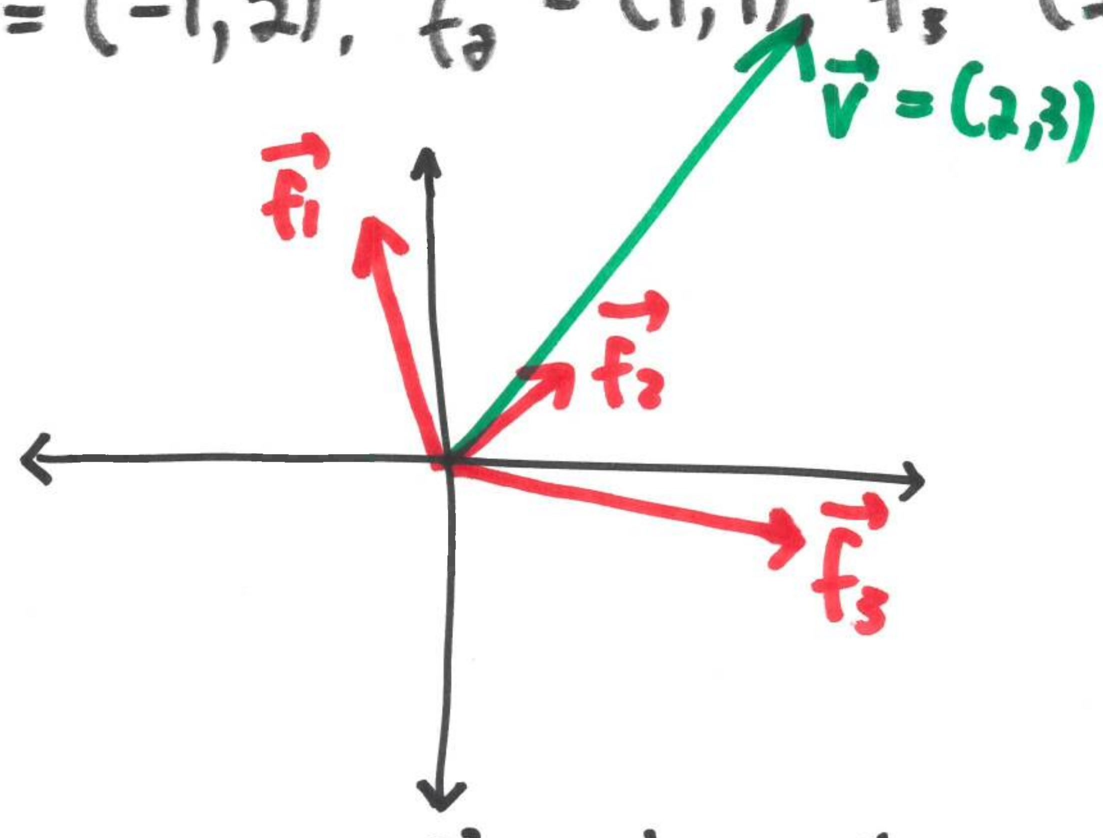
Lecture 7

Last time:

- Linear combination
- span

Example $\mathbb{R}^2$ is spanned by

$\vec{f_1} = (-1, 2)$, $\vec{f_2} = (1, 1)$, $\vec{f_3} = (2, -1)$



i.e. any vector in $\mathbb{R}^2$ can be written as a linear combination of $\vec{f_1}, \vec{f_2}, \vec{f_3}$.

In fact, in infinitely many ways!

eg. $\vec{v} = (2, 3)$:

$$\vec{v} = \frac{1}{3}\vec{f_1} + \frac{7}{3}\vec{f_2} \text{ ``} + 0.\vec{f_3} \text{''}$$

$$\vec{v} = 3\vec{f_1} - \frac{1}{3}\vec{f_2} + \frac{8}{3}\vec{f_3}$$

$$\vdots$$

2.2. Linear independence

Definition A list of vectors

$$B = \{ \vec{v}_1 , \cdots , \vec{v}_n \}$$

in a vector space V is linearly independent if the eqn

$$a_1 \vec{v}_1 + a_2 \vec{v}_2 + \cdots + a_n \vec{v}_n = \vec{0}$$

has only the trivial solution

$$a_1 = 0 , \ a_2 = 0 , \ \cdots , \ a_n = 0 .$$

Otherwise, B is linearly dependent.

Notation

$\{\text{apple}, \text{pear}, \text{dog}\} = \{\text{pear}, \text{apple}, \text{dog}\}$

set

= <u>unordered</u> collection

$\{\text{apple}, \text{pear}, \text{dog}\} \neq \{\text{pear}, \text{apple}, \text{dog}\}$

list = <u>ordered</u> set.

