▷ Quadratwurzeln

1.) von $z = 1 - j\sqrt{3}$. Verwende $z_{0,1} = \pm\left(\sqrt{\frac{|z|+x}{2}} + j\cdot \mathrm{sgn}(y)\sqrt{\frac{|z|-x}{2}}\right)$

vgl. $z = x + jy$

$x = 1,\ y = -\sqrt{3}$ mit $\mathrm{sgn}(y) = 1$ für $y \geq 0$ und $-1$ für $y < 0$

$|z| = \sqrt{1^2 + (-\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2.$

$z_{0,1} = \pm\left(\sqrt{\frac{2+1}{2}} + j\cdot(-1)\sqrt{\frac{2-1}{2}}\right)$

$\underline{z_{0,1}} = \pm\left(\sqrt{\frac{3}{2}} - j\sqrt{\frac{1}{2}}\right) = \underline{\pm\left(\sqrt{\frac{3}{2}} - j\frac{1}{\sqrt{2}}\right)}.$

2.) von $z = -5 + 12j$. Verwende dieselbe Formel

$z = x + j\cdot y$

$x = -5,\ y = 12$

$|z| = \sqrt{(-5)^2 + 12^2} = \sqrt{25+144} = \sqrt{169} = 13$

$z_{0,1} = \pm\left(\sqrt{\frac{13-5}{2}} + j\cdot 1\cdot\sqrt{\frac{13+5}{2}}\right)$

$\underline{z_{0,1} = \pm(2 + 3j).}$

Proben:

1.) $\left[\pm\left(\sqrt{\frac{3}{2}} - j\cdot\frac{1}{\sqrt{2}}\right)\right]^2 = \left(\sqrt{\frac{3}{2}} - j\cdot\frac{1}{\sqrt{2}}\right)^2 =$

$\left(\sqrt{\frac{3}{2}}\right)^2 - 2\sqrt{\frac{3}{2}}\,j\frac{1}{\sqrt{2}} + j^2\cdot\left(\frac{1}{\sqrt{2}}\right)^2 = \frac{3}{2} - j\sqrt{3} - \frac{1}{2} = \underline{1 - j\sqrt{3}}.$ ✓

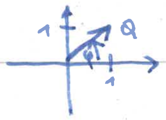
2.) $\left[\pm(2+3j)\right]^2 = (2+3j)^2 = 4 + 2\cdot 2\cdot 3j + 3^2 j^2 =$

$= 4 + 12j - 9 = \underline{-5 + 12j}.$ ✓

## D Wurzeln

1.) Bestimmen Sie alle dritten Wurzeln von $1+j$ in Polarform und in kartesischer Form. $a = 1+j = r \cdot e^{j\varphi}$

$r = \sqrt{1^2+1^2} = \sqrt{2}$,



$\varphi = 45^\circ = \frac{\pi}{4}$.

Formel $n=3$, $k=0,1,2$

$$z_k = \sqrt[n]{r} \cdot e^{j \cdot \frac{\varphi + k \cdot 2\pi}{n}}$$

$$z_0 = \sqrt[3]{\sqrt{2}} \cdot e^{j \frac{\frac{\pi}{4}+0\cdot 2\pi}{3}} = \sqrt[6]{2}\, e^{j\frac{\pi}{12}} =$$

$$= \sqrt[6]{2}\left(\cos\left(\tfrac{\pi}{12}\right) + j \cdot \sin\left(\tfrac{\pi}{12}\right)\right) \cong 1{,}0842 + 0{,}29051 j.$$

$$z_1 = \sqrt[3]{\sqrt{2}} \cdot e^{j \frac{\frac{\pi}{4}+1\cdot 2\pi}{3}} = \sqrt[6]{2} \cdot e^{j\frac{3\pi}{4}} =$$

$$= \sqrt[6]{2}\left(\cos\left(\tfrac{3\pi}{4}\right) + j \cdot \sin\left(\tfrac{3\pi}{4}\right)\right) \cong -0{,}7937 + 0{,}7937 j.$$

$$z_2 = \sqrt[3]{\sqrt{2}} \cdot e^{j \frac{\frac{\pi}{4}+2\cdot 2\pi}{3}} = \sqrt[6]{2} \cdot e^{j\frac{17\pi}{12}} =$$

$$= \sqrt[6]{2} \cdot \left(\cos\left(\tfrac{17\pi}{12}\right) + j \cdot \sin\left(\tfrac{17\pi}{12}\right)\right) \cong -0{,}29051 - 1{,}0842 j$$

