Lecture 5

1.6. Subspaces

Defn A subset $U \subseteq V$ of a vector space V is called a <u>subspace of</u> V if:

- for all $\vec{w}, \vec{w}' \in U$, we have

  $\vec{w} + \vec{w}' \in U$

- $\vec{0} \in U$
- For every $\vec{w} \in U$ and scalar $k \in \mathbb{R}$,

  $k . \vec{w} \in U$.

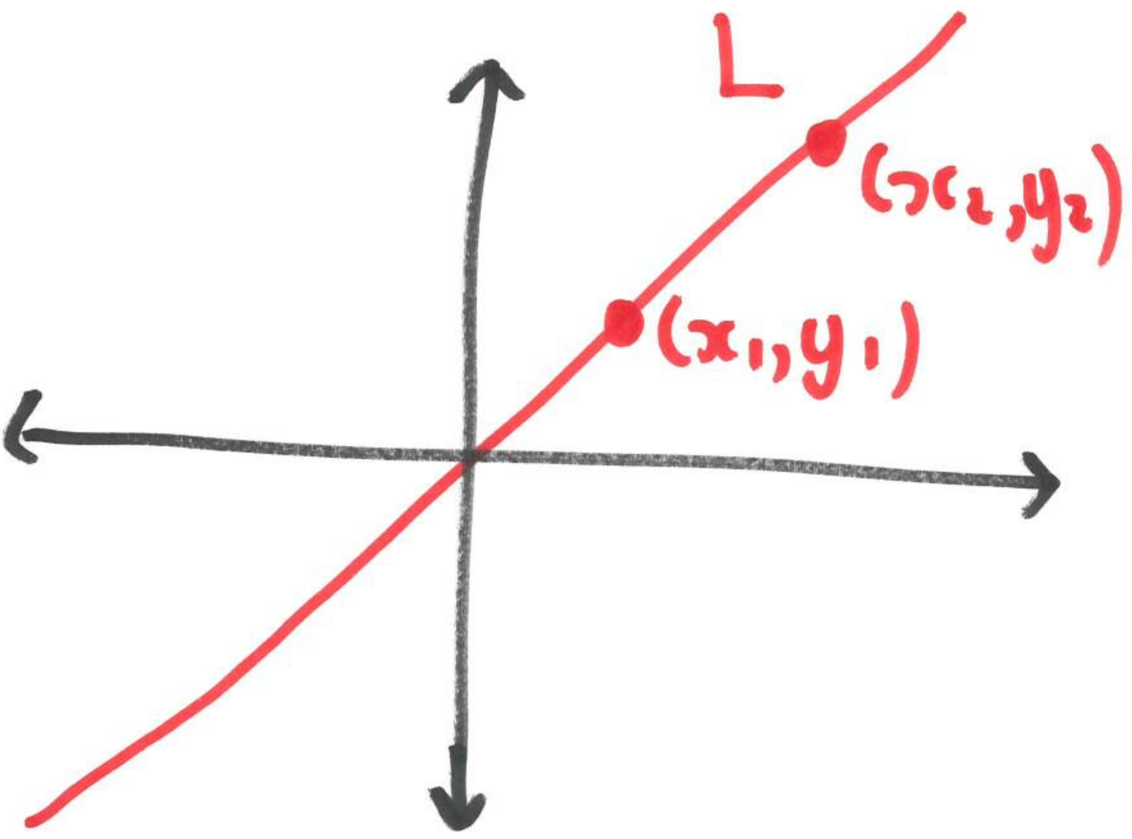

V
vector space
U
w'
w
w+w'

__Lemma__ If U is a subspace of a vector space V, then it is a vector space (when equipped with same $+, \vec{0}, \cdot$ as in V).

__Proof__ The data $+, \vec{0}, \cdot$ are well-defined since U is a subspace. The rules R1 to R8 are satisfied in V, hence also in U.

