Time-Domain Analysis of Discrete-Time Systems – Part 2

Table of Contents

A. Unit impulse response

Find characteristic modes and then compute the impulse response for

$$\boldsymbol{y[n] + \frac{1}{6}y[n-1] - \frac{1}{6}y[n-2] = \frac{1}{3}x[n]}$$

Using all initial condition are zero, setting $n = 0$

$$h[0] + \frac{1}{6}h[-1] - \frac{1}{6}h[-2] = \frac{1}{3}\delta[0]$$

$$h[0] = \frac{1}{3}$$

Setting $n = 1$

$$h[1] + \frac{1}{6}h[0] - \frac{1}{6}h[-1] = \frac{1}{3}\delta[1]$$

$$h[1] = -\frac{1}{6}h[0] = -\frac{1}{18}$$

Rewrite the D.T system above in Advance Terms:

$$y[n+2] + \frac{1}{6}y[n+1] - \frac{1}{6}y[n] = \frac{1}{3}x[n+2]$$

Convert into the form:

$$Q(E)y[n] = P(E)x[n]$$

$$\left(E^2 + \frac{1}{6}E - \frac{1}{6}E\right)y[n] = \frac{1}{3}E^2x[n]$$

The characteristic equation is then:

$$\gamma^2 + \frac{1}{6}\gamma - \frac{1}{6} = 0$$

$$\gamma = -\frac{1}{2} \; and \; \gamma = \frac{1}{3}$$

Hence, the characteristic modes: $\left(-\frac{1}{2}\right)^n \; and \; \left(\frac{1}{3}\right)^n$

The zero-input response:

$$y_0[n] = c_1\left(-\frac{1}{2}\right)^n + c_2\left(\frac{1}{3}\right)^n$$

The impulse response:

$$h[n] = \frac{b_n}{a_n}\delta[n] + \left[c_1\left(-\frac{1}{2}\right)^n + c_2\left(\frac{1}{3}\right)^n\right]u[n]$$

From the system, we have: $b_n = 0$ and $a_n = -\frac{1}{6}$, then

$$h[n] = \left[c_1\left(-\frac{1}{2}\right)^n + c_2\left(\frac{1}{3}\right)^n\right]u[n]$$

Substitute $h[0] = \frac{1}{3}$ and $h[1] = -\frac{1}{18}$ into the impulse response found above

$$\begin{cases} c_1 + c_2 = \dfrac{1}{3} \\ c_1\left(-\dfrac{1}{2}\right) + c_2\left(\dfrac{1}{3}\right) = -\dfrac{1}{18} \end{cases}$$

Solve the system of equations above, we get $c_1 = \frac{1}{5}$ and $c_2 = \frac{2}{15}$

$$\therefore h[n] = \left[\left(\frac{1}{5}\right)\left(-\frac{1}{2}\right)^n + \left(\frac{2}{15}\right)\left(\frac{1}{3}\right)^n\right]u[n]$$

Find characteristic modes and then compute the impulse response for

$$\boldsymbol{y[n] + \frac{1}{4}y[n-2] = x[n]}$$

Using all initial condition are zero, setting $n = 0$

$$h[0] + \frac{1}{4}h[-2] = \delta[0]$$
$$h[0] = 1$$

Setting $n = 1$

$$h[1] + \frac{1}{4}h[-1] = \delta[1]$$
$$h[1] = 0$$

Rewrite the D.T system above in Advance Terms:

$$y[n+2] + \frac{1}{4}y[n] = x[n+2]$$

Convert into the form:

$$Q(E)y[n] = P(E)x[n]$$
$$\left(E^2 + \frac{1}{4}E\right)y[n] = E^2x[n]$$

The characteristic equation is then:

$$\gamma^2 + \frac{1}{4} = 0$$
$$\gamma = -j\frac{1}{2} \ and\ \gamma = j\frac{1}{2}$$

Hence, the characteristic modes: $\left(-j\frac{1}{2}\right)^n \ and \ \left(j\frac{1}{2}\right)^n$

The zero-input response:

$$y_0[n] = c_1\left(-j\frac{1}{2}\right)^n + c_2\left(j\frac{1}{2}\right)^n$$

The impulse response:

$$h[n] = \frac{b_n}{a_n}\delta[n] + \left[c_1\left(-j\frac{1}{2}\right)^n + c_2\left(j\frac{1}{2}\right)^n\right]u[n]$$

