▷ Schwierige Integrale

1) $\int \frac{\arctan(x)}{1+x^2}\,dx$

Formelsammlung: $\arctan'(x) = \frac{1}{1+x^2}$

Substitution $u(x) = \arctan(x)$

$du = \arctan'(x)\,dx = \frac{1}{1+x^2}\,dx$

$= \int u\,du = \frac{u^2}{2} + C \overset{\text{Rücksubs.}}{=} \frac{1}{2}\arctan^2(x) + C.$

Probe: $\left[\frac{1}{2}\arctan^2(x) + C\right]' = \frac{1}{2}\cdot 2\cdot\arctan(x)\cdot\frac{1}{1+x^2} + 0 =$

$= \frac{\arctan(x)}{1+x^2}.$ ✓

2) $\int \sin^3(x)\,dx = \int \sin(x)\cdot\sin^2(x)\,dx = \int \sin(x)\left(1-\cos^2(x)\right)dx =$

$= \int \sin(x)\,dx - \int \sin(x)\cdot\cos^2(x)\,dx =$

Subs. $u(x) = \cos(x)\,dx$

$du = -\sin(x)\,dx$

$= -\cos(x) + \int u^2\,du = -\cos(x) + \frac{u^3}{3} = -\cos(x) + \frac{1}{3}\cos^3(x) + C.$

Probe: $\left[-\cos(x) + \frac{1}{3}\cos^3(x) + C\right]' = \sin(x) + \frac{1}{3}3\cos^2(x)(-\sin(x)) + 0 =$

$= \sin(x) - \left(1-\sin^2(x)\right)\sin(x) = \sin(x) - \sin(x) + \sin^3(x) =$

$= \sin^3(x).$ ✓

3) $\int e^{\sqrt{x}}\,dx$. Subst. $u = \sqrt{x} \Rightarrow du = \frac{1}{2}\frac{1}{\sqrt{x}}\,dx = \frac{1}{2u}\,dx$

$dx = 2u\,du$

$= \int e^u\cdot 2u\,du = 2\int u\cdot e^u\,du = 2u\cdot e^u - 2\int 1\cdot e^u\,du = 2ue^u - 2e^u =$

$\int f\cdot g' = f\cdot g - \int f'\cdot g$

$= 2\sqrt{x}\,e^{\sqrt{x}} - 2e^{\sqrt{x}} + C.$

Probe: $\left[2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + C\right]' = 2\frac{1}{2\sqrt{x}}e^{\sqrt{x}} + 2\sqrt{x}e^{\sqrt{x}}\frac{1}{2\sqrt{x}} -$

$- 2e^{\sqrt{x}}\frac{1}{2\sqrt{x}} + 0 = \frac{1}{\sqrt{x}}e^{\sqrt{x}} + e^{\sqrt{x}} - \frac{1}{\sqrt{x}}e^{\sqrt{x}} =$

$= e^{\sqrt{x}}.$ ✓

D **Partialbruchzerlegung** $\int \frac{4x-2}{x^2-2x-63}\,dx$

Nenner $x^2-2x-63$ in Linearfaktoren zerlegen, indem seine Nullstellen gefunden werden: $x^2-2x-63=0$

$$x_{1,2} = 1 \pm \sqrt{1+63} = 1 \pm 8 \quad \begin{cases} 9 = x_1 \\ -7 = x_2 \end{cases}$$

Partialbruchzerlegung: zwei einfache reelle Nullstellen

$$\frac{4x-2}{x^2-2x-63} = \frac{A}{x-9} + \frac{B}{x+7}$$

Verwende die Linearfaktorenzerlegung des Nenners

$$x^2-2x-63 = (x-9)\cdot(x+7).$$

$$\frac{4x-2}{(x-9)(x+7)} = \frac{A}{x-9} + \frac{B}{x+7} \quad \Big| \cdot (x-9)(x+7)$$

$$4x-2 = A(x+7) + B(x-9) \text{ gilt } \forall x \in \mathbb{R}, \text{z.B}$$

für $x_1 = 9$: $4\cdot 9-2 = A(9+7) + B\cdot(9-9)$

$$34 = 16A + 0, \quad \underline{A} = \frac{34}{16} = \underline{\frac{17}{8}}.$$

für $x_2 = -7$: $4(-7)-2 = A(-7+7) + B(-7-9)$

$$-30 = A\cdot 0 - 16B, \quad \underline{B} = \frac{-30}{-16} = \underline{\frac{15}{8}}.$$

Insgesamt: $\frac{4x-2}{x^2-2x-63} = \frac{17}{8}\cdot\frac{1}{x-9} + \frac{15}{8}\cdot\frac{1}{x+7}$

$$\int \frac{4x-2}{x^2-2x-63}\,dx = \frac{17}{8}\int \frac{1}{x-9}\,dx + \frac{15}{8}\int \frac{1}{x+7}\,dx =$$

$$= \underline{\frac{17}{8}\ln(|x-9|) + \frac{15}{8}\ln(|x+7|) + C.}$$

D Polynomdivision und Partialbruchzerlegung (PBZ):

$$I = \int \frac{x^3}{(x^2-1)(x+1)}\,dx .$$ Grad des Zählerpolynoms ist drei, gleich wie der Grad des Nenners $\Rightarrow$ Poly. div.