2.) $a = -1 = r \cdot e^{j\varphi}$, $r = 1$, $\varphi = \pi$, dieselbe Formel mit $n = 5$ und $k = 0,1,2,3,4$

$$z_0 = \sqrt[5]{1}\, e^{j\frac{\pi + 0\cdot 2\pi}{5}} = e^{j\pi/5} = \cos\left(\tfrac{\pi}{5}\right) + j \sin\left(\tfrac{\pi}{5}\right) \cong 0{,}809 + 0{,}588 j.$$

$$z_1 = \sqrt[5]{1}\, e^{j\frac{\pi + 1\cdot 2\pi}{5}} = e^{j\frac{3\pi}{5}} = \cos\left(\tfrac{3\pi}{5}\right) + j \sin\left(\tfrac{3\pi}{5}\right) \cong -0{,}309 + 0{,}951 j.$$

$$z_2 = \sqrt[5]{1}\, e^{j\frac{\pi + 2\cdot 2\pi}{5}} = e^{j\pi} = \cos(\pi) + j \sin(\pi) = -1.$$

$$z_3 = \sqrt[5]{1}\, e^{j\frac{\pi + 3\cdot 2\pi}{5}} = e^{j\frac{7\pi}{5}} = \cos\left(\tfrac{7\pi}{5}\right) + j \cdot \sin\left(\tfrac{7\pi}{5}\right) \cong -0{,}309 - 0{,}951 j.$$

$$z_4 = \sqrt[5]{1}\, e^{j\frac{\pi + 4\cdot 2\pi}{5}} = e^{j\frac{9\pi}{5}} = \cos\left(\tfrac{9\pi}{5}\right) + j \cdot \sin\left(\tfrac{9\pi}{5}\right) \cong 0{,}809 - 0{,}588 j.$$

Beachte $\bar{z}_0 = z_{\cancel{5}\,4}$, $\bar{z}_1 = z_{\cancel{4}\,3}$, $\bar{z}_2 = z_2$, d.h. die konj. kompl. Lösungen sind wieder Lösungen der Gleichung $z^5 = -1$, weil die Koeffizienten der Gleichung reell sind: $z^5 + 1 = 0$.

## D Quadratische Gleichung

Lösen Sie $z^2 - 3z + 3 = j$, und machen Sie die Probe.

Vgl. $x^2 + px + q = 0$ mit $x_{1,2} = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}$

$$z^2 - 3z + 3 - j = 0$$

$$z_{1,2} = \frac{3}{2} \pm \sqrt{\frac{9}{4} - 3 + j} = \frac{3}{2} \pm \underbrace{\sqrt{\underbrace{-\frac{3}{4} + j}_{=:a}}}_{\text{Quadratwurzeln von } a}$$

Formel für $a = x + jy$: $\pm\left(\sqrt{\frac{|a|+x}{2}} + j \cdot \operatorname{sgn}(y)\sqrt{\frac{|a|-x}{2}}\right)$

$$|a| = \sqrt{\left(-\frac{3}{4}\right)^2 + 1^2} = \sqrt{\frac{9}{16} + \frac{16}{16}} = \sqrt{\frac{25}{16}} = \frac{5}{4}$$

$x = -\frac{3}{4}$, $y = 1$, $\operatorname{sgn}(y) = +1$

$$\pm\left(\sqrt{\frac{5/4 - 3/4}{2}} + j \cdot 1 \cdot \sqrt{\frac{5/4 + 3/4}{2}}\right) = \pm\left(\sqrt{\frac{1}{4}} + j \cdot \sqrt{1}\right) = \pm\left(\frac{1}{2} + j\right)$$

Daher ist

$z_{1,2} = \frac{3}{2} \pm \left(\frac{1}{2} + j\right)$. $\quad z_1 = \frac{3}{2} + \frac{1}{2} + j = \underline{2 + j}$.

$z_2 = \frac{3}{2} - \frac{1}{2} - j = \underline{1 - j}$.

<u>Proben</u>: Für $z_1$: $(2+j)^2 - 3(2+j) + 3 \overset{?}{=} j$

$$4 + 4j + j^2 - 6 - 3j + 3 \overset{?}{=} j$$

$$1 + j - 1 \overset{?}{=} j$$

$$j \overset{?}{=} j \quad ✓$$

Für $z_2$: $(1-j)^2 - 3(1-j) + 3 \overset{?}{=} j$

$$1 - 2j + j^2 - 3 + 3j + 3 \overset{?}{=} j$$

$$1 + j + j^2 \overset{?}{=} j$$

$$j \overset{?}{=} j \quad ✓$$

D) **Algebraische Gleichung**

Lösen Sie $z^4-3(1+2j)z^2-8+6j=0$, und machen Sie die Probe.