From the system, we have: $b_n = 0$ and $a_n = \frac{1}{4}$, then

$$h[n] = \left[c_1\left(-j\frac{1}{2}\right)^n + c_2\left(j\frac{1}{2}\right)^n\right]u[n]$$

Substitute $h[0] = 1$ and $h[1] = 0$ into the impulse response found above

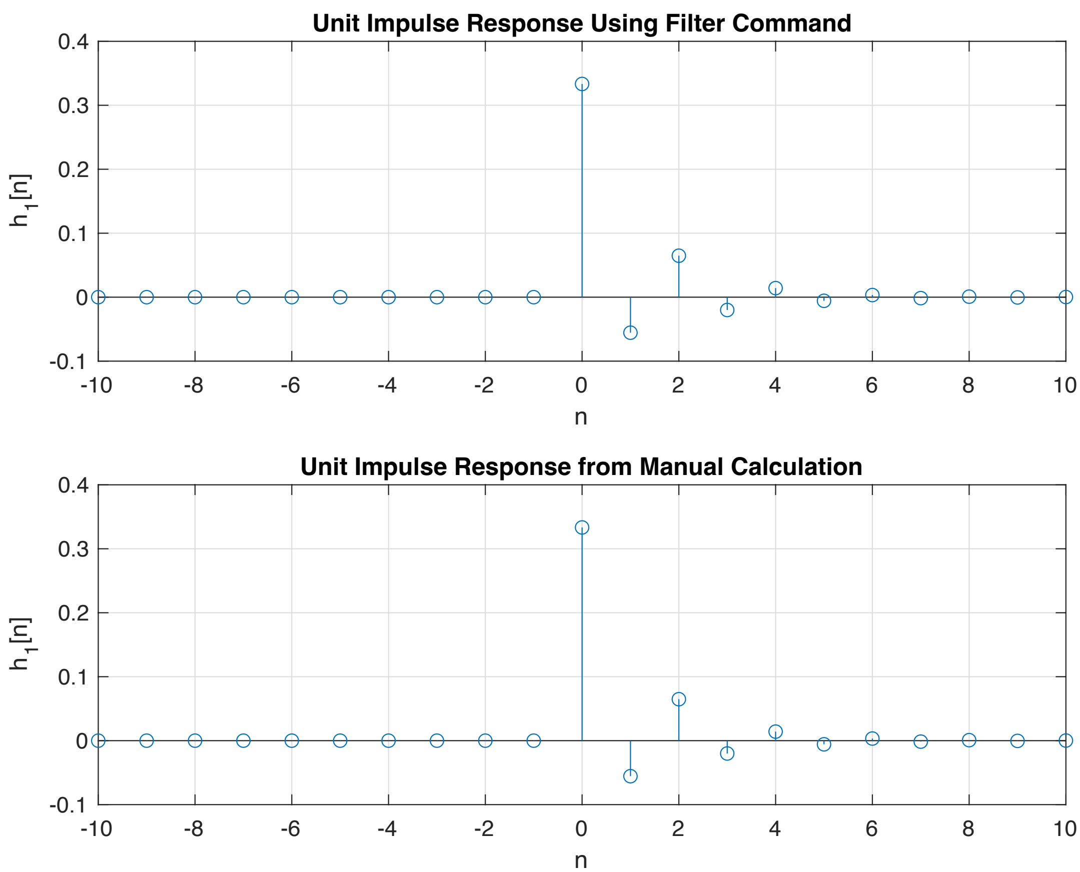
$$\begin{cases} c_1 + c_2 = 1 \\ c_1\left(-j\frac{1}{2}\right) + c_2\left(j\frac{1}{2}\right) = 0 \end{cases}$$

