# “sup” and “inf”

## Definition 8

1. A set $A$ of real numbers is *bounded above* if there exists a real number $M$ such that $x \leq M$ for all $x \in A$. We call $M$ an *upper bound* for $A$.

2. A set $A$ of real numbers is *bounded below* if there exists a real number $m$ such that $x \geq m$ for all $x \in A$. We call $m$ a *lower bound* for $A$.

3. A set $A \subset \mathbb{R}$ is *bounded* if $A$ is bounded above as well as bounded below.

4. If $A \subset \mathbb{R}$ is not bounded, then we say that $A$ is *unbounded*.

**Definition 9**
If a set $A \subset \mathbb{R}$ has a smallest upper bound $S$, then we call $S$ the *supremum* or *sup* of $A$ and write $S = \sup A$. This means:

1. $S$ is an upper bound for $A$.

2. If $T$ is any upper bound for $A$, then $S \leq T$.

**Definition 11**
If a set $A \subset \mathbb{R}$ has a greatest lower bound $I$, then we call $I$ the *infimum* or *inf* of $A$ and write $I = \inf A$. This means:

1. $I$ is a lower bound for $A$.

2. If $J$ is any lower bound for $A$, then $I \geq J$.

**Definition 9**
If a set $A \subset \mathbb{R}$ has a smallest upper bound $S$, then we call $S$ the *supremum* or *sup* of $A$ and write $S = \sup A$. This means:

1. $S$ is an upper bound for $A$.

2. If $T$ is any upper bound for $A$, then $S \leq T$.

**Theorem 10**
Let $A \subset \mathbb{R}$. Then $\sup A = S$ if and only if the following two conditions hold:

1. $S$ is an upper bound for $A$.

2. For each $\epsilon > 0$ there exists an $x \in A$ such that $x > S - \epsilon$.

**Theorem 12**
Let $A \subset \mathbb{R}$. Then $\inf A = I$ if and only if the following two conditions hold:

1. $I$ is a lower bound for $A$.

2. For each $\epsilon > 0$ there exists an $x \in A$ such that $x < I + \epsilon$.

**The completeness property of $\mathbb{R}$**
Each nonempty set of real numbers which is bounded above has a smallest upper bound.

**Corollary**
Each nonempty set of real numbers which is bounded below has a greatest lower bound.

**Theorem 13**

Suppose that $f$ is increasing on $[a, \infty)$. Then $\lim_{x\to\infty} f(x)$ exists if and only if $f$ is bounded above on $[a, \infty)$, in which case

$$\lim_{x\to\infty} f(x) = \sup\{f(x) : x \in [a, \infty)\}.$$

**Theorem 10**

Let $A \subset \mathbb{R}$. Then $\sup A = S$ if and only if the following two conditions hold:

1. $S$ is an upper bound for $A$.

2. For each $\epsilon > 0$ there exists an $x \in A$ such that $x > S - \epsilon$.

## The comparison test for improper integrals of type I

Suppose that $a \in \mathbb{R}$ and $f$ and $g$ are continuous functions on $[a, \infty)$ such that
$0 \leq g(x) \leq f(x)$ for all $x \geq a$.

1. If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is convergent.

2. If $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is divergent.

## Theorem 13

Suppose that $f$ is increasing on $[a, \infty)$. Then $\lim\limits_{x\to\infty} f(x)$ exists if and only if $f$ is bounded above on $[a, \infty)$, in which case

$$\lim_{x\to\infty} f(x) = \sup\{f(x) : x \in [a, \infty)\}.$$

## The comparison test for improper integrals of type II

Suppose that $a < b$, $f$ and $g$ are continuous but unbounded on $(a, b]$ and
$0 \leq g(x) \leq f(x)$ for all $x \in (a, b]$.

1. If $\int_a^b f(x)\,dx$ is convergent, then $\int_a^b g(x)\,dx$ is convergent.
2. If $\int_a^b g(x)\,dx$ is divergent, then $\int_a^b f(x)\,dx$ is divergent.

## The comparison test for improper integrals of type I

Suppose that $a \in \mathbb{R}$ and $f$ and $g$ are continuous functions on $[a, \infty)$ such that
$0 \leq g(x) \leq f(x)$ for all $x \geq a$.

1. If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is convergent.
2. If $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is divergent.

## Homework

Ex. 7.8 nr. 49, 53

**The quotient test for improper integrals of type I**

Suppose that $a \in \mathbb{R}$ and $f$ and $g$ are continuous functions on $[a, \infty)$ such that $f(x) \geq 0$ and $g(x) \geq 0$ for all $x \geq a$.

1. If $\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$ and $\int_a^\infty g(x)\, dx$ is convergent, then $\int_a^\infty f(x)\, dx$ is convergent.

2. If $\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)} = A > 0$, then $\int_a^\infty g(x)\, dx$ is convergent if and only if $\int_a^\infty f(x)\, dx$ is convergent.

3. If $\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$ and $\int_a^\infty g(x)\, dx$ is divergent, then $\int_a^\infty f(x)\, dx$ is divergent.

**The quotient test for improper integrals of type II**

Suppose that $a < b$, $f$ and $g$ are continuous but unbounded on $(a, b]$ and $f(x) \geq 0$ and $g(x) \geq 0$ for all $x \in (a, b]$.

1. If $\displaystyle\lim_{x \to a+} \frac{f(x)}{g(x)} = 0$ and $\int_a^b g(x)\, dx$ is convergent, then $\int_a^b f(x)\, dx$ is convergent.

2. If $\displaystyle\lim_{x \to a+} \frac{f(x)}{g(x)} = A > 0$, then $\int_a^b g(x)\, dx$ is convergent if and only if $\int_a^b f(x)\, dx$ is convergent.

3. If $\displaystyle\lim_{x \to a+} \frac{f(x)}{g(x)} = \infty$ and $\int_a^b g(x)\, dx$ is divergent, then $\int_a^b f(x)\, dx$ is divergent.

**Homework**

Exercises 2 and 3 (Sunlearn)

**Definition 14**

Suppose that $f$ is continuous on $[a, \infty)$. Then $\int_a^\infty f(x)\,dx$ is *absolutely convergent* if $\int_a^\infty |f(x)|\,dx$ is convergent.

$\int_a^\infty f(x)\,dx$ is *conditionally convergent* if $\int_a^\infty f(x)\,dx$ is convergent and $\int_a^\infty |f(x)|\,dx$ is divergent.

**Theorem 15**

If $\int_a^\infty f(x)\,dx$ is absolutely convergent, then $\int_a^\infty f(x)\,dx$ is convergent.