MATHEMATICS

Homework 3. Sequences

In the questions 1, 2 and 3 you have to choose one correct answer from the list, in the questions 4 and 5 you have to give a solution.

Question 1. Choose one true proposition.

- A All terms of a sequence must be positive.
- B All terms of a sequence must be integer.
- C A sequence can have both an explicit and a recursive definition.
- D A sequence can have at most two recursive relations.
- E Initial values are not always necessary to define a sequence recursively.

Question 2. Choose a closed-form expression for the nth term of the sequence $\{a_n\}$, where $n \in \mathbb{N}$, whose first four terms are $7, 19, 39, 67, \ldots$.

- A $a_n = 7n$.
- B $a_n = 2 \cdot 3^n + 1$.
- C $a_n = 4n^2 + 3$.
- D $a_n = 2n^3 + 1$.
- E $a_n = 2^{2n} + 5$.

Question 3. Let s_n be the number of n-digit numbers such that every digit is 0, 1, 2 or 3 and the sequence "00" never appears (note that a number cannot start with 0). Choose one true proposition.

- A $s_n = 3s_{n-1} + 3$ for all $n > 2$.
- B $s_n = 3s_{n-1} + s_{n-2}$ for all $n > 2$.
- C $s_n = s_{n-1} + 3s_{n-2}$ for all $n > 2$.
- D $s_n = 3s_{n-1} + 3s_{n-2}$ for all $n > 2$.
- E All the four previous relations are false.

Question 4. Prove that the Fibonacci sequence satisfies the following relation for every $n \geq 1$:

$$F_{n+7} = F_{n+5} + 4F_{n+3} + F_n.$$

Question 5. Solve the recurrence relation

$$2a_n = 7a_{n-2} - 3a_{n-1}$$

with $a_1 = 1$ and $a_2 = 3$.

Question 4 - Answer.

Prove that the Fibonacci sequence satisfies the following relation for every $n \geq 1$:

$$F_{n+7} = F_{n+5} + 4F_{n+3} + F_n$$

Prove by induction

When n = 1,

$F_8 = F_6 + 4F_4 + 1$

F8 = 21

F6 = 8

F4 = 3

21 = 8 + 12 + 1

Thus the formula holds for n = 1

Assume that the relation holds for some arbitrary positive integer k, n = k

That is:

$$F_{k+7} = F_{k+5} + 4F_{k+3} + F_k$$

Let us then show that it is also true for n = k + 1:

$$F_{(k+1)+7} = F_{(k+1)+5} + 4F_{(k+1)+3} + F_{(k+1)}$$

$F_{(k+1)+5} + 4F_{(k+1)+3} + F_{(k+1)} = F_{(k+6)} + 4F_{(k+4)} + F_{(k+1)}$

$F_{(k+7)} + F_{(k+6)} = F_{(k+6)} + 4F_{(k+4)} + F_{(k+1)}$

$F_{(k+7)} + F_{(k+6)} = F_{(k+5)} + F_{(k+4)} + 4F_{(k+4)} + F_{(k+1)}$

$F_{(k+1)} = F_k + F_{k-1}$

$F_{(k+4)} + 4F_{(k+4)} = 5F_{(k+4)}$

$F_{(k+7)} + F_{(k+6)} = F_{(k+5)} + 5F_{(k+4)} + F_k + F_{k-1}$

$F_{(k+7)} + F_{(k+6)} = F_{(k+5)} + 4F_{(k+3)} + F_k$

By the Principle of Mathematical Induction, we conclude that

$F_{n+7} = F_{n+5} + 4F_{n+3} + F_n$ for every integer n >= 1

Question 5 – Answer.

Rewrite the recurrence relation:

$2a_n + 3a_{n-1} – 7a_{n-2} = 0$

Form the characteristic equation:

$2x^2 + 3x - 7 = 0$

Solve for x:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-3 \pm \sqrt{3^2 - 4 \times 2 \times (-7)}}{2 \times 2}$$

The roots of the characteristic equation:

$$x = \frac{-3 + \sqrt{65}}{4}$$

$$x = \frac{-3 - \sqrt{65}}{4}$$

Recurrence relation: $a_n = Ar_1^n + Br_2^n$

When n = 1,

$$a1 = A\frac{-3 + \sqrt{65}}{4} + B\frac{-3 - \sqrt{65}}{4} = 1$$

When n = 2,

$$a2 = A \times \left(\frac{-3 + \sqrt{65}}{4}\right)^2 + B \times \left(\frac{-3 - \sqrt{65}}{4}\right)^2 = 3$$

by equations simultaneously:

$$A = \frac{4 - \sqrt{65}}{8}$$

$$B = \frac{4 + \sqrt{65}}{8}$$

So the solution to the recurrence relation is:

$$a_n = \frac{4 - \sqrt{65}}{8} \times \left(\frac{-3 + \sqrt{65}}{4}\right)^n + \frac{4 + \sqrt{65}}{8} \times \left(\frac{-3 - \sqrt{65}}{4}\right)^n$$