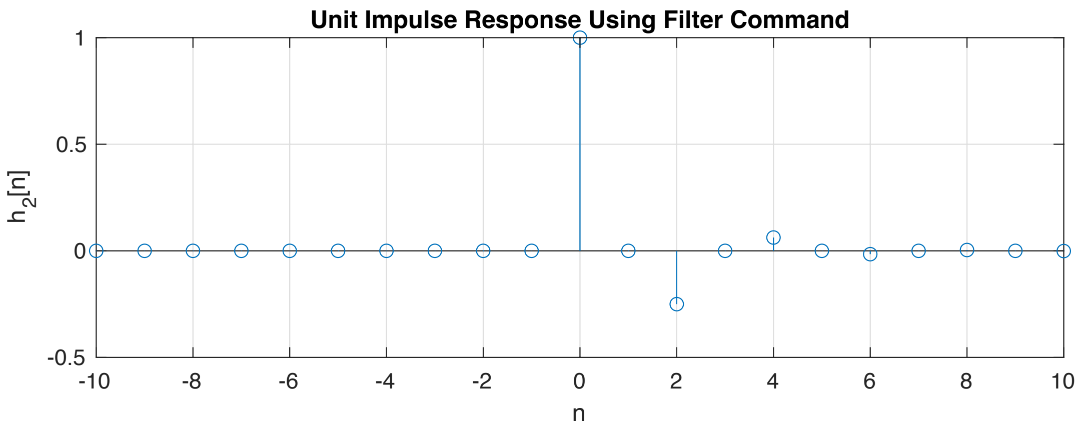
Solve the system of equations above, we get $c_1 = c_2 = \frac{1}{2}$

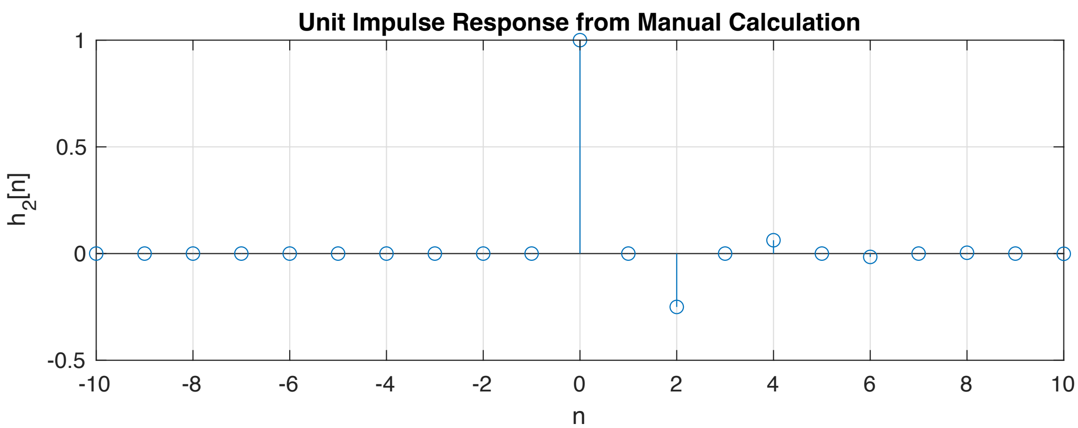
$$\therefore h[n] = \left[\left(\frac{1}{2}\right)\left(-j\frac{1}{2}\right)^n + \left(\frac{1}{2}\right)\left(j\frac{1}{2}\right)^n\right]u[n]$$

```
% A.1

n = -10:10; % Range for plotting
impulse = @(n) (n == 0) * 1.0 .* (mod(n, 1) == 0);
u = @(n) (n >= 0) * 1.0 .* (mod(n,1)==0);

% Determine h[n] using filter command
% A.1.I
b1 = [1/3 0 0]; % Numerator coefficients
a1 = [1 1/6 -1/6]; % Denominator coefficients
h1_filter = filter(b1,a1,impulse(n));

% A.1.II
b2 = [1 0 0]; % Numerator coefficients
a2 = [1 0 1/4]; % Denominator coefficients
h2_filter = filter(b2,a2,impulse(n));

% A.2.I
h1_calculated = @(n) ((2/15).*(1/3).^n + 0.2.*(-0.5).^n).*u(n);

% A.2.II
h2_calculated = @(n) 0.5*(((-j/2).^n)+((j/2).^n)).*u(n);

% Plotting

figure;
subplot(2,1,1);
stem(n,h1_filter);
grid;
title("Unit Impulse Response Using Filter Command");
xlabel('n');
ylabel('h_1[n]');

subplot(2,1,2);
stem(n,h1_calculated(n));
grid;
title("Unit Impulse Response from Manual Calculation");
xlabel('n');
ylabel('h_1[n]');
```

```
figure;
subplot(2,1,1);
stem(n,h2_filter);
grid;
title("Unit Impulse Response Using Filter Command");
xlabel('n');
ylabel('h_2[n]');

subplot(2,1,2);
stem(n,h2_calculated(n));
grid;
title("Unit Impulse Response from Manual Calculation");
xlabel('n');
ylabel('h_2[n]');
```








```
% h[n]s calculated by hand give same results as using filter
% command from Matlab.

% A.3

% Filtered data using filter command from Matlab, returned as a
% vector, matrix, or multidimensional array of the same size as
% the input data. Therefore, to extract the value of n = 3 in
% h1_filter and h2_filter, in the range [-10:10], the index is 14.
```