Remark Suppose one of the vectors $\vec{v_i}$ in $\mathcal{B}$ is the zero vector $\vec{0}$.

Then $\mathcal{B}$ is linearly dependent:

$$\boxed{0}\vec{v_1} + \boxed{0}\vec{v_2} + \cdots + \boxed{8}\underbrace{\vec{v_i}}_{=\vec{0}} + \cdots + \boxed{0}\vec{v_n} = \vec{0}$$

nontrivial soln

$\therefore$ list is lin. dep.

<u>Lemma</u> Let

$$B = \{\vec{v}_1, \cdots, \vec{v}_n\}$$

be a list of vectors in a vector space.

Then the following are equivalent:

(1) B is linearly dependent.

(2) (Linear Comb. of Other Vectors)

One of the vectors $\vec{v}_s$ is a linear combination of the other vectors in B.

(3) (Linear Comb. of Preceding Vectors)

Either $\vec{v}_1 = \vec{0}$, or for some $r \in \{2, \cdots, n\}$

$\vec{v}_r$ is a lin. comb. of $\vec{v}_1, \cdots, \vec{v}_{r-1}$.

Proof (1) $\Rightarrow$ (2)

$$\begin{tikzcd} & (1) \arrow[dl, Rightarrow, shift left] \arrow[dr, Rightarrow] & \\ (2) \arrow[ur, Rightarrow, shift left] & & (3) \arrow[ll, Rightarrow] \end{tikzcd}$$

Suppose (1) is true.

$\therefore$ there exist constants $a_1, \cdots, a_n$,

not all zero, s.t.

$$a_1\vec{v}_1 + \cdots + a_n\vec{v}_n = \vec{0}$$

Suppose $a_s \neq 0$. [we know it exists!]

$$\therefore a_s\vec{v}_s = -a_1\vec{v}_1 - \cdots - \widehat{a_s\vec{v}_s} - \cdots - a_n\vec{v}_n$$

$(\wedge = \text{"omitted"})$

$$\therefore \vec{v}_s = \frac{1}{a_s}\left(-a_1\vec{v}_1 - \cdots - \widehat{a_s\vec{v}_s} - \cdots - a_n\vec{v}_n\right)$$

$$= -a_1/a_s\,\vec{v}_1 - \cdots \widehat{a_s/a_s} \cdots - \frac{a_n}{a_s}\vec{v}_n$$

$\therefore \vec{v}_s$ is a lin. comb. of the other vectors.

(1) => (3).

Suppose (1) is true.

$\therefore$ there exist scalars $a_1, \cdots, a_n$,

not all zero, s.t. ← "such that"

$$a_1\vec{v_1} + \cdots + a_n\vec{v_n} = \vec{0}. \quad ⊛$$

Let r be the largest index s.t. $a_r \neq 0$.

Case : $r=1$ Eqn ⊛ becomes

$$a_1\vec{v_1} = \vec{0} \quad \text{with } a_1 \neq 0.$$

$$\therefore \vec{v_1} = \vec{0}$$

Case : $r \in \{2, \cdots, n\}$ Eqn ⊛ becomes

$$a_1\vec{v_1} + \cdots + a_r\vec{v_r} = \vec{0} \quad (a_r \neq 0)$$

$$\therefore \vec{v_r} = -a_1/a_r \, \vec{v_1} - \cdots - \frac{a_{r-1}}{a_r}\vec{v_{r-1}}$$

$$\begin{tikzcd}
(1) \arrow[rr, Leftrightarrow] \arrow[dr, Rightarrow, bend left] & & (2) \\
& (3) \arrow[ul, Rightarrow, bend left] \arrow[ur, Rightarrow, "\text{obvious}"'] &
\end{tikzcd}$$

<u>(3) $\Rightarrow$ (1):</u>

Suppose $\vec{v_1} = \vec{0}$.

$\therefore$ $\mathcal{B}$ is linearly dependent
(did ½ hour ago!)

Suppose $\vec{v_r} = k_1\vec{v_1} + \cdots + k_{r-1}\vec{v_{r-1}}$

$$\therefore \boxed{k_1}\vec{v_1} + \boxed{k_2}\vec{v_2} + \cdots + \boxed{-1}\vec{v_r} + \boxed{0}\cdots + \boxed{0}\vec{v_n} = \vec{0}$$

is a nontrivial soln, $\therefore$ $\mathcal{B}$ is lin. dep $\blacksquare$