$\square$

Example (Line through origin in $\mathbb{R}^2$)



$V = \mathbb{R}^2$

$$L = \{(x, y) : ax + by = 0\}$$

$(x_1, y_1) \in L$, $(x_2, y_2) \in L$

$(x_1, y_1) + (x_2, y_2) \in L$? Yes.

Check:

$$(x_1, y_1) + (x_2, y_2)$$
$$= (x_1 + x_2,\ y_1 + y_2) \in L?$$

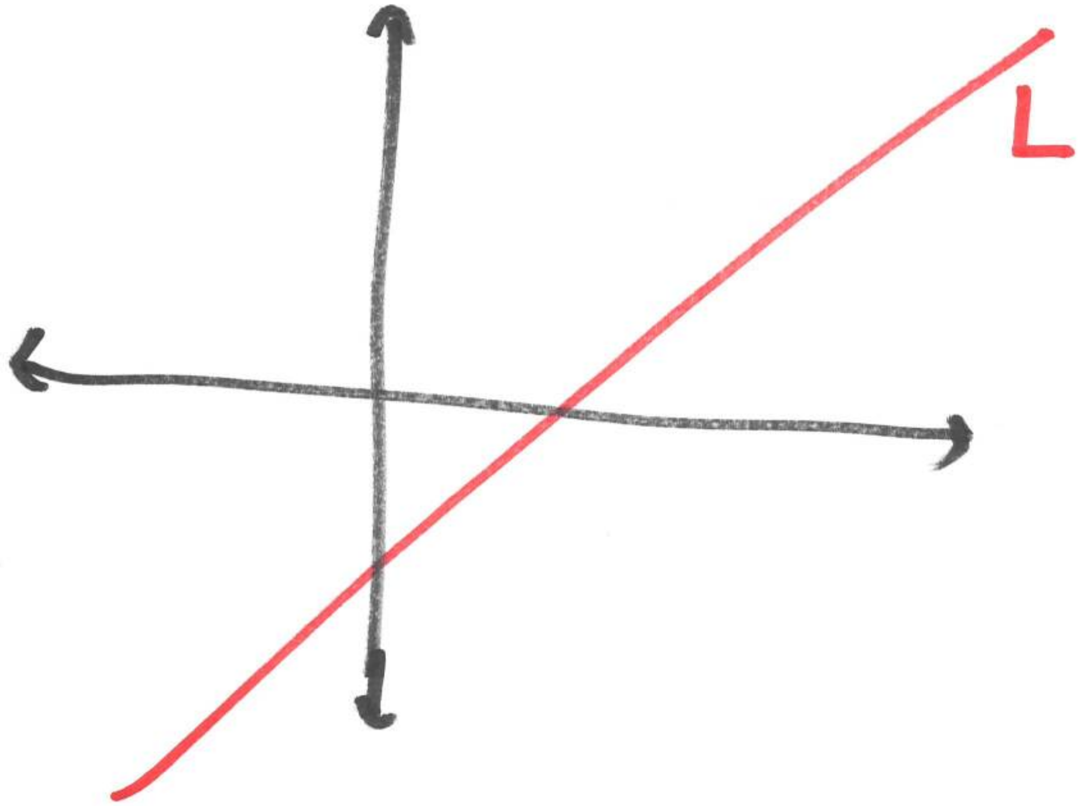
$$a(x_1 + x_2) + b(y_1 + y_2) = 0 ?$$

We must check whether

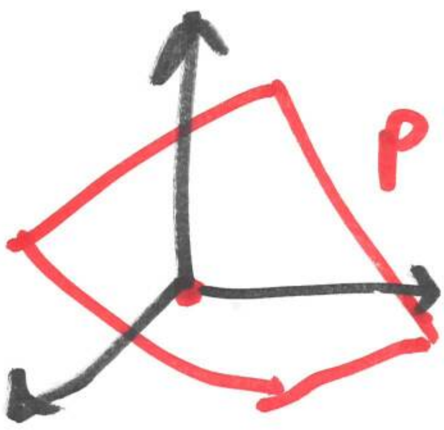
i.e. $ax_1 + ax_2 + by_1 + by_2 = 0$?

i.e. $\underbrace{(ax_1 + by_1)}_{=0} + \underbrace{(ax_2 + by_2)}_{=0} = 0$? ✓

Non-example (Line not through origin in $\mathbb{R}^2$)



Example (Plane through origin in $\mathbb{R}^3$)



Example (zero subspace)

For any vector space V,

$$\{\vec{0}\} \subseteq V$$

is a subspace.