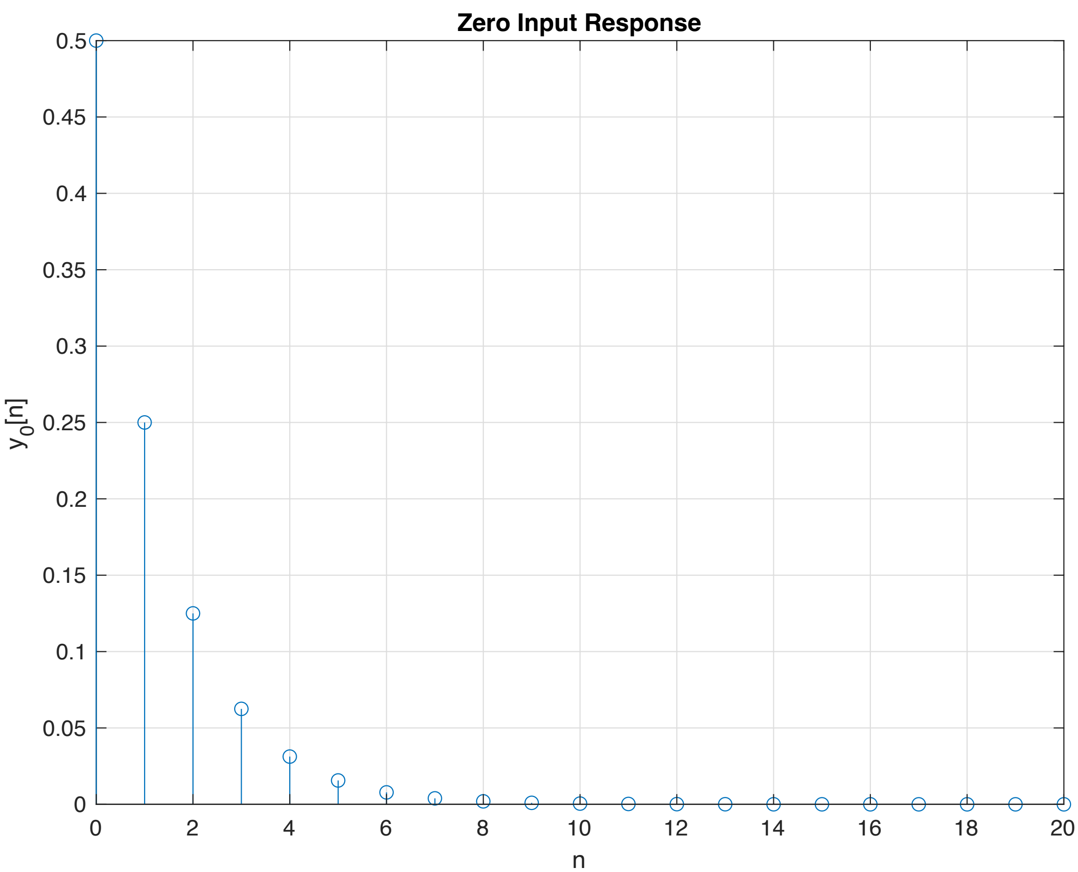
```
>> h1_filter(14)

ans =

   -0.0201

>> h1_calculated(3)

ans =

   -0.0201
```

```
>> h2_filter(14)

ans =

     0

>> h2_calculated(3)

ans =

     0
```

```
% From the plots or from Matlab command, the value of h[3] is
% the same in both methods.
```