Setze $u:=z^2$: $u^2-3(1+2j)u-8+6j=0$

Vgl. $x^2+p\,x+q=0$ mit $x_{1,2}=-\frac{p}{2}\pm\sqrt{\left(\frac{p}{2}\right)^2-q}$

$$u_{1,2}=\frac{3(1+2j)}{2}\pm\sqrt{\underbrace{\frac{9(1+2j)^2}{4}+8-6j}}$$

$$=\frac{9(1+4j+4j^2)+32-24j}{4}=\frac{9+36j-36+32-24j}{4}=$$

$$=\frac{5+12j}{4}$$

$$u_{1,2}=\frac{3+6j}{2}\pm\frac{\sqrt{5+12j}}{2}.$$

$a=5+12j$ hat Quadratwurzeln

$$\pm\left(\sqrt{\frac{|a|+x}{2}}+j\cdot\mathrm{sgn}(y)\sqrt{\frac{|a|-x}{2}}\right)=$$

$\sqrt{5^2+12^2}=\sqrt{169}=13$

$$=\pm\left(\sqrt{\frac{13+5}{2}}+j\cdot 1\cdot\sqrt{\frac{13-5}{2}}\right)=$$

$$=\pm(3+j\cdot 2).$$

$$u_1=\frac{3+6j}{2}+\frac{3+2j}{2}=\frac{3+6j+3+2j}{2}=\frac{6+8j}{2}=\underline{3+4j}.$$

$$u_2=\frac{3+6j}{2}-\frac{3+2j}{2}=\frac{3+6j-3-2j}{2}=\frac{4j}{2}=\underline{2j}.$$

$$\underline{z_{1,2}}=\pm\sqrt{3+4j}=\pm\left(\sqrt{\frac{5+3}{2}}+j\cdot\mathrm{sgn}(4)\cdot\sqrt{\frac{5-3}{2}}\right)=$$

$$=\pm(2+j\cdot 1\cdot 1)=\underline{\pm(2+j)}.$$

$$z_{3,4}=\pm\sqrt{2j}=\pm\left(\sqrt{\frac{2+0}{2}}+j\cdot\mathrm{sgn}(2)\cdot\sqrt{\frac{2-0}{2}}\right)=$$

$$=\pm(1+j\cdot 1\cdot 1)=\underline{\pm(1+j)}.$$

Probe: $z_{1,2}$ einsetzen: $(2+j)^4-3(1+2j)(2+j)^2-8+6j=$

$=(4+4j+j^2)^2-3(1+2j)(4+4j+j^2)-8+6j=$

$=(3+4j)^2-3(1+2j)(3+4j)-8+6j=$

$=9+24j+16j^2-3(3+4j+6j+8j^2)-8+6j=$

$=9+24j-16-9-30j+24-8+6j=0+0j=0.$ ✓

$z_{3,4}$ einsetzen: $(1+j)^4 - 3(1+2j)(1+j)^2 - 8 + 6j =$

$= (1+2j+j^2)^2 - 3(1+2j)(1+2j+j^2) - 8 + 6j =$

$= (2j)^2 - 3(1+2j)\cdot 2j - 8 + 6j =$

$= 4j^2 - 6j - 12j^2 - 8 + 6j =$

$= -4 + 12 - 8 = 0.$ ✓

D <u>Wechselstromrechnung</u>: $y_1(t) = 20\cdot\sin(\omega t + \frac{\pi}{10})$

$y_2(t) = 15\cdot\sin(\omega t + \frac{\pi}{6})$

$y_1(t) = \mathrm{Im}\left(20\cdot\cos(\omega t + \frac{\pi}{10}) + j\cdot 20\sin(\omega t + \frac{\pi}{10})\right) = \mathrm{Im}\left(20\, e^{j(\omega t + \frac{\pi}{10})}\right) =$

$= \mathrm{Im}\left(20\cdot e^{j\frac{\pi}{10}}\cdot e^{j\omega t}\right)$

$y_2(t) = \mathrm{Im}\left(15\cdot\cos(\omega t + \frac{\pi}{6}) + j\cdot 15\cdot\sin(\omega t + \frac{\pi}{6})\right) = \mathrm{Im}\left(15\, e^{j(\omega t + \frac{\pi}{6})}\right) =$

