Lecture 5
Time-Frequency Analysis and Wavelets

Jason J. Bramburger

With the Gabor transform we saw how to recover information about a signal in both the time and frequency domain. This was achieved by introducing the two key principles for joint time-frequency analysis: *translation* of a short time window and *scaling* of the short-time window to capture finer time resolution. The shortcoming of this method is that it trades off accuracy in time for accuracy in frequency and vice-versa. That is, a large window is great for the frequency resolution, but not for the time resolution. A small window is good for time resolution, but not for frequency.

One way to try to improve this method is to use windows of different sizes. We can start by extracting the low frequency components using very large windows. Then we can use successively smaller windows to get higher frequencies that are more localized in time. By keeping a catalogue of the extracting process, both excellent time and frequency resolution of a given signal can be obtained. This is the idea behind wavelets.

Mother Wavelet

We begin by considering a function to be shifted and scaled. There are many choices that for this function - infinitely many in fact - which we refer to as the **mother wavelet**. We will denote the mother wavelet by $\psi(t)$. The shifted and scaled version are written:

$$\psi_{a,b}(t) = \frac{1}{\sqrt{a}}\psi\left(\frac{t-b}{a}\right).$$

Here $a > 0$ is the **scaling parameter** and b is the **translation parameter**. The specific mother wavelet that is used it typically application dependent, but we will highlight some important ones here.

Haar Wavelet

Historically, the first wavelet was constructed by Haar in 1910. The **Haar wavelet** is named after him and given by the piecewise constant function

$$\psi(t) = \begin{cases} 1, & 0 \le t < 1/2 \\ -1, & 1/2 \le t < 1 \\ 0 & \text{otherwise} \end{cases}$$

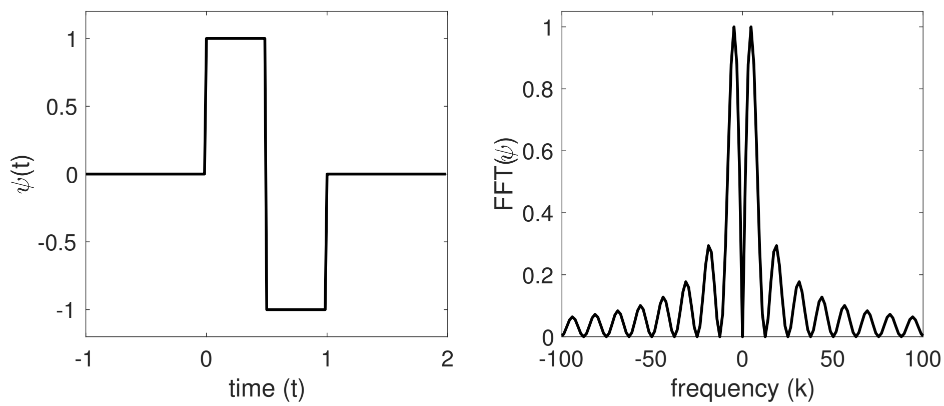
Note that $\int_{-\infty}^{\infty} \psi(t)\mathrm{d}t = 0$ and $\|\psi(t)\|_2^2 = \int_{-\infty}^{\infty} |\psi(t)|^2\mathrm{d}t = 1$. You should note that the Haar wavelet has compact support. That is, it is only nonzero on a bounded range of t values. This makes it ideal for describing localized signals in time (or space). However, from the Heisenberg/Fourier Uncertainty Principle, this localization in time gives that the Fourier transform of the Haar wavelet has poor localization properties in frequency space. This can be observed by computing the Fourier transform of the Haar wavelet, $\hat{\psi}(k)$, which is given by