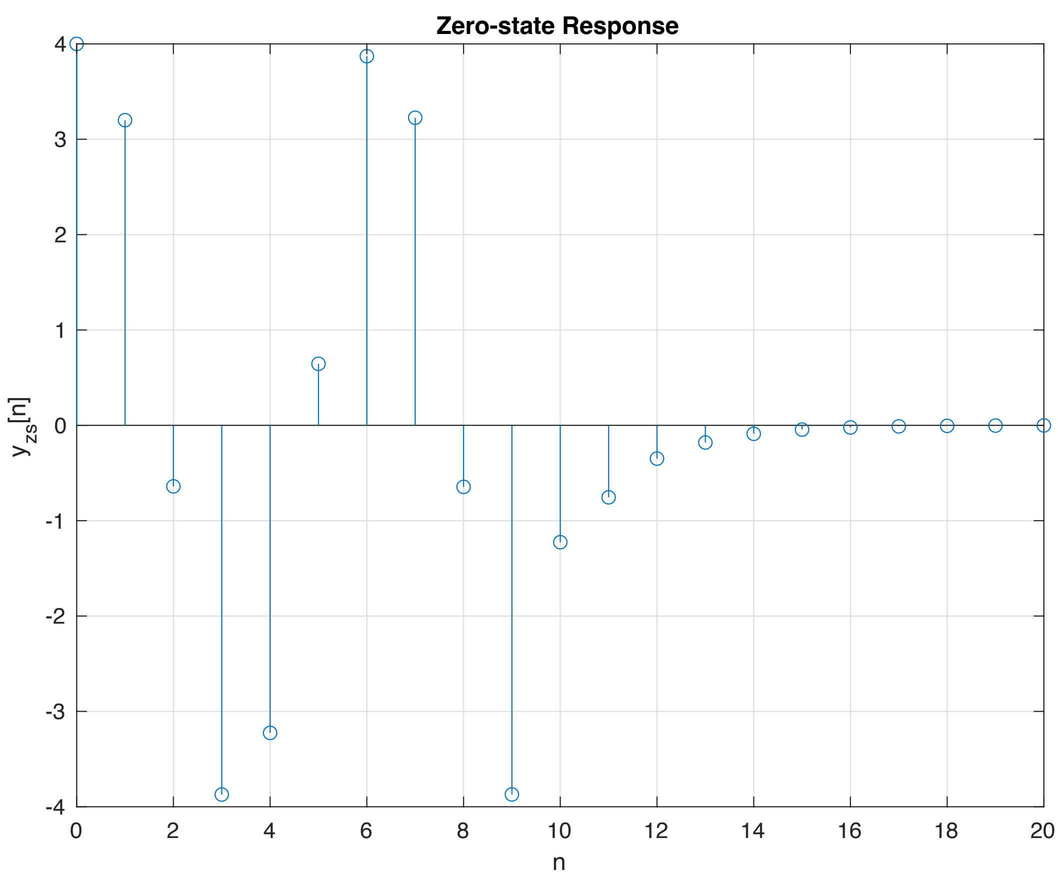
B. Zero input response

```
n = 0:20; % Range for plotting
b = [2]; % Numerator coefficients
a = [1, -3/10, -1/10]; % Denominator coefficients
y = [1, 2]; % Initial conditions for output
x_ic = filtic(b, a, y); % Finding initial conditions
m = zeros(1, length(n)); % The input is assigned as zeros
y_zi = filter(b, a, m, x_ic) ; % Zero input response

% Plotting
stem(n, y_zi);
grid;
title('Zero Input Response');
xlabel('n');
ylabel('y_0[n]');
```