$= \mathrm{Im}\left(15\cdot e^{j\frac{\pi}{6}}\cdot e^{j\omega t}\right)$

$y_1(t) + y_2(t) = \mathrm{Im}\left(20\, e^{j\frac{\pi}{10}}\cdot e^{j\omega t} + 15\cdot e^{j\frac{\pi}{6}}\cdot e^{j\omega t}\right) =$

$= \mathrm{Im}\left[\underbrace{\left(20 e^{j\frac{\pi}{10}} + 15 e^{j\frac{\pi}{6}}\right)}_{=A} e^{j\omega t}\right]$

$A = 20\left(\cos(\frac{\pi}{10}) + j\cdot\sin(\frac{\pi}{10})\right) + 15\left(\cos(\frac{\pi}{6}) + j\cdot\sin(\frac{\pi}{6})\right) \cong 32{,}01 + 13{,}68\, j$

in Polarform: $r = \sqrt{32{,}01^2 + 13{,}68^2} \cong 34{,}81$

$\varphi = \arctan\left(\frac{13{,}68}{32{,}01}\right) \cong 0{,}40 = 23{,}14°$

$y_1(t) + y_2(t) = \mathrm{Im}\left[34{,}81\, e^{j\cdot 0{,}40}\cdot e^{j\omega t}\right] = \mathrm{Im}\left[34{,}81\cdot e^{j(\omega t + 0{,}40)}\right]$

$= 34{,}81\cdot\sin(\omega t + 0{,}40) =$

$= 34{,}81\cdot\cos(\omega t + 0{,}40 - \frac{\pi}{2}) =$

$= \underline{34{,}81\cdot\cos(\omega t - 1{,}17)} = \underline{34{,}81\cos(\omega t - 66{,}86°)} =$

$= \underline{34{,}81\cdot\cos(\omega t + 293{,}14°)}.$

▷ <u>Wechselstromrechnung</u>: $u_1(t) = 100 \sin(\omega t)$

$$u_2(t) = 150 \cos(\omega t - \tfrac{\pi}{4})$$
$$= 150 \sin(\omega t - \tfrac{\pi}{4} + \tfrac{\pi}{2}) = 150 \sin(\omega t + \tfrac{\pi}{4})$$

$$u_1(t) = \operatorname{Im}\left(100 \cos(\omega t) + j \cdot 100 \sin(\omega t)\right) = \operatorname{Im}\left(100\, e^{j\omega t}\right).$$

$$u_2(t) = \operatorname{Im}\left(150 \cos(\omega t + \tfrac{\pi}{4}) + j \cdot 150 \sin(\omega t + \tfrac{\pi}{4})\right) = \operatorname{Im}\left(150\, e^{j(\omega t + \frac{\pi}{4})}\right) =$$
$$= \operatorname{Im}\left(150\, e^{j\frac{\pi}{4}} e^{j\omega t}\right).$$

$$u_1(t) + u_2(t) = \operatorname{Im}\left[100\, e^{j\omega t} + 150\, e^{j\frac{\pi}{4}} e^{j\omega t}\right] =$$
$$= \operatorname{Im}\left[\underbrace{(100 + 150\, e^{j\frac{\pi}{4}})}_{=A} \cdot e^{j\omega t}\right]$$

$$A = 100 + 150\, e^{j\frac{\pi}{4}} = 100 + 150\left(\cos(\tfrac{\pi}{4}) + j \sin(\tfrac{\pi}{4})\right) =$$
$$= 100 + 106{,}07 + j \cdot 106{,}07 = 206{,}07 + j \cdot 106{,}07$$

in Polarform: $r = \sqrt{206{,}07^2 + 106{,}07^2} = 231{,}77$

$$\varphi = \arctan\left(\frac{106{,}07}{206{,}07}\right) \approx 0{,}4754$$

$$A = 231{,}77 \cdot e^{j \cdot 0{,}4754}$$

$$u_1(t) + u_2(t) = \operatorname{Im}\left[231{,}77 \cdot e^{j \cdot 0{,}4754} \cdot e^{j\omega t}\right] =$$
$$= \operatorname{Im}\left[231{,}77 \cdot e^{j(\omega t + 0{,}4754)}\right] =$$
$$= \underline{231{,}77 \cdot \sin(\omega t + 0{,}4754)}.$$