$$\hat{\psi(k)} = \mathrm{i}e^{-\mathrm{i}k/2}\frac{\sin^2(k/4)}{k/4},$$

which behaves like the slowly decaying function $1/|k|$ as $|k| \to \infty$.

```
L = 4; n = 256;
t2 = linspace(-L/2,L/2,n+1); t = t2(1:n);
k = (2*pi/L)*[0:n/2-1 -n/2:-1];
ks = fftshift(k);

% Create Haar wavelet
```

```
H = zeros(length(t),1);
for j = 1:length(t)
    if t(j) ≥ 0 && t(j) < 0.5
        H(j) = 1;
    elseif t(j) ≥ 0.5 && t(j) < 1
        H(j) = -1;
    end
end

% Fourier transform of Haar wavelet
Ht = fft(H);

% Plot Haar wavelet in both time and frequency domain
figure(1)
subplot(1,2,1) % time domain
plot(t,H,'k','Linewidth',2)
set(gca,'Fontsize',16); xlabel('time (t)'); ylabel('\psi(t)')
axis([-1 2 -1.2 1.2])

subplot(1,2,2) % frequency domain
plot(ks,abs(fftshift(Ht))/max(abs(Ht)),'k','Linewidth',2)
set(gca,'Fontsize',16); xlabel('frequency (k)'); ylabel('FFT(\psi)')
axis([-100 100 0 1.05])
```



Haar Wavelet Basis

The Haar wavelet above can be used as the mother wavelet to create the Haar wavelet basis. That is, our $\psi(t)$ denotes $\psi_{1,0}(t)$ since the scaling is unity and the translation is zero. Then, the shifted and scaled Haar wavelets become

$$\psi_{a,b}(t) = \begin{cases} 1/\sqrt{a}, & b \le t < b + a/2 \\ -1/\sqrt{a}, & b + a/2 \le t < b + a \\ 0 & \text{otherwise} \end{cases}$$

when $a > 0$. For $0 < a < 1$ the wavelet is a compressed version of $\psi_{1,0}$, whereas for $a > 1$ the wavelet is a dilated version of $\psi_{1,0}$. In practice, the scaling parameter a is typically taken to be a power of 2, so that $a = 2^j$ for some integer j. The compressed wavelet allows for finer scale resolution of a given signal while the dilated wavelet captures low-frequency components of a signal having a broad range in time.

```
function H = Haar(t,a,b)
    % INPUTS:
    % t is the independent variable, input as a vector
    % a is the scale parameter
    % b is the translation parameter

    % OUTPUT:
    % H is the scaled and shifted Haar wavelet
```

```

    H = zeros(length(t),1);
    for j = 1:length(t)
        if t(j) ≥ b && t(j) < b + 0.5*a
            H(j) = 1/sqrt(abs(a));
        elseif t(j) ≥ b + 0.5*a && t(j) < b + a
            H(j) = -1/sqrt(abs(a));
        end
    end

end
```

```
% Create the scaled and shifted Haar wavelets
H1 = Haar(t,0.5,0);
H1t = fft(H1);
H2 = Haar(t,2,0);
H2t = fft(H2);
H3 = Haar(t,2,-0.5);
H3t = fft(H3);

figure(1)
subplot(3,2,1) % time domain
plot(t,H1,'k','Linewidth',2)
set(gca,'Fontsize',16); xlabel('time (t)'); ylabel('\psi(t)')
axis([-1 3 -2 2])

subplot(3,2,2) % frequency domain
plot(ks,abs(fftshift(H1t))/max(abs(H1t)),'k','Linewidth',2)
set(gca,'Fontsize',16); xlabel('frequency (k)'); ylabel('FFT(\psi)')
axis([-100 100 0 1.05])

subplot(3,2,3) % time domain
plot(t,H2,'k','Linewidth',2)
set(gca,'Fontsize',16); xlabel('time (t)'); ylabel('\psi(t)')
axis([-1 3 -2 2])

subplot(3,2,4) % frequency domain
plot(ks,abs(fftshift(H2t))/max(abs(H2t)),'k','Linewidth',2)
set(gca,'Fontsize',16); xlabel('frequency (k)'); ylabel('FFT(\psi)')
axis([-100 100 0 1.05])

subplot(3,2,5) % time domain
plot(t,H3,'k','Linewidth',2)
set(gca,'Fontsize',16); xlabel('time (t)'); ylabel('\psi(t)')
axis([-1 3 -2 2])

subplot(3,2,6) % frequency domain
plot(ks,abs(fftshift(H3t))/max(abs(H3t)),'k','Linewidth',2)
set(gca,'Fontsize',16); xlabel('frequency (k)'); ylabel('FFT(\psi)')
axis([-100 100 0 1.05])
```