Example (Polynomials)

A polynomial is a function

$$p: \mathbb{R} \longrightarrow \mathbb{R}$$

of the form

$$p(x) = a_n x^n + \cdots + a_1 x + a_0.$$

The degree of a polynomial is the highest power of x that has a nonzero coefficient.

eg. $2x^3 + 1$ ... degree 3

$0x^4 + 5x^2 - 7$ ... degree 2

We write

$$\mathrm{Poly} := \{\text{all polynomials}\}.$$

Note that $\mathrm{Poly} \subseteq \mathrm{Fun}(\mathbb{R})$,

in fact a <u>subspace</u>.

$\therefore$ By Lemma, Poly is a vector space.

Also, we write

$$\mathrm{Poly}_n := \{\text{all polynomials of degree} \leq n\}$$

Poly_n is a subspace of Poly.

Example (Solutions to homogenous linear ODE's)

An n^{th} order homogenous linear ordinary differential eqn is one of the form

$$a_n(x)\, y^{(n)} + a_{n-1}(x)\, y^{(n-1)} + \cdots + a_1(x)\, y + a_0(x) = 0$$

where $y^{(n)}$ = n^{th} derivative of y.

eg. $x^2 y''' + \cos x = 0$

... 3rd order.

eg. Consider

$$y'' + 2y' + y = 0$$

Note :

$$y_1 = e^{-x}$$

$$y_2 = x e^{-x}$$

are solutions.

Is $y_1 + y_2$ a solution? <u>Yes</u>

Check:

$$(y_1 + y_2)'' + 2(y_1 + y_2)' + (y_1 + y_2)$$

$$= y_1'' + y_2'' + 2y_1' + 2y_2' + y_1 + y_2$$

$$= (y_1'' + 2y_1' + y_1) + (y_2'' + 2y_2' + y_2)$$

$$= 0 + 0 = 0.$$

In summary,

$$V := \Big\{ y : a_n(x) y^{(n)} + a_{n-1}(x) y^{(n-1)} + \cdots + a_1(x) y_1 + a_0(x) = 0 \Big\}$$

is a subspace of Fun($\mathbb{R}$).

Non-example

$$W := \{ y : y' = y^2 \}$$

Solve:

$$\frac{dy}{dx} = y^2$$

$$\cancel{\frac{dy}{dy^2}} \quad \int \frac{dy}{y^2} = \int dx$$

$$\therefore \; -y^{-1} = x + C'$$

$$\therefore \; y = \frac{1}{C - x}$$

eg. $y_1 = \dfrac{1}{2-x}$ $\qquad$ $y_2 = \dfrac{1}{3-x}$

Is $y_1 + y_2$ a soln ?

$$(y_1+y_2)' = (y_1+y_2)^2 \quad ?$$

$$\begin{aligned}
\text{LHS} &= y_1' + y_2' \\
&= y_1^2 + y_2^2 \\
&\neq (y_1+y_2)^2 \\
&= \text{RHS}.
\end{aligned}$$

__Example__ (Trigonometric polynomials)

A trigonometric polynomial of degree n is a function of the form

$$T(x) = a_0 + a_1 \cos x + b_1 \sin x$$
$$+ a_2 \cos 2x + b_2 \sin 2x$$
$$+ \cdots$$
$$+ a_n \cos nx + b_n \sin x$$

$$\mathrm{Trig}_n := \left\{ \begin{array}{l} \text{trig polynomials} \\ \quad \text{of degree} \leq n \end{array} \right\}$$

$$\mathrm{Trig}_n \overset{\text{subspace}}{\subseteq} \mathrm{Fun}(\mathbb{R})$$