Mathematics: analysis and approaches
Higher Level
Paper 1

Name

Date: ____________________

2 hours

Instructions to candidates

- Write your name in the box above.
- Do not open this examination paper until instructed to do so.
- You are not permitted access to any calculator for this paper.
- Section A: answer all of Section A in the spaces provided.
- Section B: answer all of Section B on the answer sheets provided. Write your name on each answer sheet and attach them to this examination paper.
- Unless otherwise stated in the question, all numerical answers must be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

exam: 13 pages

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

Section A

Answer **all** questions in the boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 5]

Let A and B be events such that $\mathrm{P}(A \cap B) = \frac{1}{5}$, $\mathrm{P}(B \mid A) = \frac{1}{2}$ and $\mathrm{P}(A \mid B) = \frac{3}{10}$.

Find $\mathrm{P}(A \cup B)$.

2. [Maximum mark: 4]

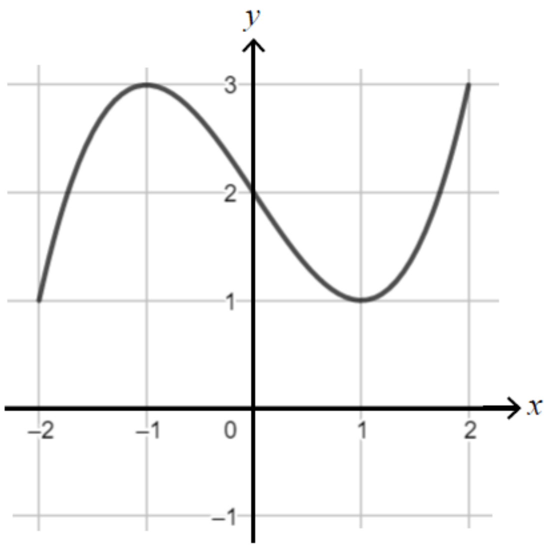
Use a proof by contradiction to prove that there are no positive integers a and b such that $a^2 - b^2 = 2$.

3. [Maximum mark: 5]

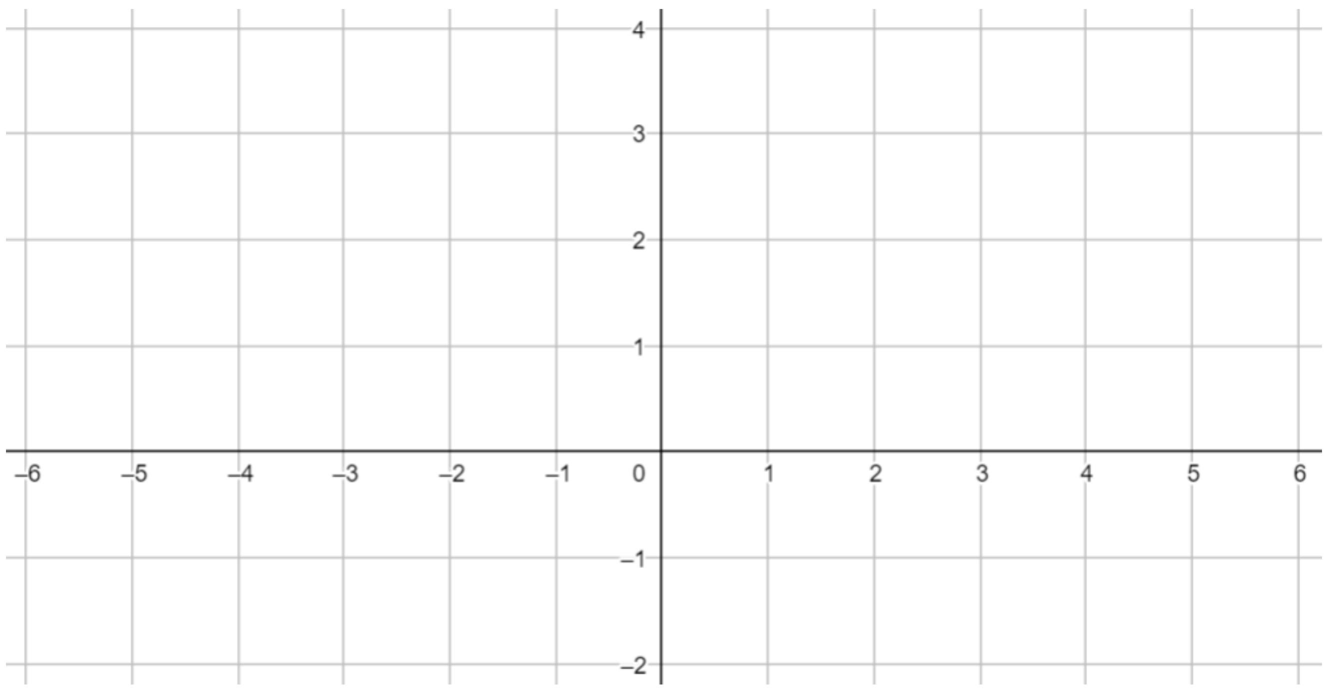
Let $g'(x)=\frac{(\ln x)^2}{x}$. Given that $g(1)=2$, find $g(x)$.