Admissible Mother Wavelets

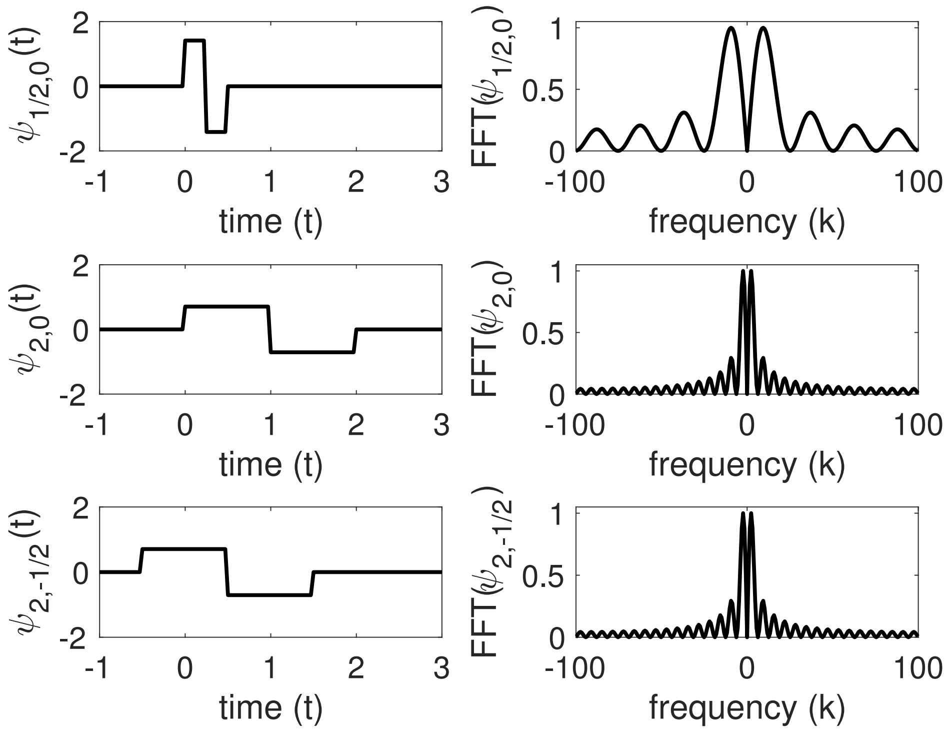
Not all functions can be used as a mother wavelet. We require the following for a wavelet to be admissible:

1. $\int_{-\infty}^{\infty} \psi(t)\mathrm{d}t = 0$. In words, the average of the wavelet is zero.

2. $\|\psi(t)\|_2^2 = \int_{-\infty}^{\infty} |\psi(t)|^2 \mathrm{d}t = 1$. This is more convention than actually a criteria. We do require mother wavelet to be such that $\|\psi(t)\|_2^2 < \infty$, and then we can re-scale the wavelet so that this condition holds.

3. We refer to the following as the **admissibility condition**:

$$C_\psi := \int_{-\infty}^{\infty} \frac{|\hat{\psi}(k)|^2}{k} \mathrm{d}k < \infty.$$



Essentially, this means that the Fourier transform of ψ has to decay fast enough as $|k| \to \infty$. Notice that this requires $\hat{\psi}(0) = 0$, which therefore implies the first condition. Indeed, using the Fourier transform we get

$$\hat{\psi}(0) = \int_{-\infty}^{\infty} \psi(t) \mathrm{d}k,$$

which must be zero.

4. Although not required, it is always nice to have a wavelet with compact support. That means that $\psi(t)$ is zero outside of some interval. You should note that the Gaussian function function used in the Gabor transform does not have compact support.

