{v}]_C = [a_1 \vec{b_1} + \cdots + a_n \vec{b_n}]_C$$

$$= a_1 [\vec{b_1}]_C + \cdots + a_n [b_n]_C$$

(using Lemma 1)

$$= \begin{bmatrix} [\vec{b}_1]_C & \cdots & [\vec{b}_n]_C \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \quad \text{(by Lemma 2)}$$

$$= P_{C \leftarrow B} [\vec{v}]_B$$

$\square$

In our example,

$$P_{C \leftarrow B} = \begin{array}{c} \left[\begin{array}{cc} \boxed{\begin{matrix} 1 \\ 3 \end{matrix}} & \boxed{\begin{matrix} 2 \\ 3 \end{matrix}} \end{array}\right] \\ \begin{array}{cc} \uparrow & \uparrow \\ [\vec{b}_1]_C & [\vec{b}_2]_C \end{array} \end{array}$$

In our example,

$$[\vec{v}]_C = \begin{bmatrix} 1 & 2 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -3 \end{bmatrix} \checkmark$$