Lecture 9

Example (illustrate proof of Bumping Off Prop.)

$V = \mathbb{R}^2$

$\vec{l_1} = (1,0)$, $\vec{l_2} = (0,1)$

$\vec{s_1} = (-1,2)$, $\vec{s_2} = (1,1)$, $\vec{s_3} = (2,-1)$

$\mathcal{L} = \{\vec{l_1}, \vec{l_2}\}$ $\quad S = \{\vec{s_1}, \vec{s_2}, \vec{s_3}\}$

$\mathcal{L}_1 = \{\vec{l_2}\}$ $\quad S_1 = \{\vec{l_1}, \vec{s_1}, \vec{s_3}\}$

$\mathcal{L}_2 = \{\ \}$ $\quad S_2 = \{\vec{l_2}, \vec{l_1}, \vec{s_3}\}$

- $S' = \{\vec{l}_1, \vec{s}_1, \vec{s}_2, \vec{s}_3\}$

One of the $\vec{s}$-vectors is a lin. comb. of preceding vectors.

- $\vec{s}_1 = k\vec{l}_1$? No.
- $\vec{s}_2 = \boxed{3/2}\,\vec{l}_1 + \boxed{1/2}\,\vec{s}_1$

Bump off $\vec{s}_2$.

- $S_1' = \{\vec{l}_2, \vec{l}_1, \vec{s}_1, \vec{s}_3\}$

One of the $\vec{s}$-vectors is a lin. comb. of preceding vectors.

- $\vec{s}_1 = \boxed{2}\,\vec{\ell}_2 + \boxed{-1}\,\vec{\ell}_1$?

  Yes.

Bump off $\vec{s}_1$.

2.3. Basis and Dimension

<u>Defn</u> A list of vectors

$$\{\vec{v_1}, \cdots, \vec{v_n}\}$$

in a vector space V is called a <u>basis</u> for V if it is <u>linearly independent</u> and <u>spans</u> V.

<u>Theorem</u> (Invariance of dimension)

If $B = \{\vec{e_1}, \cdots, \vec{e_m}\}$ and $C = \{\vec{f_1}, \cdots, \vec{f_n}\}$ are bases for a vector space V, then $m = n$.

Proof B is lin. ind. and C spans V.

$\therefore$ By Bumping Off Prop., $m \leq n$.

C is lin ind. and B spans V.

$\therefore$ By Bumping Off Prop., $n \leq m$

$\therefore$ $m = n$.

$\square$

<u>Defn</u> We say that a V. space is <u>finite-dimensional</u> if it has a basis $\{\vec{e}_1, \cdots, \vec{e}_n\}$, or ~~the~~ if it is the zero vector space.

Defn For a finite-dimensional vector space V, we define the <u>dimension of V</u> as

$$\dim V := \text{number of vectors in } \underline{\text{any}} \text{ basis for } V.$$

(Also, $\dim\begin{pmatrix}\text{zero vector}\\ \text{space}\end{pmatrix} := 0$).

Example (Dimension of $\mathbb{R}^n$).

A basis for $\mathbb{R}^n$ is:

$$\vec{e_1} = (1,0,\cdots,0),\ \vec{e_2} = (0,1,\cdots,0),$$
$$\cdots,\ \vec{e_n} = (0,\cdots,0,1).$$

Lin independent Consider

$$a_1\vec{e_1} + a_2\vec{e_2} + \cdots + a_n\vec{e_n} = \vec{0}$$

i.e. $(a_1, a_2, \cdots, a_n) = (0,0,\cdots,0)$

$\therefore a_1 = 0,\ a_2 = 0,\ \cdots,\ a_n = 0$

$\therefore$ list is lin. ind.

Spans $\mathbb{R}^n$: Let

$$\vec{v} = (v_1, v_2, \cdots, \vec{v}_n)$$

be an arbitrary vector in $\mathbb{R}^n$.

We can write

$$\vec{v} = \boxed{v_1}\,\vec{e}_1 + \boxed{v_2}\,\vec{e}_2 + \cdots + \boxed{v_n}\,\vec{e}_n$$

$\therefore$ $\{\vec{e}_1, \cdots, \vec{e}_n\}$ spans $\mathbb{R}^n$.

$\therefore$ a basis

$\therefore$ $\mathrm{Dim}(\mathbb{R}^n) = n$.

Example (Another basis for $\mathbb{R}^4$).

$\vec{v_1} = (1,0,2,-3)$, $\vec{v_2} = (1,3,-1,2)$

$\vec{v_3} = (0,1,2,-1)$, $\vec{v_4} = (1,2,3,4)$

Basis for $\mathbb{R}^4$?

- Lin. ind. Consider

$$a_1\vec{v_1} + a_2\vec{v_2} + a_3\vec{v_3} + a_4\vec{v_4} = \vec{0}$$

$$(a_1 + a_2 + a_4,\ 3a_2 + a_3 + 2a_4,\ 2a_1 - a_2 + 2a_3 + 3a_4,\ -3a_1 + 2a_2 - a_3 + 4a_4) = (0,0,0,0)$$

$$\therefore \quad \begin{aligned} a_1 + a_2 \qquad\quad + a_4 &= 0 \\ 3a_2 + a_3 + 2a_4 &= 0 \\ 2a_1 - a_2 + 2a_3 + 3a_4 &= 0 \\ -3a_1 + 2a_2 - a_3 + 4a_4 &= 0 \end{aligned}$$

$$\Rightarrow a_1 = 0,\ a_2 = 0,\ a_3 = 0,\ a_4 = 0$$

$\therefore$ lin ind ✓

- <u>span $\mathbb{R}^4$</u> Let $\vec{w} = (w_1, w_2, w_3, w_4) \in \mathbb{R}^4$.

Can we write

$$\vec{w} = a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3 + a_4\vec{v}_4 ?$$

$\vdots$

$$\begin{aligned} a_1 + a_2 + a_4 &= w_1 \\ 3a_2 + a_3 + 2a_4 &= w_2 \\ 2a_1 - a_2 + 2a_3 + 3a_4 &= w_3 \\ -3a_1 + 2a_2 - a_3 + 4a_4 &= w_4 \end{aligned}$$

Has soln ✓

$\therefore$ span $\mathbb{R}^4$ ✓

$\therefore$ a basis.