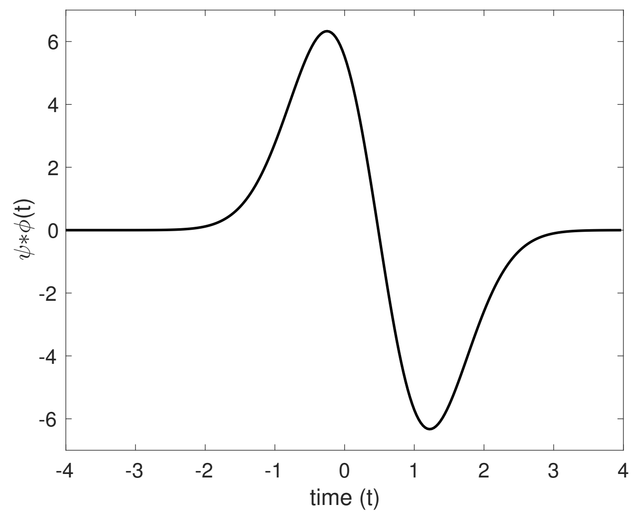
Creating New Wavelets

An interesting property of wavelets is that you can use them to create new wavelets. That is, if ψ is a wavelet and φ is a bounded function satisfying $\int_{-\infty}^{\infty} |\varphi(t)| \mathrm{d}t < \infty$, then the convolution $\psi * \varphi$ is also a wavelet. By saying that $\psi * \varphi$ is a wavelet, we mean that it satisfies (at least) properties (1)-(3) above.

Let's use the Haar wavelet ψ to construct a new wavelet by finding the convolution with the Gaussian $\varphi(t) = \mathrm{e}^{-t^2}$. To find the convolution $\psi * \varphi$ we can take advantage of the fact that the Fourier transform of a convolution is the product of the Fourier transforms. Therefore, we can find $\hat{\psi}$ and $\hat{\varphi}$ individually, multiply them together and then take the inverse Fourier transform to find $\psi * \varphi$.

```
% Create the Gaussian and fft
phi = exp(-t.^2)';
phit = fft(phi);

% Multiply transforms and apply ifft
convt = Ht.*phit;
conv = ifft(convt);

figure(3)
plot(t,ifftshift(conv),'k','Linewidth',2)
```

```
set(gca,'Fontsize',16); xlabel('time (t)'); ylabel('\psi(t)')
axis([-4 4 -7 7])
```



Continuous Wavelet Transform (CWT)

Once you have a mother wavelet, the strategy of the wavelet transform is to expand your function (or signal) as the sum of shifted and scaled versions of the mother wavelet. This is the same idea as what we did with the Fourier and Gabor transforms, and leads to the **continuous wavelet transform (CWT)**:

$$\mathcal{W}_\psi[f](a,b) = \int_{-\infty}^{\infty} f(t)\psi_{a,b}(t)\mathrm{d}t.$$

The **inverse CWT** is given by

$$f(t) = \frac{1}{C_\psi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\mathcal{W}_\psi[f](a,b)\psi_{a,b}(t)}{a^2}\mathrm{d}b\mathrm{d}a.$$

The CWT has the following properties:

1. Linearity: $\mathcal{W}_\psi[\alpha f + \beta g](a,b) = \alpha\mathcal{W}_\psi[f](a,b) + \beta\mathcal{W}_\psi[g](a,b)$.
2. Translation: $\mathcal{W}_\psi[f(t-c)](a,b) = \mathcal{W}_\psi[f](a,b-c)$ for any $c \in \mathbb{R}$.
3. Dilation: $\mathcal{W}_\psi[(1/c)f(t/c)](a,b) = (1/\sqrt{c})\mathcal{W}_\psi[f](a/c,b/c)$ for any $c > 0$.

The Mexican Hat Wavelet

Another commonly used wavelet is the Mexican hat wavelet:

