Chapter 2 Analytic Functions

Section 13 Functions and Mappings

Definition

Let S be a set in $\mathbb{C}$. A *function* f defined on S is a rule that assigns to each $z \in S$ a complex number w, called the *value* of f at z.

We write $w = f(z)$. The set S is called the *domain of definition (definisieversameling)* of f (and is not necessarily a domain in the sense of Section 12). When the domain of definition is not specified, we agree that the largest possible set is to be taken. Note that, if $z = x + iy$, then $f(z) = u(x, y) + iv(x, y)$, and if $z = re^{i\theta}$, then $f(z) = u_1(r, \theta) + iv_1(r, \theta)$.

Examples

$f(z) = z^2$, $g(z) = |z|^2$, polynomials, rational functions.

Definition
A *multiple-valued function (meerwaardige funksie)* is a rule that assigns more than one value to each point in the domain of definition.

In this case we usually take only one of the possible values assigned to each point, in a systematic manner, and a (single-valued) function is constructed from the multiple-valued one.

Example
Considering the multiple-valued function

$$z^{\frac{1}{2}} = \sqrt{r}e^{i\left(\frac{\theta+2k\pi}{2}\right)},\ k = 0, 1,$$

where $z = re^{i\theta}$ with $-\pi < \theta \leq \pi$, i.e. $z^{\frac{1}{2}} = \{\sqrt{r}e^{\frac{i\theta}{2}}, -\sqrt{r}e^{\frac{i\theta}{2}}\}$, we can define the single-valued function

$$f(z) = \begin{cases} \sqrt{r}e^{\frac{i\theta}{2}} & \text{if} \quad z \neq 0\ (r > 0, -\pi < \theta \leq \pi) \\ 0 & \text{if} \quad z = 0. \end{cases}$$

Read the last part of Section 13. Leave out Section 14.

Tutorial: pp.43–44 nr. 1(b),(c), 2(b), 3, 4.

Sections 15–16 Limits

Definition
Let f be a complex valued function with domain D_f and let $z_0 \in \mathbb{C}$. Then f has a *limit* w_0 at z_0 if for every $\epsilon > 0$ there exists $\delta > 0$ such that if $z \in D_f$ and $0 < |z - z_0| < \delta$, then $|f(z) - w_0| < \epsilon$.

We write $\lim_{z \to z_0} f(z) = w_0$. Note that once a suitable δ has been found, it can be replaced by any smaller positive number (e.g. $\frac{\delta}{2}$).

Theorem p.45
If the limit of a complex valued function f exists, then it is unique.

Examples
Find the limits, if they exist, of $f(z) = \frac{i\overline{z}}{2}$ at 1, and of $g(z) = \frac{z}{\overline{z}}$ at 0.

Theorem 1, p.47

Suppose that $f(z) = u(x, y) + iv(x, y)$ where $z = x + iy$, $z_0 = x_0 + iy_0$ and $w_0 = u_0 + iv_0$. Then

$$\lim_{(x,y)\to(x_0,y_0)} u(x, y) = u_0$$

and

$$\lim_{(x,y)\to(x_0,y_0)} v(x, y) = v_0$$

if and only if

$$\lim_{z\to z_0} f(z) = w_0.$$

Theorem 2, p.48
Suppose that

$$\lim_{z \to z_0} f(z) = w_0 \text{ and } \lim_{z \to z_0} F(z) = W_0.$$

Then

$$\lim_{z \to z_0} [f(z) + F(z)] = w_0 + W_0,$$

$$\lim_{z \to z_0} [f(z)F(z)] = w_0 W_0$$

and if $W_0 \neq 0$, then

$$\lim_{z \to z_0} \frac{f(z)}{F(z)} = \frac{w_0}{W_0}.$$

Read the rest of Section 16.