4. [Maximum mark: 5]

The following diagram shows the graph of $y = f(x), -2 \le x \le 2$. The graph has a horizontal tangent at the points $(-1, 3)$ and $(1, 1)$.



On the set of axes below, sketch the graph of $y = f(2x - 4)$, clearly indicating the coordinates of any local maxima or minima.



5. [Maximum mark: 7]

Consider the function $h(x)=\dfrac{2x+3}{x-4}$, $x \neq 4$.

(a) Find an expression for the inverse function $h^{-1}(x)$. [4]

(b) There is a function k such that $(h^{-1} \circ k)(x)=\dfrac{4x+8}{x-9}$, $x \neq 9$. Show that $k(x)=\dfrac{x-1}{4}$. [3]

6. [Maximum mark: 8]

Solve for x in each equation.

(a) $\ln x + \ln(x-2) - \ln(x+4) = 0$ [4]

(b) $\log_3(4x^2 - 5x - 6) = 1 + 2\log_3 x$ [4]

7. [Maximum mark: 7]

A continuous random variable X has the probability density function f given by

$$f(x) = \begin{cases} a\sin\left(\frac{\pi x}{2}\right), & 0 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$$

(a) Show that $a = \frac{\pi}{4}$. [3]

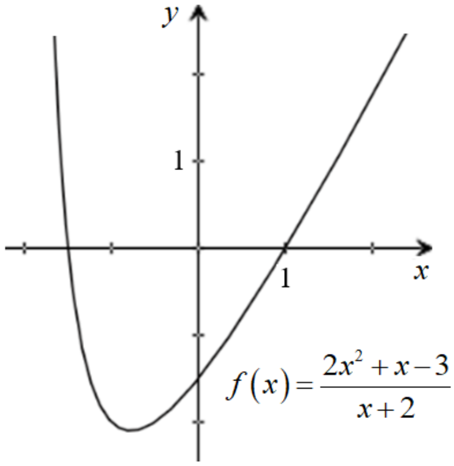
(b) Given that $P(0 \le X \le q) = 0.25$, find the value of q. [4]

8. [Maximum mark: 6]

The angle between unit vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ is $60°$. Calculate $|3\boldsymbol{v}+4\boldsymbol{w}|$.

9. [Maximum mark: 8]

Consider the function $f(x) = \dfrac{2x^2 + x - 3}{x + 2}$. A portion of the graph of f is shown below.



(a) Given that f can be expressed in the form $ax + b + \dfrac{c}{x+2}$, find the value of a, the value of b and the value of c. [4]

(b) The graph of f has two asymptotes. One of the asymptotes is the vertical line $x = -2$. Write down the equation of the other asymptote. [1]

(c) Show that the area of the region bounded by the graph of f and the x-axis is $\left|3\ln 6 - \dfrac{35}{4}\right|$. [3]

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(Question 9 continued)

Do **not** write solutions on this page.

Section B

Answer **all** the questions on the answer sheets provided. Please start each question on a new page.

10. [Maximum mark: 17]

Consider the complex number $z = \cos\theta + \mathrm{i}\sin\theta$.

(a) Prove, using mathematical induction, that for a positive integer *n*,

$$(\cos\theta + \mathrm{i}\sin\theta)^n = \cos n\theta + \mathrm{i}\sin n\theta \quad \text{where } \mathrm{i}^2 = -1$$ [5]

(b) Using De Moivre's theorem show that $z^n + \dfrac{1}{z^n} = 2\cos n\theta$. [3]

(c) By expanding $\left(z + \dfrac{1}{z}\right)^4$ show that $\cos^4\theta = \dfrac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$. [4]

(d) Let $h(c) = \int_0^c \cos^4\theta \, \mathrm{d}\theta$ where $0 < c < \pi$.

(i) Find $h(c)$.

(ii) Solve $h(c) = \dfrac{3}{8}c$. [5]

11. [Maximum mark: 16]

(a) The plane Π_1 has equation $\boldsymbol{r} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ 8 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -3 \\ -9 \end{pmatrix}$.

The plane Π_2 has equation $\boldsymbol{r} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + s \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

(i) For points which lie in both Π_1 and Π_2, show that $\lambda = \mu$.

(ii) Hence, find a vector equation of the line of intersection of Π_1 and Π_2. [7]

(b) The plane Π_3 contains the line $\dfrac{2-x}{3} = \dfrac{y}{-4} = z + 1$ and is perpendicular to $3\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$.

Find the cartesian equation of Π_3. [5]

(c) Find the intersection of Π_1, Π_2 and Π_3. [4]

Do **not** write solutions on this page.

12. [Maximum mark: 22]

Consider the differential equation $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{y^2}{1+x}$, where $x>-1$ and $y=1$ when $x=0$.

(a) Use Euler's method, with a step length of 0.5, to find an approximate value of y when $x=1$. [7]

(b) (i) Show that $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=\dfrac{2y^3-y^2}{(1+x)^2}$.

(ii) Hence, find the Maclaurin series for y, up to and including the term in x^2. [8]

(c) (i) Solve the differential equation.

(ii) Given that $\lim\limits_{x\to a} y=\pm\infty$ find the value of a. [7]