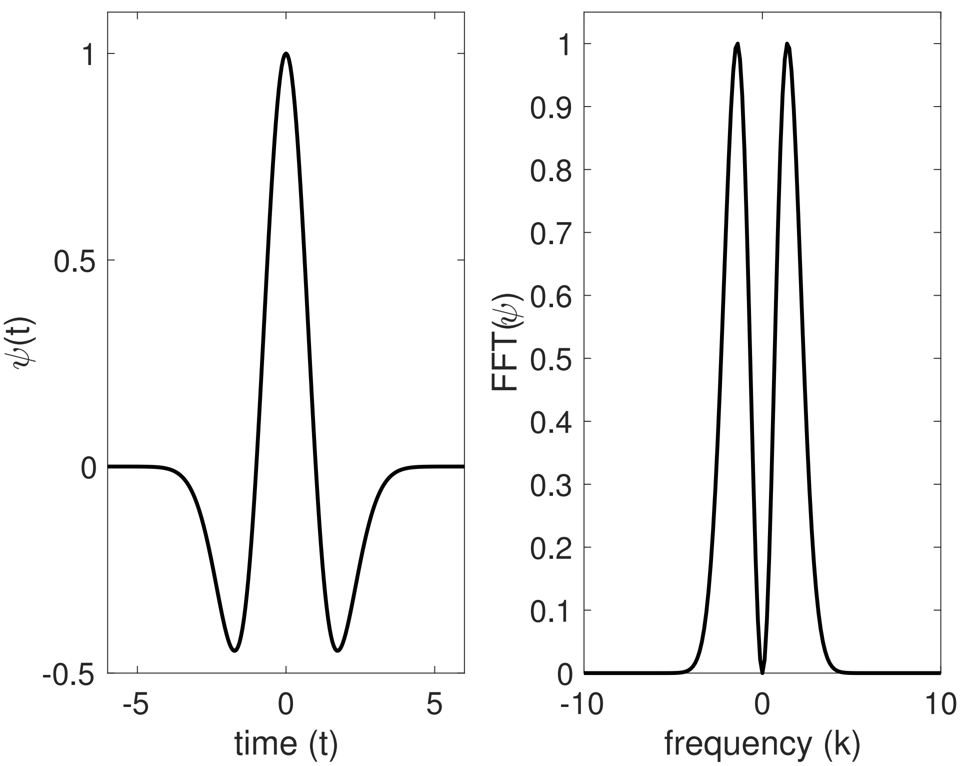
$$\psi(t) = (1-t^2)\mathrm{e}^{-t^2/2},$$

which is the second derivative of the Gaussian $\mathrm{e}^{-t^2/2}$. It has excellent localization properties, but you should notice that it is not compactly supported. This is true in both time and frequency space since the Fourier transform is given by

$$\hat{\psi} = \sqrt{2\pi}k^2\mathrm{e}^{-k^2/2}.$$

```
L = 50; n = 1024;
t2 = linspace(-L/2,L/2,n+1); t = t2(1:n);
k = (2*pi/L)*[0:n/2-1 -n/2:-1];
ks = fftshift(k);

mex = (1 - t.^2).*exp(-0.5*t.^2);
```

```
mext = fft(mex);

% Plot Mexican hat wavelet in both time and frequency domain
figure(1)
subplot(1,2,1) % time domain
plot(t,mex,'k','Linewidth',2)
set(gca,'Fontsize',16); xlabel('time (t)'); ylabel('\psi(t)')
axis([-6 6 -0.5 1.2])

subplot(1,2,2) % frequency domain
plot(ks,abs(fftshift(mext))/max(abs(mext)),'k','Linewidth',2)
set(gca,'Fontsize',16); xlabel('frequency (k)'); ylabel('FFT(\psi)')
axis([-10 10 0 1.05])
```



Discretization on a Computational Grid

As with the Fourier and Gabor transforms, we need to consider discretizing the wavelet transform on a computational grid. If we consider pairs of integers (m, n) on a computational lattice of shift and scale parameters we get

$$\psi_{m,n}(t) = a_0^{-m/2}\phi(a_0^{-m}t - nb_0)$$

where $a_0, b_0 > 0$ are the base shift and scale parameters. The **discrete wavelet transform** is then defined by

$$\begin{aligned}\mathcal{W}_\phi[f](m,n) &= \int_{-\infty}^{\infty} f(t)\psi_{m,n}(t)\mathrm{d}t \\ &= a_0^{-m/2}\int_{-\infty}^{\infty} f(t)\psi(a_0^{-m}t - nb_0)\mathrm{d}t.\end{aligned}$$

If the $\psi_{m,n}$ form a complete basis, then a given signal or function can be expanded in the wavelet basis:

$$f(t) = \sum_{m,n=-\infty}^{\infty} \mathcal{W}_\phi[f](m,n)\psi_{m,n}(t).$$

This is just like how the Fourier series expands a function as the sum of sines and cosines, except now the expansion is done in wavelets of varying window size and shifts. In practice one typically takes $a_0 = 2$ and $b = 1$, corresponding to dilations of 2^{-m} and translations of $n2^m$.