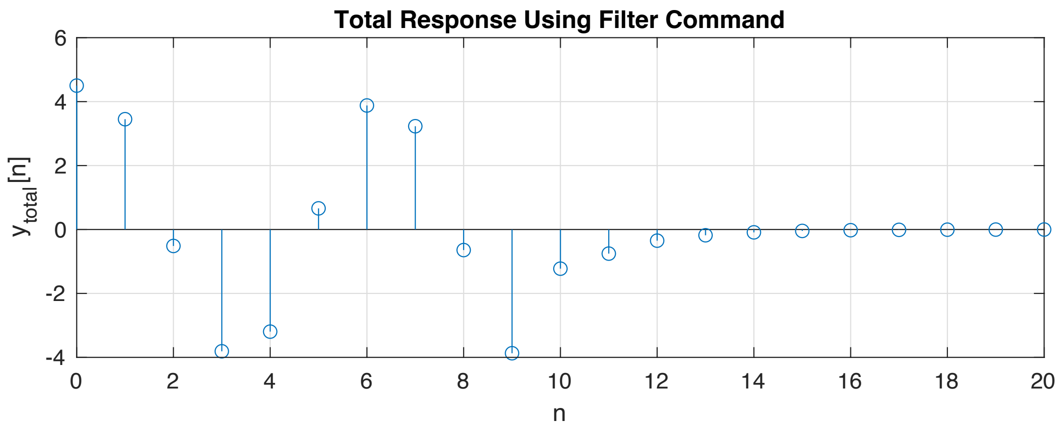
C. Zero-state response

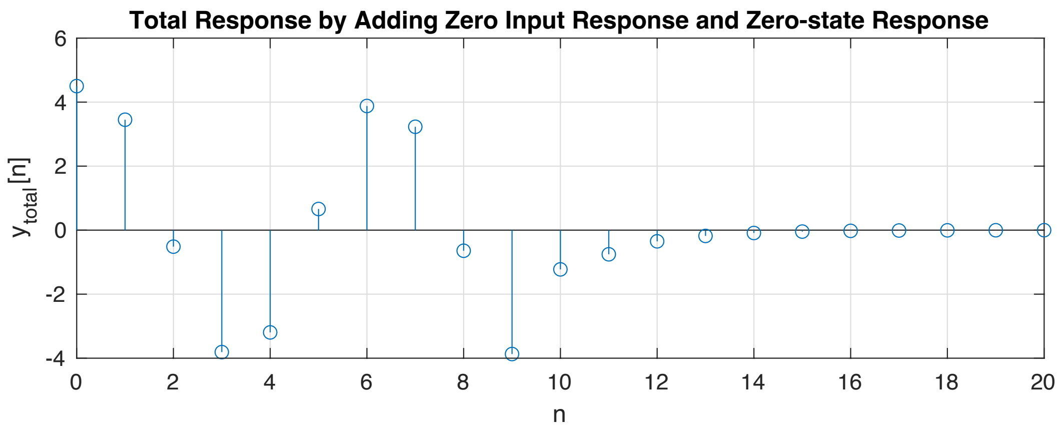
```
n = 0:20; % Range for plotting
u = @(n) (n >= 0) * 1.0 .* (mod(n,1)==0); % Unit step function
x = @(n) 2*cos((2/6)*pi*n).*(u(n) - u(n-10)); % Input function

b = [2]; % Numerator coefficients
a = [1, -3/10, -1/10]; % Denominator coefficients

y_zs = filter(b,a,x(n)); % Zero-state response
stem(n,y_zs)
title('Zero-state Response');
grid;
xlabel('n');
ylabel('y_{zs}[n]');
```



D. Total response

```
n = 0:20; % Range for plotting

% Input x[n] from part C
u = @(n) (n >= 0) * 1.0 .* (mod(n,1)==0);
x = @(n) 2*cos((2/6)*pi*n).*(u(n) - u(n-10));

% Zero input response from part B and zero-state response from part C
b = [2];
a = [1, -3/10, -1/10];
y = [1, 2];
x_ic = filtic(b, a, y);
m = zeros(1, length(n));
y_zi = filter(b, a, m, x_ic) ;
y_zs = filter(b,a,x(n));

% Total response using filter command
y_total = filter(b,a,x(n),x_ic);

% Total response by adding zero input response and zero-state response
y_add = y_zi + y_zs;

% Plotting
figure;
subplot(2,1,1);
stem(n,y_total);
grid;
title("Total Response Using Filter Command");
xlabel('n');
ylabel('y_{total}[n]');

subplot(2,1,2);
stem(n,y_add);
grid;
title("Total Response by Adding Zero Input Response and Zero-state
Response");
xlabel('n');
ylabel('y_{total}[n]');
```



