# Improper Integrals §7.8

**Definition 1**

Suppose that $a \in \mathbb{R}$ and $f$ is continuous on $[a, \infty)$. Then

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

provided that this limit exists.

Suppose that $b \in \mathbb{R}$ and $f$ is continuous on $(-\infty, b]$. Then

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

provided that this limit exists.

*improper integrals of type I*

*convergent* if limit exists
*divergent* if limit does not exist

If *both* $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ are convergent, then

$$\int_{-\infty}^\infty f(x)\,dx := \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx.$$

## Lemma 4

1. Suppose that $a < b$. If $f$ is continuous on $[a, \infty)$, then $\int_a^\infty f(x)\,dx$ is convergent if and only if $\int_b^\infty f(x)\,dx$ is convergent.

2. If both $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ converge and $\alpha$ and $\beta$ are numbers, then $\int_a^\infty [\alpha f(x) + \beta g(x)]\,dx$ also converges.

## Lemma 5

Suppose that $a < b$. Then $\int_b^\infty \frac{1}{(x-a)^p}\,dx$ is convergent for $p > 1$ and divergent for $p \leq 1$.

## Lemma 6

Suppose that $a \in \mathbb{R}$. Then $\int_a^\infty e^{-px}\,dx$ is convergent for $p > 0$ and divergent for $p \leq 0$.

## Homework

Ex. 7.8 nr. 9, 13, 15

**Definition 2**

Suppose that $f$ is continuous but unbounded on $[a,b)$. Then

$$\int_a^b f(x)\,dx = \lim_{t \to b-} \int_a^t f(x)\,dx$$

provided that this limit exists.

Suppose that $f$ is continuous but unbounded on $(a,b]$. Then

$$\int_a^b f(x)\,dx = \lim_{t \to a+} \int_t^b f(x)\,dx$$

provided that this limit exists.

*improper integrals of type II*

**Lemma 7**
Suppose that $a < b$. Then $\int_a^b \frac{1}{(x-a)^p}\,dx$ and $\int_a^b \frac{1}{(x-b)^p}\,dx$ are convergent if $p < 1$ and divergent if $p \geq 1$.

**Definition 2 (last part)**
Suppose that $a < c < b$, $f$ is discontinuous at $c$ and $f$ is continuous but unbounded on $[a, c)$ and on $(c, b]$. If *both* $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then

$$\int_a^b f(x)\,dx := \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

**Definition 3**
Suppose that $a \in \mathbb{R}$, $f$ is continuous on $(a, \infty)$ and for any $c > a$ we have that $f$ is unbounded on $(a, c]$. If *both* the improper integrals $\int_a^c f(x)\,dx$ and $\int_c^\infty f(x)\,dx$ converge, then

$$\int_a^\infty f(x)\,dx := \int_a^c f(x)\,dx + \int_c^\infty f(x)\,dx.$$

*improper integral of type III*

**Homework**
Ex. 7.8 nr. 29, 33
Exercises 1 (Sunlearn)