$$(x^2-1)(x+1) = x^3 + x^2 - x - 1$$

$$x^3 : (x^3+x^2-x-1) = 1 + \frac{-x^2+x+1}{x^3+x^2-x-1}$$

$$\underline{-x^3 - x^2 + x + 1}$$

$0 \quad -x^2 + x + 1$ Rest

$$I = \underbrace{\int 1\,dx}_{=x} + \underbrace{\int \frac{-x^2+x+1}{x^3+x^2-x-1}\,dx}_{PBZ} .$$ Nenner $x^3+x^2-x-1 = \underbrace{(x^2-1)}_{=0}\ \underbrace{(x+1)}_{=0}$ hat

Nullstellen $x^2 - 1 = 0$

$$x^2 = 1 \qquad\qquad x+1=0$$

$$x_{1,2} = \pm 1 \qquad\qquad x_3 = -1$$

D.h. $+1$ ist eine einfache Nullstelle, und -1 eine doppelte.

$$\frac{-x^2+x+1}{(x^2-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2} \quad \Big| \cdot \underbrace{(x^2-1)}_{(x+1)(x-1)}(x+1)$$

$$-x^2+x+1 = A(x+1)^2 + B(x-1)(x+1) + C(x-1)$$

$x=1$: $\quad -1+1+1 = 4A + 0 + 0 \Rightarrow A = \frac{1}{4}$.

$x=-1$: $\quad -1-1+1 = 0 + 0 - 2C \Rightarrow C = \frac{1}{2}$.

$x=0$: $\quad 1 = A - B - C$

$$B = A - C - 1 = \frac{1}{4} - \frac{1}{2} - 1 = \frac{1-2-4}{4} = -\frac{5}{4}.$$

$$I = x + \frac{1}{4}\int \frac{1}{x-1}\,dx - \frac{5}{4}\int \frac{1}{x+1}\,dx + \frac{1}{2}\underbrace{\int \frac{1}{(x+1)^2}\,dx}_{\frac{(x+1)^{-2+1}}{-2+1}} =$$

$$= \underline{x + \frac{1}{4}\ln(|x-1|) - \frac{5}{4}\ln(|x+1|) - \frac{1}{2}\cdot\frac{1}{x+1} + C}.$$

▷ Schwerpunkt: $y(x) = \sqrt{\cos(x)}$, $0 \le x \le \frac{\pi}{2}$, Rotation um x-Achse

$$V = \pi \int_a^b y(x)^2 \, dx = \pi \int_0^{\pi/2} \cos(x) \, dx = \pi \sin(x) \Big|_0^{\pi/2} =$$
$$= \pi \sin(\tfrac{\pi}{2}) - \pi \sin(0) = \pi$$

$$x_s = \frac{\pi}{V} \int_a^b x \cdot y(x)^2 \, dx = \frac{\pi}{\pi} \int_0^{\pi/2} \underbrace{x}_{f} \cdot \underbrace{\cos(x)}_{g'} \, dx =$$
$$= \underbrace{x \cdot \sin(x)}_{f \cdot g} \Big|_0^{\pi/2} - \underbrace{\int_0^{\pi/2} 1 \cdot \sin(x) \, dx}_{\int f' \cdot g} = \frac{\pi}{2} \cdot \sin(\tfrac{\pi}{2}) - 0 \cdot \sin(0) + \cos(x) \Big|_0^{\pi/2} =$$
$$= \frac{\pi}{2} + \cos(\tfrac{\pi}{2}) - \cos(0) = \frac{\pi}{2} - 1 \approx \underline{0{,}571.}$$

▷ Bogenlänge: $y(x) = x^{3/2}$, $1 \le x \le 7{,}45$. $y'(x) = \frac{3}{2} x^{1/2} = \frac{3}{2}\sqrt{x}$

$$S = \int_1^{7{,}45} \sqrt{1 + (\tfrac{3}{2}\sqrt{x})^2} \, dx = \int_1^{7{,}45} \sqrt{1 + \tfrac{9}{4} x} \, dx =$$
$$= \frac{(1 + \frac{9}{4} x)^{\frac{1}{2}+1}}{\frac{1}{2}+1} \cdot \frac{1}{\frac{9}{4}} \Bigg|_1^{7{,}45} = \frac{4}{9} \cdot \frac{2}{3} (1 + \tfrac{9}{4} x)^{3/2} \Big|_1^{7{,}45} =$$
$$= \frac{8}{27} \sqrt{(1 + \tfrac{9}{4} x)^3} \Big|_1^{7{,}45} = \underline{20{,}445.}$$

▷ Uneigentliches Integral:

$$\int_0^1 \frac{1}{x} \, dx = \lim_{\lambda \to 0, \lambda > 0} \int_\lambda^1 \frac{1}{x} \, dx = \lim_{\lambda \to 0, \lambda > 0} \ln(|x|) \Big|_\lambda^1 =$$
$$= \lim_{\lambda \to 0, \lambda > 0} \left[\underbrace{\ln(1)}_{=0} - \ln(\lambda) \right] = \lim_{\lambda \to 0, \lambda > 0} (\underbrace{-\ln(\lambda)}_{\to -\infty}) = \underline{\infty.}$$