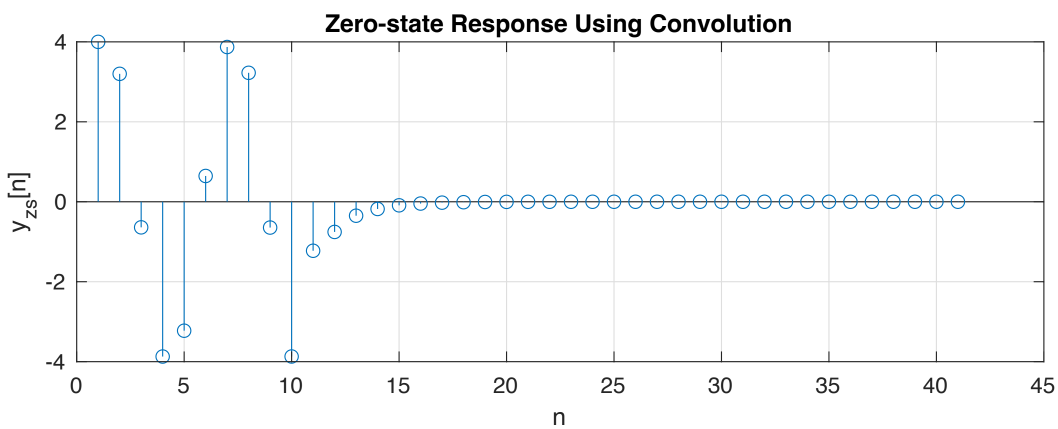


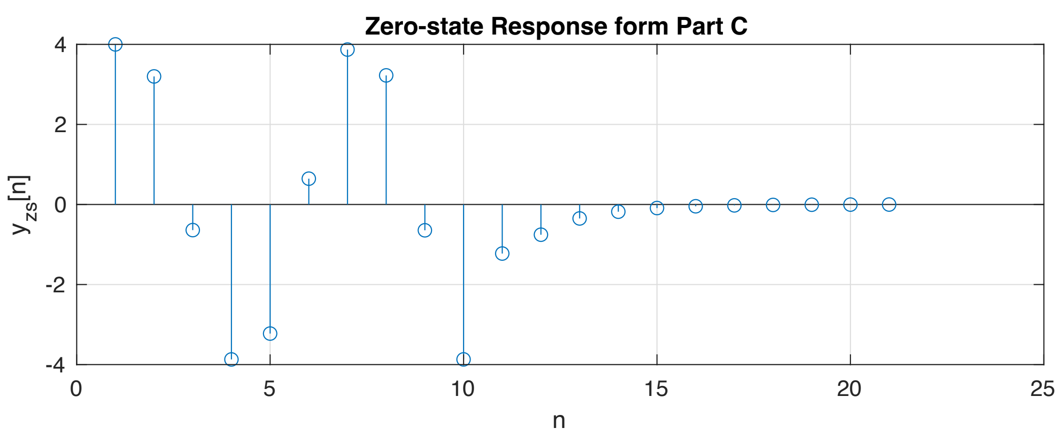

```
% Both methods give the same result for total response.
```

E. Convolution and system stability

```
% E.1

% System in part B
impulse = @(n) (n == 0) * 1.0 .* (mod(n, 1) == 0);
b = [2];
a = [1, -3/10, -1/10];
h = filter(b,a,impulse(n));   % Unit impulse response

% Input x[n] from part C
u = @(n) (n >= 0) * 1.0 .* (mod(n,1)==0);
x = @(n) 2*cos((2/6)*pi*n).*(u(n) - u(n-10));

% Zero-state response from part C
y_zs = filter(b,a,x(n));

% Zero-state response using convolution
y = @(n) conv(x(n), h);

% Plotting
figure;
```

```
subplot(2,1,1);
stem(y(n));
grid;
title("Zero-state Response Using Convolution");
xlabel('n');
ylabel('y_{zs}[n]');

subplot(2,1,2);
stem(y_zs)
title('Zero-state Response form Part C');
grid;
xlabel('n');
ylabel('y_{zs}[n]');
```






```
% E.2

% Both methods give the same result for zero-state response.

% E.3

% The system is asymptotically stable since the bended input converges to 0.
```

F. Moving average filter

Determine a constant coefficient difference equation that has impulse response

$$\boldsymbol{h[n] = \frac{u[n] - u[n-N]}{N}}$$

First, $h[n]$ can be rewritten as summation below

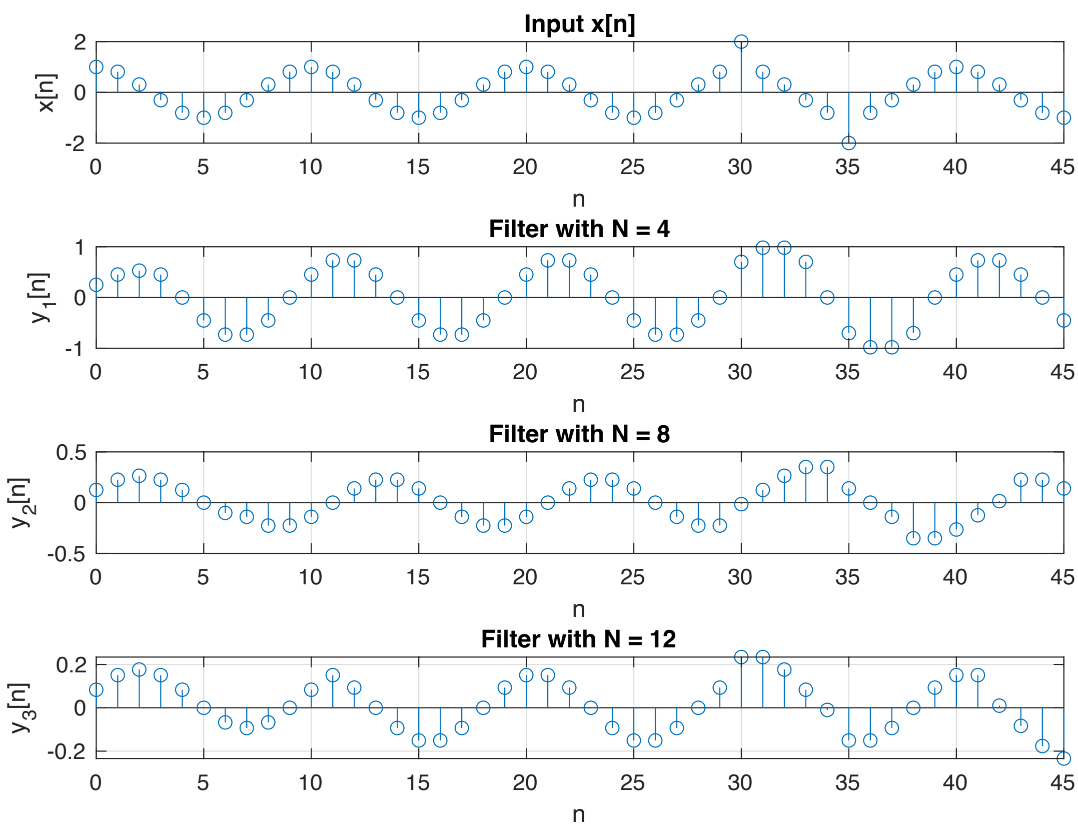
$$h[n] = \frac{u[n] - u[n-N]}{N} = \frac{1}{N}\sum_{k=0}^{N-1} \delta(n-k)$$

Replacing $h[n]$ with $y[n]$ and $\delta[n]$ with $x[n]$ gives the difference equation

$$y[n] = \frac{1}{N}\sum_{k=0}^{N-1} x(n-k)$$

```
% F.2

function [a,b] = params(N)
% This function will return the generated parameters a and b which
% correspond to the coefficients of the y[n] and x[n] respectively.

a = 1;
b = ones(N,1)/N;

% F.3

n = 0:45;

impulse = @(n) (n == 0) * 1.0 .* (mod(n, 1) == 0);
x = @(n) cos(pi*n/5)+impulse(n-30)-impulse(n-35);

[a1,b1] = params(4);
[a2,b2] = params(8);
[a3,b3] = params(12);

figure;
subplot(4,1,1);
stem(n,x(n));
grid;
title("Input x[n]");
xlabel('n');
ylabel('x[n]');

subplot(4,1,2);
stem(n,filter(b1,a1,x(n)));
grid;
title("Filter with N = 4");
xlabel('n');
ylabel('y_1[n]');

subplot(4,1,3);
stem(n,filter(b2,a2,x(n)));
```

```
grid;
title("Filter with N = 8");
xlabel('n');
ylabel('y_2[n]');

subplot(4,1,4);
stem(n,filter(b3,a3,x(n)));
grid;
title("Filter with N = 12");
xlabel('n');
ylabel('y_3[n]');
```




```
% The moving average filter is a simple Low Pass FIR filter. Therefore,
% a sinusoidal input reaches steady-state after N-1 sample. In addition,
% larger value of N results in filtered signal approaching to 0 and the
% effect of the impulse on the size of the output signal decreasing.
```